CSOR W4231 Analysis of algorithms I
Assignment 3

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Problem 1

We will use greedy algorithm here. The algorithm will be as follow:

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Initialize an empty list A to store the placement of guards
for x_i in set X
  if A is empty or x_i > A[-1]:
    place a guard a on 1 + x_i and append a to the end of A
  else:
    continue to x_{i+1}
```

Prove the optimality of the algorithm:
Let $A = \{a_1, a_2, ..., a_k\}$ denotes the solution generated by the above algorithm, and let $O = \{o_1, o_2, ..., o_m\}$ be an optimal solution.
Property 1:
Based on our strategy, $a_1 = x_1 + 1$. If the first placement o_1 of O is on the right side of a_1, i.e., $a_1 < o_1$, then it cannot guard x_1, since $x_1 = a_1 - 1 < o_1 - 1$. This leads to a contradiction, so o_1 must be on the left side of a_1. Therefore, replacing o_1 in O with a_1 yields another feasible (and optimal) solution, since every x_i guarded by o_1 can be guarded by a_1, too. Proof:

$$\begin{aligned}
&\text{Given } \ o_1 - 1 \leq x_i \leq o_1 + 1, \\
&\text{we have } \ o_1 - 1 < a_i - 1 = x_1 \leq x_i \leq o_1 + 1 < a_1 + 1, \\
&\text{thus, } \ a_1 - 1 \leq x_i < a_1 + 1
\end{aligned}$$

Therefore, there is an optimal solution whose first guard agrees with the greedy algorithm above.
Property 2:
Once we placed the first guard, suppose that it can guard $x_1 ... x_k$. So the remaining guards must be placed to protect $X' = \{x_{k+1} ... x_i\}$. This is a subproblem of the original one, and we can again use the above steps to prove that at each placement i, replacing the choice o_i of an arbitrary optimal solution with the choice a_i of greedy can yield another feasible optimal solution.
Therefore, we have proved the optimality of out greedy algorithm.