

Examples of Divide and Conquer and the Master theorem

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Divide and Conquer

Reduce to any number of smaller instances:

1. **Divide** the given problem instance into subproblems
2. **Conquer** the subproblems by solving them recursively
3. **Combine** the solutions for the subproblems to a solution for the original problem

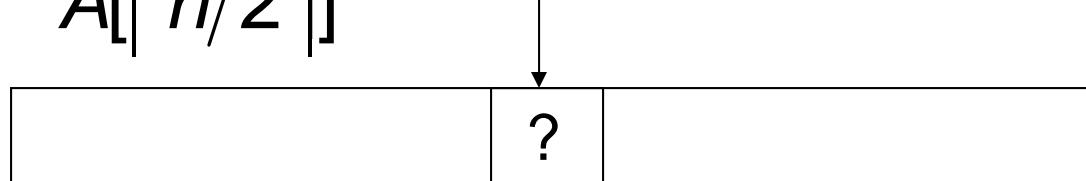
Search Problem

- **Input:** A set of numbers a_1, a_2, \dots, a_n and a number x
- **Question:** Is x one of the numbers a_i in the given set?

If sorted array A of numbers a_1, a_2, \dots, a_n then **Binary Search**

Binary Search

- Compare x to the middle element of the array $A[\lceil n/2 \rceil]$



- If $x = A[\lceil n/2 \rceil]$ then done
- If $x < A[\lceil n/2 \rceil]$ then
 - recursively Search $A[1, \dots, \lceil n/2 \rceil - 1]$
- If $x > A[\lceil n/2 \rceil]$ then
 - recursively Search $A[\lceil n/2 \rceil + 1, \dots, n]$

Binary Search Analysis

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + \Theta(1) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Can use Master Theorem

$$a=1, b=2, f(n)=\Theta(1), n^{\log_b a} = n^{\log_2 1} = n^0 = 1$$

Case 2 $\Rightarrow T(n) = \Theta(\log n)$

Merge Sort

1. **Divide:** Divide the given n -element sequence to be sorted into two sequences of length $n/2$
2. **Conquer:** Sort recursively the two subsequences using Merge Sort
3. **Combine:** Merge the two sorted subsequences to produce the sorted answer

Merge

- **Input:** Sorted arrays $K[1..n_1]$, $L[1..n_2]$
- **Output:** Merged sorted array $M[1.. n_1+n_2]$

```
i = 1, j = 1
for t = 1 to  $n_1 + n_2$ 
{ if ( $i \leq n_1$  and (  $j > n_2$  or  $K[i] < L[j]$  ) )
  then { M[t] = K[i], i = i + 1 }
  else { M[t] = L[j], j = j + 1 }
}
```

Linear Time Complexity: $\Theta(n_1 + n_2)$

What if inputs, output in same array?

- **Input:** Sorted array segments
 $A[1..n_1]$, $A[n_1+1..n_1+n_2]$
- **Output:** Merged sorted array $A[1..n_1+n_2]$

Copy $A[1..n_1]$ into new array $K[1..n_1]$

Copy $A[n_1+1..n_1+n_2]$ into $L[1..n_2]$

Merge $K[1..n_1]$ and $L[1..n_2]$ into $A[1..n_1+n_2]$

Linear Time Complexity: $\Theta(n_1 + n_2)$

Merge-Sort

Merge-Sort $A[1 \dots n]$

If $n > 1$ then

1. Recursively merge-sort $A[1 \dots \lfloor n/2 \rfloor]$
and $A[\lfloor n/2 \rfloor + 1 \dots n]$
2. Merge the two sorted subsequences

Analysis of Merge-Sort

$$T(n) = \begin{cases} T(\lfloor n/2 \rfloor) + T(\lceil n/2 \rceil) + \Theta(n) & \text{for } n > 1 \\ \Theta(1) & \text{for } n = 1 \end{cases}$$

Assume for simplicity that n is a power of 2

$$T(n) = 2T(n/2) + cn$$

Can use Master Theorem

$$a=2, b=2, f(n)=\Theta(n), n^{\log_b a} = n^{\log_2 2} = n$$

Case 2 $\Rightarrow T(n) = \Theta(n \log n)$

Maximum Sum Subarray Problem

- **Input:** Array $A[1\dots n]$ of integers (positive and negative)
- **Problem:** Compute a subarray $A[i^* \dots j^*]$ with maximum sum

i.e., if $s(i,j)$ denotes the sum of the elements of a subarray $A[i\dots j]$,

$$s(i, j) = \sum_{k=i}^j A[k]$$

We want to compute indices $i^* \leq j^*$ such that

$$s(i^*, j^*) = \max\{s(i, j) \mid 1 \leq i \leq j \leq n\}$$

Example: 3 -4 5 -2 -2 6 -3 5 -3 2


$$\text{max sum} = 9$$

Brute force solution

- Compute the sum of every subarray and pick the maximum
Try every pair of indices i, j with $1 \leq i \leq j \leq n$, and for each one compute $s(i, j) = \sum_{k=i}^j A[k]$
- Time complexity $\Theta(n^3)$
- With a little more care, can improve to $\Theta(n^2)$:
can compute the sums of all the subarrays in time $\Theta(n^2)$.

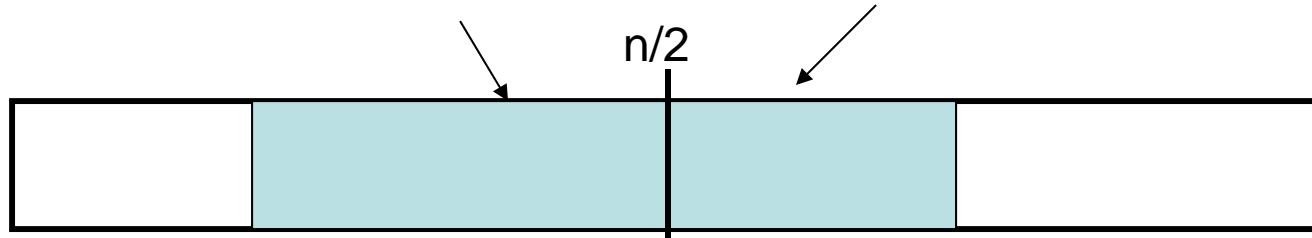
Brute force solution - improved

- With a little more care, can improve to $\Theta(n^2)$:
- Can compute the sums for all subarrays with same left end in $O(n)$ time \Rightarrow compute the sums of all the subarrays (there are $n(n-1)/2 + n$ subarrays) in time $O(n^2)$

```
for i =1 to n
{ s(i,i)=A[i]
  for j=i+1 to n
    s(i,j) = s(i,j-1)+A[j]
}
```

Divide and Conquer

- A subarray $A[i^* \dots j^*]$ with maximum sum is
 - Either contained entirely in the first half , i.e. $j^* \leq n/2$
 - Or contained entirely in the right half , i.e. $i^* \geq n/2$
 - Or overlaps both halves: $i^* \leq n/2 \leq j^*$
- We can compute the best subarray of the first two types with recursive calls on the left and right half.
- The best subarray of the third type consists of the best subarray that ends at $n/2$ and the best subarray that starts at $n/2$. We can compute these in $O(n)$ time.



Divide and Conquer analysis

- Recurrence: $T(n) = 2T(n/2) + \Theta(n)$
- Solution: $T(n) = \Theta(n \log n)$
- It is possible to do better: can compute the maximum sum subarray in $\Theta(n)$ time.
(Not divide and conquer)

For a nice paper on this problem see

J. Bentley, Programming Pearls, Addison-Wesley,
chapter 8 (Algorithm Design Techniques)

Also in Communications of the ACM, 27(9), 1984.

Multiplication of Big integers

Count # bit operations

- Given integers A, B with n bits each, can + , - in O(n) time.
- Ordinary multiplication: n^2 time (n additions)
- D&C: partition into high n/2 and low n/2 bits

$$A \quad \boxed{A_h \quad A_l} \quad A = A_h \cdot 2^{n/2} + A_l$$

$$B \quad \boxed{B_h \quad B_l} \quad B = B_h \cdot 2^{n/2} + B_l$$

$$\begin{aligned} A \cdot B &= (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l) \\ &= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l \end{aligned}$$

Multiplication of Big integers

$$\begin{aligned} A \cdot B &= (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l) \\ &= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l \end{aligned}$$

4 multiplications of $n/2$ -bit numbers: $A_h B_h$, $A_h B_l$, $A_l B_h$, $A_l B_l$,
additions and shifts.

Note: multiplications by powers of 2 are just shifts

Recurrence: $T(n) = 4T(n/2) + cn$
(last term for additions and shifts)

Solution: $T(n) = O(n^2)$

Multiplication of Big integers – Karatsuba'60

$$\begin{aligned} A \cdot B &= (A_h \cdot 2^{n/2} + A_l) \cdot (B_h \cdot 2^{n/2} + B_l) \\ &= A_h B_h \cdot 2^n + A_h B_l \cdot 2^{n/2} + A_l B_h 2^{n/2} + A_l B_l \end{aligned}$$

$$\begin{aligned} (A_h + A_l)(B_h + B_l) &= A_h B_l + A_h B_h + A_l B_l + A_l B_h \Rightarrow \\ A \cdot B &= A_h B_h \cdot 2^n + [(A_h + A_l)(B_h + B_l) - A_h B_h - A_l B_h] \cdot 2^{n/2} + A_l B_l \end{aligned}$$

3 multiplications of n/2-bit numbers:

$$A_h B_h, A_l B_l, (A_h + A_l)(B_h + B_l)$$

+ additions, subtractions and shifts.

Recurrence: $T(n) = 3T(n/2) + cn$

Solution: $T(n) = n^{\log_2 3} = n^{1.585}$

Multiplication of Big integers – Karatsuba'60

Recursive Algorithm MULT(A,B)

Write $A = A_h 2^{n/2} + A_l$ and $B = B_h 2^{n/2} + B_l$

Compute $a = A_h + A_l$ and $b = B_h + B_l$

$C = \text{MULT}(a,b)$

$D_h = \text{MULT}(A_h, B_h)$

$D_l = \text{MULT}(A_l, B_l)$

Return $D_h \cdot 2^n + [C - D_h - D_l] \cdot 2^{n/2} + D_l$

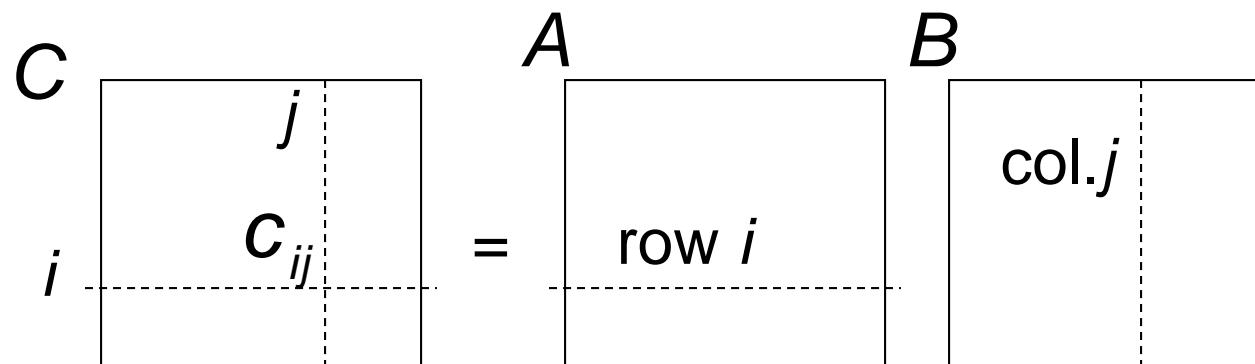
Time: $T(n) = n^{\log_2 3} = n^{1.585}$

FFT-based method: $n \log n$

Matrix Multiplication

Input: Matrices $A = [a_{ij}]$, $B = [b_{ij}]$, $i, j = 1, \dots, n$

Output: $C = [c_{ij}] = A \cdot B$



$$c_{ij} = \sum_{k=1}^n a_{ik} b_{kj}$$

Standard Matrix Multiplication algorithm

for $i = 1$ to n

for $j = 1$ to n

{ $C_{ij} = 0$

for $k = 1$ to n

$C_{ij} = C_{ij} + a_{ik} b_{kj}$

}

Time Complexity: $\Theta(n^3)$

Divide and Conquer

Partition matrices A, B, C into 4 $n/2 \times n/2$ submatrices

$$\begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix}$$

$$C_{11} = A_{11} B_{11} + A_{12} B_{21}$$

$$C_{12} = A_{11} B_{12} + A_{12} B_{22}$$

$$C_{21} = A_{21} B_{11} + A_{22} B_{21}$$

$$C_{22} = A_{21} B_{12} + A_{22} B_{22}$$

8 recursive multiplications of

$n/2 \times n/2$ matrices

4 additions (direct – no recursion)

$$T(n) = 8T(n/2) + \Theta(n^2)$$

$$a = 8, b = 2, f(n) = \Theta(n^2), n^{\log_b a} = n^3$$

Case 1 $\Rightarrow T(n) = \Theta(n^3)$

Same as standard MM algorithm

Strassen's algorithm

- $P = (A_{11} + A_{22})(B_{11} + B_{22})$
 $Q = (A_{21} + A_{22})B_{11}$
 $R = A_{11}(B_{12} - B_{22})$
 $S = A_{22}(B_{21} - B_{11})$
 $T = (A_{11} + A_{12})B_{22}$
 $U = (A_{21} - A_{11})(B_{11} + B_{12})$
 $V = (A_{12} - A_{22})(B_{21} + B_{22}).$
- $C_{11} = P + S - T + V$
 $C_{12} = R + T$
 $C_{21} = Q + S$
 $C_{22} = P + R - Q + U$

Strassen's algorithm

- Can multiply 2×2 matrices with 7 multiplications, and 18 additions and subtractions. The method does not assume commutativity of multiplication
- Method applies to multiplication of 2×2 block matrices.
- Can be used in divide and conquer scheme with 7 recursive multiplications of $n/2 \times n/2$ submatrices.

$$T(n) = 7 T(n/2) + \Theta(n^2)$$

Strassen's Algorithm

$$T(n) = 7 (n/2) + \Theta(n^2)$$

$$a = 7, b = 2, f(n) = \Theta(n^2), n^{\log_b a} = n^{\log_2 7} \approx n^{2.81}$$

Case 1 $\Rightarrow T(n) = \Theta(n^{\log 7})$

Best current (theoretical) result: $\Theta(n^{2.373})$