

# Randomization in Algorithms

- Randomization is a *tool* for designing good algorithms.
- Two kinds of algorithms
  - **Las Vegas** - always correct, running time is random.
  - **Monte Carlo** - may return incorrect answers, but running time is deterministic.

# Hiring Problem

*Hire – Assistant(n)*

```
1   best ← 0           ▷ candidate 0 is a least-qualified dummy candidate
2   for  $i \leftarrow 1$  to  $n$ 
3       do interview candidate  $i$ 
4       if candidate  $i$  is better than candidate  $best$ 
5           then  $best \leftarrow i$ 
6           hire candidate  $i$ 
```

How many times is a new person hired?

## Analysis

- A **random variable**  $X$  takes on values from some set, each with a certain probability.
- Expected value:  $E[X] = \sum_{\text{values } x} \Pr(X = x) \cdot x$
- Example: rolling a die.

## Expected number of hirings

- Assume that all orderings of candidates are equally likely.
- $n!$  orderings,  $\pi_1, \pi_2, \dots, \pi_{n!}$
- $H$  is the total number of hirings.
- $h(\pi_i)$  is the number of hirings for permutation  $\pi_i$ .

$$E[H] = \sum_{\pi_i} \frac{1}{n!} h(\pi_i)$$

How do we compute  $E[H]$ ?

## Indicator random variables

- Let  $A$  be an event.
- The indicator variable  $I\{A\}$  is defined by:

$$I\{A\} = \begin{cases} 1 & \text{if } A \text{ occurs ,} \\ 0 & \text{if } A \text{ does not occur .} \end{cases} \quad (1)$$

What is the expected number of heads when I flip a coin?

- Let  $Y$  be a random variable that denotes heads or tails.
- Let  $X_H$  be the i.r.v. that counts the number of heads.

$$X_H = I\{Y \text{ is heads}\} = \begin{cases} 1 & \text{if Y is heads} \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} E[X_H] &= \Pr(X_H = 1) \cdot 1 + \Pr(X_H = 0) \cdot 0 \\ &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 \\ &= \frac{1}{2} \end{aligned}$$

## Linearity of Expectation

Let  $X$  and  $Y$  be two random variables

$$E[X + Y] = E[X] + E[Y]$$

Linearity of expectation holds even if  $X$  and  $Y$  are dependent.

## $n$ coin flips

- What is  $E[\text{number of heads}]$  when you flip  $n$  coins.
- Different events are:
  - 0 heads
  - 1 head
  - 2 heads
  - 3 heads
  - ...

$$E[\text{number of heads}] = \sum_{i=0}^n \Pr(\text{i heads in n flips}) \cdot i$$

- Complicated calculation
- Is there another way?

## Use indicator random variables

- Divide events not by number of heads overall, but by heads in  $i$ th flip.
- Let  $X_i$  be the indicator random variable associated with the event in which the  $i$ th flip comes up heads:
- $X_i = I\{\text{the } i\text{th flip results in the event } H\}$ .
- Let  $X$  be the random variable denoting the total number of heads in the  $n$  coin flips
- $X = \sum_{i=1}^n X_i$  .
- We take the expectation of both sides  $E[X] = E[\sum_{i=1}^n X_i]$  .

$$\begin{aligned}E[X] &= E\left[\sum_{i=1}^n X_i\right] \\&= \sum_{i=1}^n E[X_i] \\&= \sum_{i=1}^n 1/2 \\&= n/2 .\end{aligned}$$

## Hiring

- Divide events not by number of hires overall, but by hires in  $i$ th flip.
- Let  $X_i$  be the indicator random variable associated with the event in which the  $i$ th person is hired
- $X_i = I\{\text{the } i\text{th person is hired}\}$ .
- Let  $X$  be the random variable denoting the total number of people hired.
- $X = \sum_{i=1}^n X_i$  .
- We take the expectation of both sides  $E[X] = E[\sum_{i=1}^n X_i]$  .

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^n X_i\right] \\ &= \sum_{i=1}^n E[X_i] \\ &= \sum_{i=1}^n \Pr(X_i = 1) \end{aligned}$$

What is  $\Pr(X_i = 1)$ ?

## Analysis

What is  $\Pr(X_j = 1)$  , the probability that we hire on the  $j$  th day?

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$$\Pr(X_1 = 1) = 1$$

$$\Pr(X_2 = 1) = 1/2$$

$$\Pr(X_j = 1) = 1/j$$

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^n X_i\right] \\ &= \sum_{i=1}^n E[X_i] \\ &= \sum_{i=1}^n \Pr(X_i = 1) \\ &= \sum_{i=1}^n \frac{1}{i} \\ &\approx \ln n \end{aligned}$$

## Randomized algorithms vs. Probabilistic Analysis

- We have assumed that the candidates come in a random order.
- Can we remove this assumption?

# Randomized algorithms vs. Probabilistic Analysis

- We have assumed that the candidates come in a random order.
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**Randomize the algorithm:**

- Force the candidates to come in a random order by randomly permuting the data, before we start.
- We have now eliminated an adversarial-chosen bad case, the only bad case is to be extremely unlucky in our coin flips.

## Case of Sorting

**Scenario** Imagine a sorting algorithm whose bad case is when the data comes in reverse sorted order.

- **Data is “random”:** Bad case is reverse sorted order.
- **Algorithm is random:** some set of coin flips that occur with probability  $1/n!$  makes the algorithm slow

# Producing a Uniform Random Permutation

**Def:** A uniform random permutation is one in which each of the  $n!$  possible permutations are equally likely.

RANDOMIZE-IN-PLACE( $A$ )

```
1   $n \leftarrow \text{length}[A]$ 
2  for  $i \leftarrow 1$  to  $n$ 
3      do swap  $A[i] \leftrightarrow A[\text{RANDOM}(i, n)]$ 
```

**Lemma** Procedure RANDOMIZE-IN-PLACE computes a uniform random permutation.

**Def** Given a set of  $n$  elements, a  $k$ -permutation is a sequence containing  $k$  of the  $n$  elements.

There are  $n!/(n - k)!$  possible  $k$ -permutations of  $n$  elements

## Proof via Loop invariant

We use the following loop invariant:

Just prior to the  $i$ th iteration of the for loop of lines 2–3, for each possible  $(i-1)$ -permutation, the subarray  $A[1..i-1]$  contains this  $(i-1)$ -permutation with probability  $(n-i+1)!/n!$ .

# Initialization

RANDOMIZE-IN-PLACE( $A$ )

```
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```

Just prior to the  $i$ th iteration of the for loop of lines 2–3, for each possible  $(i-1)$ -permutation, the subarray  $A[1..i-1]$  contains this  $(i-1)$ -permutation with probability  $(n-i+1)!/n!$ .

**Initialization** Consider the situation just before the first loop iteration, so that  $i = 1$ . The loop invariant says that for each possible 0-permutation, the subarray  $A[1..0]$  contains this 0-permutation with probability  $(n-i+1)!/n! = n!/n! = 1$ . The subarray  $A[1..0]$  is an empty subarray, and a 0-permutation has no elements. Thus,  $A[1..0]$  contains any 0-permutation with probability 1, and the loop invariant holds prior to the first iteration.

# Maintenance

RANDOMIZE-IN-PLACE( $A$ )

```
1   $n \leftarrow \text{length}[A]$ 
2  for  $i \leftarrow 1$  to  $n$ 
3      do swap  $A[i] \leftrightarrow A[\text{RANDOM}(i, n)]$ 
```

Just prior to the  $i$ th iteration of the for loop of lines 2–3, for each possible  $(i-1)$ -permutation, the subarray  $A[1..i-1]$  contains this  $(i-1)$ -permutation with probability  $(n-i+1)!/n!$ .

**Maintenance** We assume that just before the  $(i-1)$ st iteration, each possible  $(i-1)$ -permutation appears in the subarray  $A[1..i-1]$  with probability  $(n-i+1)!/n!$ , and we will show that after the  $i$ th iteration, each possible  $i$ -permutation appears in the subarray  $A[1..i]$  with probability  $(n-i)!/n!$ . Incrementing  $i$  for the next iteration will then maintain the loop invariant.

Let us examine the  $i$ th iteration. Consider a particular  $i$ -permutation, and denote the elements in it by  $\langle x_1, x_2, \dots, x_i \rangle$ . This permutation consists of an  $(i - 1)$ -permutation  $\langle x_1, \dots, x_{i-1} \rangle$  followed by the value  $x_i$  that the algorithm places in  $A[i]$ . Let  $E_1$  denote the event in which the first  $i - 1$  iterations have created the particular  $(i - 1)$ -permutation  $\langle x_1, \dots, x_{i-1} \rangle$  in  $A[1..i-1]$ . By the loop invariant,  $\Pr(E_1) = (n-i+1)!/n!$ . Let  $E_2$  be the event that  $i$ th iteration puts  $x_i$  in position  $A[i]$ . The  $i$ -permutation  $\langle x_1, \dots, x_i \rangle$  is formed in  $A[1..i]$  precisely when both  $E_1$  and  $E_2$  occur, and so we wish to compute  $\Pr(E_2 \cap E_1)$ . Using equation ??, we have

$$\Pr(E_2 \cap E_1) = \Pr(E_2 | E_1)\Pr(E_1) .$$

The probability  $\Pr(E_2 | E_1)$  equals  $1/(n-i+1)$  because in line 3 the algorithm chooses  $x_i$  randomly from the  $n - i + 1$  values in positions  $A[i..n]$ . Thus, we have

$$\begin{aligned}\Pr(E_2 \cap E_1) &= \Pr(E_2 | E_1)\Pr(E_1) \\ &= \frac{1}{n-i+1} \cdot \frac{(n-i+1)!}{n!} \\ &= \frac{(n-i)!}{n!} .\end{aligned}$$

# Termination

RANDOMIZE-IN-PLACE( $A$ )

```
1   $n \leftarrow \text{length}[A]$ 
2  for  $i \leftarrow 1$  to  $n$ 
3      do swap  $A[i] \leftrightarrow A[\text{RANDOM}(i, n)]$ 
```

Just prior to the  $i$ th iteration of the for loop of lines 2–3, for each possible  $(i - 1)$ -permutation, the subarray  $A[1..i - 1]$  contains this  $(i - 1)$ -permutation with probability  $(n - i + 1)!/n!$ .

**Termination** At termination,  $i = n + 1$ , and we have that the subarray  $A[1..n]$  is a given  $n$ -permutation with probability  $(n - n)!/n! = 1/n!$ .

# Birthday Paradox

Setup:

- $n$  people
- Do two people have the same birthday?
- Compute expected number of pairs of people that have the same birthday.
- $X_{ij}$  is indicator random variable associated with  $i$  and  $j$  having the same birthday.
- $X$  is the expected number of pairs that have the same birthday

$$\begin{aligned} X &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij} \\ E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}\right] \end{aligned}$$

# Birthday Paradox

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{n-1} \sum_{j=i+1}^n X_{ij}\right] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n E[X_{ij}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \Pr[i \text{ and } j \text{ have the same birthday}] \\ &= \sum_{i=1}^{n-1} \sum_{j=i+1}^n \frac{1}{365} \\ &= \binom{n}{2} \frac{1}{365} \\ &= \frac{n(n-1)}{730} \end{aligned}$$

## Values

$$n = 23 \quad .69$$

$$n = 28 \quad 1.03$$

$$n = 64 \quad 5.5$$

$$n = 90 \quad 10.9$$

$$n = 140 \quad 26.6$$

## Streaks

**Question:** Suppose we flip  $n$  coins, what is the longest streak of heads?

**Answer:**

- Use indicator random variables.
- Let  $X_{ik}$  be the event that there is a streak of length  $k$  starting at position  $i$ . ( $A[i \dots i+k-1]$  are all heads.)
- Let  $X_k$  be the number of streaks of length  $k$ .
- $X_k = \sum_{i=1}^{n-k+1} X_{ik}$

$$\begin{aligned} E[X_k] &= E\left[\sum_{i=1}^{n-k+1} X_{ik}\right] \\ &= \sum_{i=1}^{n-k+1} E[X_{ik}] \\ &= \sum_{i=1}^{n-k+1} \Pr(\text{streak of length } k \text{ starting at position } i) \\ &= \sum_{i=1}^{n-k+1} 2^{-k} \\ &= \frac{n - k + 1}{2^k} \end{aligned}$$

What is the behavior of

$$\frac{n - k + 1}{2^k}$$

? What is it around 1?

## When do we have 1 streak of length k

Think about?

$$n - k + 1 = 2^k$$

so if  $k = c \lg n$  for some  $c$ , we have

$$\frac{n - k + 1}{2^k} = \frac{n - c \lg n + 1}{2^{c \lg n}} = \frac{n - c \lg n + 1}{n^c}$$

- if  $c = 1$ , then the expected number is around 1.
- if  $c >> 1$ , then the expected number starts to decrease rapidly.
- if  $c << 1$ , then the expected number starts to increase rapidly.
- so the longest streak should be around length  $\lg n$ .