Technical Notes and Correspondence

On Undershoot and Nonminimum Phase Zeros

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Abstract—**In this note we derive a simple necessary and sufficient condition for a stable system to exhibit an undershooting step response. Specifically, we show that undershoot occurs if and only if the plant has an odd number of real right-half plane zeros.**

Consider a lumped scalar system with a strictly proper transfer function $p(s)$, and suppose the system is stable. Let $y(\cdot)$ denote the step response of the system. Then by stability, the limit $y(\infty)$ is well-defined and equals $p(0)$. Let r denote the relative degree of p. Then y and its first $r-1$ derivatives are zero at $t=0$, and $y^{(r)}(0)$ is the first nonzero derivative. If $r=1$ then the step response is continuous at $t=0$ but $\dot{y}(0)\neq 0$. Within the control community, a folklore definition is that the step response exhibits "undershoot" if it "initially starts off in the wrong direction." However, to date no precise definition is available. For the purposes of this note, we adopt the definition that the step response exhibits undershoot if its steady-state value has a sign opposite from that of its first nonzero derivative at time $t=0$. Thus, we define a system to *have undershoot* if $y^{(r)}(0)y(\infty)<0$. Clearly, this definition only makes sense if $p(0)=y(\infty)\neq 0$. This is a natural mathematical version of "the step response initially starts in the wrong direction." Then we have the following result, which is very easy to prove but does not seem to appear anywhere.

Proposition: The system has undershoot if and only if its transfer function has an odd number of real RHP zeros.

Proof: We can assume that $p(0)=1$ without loss of generality, since the presence or absence of undershoot is not affected by dividing $p(s)$ by a nonzero constant. As for $y^{(r)}(0)$, the initial value theorem tells us that

$$y^{(r)}(0)=\lim_{s\to\infty} s^r p(s). \tag{1}$$

Now write $p(s)$ in the form

$$p(s)=\frac{\prod_{i=1}^{n}\left[1-\frac{s}{z_i}\right]}{\prod_{i=1}^{n+r}\left[1-\frac{s}{p_i}\right]}. \tag{2}$$

The numerator terms can be grouped into three types: i) those corresponding to positive real zeros, ii) those corresponding to negative real zeros, and iii) those corresponding to complex zeros. Now the first are of the form

$$1-\alpha_i s. \tag{3}$$

For some positive α_i, while the second are of the form

$$1+\alpha_i s \tag{4}$$

for some positive α_i. The third terms are of the form

$$1+\beta_i s+\alpha_i s^2, \tag{5}$$

for some positive α_i, although of course β_i could be negative. Also, we must have $\beta_i^2<4\alpha_i$ to ensure that the corresponding zero is complex. On the other hand, by the assumption that p is stable, all terms in the denominator are of the form (4) or (5). Since

$$y^r(0)=\lim_{s\to\infty} s^r p(s), \tag{6}$$

we see that the sign of $y^{(r)}(0)$ is determined solely by the number of terms of the type (3). Specifically, if their number is odd, then $y^{(r)}(0)$ is negative, whereas if their number is even, then $y^{(r)}(0)$ is positive. Since $y(\infty)=p(0)=1$, we have the desired conclusion.

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Simultaneous Partial Pole Placement: A New Approach to Multimode System Design

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Abstract—**Simultaneous partial pole placement of a family of single-input single-output plants is proposed as a generalization of the classical pole placement and stabilization problems. This problem finds application in the design of a compensator for a family of linear dynamical systems. In this note we show that the proposed problem is equivalent to a new class of transcendental problem using stable, minimum phase rational functions with real coefficients. A necessary condition for the solvability of the associated transcendental problem is obtained. Finally, a counterexample to the following conjecture is obtained—"pairs of simultaneously stabilizable plants of bounded McMillan degree have simultaneously stabilizing compensators of bounded McMillan degree."**

Notations

- $\mathfrak{C}$ The complex plane.
- $\mathfrak{C}_s$ A self conjugate open subset of the complex plane which intersects the real axis $\mathfrak{R}$.
- $\mathfrak{C}_u$ $[\mathfrak{C}\cup\{\infty\}]-\mathfrak{C}_s$.
- C^- Open left half of the complex plane.
- $\mathfrak{C}^+$ Closed right half of the complex plane including infinity.
- ID^- Open interior of the unit disk.
- ID^+ Closed exterior of the unit disk including infinity.
- H Ring of proper rational functions with real coefficients with poles in $\mathfrak{C}_s$.
- J Set of multiplicative units in H.
- Λ A parameter set.

I. Statement of the Simultaneous Partial Pole Placement Problem

The simultaneous partial pole placement problem consists of answering the following question.

"Given a family $g_\lambda(s)$ of single-input single-output proper transfer functions of degree n_λ, $\lambda\in\Lambda$, respectively. Given a family of nonzero biproper (i.e., proper but not strictly proper) transfer functions $\psi_\lambda(s)$ of degree d_λ, $\lambda\in\Lambda$, respectively, with poles in a suitable open subset $\mathfrak{C}_s$ of the complex plane $\mathfrak{C}$ and zeros in $\mathfrak{C}_u$. Does there exist a proper compensator $k(s)$ of degree $q\geq\max\,[d_\lambda-n_\lambda]$ such that the closed-loop systems $g_\lambda(s)[1+k(s)g_\lambda(s)]^{-1}$, $\lambda\in\Lambda$ have, respectively, d_λ poles in $\mathfrak{C}_u$ where $\psi_\lambda(s)$ vanishes and all but the above d_λ poles are in $\mathfrak{C}_s$ for $\lambda\in\Lambda$, respectively?"

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