Week 5 at a glance

Textbook reading: Chapter 2
Before Monday, read Introduction to Section 2.1 (pages 101-102).

Before Wednesday, read Section 2.1

Before Friday, read Theorem 2.20.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Describe and use models of computation that don't involve state machines.
    - **Identify the components of a formal definition of a context-free grammar (CFG)**
    - **Derive strings in the language of a given CFG**
    - **Determine the language of a given CFG**
    - **Design a CFG generating a given language**
    - **Use context-free grammars and relate them to languages and pushdown automata.**
  - Use precise notation to formally define the state diagram of a Turing machine
  - Use clear English to describe computations of Turing machines informally.
    - **Design a PDA that recognizes a given language.**
  - Give examples of sets that are context-free (and prove that they are).
    - **State the definition of the class of context-free languages**
    - **Explain the limits of the class of context-free languages**
    - **Identify some context-free sets and some non-context-free sets**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Describe and prove closure properties of classes of languages under certain operations.
    - **Apply a general construction to create a new PDA or CFG from an example one.**
    - **Formalize a general construction from an informal description of it.**
    - **Use general constructions to prove closure properties of the class of context-free langugages.**
    - **Use counterexamples to prove non-closure properties of the class of context-free langugages.**

TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (http://us.prairietest.com) . The first Test 1 sessions are next week!

Review Quiz 4 on PrairieLearn (http://us.prairielearn.com), due 2/5/2025

Homework 3 submitted via Gradescope (https://www.gradescope.com/), due 2/6/2025

Review Quiz 5 on PrairieLearn (http://us.prairielearn.com), due 2/12/2025

Monday: More Pushdown Automata

Definition A **pushdown automaton** (PDA) is specified by a 6-tuple $(Q, \Sigma, \Gamma, \delta, q_0, F)$ where Q is the finite set of states, Σ is the input alphabet, Γ is the stack alphabet,

Sigma ↗ ↑ Gamma

$$\delta : Q \times \boxed{\Sigma_\varepsilon} \times \boxed{\Gamma_\varepsilon} \to \mathcal{P}(Q \times \Gamma_\varepsilon)$$

is the transition function, $q_0 \in Q$ is the start state, $F \subseteq Q$ is the set of accept states.

For the PDA state diagrams below, $\Sigma = \{0, 1\}$.

Mathematical description of language | State diagram of PDA recognizing language

$\{0^n 1^{2n} \mid n \geq 0\}$

$\varepsilon = 0^0 1^{2\cdot 0} = 0^0 1^0$ ← repeat 0 zero times ← repeat 1 zero times

q_0, q_1, q_2, q_4 ACC

$\sqcup$ $\sqcup_{\$}$ $\sqcup_{\$}$ $\sqcup$

011↓

q_0, q_1, q_1, q_2, q_3, q_2, q_4

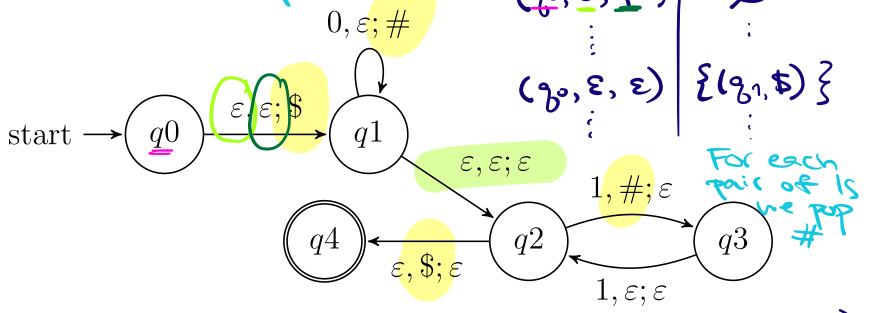
$\sqcup$ $\sqcup_{\$}$ $\sqcup_{\#\,\$}$ $\sqcup_{\#\,\$}$ $\sqcup_{\$}$ $\sqcup_{\$}$ $\sqcup$

$\Gamma = \{\$, \#\}$

For each 0, push one #

δ

Input	Output
$(q_0, 0, \$)$	$\emptyset$
⋮	⋮
$(q_0, \varepsilon, \varepsilon)$	$\{(q_1, \$)\}$
⋮	⋮



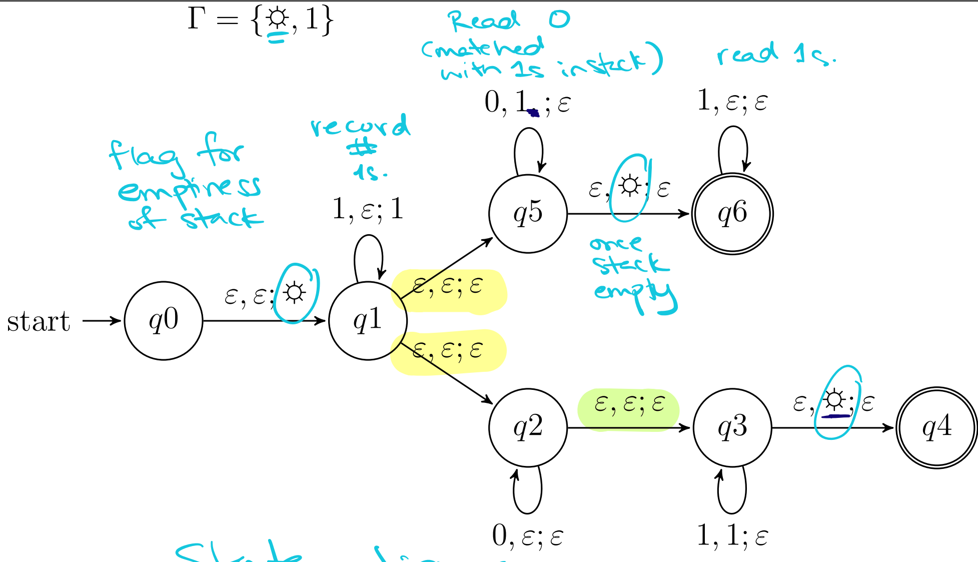
$(\{q_0, q_1, q_2, q_3, q_4\}, \{0,1\}, \{\$, \#\}, \delta, q_0, \{q_4\})$

$\Gamma = \{☼, 1\}$

$\{1^n 0^n 1^m \mid n, m \geq 0\}$

$\cup$

$\{1^n 0^m 1^n \mid n, m \geq 0\}$

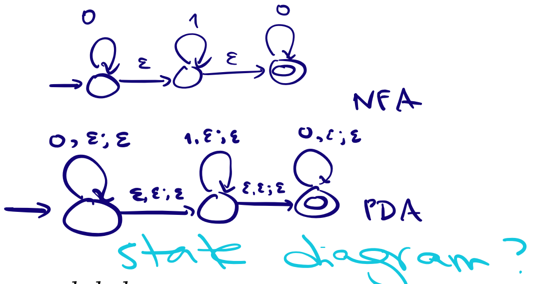


language?

State diagram

$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$

language



Note: alternate notation is to replace ; with → on arrow labels.

Corollary: for each language L over Σ, if there is an NFA N with $L(N) = L$ then there is a PDA M with $L(M) = L$

Proof idea: Declare stack alphabet to be $\Gamma = \Sigma$ and then don't use stack at all.

Given NFA $(Q, \Sigma, \delta, q_0, F)$ define PDA $(Q, \Sigma, \Sigma, \delta_{new}, q_0, F)$ with transition function $\delta_{new} : Q \times \Sigma_\varepsilon \times \Sigma_\varepsilon \to \mathcal{P}(Q \times \Sigma_\varepsilon)$ given by

$$\delta_{new}((q, x, y)) = \begin{cases} \{(q', \varepsilon) \mid q' \in \delta((q, x))\} & \text{if } q \in Q, x \in \Sigma_\varepsilon, y = \varepsilon \\ \emptyset & \text{if } q \in Q, x \in \Sigma_\varepsilon, y \in \Sigma \end{cases}$$

Big picture: PDAs are motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

Context-free grammars: Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

Wednesday: Context-free Grammars and Languages

Definitions below are on pages 101-102.

Term	Typical symbol or Notation	Meaning
Context-free grammar (CFG)	G	$G = (V, \Sigma, R, S)$
The set of **variables**	V	Finite set of symbols that represent phases in production pattern
The set of **terminals**	Σ	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of **rules**	R	Each rule is $A \to u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The **start** variable	S	Usually on left-hand-side of first/ topmost rule
Derivation	$S \Rightarrow \cdots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of variables to apply a rule to.
Language **generated** by the context-free grammar G	$L(G)$	The set of strings for which there is a derivation in G. Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e. $\{w \in \Sigma^* \mid \text{there is } \underline{\text{derivation}} \text{ in } G \text{ that ends in } w\}$
Context-free language		A language that is the language generated by some context-free grammar
Regular language		A language that is the language described by some regular expressions

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\underbrace{\{S\}}_{V}, \underbrace{\{0\}}_{\Sigma}, R, S)$ with rules

$$L(G_1) = L(0^+) = \{0^i \mid i > 0\}$$

$$\boxed{\begin{aligned} &\textcircled{1}\quad S \to 0S \\ &\textcircled{2}\quad S \to 0 \end{aligned}}$$

Examples of strings in Σ^*

In $L(G_1)$ … $S \overset{2}{\Rightarrow} 0$ is a derivation that proves $0 \in L(G_1)$

Not in $L(G_1)$ … ε because each derivation in G_1 must start with S and all (two) of the rules in G_1 that have S on LHS add a 0 to final string so each string in $L(G_1)$ must have at least one 0.

$S \overset{1}{\Rightarrow} 0S \overset{1}{\Rightarrow} 00S \overset{1}{\Rightarrow} 000S$

$\overset{V}{}$ $\overset{\Sigma}{}$

$G_2 = (\{S\}, \{0,1\}, R, S)$ $\qquad S \to \overset{①}{0S} \mid \overset{②}{1S} \mid \overset{③}{\varepsilon}$

abbreviating

① $S \to 0S$

② $S \to 1S$

③ $S \to \varepsilon$

In $L(G_2)$ . . .

$$S \overset{①}{\Rightarrow} 0S \overset{②}{\Rightarrow} 01S \overset{②}{\Rightarrow} 011S \overset{①}{\Rightarrow} 0110S \overset{③}{\Rightarrow} 0110$$

Notice $\quad L(G_2) = \{0,1\}^*$

Not in $L(G_2)$ . . . None!

$(\{S,T\}, \{0,1\}, R, S)$ with rules
$\underset{V}{}\quad\underset{\Sigma}{}$

① $S \to T1T1T1T$

$T \to \overset{}{\underset{②}{0T}} \mid \underset{③}{1T} \mid \underset{④}{\varepsilon}$

all derivations will need to start with application of rule ①

In $L(G_3)$ . . .

$$S \overset{①}{\Rightarrow} T1T1T1T \overset{④}{\Rightarrow} 1T1T1T \overset{④}{\Rightarrow} 11T1T \overset{④}{\Rightarrow} 111T \overset{④}{\Rightarrow} 111$$

$L(G_3) = L(\Sigma^* 1 \Sigma^* 1 \Sigma^* 1 \Sigma^*) = \{ w \in \{0,1\}^* \mid w \text{ has at least three 1s} \}$

Not in $L(G_3)$ . . .

ε , 1 , 00001

$G_4 = (\{A, B\}, \{0,1\}, R, A)$ with rules
$\underset{V}{}\quad\underset{\Sigma}{}\quad$ start

$$A \to \overset{①}{0A0} \mid \overset{②}{0A1} \mid \overset{③}{1A0} \mid \overset{④}{1A1} \mid \overset{⑤}{1}$$

ok (but not helpful) to have variable we don't use

In $L(G_4)$ . . .

$$A \overset{⑤}{\Rightarrow} 1$$
$$A \overset{①}{\Rightarrow} 0A0 \overset{⑤}{\Rightarrow} 010$$
$$A \overset{②}{\Rightarrow} 0A1 \overset{⑤}{\Rightarrow} 011$$

$L(G_4) = \{ w \in \{0,1\}^* \mid w$ has odd length middle character is $1 \}$

- nonregular
- context-free

Not in $L(G_4)$ . . .

01 , 100

Design a CFG to generate the language $\{a^n b^n \mid n \geq 0\}$

– nonregular
– context-free?

$(\{S\}, \{a, b\}, R, S)$

R given by $S \to aSb \mid \varepsilon$

Sample derivation: $S \Rightarrow aSb \Rightarrow aaSbb \Rightarrow aabb$

Design a CFG to generate the language $\{a^i b^j \mid j \geq i \geq 0\}$

(Bonus)

Idea: can add to strings in two ways

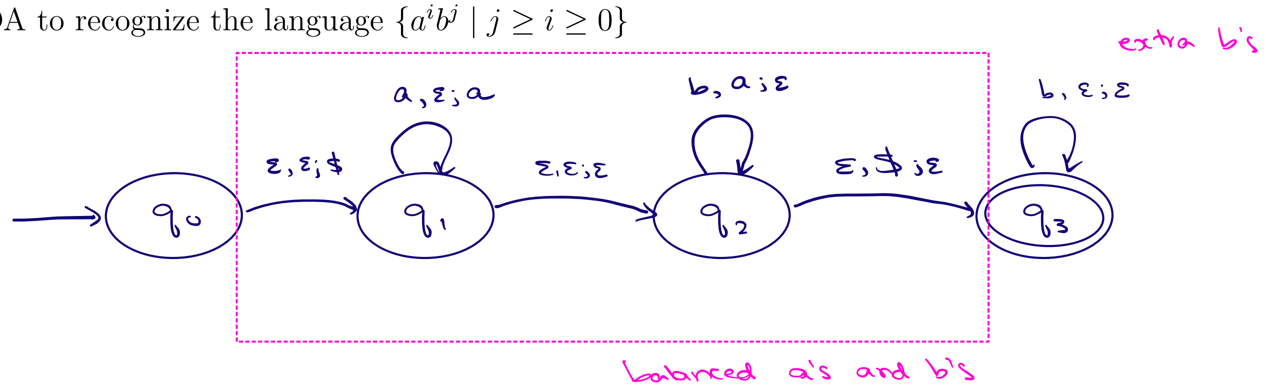
$a^i \underbrace{b^{j-i}}_{\text{extra}} b^i$ balanced

$(\{S\}, \{a, b\}, R, S)$

where R is given by

$S \to \varepsilon \mid aSb \mid Sb$

Design a PDA to recognize the language $\{a^i b^j \mid j \geq i \geq 0\}$



Friday: Context-free and non-context-free languages

Theorem 2.20: A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free language over a given alphabet Σ is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a descripton. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can "test for emptiness of stack" without providing details. *How?* We can always push a special end-of-stack symbol, \$, at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can "test for end of input" without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \cup L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Suppose L_1 and L_2 are context-free languages over Σ. **Goal**: $L_1 \circ L_2$ is also context-free.

Approach 1: with PDAs

Let $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$ and $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$ be PDAs with $L(M_1) = L_1$ and $L(M_2) = L_2$.

Define $M =$

Approach 2: with CFGs

Let $G_1 = (V_1, \Sigma, R_1, S_1)$ and $G_2 = (V_2, \Sigma, R_2, S_2)$ be CFGs with $L(G_1) = L_1$ and $L(G_2) = L_2$.

Define $G =$

Summary

Over a fixed alphabet Σ, a language L is **regular**

iff it is described by some regular expression
iff it is recognized by some DFA
iff it is recognized by some NFA

Over a fixed alphabet Σ, a language L is **context-free**

iff it is generated by some CFG
iff it is recognized by some PDA

Fact: Every regular language is a context-free language.

Fact: There are context-free languages that are not nonregular.

Fact: There are countably many regular languages.

Fact: There are countably infinitely many context-free languages.

Consequence: Most languages are **not** context-free!

Examples of non-context-free languages

$$\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\}$$
$$\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\}$$
$$\{ww \mid w \in \{0,1\}^*\}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If A is a context-free language, there is a number p where, if s is any string in A of length at least p, then s may be divided into five pieces $s = uvxyz$ where (1) for each $i \geq 0$, $uv^ixy^iz \in A$, (2) $|uv| > 0$, (3) $|vxy| \leq p$. *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set X is said to be **closed** under an operation OP if, for any elements in X, applying OP to them gives an element in X.

True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y\, (\, (x \in \mathbb{Z} \land y \in \mathbb{Z}) \to xy \in \mathbb{Z}\,)$
True	For each set A, the power set of A is closed under intersection. $\forall A_1 \forall A_2\, (\, (A_1 \in \mathcal{P}(A) \land A_2 \in \mathcal{P}(A) \in \mathbb{Z}) \to A_1 \cap A_2 \in \mathcal{P}(A)\,)$
	The class of regular languages over Σ is closed under complementation.
	The class of regular languages over Σ is closed under union.
	The class of regular languages over Σ is closed under intersection.
	The class of regular languages over Σ is closed under concatenation.
	The class of regular languages over Σ is closed under Kleene star.
	The class of context-free languages over Σ is closed under complementation.
	The class of context-free languages over Σ is closed under union.
	The class of context-free languages over Σ is closed under intersection.
	The class of context-free languages over Σ is closed under concatenation.
	The class of context-free languages over Σ is closed under Kleene star.