NP-Completeness

Goal: We want some way to classify problems that are hard to solve, i.e. problems for which we can not find polynomial time algorithms.

For many interesting problems

- we cannot find a polynomial time algorithm
- we cannot prove that no polynomial time algorithm exists
- the best we can do is formalize a class of NP-complete problems that either all have polynomial time algorithms or none have polynomial time algorithms

NP-completeness arises in many fields including

- biology
- chemistry
- economics
- physics
- engineering
- sports
- etc.

Goal in class:

To learn how to prove that problems are NP-complete.

We need a formalism for proving problems hard.

Turing Machine (simplified description)

A Turing Machine has

- **Finite state control**
- **Infinite tape (each square can hold 0, 1, $, or be blank.**
- **Read-Write head**

Each step of the finite state control is a function

$$f(\textbf{current state}, \textbf{tape symbol}) \rightarrow (\textbf{new state}, \textbf{symbol to write}, \textbf{movement of head})$$

Example

Program to test if a binary number is even. Input is $ terminated. Output is written immediately after $, 1 for yes, 0 for no.

- **Read until $ (state q_0)**
- **Back up, check last digit (state q_1)**
- **if even, write a 1 (states q_2, q_3, q_F)**
- **if odd, write a 0 (states q_4, q_5, q_F)**

Here is a program. Each cell is (new state, write symbol move)

state	input 0	input 1	input $
(q_0)	$(q_0, -, R)$	$(q_0, -, R)$	$(q_1, -, L)$
(q_1)	$(q_2, -, R)$	$(q_4, -, R)$	**error**
(q_2)	**error**	**error**	$(q_3, -, R)$
(q_3)	$(q_F, 1, -)$	$(q_F, 1, -)$	$(q_F, 1, -)$
(q_4)	**error**	**error**	$(q_5, -, R)$
(q_5)	$(q_F, 0, -)$	$(q_F, 0, -)$	$(q_F, 0, -)$
(q_F)	**halt**	**halt**	**halt**

Church Turing Thesis **The set of things that can be computed on a TM is the same as the set of things that can be computed on any digital computer.**

P

Definition Let P be defined as the set of problems that can be solved in polynomial time on a TM (On an input of size n, they can be solved in time $O(n^k)$ for some constant k)

Theorem P is the set of problems that can be solved in polynomial time on the model of computation used in CSOR 4231 and on every modern non-quantum digital computer.

Technicalities

- We assume a reasonable (binary) encoding of input
- Note that all computers are related by a polynomial time transformation. Think of this as a "compiler"

Further details

- We restrict attention to "yes-no" questions
- Shortest path is now "Given a graph G and a number b does the shortest path from s to t have length at most b.
- We do not use the language framework from the book in class

Verification

Verification Given a problem X and a possible solution S, is S a solution to X.

Example X is shortest paths and S is an s-t path in S that is claimed to have length at most b, check whether the path really is of length at most b

Example X is sorting and S is an allegedly sorted list. Is the list really sorted?

Claim Verification is no harder than solving a problem from scratch.
We write

$$\text{Verification} \leq \text{Solving}$$

Def: NP is the set of problems that can be verified in polynomial time

Formally: Problem X with input of size n is in NP if there exists a "certificate" y, $|y| = \text{poly}(n)$ such that, using y, one can verify whether a solution x is really a solution in polynomial time. (Think of y as the "answer")

Some problems

Longest Path **Given a graph G, and number k is the longest simple path from s to t of length $\geq k$.**

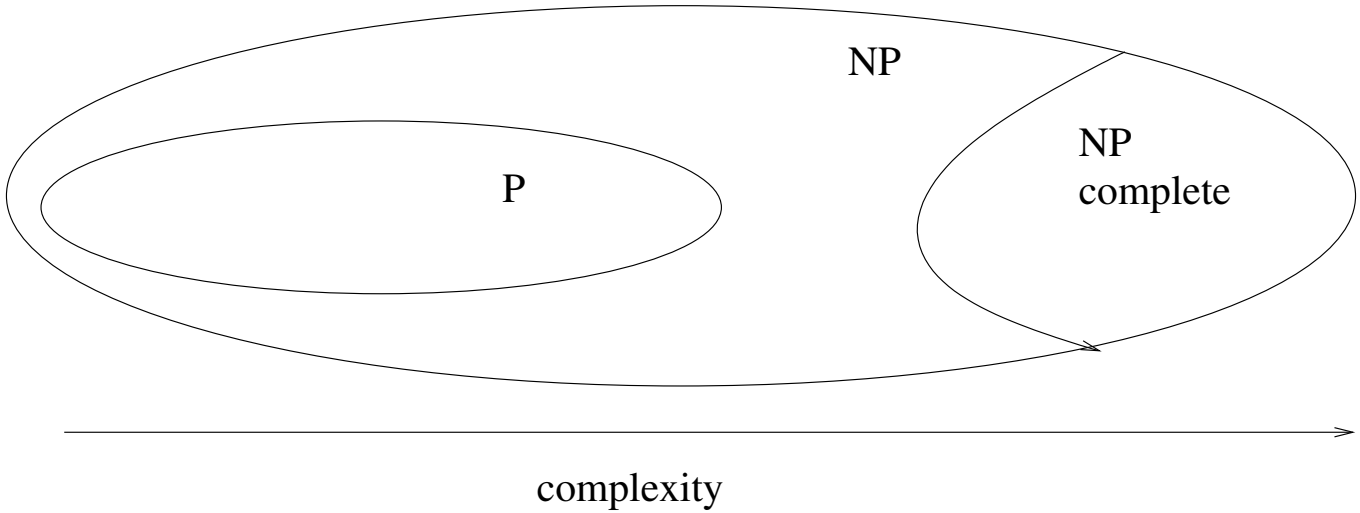
Satisfiability **Given a formula Φ in CNF (conjunctive normal form), does there exist a satisfying assignment to Φ, i.e. an assignment of the variables that evaluates to true.**

Big Question

$$P = NP??$$

Is solving a problem no harder than verifying?

**Don't know answer. Instead we will identify "hardest" problems in NP
If any of these are in P then all of NP is in P.**



NP-complete

Definition **Problem X is NP-complete if**

1. $X \in NP$

2. $Y \leq X \ \ \forall Y \in NP$

Definition $Y \leq X$ **means**

- **Y is polynomial time reducible to X, which means**

there exists a polynomial time computable function f **that maps inputs to** Y **to inputs to** X**, such that**

input y **to problem** Y **returns "Yes" iff input** $f(y)$ **to problem** X **returns "Yes"**

Informally $Y \leq X$ **means that Y is "not much harder than" ("easier than") X**

Theorem

If $Y \leq X$ **then** $X \in P \Rightarrow Y \in P$

Contrapositive

If $Y \leq X$ **then** $Y \notin P \Rightarrow X \notin P$

SAT

Theorem **SAT is NP-complete**

Proof idea: **The turing machine program for any problem in NP can be verified by a polynomial sized SAT instance that encodes that the input is well formed and that each step follows legally from the next.**

Implication **We now have one NP-complete problem. We will now reduce other problems to it.**

Reductions

- **If I want to show that X is easy, I show that in polynomial time I can reduce X to Y, where I already know that Y is easy.**
- **If I want to show that X is hard, then I reduce Y to X, where I already know that Y is hard.**
- **So if SAT $\leq X$, then X is hard.**

Showing X is NP-complete

To show that X is NP-complete, I show:

1. $X \in NP$

2. For some problem Z that I know to be NP-complete $Z \leq X$

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Expanded version: **To show that X is NP-complete, I show:**

1. $X \in NP$
2. **Find a known NP-complete problem Z.**
3. **Describe f, which maps input z to Z to input $f(z)$ to X.**
4. **Show that Z with input z returns "yes" iff X with input $f(z)$ returns "yes'**
5. **Show that f runs in polynomial time.**

3SAT

3SAT is SAT with exactly 3 literals per clause

Example:

$$(x_1 \vee x_2 \vee x_3) \wedge (\overline{x_1} \vee x_4 \vee \overline{x_5}) \wedge (x_1 \vee x_3 \vee \overline{x_4}) \wedge (x_2 \vee \overline{x_3} \vee \overline{x_5})$$

Comments

- n **variables,** m **clauses**
- **3SAT is a special case of SAT**
- **If SAT is easy, then 3SAT must be easy**
- **IS SAT is hard, then ???**
- **1-SAT is easy.**
- **2-SAT is easy.**

3SAT is NP-complete

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1. $X \in NP$
2. **Find a known NP-complete problem Z.**
3. **Describe f, which maps input z to Z to input $f(z)$ to X.**
4. **Show that Z with input z returns "yes" iff X with input $f(z)$ returns "yes'**
5. **Show that f runs in polynomial time.**

1) 3SAT is in NP. becasue SAT is in NP and 3SAT is a special case of SAT.

2) SAT

3,4, 5) Next slde..

Reduction

Approach **We need to show how to convert an input to SAT into an input to 3SAT, while preserving yes/no instances. We will give a clause by clause conversion. Let** k **be the number of literals in a clause**

Easy cases:

- $k = 1$. $x_1 \Rightarrow (x_1 \vee x_1 \vee x_1)$
- $k = 2$. $(x_1 \vee x_2) \Rightarrow (x_1 \vee x_1 \vee x_2)$
- $k = 3$. $(x_1 \vee x_2 \vee x_3) \Rightarrow (x_1 \vee x_2 \vee x_3)$

Easy to verify that transformation preserves satisfiability

k=4

- **Need to convert** $x_1 \vee x_2 \vee x_3 \vee x_4$ **to a 3SAT expression.**
- **Will need more than one clause**

First try:

$$(x_1 \vee x_2 \vee x_3) \wedge (x_2 \vee x_3 \vee x_4)$$

Is this true for exactly the same settings as $x_1 \vee x_2 \vee x_3 \vee x_4$ **?**

k=4

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Is this true for exactly the same settings as $x_1 \vee x_2 \vee x_3 \vee x_4$ **?**

No: **Consider**

$$\begin{aligned} x_1 &= T \\ x_2 &= F \\ x_3 &= F \\ x_4 &= F \end{aligned}$$

Lesson: **Need additional variables**

k=4

- **Need to convert** $\Phi = x_1 \vee x_2 \vee x_3 \vee x_4$ **to a 3SAT expression.**
- **Will need more than one clause**
- **Will need extra variables**

3SAT Expression:

$$\Phi' = (x_1 \vee x_2 \vee y_1) \wedge (\overline{y_1} \vee x_3 \vee x_4)$$

Claim: **There is a setting of** x_1, x_2, x_3, x_4 **that makes** Φ **true iff there is a setting of** x_1, x_2, x_3, x_4, y_1 **that makes** Φ' **true.**

k=5

- **Need to convert** $\Phi = x_1 \vee x_2 \vee x_3 \vee x_4 \vee x_5$ **to a 3SAT expression.**
- **Will need more than one clause**
- **Will need extra variables**

3SAT Expression:

$$\Phi' = (x_1 \vee x_2 \vee y_1) \wedge (\overline{y_1} \vee x_3 \vee y_2) \wedge (\overline{y_2} \vee x_4 \vee x_5)$$

Claim: **There is a setting of** x_1, x_2, x_3, x_4, x_5 **that makes** Φ **true iff there is a setting of** $x_1, x_2, x_3, x_4, x_5, y_1, y_2$ **that makes** Φ' **true.**

General k

- **Need to convert** $\Phi = x_1 \vee x_2 \vee \ldots \vee x_k$ **to a 3SAT expression.**
- **Will need more than one clause**
- **Will need extra variables**

3SAT Expression:

$$
\begin{aligned}
\Phi' = \quad & (x_1 \vee x_2 \vee y_1) \\
\wedge \quad & (\overline{y_1} \vee x_3 \vee y_2) \\
\wedge \quad & \ldots \\
\wedge \quad & (\overline{y_{i-2}} \vee x_i \vee y_{i-1}) \\
\wedge \quad & \ldots \\
\wedge \quad & (\overline{y_{k-4}} \vee x_{k-2} \vee y_{k-3}) \\
\wedge \quad & (\overline{y_{k-3}} \vee x_{k-1} \vee x_k))
\end{aligned}
$$

Claim: **There is a setting of** $x_1, x_2, \ldots, x_k$ **that makes** Φ **true iff there is a setting of** $x_1, x_2, \ldots, x_k, y_1, \ldots, y_{k-3}$ **that makes** Φ' **true.**

Recap

- Described f
- f is polynomial time
  - A clause with k variables is mapped to $k-2$ clauses of 3 variables each.
  - Clauses blow up by a factor of at most n
  - Variables blow up by a factor of at most n
- We argued (clause by clause) that Φ is a yes instance to SAT iff Φ' is a yes instance to 3SAT.

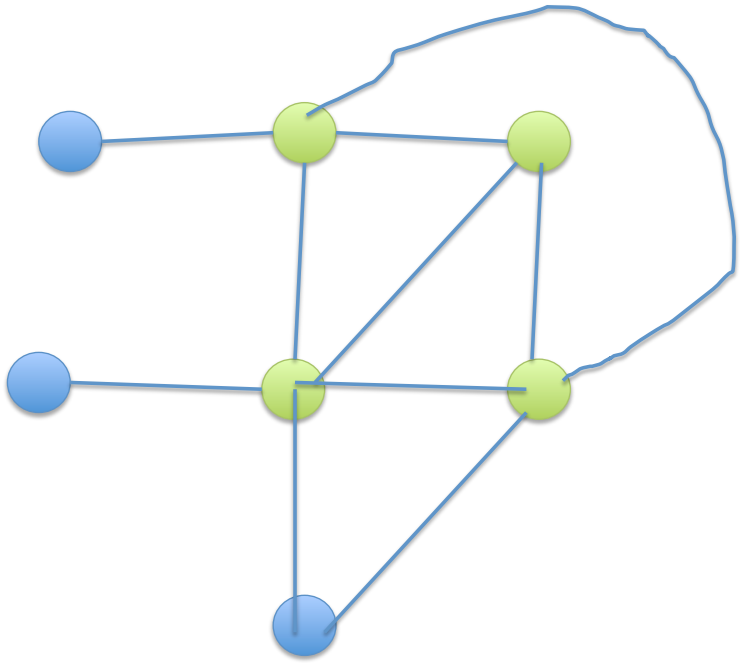
Sanity Checks

- **Why can't we prove that 2SAT is NP-complete via this reduction?**
- **What does the reduction from 2SAT to 3SAT tell us?**

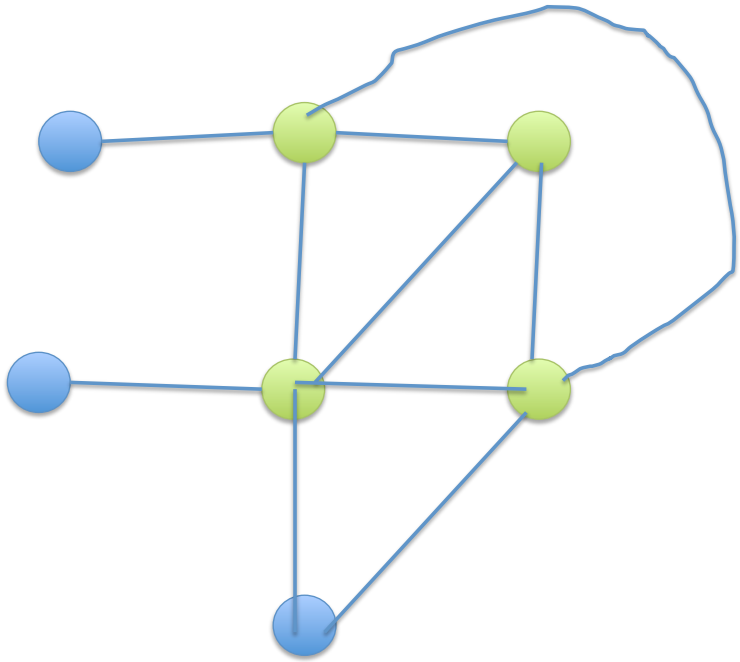
Clique

Definition **A k -clique is a set of k vertices with all $\binom{k}{2}$ edges between them.**

Clique **Given a graph $G = (V, E)$ and an integer k , does G have a k -clique?**



Clique



- **G has a 4-clique**
- **G has no 5-clique.**

Reduction

Goal We need to describe a function f that takes an instance Φ of 3SAT and produces instances $f(\Phi) = (G, k)$ of k-clique such that Φ is satisfiable iff $f(\Phi)$ has a k-clique.

Observation To make a 3SAT instance true, we need to make at least one literal in each clause true **Strategy:**

- A node for each appearance of a literal (a literal is a variable or its negation)
- An edge between literals that can be simultaneously true and in different clauses
- A k-clique will be a set of literals, one per clause, that can all be true simultaneously.

Example

$$\Phi = (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee \overline{x_3}) \wedge (x_1 \vee x_2 \vee x_3)$$

Proof

Claim Φ **is satisfiable iff** G **has a** k **-clique.**

Proof

($\Rightarrow$) **If** Φ **is satisfiable, then there is a setting of the variables with at least one literal per clause set to true. Let** Z **be such a set of literals. This set** Z **cannot contain both** x_1 **and** $\overline{x_i}$ **, so in the graph** G **, the nodes in** Z **have an edge between each pair and therefore form a** k **-clique.**

(($\Leftarrow$) **If** G **has a** k **-clique, the clque must consist of** k **nodes, and they must be 1 per clause and must not have any pairs** x_i **and** $\overline{x_i}$ **. Therefore you can set the corresponding literals to true and satisfy** Φ

Reflections

- We have actually shown that a "special case" with nodes in groups of 3 of clique is NP-complete. But if a special case is hard, there can't be a general algorithm for clique.
- In the proof, the function f goes one way, from 3SAT to clique, but the proof about yes instances has to go both ways.
- If the proof only went one way, it would be very easy (and incorrect)

Vertex Cover

Defintion Given a graph $G = (V, E)$ and an integer k, a **vertex cover** $V' \subseteq V$ is a subset of the vertices such that for all edges (v, w), at least one of v and w is in V'. The vertex cover problem asks whether a graph G has a vertex cover of size at most k.

Claim Vertex cover is NP-complete.

- Vertex cover is in NP
- We will reduce from clique.
- What is the relationship between vertex covers and cliques, i.e. what does the vertex cover of a clique look like.

Reduction

Definition **Given a graph $G = (V, E)$ the complement G' is the graph in which edges are replaces by non-edges and vica versa.**

Claim: **G has a k -clique iff G' has a vertex cover of size $|V| - k$.**

Subset Sum

Definition **Given a set of integers** $S = \{s_1, s_2, \ldots, s_n\}$ **and a target value** t **, is there a subset** $S' \subseteq S$ **such that** $\Sigma_{s_i \in S'} = t$ **.**

Example

$$S = \{1, 4, 16, 64, 256, 1040, 1041, 1093, 1284, 1344\} \quad t = 3754$$

Solution

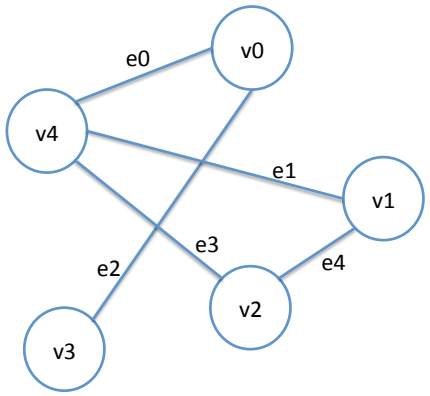
$$S' = \{1, 16, 64, 256, 1040, 1093, 1284\}$$

Question **What about** $t = 3755$ **?**

Reduction

Claim **Vertex cover reduces to Subset Sum.**

Idea 1: **Look at the vertex edge adjacency matrix**



	e_4	e_3	e_2	e_1	e_0
v_0	**0**	**0**	**1**	**0**	**1**
v_1	**1**	**0**	**0**	**1**	**0**
v_2	**1**	**1**	**0**	**0**	**0**
v_3	**0**	**0**	**1**	**0**	**0**
v_4	**0**	**1**	**0**	**1**	**1**

We now have numbers!

Ideas

- A vertex cover is a subset R of rows, such that each column has at least one 1 in a row of R.
- Maybe we can think of the rows as binary numbers, can we say something about the sum of the numbers in R.
- Example, $R = \{v_1, v_3, v_4\}$

	e_4	e_3	e_2	e_1	e_0
v_0	**0**	**0**	**1**	**0**	**1**
v_1	**1**	**0**	**0**	**1**	**0**
v_2	**1**	**1**	**0**	**0**	**0**
v_3	**0**	**0**	**1**	**0**	**0**
v_4	**0**	**1**	**0**	**1**	**1**
$v_1 + v_3 + v_4$	**1**	**1**	**1**	**2**	**1**

- Sort of works:
  - If every edge had exactly one endpoint in R, then the binary sum would be 11111, and we would choose $t = 11111$.
- Problems:
  - Edges can have one or two endpoints in R, which generates carries in base 2.
  - What should t be?
  - We ignored k.

Fixes

Problems:

- **Edges can have one or two endpoints in R, which generates carries in base 2. Use base 4, and there won't be any carries**
- **What should t be?**
- **We ignored k. Add an extra column to "count". It will be the left-most column, so carries don't matter**

	vert	e_4	e_3	e_2	e_1	e_0
x_0	**1**	**0**	**0**	**1**	**0**	**1**
x_1	**1**	**1**	**0**	**0**	**1**	**0**
x_2	**1**	**1**	**1**	**0**	**0**	**0**
x_3	**1**	**0**	**0**	**1**	**0**	**0**
x_4	**1**	**0**	**1**	**0**	**1**	**1**
$xv_1 + x_3 + x_4$	**(3)**	**1**	**1**	**1**	**2**	**1**

- **Still have a problem, what should t be?**
- **We will introduce dummy rows to allow us to say that columns should sum to exactly 2.**

Final reduction

	vert	e_4	e_3	e_2	e_1	e_0	**number converted to base 10**
x_0	**1**	**0**	**0**	**1**	**0**	**1**	**1041**
x_1	**1**	**1**	**0**	**0**	**1**	**0**	**1284**
x_2	**1**	**1**	**1**	**0**	**0**	**0**	**1344**
x_3	**1**	**0**	**0**	**1**	**0**	**0**	**1044**
x_4	**1**	**0**	**1**	**0**	**1**	**1**	**1093**
y_0	**0**	**0**	**0**	**0**	**0**	**1**	**1**
y_1	**0**	**0**	**0**	**0**	**1**	**0**	**4**
y_2	**0**	**0**	**0**	**1**	**0**	**0**	**16**
y_3	**0**	**0**	**1**	**0**	**0**	**0**	**64**
y_4	**0**	**1**	**0**	**0**	**0**	**0**	**256**
t	**(3)**	**2**	**2**	**2**	**2**	**2**	**3754**

Claim **G has a verex cover of size k iff the subset sum instance has a set that sums to t.**

Hamiltonian Cycle

Definition **Given a graph $G = (V, E)$, is there cycle visiting each vertex exactly once?**

Fact **Hamiltonian Cycle is NP-complete. See book for reduction.**

Travelling Salesman Problem **Given a graph $G = (V, E)$ with edge weights w and an integer B . Is there a Hamilonian Cycle C s.t.**

$$\sum_{e \in C} w(e) \leq B$$

.

Claim **Travelling Salesman Problem is NP-complete, via a reduction from Hamiltonian Cycle.**

More NP-complete problems

Minimum Makespan Scheduling Given n jobs with processing times $p_1, \ldots, p_n$, and m identical machines and a number B. a schedule assigns each job to a machine. If J_i is the set of jobs assigned to machine i, then the load on machine i, $L_i = \Sigma_{j \in J_i} p_j$. The makespan of the schedule is the maximum machine load $M = \max_i L_i$. You want to know if there exists a schedule with makespan at most B.

3-partition Given a set of $3n$ numbers $x_1, \ldots, x_{3n}$, with $\Sigma_{i=1}^{3n} x_i = nB$, can you partition the numbers into n groups, each with 3 elements and each summing to B.