COMS W4701: Artificial Intelligence

Lecture 10: Bayesian Networks

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Review: Inference in HMMs

- **Forward algorithm** for state estimation: Find $\boldsymbol{f}_t = P(X_t|e_{1:t})$
- Start from $\boldsymbol{f}_0 = P(X_0)$ and compute:

$$\boxed{\begin{array}{l} \boldsymbol{f}'_{t+1} = T\boldsymbol{f}_t \\ \boldsymbol{f}_{t+1} \propto_{X_{t+1}} O_{t+1}\boldsymbol{f}'_{t+1} \end{array}}$$

$$T_{ij} = P(X_{t+1} = i|X_t = j)$$

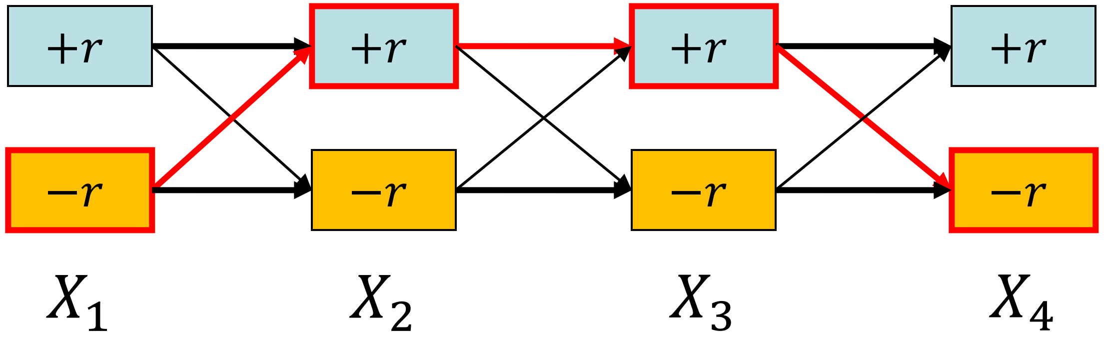
$$(O_t)_{ii} = P(E_t = e_t|X_t = i)$$

- **Viterbi algorithm** to find most likely sequence of states: $\text{argmax}_{x_{1:t}} P(x_{1:t}|e_{1:t})$
- Compute the joint distribution $\boldsymbol{m}_t = \max_{x_1 \ldots x_{t-1}} P(x_{1:t-1}, X_t, e_{1:t})$
- Start from $\boldsymbol{m}_0 = P(X_0)$ and compute:

$$\boxed{\begin{array}{c} \boldsymbol{m}'_{t+1}(x_{t+1}) = \max_{x_t} P(x_{t+1}|x_t)\boldsymbol{m}_t(x_t) \\ \boldsymbol{m}_{t+1} = O_{t+1}\boldsymbol{m}'_{t+1} \end{array}}$$

$$\boldsymbol{m}'_{t+1} = \begin{pmatrix} \max(T_{11}\boldsymbol{m}_t(1), T_{12}\boldsymbol{m}_t(2), \ldots, T_{1n}\boldsymbol{m}_t(n)) \\ \max(T_{21}\boldsymbol{m}_t(1), T_{22}\boldsymbol{m}_t(2), \ldots, T_{2n}\boldsymbol{m}_t(n)) \\ \vdots \\ \max(T_{n1}\boldsymbol{m}_t(1), T_{n2}\boldsymbol{m}_t(2), \ldots, T_{nn}\boldsymbol{m}_t(n)) \end{pmatrix}$$

Review: Viterbi Algorithm



$\begin{bmatrix}-r\\-r\end{bmatrix}$ $\begin{bmatrix}+r\\-r\end{bmatrix}$ $\begin{bmatrix}-r\\+r\end{bmatrix}$

$Pointer_2$ $Pointer_3$ $Pointer_4$

- Also need to keep track of most likely prior state when computing each $\boldsymbol{m}_{t+1}$

$$Pointer_{t+1}(x_{t+1}) = \underset{x_t}{\operatorname{argmax}}\, P(x_{t+1}|x_t)\boldsymbol{m}_t(x_t)$$

- $\underset{x_T}{\operatorname{argmax}}\, \boldsymbol{m}_T$ is the last state in most likely sequence

- Backward pass: Follow pointers backward to iteratively obtain each prior state

Today

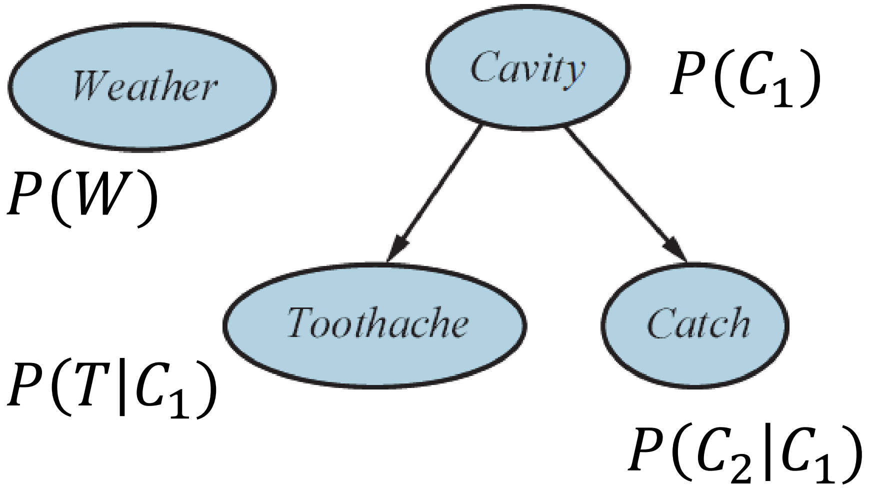
- Last time: Hidden Markov models help us reason about belief states that evolve over space or time while incorporating observations
- This time: **Bayesian networks** are general graph structures that can represent arbitrary relationships and independences
- Naïve Bayes models and maximum likelihood learning

Bayesian Networks

- Recall: A set of RVs is described by their joint distribution
- **Bayesian network**: A directed acyclic graph (DAG) representation of a distribution

- Each node corresponds to a random variable

- Each edge indicates influence or correlation
- May also be causation, but not always



- **Parameters** of the Bayes net: A conditional probability table for each node
- The table for a node X_i contains the values $P(X_i|parents(X_i))$

Joint Distribution

- Recall that the *chain rule* gives us a general way to compute joint distributions:

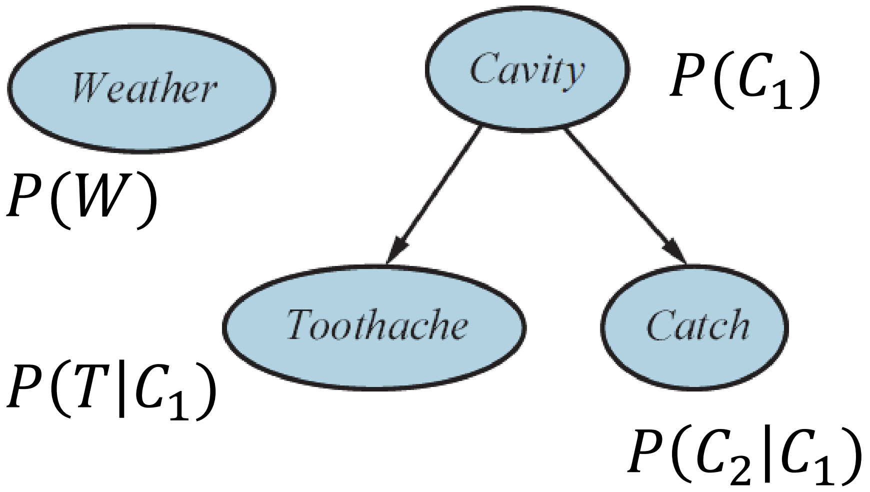
$$P(x_1, \dots, x_n) = P(x_1)P(x_2|x_1)\cdots P(x_n|x_1, \dots, x_{n-1}) = \prod_{i=1}^{n} P(x_i|x_1, \dots, x_{i-1})$$

- By definition, a joint distribution in a Bayes net can be written as

$$P(x_1, \dots, x_n) = \prod_{i=1}^{n} P(x_i|parents(X_i))$$

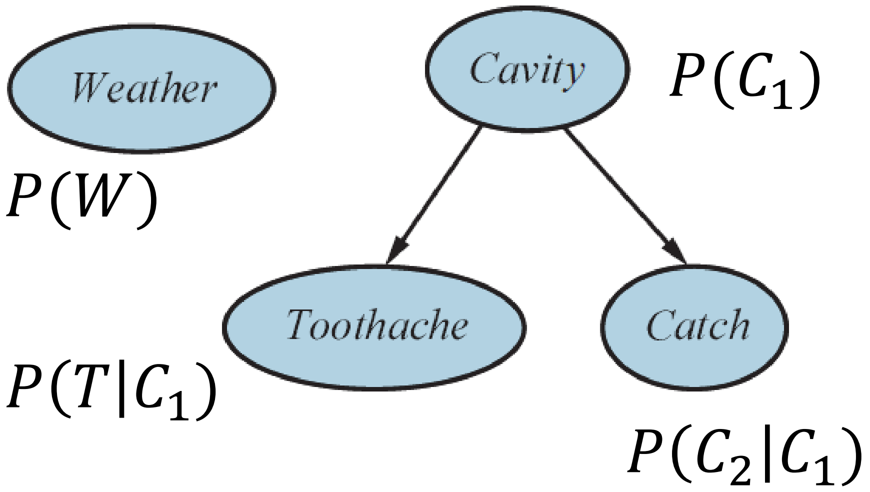
- This can simplify a lot of computation!

- Example: $P(w, c_1, t, c_2)$
  $= P(w)P(c_1)P(t|c_1)P(c_2|c_1)$



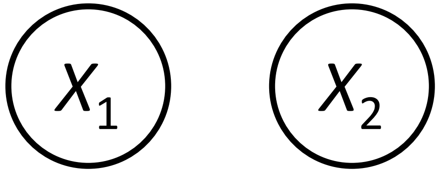
Topological Ordering

- We asserted that $$P(x_1, \dots, x_n) = \prod_{i=1}^{n} P(x_i | parents(X_i))$$
- This implies that a node is conditionally independent of predecessors given its parents
- *Numbering* must correspond to **topological ordering**: parents before children
- One possible ordering: W, C_1, T, C_2
- Another one: C_1, C_2, T, W
- Another one: C_1, C_2, W, T
- Many more…



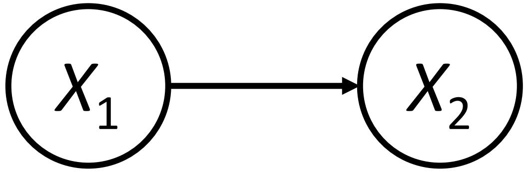
Constructing Bayes Nets

- Bayes net representations are not unique!



$P(X_1)$ $P(X_2)$

- Example: 2 (independent) coin flips
- Can use either $P(X_2)$ or $P(X_2|X_1)$



$P(X_1)$ $P(X_2 \mid X_1)$

- Independence is deduced from probabilities
- No path *guarantees* independence between two RVs

- Presence of a path *may* indicate correlation between two RVs
- May also be independent, depending on probabilities
- Ex: $P(X_2|X_1)$ is storing redundant information

X_1	X_2	$P(X_2\|X_1)$
+x	+x	0.5
+x	-x	0.5
-x	+x	0.5
-x	-x	0.5

Compactness

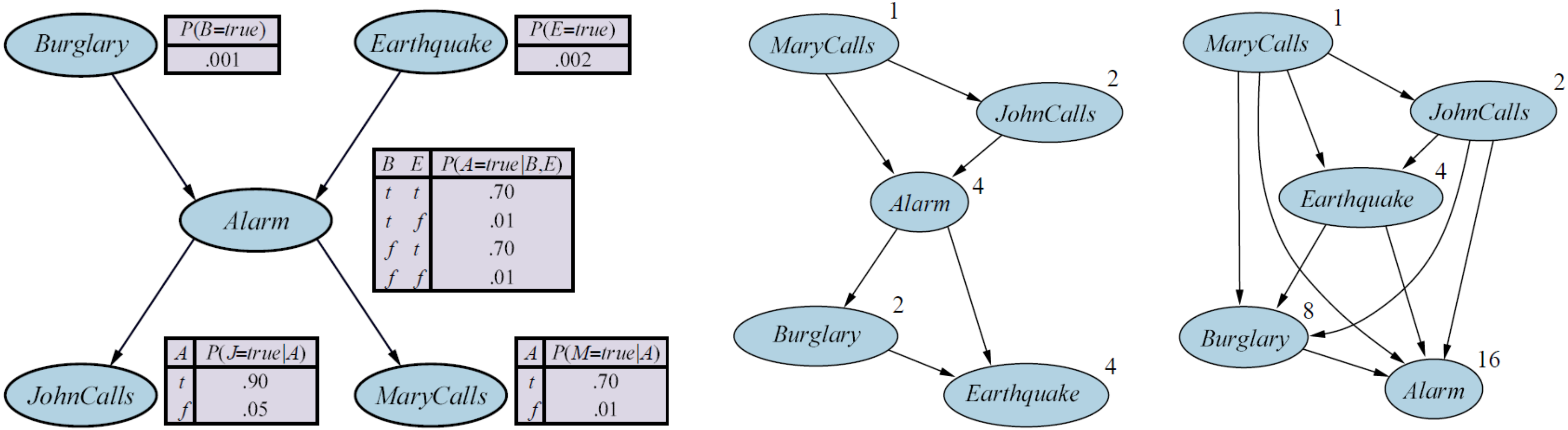
- When constructing a Bayes net, **compact** structures are more efficient
- The more independences that we can assert, the fewer the connections and parameters that we need to store

- Consider a distribution of n RVs, each with domain size d
- Full joint distribution is specified by a table of d^n numbers

- If any node has at most k parents, then its distribution is $O(d^{k+1})$
- We only need $\mathrm{O}(n \times d^{k+1})$ parameters to store all information

Example: Alarm Network

- Boolean random variables: Each table has 2^k parameters
- All three structures can correctly represent same joint distribution
- Optimal structure has 10 parameters in total; others have 13 and 31



Conditional Independences

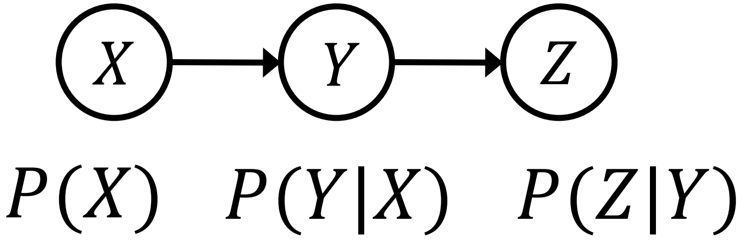
- Given a set of evidence variables, what conditional independences arise?
- We know that a node is independent of its “ancestors” given all its parents

- More generally, two nodes are conditionally independent if they are **d-separated** with *colliders* in all paths between them

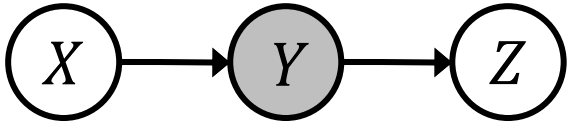
- Here, “paths” do not depend on edge directionality!
- We identify three scenarios, which can generalize to any situation

Causal Chain

- Nodes in a *causal chain* (edges pointing in same direction) generally not independent



- But an observed node acts a *collider*
- Nodes on one side become independent of nodes on the other

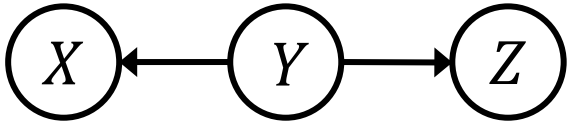


- Proof: $$P(x,z|y) = \frac{P(x)P(y|x)P(z|y)}{P(y)}$$

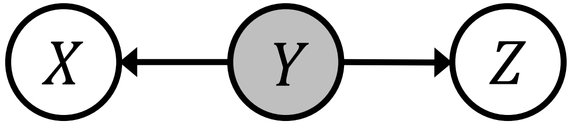
$$= \frac{P(x)P(z|y)}{P(y)} \frac{P(x|y)P(y)}{P(x)} = P(z|y)P(x|y)$$

Common Cause

- Nodes connected by a *common cause* (arc heads pointing away from each other) are generally not independent



$P(X|Y)$ $P(Y)$ $P(Z|Y)$

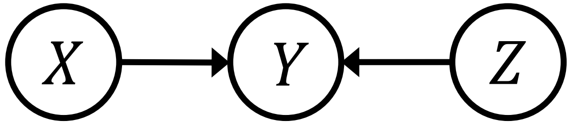


- But an observed node cause acts a *collider*
- Nodes on one side become independent of nodes on the other

- Proof: $P(x,z|y) = \dfrac{P(x,y,z)}{P(y)} = \dfrac{P(y)P(x|y)(z|y)}{P(y)} = P(x|y)P(z|y)$

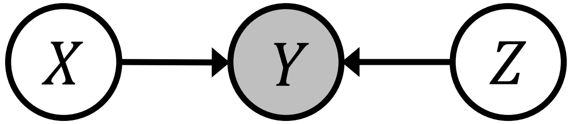
Common Effect

- A *common effect* (arc heads pointing toward each other) between two nodes acts a *collider*
- Nodes on either side are independent

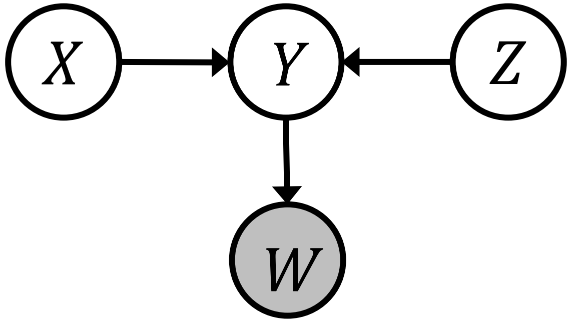


$P(X)$ $P(Y|X,Z)$ $P(Z)$

- But observing the effect or any of its descendants removes the collider!



- Nodes are all connected; independence no longer guaranteed



Example: Common Effect

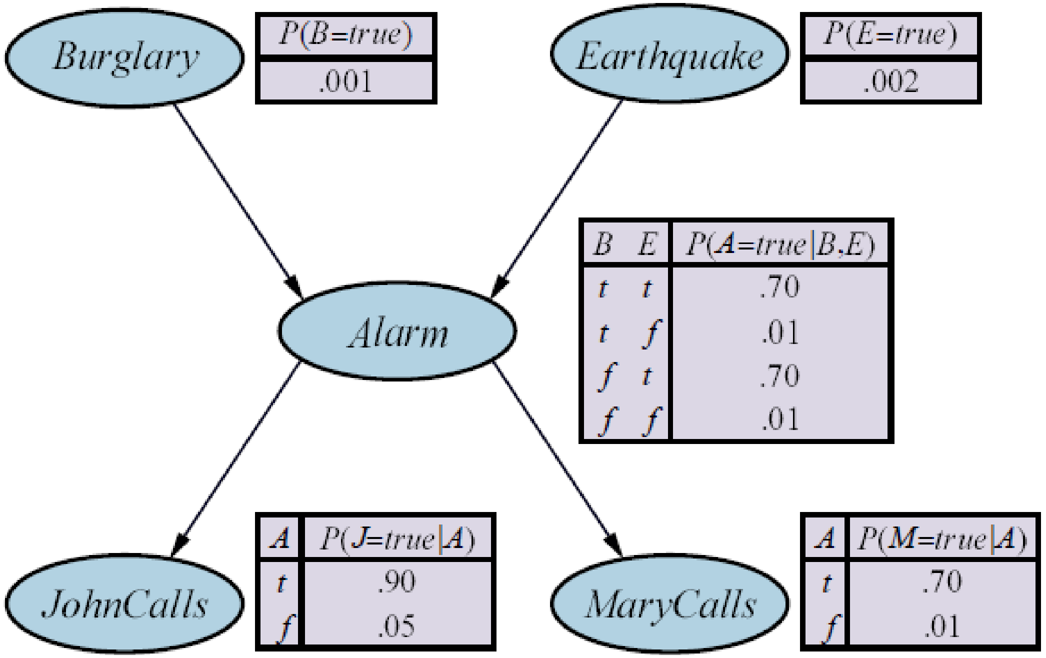
$$P(+b,+e) = P(+b)P(+e) = 0.001 \times 0.002 = 2 \times 10^{-6}$$

$$P(+b,+e|+a) = \frac{P(+b,+e,+a)}{P(+a)}$$

$$= \frac{P(+b)P(+e)P(+a|+b,+e)}{\sum_{b,e} P(b)P(e)P(+a|b,e)}$$

$$= \frac{0.001 \times 0.002 \times 0.7}{.001(.002)(.7) + .001(.998)(.01) + .999(.002)(.7) + .999(.998)(.01)}$$

$$= 1.23 \times 10^{-4}$$



P(B=true)
.001

P(E=true)
.002

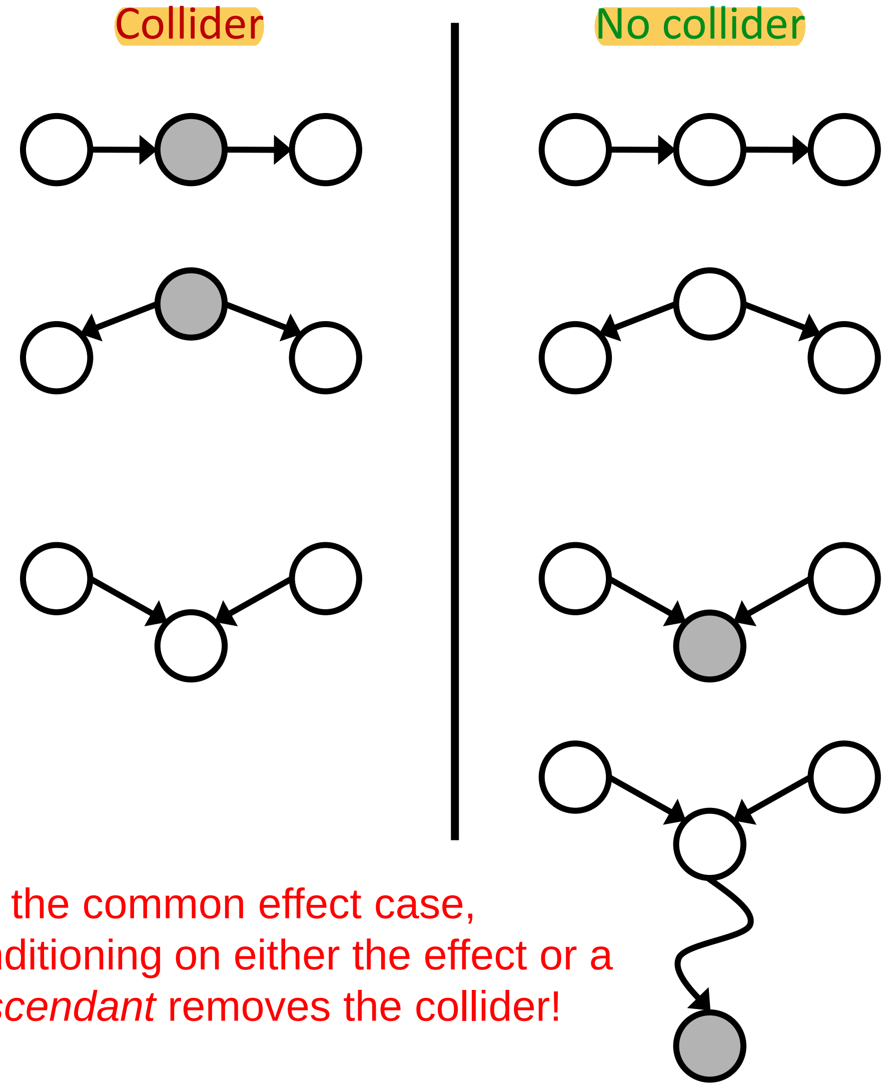
B	E	P(A=true|B,E)
t	t	.70
t	f	.01
f	t	.70
f	f	.01

A	P(J=true|A)
t	.90
f	.05

A	P(M=true|A)
t	.70
f	.01

D-Separation

- Question: Are two nodes X_i and X_j independent conditioned on a set of nodes E?
- First identify "conditioned" nodes (shade them in)
- If *at least one path* between X_i and X_j is unblocked (no colliders): not conditionally independent
- If *every path* between X_i and X_j contains a collider: conditionally independent!



For the common effect case, conditioning on either the effect or a *descendant* removes the collider!

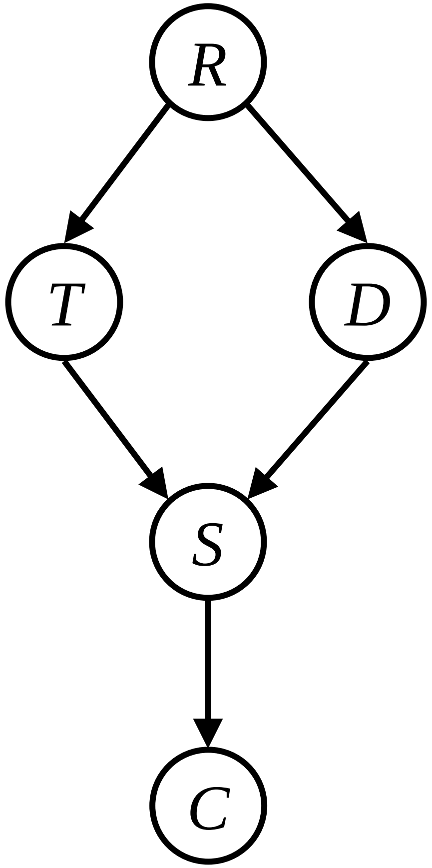
Example: D-Separation

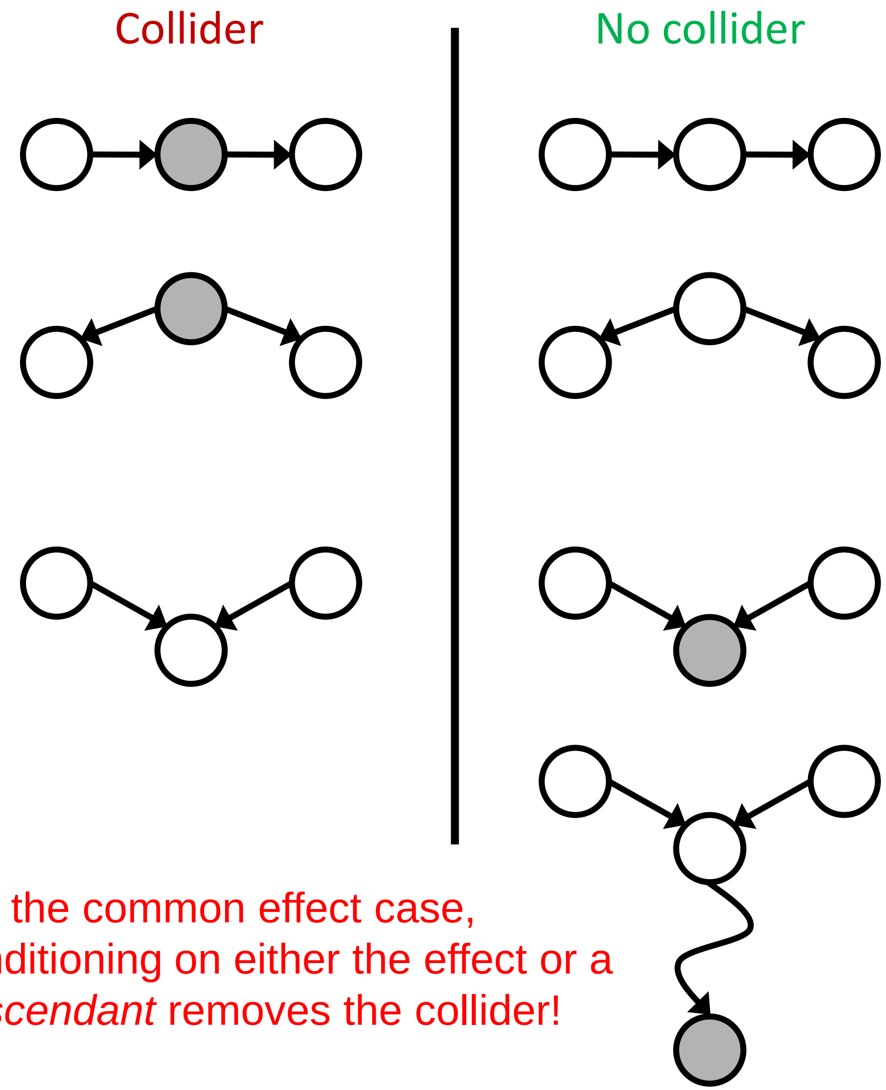
Which nodes are independent…

- Given S?
- Given R?

- Given T or D?
- Given T and D?

- Given R and S?
- Given R and C?



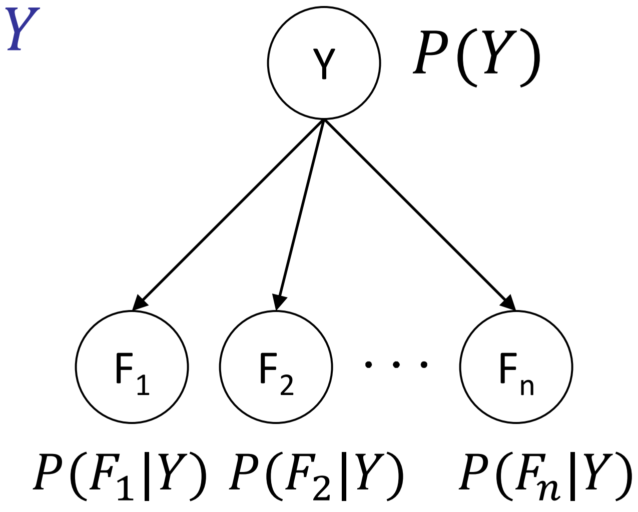


For the common effect case, conditioning on either the effect or a *descendant* removes the collider!

Naïve Bayes Models

- In a naïve Bayes model, there is an underlying "cause" variable Y
- Y then influences a number of different effects F_i
- Given Y, all the F_i are conditionally independent!
- Joint distribution: $P(y, f_1, \dots, f_n) = P(y) \prod_{i=1}^{n} P(f_i|y)$
- How to *infer* the cause given observed effects?

$$P(Y|f_1, \dots, f_n) = \frac{P(Y, f_1, \dots, f_n)}{P(f_1, \dots, f_n)} \propto_Y P(Y) \prod_{i=1}^{n} P(f_i|Y)$$



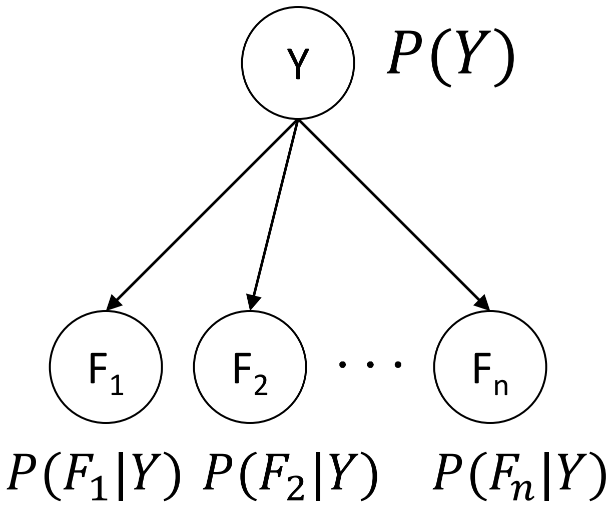
Classification

- Naïve Bayes can be used for **classification** tasks
- Given a data input, predict an output *label* or *class*

- Inputs are *features* f_i and output is most likely class y
- Predicted class:

$$y = \operatorname*{argmax}_y P(y|f_1, \dots, f_n) = \operatorname*{argmax}_y P(y) \prod_{i=1}^{n} P(f_i|y)$$

- We don't even need to normalize—exact probabilities are not required
- Problem: If we have too many features, product will underflow



Log Probabilities

- As with Viterbi, NB classification only looks at relative ordering of likelihoods, not the actual probabilities
- *Logarithm* function preserves ordering and can address underflow issues

$$\underset{y}{\operatorname{argmax}} P(y) \prod_{i=1}^{n} P(f_i|y) = \underset{y}{\operatorname{argmax}} \log\left(P(y) \prod_{i=1}^{n} P(f_i|y)\right)$$

- Log of a product = sum of logs

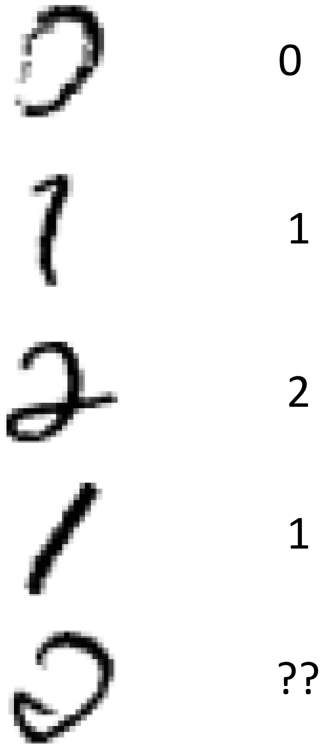
$$\log\left(P(y) \prod_{i=1}^{n} P(f_i|y)\right) = \log(P(y)) + \sum_{i=1}^{n} \log(P(f_i|y))$$

- Since probabilities ≤ 1, log probabilities are always ≤ 0

Classification Examples

- Spam filter: Inputs are emails, class/label is spam/ham
- Emails contain **features** that will help classify
  - Words: “wire money”, “bank account”, “supplements”
  - Text patterns: all caps, unusual font, dollar signs
  - Other flags: Sender not in contacts, geographic origin

- Digit recognition: Inputs are pixels, class/label is a digit
- Pixel grids contain **features** that will help classify
  - Whether specific pixels are ON or OFF
  - Patterns: Size, number of loops, number of lines



Example: Spam Filtering

- Labels $Y = \{\text{spam}, \text{ham}\}$
- Features W_i for position i in the email over common dictionary words

- **"Bag-of-words"** model: Each W_i is independent, identically distributed
  - No dependence on ordering or other nearby words!

Word	$\boldsymbol{P(Y)}$	**Hey**	**would**	**you**	**like**	**to**	**lose**	**weight**
$\boldsymbol{P(w\|\text{spam})}$	0.33333	0.00002	0.00069	0.00881	0.00086	0.01517	0.00008	0.00016
$\boldsymbol{P(w\|\text{ham})}$	0.66666	0.00021	0.00084	0.00304	0.00083	0.01339	0.00002	0.00002

- $\log P(spam) + \sum \log P(w|spam) = -53.3$
- $\log P(ham) + \sum \log P(w|ham) = -55.0$

Supervised Learning

- Where do the probabilities (*parameters*) of our models come from?

- **Supervised learning**: Given *training data* with input-output pairs $(x_1, y_1), \dots, (x_N, y_N)$ generated by an unknown function f, find a *hypothesis* to best approximate f

- Expressiveness-complexity tradeoff: The more expressive the hypothesis space, the higher the computational complexity of searching for a good hypothesis

- Bias-variance tradeoff: More complex hypotheses that fit training data well, or simpler hypotheses that generalize to new data better?

Learning Naïve Bayes Models

- If our data are identical and independently distributed (i.i.d.), the likelihood of all of them occurring given a hypothesis h is $P(data|h) = \prod_j P(data_j|h)$

- **Maximum likelihood learning**: Find the hypothesis that maximizes $P(data|h)$
- More specifically, find the hypothesis *parameters* θ that maximize $L(\theta) = P(data|h_\theta)$

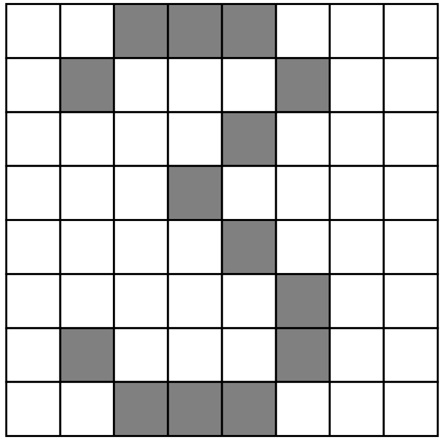
- For Naïve Bayes, the parameters are the individual model probabilities $P(y)$, $P(f_i|y)$

- $L(\theta)$ is maximized when the probabilities are chosen by simply *counting the occurrences of each event* in the data

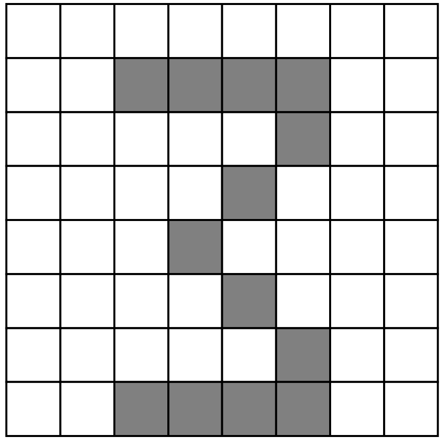
Example: Digit Recognition

- Labels $Y = \{0,1,2,3,4,5,6,7,8,9\}$
- Binary features F_{ij} for each pixel (i, j) in the input image

- Suppose we have two training examples of 3s



$$\begin{aligned} Y &= 3 \\ F_{00} &= 0 \\ F_{01} &= 0 \\ F_{02} &= 1 \\ F_{03} &= 1 \\ &\vdots \\ F_{77} &= 0 \end{aligned}$$



$$\begin{aligned} Y &= 3 \\ F_{00} &= 0 \\ &\vdots \\ F_{12} &= 1 \\ F_{13} &= 1 \\ &\vdots \\ F_{77} &= 0 \end{aligned}$$

$$\begin{aligned} P(F_{00} = 1|Y = 3) &= 0 \\ P(F_{01} = 1|Y = 3) &= 0 \\ P(F_{02} = 1|Y = 3) &= 0.5 \\ &\vdots \\ P(F_{72} = 1|Y = 3) &= 1 \\ &\vdots \\ P(F_{77} = 1|Y = 3) &= 0 \end{aligned}$$

- In practice, we will be using much more than just 2 training samples

Overfitting: Digit Recognition

- Suppose we learned a model to classify 2 or 3, but our training data has no 3s with the bottom-right pixel on

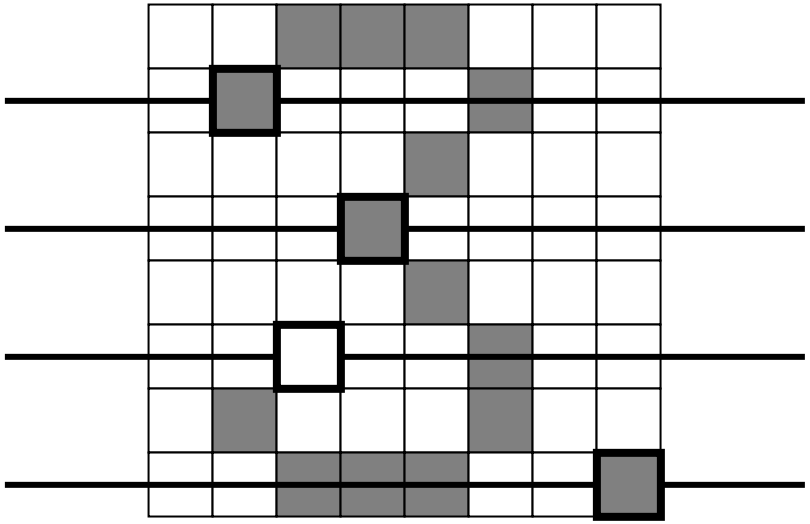
$P(C=2)=0.1$ | $P(C=3)=0.1$

$P(\text{on}|C=2)=0.8$ | $P(\text{on}|C=3)=0.8$

$P(\text{on}|C=2)=0.1$ | $P(\text{on}|C=3)=0.9$

$P(\text{off}|C=2)=0.1$ | $P(\text{off}|C=3)=0.7$

$P(\text{on}|C=2)=0.01$ | $P(\text{on}|C=3)=0.0$



$\prod = 8 \times 10^{-6}$

So we predict "2".
What went wrong here?

$\prod = 0$

Overfitting: Spam Detection

- We will not see every dictionary word in training data—0s all over the place in the model!
- If any word appears "ham" data but never in "spam", then any email with that word will be classified as "ham" (and vice-versa)
- "`Dear Sir/Madam, I would like to` **`seriously`** `offer you an exclusive opportunity to win $1M by simply sending us your bank info`" -> HAM

$P(W|\text{spam})$

```
south-west  : 0
nation      : 0
morally     : 0
nicely      : 0
extent      : 0
seriously   : 0
...
```

$P(W|\text{ham})$

```
screens    : 0
minute     : 0
guaranteed : 0
$205.00    : 0
delivery   : 0
signature  : 0
...
```

Additive Smoothing

- Problem : If a specific class/feature does not appear in training data, probability assumed to be 0!
  - This will automatically zero out $P(Y, F_1, \dots, F_n) = P(Y) \prod_i P(F_i|Y)$
  - To avoid overfitting, we need to **generalize** our models
- Idea: **Smooth** our estimates by adding a “pseudocount” k

$$\hat{\theta} = \frac{c(x) + k}{N + k|X|} \xrightarrow{k \to 0} \hat{\theta} = \frac{c(x)}{N}$$

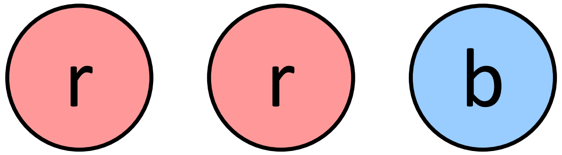
Original maximum likelihood estimator (average)

$$\hat{\theta} = \frac{c(x) + k}{N + k|X|} \xrightarrow{k \to \infty} \hat{\theta} = \frac{1}{|X|}$$

Uniform prior over all possibilities of X (ignore all samples)

Example: Additive Smoothing

- Estimate probabilities of red, blue, and green marbles



Color	$\hat{P}_{MLE}$(color) $= \hat{P}_{k0}$(color)
Red	0.667
Blue	0.333
Green	0

Color	$\hat{P}_{k1}$(color)
Red	(2+1)/(3+3)=0.5
Blue	(1+1)/(3+3)=0.333
Green	(0+1)/(3+3)=0.167

Color	$\hat{P}_{k100}$(color)
Red	(2+100)/(3+300)=0.337
Blue	(1+100)/(3+300)=0.333
Green	(0+100)/(3+300)=0.330

$$\hat{\theta} = \frac{c(x) + k}{N + k|X|}$$

$$\xrightarrow{k \to 0} \quad \hat{\theta} = \frac{c(x)}{N}$$

Original maximum likelihood estimator (average)

$$\xrightarrow{k \to \infty} \quad \hat{\theta} = \frac{1}{|X|}$$

Uniform prior over all possibilities of X (ignore all samples)

Summary

- Bayesian networks graphically encode independence assumptions about joint distributions in a compact way

- D-separation rules can help infer additional independences given evidence

- Naïve Bayes models are simple ways of thinking about cause-effect or class-label relationships

- Supervised learning of model parameters can be done using maximum likelihood learning (for Naïve Bayes, simple counting)