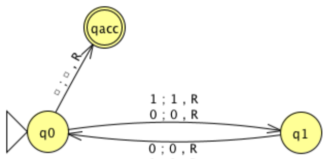
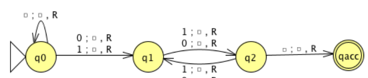
Monday: A_{TM} is recognizable but undecidable

string encoding

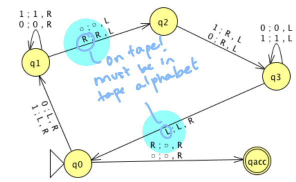
Acceptance problem for Turing machines	A_{TM}	$\{\langle M, w\rangle \mid M \text{ is a Turing machine that accepts input string } w\}$
Language emptiness testing for Turing machines	E_{TM}	$\{\langle M\rangle \mid M \text{ is a Turing machine and } L(M) = \emptyset\}$
Language equality testing for Turing machines	EQ_{TM}	$\{\langle M_1, M_2\rangle \mid M_1 \text{ and } M_2 \text{ are Turing machines and } L(M_1) = L(M_2)\}$



M_1



M_2



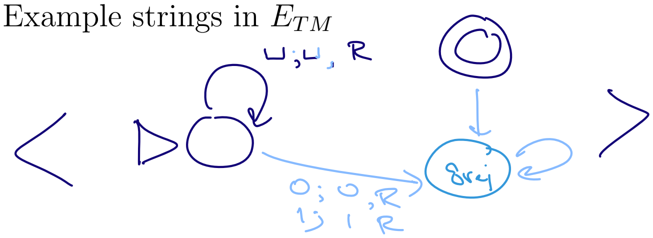
M_3

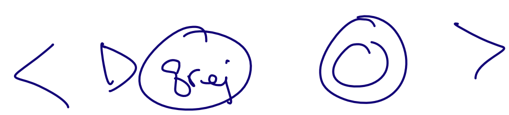
(For these TMs, $\Sigma = \{0,1\}$ $\Gamma = \{0,1,\sqcup,L,R\}$

Example strings in A_{TM}

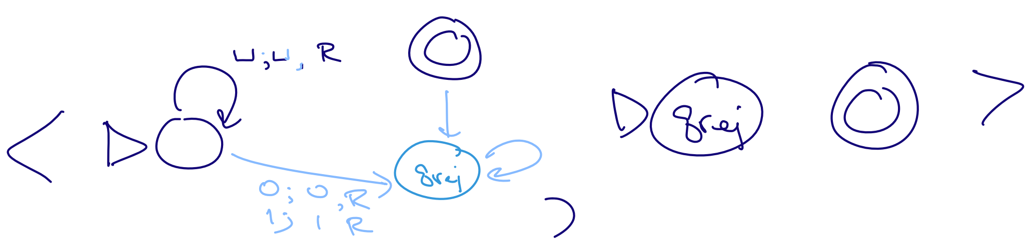
$\langle M_1, 01\rangle$

Example strings in E_{TM}





Example strings in EQ_{TM}



Theorem: A_{TM} is Turing-recognizable.

Strategy: To prove this theorem, we need to define a Turing machine R_{ATM} such that $L(R_{ATM}) = A_{TM}$.

Define R_{ATM} = " On input x

1. If $x \neq \langle M, w \rangle$ M Turing machine, w a string
   reject
2. Otherwise $x = \langle M, w \rangle$ M Turing machine, w a string
   Simulate M on w
3. If M accepts w, accept x.
4. If M rejects w, reject x "

Proof of correctness:

Need two subset inclusions to prove $L(R_{ATM}) = A_{TM}$.

① WTS for each $x \in \Sigma^*$ if $x \in A_{TM}$ then R_{ATM} accepts x.

Consider arbitrary $x \in \Sigma^*$ and assume $x \in A_{TM}$. By definition of A_{TM} $x = \langle M, w \rangle$ for some TM M and string w with $w \in L(M)$. Tracing R_{ATM} on x, step 1 type check passes and in step 2 simulate M on w so since M accepts w, in step 3 R_{ATM} accepts x.

② WTS for each $x \in \Sigma^*$ if $x \notin A_{TM}$ then R_{ATM} does not accept x.

Consider arbitrary $x \in \Sigma^*$ and assume $x \notin A_{TM}$

Case 2a: $x \neq \langle M, w \rangle$ for any M TM, w string. Then R_{ATM} reject x in step 1.

Case 2b: $x = \langle M, w \rangle$, M rejects w. Then R_{ATM} rejects x in step 4.

Case 2c: $x = \langle M, w \rangle$, M loops on w. Then R_{ATM} loops on x in step 2.

We will show that A_{TM} is undecidable. *First, let's explore what that means.*

To prove that a computational problem is **decidable**, we find/ build a Turing machine that recognizes the language encoding the computational problem, and that is a decider.

How do we prove a specific problem is **not decidable**?

How would we even find such a computational problem?

Counting arguments for the existence of an undecidable language:

- The set of all Turing machines is countably infinite.
- Each recognizable language has at least one Turing machine that recognizes it (by definition), so there can be no more Turing-recognizable languages than there are Turing machines.
- Since there are infinitely many Turing-recognizable languages (think of the singleton sets), there are countably infinitely many Turing-recognizable languages.
- Such the set of Turing-decidable languages is an infinite subset of the set of Turing-recognizable languages, the set of Turing-decidable languages is also countably infinite.

Since there are uncountably many languages (because $\mathcal{P}(\Sigma^*)$ is uncountable), there are uncountably many unrecognizable languages and there are uncountably many undecidable languages.

Thus, there's at least one undecidable language!

What's a specific example of a language that is unrecognizable or undecidable?

To prove that a language is undecidable, we need to prove that there is no Turing machine that decides it.

Key idea: proof by contradiction relying on self-referential disagreement.

Theorem: A_{TM} is not Turing-decidable.

Proof: Suppose **towards a contradiction** that there is a Turing machine that decides A_{TM}. We call this presumed machine M_{ATM}.

By assumption, for every Turing machine M and every string w

$\langle M, w\rangle \in A_{TM}$

- If $w \in L(M)$, then the computation of M_{ATM} on $\langle M, w\rangle$ ____accepts____
- If $w \notin L(M)$, then the computation of M_{ATM} on $\langle M, w\rangle$ ____rejects____

M rejects w   M loops on w $\langle M, w\rangle \notin A_{TM}$

Define a **new** Turing machine using the high-level description:

string

$D =$ " On input $\langle M\rangle$, where M is a Turing machine:

Turing machine

1. Run M_{ATM} on $\langle M, \langle M\rangle\rangle$.   string representing Turing machine
2. If M_{ATM} accepts, reject; if M_{ATM} rejects, accept."

Is D a Turing machine?   Yes.

Is D a decider?   step 1: finite time b/c M_{ATM} is a decider
step 2: Boolean so it also takes finite time

Yes.

What is the result of the computation of D on $\langle D\rangle$?

Case 1 D halts and accept $\langle D\rangle$
Then $\langle D, \langle D\rangle\rangle \in A_{TM}$ i.e. M_{ATM} accepts $\langle D, \langle D\rangle\rangle$
Tracing D: Step 1 Run M_{ATM} on $\langle D, \langle D\rangle\rangle$ Step 2. Reject $\langle D\rangle$! $\rightarrow\leftarrow$

Case 2 D halts and rejects $\langle D\rangle$

Then $\langle D, \langle D \rangle \rangle \notin A_{TM}$ i.e. M_{ATM} rejects $\langle D, \langle D \rangle \rangle$

Tracing D: step 1 Run M_{ATM} on $\langle D, \langle D \rangle \rangle$ Step 2: Accept $\langle D \rangle$!

Definition: A language L over an alphabet Σ is called **co-recognizable** if its complement, defined as $\Sigma^* \setminus L = \{x \in \Sigma^* \mid x \notin L\}$, is Turing-recognizable.

A_{TM} is recognizable and undecidable.

$\overline{A_{TM}}$ is co-recognizable b/c $\overline{\overline{A_{TM}}} = A_{TM}$

once we have theorem 4.22, putting these together gives that $\overline{A_{TM}}$ is not recognizable.

But: Σ^*, $\emptyset$, A_{DFA} recognizable and co-recognizable.

Theorem (Sipser Theorem 4.22): A language is Turing-decidable if and only if both it and its complement are Turing-recognizable.

Proof, first direction: Suppose language L is Turing-decidable. WTS that both it and its complement are Turing-recognizable.

Let L be decidable. By definition, there is a decider, call it M_L that recognizes L. Then M_L witnesses that L is recognizable. By closure of the class of decidable languages under complementation, $\overline{L}$ is also decidable, hence also recognizable.

Proof, second direction: Suppose language L is Turing-recognizable, and so is its complement. WTS that L is Turing-decidable.

Suppose L is arbitrary language that is recognizable and corecognizable. Let M and M_{comp} be TMs recognizing L and $\overline{L}$, respectively. Define the new TM

D = " On input x

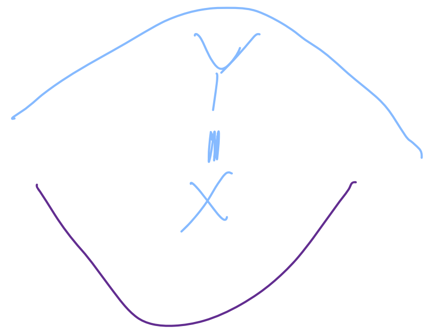
1. For $n = 1, 2, 3, \ldots$
2. Run M on x for at most n steps. If it halts and accepts, accept; if it halts and rejects, reject.
3. Run M_{comp} on x for at most n steps. If it halts and accepts, reject; if it halts and rejects, accept.
4. increment n and go to next loop iteration."

Notation: The complement of a set X is denoted with a superscript c, X^c, or an overline, $\overline{X}$.

Wednesday: Computable functions and reduction

Mapping reduction

Motivation: Proving that A_{TM} is undecidable was hard. How can we leverage that work? Can we relate the decidability / undecidability of one problem to another?



> If problem X is **no harder than** problem Y
>
> …and if Y is easy,
>
> …then X must be easy too.

> If problem X is **no harder than** problem Y
>
> …and if X is hard,
>
> …then Y must be hard too.

"Problem X is no harder than problem Y" means "Can answer questions about membership in X by converting them to questions about membership in Y".

Definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

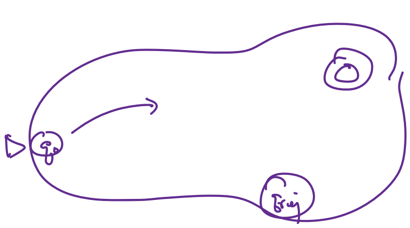
$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

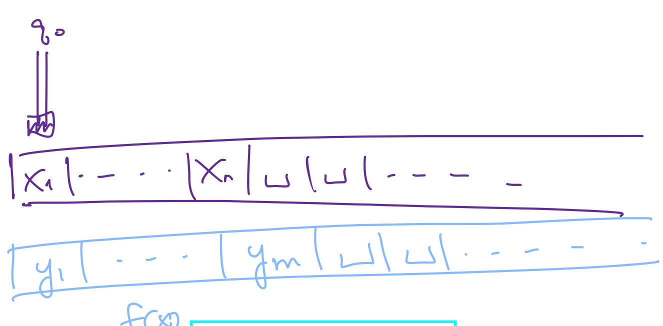
Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Intuition: $A \leq_m B$ means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

To do

1. What is a computable function? ✓
2. How do mapping reductions help? ←





Use Turing machines, interpreted a little differently

Computable functions

a Turing machine computing f(x)

Definition: A function $f : \Sigma^* \to \Sigma^*$ is a **computable function** means there is some Turing machine such that, for each x, on input x the Turing machine halts with exactly $f(x)$ followed by all blanks on the tape

Examples of computable functions:

The function that maps a string to a string which is one character longer and whose value, when interpreted as a fixed-width binary representation of a nonnegative integer is twice the value of the input string (when interpreted as a fixed-width binary representation of a non-negative integer)

$$f_1 : \Sigma^* \to \Sigma^* \qquad f_1(x) = x0$$

shift in binary multiplies by 2

To prove f_1 is computable function, we define a Turing machine computing it.

High-level description

"On input w

1. Append 0 to w.
2. Halt."

Test cases

$f_1(0) = 00$

$f_1(10) = 100$

$f_1(\varepsilon) = 0$

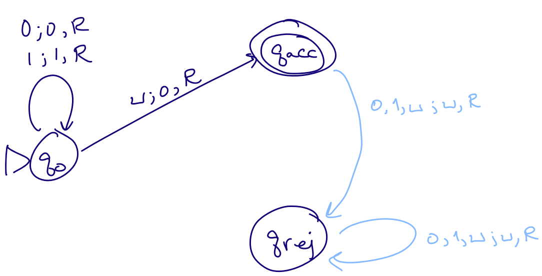
Implementation-level description

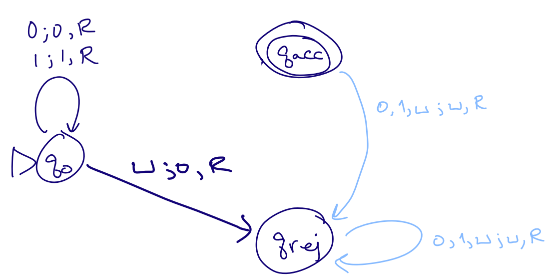
"On input w

1. Sweep read-write head to the right until find first blank cell.
2. Write 0.
3. Halt."

3 states Σ Γ

Formal definition $(\{q0, qacc, qrej\}, \{0,1\}, \{0,1,\sqcup\}, \delta, q0, qacc, qrej)$ where δ is specified by the state diagram:





OR

The function that maps a string to the result of repeating the string twice.

$$f_2 : \Sigma^* \to \Sigma^* \qquad f_2(x) = xx$$

Extra practice: state diagram.

" On input x

1. Output xx ".

The function that maps strings that are not the codes of Turing machines to the empty string and that maps strings that code Turing machines to the code of the related Turing machine that acts like the Turing machine coded by the input, except that if this Turing machine coded by the input tries to reject, the new machine will go into a loop.

default output

one more state

$$f_3 : \Sigma^* \to \Sigma^* \qquad f_3(x) = \begin{cases} \varepsilon & \text{if } x \text{ is not the code of a TM} \\ \langle (Q \cup \{q_{trap}\}, \Sigma, \Gamma, \delta', q_0, q_{acc}, q_{rej}) \rangle & \text{if } x = \langle (Q, \Sigma, \Gamma, \delta, q_0, q_{acc}, q_{rej}) \rangle \end{cases}$$

piecewise definition

string

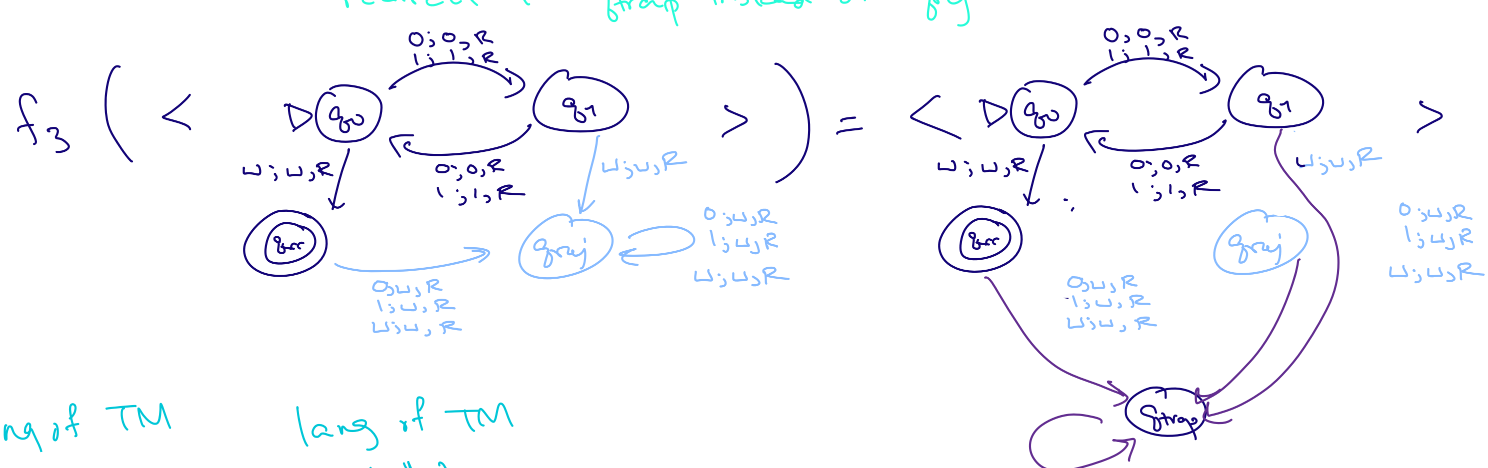
string representing a TM

where $q_{trap} \notin Q$ and

$$\delta'((q,x)) = \begin{cases} (r, y, d) & \text{if } q \in Q,\ x \in \Gamma,\ \delta((q,x)) = (r,y,d), \text{ and } r \neq q_{rej} \\ (q_{trap}, \llcorner\!\lrcorner, R) & \text{otherwise} \end{cases}$$

arrow doesn't point to reject state don't get changed.

redirect to q_{trap} instead of q_{rej}



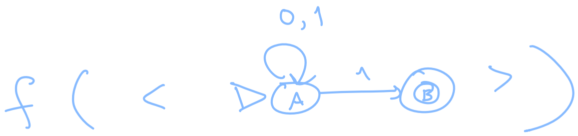
lang of TM coded by input = lang of TM coded by output of f

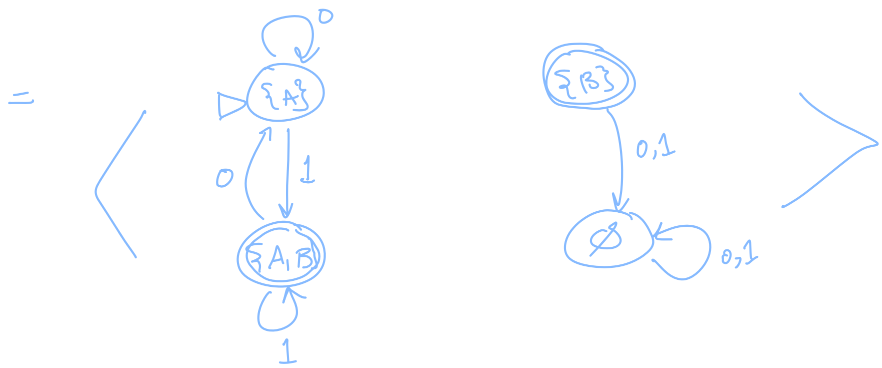
f_3 is computable!

" On input x

1. If $x \neq \langle M \rangle$ for any TM M, output ε.
2. If $x = \langle M \rangle$ output $\langle \_\_ \_\_ \_\_ . \rangle$ by adapting set of states and transition function of M "

The function that maps strings that are not the codes of ~~CFGs~~ NFAs to the empty string and that maps strings that code ~~CFGs~~ NFAs to the code of a ~~PDA~~ DFA that recognizes the language ~~generated by the CFG~~ recognized by the NFA. produced by the macro state construction





"On input x

1. If x is not the code of an NFA, output ε.
2. If $x = \langle N \rangle$ for some NFA, use the macro state construction from Ch 1 to produce DFA D with $L(D) = L(N)$
3. Output $\langle D \rangle$"

The function that maps strings that are not the codes of CFGs to the empty string and that maps strings that code CFGs to the code of a PDA that recognizes the language generated by the CFG.

extra ex.

Other examples?

Definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

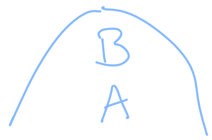
$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Intuition: $A \leq_m B$ means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

Theorem (Sipser 5.22): If $A \leq_m B$ and B is decidable, then A is decidable.

Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.



Pf of 5.22.

Given languages A, B with $A \leq_m B$ and B decidable

Given TM F that computes function $f: \Sigma^* \to \Sigma^*$ w/ $x \in A$ iff $f(x) \in B$.

Given M_B decides B

WTS A is decidable.

Need TM decides A.

"On input x
1. Use TM F to compute $f(x)$.
2. Run M_B on $f(x)$.
3. If M_B accepts $f(x)$, accept.
4. If M_B rejects $f(x)$, reject"

Claim this TM works . . . .

$\boxed{\Xi}$

Friday: The Halting problem

Recall definition: A is **mapping reducible to** B means there is a computable function $f : \Sigma^* \to \Sigma^*$ such that *for all* strings x in Σ^*,

$$x \in A \qquad \text{if and only if} \qquad f(x) \in B.$$

Notation: when A is mapping reducible to B, we write $A \leq_m B$.

Intuition: $A \leq_m B$ means A is no harder than B, i.e. that the level of difficulty of A is less than or equal the level of difficulty of B.

Example: $A_{TM} \leq_m A_{TM}$

Example: $A_{DFA} \leq_m \{ww \mid w \in \{0,1\}^*\}$

Theorem (Sipser 5.22): If $A \leq_m B$ and B is decidable, then A is decidable.

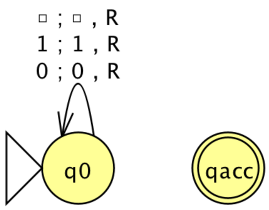
Theorem (Sipser 5.23): If $A \leq_m B$ and A is undecidable, then B is undecidable.

Halting problem

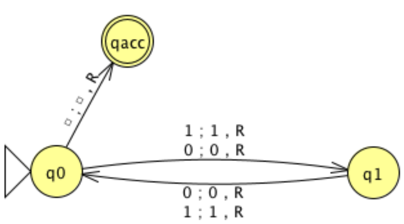
$$HALT_{TM} = \{\langle M, w\rangle \mid M \text{ is a Turing machine, } w \text{ is a string, and } M \text{ halts on } w\}$$

Define $F : \Sigma^* \to \Sigma^*$ by

$$F(x) = \begin{cases} const_{out} & \text{if } x \neq \langle M, w\rangle \text{ for any Turing machine } M \text{ and string } w \text{ over the alphabet of } M \\ \langle M', w\rangle & \text{if } x = \langle M, w\rangle \text{ for some Turing machine } M \text{ and string } w \text{ over the alphabet of } M. \end{cases}$$

where $const_{out} = \langle$ 

$, \varepsilon\rangle$ and M' is a Turing machine that computes like M except, if the computation ever were to go to a reject state, M' loops instead.

$F(\langle$ 

$, \varepsilon\rangle) =$

To use this function to prove that $A_{TM} \leq_m HALT_{TM}$, we need two claims:

Claim (1): F is computable

Claim (2): for every x, $x \in A_{TM}$ iff $F(x) \in HALT_{TM}$.

Week 8 at a glance

Textbook reading: Section 4.1, 4.2, 5.3

For Monday: An undecidable language, Sipser pages 207-209.

For Wednesday: Definition 5.20 and figure 5.21 (page 236)

For Friday: Example 5.24 (page 236)

For Monday of Week 9: Example 5.26 (page 237)

Make sure you can:

- Classify the computational complexity of a set of strings by determining whether it is decidable or undecidable and recognizable or unrecognizable.
  - State, prove, and use theorems relating decidability, recognizability, and co-recognizability.
  - Prove that a language is decidable or recognizable by defining and analyzing a Turing machines with appropriate properties.
- Use diagonalization to prove that there are 'hard' languages relative to certain models of computation.
- Use mapping reduction to deduce the complexity of a language by comparing to the complexity of another.
  - Define computable functions, and use them to give mapping reductions between computational problems
  - Define and explain A_{TM} and $HALT_{TM}$
  - Build and analyze mapping reductions between computational problems

TODO:

Review quizzes based on class material each day.

Homework assignment 4 due this Thursday.

Test 2 next Friday.

