Greedy Algorithms

Informal Definition A greedy algorithm makes its next step based only on the current "state" and "simple" calculations on the input.

- "easy" to design
- not always correct
- challenge is to identify when greedy is the correct solution

Examples

- Rod cutting is not greedy. e.g. $\mathit{profit} = (5, 10, 11, 15)$
- Matrix Chain is not greedy.
- Change with U.S. coins is greedy
- Shortest paths with non-negative edge lengths is greedy, but not in the obvious way.

Greedy

Consider a set of requests for a room. Only one person can reserve the room at a time, and you want to allow the maximum number of requests. The requests for periods (s_i, f_i) are:

$$(1,4),(3,5),(0,6),(5,7),(3,8),(5,9),(6,10),(8,11),(8,12),(2,13),(12,14)$$

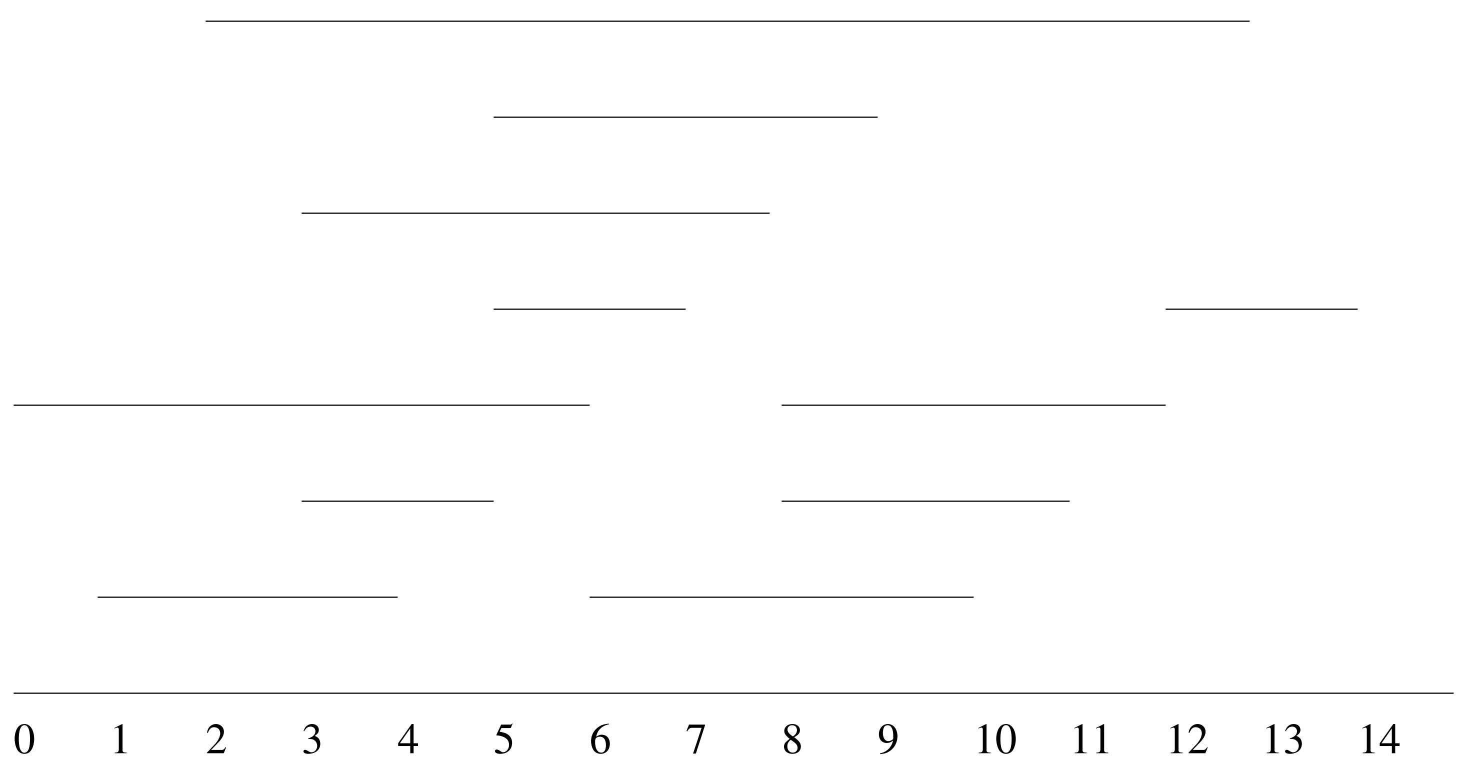
Which ones should we schedule?

Greedy

Consider a set of requests for a room. Only one person can reserve the room at a time, and you want to allow the maximum number of requests. The requests for periods (s_i, f_i) are:

$$(1,4),(3,5),(0,6),(5,7),(3,8),(5,9),(6,10),(8,11),(8,12),(2,13),(12,14)$$

Which ones should we schedule?



Code

Sort by finishing time, renumber with 1 having earliest finishing time
Output 1
$last = f_1$
for $i = 2$ **to** n
    do if $(s_i \geq last)$
        then Output i
            $last = f_i$

Proving a Greedy Algorithm is Optimal

Two components:

1. **Optimal substructure**
2. **Greedy Choice Property: There exists an optimal solution that is consistent with the greedy choice made in the first step of the algorithm.**

Optimal Substructure

- **Let $c[i, j]$ be the number of activities scheduled from time i to time j**

$$c[i,j] = \begin{cases} 0 & \textbf{if } S_{ij} = \emptyset \ , \\ \max_{a_k \in S_{ij}} \{c[i, s_k] + c[f_k, j] + 1\} & \textbf{if } S_{ij} \neq \emptyset \end{cases} . \tag{1}$$

Greedy Choice

Greedy Choice Property

1. Let S_k be a nonempty subproblem containing the set of activities that finish after activity a_k.
2. Let a_m be an activity in S_k with the earliest finish time.
3. Then a_m is included in some maximum-size subset of mutually compatible activities of S_k.

Proof

- Let A_k be a maximum-size subset of mutually compatible activities in S_k,
- let a_j be the activity in A_k with the earliest finish time.
- If $a_j = a_m$, we are done, since we have shown that a_m is in some maximum-size subset of mutually compatible activities of S_k.
- If $a_j \neq a_m$, let the set $A'_k = A_k - \{a_j\} \cup \{a_m\}$
- The activities in A'_k are disjoint, because
  - the activities in A_k are disjoint,
  - a_j is the first activity in A_k to finish,
  - $f_m \leq f_j$.
- Since $|A'_k| = |A_k|$, we conclude that A'_k is a maximum-size subset of mutually compatible activities of S_k, and it includes a_m.

Procedure for Designing a Greedy Algorithm

1. **Identify optimal substructure**
2. **Cast the problem as a greedy algorithm with the greedy choice property**
3. **Write a simple iterative algorithm**

Robbery

- I want to rob a house and I have a knapsack which holds B pounds of stuff
- I want to fill the knapsack with the most profitable items

item	1	2	3
weight	10	20	30
value	60	100	120
value/weight	6	5	4

Two variants

- **integral knapsack**: Take an item or leave it
- **fractional knapsack**: Can take a fraction of an item (infinitely divisible)

Fractional vs. Integral Knapsack

- **Both fractional and integral knapsack have optimal substructure.**
- **Only fractional knapsack has the greedy choice property.**

Fractional Knapsack

Greedy Choice Property: **Let j be the item with maximum v_i/w_i. Then there exists an optimal solution in which you take as much of item j as possible.**

Proof

- **Suppose fpoc, that there exists an optimal solution in you didn't take as much of item j as possible.**
- **If the knapsack is not full, add some more of item j, and you have a higher value solution. Contradiction**
- **We thus assume the knapsack is full.**
- **There must exist some item $k \neq j$ with $\frac{v_k}{w_k} < \frac{v_j}{w_j}$ that is in the knapsack.**
- **We also must have that not all of j is in the knapsack.**
- **We can therefore take a piece of k, with ϵ weight, out of the knapsack, and put a piece of j with ϵ weight in.**
- **This increases the knapsack's value by**

$$\epsilon \frac{v_j}{w_j} - \epsilon \frac{v_k}{w_k} = \epsilon \left(\frac{v_j}{w_j} - \frac{v_k}{w_k} \right) > 0$$

Contradition to the original solution being optimal.

Algorithm

1. **Sort items by v_j/w_j, renumber.**
2. **For $i = 1$ to n**
   - **Add as much of item i as possible**

Question **Why does this fail for integer knapsack?**

Dynamic Programming Algorithm

- Let $A[x, W]$ be the maximum value obtainable from items $1, \ldots, x$ using at most W weight
- To compute $A[x, W]$, either
  1. item x is in the best solution
  2. item x is not.

Dynamic Programming Algorithm

- **Let $A[x, W]$ be the maximum value obtainable from items $1, \ldots, x$ using at most W weight**
- **To compute $A[x, W]$, either**
  1. **item x is in the best solution – include x, along with the best solution from $1, \ldots, x-1$ that, along with x has weight at most $W - w_x$.**
  2. **item x is not – then just use the best solution from $1, \ldots, x-1$ that has weight at most W.**

Dynamic Programming Algorithm

- **Let $A[x, W]$ be the maximum value obtainable from items $1, \ldots, x$ using at most W weight**
- **To compute $A[x, W]$, either**
  1. **item x is in the best solution – include x , along with the best solution from $1, \ldots, x-1$ that, along with x has weight at most W .**
  2. **item x is not – then just use the best solution from $1, \ldots, x-1$ that has weight at most W .**

$$A[x, W] = \max\{A[x-1, W-w_i] + v_i, A[x-1, W]\}$$