C3-SDP: Multi-Contact Consensus Complementarity Control via ADMM and Moment Relaxation

Xihang Yu*

College of Literature, Science, and the Arts, University of Michigan.

Abstract

Aydinoglu's pioneering work [1] solves Model Predictive Control (MPC) via ADMM which achieves real-time performance. However, a nonconvex projection subproblem in ADMM remains the hardest part and a bottle neck of real-time implementation for higher dimension control. In [1] proposes three methods to solve a projection subproblem in ADMM - Mixed Integer Quadratic Programming (MIQP) projection, ADMM projection, LCP projection. Despite faster performance in ADMM and LCP projections, these methods suffer from robustness [1]. MIQP prjection, on the other hand, is robust to all experiments as shown in [1] thanks to optimal guarantee. However, MIQP projection is slow due to the exponential worst run-time. In the pursuit of enhancing the efficiency of MIQP while keeping optimal guarantee with rank certificate, this thesis proposes an innovative approach leveraging semidefinite relaxation. The study delves deep into the integration of Semidefinite Programming (SDP) and moment relaxation to formulate complex mathematical problems, aiming at transcending the computational barriers of existing methodologies. Preliminary efforts have culminated in the successful derivation of a semidefinite formulation for MIQP projection. The ongoing phase of the study is dedicated to the meticulous implementation and evaluation of the SDP code in MOSEK, benchmarked against the cube pivoting example.

1 Problem Formulation

1.1 Model Predictive Control of Multi-Contact Systems

$$
\begin{aligned}
f^\star = \quad & \min_{x_k,\lambda_k,u_k} \textstyle\sum_{k=0}^{N-1}(x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N && (1)\\
\text{subject to} \quad & x_{k+1} = Ax_k + Bu_k + D\lambda_k + d, && (2)\\
& Ex_k + F\lambda_k + Hu_k + c \geq 0, && (3)\\
& \lambda_k \geq 0, && (4)\\
& \lambda_k^T(Ex_k + F\lambda_k + Hu_k + c) = 0, && (5)\\
& (x,\lambda,u) \in \mathcal{C}, \ \text{for } k = 0,...,N-1, \ \text{given } x_0 && (6)
\end{aligned}
$$

1.2 Linear Complementarity System

$$x_{k+1} = Ax_k + Bu_k + D\lambda_k + d \tag{7}$$

$$0 \leq \lambda_k \perp Ex_k + F\lambda_k + Hu_k + c \geq 0 \tag{8}$$

*Course project in EECS563 Hybrid Control Systems at University of Michigan. Email: `xihangyu@umich.edu`

1.3 Mixed Integer Quadratic Programming formulation of Hybrid Model Predictive Control

$$\begin{aligned} f^\star = \quad & \min_{x_k,\lambda_k,u_k} \textstyle\sum_{k=0}^{N-1} (x_k^T Q_k x_k + u_k^T R_k u_k) + x_N^T Q_N x_N & (9)\\ \text{subject to} \quad & x_{k+1} = Ax_k + Bu_k + D\lambda_k + d, & (10)\\ & Ms_k \geq Ex_k + F\lambda_k + Hu_k + c \geq 0, & (11)\\ & M(1-s_k) \geq \lambda_k \geq 0, & (12)\\ & (x,\lambda,u) \in \mathcal{C}, s_k \in \{0,1\}^{n_\lambda}, & (13)\\ & \text{for } k = 0,...,N-1, \text{ given } x_0 & (14) \end{aligned}$$

1.4 Consensus Form

$$\begin{aligned} & \min_z c(z) + \mathcal{I}_{\mathcal{D}}(z) + \mathcal{I}_{\mathcal{C}}(z) + \textstyle\sum_{k=0}^{N-1} \mathcal{I}_{\mathcal{H}_k}(\delta_k) & (15)\\ \text{subject to} \quad & z_k = \delta_k, \forall k & (16) \end{aligned}$$

where

$$\begin{aligned} \mathcal{D} = \quad & \textstyle\bigcap_{k=0}^{N-2} \{z : x_{k+1} = Ax_k + Bu_k + D\lambda_k + d\} & (17)\\ \mathcal{H}_k = \quad & \{(x_k,\lambda_k,u_k) : Ex_k + F\lambda_k + Hu_k + c \geq 0, \lambda_k \geq 0, \lambda_k^T(Ex_k + F\lambda_k + Hu_k + c) = 0\}. & (18) \end{aligned}$$

1.5 General augmented Lagrangian

$$\mathcal{L}_\rho(z,\delta,w) = c(z) + \mathcal{I}_{\mathcal{D}}(z) + \mathcal{I}_{\mathcal{C}}(z) + \sum_{k=0}^{N-1} (\mathcal{I}_{\mathcal{H}_k}(\delta_k) + \rho(r_k^T G_k r_k - w_k^T G_k w_k)) \tag{19}$$

1.6 ADMM method

$$z^{i+1} = \arg\min_z L_\rho(z,\delta_i,w_i), \tag{20}$$

$$\delta_{i+1}^k = \arg\min_{\delta_k} L_\rho^k(z_{i+1}^k,\delta_k,w_i^k), \quad \forall k, \tag{21}$$

$$w_{i+1}^k = w_i^k + z_{i+1}^k - \delta_{i+1}^k, \quad \forall k \tag{22}$$

1.6.1 quadratic step

$$\begin{aligned} \min_z \quad & c(z) + \sum_{k=0}^{N-1} (z_k - \delta_i^k + w_i^k)^T \rho G_k (z_k - \delta_i^k + w_i^k)\\ \text{s.t.} \quad & z \in D \cap C \end{aligned} \tag{23}$$

1.6.2 projection step

$$\begin{aligned} \min_{\delta_k} \quad & (\delta_k - (z_{i+1}^k + w_i^k))^T \rho G_k (\delta_k - (z_{i+1}^k + w_i^k))\\ \text{s.t.} \quad & \delta_k \in H_k \end{aligned} \tag{24}$$

1.7 C3

In ADMM, the projection step is to project the solution back to the nonconvex constraint after solving a relaxed augmented Lagrangian problem. The projection problem we are interested in is in the following form:

$$\begin{aligned}\rho^\star_{\mathsf{LCP}} = \min_{\delta_k} & (\delta_k - z)^T U(\delta_k - z) \\ \text{subject to} \quad & E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c \geq 0, \\ & \delta_k^\lambda \geq 0, {\delta_k^\lambda}^T (E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c) = 0.\end{aligned} \tag{LCP}$$

where $z \in \mathbb{R}^{n_\lambda + n_x + n_u}, U \in \mathbb{S}^{n_\lambda + n_x + n_u}, E \in \mathbb{R}^{n_\lambda \times n_x}, F \in \mathbb{R}^{n_\lambda \times n_\lambda}, H \in \mathbb{R}^{n_\lambda \times n_u}, c \in \mathbb{R}^{n_\lambda}$ are constants. $\delta_k := (\delta_k^x, \delta_k^\lambda, \delta_k^u) \in \mathbb{R}^{n_\lambda + n_x + n_u}$ where δ_k^x is the state, δ_k^λ is the contact force and δ_k^u is the control input. We observe that the objective function and constraints are polynomials. The equivalent formulation using linear complementarity problem (LCP) is

$$\rho^\star_{\mathsf{LCP}} = \min_{\delta_k} (\delta_k - z)^T U(\delta_k - z) \tag{25}$$

$$\text{subject to } \delta_k \in \{(\delta_k^x, \delta_k^\lambda, \delta_k^u) : 0 \leq \delta_k^\lambda \perp (E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c) \geq 0\}. \tag{26}$$

This optimization has a MIQP formulation using big $\mathbf{M}$ method:

$$\begin{aligned}\rho^\star_{\mathsf{MIQP}} = \min_{\delta_k, s_k} & (\delta_k - z)^T U(\delta_k - z) \\ \text{subject to} \quad & M s_k \geq E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c \geq 0, \\ & M(1 - s_k) \geq \delta_k^\lambda \geq 0, \\ & s_k \in \{0, 1\}^{n_\lambda}.\end{aligned} \tag{MIQP}$$

where M is a matrix of the $M = mI$ where m is a pre-specified large number. We can not apply SDP relaxation to (MIQP) directly since the decision variable s_k is integer. However, the integer constraints can be written in equivalent form:

$$\begin{aligned}\rho^\star_{\mathsf{MIQP}} = \min_{\delta_k, s_k} & (\delta_k - z)^T U(\delta_k - z) \\ \text{subject to} \quad & M s_k \geq E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c \geq 0, \\ & M(1 - s_k) \geq \delta_k^\lambda \geq 0, \\ & s_k(s_k - 1) = 0.\end{aligned} \tag{27}$$

Now the objective function and constraints are polynomials and we can apply SDP relaxation.

2 Semidefinite Relaxation

The previous section has formulated the (LCP) and (MIQP). We can now design the following two semidefinite relaxations.

Proposition 1 (SDP Relaxation form 1)**.** *The following semidefinite program (SDP)*

$$f^\star = \quad \min_{X \in \mathbb{S}^{1 + n_\lambda + n_x + n_u}} \operatorname{tr}(CX) \tag{28}$$

$$\textit{subject to} \quad l_i \leq \textstyle\sum_i \operatorname{tr}(A_i, X) \leq u_i, i = 1, ..., t \tag{29}$$

$$X = \begin{bmatrix} 1 & {\delta_k^x}^T & {\delta_k^\lambda}^T & {\delta_k^u}^T \\ \delta_k^x & \delta_k^x {\delta_k^x}^T & \delta_k^x {\delta_k^\lambda}^T & \delta_k^x {\delta_k^u}^T \\ \delta_k^\lambda & \delta_k^\lambda {\delta_k^x}^T & \delta_k^\lambda {\delta_k^\lambda}^T & \delta_k^\lambda {\delta_k^u}^T \\ \delta_k^u & \delta_k^u {\delta_k^x}^T & \delta_k^u {\delta_k^\lambda}^T & \delta_k^u {\delta_k^u}^T \end{bmatrix} \succeq 0 \tag{30}$$

is a convex relaxation to (LCP) *and* $f^\star \leq \rho^\star_{\mathsf{LCP}}$. *Let* $X^\star$ *be a global minimizer of* (28)*. If* $\operatorname{rank}(X^\star) = 1$*, then* $X^\star$ *can be factorized as* $X^\star = (x_k^\star)^\mathsf{T} x_k^\star$*, where* $x_k^\star \in \mathbb{R}^{1+n_\lambda+n_x+n_u}$ *is a global optimizer to* (LCP)*.*

Proof. It is easy to verify that the objective function can be written as:

$$(\delta_k - z)^T U(\delta_k - z) = \operatorname{tr}(CX) \tag{31}$$

where

$$C = \begin{bmatrix} z^T U z & (-Uz)^T \\ -Uz & U \end{bmatrix} \tag{32}$$

For the constraint $E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c \geq 0$, it can be written in n_λ constraints:

$$-c_i \leq \operatorname{tr}(A_{1,i}X) \tag{33}$$

where c_i is i^{th} element of c and $A_{1,i}$ is as follows:

$$A_{1,i} = \begin{bmatrix} 0 & \frac{1}{2}E_i & \frac{1}{2}F_i & \frac{1}{2}H_i \\ (\frac{1}{2}E_i)^T & \mathbf{0}_{n_x} & \mathbf{0}_{n_x \times n_\lambda} & \mathbf{0}_{n_x \times n_u} \\ (\frac{1}{2}F_i)^T & \mathbf{0}_{n_\lambda \times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda \times n_u} \\ (\frac{1}{2}H_i)^T & \mathbf{0}_{n_u \times n_x} & \mathbf{0}_{n_u \times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{34}$$

with E_i, F_i, H_i are i^{th} row of E, F, H. For the constraint $\delta_k^\lambda \geq 0$, it can be written in n_λ constraints:

$$0 \leq \operatorname{tr}(A_{2,i}X) \tag{35}$$

where $A_{2,i}$ is as follows:

$$A_{2,i} = \begin{bmatrix} 0 & \mathbf{0}_{1 \times n_x} & \frac{1}{2}e^T_{n_\lambda,i} & \mathbf{0}_{1 \times n_u} \\ \mathbf{0}_{n_x \times 1} & \mathbf{0}_{n_x} & \mathbf{0}_{n_x \times n_\lambda} & \mathbf{0}_{n_x \times n_u} \\ \frac{1}{2}e_{n_\lambda,i} & \mathbf{0}_{n_\lambda \times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda \times n_u} \\ \mathbf{0}_{n_u \times 1} & \mathbf{0}_{n_u \times n_x} & \mathbf{0}_{n_u \times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{36}$$

with $e_{n_\lambda,i} \in \mathbb{R}^{n_\lambda}$ be unit vector where i^{th} entry is 1 and other entries are all zeros. For the constraint $\delta_k^{\lambda T}(E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c) = 0$, it can be written as:

$$\operatorname{tr}(A_{3,i}X) = 0 \tag{37}$$

where $A_{3,i}$ is as follows:

$$A_{3,i} = \begin{bmatrix} 0 & \mathbf{0}_{1 \times n_x} & (\frac{1}{2}c)^T & \mathbf{0}_{1 \times n_u} \\ \mathbf{0}_{n_x \times 1} & \mathbf{0}_{n_x} & (\frac{1}{2}E)^T & \mathbf{0}_{n_x \times n_u} \\ \frac{1}{2}c & \frac{1}{2}E & \frac{1}{2}(F + F^T) & \frac{1}{2}H \\ \mathbf{0}_{n_u \times 1} & \mathbf{0}_{n_u \times n_x} & (\frac{1}{2}H)^T & \mathbf{0}_{n_u} \end{bmatrix} \tag{38}$$

$\square$

Proposition 2 (SDP Relaxation form 2)**.** *The following semidefinite program (SDP)*

$$f^\star = \quad \min_{Y \in \mathbb{S}^{1+2n_\lambda+n_x+n_u}} \operatorname{tr}(BY) \tag{39}$$

$$\textit{subject to} \quad l_i \leq \textstyle\sum_i \operatorname{tr}(D_i, Y) \leq u_i, i = 1, ..., q \tag{40}$$

$$Y = \begin{bmatrix} 1 & s_k^T & \delta_k^{xT} & \delta_k^{\lambda T} & \delta_k^{uT} \\ s_k & s_k s_k^T & s_k \delta_k^{xT} & s_k \delta_k^{\lambda T} & s_k \delta_k^{uT} \\ \delta_k^x & \delta_k^x s_k^T & \delta_k^x \delta_k^{xT} & \delta_k^x \delta_k^{\lambda T} & \delta_k^x \delta_k^{uT} \\ \delta_k^\lambda & \delta_k^\lambda s_k^T & \delta_k^\lambda \delta_k^{xT} & \delta_k^\lambda \delta_k^{\lambda T} & \delta_k^\lambda \delta_k^{uT} \\ \delta_k^u & \delta_k^\lambda s_k^T & \delta_k^u \delta_k^{xT} & \delta_k^u \delta_k^{\lambda T} & \delta_k^u \delta_k^{uT} \end{bmatrix} \succeq 0 \tag{41}$$

is a convex relaxation to (MIQP) *and* $f^\star \leq \rho^\star_{\mathsf{MIQP}}$. *Let* $Y^\star$ *be a global minimizer of* (39). *If* $\operatorname{rank}(Y^\star) = 1$, *then* $Y^\star$ *can be factorized as* $Y^\star = (y_k^\star)^\mathsf{T} y_k^\star$, *where* $y_k^\star \in \mathbb{R}^{1+2n_\lambda+n_x+n_u}$ *is a global optimizer to* (MIQP).

Proof. It is easy to verify that the objective function can be written as:

$$(\delta_k - z)^T U(\delta_k - z) = \operatorname{tr}(BY) \tag{42}$$

where

$$B = \begin{bmatrix} z^T U z & \mathbf{0}_{1\times n_\lambda} & (-Uz)^T \\ \mathbf{0}_{n_\lambda \times 1} & \mathbf{0}_{n_\lambda\times n_\lambda} & \mathbf{0}_{n_\lambda\times(n_\lambda+n_x+n_u)} \\ -Uz & \mathbf{0}_{(n_\lambda+n_x+n_u)\times n_\lambda} & U \end{bmatrix} \tag{43}$$

For the constraint $E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c - Ms_k \leq 0$, it can be written in n_λ constraints:

$$\operatorname{tr}(D_{1,i}X) \leq -c_i \tag{44}$$

where c_i is i^{th} element of c and $D_{1,i}$ is as follows:

$$D_{1,i} = \begin{bmatrix} 0 & -\frac{1}{2}M_i & \frac{1}{2}E_i & \frac{1}{2}F_i & \frac{1}{2}H_i \\ (-\frac{1}{2}M_i)^T & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ (\frac{1}{2}E_i)^T & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x\times n_u} \\ (\frac{1}{2}F_i)^T & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ (\frac{1}{2}H_i)^T & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u\times n_x} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{45}$$

with M_i, E_i, F_i, H_i are i^{th} row of M, E, F, H. For the constraint $E\delta_k^x + F\delta_k^\lambda + H\delta_k^u + c \geq 0$, it can be written in n_λ constraints:

$$-c_i \leq \operatorname{tr}(D_{2,i}X) \tag{46}$$

where c_i is i^{th} element of c and $D_{2,i}$ is as follows:

$$D_{2,i} = \begin{bmatrix} 0 & \mathbf{0}_{1\times n_\lambda} & \frac{1}{2}E_i & \frac{1}{2}F_i & \frac{1}{2}H_i \\ \mathbf{0}_{n_\lambda\times 1} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ (\frac{1}{2}E_i)^T & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x\times n_u} \\ (\frac{1}{2}F_i)^T & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ (\frac{1}{2}H_i)^T & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u\times n_x} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{47}$$

with E_i, F_i, H_i are i^{th} row of E, F, H. For the constraint $\delta_k^\lambda + Ms_k \leq M$, it can be written in n_λ constraints:

$$\operatorname{tr}(D_{3,i}X) \leq M \tag{48}$$

where $D_{3,i}$ is as follows:

$$D_{3,i} = \begin{bmatrix} 0 & \frac{1}{2}M_i & \mathbf{0}_{1\times n_x} & \frac{1}{2}e_{n_\lambda,i}^T & \mathbf{0}_{1\times n_u} \\ (\frac{1}{2}M_i)^T & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ \mathbf{0}_{n_x\times 1} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x\times n_u} \\ \frac{1}{2}e_{n_\lambda,i} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ \mathbf{0}_{n_u\times 1} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u\times n_x} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{49}$$

For the constraint $\delta_k^\lambda \geq 0$, it can be written in n_λ constraints:

$$0 \leq \operatorname{tr}(D_{4,i}X) \tag{50}$$

where $D_{4,i}$ is as follows:

$$D_{4,i} = \begin{bmatrix} 0 & \mathbf{0}_{1\times n_\lambda} & \mathbf{0}_{1\times n_x} & \frac{1}{2}e_{n_\lambda,i}^T & \mathbf{0}_{1\times n_u} \\ \mathbf{0}_{n_\lambda\times 1} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ \mathbf{0}_{n_x\times 1} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x} & \mathbf{0}_{n_x\times n_\lambda} & \mathbf{0}_{n_x\times n_u} \\ \frac{1}{2}e_{n_\lambda,i} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_x} & \mathbf{0}_{n_\lambda} & \mathbf{0}_{n_\lambda\times n_u} \\ \mathbf{0}_{n_u\times 1} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u\times n_x} & \mathbf{0}_{n_u\times n_\lambda} & \mathbf{0}_{n_u} \end{bmatrix} \tag{51}$$

For the constraint $s_k(s_k-1)=0$, it can be written as:

$$\operatorname{tr}(D_{5,i}X) = 0 \tag{52}$$

where $D_{5,i}$ is as follows:

$$D_{5,i} = \begin{bmatrix} 0 & \frac{1}{2}e_{n_\lambda,i}^T & \mathbf{0}_{1\times(n_x+n_\lambda+n_u)} \\ \frac{1}{2}e_{n_\lambda,i} & e_{n_\lambda,i}e_{n_\lambda,i}^T\mathbf{I}_{n_\lambda} & \mathbf{0}_{n_\lambda\times(n_x+n_\lambda+n_u)} \\ \mathbf{0}_{(n_x+n_\lambda+n_u)\times 1} & \mathbf{0}_{(n_x+n_\lambda+n_u)\times n_\lambda} & \mathbf{0}_{n_x+n_\lambda+n_u} \end{bmatrix} \tag{53}$$

where $I_{n_\lambda} \in \mathbb{R}^{n_\lambda}$ is identity matrix.

□

3 Experiments

In the experiments section, we will first investigate the suboptimality comparison of two SDP solvers. Then, we will compare the running time of MIQP and SDP solvers. For each of the experiments above, we will vary noise.

References

[1] Alp Aydinoglu, Adam Wei, and Michael Posa. Consensus complementarity control for multi-contact mpc. *arXiv preprint arXiv:2304.11259*, 2023. 1