Proving a bound by Induction

- **Recurrence to solve:** $T(n) = 3T(n/3) + n$
- **Guess at a solution:** $T(n) = O(n \lg n)$

$Proof steps:$

- **Rewrite claim to remove big-O:** $T(n) \leq cn \lg n$ **for some** $c \geq 0$ **.**
- **"Assume"** $T(n') \leq cn' \lg n'$ **for all** $n' < n$ **.**
- **Prove that the claim holds for** n **. Here is the proof**

$$
\begin{aligned}
T(n) &= 3T(n/3) + n \\
&\leq 3(c(n/3)\lg(n/3)) + n \quad \textbf{(by inductive hypothesis since } n/3 < n\textbf{)} \\
&= cn(\lg n - \lg 3) + n \\
&= cn \lg n + n - cn \lg 3
\end{aligned}
$$

- **Now we really want to choose c so that this last line is** $\leq cn \lg n$
- **Equivalently, we really want to choose c so that** $n - cn \lg 3 < 0$
- **Equivalently, we really want to choose c so that** $c \lg 3 > 1$
- $c = 1$ **works and completes the proof, as now** $n \lg n + n(1 - \lg 3) \leq n \lg n$

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Recurrences with a big-O in the $f(n)$

- Recurrences describing running times often have a big-O in the non-recursive term
- Consider $T(n) = 2T(n/2) + O(n)$
- What does the $O(n)$ mean?

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- **What does the** $O(n)$ **mean?**
- **The** $O(n)$ **is desribing an algoithm e.g. merge, that runs in time** kn **for some** $k > 0$ **that we don't get to pick.**

Claim: $T(n) = O(n \lg n)$

Question: **What do this big-O mean?**

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Claim: $T(n) = O(n \lg n)$

Question: What do this big-O mean?

Anwer: $T(n) \leq cn \lg n$ **for some** $c > 0$ **, which we do get to pick**

Mechanics of Proof

Claim: **The recurrence** $T(n) = 2T(n/2) + kn$ **has solution** $T(n) \leq cn \lg n$ **.**

Proof: **Use mathematical induction. The base case (implicitly) holds (we didn't even write the base case of the recurrence down).**

Inductive step:

$$\begin{aligned} T(n) &= 2T(n/2) + kn \\ &\leq 2\left(c\frac{n}{2}\lg\left(\frac{n}{2}\right)\right) + kn \\ &= cn(\lg n - 1) + kn \\ &= cn \lg n + kn - cn \end{aligned}$$

Now we want this last term to be

$$\leq cn \lg n$$

, so we need $kn - cn \leq 0$

$$\begin{aligned} kn - cn &\leq 0 \\ \Leftrightarrow (k - c)n &\leq 0 \\ \Leftrightarrow (k - c) &\leq 0 \\ \Leftrightarrow k &\leq c \end{aligned}$$

Is $k \leq c$

- **Recall that k is given to us (we don't choose it)**
- **We get to choose c.**
- **So if we choose $c = k$, then we have satisfied $c \leq k$, and the proof is complete.**

Proof subtlety

$n^{\lg_2 4}$

$O(n^2)$

Sometimes we have the correct solution, but the proof by induction doesn't work

- **Consider** $T(n) = 4T(n/2) + n$
- **By the master theorem, the solution is** $O(n^2)$

Proof by induction that $T(n) \le cn^2$ **for some** $c > 0$ **.**

$q \le 10$

$q \le 100$

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\le 4\left(c\left(\frac{n}{2}\right)^2\right) + n \\ &= cn^2 + n \end{aligned}$$

Now we want this last term to be

$$\le cn^2$$

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UhOh No way is $n \le 0$. What went wrong?

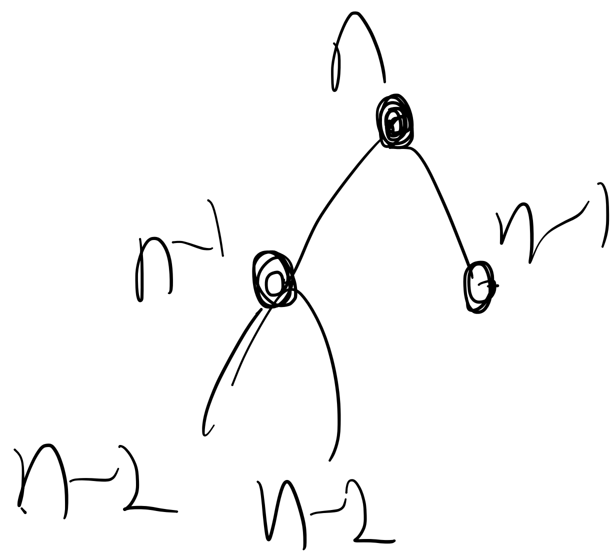
General Issue with proofs by induction

- Sometimes, you can't prove something by induction because it is too weak. So your inductive hypothesis is not strong enough.
- The fix is to prove something stronger
- We will prove that $T(n) \le cn^2 - dn$ for some positive constants c, d that we get to chose.
- We chose to add the $-dn$ because we noticed that there was an extra n in the previous proof.

$T(n) = 2T(n-1) + n$

n

$2(n-1)$



2

4

The proof

Claim: $T(n) \leq cn^2 - dn$ **for some positive constants** c, d

Proof:

$$\begin{aligned} T(n) &= 4T(n/2) + n \\ &\leq 4\left(c\left(\frac{n}{2}\right)^2 - d\frac{n}{2}\right) + n \\ &= cn^2 - 2dn + n \\ &= (cn^2 - dn) + (n - dn) \\ &= (cn^2 - dn) + (1 - d)n \end{aligned}$$

Now we want this last term to be

$$\leq cn^2 - dn$$

, so we need $(1-d)n \leq 0$ **. Just choose** $d = 2$ **. We can choose** c **to be anything, say** 1

Conclusion

$$T(n) \leq cn^2 - 2n = O(n^2)$$

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