Week 9 at a glance

Textbook reading: Section 5.3, Section 5.1, Section 3.2

For Monday, Example 5.26 (page 237).

For Wednesday, Theorem 5.30 (page 238)

For Friday, skim section 3.2.

For Monday of Week 10: Definition 7.1 (page 276)

We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Give examples of sets that are regular, context-free, decidable, or recognizable (and prove that they are).
    - **Define and explain computational problems, including A_{**}, E_{**}, EQ_{**}, (for ** DFA or TM) and $HALT_{TM}$**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Use mapping reduction to deduce the complexity of a language by comparing to the complexity of another.
    - **Explain what it means for one problem to reduce to another**
    - **Define computable functions, and use them to give mapping reductions between computational problems**
    - **Build and analyze mapping reductions between computational problems**
  - Classify the computational complexity of a set of strings by determining whether it is regular, context-free, decidable, or recognizable.
    - **State, prove, and use theorems relating decidability, recognizability, and co-recognizability.**
    - **Prove that a language is decidable or recognizable by defining and analyzing a Turing machines with appropriate properties.**
  - Describe several variants of Turing machines and informally explain why they are equally expressive.
    - **Define an enumerator**
    - **Define nondeterministic Turing machines**
    - **Use high-level descriptions to define and trace machines (Turing machines and enumerators)**
    - **Apply dovetailing in high-level definitions of machines**

TODO:

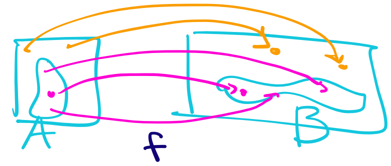
Review Quiz 8 on PrairieLearn (http://us.prairielearn.com), due 3/5/2025

Review Quiz 9 on PrairieLearn (http://us.prairielearn.com), due 3/12/2025

Homework 6 submitted via Gradescope (https://www.gradescope.com/), due 3/13/2025

Project submitted via Gradescope (https://www.gradescope.com/), due 3/19/2025

Monday: Mapping reductions and recognizability



Recall definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

$x \notin A$ if and only if $f(x) \notin B$

$P \leftrightarrow Q \equiv \neg P \leftrightarrow \neg Q$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.

Last time we proved that $A_{TM} \leq_m HALT_{TM}$ where

$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$$

and since A_{TM} is undecidable, $HALT_{TM}$ is also undecidable. The function witnessing the mapping reduction mapped strings in A_{TM} to strings in $HALT_{TM}$ and strings not in A_{TM} to strings not in $HALT_{TM}$ by changing encoded Turing machines to ones that had identical computations except looped instead of rejecting.

True ~~or False~~: $\overline{A_{TM}} \leq_m \overline{HALT_{TM}}$

In general, if f witnesses $A \leq_m B$ then it also witnesses $\overline{A} \leq_m \overline{B}$.

True ~~or False~~: $HALT_{TM} \leq_m A_{TM}$.

but requires proof (the function that witnesses $A_{TM} \leq_m HALT_{TM}$ doesn't witness this reduction)

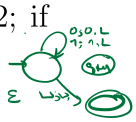
Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F(x) \in A_{TM}$. Define

$F =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$ $\to q_{acc}$ q_{rej} $\rangle$
2. Construct the following machine M'_x: $x = \langle M, w \rangle$

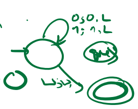
   $M'_x =$ " On input y
   1. Run M on w
   2. If M halts on w, accept y."

3. Output $\langle M'_x, w\rangle$."

Notice when $M =$  $w = \varepsilon$

$L(M'_x) = \Sigma^* = \Sigma^* \cup \{\varepsilon\}$

$L(M) = \{\varepsilon\}$

Notice when $M =$  $w = 0$

$L(M'_x) = \emptyset$

$L(M) = \{\varepsilon\}$

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in A_{TM}?

	Input string	Output string	Goal
$x \in HALT_{TM}$	$\langle M, w\rangle$ where M halts on w	$\langle M'_x, w\rangle$ where M'_x's language is Σ^* so w is in the language of M'_x	$F(x) \in A_{TM}$
$x \notin HALT_{TM}$	$\langle M, w\rangle$ where M does not halt on w	$\langle M'_x, w\rangle$ where M'_x's language is $\emptyset$ so w is not in the language of M'_x	$F(x) \notin A_{TM}$
	x not encoding any pair of TM and string	$\langle \to q_{acc} \quad q_{rej} \rangle \neq \langle M, w\rangle$ for any M, w so not in A_{TM}	

Notice: $A_{TM} \leq_m HALT_{TM}$ and $HALT_{TM} \leq_m A_{TM}$ BUT $A_{TM} \neq HALT_{TM}$

Theorem (Sipser 5.28): If $A \leq_m B$ and B is recognizable, then A is recognizable.

Proof:

Identical proof to before (Thm 5.22)

Suppose A, B are arbitrary and $A \leq_m B$ and B is recognizable. By definition we therefore have TMs F and M_B with F computing a witnessing function for $A \leq_m B$ and $L(M_B) = B$. Define

M = " On input x
1. Calculate $F(x)$.
2. Run M_B on $F(x)$.
3. If M_B accepts $F(x)$, accept x.
4. If M_B rejects $F(x)$, reject x "

Observe that $L(M) = A$ because for any x, 1 if $x \in A$ then $F(x) \in B$ so running M on x will first compute $F(x)$ (with finitely many steps) and then run M_B on $F(x)$, which halts and accepts by case assumption, so M accepts x as needed; 2 if $x \notin A$ then $F(x) \notin B$ so M_B doesn't accept $F(x)$: 2a if M_B rejects $F(x)$, running M on x first computes $F(x)$ (with finitely many steps) and then runs M_B on $F(x)$, which halts and rejects by case assumption, so M rejects x; 2b if M_B loops on $F(x)$, running M on x first computes $F(x)$ (still using finitely many steps, by definition of F computing a function) then runs M_B on $F(x)$ and looping by case assumption. Thus, for all x, $x \in A$ iff M accepts x ∎

Corollary: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable.

Proof by contradiction.

Identical proof to before (Thm 5.23)

Strategy:

(i) To prove that a recognizable language R is undecidable, prove that $A_{TM} \leq_m R$. (ex $HALT_{TM}$)

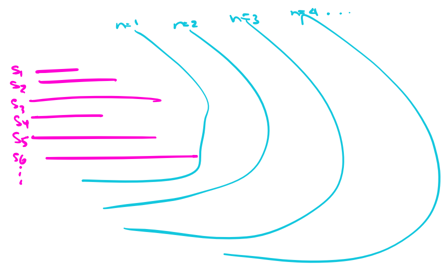
(ii) To prove that a co-recognizable language U is undecidable, prove that $\overline{A_{TM}} \leq_m U$, i.e. that $A_{TM} \leq_m \overline{U}$.

$$E_{TM} = \{\langle M \rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$$

Can we find algorithms to recognize

E_{TM} ? No

$\overline{E_{TM}}$? Yes via dovetailing



Hunch: E_{TM} is not recognizable (not decidable) but is co-recognizable

Claim: $\boxed{A_{TM} \leq_m \overline{E_{TM}}}$. *And hence also* $\overline{A_{TM}} \leq_m E_{TM}$

Recall these are equivalent $X \leq_m Y$ iff $\overline{X} \leq_m \overline{Y}$

Proof: Need computable function $F : \Sigma^* \to \Sigma^*$ such that $x \in A_{TM}$ iff $F(x) \notin E_{TM}$. Define

F = " On input x,

high level description

1. Type-check whether $x = \langle M, w \rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$ → (q0,rej) ◎ $\rangle$
2. Construct the following machine $\boxed{M'_x}$: $x = \langle M, w \rangle$

   " On input y
   1. Run M on w
   2. If M accepts w, accept y
   3. If M rejects w, reject y."
3. Output $\langle M'_x \rangle$." $F(x)$

When y = ε, 0: 1. Run M on w, HALT AND ACCEPT 2. Accept ε, 0 for M'_x built when $x \in A_{TM}$

OR: M halts and rejects w, M'_x on ε; M loops on w, M'_x doesn't accept ε, 0, 00, 01, 111, for M'_x built when $x \notin A_{TM}$

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in A_{TM}$ iff its image is **not** in E_{TM} ?

Case	Input string	Output string	Goal
$x \in A_{TM}$	$\langle M, w \rangle$ where $w \in L(M)$	$F(x) = \langle M'_x \rangle$ with $L(M'_x) = \Sigma^*$	$F(x) \notin E_{TM}$
$x \notin A_{TM}$	$\langle M, w \rangle$ where $w \notin L(M)$	$F(x) = \langle M'_x \rangle$ with $L(M'_x) = \emptyset$ whether M loops on w or M halts and reject w	$F(x) \in E_{TM}$
	x not encoding any pair of TM and string	$\langle$ → (q0,rej) ◎ $\rangle$ so represents a TM with empty language	

Corollary: $\overline{E_{TM}}$ is undecidable (and recognizable)

Corollary: E_{TM} is undecidable

Corollary: E_{TM} is unrecognizable

Wednesday: More mapping reductions

Recall: A is **mapping reducible to** B, written $A \leq_m B$, means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

So far:

Can be used as reference / benchmark sets for calibrating "difficulty" of other problems

- A_{TM} is recognizable, undecidable, and not-co-recognizable.
- $\overline{A_{TM}}$ is unrecognizable, undecidable, and co-recognizable.
- $HALT_{TM}$ is recognizable, undecidable, and not-co-recognizable.
- $\overline{HALT_{TM}}$ is unrecognizable, undecidable, and co-recognizable.
- E_{TM} is unrecognizable, undecidable, and co-recognizable.
- $\overline{E_{TM}}$ is recognizable, undecidable, and not-co-recognizable.

$$EQ_{TM} = \{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are both Turing machines and } L(M_1) = L(M_2)\}$$

Can we find algorithms to recognize

EQ_{TM} ? Even harder than E_{TM} . . .

$\overline{EQ_{TM}}$? Even one counterexample may be hard . . .

Goal: Show that EQ_{TM} is not recognizable and that $\overline{EQ_{TM}}$ is not recognizable.

Using Corollary to **Theorem 5.28**: If $A \leq_m B$ and A is unrecognizable, then B is unrecognizable, it's enough to prove that

$\overline{HALT_{TM}} \leq_m EQ_{TM}$ aka $HALT_{TM} \leq_m \overline{EQ_{TM}}$

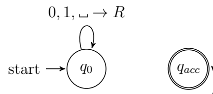
$\overline{HALT_{TM}} \leq_m \overline{EQ_{TM}}$ aka $HALT_{TM} \leq_m EQ_{TM}$

New benchmarks!

WTS $HALT_{TM} \leq_m \overline{EQ_{TM}}$

Need computable function $F_1 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_1(x) \notin EQ_{TM}$.

Strategy:

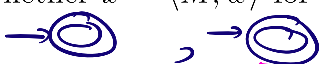
Map strings $\langle M, w\rangle$ to strings $\langle M'_x,$  $\rangle$. This image string is not in EQ_{TM} when $L(M'_x) \neq \emptyset$.

We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

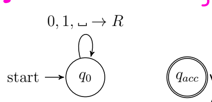
Thus: when $\langle M, w\rangle \in HALT_{TM}$ it gets mapped to a string not in EQ_{TM} and when $\langle M, w\rangle \notin HALT_{TM}$ it gets mapped to a string that is in EQ_{TM}.

Define

$F_1 =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle$  $\rangle$
2. Construct the following machine $\boxed{M'_x}$: $x = \langle M, w \rangle$

   " On input y
   1. Run M on w
   2. If M accepts w, accept y.
   3. If M rejects w, accept y. "

3. Output $\langle \boxed{M'_x},$  $\rangle$ "

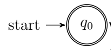
   recognizes $\emptyset$

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is **not** in EQ_{TM} ?

Case	Input string	Output string	Goal
$x \in HALT_{TM}$	$\langle M, w\rangle$ where M halts on w	$\langle M'_x, \ldots \rangle$ with $L(M'_x) = \Sigma^*$ which is not $\emptyset$, the language of the reference set	$F(x) \notin EQ_{TM}$
$x \notin HALT_{TM}$	$\langle M, w\rangle$ where M loops on w	$\langle M'_x, \ldots \rangle$ with $L(M'_x) = \emptyset$ so with same language as the reference	$F(x) \in EQ_{TM}$
	x not encoding any pair of TM and string	$F(x) = \langle$  $\rangle \in EQ_{TM}$	

Conclude: $HALT_{TM} \leq_m \overline{EQ_{TM}}$

Need computable function $F_2 : \Sigma^* \to \Sigma^*$ such that $x \in HALT_{TM}$ iff $F_2(x) \in EQ_{TM}$.

Strategy:
Map strings $\langle M, w\rangle$ to strings $\langle M'_x,$  $\rangle$. This image string is in EQ_{TM} when $L(M'_x) = \Sigma^*$.

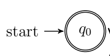
We will build M'_x so that $L(M'_x) = \Sigma^*$ when M halts on w and $L(M'_x) = \emptyset$ when M loops on w.

Thus: when $\langle M, w\rangle \in HALT_{TM}$ it gets mapped to a string in EQ_{TM} and when $\langle M, w\rangle \notin HALT_{TM}$ it gets mapped to a string that is not in EQ_{TM}.

Define

$F_2 =$ " On input x,

1. Type-check whether $x = \langle M, w\rangle$ for some TM M and string w. If so, move to step 2; if not, output $\langle \qquad\qquad\qquad \rangle$
2. Construct the following machine M'_x:

3. Output $\langle M'_x,$  $\rangle$ "

Verifying correctness: (1) Is function well-defined and computable? (2) Does it have the translation property $x \in HALT_{TM}$ iff its image is in EQ_{TM} ?

Input string	Output string
$\langle M, w\rangle$ where M halts on w	
$\langle M, w\rangle$ where M loops on w	
x not encoding any pair of TM and string	

Conclude: $HALT_{TM} \leq_m EQ_{TM}$

Friday: Other models of computation

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

True / False: NFAs and PDAs are equally expressive.

True / False: Regular expressions and CFGs are equally expressive.

Church-Turing Thesis (Sipser p. 183): The informal notion of algorithm is formalized completely and correctly by the formal definition of a Turing machine. In other words: all reasonably expressive models of computation are equally expressive with the standard Turing machine.

Some examples of models that are **equally expressive** *with deterministic Turing machines:*

May-stay machines The May-stay machine model is the same as the usual Turing machine model, except that on each transition, the tape head may move L, move R, or Stay.

Formally: $(Q, \Sigma, \Gamma, \delta, q_0, q_{accept}, q_{reject})$ where

$$\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R, S\}$$

Claim: Turing machines and May-stay machines are equally expressive. *To prove ...*

To translate a standard TM to a may-stay machine: never use the direction S!

To translate one of the may-stay machines to standard TM: any time TM would Stay, move right then left.

Multitape Turing machine A multitape Turing machine with k tapes can be formally represented as $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where Q is the finite set of states, Σ is the input alphabet with $\sqcup \notin \Sigma$, Γ is the tape alphabet with $\Sigma \subsetneq \Gamma$, $\delta : Q \times \Gamma^k \to Q \times \Gamma^k \times \{L, R\}^k$ (where k is the number of states)

If M is a standard TM, it is a 1-tape machine.

To translate a k-tape machine to a standard TM: Use a new symbol to separate the contents of each tape and keep track of location of head with special version of each tape symbol. Sipser Theorem 3.13

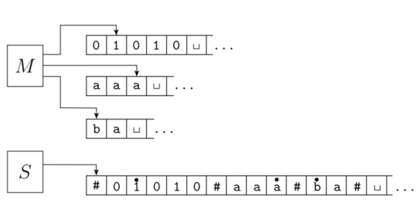


FIGURE 3.14
Representing three tapes with one

Enumerators Enumerators give a different model of computation where a language is **produced, one string at a time**, rather than recognized by accepting (or not) individual strings.

Each enumerator machine has finite state control, unlimited work tape, and a printer. The computation proceeds according to transition function; at any point machine may "send" a string to the printer.

$$E = (Q, \Sigma, \Gamma, \delta, q_0, q_{print})$$

Q is the finite set of states, Σ is the output alphabet, Γ is the tape alphabet ($\Sigma \subsetneq \Gamma, \llcorner\!\lrcorner \in \Gamma \setminus \Sigma$),

$$\delta : Q \times \Gamma \times \Gamma \to Q \times \Gamma \times \Gamma \times \{L, R\} \times \{L, R\}$$

where in state q, when the working tape is scanning character x and the printer tape is scanning character y, $\delta((q, x, y)) = (q', x', y', d_w, d_p)$ means transition to control state q', write x' on the working tape, write y' on the printer tape, move in direction d_w on the working tape, and move in direction d_p on the printer tape. The computation starts in q_0 and each time the computation enters q_{print} the string from the leftmost edge of the printer tape to the first blank cell is considered to be printed.

The language **enumerated** by E, $L(E)$, is $\{w \in \Sigma^* \mid E \text{ eventually, at finite time, prints } w\}$.

Theorem 3.21 A language is Turing-recognizable iff some enumerator enumerates it.

Proof, part 1: Assume L is enumerated by some enumerator, E, so $L = L(E)$. We'll use E in a subroutine within a high-level description of a new Turing machine that we will build to recognize L.

Goal: build Turing machine M_E with $L(M_E) = L(E)$.

Define M_E as follows: $M_E =$ "On input w,

1. Run E. For each string x printed by E.
2. Check if $x = w$. If so, accept (and halt); otherwise, continue."

Proof, part 2: Assume L is Turing-recognizable and there is a Turing machine M with $L = L(M)$. We'll use M in a subroutine within a high-level description of an enumerator that we will build to enumerate L.

Goal: build enumerator E_M with $L(E_M) = L(M)$.

Idea: check each string in turn to see if it is in L.

How? Run computation of M on each string. *But*: need to be careful about computations that don't halt.

Recall String order for $\Sigma = \{0, 1\}$: $s_1 = \varepsilon$, $s_2 = 0$, $s_3 = 1$, $s_4 = 00$, $s_5 = 01$, $s_6 = 10$, $s_7 = 11$, $s_8 = 000$, $\ldots$

Define E_M as follows: $E_M =$ " *ignore any input.* Repeat the following for $i = 1, 2, 3, \ldots$

1. Run the computations of M on s_1, s_2, $\ldots$, s_i for (at most) i steps each
2. For each of these i computations that accept during the (at most) i steps, print out the accepted string."

Nondeterministic Turing machine

At any point in the computation, the nondeterministic machine may proceed according to several possibilities: $(Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej})$ where

$$\delta : Q \times \Gamma \to \mathcal{P}(Q \times \Gamma \times \{L, R\})$$

The computation of a nondeterministic Turing machine is a tree with branching when the next step of the computation has multiple possibilities. A nondeterministic Turing machine accepts a string exactly when some branch of the computation tree enters the accept state.

Given a nondeterministic machine, we can use a 3-tape Turing machine to simulate it by doing a breadth-first search of computation tree: one tape is "read-only" input tape, one tape simulates the tape of the nondeterministic computation, and one tape tracks nondeterministic branching. Sipser page 178

Summary

Two models of computation are called **equally expressive** when every language recognizable with the first model is recognizable with the second, and vice versa.

To prove the existence of a Turing machine that decides / recognizes some language, it's enough to construct an example using any of the equally expressive models.

But: some of the **performance** properties of these models are not equivalent.

