Basics of Algorithm Analysis

- We measure running time as a function of n, the size of the input (in bytes assuming a reasonable encoding).
- We work in the RAM model of computation. All "reasonable" operations take "1" unit of time. (e.g. +, *, -, /, array access, pointer following, writing a value, one byte of I/O...)

What is the running time of an algorithm

- Best case (seldom used)
- Average case (used if we understand the average)
- Worst case (used most often)

Example

```
input: A[n]
for i = 1 to n
    if (A[i] == 7)
        for j = 1 to n
            for k = 1 to n
                Print "hello"
```

- **What is the worst case running time?**
- **What is the best case running time?**
- **What is the average case running time?**

Example

```
input: A[n]
for i = 1 to n
        if (A[i] == 7)
                for j = 1 to n
                        for k = 1 to n
                                Print "hello"
```

- **What is the worst case running time?** $O(n^3)$
- **What is the best case running time?** $O(n)$
- **What is the average case running time?** **What is an average array?**

How do we measure the running time?

We measure as a function of n, and ignore low order terms.

- $5n^3 + n - 6$ **becomes** n^3
- $8n \log n - 60n$ **becomes** $n \log n$
- $2^n + 3n^4$ **becomes** 2^n

Asymptotic notation

big-O

$$O(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}\ .$$

Alternatively, we say

$$f(n) = O(g(n)) \text{ if there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le f(n) \le cg(n) \text{ for all } n \ge n_0\}$$

Informally, $f(n) = O(g(n))$ means that $f(n)$ is asymptotically less than or equal to $g(n)$.

big-Ω

$$\Omega(g(n)) = \{f(n) : \text{ there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}\ .$$

Alternatively, we say

$$f(n) = \Omega(g(n)) \text{ if there exist positive constants } c \text{ and } n_0 \text{ such that } 0 \le cg(n) \le f(n) \text{ for all } n \ge n_0\}\ .$$

Informally, $f(n) = \Omega(g(n)$ means that $f(n)$ is asymptotically greater than or equal to $g(n)$.

big-Θ

$f(n) = \Theta(g(n))$ **if and only if** $f(n) = O(g(n))$ **and** $f(n) = \Omega(g(n))$**.**

Informally, $f(n) = \Theta(g(n)$ **means that** $f(n)$ **is asymptotically equal to** $g(n)$**.**

INFORMAL summary

- $f(n) = O(g(n))$ **roughly means** $f(n) \leq g(n)$
- $f(n) = \Omega(g(n))$ **roughly means** $f(n) \geq g(n)$
- $f(n) = \Theta(g(n))$ **roughly means** $f(n) = g(n)$
- $f(n) = o(g(n))$ **roughly means** $f(n) < g(n)$
- $f(n) = w(g(n))$ **roughly means** $f(n) > g(n)$

Big-O proofs

- $3n = O(n^2)$
- $2n + 7 = O(n)$
- $n^{\log n} = O(2^n)$

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Use of big-O

$$2n + 7 = O(n)$$

$$2n + 7 = O(n^3)$$

$$2n + 7 = O(n^{4.5} \log n)$$

$$2n + 7 = O(2^n)$$

Which of these do we care about?

Use of big-O

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$$2n + 7 = O(2^n)$$

Which of these do we care about?

- **Given a function** $f(n)$ **, we want to know the "smallest"** $g(n)$ **such that** $f(n) = O(g(n))$ **and** $g(n)$ **is "simple"**

Simple Functions

- **Given a function $f(n)$, we want to know the "smallest" $g(n)$ such that $f(n) = O(g(n))$ and $g(n)$ is "simple"**
- **Typical simple functions include (but are not limited to)**
  - 1
  - $\log \log n$
  - $\log n$
  - $\log^2 n$
  - n
  - $n \log n$
  - n^2
  - n^3
  - 2^n
  - $n!$
- **We use these to classify algorithms into classes**

See chart for justification

Polynomial Time

An algorithm runs in polynomial time if, on an input of size n , its running time is $O(n^k)$ for some constant k .

2^n is NOT polynomial. Let's try to prove that it is polynomial and see what goes wrong.

Proving Omega and Theta

$f(n) = \Omega(g(n))$ **if there exist positive constants** c **and** n_0 **such that** $0 \le cg(n) \le f(n)$ **for all** $n \ge n_0\}$.

$f(n) = \Theta(g(n))$ **if and only if** $f(n) = O(g(n))$ **and** $f(n) = \Omega(g(n))$**.**

3 useful formulas

Arithmetic series

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

Geometric series

$$\sum_{i=0}^{\infty} a^i = \frac{1}{1-a} \qquad \textbf{for } 0 < a < 1$$

Harmonic series

$$\sum_{i=1}^{n} \frac{1}{i} = \ln n + O(1) = \Theta(\ln n)$$

Arithmetic Series in PseudoCode

for $i = 1$ **to** n
  for $j = 1$ **to** n
    Jump up and down

compared to

for $i = 1$ **to** n
  for $j = 1$ **to** i
    Jump up and down

Geometric Series

for $i = 1$ **to** $\log n$
    for $j = 1$ **to** 2^i
        Jump up and down

or

JUMP**(n)**
if $n = 1$
    Jump up and down once
else
    Jump up and down n **times**
    JUMP$(\lfloor n/2 \rfloor)$

A few facts about logs

- $\log_b a = \frac{\log_c a}{\log_c b}$ **for any** $c > 1$
- **therefore** $\ln n = O(\log n)$
- **in general, the base of the logarithm in a big-O statement is not important**

$$\begin{aligned} n + \frac{n}{2} + \frac{n}{3} + \frac{n}{4} + \frac{n}{5} + \ldots + 1 &= n\left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{n}\right) \\ &= O(n \log n) \end{aligned}$$

Algorithmic Correctness

- **Very important, but we won't typically prove correctness from first principles.**
- **We will use loop invariants**
- **We will use other problem specific methods**

Divide and Conquer

- **Divide a problem into pieces**
- **Recursively solve the pieces**
- **Combine the solutions to the subproblems**

MergeSort

$Merge - Sort(A, p, r)$
if $p < r$
    $q = \lfloor (p + r)/2 \rfloor$
    $\textsc{Merge-Sort}(A, p, q)$
    $\textsc{Merge-Sort}(A, q + 1, r)$
    $\textsc{Merge}(A, p, q, r)$

Let $T(n)$ be the running time of MergeSort on n items. Merge takes $O(n)$ time.

$$T(n) = \begin{cases} \Theta(1) & \textbf{if } n = 1 \ , \\ 2T(n/2) + \Theta(n) & \textbf{if } n > 1 \ . \end{cases}$$

3 Recurrence Trees

1. $T(n) = 2T(n/2) + n$

2. $T(n) = 2T(n/2) + 1$

3. $T(n) = 2T(n/2) + n^2$

$$T(n) = 2T(n/2) + n$$

$$T(n) = 2T(n/2) + 1$$

$$T(n) = 2T(n/2) + n^2$$

Master Theorem

Master Theorem for Recurrences **Let $a \geq 1$ and $b > 1$ be constants, let $f(n)$ be a function, and let $T(n)$ be defined on the non-negative integers by the recurrence**

$$T(n) = aT(n/b) + f(n) \ ,$$

where we interpret n/b to mean either $\lfloor n/b \rfloor$ or $\lceil n/b \rceil$. Then $T(n)$ can be bounded asymptotically as follows.

1. **If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.**
2. **If $f(n) = \Theta(n^{\log_b a})$, then $T(n) = \Theta(n^{\log_b a} \lg n)$.**
3. **If $f(n) = \Omega(n^{\log_b a + \epsilon})$ for some constant $\epsilon > 0$, and if $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n, then $T(n) = \Theta(f(n))$.**