Disjoint Sets

- Set of items - X .
- Maintain disjoint sets $S_1, \ldots, S_k$; i.e. $S_i \cap S_j = \emptyset \;\; \forall i \neq j$
- Operations:
  - MakeSet(x) - create a one-element set with x
  - Find-Set(x) - return the "name" of the set containing x
  - Union(x , y) - merge the set containing x and the set containing y into one set.

Representation

- Represent set as a rooted tree, with name being root
- Time per operation is proportional to height of tree.
- Two good heuristics
  - Union by Rank - make shallow tree a child of root of big tree
  - Path Compression - every time you touch a node, make it a child of root
- Union by Rank gives $\log V$ time per operation
- Union by Rank and path compression give better performance.

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Disjoint Set Code

One example

10 million items

max depth: 5 or 6

$O(\lg^* n)$ time

Make-Set(x)

$p[x] \leftarrow x$
$rank[x] \leftarrow 0$

Union(x, y)

$\textsc{Link}(\textsc{Find-Set}(x), \textsc{Find-Set}(y))$

Link(x, y)

if $rank[x] > rank[y]$
    then $p[y] \leftarrow x$
    else $p[x] \leftarrow y$
        if $rank[x] = rank[y]$
            then $rank[y] \leftarrow rank[y] + 1$

Find-Set(x)

if $x \neq p[x]$
    then $p[x] \leftarrow \textsc{Find-Set}(p[x])$
return $p[x]$

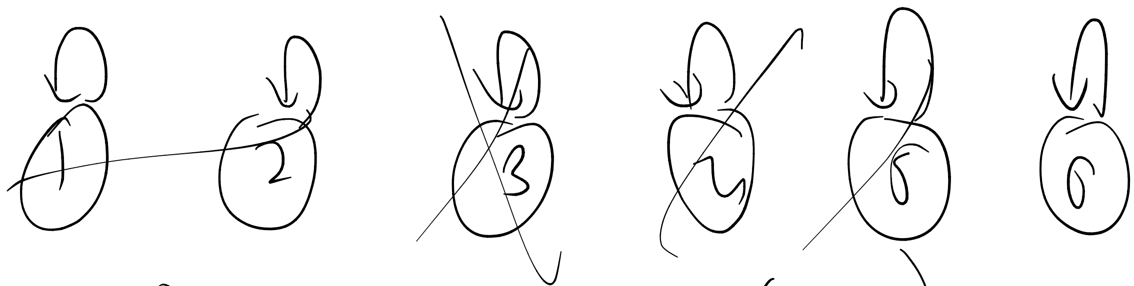
5 1 ≠ 5   4, 2
4

4   9

Example

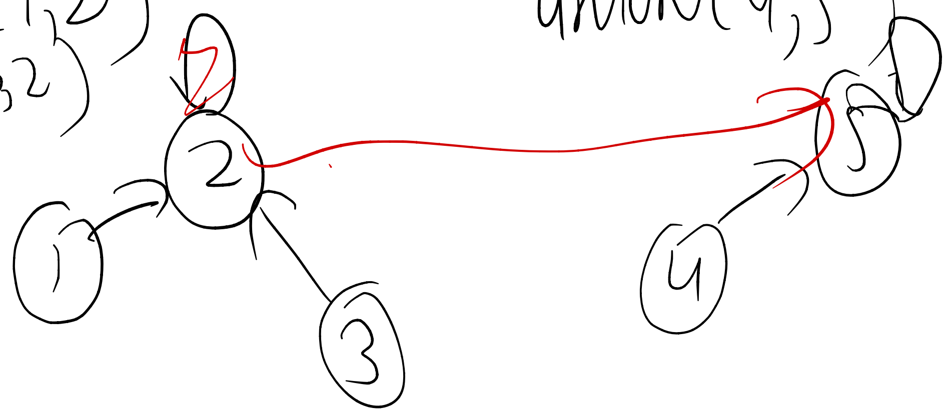
root = "name"

1, 2, 3, 4, 5, 6



union(1,2)
union(3,2)

union(4,5)



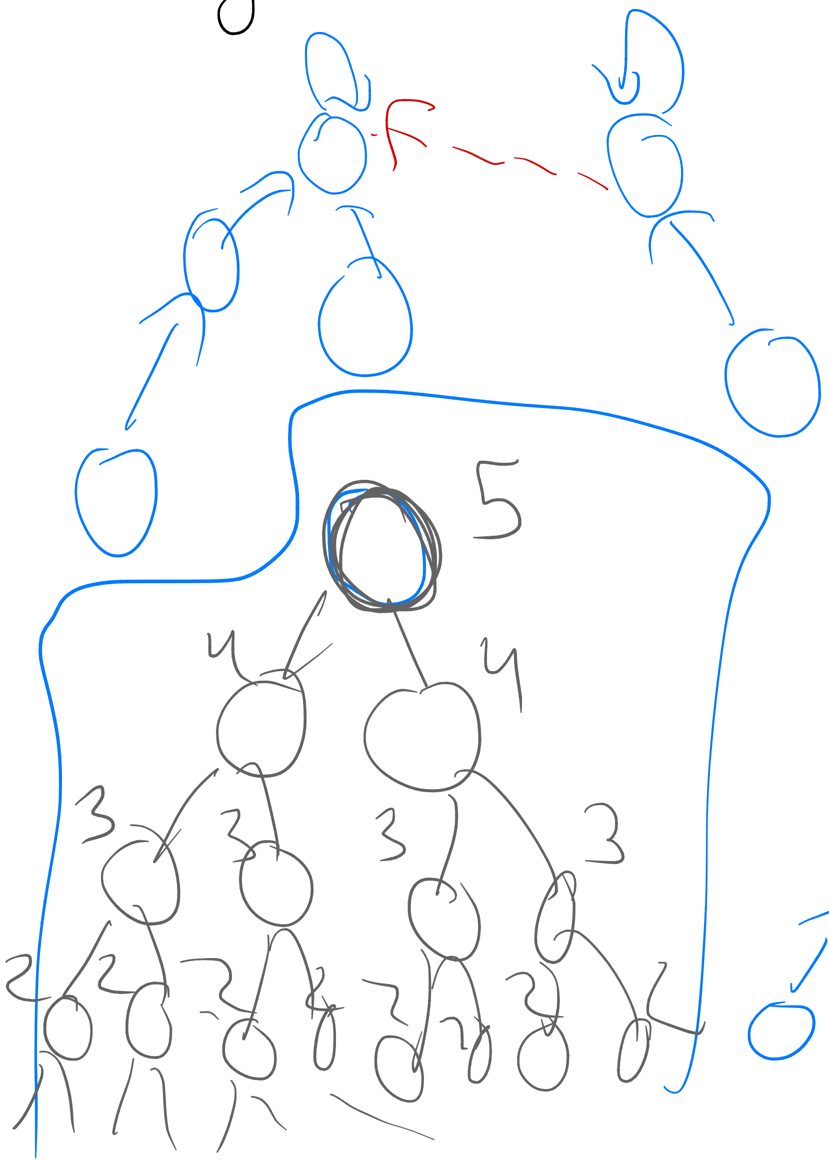
time = ht of tree

union(3,4)

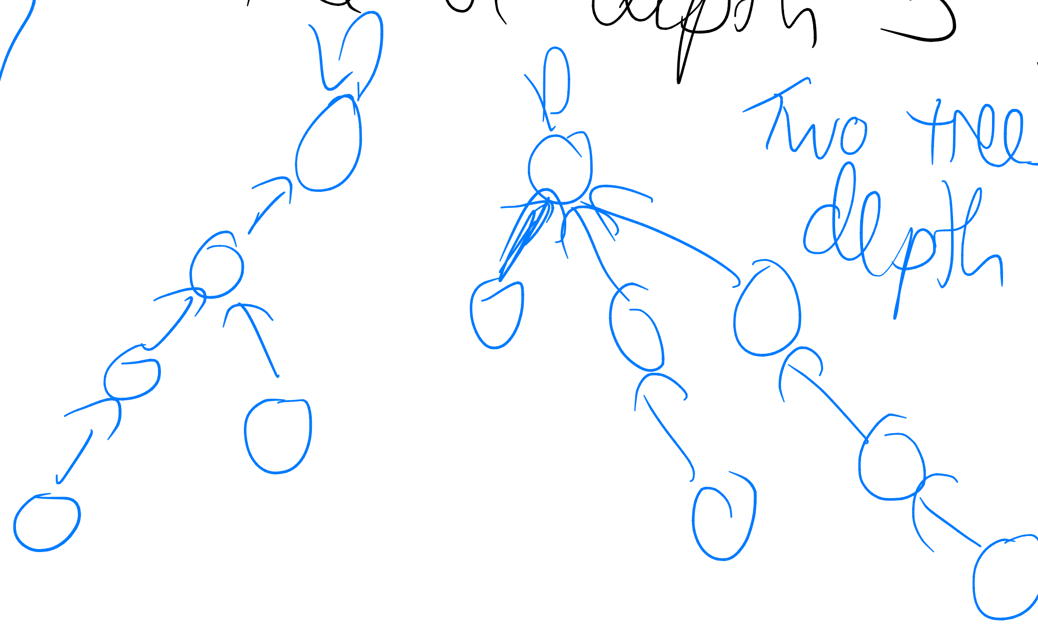
Example

Union by Rank – make shallower tree child of larger tree

Claim: max depth $\leq \lg n$

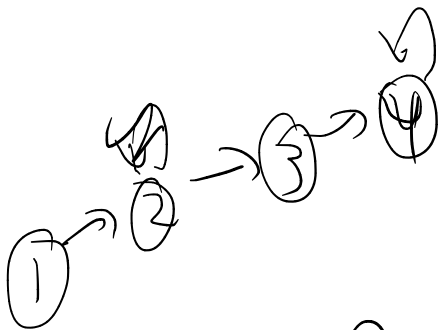


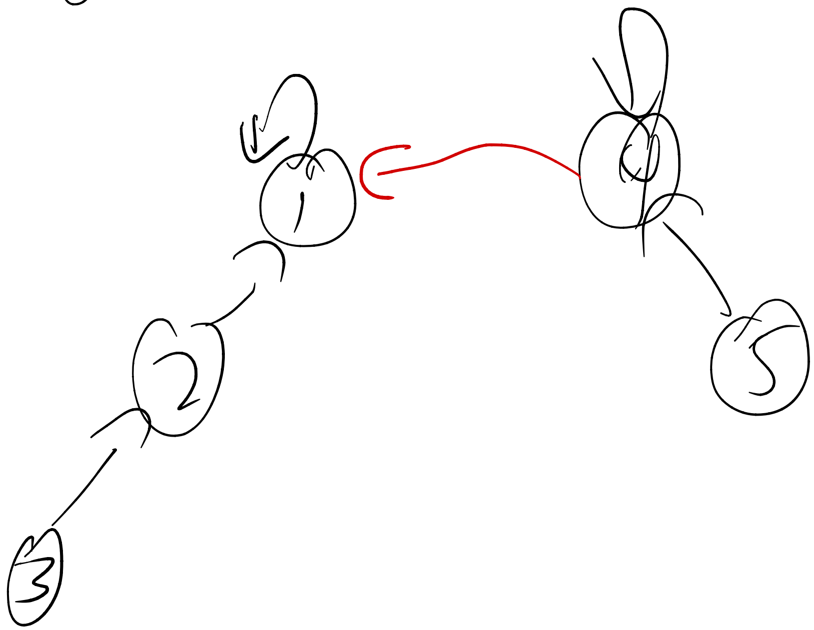
How can we get a tree of depth 5?

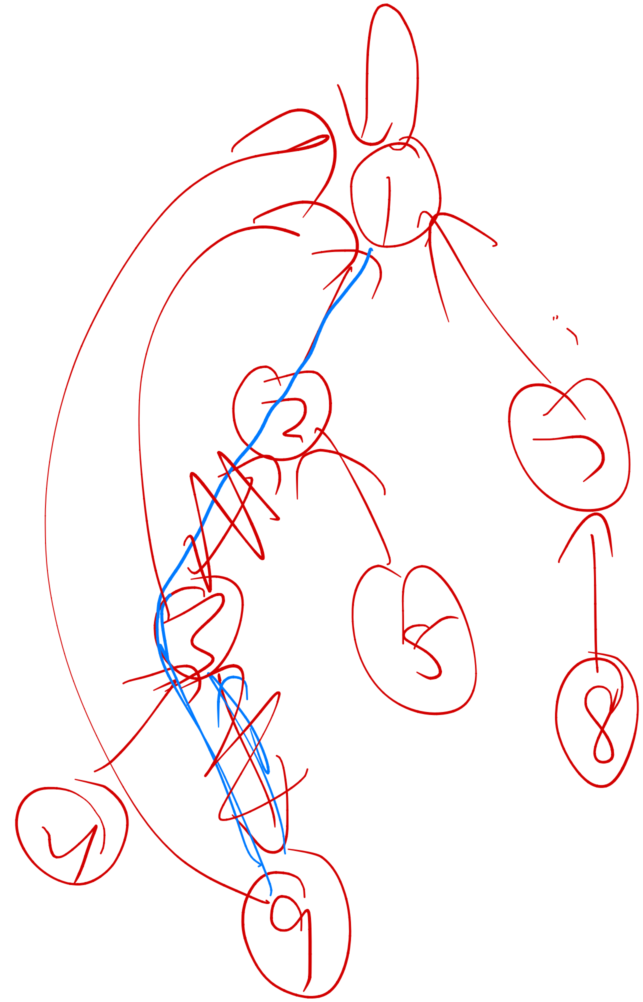


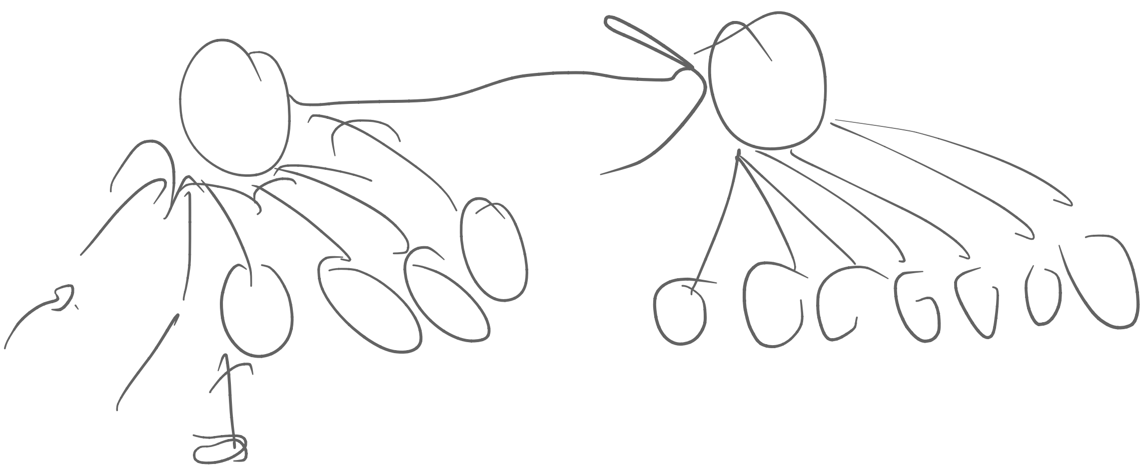
Two trees of depth 4.

Example









Ackerman's Function

$$A_k(j) = \begin{cases} j+1 & \textbf{if } k = 0 \\ A_{k-1}^{(j+1)}(j) & \textbf{if } k \geq 1 \end{cases}$$

$$\alpha(n) = \min\{k : A_k(1) \geq n\}$$

$$\begin{aligned} A_0(j) &= j+1 \\ A_1(j) &= A_0^{(j+1)}(j) \\ &= 2j+1 \\ A_2(j) &= A_1^{(j+1)}(j) \\ &= 2(2(\cdots(2j+1)\cdots)+1)+1 \\ &= 2^{j+1}(j+1)-1 \end{aligned}$$

Ackerman

$$
\begin{aligned}
A_3(1) &= A_2^{(2)}(1) \\
&= A_2(A_2(1)) \\
&= A_2(7) \\
&= 2^8 \cdot 8 - 1 \\
&= 2^{11} - 1 \\
&= 2047
\end{aligned}
$$

$$
\begin{aligned}
A_4(1) &= A_3^{(2)}(1) \\
&= A_3(A_3(1)) \\
&= A_3(2047) \\
&= A_2^{(2048)}(2047) \\
&\gg A_2(2047) \\
&= 2^{2048} \cdot 2048 - 1 \\
&> 2^{2048} \\
&= (2^4)^{512} \\
&= 16^{512} \\
&\gg 10^{80} \ ,
\end{aligned}
$$

Summary

- **Amortized time per operation is** $\alpha(V)$ **.**
- **Can think of it as** $\lg^* V$ **, which is slightly bigger.**

n	$\lg^* n$
$1-2$	1
$2-4$	2
$4-16$	3
$16-2^{16}$	4
2^{16} 2^{65000}	5