Finite Diff Approach 1.2)

equation: $\frac{d^2y}{dx^2} + \alpha \frac{dy}{dx} + \beta y = f(x)$

BC: $y(a) = y_a \quad y(b) = y_b$

finite diff.

$$\left.\frac{dy}{dx}\right|_i = \frac{y_{i+1} - y_{i-1}}{2h}$$

$$\left.\frac{d^2y}{dx^2}\right|_i = \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2}$$

$$\Rightarrow \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} + \alpha \cdot \frac{y_{i+1} - y_{i-1}}{2h} + \beta y_i = f(x_i)$$

rearange and multiply by h^2

$$y_{i-1}\left(1 - \frac{h}{2}\alpha\right) + y_i\left(h^2\beta - 2\right) + y_{i+1}\left(1 + \frac{h}{2}\alpha\right) = h^2 f(x_i)$$

In case of the assignment:

Question 1.2:

$$\frac{d^2\phi^*}{dx^{*2}} = \phi^*(x^*)$$

domain: $0 < y < 10$

Boundy condition:

- $\phi^*(0) = \frac{\phi_0}{V_T}$ where $\phi^* = \frac{\phi}{V_T}$ with $\phi = \phi_0 \cdot \exp\left(-\frac{x}{\lambda}\right)$
- $\frac{d\phi^*}{dx^*}(x^* = 10) = 0$

Centrantral Diff method:

$$\frac{1 \cdot \phi^*[i+1] - 2 \cdot \phi^*[i] + 1 \cdot \phi^*[i-1]}{h^2} = \phi^*(x^*)[i]$$

rearange: $1 \cdot \phi^*[i+1] - 2\phi^*[i] + 1 \cdot \phi^*[i-1] = 0$ notice it is in range $[1, N-2]$

$$\phi^*[i+1] - (2 + h^2)\phi^*[i] + \phi^*[i-1] = 0$$

Constructing matrix A: $(Ay = b)$

case $i = 1$: $\phi^*_{i-1} = \phi^*(0) = \frac{\phi_0}{V_T}$

$$1 \cdot \phi^*(0) - (2 + h^2)\phi^*[i] + \phi^*[i-1] = 0$$

$$-(2 + h^2)\phi^*[i] + \phi^*[i-1] = 0 - 1 \cdot \phi^*(0)$$

case $2 \leq i \leq N-3$

$$1 \cdot \phi^*[i-1] - (2 + h^2)\phi^*[i] + \phi^*[i-1] = 0$$

case $i = N-2$: (far right case)

Neumann boundary:

$$\frac{d\phi^*}{dx^*} = 0$$

$$\Rightarrow \frac{y[i] - y[i-2]}{2h} = 0 \Rightarrow y[i] = y[i-2]$$

in matrix A:

$$1\,\phi[i-2] - (2+h^2)\,\phi[i-1] + 1\,\phi[i] = 0$$

plug in for $\phi[i] = \phi[i-2]$

$$\Rightarrow 2\,\phi[i-2] - (2+h^2)\,\phi^*[i-1] = 0$$

Question 2:

$$\frac{d^2\phi}{dx^2} = \frac{2c_0 e}{\varepsilon} \sinh\left(\frac{e\phi(x)}{k_B T}\right)$$

$$\frac{d^2\phi^*}{dx^{*2}} = \sinh\left(\phi^*(x^*)\right)$$

$x^* = \frac{x}{\lambda}$ $\phi^* = \frac{\phi}{V_T}$ $V_T = \frac{k_B T}{e}$

$$\frac{1\cdot\phi^*[i-1] - 2\cdot\phi^*[i] + 1\cdot\phi^*[i+1]}{h^2} = \sin\left(\phi^*(x^*)\right)$$

$$1\,\phi^*[i-1] - 2\cdot\phi^*[i] - \underbrace{h^2\sin(\phi^*[i])}_{f(x)\,\rightarrow\,\text{can be solved with Newton Raphson}} + 1\,\phi^*[i+1] = 0$$

where Newton Raphson will guess for the value of ϕ^*. It does so by expanding F around an estimate $\phi[i]$ using taylor series:

$$F(\phi[i] + \Delta\phi) \approx F\,\phi[i] + J\,\phi[i]\cdot\Delta\phi$$

$\uparrow$ Jacobian

making a the Jacobian:

$$F[i] = \phi[i-1] - 2\phi[i] - h^2\cdot\sinh(\phi[i]) + \phi[i+1]$$

Term:	Derivative
$\phi[i-1]$	$\frac{dF_i}{d\phi[i-1]} = 1$
$\phi[i]$	$\frac{dF_i}{d\phi[i]} = -2 - h^2\cosh(\phi[i])$
$\phi[i+1]$	$\frac{dF_i}{d\phi[i+1]} = 1$

note any other derivative = 0

$\Rightarrow$ only diagonal has values

note only Neumann conditions must be applied to the Jacobian

the Dirichlet boundary $\phi[0] =$ phi_0/V_T will be defined in solution

however Neumann must be defined

$\therefore$ @ $N = -1$ $\frac{d\phi^*}{dx^2} = 0 \Rightarrow$

$$F[N-1] = 2\,\phi[i-2] - 2\,\phi[i-1] - h^2\cdot\cosh(\phi[i-1])$$