

## Week 5 at a glance

### Textbook reading: Chapter 2

Before Monday, read Introduction to Section 2.1 (pages 101-102).

Before Wednesday, read Section 2.1

Before Friday, read Theorem 2.20.

For Week 6 Monday: Page 165-166 Introduction to Section 3.1.

### We will be learning and practicing to:

- Clearly and unambiguously communicate computational ideas using appropriate formalism. Translate across levels of abstraction.
  - Describe and use models of computation that don't involve state machines.
    - \* **Identify the components of a formal definition of a context-free grammar (CFG)**
    - \* **Derive strings in the language of a given CFG**
    - \* **Determine the language of a given CFG**
    - \* **Design a CFG generating a given language**
    - \* **Use context-free grammars and relate them to languages and pushdown automata.**
  - Use precise notation to formally define the state diagram of a Turing machine
  - Use clear English to describe computations of Turing machines informally.
    - \* **Design a PDA that recognizes a given language.**
  - Give examples of sets that are context-free (and prove that they are).
    - \* **State the definition of the class of context-free languages**
    - \* **Explain the limits of the class of context-free languages**
    - \* **Identify some context-free sets and some non-context-free sets**
- Know, select and apply appropriate computing knowledge and problem-solving techniques. Reason about computation and systems.
  - Describe and prove closure properties of classes of languages under certain operations.
    - \* **Apply a general construction to create a new PDA or CFG from an example one.**
    - \* **Formalize a general construction from an informal description of it.**
    - \* **Use general constructions to prove closure properties of the class of context-free languages.**
    - \* **Use counterexamples to prove non-closure properties of the class of context-free languages.**

## TODO:

Schedule your Test 1 Attempt 1, Test 2 Attempt 1, Test 1 Attempt 2, and Test 2 Attempt 2 times at PrairieTest (<http://us.prairietest.com>) . The first Test 1 sessions are next week!

Review Quiz 4 on PrairieLearn (<http://us.prairielearn.com>), due 2/5/2025

Homework 3 submitted via Gradescope (<https://www.gradescope.com/>), due 2/6/2025

Review Quiz 5 on PrairieLearn (<http://us.prairielearn.com>), due 2/12/2025

# Monday: More Pushdown Automata

**Definition** A **pushdown automaton** (PDA) is specified by a 6-tuple  $(Q, \Sigma, \Gamma, \delta, q_0, F)$  where  $Q$  is the finite set of states,  $\Sigma$  is the input alphabet,  $\Gamma$  is the stack alphabet,

$$\delta : Q \times \Sigma_{\epsilon} \times \Gamma_{\epsilon} \rightarrow \mathcal{P}(Q \times \Gamma_{\epsilon})$$

is the transition function,  $q_0 \in Q$  is the start state,  $F \subseteq Q$  is the set of accept states.

For the PDA state diagrams below,  $\Sigma = \{0, 1\}$ .

Mathematical description of language

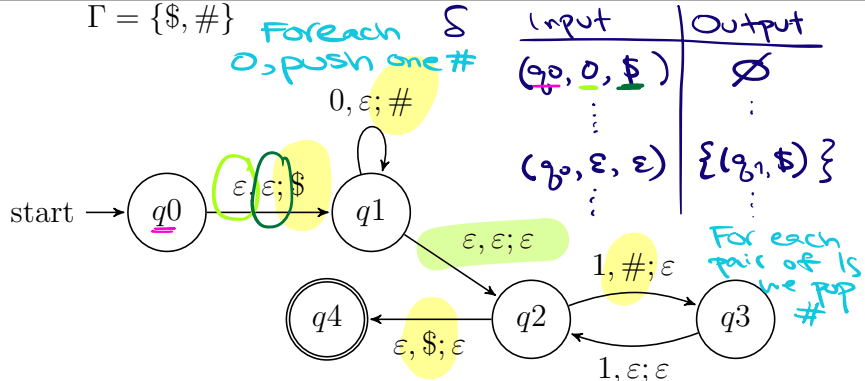
State diagram of PDA recognizing language

$$\{0^n 1^{2n} \mid n \geq 0\}$$

$$\epsilon = 0^0 1^{2 \cdot 0} = 0^0 1^0$$

$q_0, q_1, q_2, q_4$  Acc

$$\Gamma = \{\$, \#\}$$



011↓

$q_0, q_1, q_2, q_3, q_4$

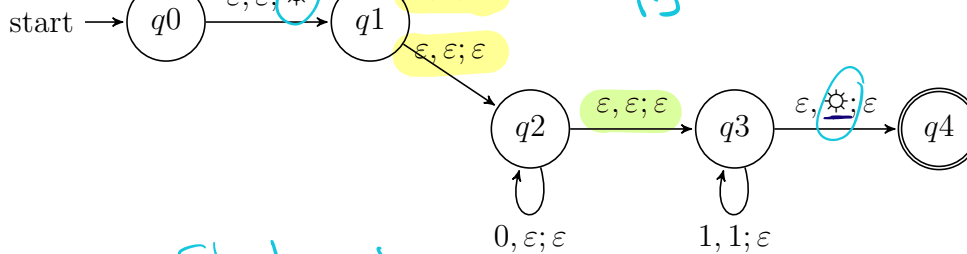
$$(\{q_0, q_1, q_2, q_3, q_4\}, \{0, 1\}, \{\$, \#\}, \delta, q_0, \{q_4\})$$

$$\Gamma = \{\$, 1\}$$

$$\{1^n 0^n 1^m \mid n, m \geq 0\}$$

U

$$\{1^n 0^m 1^n \mid n, m \geq 0\}$$

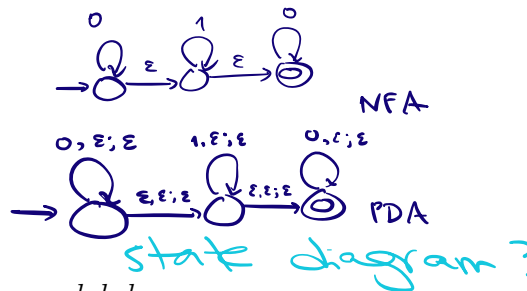


language?

State diagram

$$\{0^i 1^j 0^k \mid i, j, k \geq 0\}$$

language



Note: alternate notation is to replace ; with  $\rightarrow$  on arrow labels.

Corollary: for each language  $L$  over  $\Sigma$ , if there is an NFA  $N$  with  $L(N) = L$  then there is a PDA  $M$  with  $L(M) = L$

Proof idea: Declare stack alphabet to be  $\Gamma = \Sigma$  and then don't use stack at all.

Given NFA  $(Q, \Sigma, \delta, q_0, F)$  define  
PDA  $(Q, \Sigma, \Sigma, \delta_{\text{new}}, q_0, F)$  with  
transition function  $\delta_{\text{new}}: Q \times \Sigma_\epsilon \times \Sigma_\epsilon \rightarrow \mathcal{P}(Q \times \Sigma_\epsilon)$   
given by  
$$\delta_{\text{new}}(q, x, y) = \begin{cases} \{(q', \epsilon) \mid q' \in \delta(q, x)\} & \text{if } q \in Q, x \in \Sigma_\epsilon, y = \epsilon \\ \emptyset & \text{if } q \in Q, x \in \Sigma_\epsilon, y \in \Sigma \end{cases}$$

*Big picture:* PDAs are motivated by wanting to add some memory of unbounded size to NFA. How do we accomplish a similar enhancement of regular expressions to get a syntactic model that is more expressive?

DFA, NFA, PDA: Machines process one input string at a time; the computation of a machine on its input string reads the input from left to right.

Regular expressions: Syntactic descriptions of all strings that match a particular pattern; the language described by a regular expression is built up recursively according to the expression's syntax

**Context-free grammars:** Rules to produce one string at a time, adding characters from the middle, beginning, or end of the final string as the derivation proceeds.

# Wednesday: Context-free Grammars and Languages

Definitions below are on pages 101-102.

Term	Typical symbol or Notation	Meaning
<b>Context-free grammar</b> (CFG)	$G$	$G = (V, \Sigma, R, S)$
The set of <b>variables</b>	$V$	Finite set of symbols that represent phases in production pattern
The set of <b>terminals</b>	$\Sigma$	Alphabet of symbols of strings generated by CFG $V \cap \Sigma = \emptyset$
The set of <b>rules</b>	$R$	Each rule is $A \rightarrow u$ with $A \in V$ and $u \in (V \cup \Sigma)^*$
The <b>start</b> variable	$S$	Usually on left-hand-side of first/ topmost rule
<b>Derivation</b>	$S \Rightarrow \dots \Rightarrow w$	Sequence of substitutions in a CFG (also written $S \Rightarrow^* w$ ). At each step, we can apply one rule to one occurrence of a variable in the current string by substituting that occurrence of the variable with the right-hand-side of the rule. The derivation must end when the current string has only terminals (no variables) because then there are no instances of variables to apply a rule to.
Language <b>generated</b> by the context-free grammar $G$	$L(G)$	The set of strings for which there is a derivation in $G$ . Symbolically: $\{w \in \Sigma^* \mid S \Rightarrow^* w\}$ i.e.  $\{w \in \Sigma^* \mid \text{there is derivation in } G \text{ that ends in } w\}$
<b>Context-free language</b>		A language that is the language generated by some context-free grammar

Examples of context-free grammars, derivations in those grammars, and the languages generated by those grammars

$G_1 = (\{S\}, \{0\}, R, S)$  with rules

$$\begin{aligned} S &\rightarrow 0S \\ S &\rightarrow 0 \end{aligned}$$

In  $L(G_1)$  ...

Not in  $L(G_1)$  ...

$$G_2 = (\{S\}, \{0, 1\}, R, S)$$

$$S \rightarrow 0S \mid 1S \mid \varepsilon$$

In  $L(G_2) \dots$

Not in  $L(G_2) \dots$

$(\{S, T\}, \{0, 1\}, R, S)$  with rules

$$S \rightarrow T1T1T1T$$

$$T \rightarrow 0T \mid 1T \mid \varepsilon$$

In  $L(G_3) \dots$

Not in  $L(G_3) \dots$

$G_4 = (\{A, B\}, \{0, 1\}, R, A)$  with rules

$$A \rightarrow 0A0 \mid 0A1 \mid 1A0 \mid 1A1 \mid 1$$

In  $L(G_4) \dots$

Not in  $L(G_4) \dots$

Design a CFG to generate the language  $\{a^n b^n \mid n \geq 0\}$

Design a CFG to generate the language  $\{a^i b^j \mid j \geq i \geq 0\}$

Design a PDA to recognize the language  $\{a^i b^j \mid j \geq i \geq 0\}$

## Friday: Context-free and non-context-free languages

**Theorem 2.20:** A language is generated by some context-free grammar if and only if it is recognized by some push-down automaton.

Definition: a language is called **context-free** if it is the language generated by a context-free grammar. The class of all context-free languages over a given alphabet  $\Sigma$  is called **CFL**.

Consequences:

- Quick proof that every regular language is context free
- To prove closure of the class of context-free languages under a given operation, we can choose either of two modes of proof (via CFGs or PDAs) depending on which is easier
- To fully specify a PDA we could give its 6-tuple formal definition or we could give its input alphabet, stack alphabet, and state diagram. An informal description of a PDA is a step-by-step description of how its computations would process input strings; the reader should be able to reconstruct the state diagram or formal definition precisely from such a description. The informal description of a PDA can refer to some common modules or subroutines that are computable by PDAs:
  - PDAs can “test for emptiness of stack” without providing details. *How?* We can always push a special end-of-stack symbol,  $\$$ , at the start, before processing any input, and then use this symbol as a flag.
  - PDAs can “test for end of input” without providing details. *How?* We can transform a PDA to one where accepting states are only those reachable when there are no more input symbols.



Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . **Goal:**  $L_1 \cup L_2$  is also context-free.

*Approach 1: with PDAs*

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define  $M =$

*Approach 2: with CFGs*

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define  $G =$

Suppose  $L_1$  and  $L_2$  are context-free languages over  $\Sigma$ . **Goal:**  $L_1 \circ L_2$  is also context-free.

*Approach 1: with PDAs*

Let  $M_1 = (Q_1, \Sigma, \Gamma_1, \delta_1, q_1, F_1)$  and  $M_2 = (Q_2, \Sigma, \Gamma_2, \delta_2, q_2, F_2)$  be PDAs with  $L(M_1) = L_1$  and  $L(M_2) = L_2$ .

Define  $M =$

*Approach 2: with CFGs*

Let  $G_1 = (V_1, \Sigma, R_1, S_1)$  and  $G_2 = (V_2, \Sigma, R_2, S_2)$  be CFGs with  $L(G_1) = L_1$  and  $L(G_2) = L_2$ .

Define  $G =$

*Summary*

Over a fixed alphabet  $\Sigma$ , a language  $L$  is **regular**

iff it is described by some regular expression  
iff it is recognized by some DFA  
iff it is recognized by some NFA

Over a fixed alphabet  $\Sigma$ , a language  $L$  is **context-free**

iff it is generated by some CFG  
iff it is recognized by some PDA

**Fact:** Every regular language is a context-free language.

**Fact:** There are context-free languages that are not nonregular.

**Fact:** There are countably many regular languages.

**Fact:** There are countably infinitely many context-free languages.

*Consequence:* Most languages are **not** context-free!

## Examples of non-context-free languages

$$\begin{aligned} &\{a^n b^n c^n \mid 0 \leq n, n \in \mathbb{Z}\} \\ &\{a^i b^j c^k \mid 0 \leq i \leq j \leq k, i \in \mathbb{Z}, j \in \mathbb{Z}, k \in \mathbb{Z}\} \\ &\{ww \mid w \in \{0,1\}^*\} \end{aligned}$$

(Sipser Ex 2.36, Ex 2.37, 2.38)

There is a Pumping Lemma for CFL that can be used to prove a specific language is non-context-free: If  $A$  is a context-free language, there is a number  $p$  where, if  $s$  is any string in  $A$  of length at least  $p$ , then  $s$  may be divided into five pieces  $s = uvxyz$  where (1) for each  $i \geq 0$ ,  $uv^i xy^i z \in A$ , (2)  $|uv| > 0$ , (3)  $|vxy| \leq p$ . *We will not go into the details of the proof or application of Pumping Lemma for CFLs this quarter.*

Recall: A set  $X$  is said to be **closed** under an operation  $OP$  if, for any elements in  $X$ , applying  $OP$  to them gives an element in  $X$ .

True/False	Closure claim
True	The set of integers is closed under multiplication. $\forall x \forall y ( (x \in \mathbb{Z} \wedge y \in \mathbb{Z}) \rightarrow xy \in \mathbb{Z} )$
True	For each set $A$ , the power set of $A$ is closed under intersection. $\forall A_1 \forall A_2 ( (A_1 \in \mathcal{P}(A) \wedge A_2 \in \mathcal{P}(A)) \rightarrow A_1 \cap A_2 \in \mathcal{P}(A) )$
	The class of regular languages over $\Sigma$ is closed under complementation.
	The class of regular languages over $\Sigma$ is closed under union.
	The class of regular languages over $\Sigma$ is closed under intersection.
	The class of regular languages over $\Sigma$ is closed under concatenation.
	The class of regular languages over $\Sigma$ is closed under Kleene star.
	The class of context-free languages over $\Sigma$ is closed under complementation.
	The class of context-free languages over $\Sigma$ is closed under union.
	The class of context-free languages over $\Sigma$ is closed under intersection.
	The class of context-free languages over $\Sigma$ is closed under concatenation.
	The class of context-free languages over $\Sigma$ is closed under Kleene star.