COMS W4701: Artificial Intelligence

Lecture 2: Search Problems

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Today

- Search problem formulation
- State space graphs and search trees

- Uninformed search: DFS, BFS, UCS
- Informed search: Greedy, A*

- Search heuristics: Admissibility, design

Problem-Solving Agents

- Goal-based agent that defines goals as a set of *world states*—descriptions of the task environment at the current time
- Assume our environments are **observable**, **discrete**, **deterministic**, **static**
  - Percepts are trivial, since we see entire environment
  - Action results always known, go from one state to another state
- We will find an *action sequence* that will result in a *state sequence* to a goal state
- This is the agent's solution to a **search problem**

Search Problems

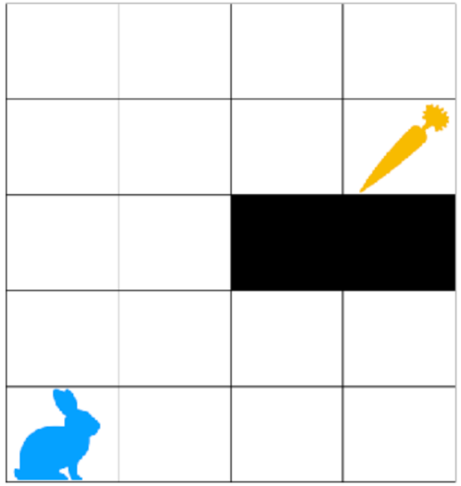
- State space S: Set of descriptions of the agent and environment
- Actions: (Finite) set of available actions in a state
  - Ex: $Actions(s_1) = \{a_1, a_2, a_3\}$

- Transition model: Mapping from (state, action) to a new state
  - Ex: $Result(s_1, a_1) = s_2$
- Action costs: Numerical cost for a (state, action, new state) transition
  - Ex: $Cost(s_1, a_1, s_2) = 10$

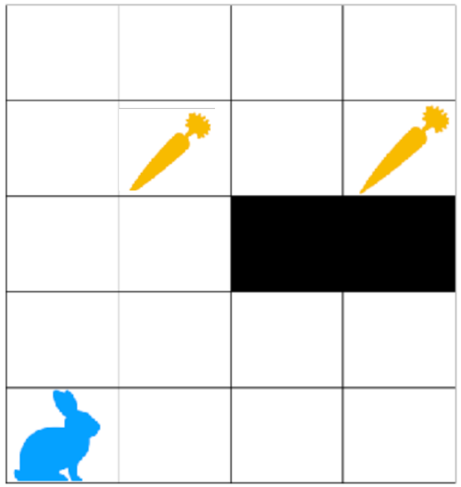
- Goal test (for goal states)
  - Ex: $IsGoal(s_1) = \text{False}$, $IsGoal(s_2) = \text{True}$

Example: Grid World Path Finding

- State space: Current coordinates of the rabbit
  - $S = \{(x, y) \mid 0 \leq x \leq 3, 0 \leq y \leq 4\}$
- Actions: $Actions\big((x, y)\big) = \{\text{Up}, \text{Down}, \text{Left}, \text{Right}\}$
- Costs: $Cost(s, a, s') = 1, \forall s, a, s'$
- Transition model: $Result((x, y), \text{Up}) = (x, y + 1)$, $Result((x, y), \text{Down}) = \cdots$
  - Should also account for walls and boundaries, e.g. $Result\big((0,0), Left\big) = (0,0)$
- Goal test: $In((3,3))$?

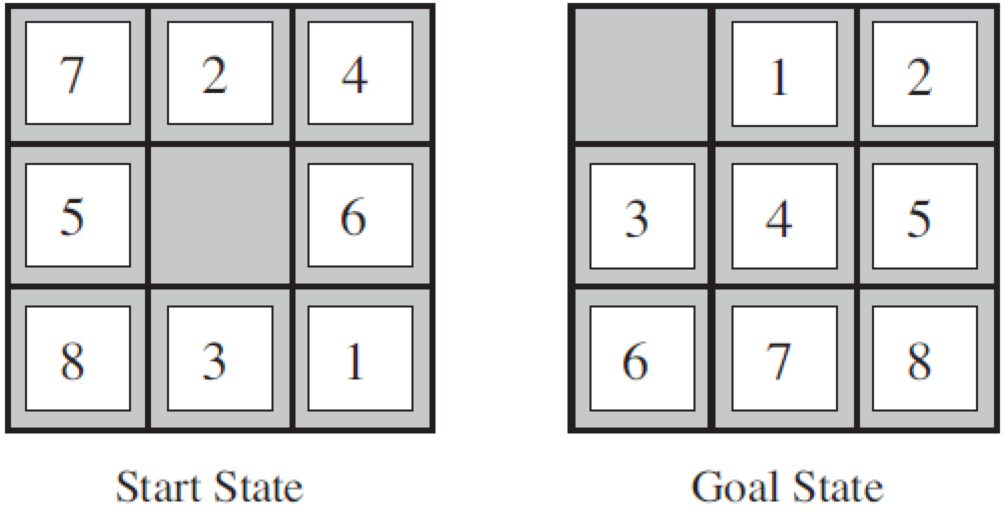


Multiple Carrots?



- What has changed about the problem?
- State space
  - Location of rabbit, Booleans indicating carrots eaten?
- Transition model
  - Update both rabbit location as well as carrot Boolean if locations match
- Goal test
  - Are all carrots eaten? Are all Boolean indicators True?

Search Problem Example: n-puzzle



- **State**:
- **Action**:
- **Action cost**:
- **Goal test**:

More Search Problem Examples

- Route-finding (e.g., vehicle navigation), robot navigation in the real world
- Touring problems (traveling salesperson)
- Layout and assembly sequencing problems

- Mathematical puzzles and proofs: Infinitely large state spaces!

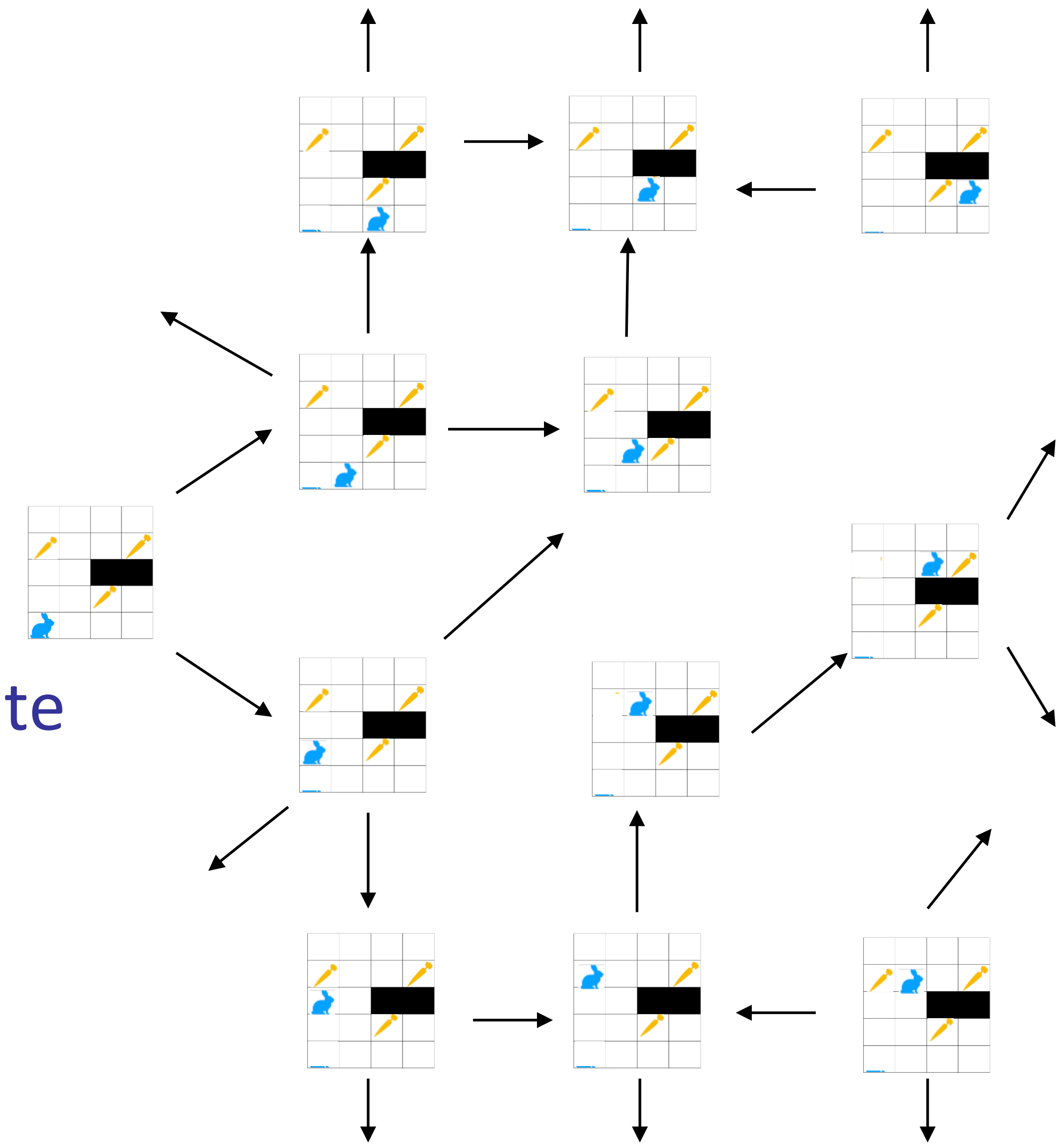
- Knuth’s conjecture (1964): Starting with the number 4, use a combination of factorial, floor, and sqrt operations to reach any other desired integer
- States: All nonnegative integers

State Space Graphs

- **State space graph**: A mathematical representation of a search problem
  - *Vertices* are states; *edges* are actions
  - Each state occurs only once!

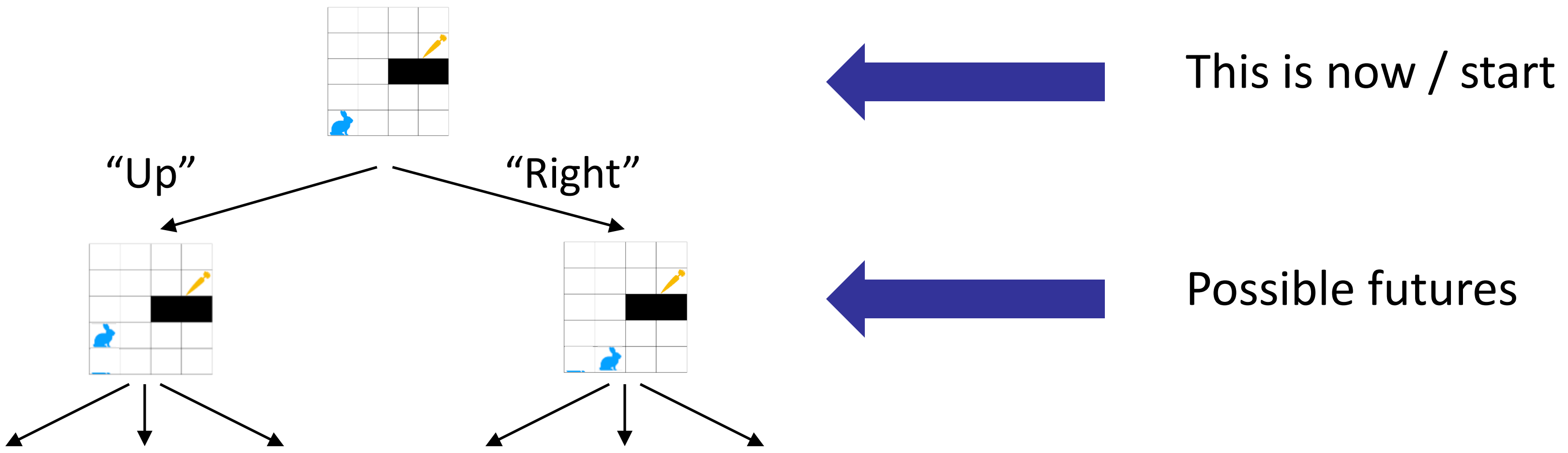
- *Paths* are sequences of actions/states
- A *solution* is a path from initial to goal state

- We can rarely build this full graph in memory—it can be very large or infinite



Search Trees

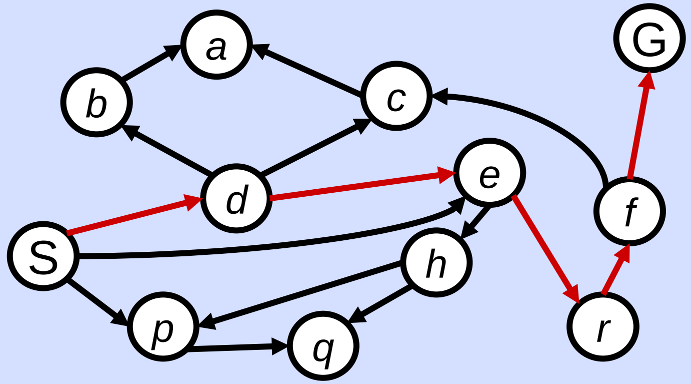
- Need a systematic way of performing search over a state space graph
- **Search tree**: Nodes are states, edges are actions; root is initial state



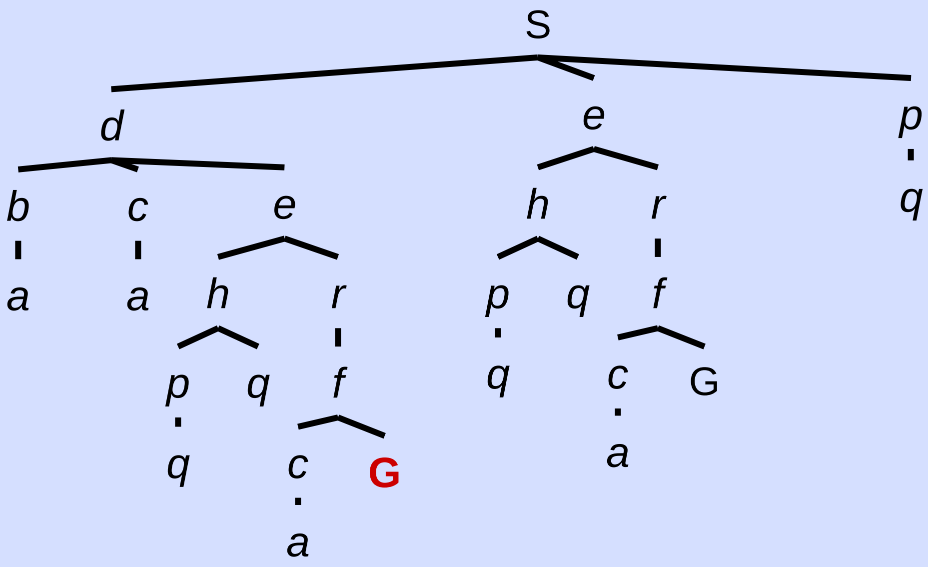
- Unlike state space graph, states can occur more than once
- Each node corresponds to a *unique* path from initial state

State Space Graphs vs. Search Trees

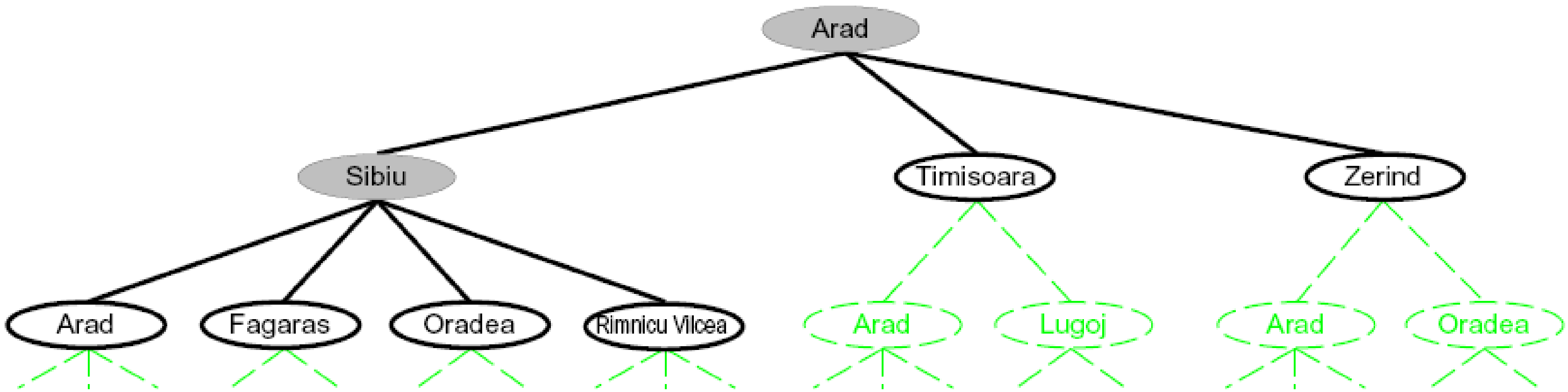
State Space Graph



Search Tree



General Search Ideas



- From current node, **expand** and consider all possible actions
- Generate successor **nodes** for each resultant state according to transition function
  - Each node should track its corresponding state, parent, prior action, and total cost so far

- Successors are added to a **frontier** of possible next nodes to expand
- Frontier forms a boundary between explored and unexplored parts of tree

Implementation Details

- What does *expansion* of children nodes entail?
- Find new state using transition function; store parent, action, and new *cumulative* cost

- How to *select* a node to expand from the frontier?
- **Best-first search**: Use an **evaluation function** $f(n)$ assigning each node a priority
- **Uninformed search**: $f(n)$ has no knowledge about how close a state is to goal

- What to do with states appearing more than once in search tree?
- Idea: Keep track of all *reached* states and costs in a lookup table
- A reached state should only be reconsidered if we find a cheaper path to it!

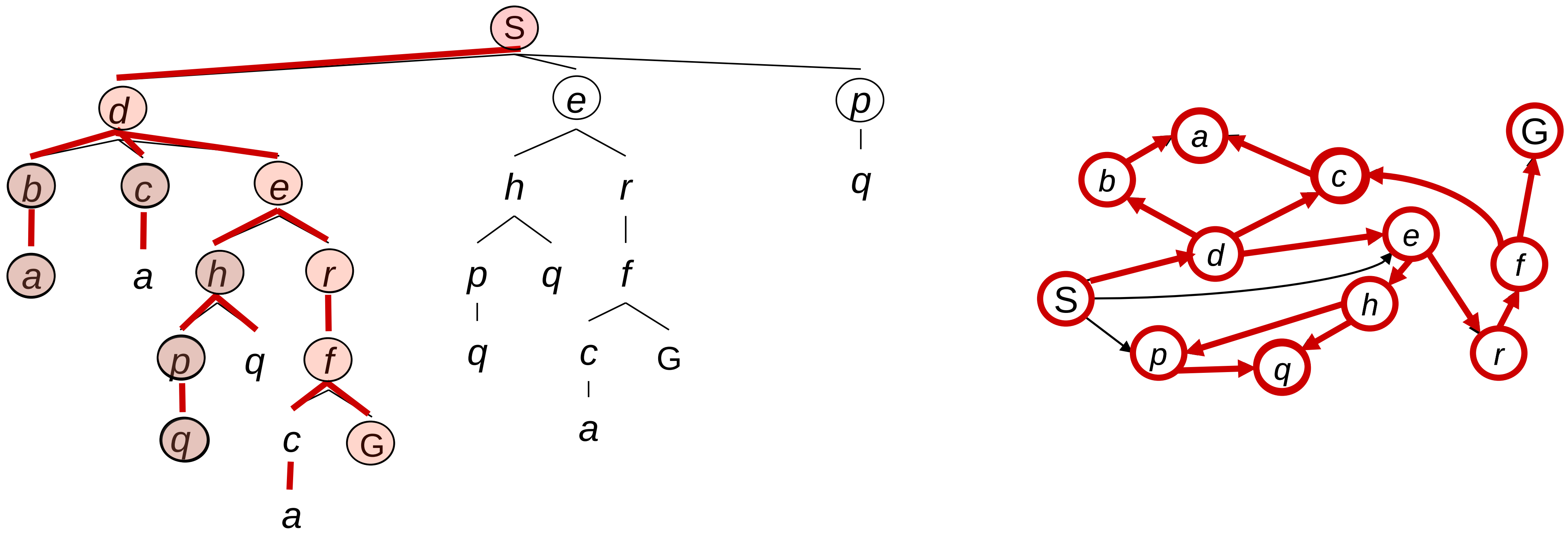
Best-First Search

```
function BEST-FIRST-SEARCH(problem, f) returns a solution node or failure
  node ← NODE(STATE=problem.INITIAL)
  frontier ← a priority queue ordered by f, with node as an element
  reached ← a lookup table, with one entry with key problem.INITIAL and value node
  while not IS-EMPTY(frontier) do
    node ← POP(frontier)
    if problem.IS-GOAL(node.STATE) then return node
    for each child in EXPAND(problem, node) do
      s ← child.STATE
      if s is not in reached or child.PATH-COST < reached[s].PATH-COST then
        reached[s] ← child
        add child to frontier
  return failure

function EXPAND(problem, node) yields nodes
  s ← node.STATE
  for each action in problem.ACTIONS(s) do
    s′ ← problem.RESULT(s, action)
    cost ← node.PATH-COST + problem.ACTION-COST(s, action, s′)
    yield NODE(STATE=s′, PARENT=node, ACTION=action, PATH-COST=cost)
```

Depth-First Search

- Idea: Expand a *deepest* node first, implement frontier as a **stack** (LIFO)
- Behavior: Frontier expands toward tree leaves
- **Early goal test** can be done when *adding to* rather than *popping* from frontier



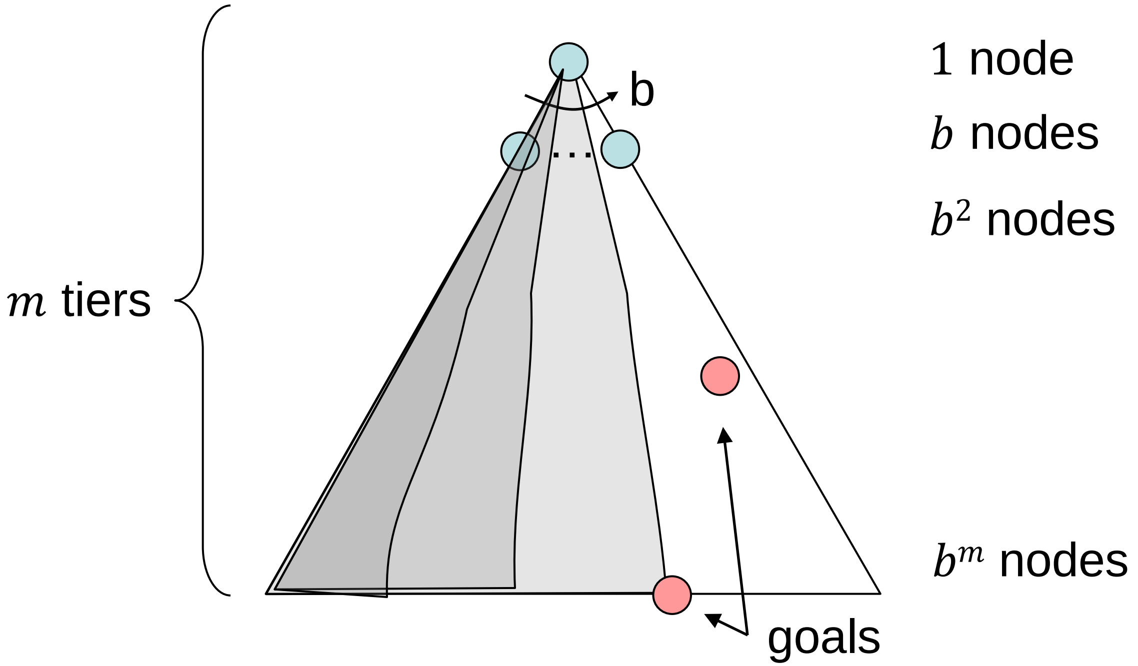
DFS Properties

- **Time complexity**: How many nodes to explore in the worst case? $O(b^m)$

- **Space complexity**: How many frontier nodes to keep in memory? $O(bm)$

- **Completeness**: Guaranteed to find solution? Not if state space is infinite

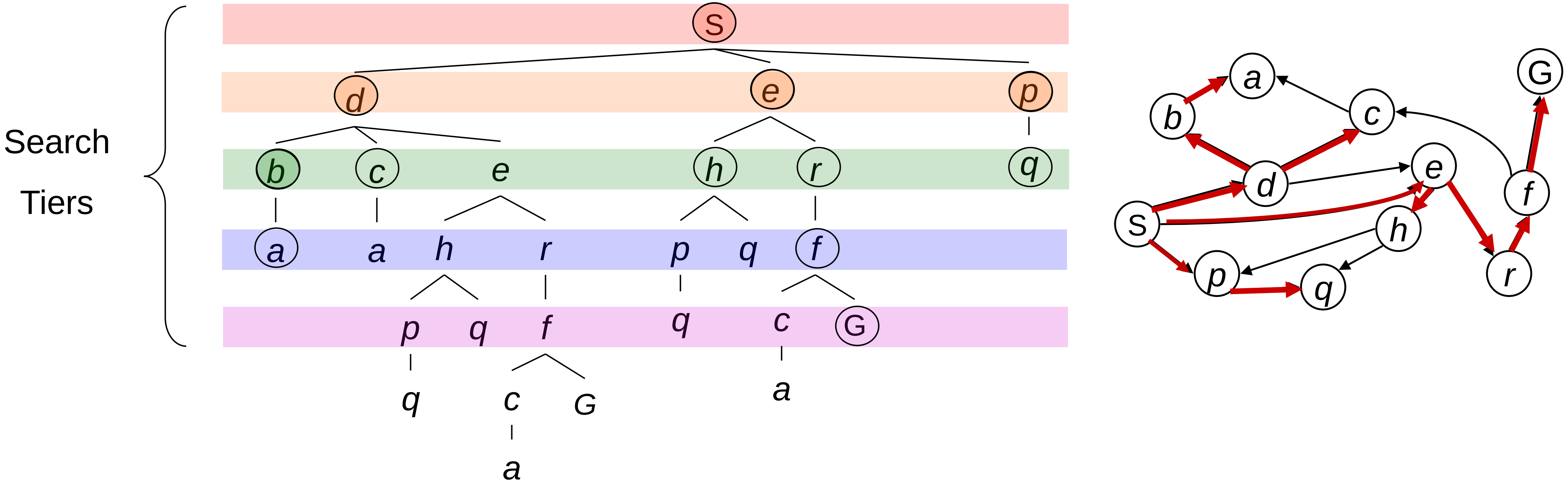
- **Optimality**: Solution guaranteed to be lowest cost? No, only returns first solution



- b is the *branching factor*
- m is the *maximum depth*
- Total nodes: $O(1 + b + b^2 + \cdots + b^m)$

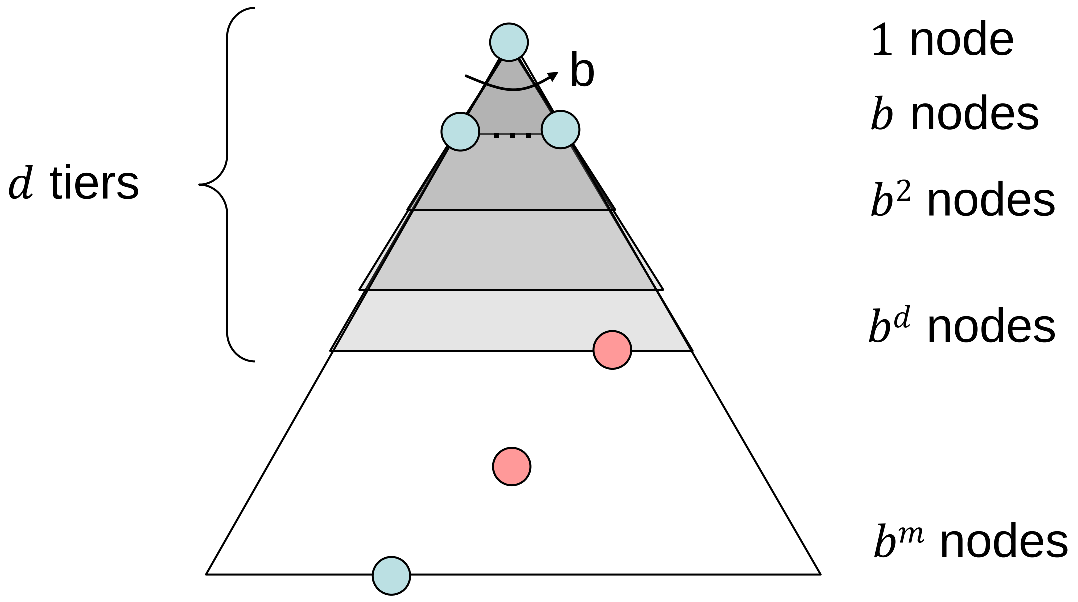
Breadth-First Search

- Idea: Expand a *shallowest* node first, implement frontier as a **queue** (FIFO)
- Behavior: Frontier expands layer by layer starting from tree root
- Only need to consider first (shallowest) encounter of a state
- **Early goal test** can be done when *adding to* rather than *popping* from frontier



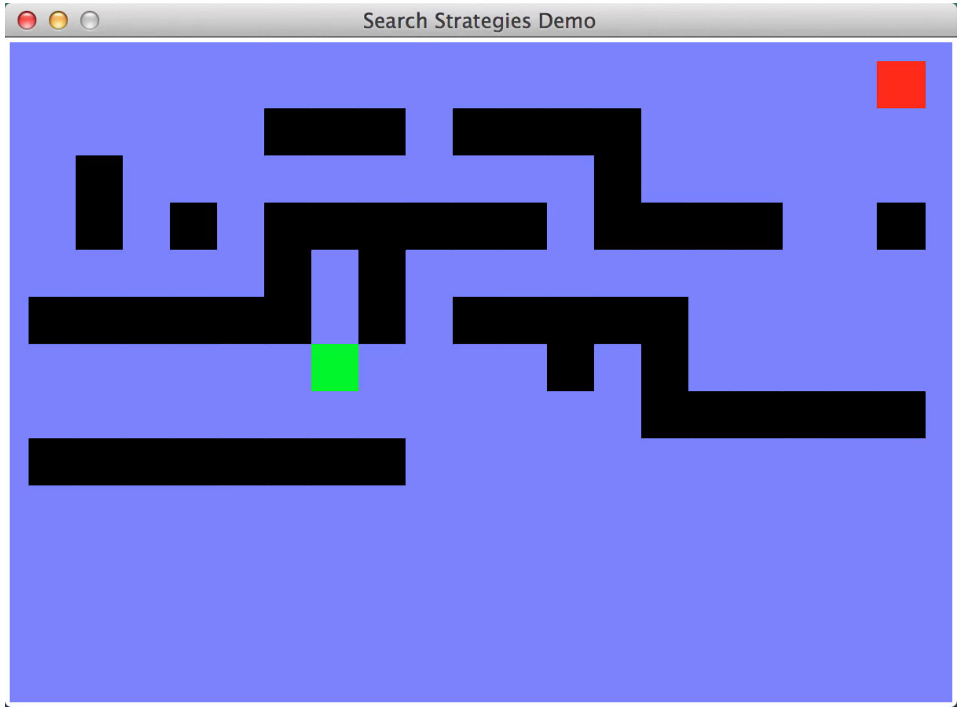
BFS Properties

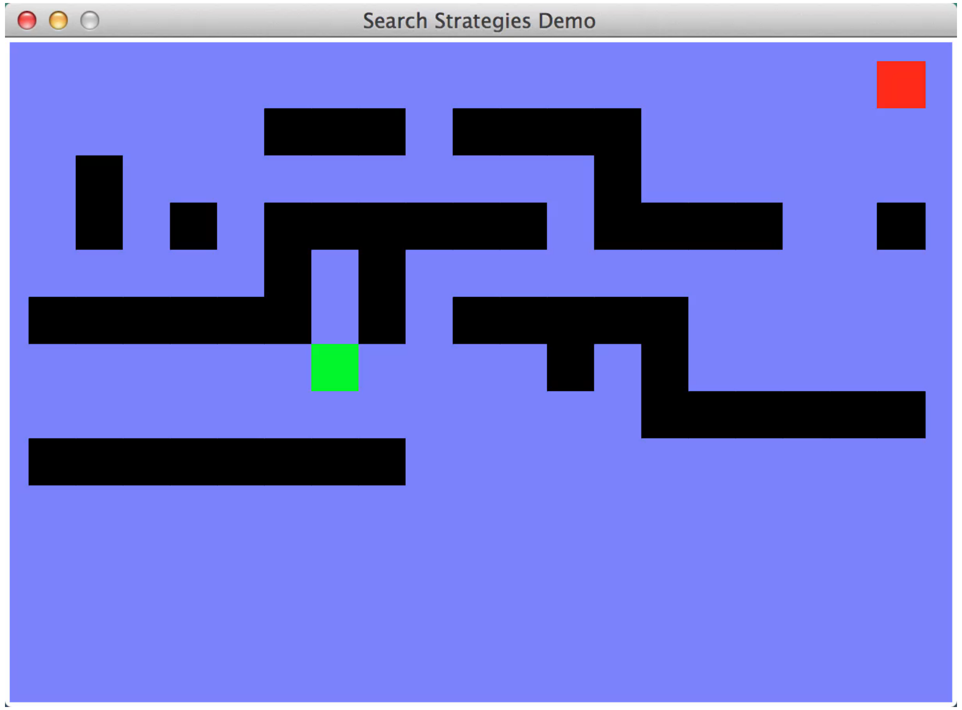
- **Time complexity**: How many nodes to explore in the worst case? $O(b^d)$
- **Space complexity**: How many frontier nodes to keep in memory? $O(b^d)$
- **Completeness**: Guaranteed to find solution? If solution exists, yes!
- **Optimality**? Solution guaranteed to be lowest cost? Only if costs are uniform



- d is depth of the shallowest solution
- May be significantly smaller than m
- Max frontier size is $O(b^d)$

BFS vs DFS



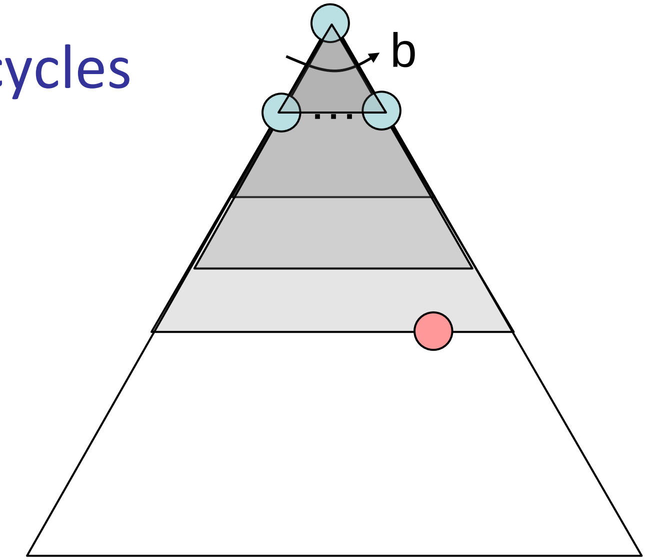


Improving DFS and BFS

- **Depth-limited DFS**: Prevent DFS from going past a set depth l
- Time complexity $O(b^l)$, space complexity $O(bl)$
- Best if we know *diameter* of state space and check for short cycles

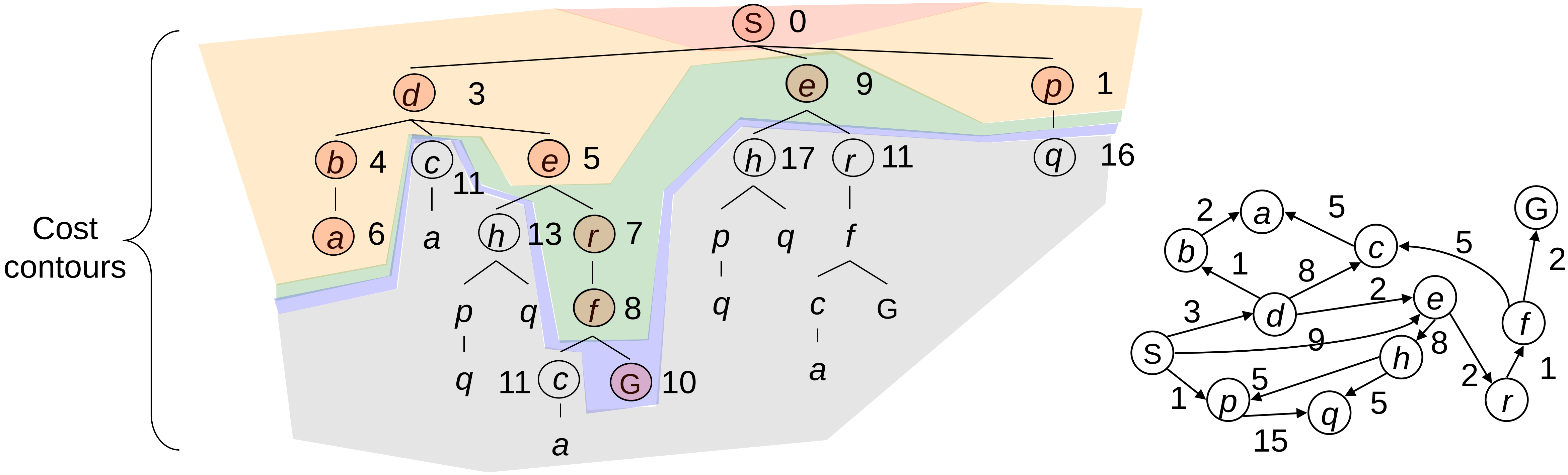
- **Iterative-deepening:** Iteratively do depth-limited search with increasing l: try $l = 0$, then $l = 1$, …
- Ends when l reaches d (depth of shallowest solution)
- Time complexity $O(b^d)$, space complexity $O(bd)$

- Why is wasted effort in upper levels of search tree not a concern?

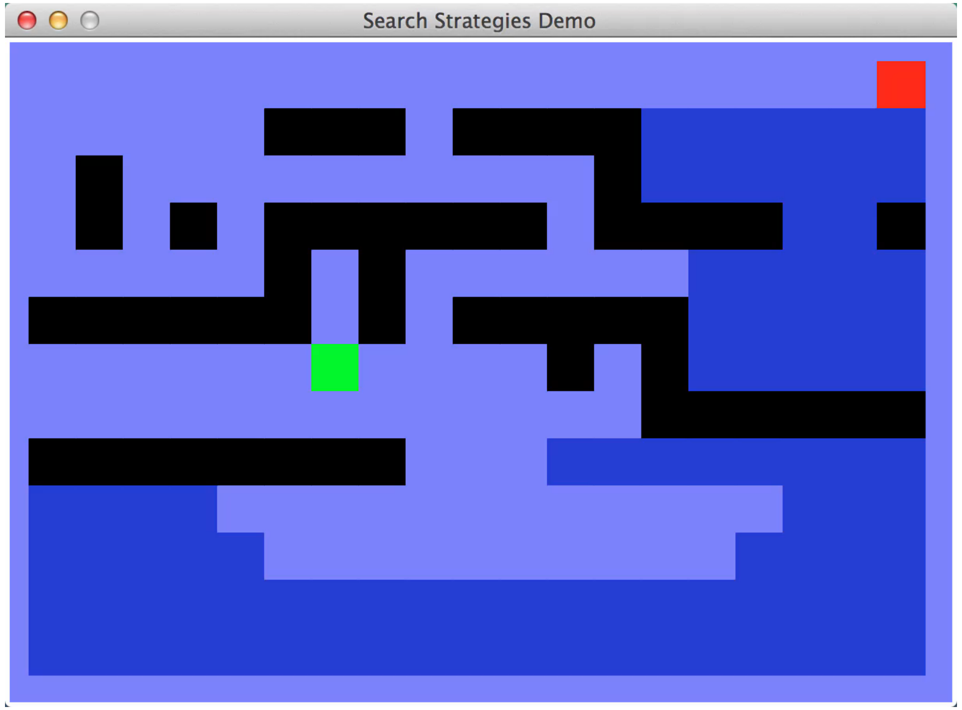


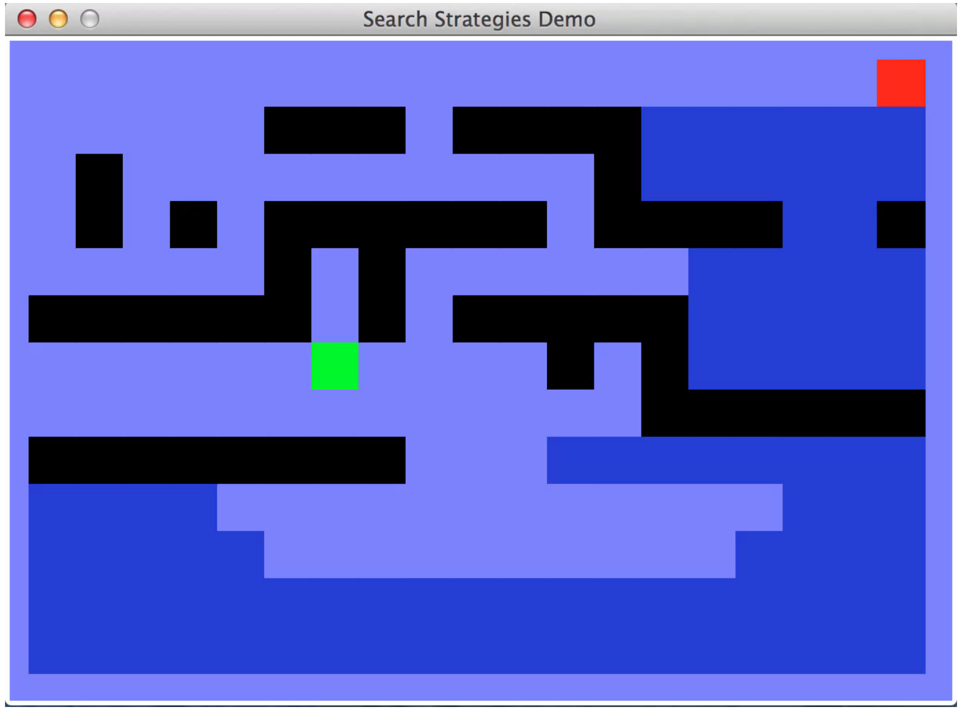
Uniform-Cost Search (Dijkstra)

- Idea: Expand node that least increases total cost, implement frontier as priority queue
- Evaluation function $f(n)$ is the *total path cost so far*
- States may be added and replaced multiple times!
- Goal test must be done when *popping* from frontier, not when adding in



BFS vs UCS





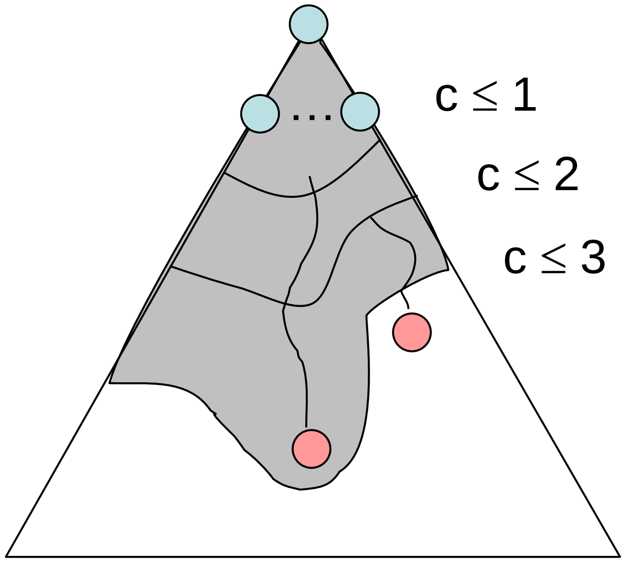
UCS Properties

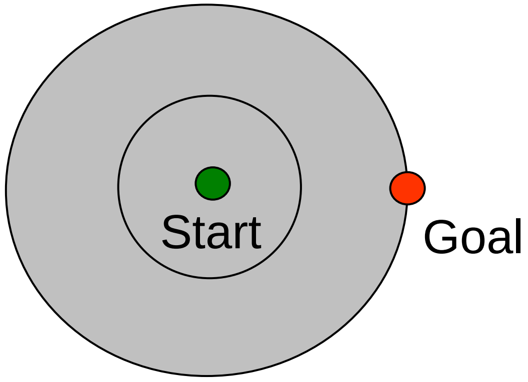
- Let C^* be the cost of optimal solution
- Let ϵ be lower bound on all possible costs

- $1 + C^*/\epsilon$ is the max depth to traverse before finding optimal solution

- **Time and space complexity**: $O(b^{1+C^*/\epsilon})$

- UCS is both **complete** and **optimal**



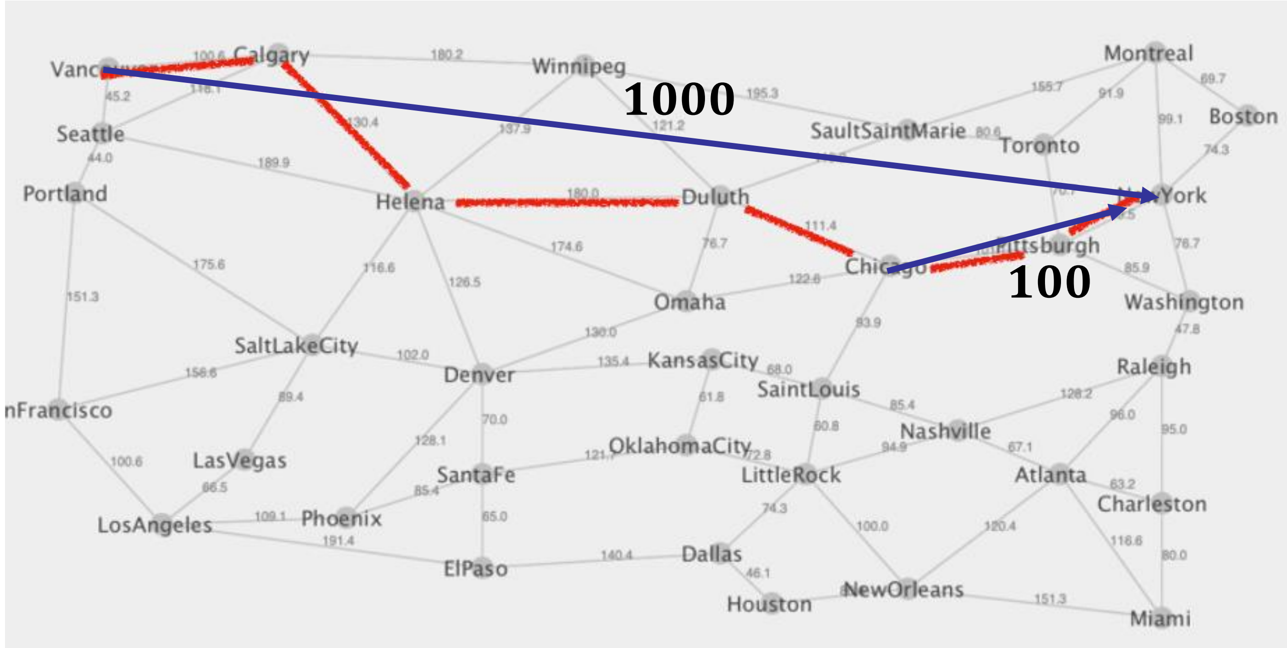


Informed (Heuristic) Search

- Oftentimes we have additional, *domain-specific* **heuristics** that tell us how close a state is to a goal

- **Heuristic function** $h(n)$: Estimated cost of cheapest path from state at node n to a goal state

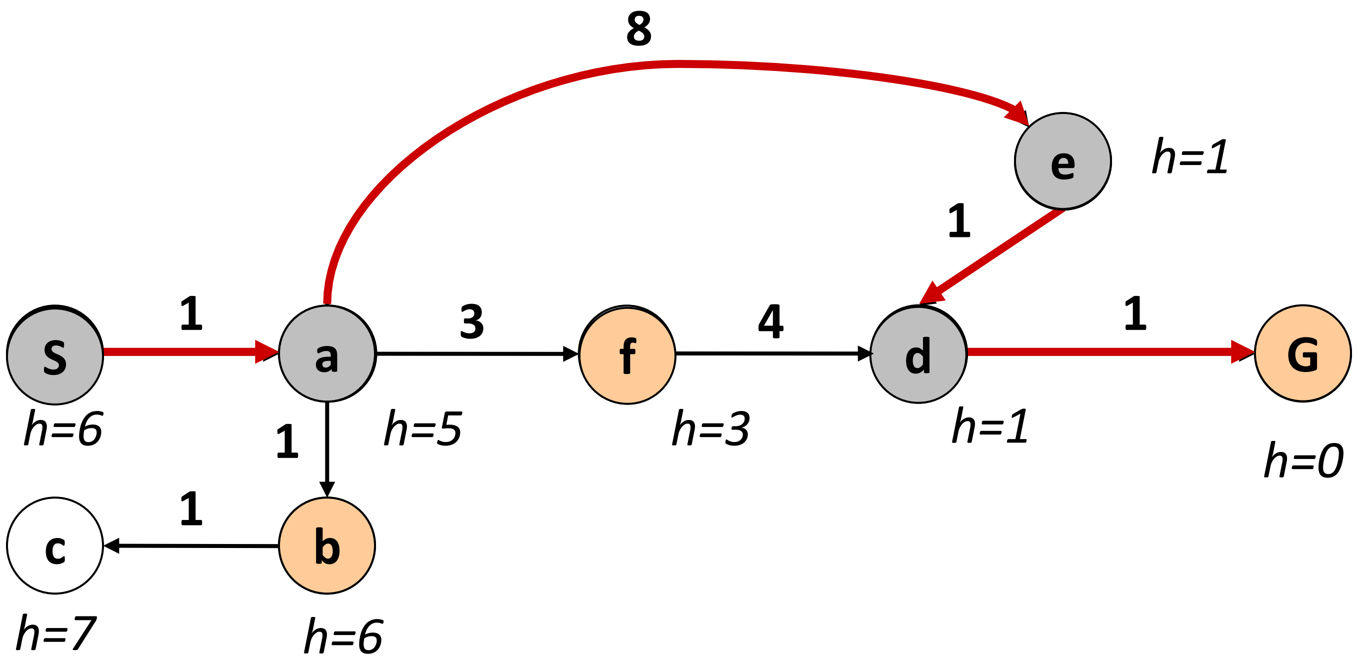
- Often come from *relaxed problems*, precomputed *subproblem* solutions, or learning from experience



Example: Euclidean or Manhattan distance on a map

Greedy Best-First Search

- Idea: Expand node that *appears* closest to goal according to heuristic function
- Evaluation function is the heuristic function! $f(n) = h(n)$
- Again implement frontier using priority queue



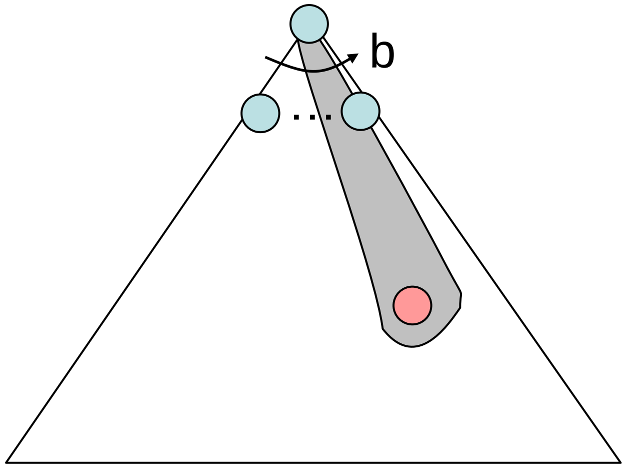
Greedy Search Properties

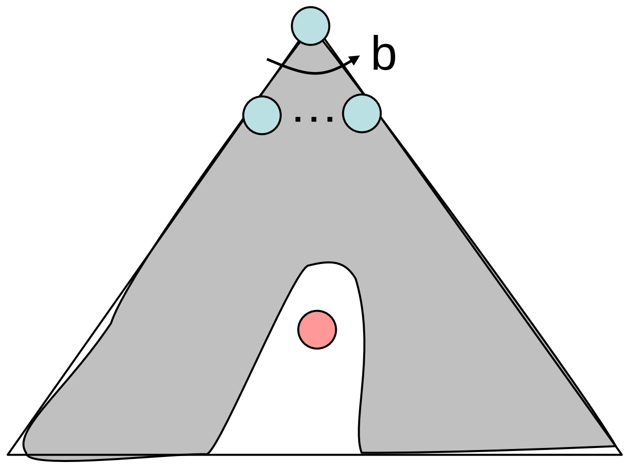
- Performance depends entirely on usefulness of heuristic function

- Best case: Go straight toward the goal
- Worst case: Like a badly misguided DFS

- Complete in finite state spaces

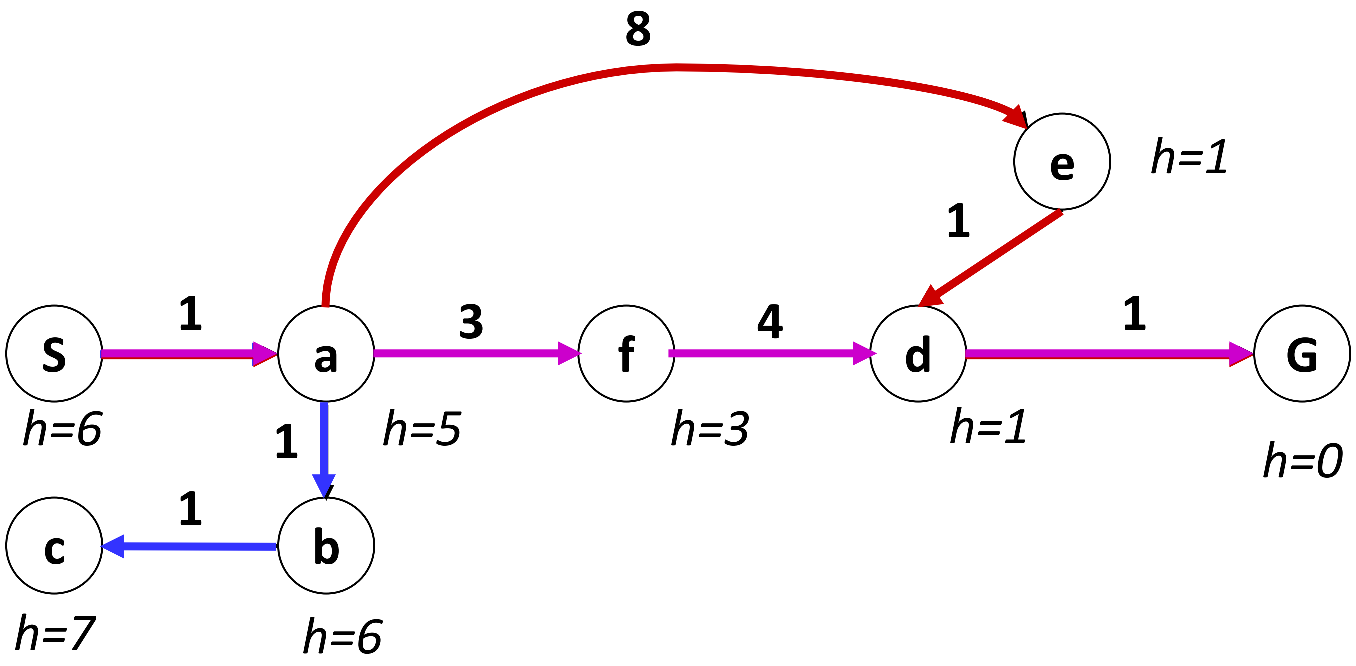
- No guarantee of optimality since true costs are never considered





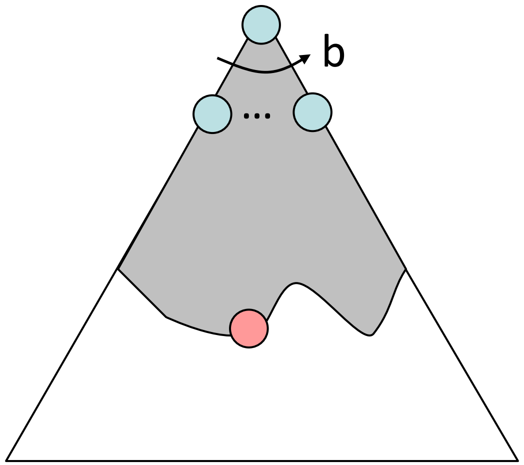
A* Search

- Idea: From a given node, estimate the *best* path that *continues* to the goal
- $f(n) = g(n) + h(n)$: Sum of path cost to n and estimated cost from n to goal
- Benefits of both UCS and greedy best-first search

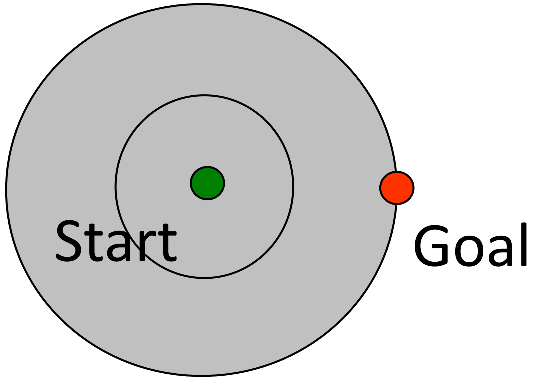


A* vs UCS vs BFS

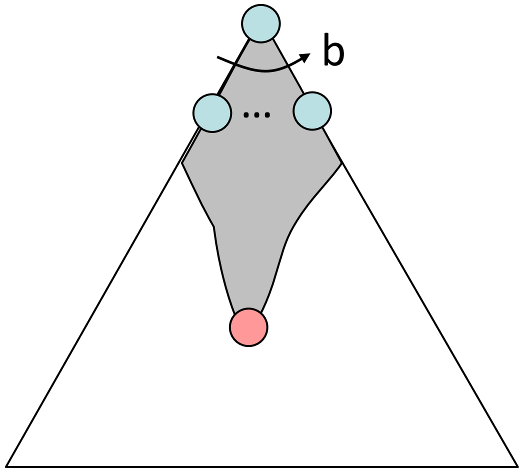
- BFS expands search tree by increasing depth

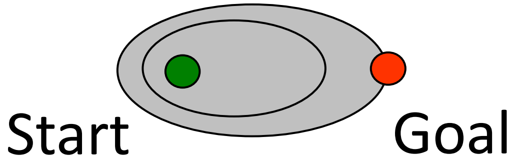


- UCS expands acc. to increasing g-cost
- Contours are “circular” around start state, if normalized by path costs

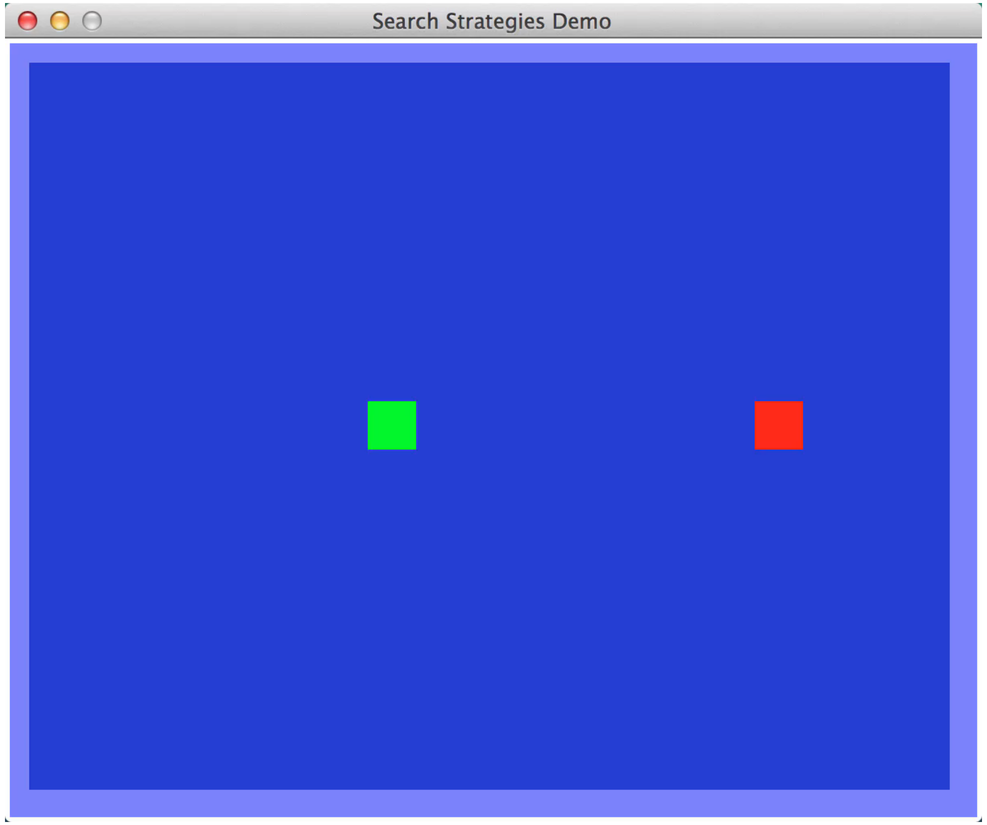


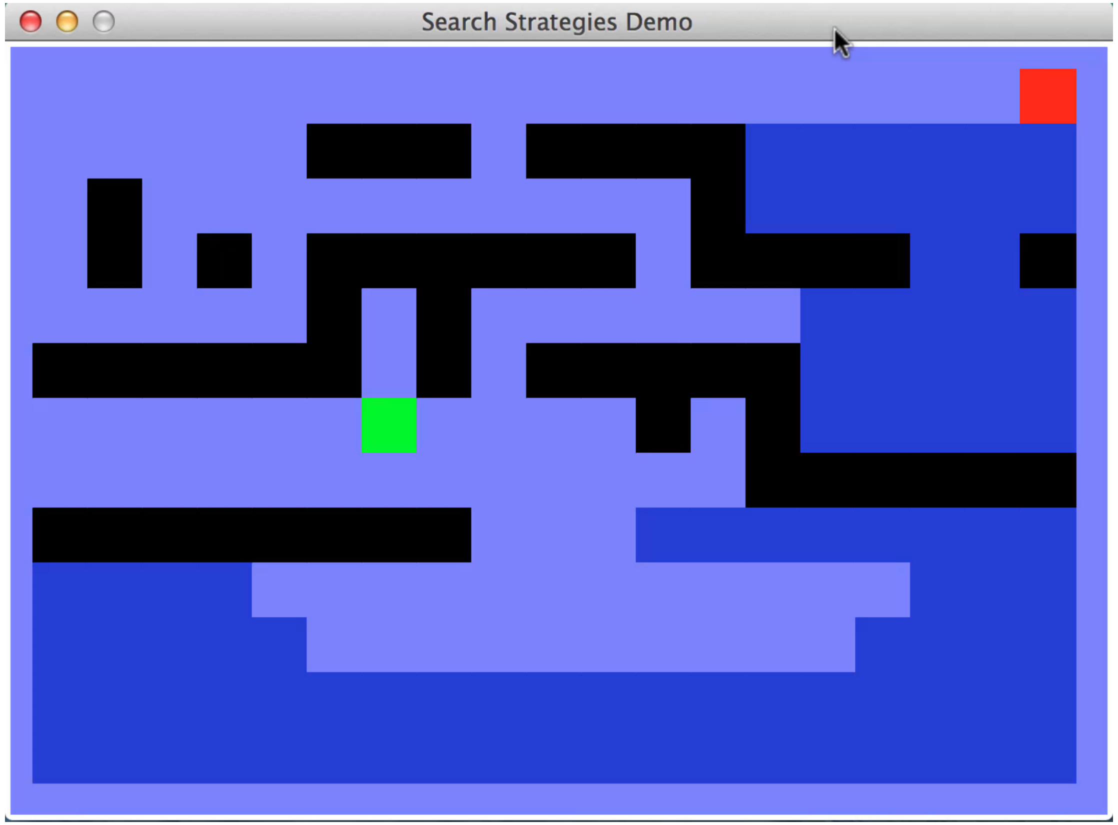
- A* expands acc. to increasing $g + h$ cost
- If heuristic is good, expanded states should show preference toward goal



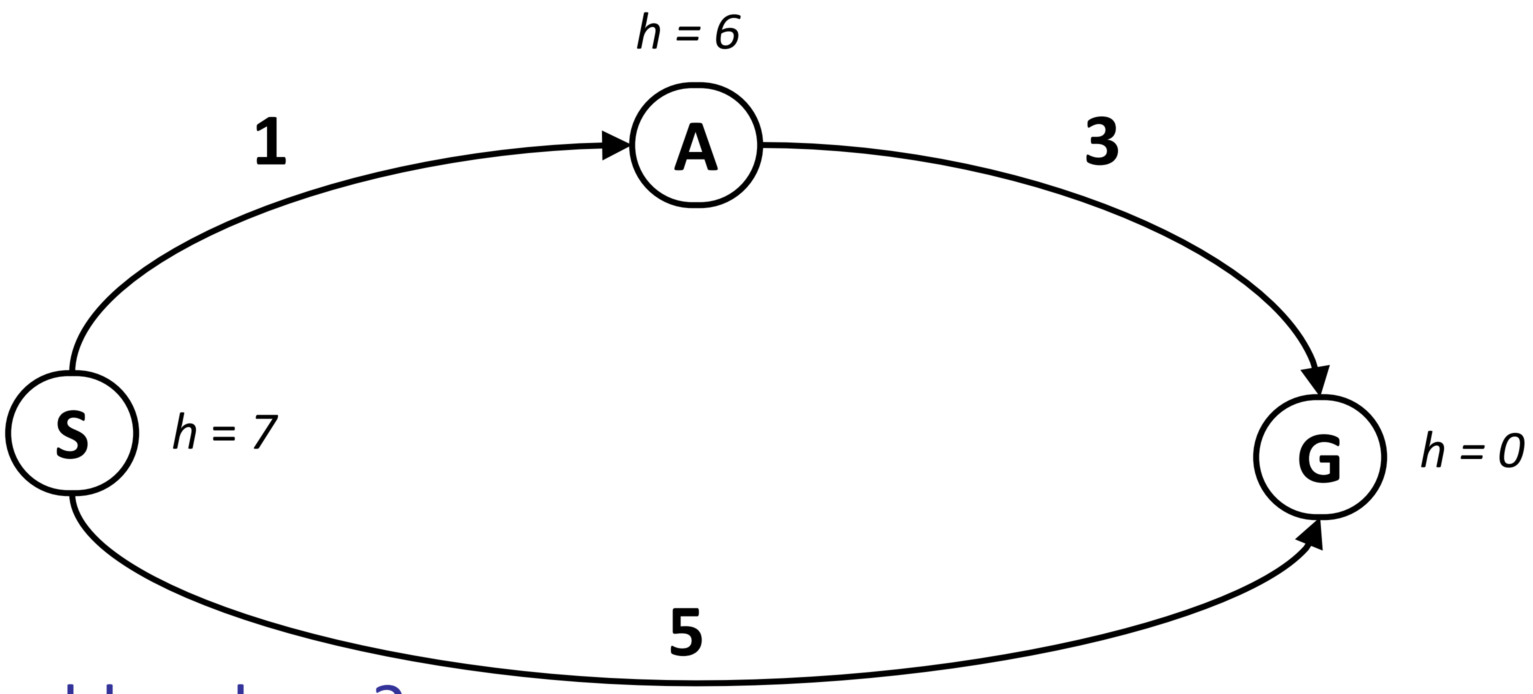


A* Examples





When is A* Optimal?



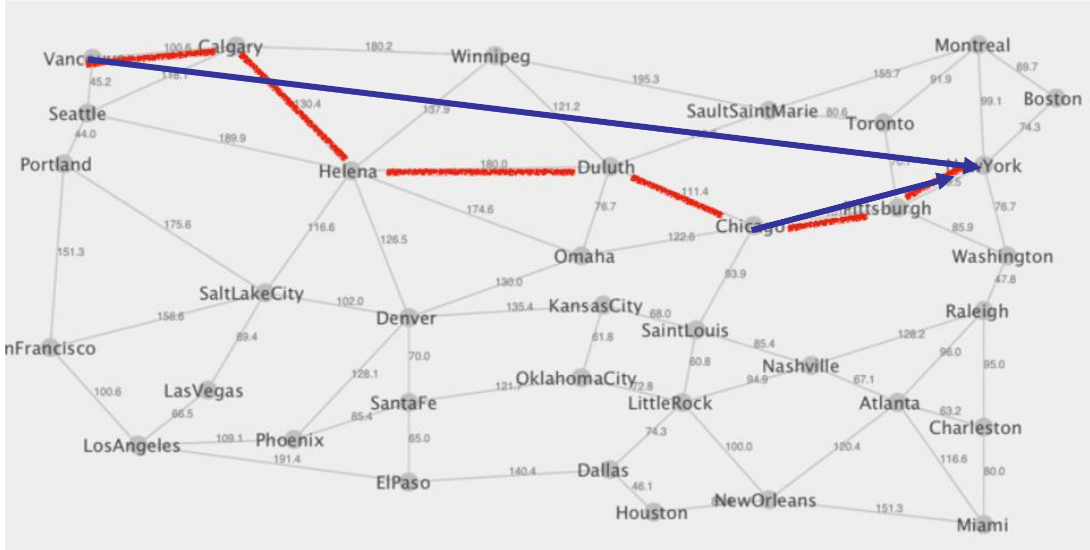
- What is the problem here?
- Heuristic along optimal path overestimated the true cost!
- Good heuristics should be optimistic—never overestimate true costs

Admissible Heuristics

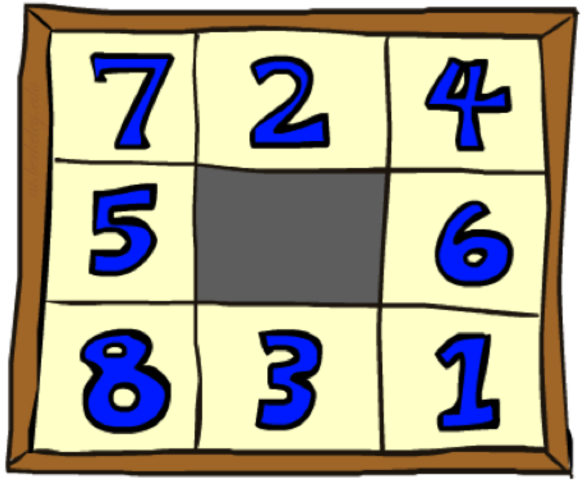
- A heuristic h is **admissible** if $0 \leq h(n) \leq h^*(n)$ where $h^*(n)$ is true cost from n to goal

- Most heuristics derived from relaxed problems are admissible
- Same state space graph, but with added edges
- With fewer constraints or restrictions, problems are easier to solve

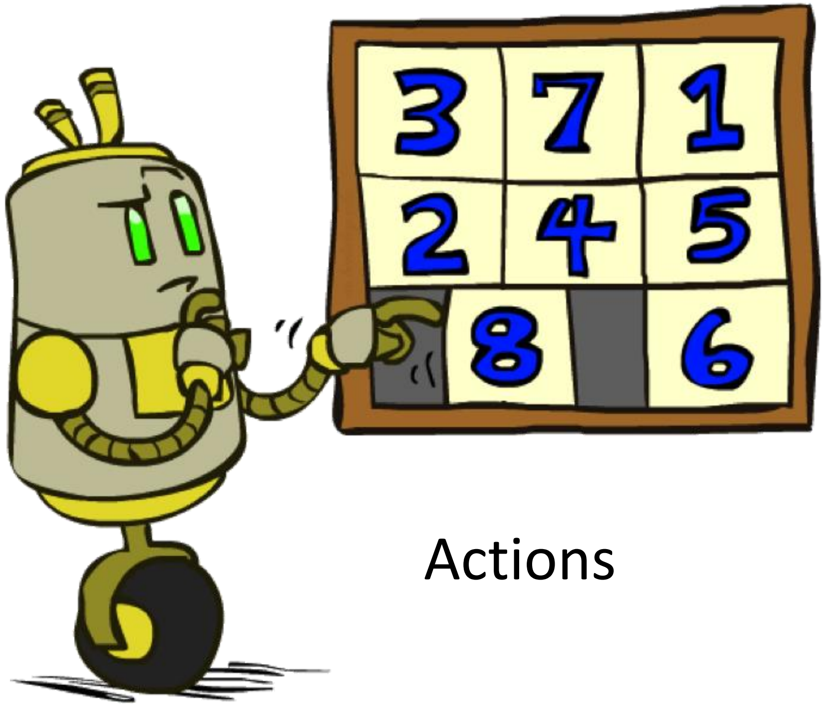
- Example: Euclidean distances



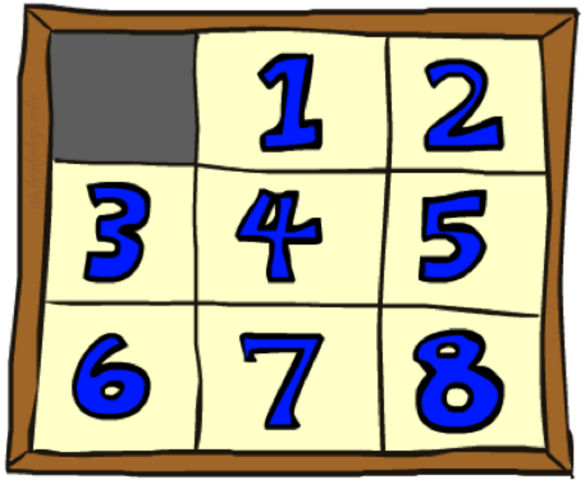
Example: 8-Puzzle



Start State



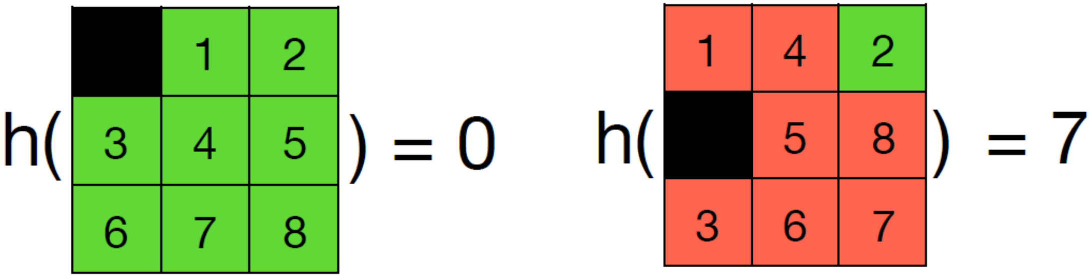
Actions



Goal State

Misplaced Tiles Heuristic

- $h(n)$ = number of misplaced tiles, not including blank



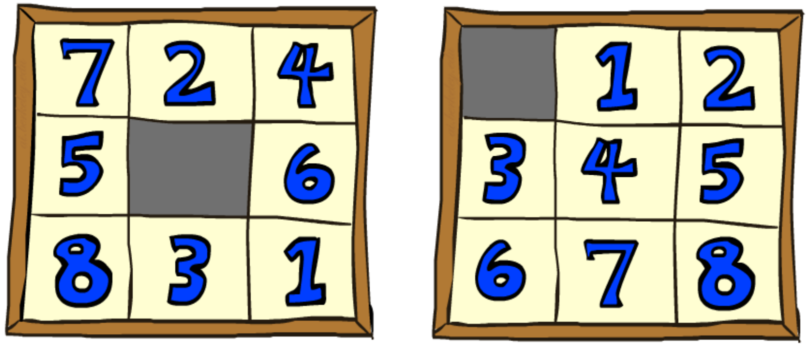
- Relaxed problem: Any tile can be correctly replaced with just one move

- Admissible because misplaced tiles will always require *at least* one move

Manhattan Distance Heuristic

- $h(n)$ = sum of Manhattan distances between current tile positions and goal positions

$h(start) = 18$



Start State Goal State

- Relaxed problem: Multiple tiles can simultaneously occupy same space
- Admissible because misplaced tiles will always require *at least* number of moves equal to Manhattan distance

Heuristic Domination

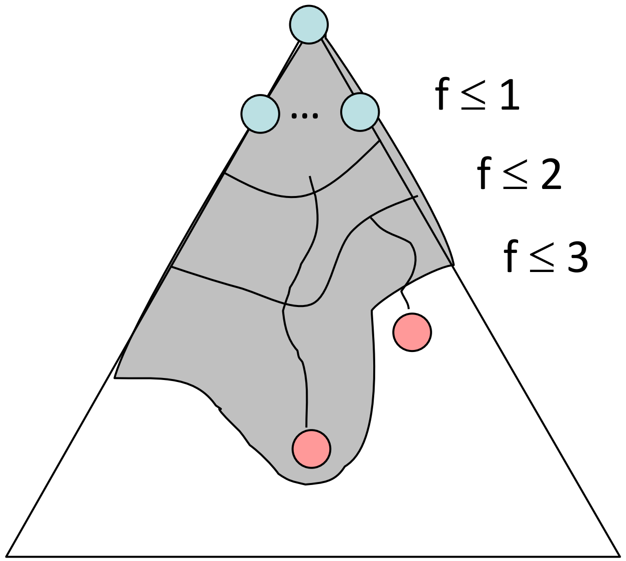
- For any node n, Manhattan distance heuristic $h_2(n)$ > misplaced tiles heuristic $h_1(n)$
- h_2 **dominates** h_1 if $h_2(n) \geq h_1(n)$ for all n

- A* search using h_2 will be more efficient and never expand more nodes than h_1
- h_2 reflects true costs more accurately

- Suppose we have collection of admissible heuristics $h_1, h_2, \ldots, h_m$
- The composite heuristic $h(n) = \max\{h_1(n), \ldots, h_m(n)\}$ is admissible and dominates all other heuristics!

Completeness and Optimality of A*

- A* is complete (same reason as greedy, UCS, BFS)
- If heuristic function is admissible, A* is also optimal!



- Proof by contradiction:
- Assume A* returns a suboptimal solution with cost $C > C^*$
- Then there exists some unexpanded node n on optimal path

- Since n was not expanded, $f(n) > C^*$
- However, $f(n) = g(n) + h(n) = g^*(n) + h(n) \leq g^*(n) + h^*(n) = C^*$ Contradiction!

By definition | Since n is on optimal path | Since h is admissible | By definition

Satisficing Solutions

- Like BFS or UCS, A* may suffer computationally intractable memory requirements
- Idea: Trade off admissibility for more accurate heuristics to reduce computation
- Return **satisficing solutions**—suboptimal, but "good enough"

- **Weighted A* search**: $f(n) = g(n) + \alpha h(n)$
- We can choose to place higher weight α on the heuristic
- Generalizes A* ($\alpha = 1$), UCS ($\alpha = 0$), and greedy best-first ($\alpha = \infty$)

- Suboptimality: If optimal solution has cost C^*, weighted A* solution may cost up to αC^*

Memory-Bounded Search

- We can also consider A* variants that are more memory-efficient

- **Beam search**: Limit frontier size by discarding worst nodes past a given limit
- Alternatively, discard nodes with scores much smaller than best one

- **Iterative-deepening A*** (IDA*): Repeatedly run A* with increasing depth limit
- Nodes with higher f-cost than limit are treated as leaves
- Increment depth limit by smallest f-cost of "leaves" from previous iteration

- IDA* worst case: Each node has different f-cost, num iterations equal to num states

Summary

- Objective of search problems is to find action/state sequence to reach a goal state
- Represented by state space graphs; search algorithms follow a tree structure

- Uninformed search: No usage of information indicating closeness to goal
- Examples: Depth-first, breadth-first, depth-limited, iterative deepening, uniform-cost
- Generally suffer from lack of completeness or intractable memory usage

- Informed search: Domain-specific heuristics guide search toward goal
- Greedy best-first and A* search use a heuristic function to evaluate frontier nodes
- Optimal if heuristics are admissible: good, optimistic estimates of true costs