Computational practicum: Lecture 5 Ordinary Differential Equations (ODEs) and Initial Value Problems (IVPs)

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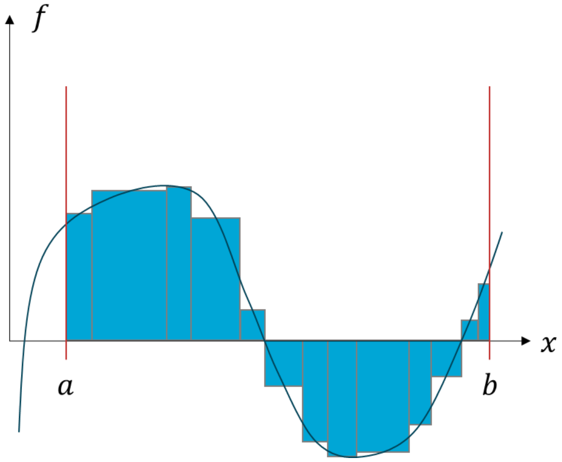
Recap last lecture

Numerical integration

- Quadrature rules
  - Rectangle method
  - Newton-Cotes rules



Numerical differentiation

- Finite difference method
  - First order difference schemes

$$f'(x) = \frac{df}{dx} \to \left.\frac{df}{dx}\right|_{x_i} \quad i = 0, \dots, N-1$$

  - Second order difference schemes

$$f''(x) = \frac{d^2 f}{dx^2} \to \left.\frac{d^2 f}{dx^2}\right|_{x_i} \quad i = 0, \dots, N-1$$

Learning goals of this lecture

After successfully completing this lecture, you are able to…

- categorize ordinary differential equations (ODEs).
- derive the linear, 1st order, autonomous form of an ODE.
- implement different numerical solution approaches to ODEs from scratch, namely,
  - Backward Euler.
  - Forward Euler.
- use Python libraries' built-in functions for numerical solution approaches to ODEs.
- discuss numerical errors and stability of numerical solution approaches to ODEs.

Agenda

- **Ordinary differential equations (ODEs)**
  - Classification of ODEs
  - System of linear ODEs
- **Numerical solution methods for Initial value problems (IVPs)**
  - Forward Euler
  - Backward Euler
  - Errors in numerical solution of ODEs and stability
  - ODE solver in scipy

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Ordinary differential equations in reaction engineering

- Take the reaction in a batch reactor with const. density:

First order elementary reaction

$$A \xrightarrow{k} B$$

- Component mass balances gives the corresponding **ordinary differential equations** (ODE), which describe the evolution of the concentrations of A.

concentration

$$\frac{dC_A}{dt} = -kC_A$$

time

rate constant

- The concentration of B follows from the following algebraic relationship assuming constant total concentration equal to 1:

Total concentration, const.

$$C_B = 1 - C_A$$

- Initial condition for concentration C_A is needed, otherwise there exist infinitely many solutions.

Ordinary differential equations in reaction engineering

- Take the reaction in a batch reactor with const. density:

$$A \xrightarrow{k} B$$

First order elementary reaction

- Component mass balances gives the corresponding ODE for C_A and the algebraic equation for C_B, which describe the evolution of the concentrations of A and B.

$$\frac{dC_A}{dt} = -kC_A, \qquad C_B = 1 - C_A$$

- Known initial condition

$$C_A(t=0) = 1$$

→ Called **initial value problem** (short: IVP)

- Analytical solutions

$$C_A(t) = e^{-kt}, \qquad C_B(t) = 1 - e^{-kt}$$

Try at home to verify that this solves the ODEs by inserting the solution in the original ODE system

See PDC course module 2, lecture 3

Ordinary differential equations in reaction engineering

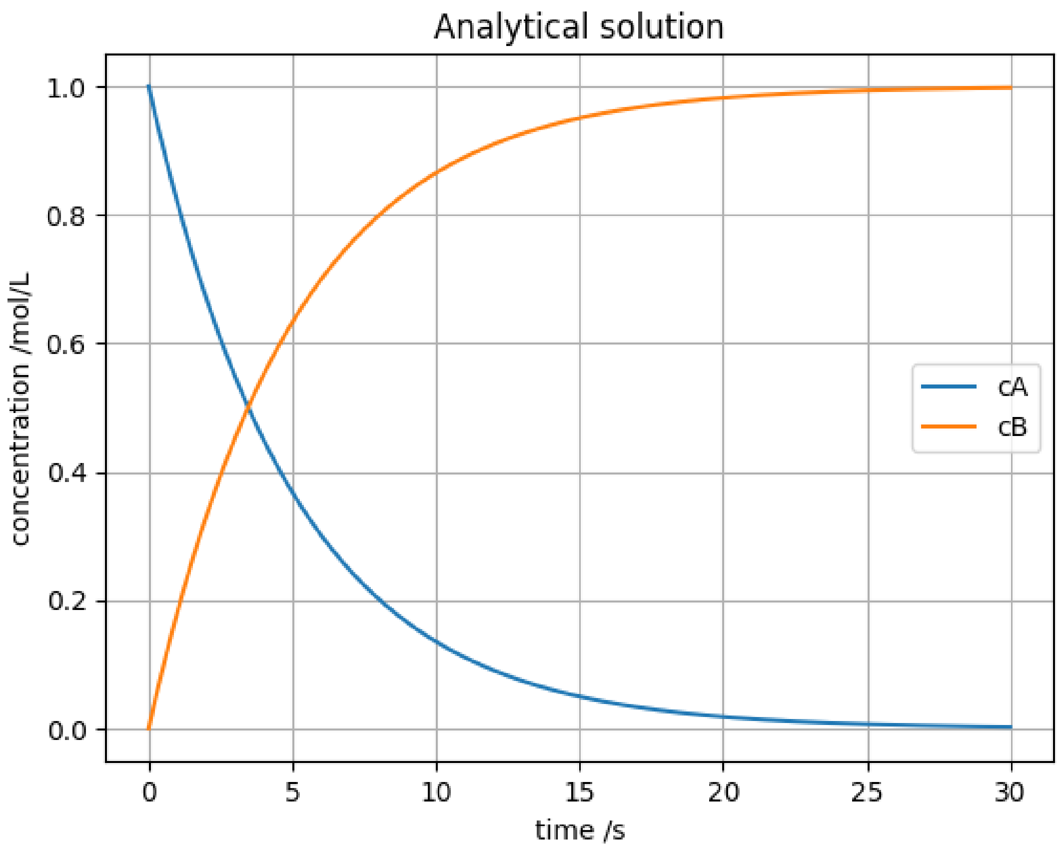
```python
import numpy as np
import matplotlib. pyplot as plt

# Define time vector t
t = np.linspace(0, 30, 100)

# Define kinetic constant k
k = 0.2

# Define concentration function
def cA(t:float, k:float) -> float:
    return np.exp(-k*t)
def cB(t:float, k:float) -> float:
    return 1 - np.exp(-k*t)

# Create the plot
fig, ax = plt.subplots()
ax.plot(t, cA(t,k), label='cA')
ax.plot(x, cB(t,k), label='cB')
ax.set_xlabel('time /s')
ax.set_ylabel('concentration /mol/L')
ax.set_title('Analytical solution')
ax.legend()
ax.grid()
```



Classification of ODEs – Definitions

Definition

An ODE of order n is an equation of the form

$$g\left(t, y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}\right) = 0, \qquad \text{Implicit form}$$

or

$$\frac{d^n y}{dt^n} = f\left(t, y, \frac{dy}{dt}, \dots, \frac{d^{n-1} y}{dt^{n-1}}\right). \qquad \text{Explicit form}$$

where

- t : independent variable
- $y(t)$: dependent variable

Classification of ODEs – Definitions

Definition

An ODE is linear, if it is linear in the dependent variable $y(t)$ and its derivatives (i.e. if $y(t)$ and its derivatives appear only to the first power and are never multiplied together). Else it is non-linear.

An ODE is autonomous if it does not include the independent variable t, i.e.

$$g\left(t, y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}\right) = g\left(y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}\right) = 0,$$

Or

$$f\left(t, y, \frac{dy}{dt}, \dots, \frac{d^n y}{dt^n}, \frac{d^{n-1} y}{dt^{n-1}}\right) = f\left(y, \frac{dy}{dt}, \dots, \frac{d^{n-1} y}{dt^{n-1}}\right).$$

Else it is non-autonomous.

Classification of ODEs – Examples

- Examples of different forms of ODE equations:
  - 1st order $\frac{dy}{dt} = f(y)$ Dependent variable Independent variable
  - 2nd order $\frac{d^2y}{dt^2} + y\frac{dy}{dt} = f(y)$
  - 3rd order $\frac{d^3y}{dt} + a\frac{d^2y}{dt^2} + b\frac{dy}{dt} = f(y)$
  - Non-autonomous $\frac{dy}{dt} = f(y, t)$
  - Linear $\frac{dy}{dt} = t$
  - Non-linear $\mathrm{y} \cdot \frac{dy}{dt} = t$

ODE form

See PDC course week 2

- ODEs are easiest to solve if they are **linear**, **1st order**, **autonomous** ODEs!

Original ODE:
(3rd order, linear, autonomous
→ not easy to solve)

$$\frac{d^3y}{dt^3} + a\underbrace{\frac{d^2y}{dt^2}}_{y_2} + b\underbrace{\frac{dy}{dt}}_{y_1} + y = 0$$

New variables:

$$y_1 \equiv \frac{dy}{dt}$$

$$y_2 \equiv \frac{dy_1}{dt} = \frac{d^2y}{dt^2}$$

differentiate

Hint: Insert in original ODE with $\frac{dy_2}{dt} = \frac{d^3y}{dt^3}$ and re-arrange

New system of ODEs:

$$\frac{dy_1}{dt} = y_2$$

$$\frac{dy_2}{dt} = -ay_2 - by_1 - y$$

→ All are linear, 1st order and autonomous.

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Linear ordinary differential equations

- Take the reaction

Series of reversible reactions

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \underset{k_4}{\overset{k_3}{\rightleftharpoons}} C$$

- Corresponding ODEs giving the evolution of the concentrations of A, B and C:

$$\frac{dC_A(t)}{dt} = -k_1 C_A + k_2 C_B$$
$$\frac{dC_B(t)}{dt} = k_1 C_A - (k_2 + k_3) C_B + k_4 C_C$$
$$\frac{dC_C(t)}{dt} = -k_4 C_C + k_3 C_B$$

Writing linear ordinary differential equations

$$\frac{dC_A(t)}{dt} = -k_1 C_A + k_2 C_B$$
$$\frac{dC_B(t)}{dt} = k_1 C_A + k_4 C_C - (k_2 + k_3) C_B$$
$$\frac{dC_C(t)}{dt} = -k_4 C_C + k_3 C_B$$

- Collect into matrix form:

$$\longrightarrow \begin{pmatrix} dC_A/dt \\ dC_B/dt \\ dC_C/dt \end{pmatrix} = \begin{pmatrix} -k_1 & k_2 & 0 \\ k_1 & -(k_2 + k_3) & k_4 \\ 0 & k_3 & -k_4 \end{pmatrix} \cdot \begin{pmatrix} C_A \\ C_B \\ C_C \end{pmatrix}$$

$$\longrightarrow \frac{d}{dt}\mathbf{c} = \mathbf{K} \cdot \mathbf{c}$$

HINT

We commonly denote vectors and matrices in bold. Vectors are lower case and matrices are upper case.

Analytic solution of linear ODEs

See PDC course module 3, lecture 2

- From process dynamics and control (PDC) course, you know that we can solve linear ODE systems analytically.

$$\frac{d}{dt}\mathbf{c} = \mathbf{K}\cdot\mathbf{c}$$
$$\Rightarrow \mathbf{c} = e^{\mathbf{K}t}\mathbf{c}_0$$

Exponential of a matrix!

As taught in CP lecture 3

- In general, the exponential of a matrix is defined by the power series: $e^{\mathbf{K}t} = \sum_{k=0}^{\infty}\frac{(\mathbf{K}t)^k}{k!}$.
- However, if $\mathbf{K}$ is **diagonalizable**:
  - It can be expressed as $\mathbf{K} = \mathbf{U}\boldsymbol{\Lambda}\,\mathbf{U}^{-1}$, where $\boldsymbol{\Lambda} = \mathbf{diag}(\boldsymbol{\lambda}_1, \dots, \boldsymbol{\lambda}_n)$ is the diagonal matrix of eigenvalues and $\mathbf{U}$ is the matrix whose columns are eigenvectors of $\mathbf{K}$.
  - Then the exponential of a matrix becomes $e^{\mathbf{K}t} = \mathbf{U}e^{\boldsymbol{\Lambda}t}\mathbf{U}^{-1}$.

As taught in CP lecture 3

Solving a system of linear ODEs

$$\begin{array}{c} k_1 \quad k_3 \\ A \rightleftharpoons B \rightleftharpoons C, \\ k_2 \quad k_4 \end{array} \qquad \mathbf{c} = e^{\mathbf{K}t}\mathbf{c}_0$$

Series of reactions

System of ODEs

- Initial condition

$$C_A(0) = 1, \qquad C_B(0) = 0, \qquad C_C(0) = 0$$

- Rates

$$k_1 = 1\ \mathrm{min}^{-1}, \qquad k_2 = 0\ \mathrm{min}^{-1}, \qquad k_3 = 2\ \mathrm{min}^{-1}, \qquad k_4 = 3\ \mathrm{min}^{-1}$$

$$\begin{pmatrix} C_A(t_p) \\ C_B(t_p) \\ C_C(t_p) \end{pmatrix} = exp\left[\begin{pmatrix} -1 & 0 & 0 \\ 1 & -2 & 3 \\ 0 & 2 & -3 \end{pmatrix} t_p\right] \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Solving a system of linear ODEs

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \underset{k_4}{\overset{k_3}{\rightleftharpoons}} C, \qquad \mathbf{c} = e^{\mathbf{K}t}\mathbf{c}_0$$

- Time discretization

$$T = [0, \Delta t, 2\Delta t, 3\Delta t, \dots, (N-1)\Delta t]$$

equidistant grid

- Solution

$$\begin{aligned}\mathbf{c}(t_{N-1} = (N-1)\Delta t) &= e^{\mathbf{K}(\mathrm{N}-1)\Delta t}\mathbf{c}_0 = \left(e^{\mathbf{K}\Delta t}\right)^{(N-1)}\mathbf{c}_0 \\ &= \left(e^{\mathbf{K}\Delta t}\right)\left(e^{\mathbf{K}\Delta t}\right)\dots\underbrace{\left(e^{\mathbf{K}\Delta t}\right)\mathbf{c}_0}_{\mathbf{c}(t_1 = \Delta t)}\end{aligned}$$

HINT

This solution is only possible if we choose equidistant grid points.

Solving a system of linear ODEs

$$A \underset{k_2}{\overset{k_1}{\rightleftharpoons}} B \underset{k_4}{\overset{k_3}{\rightleftharpoons}} C$$

- Initial condition

$$C_A(0) = 1, \qquad C_B(0) = 0, \qquad C_C(0) = 0$$

- Rates

$$k_1 = 1 \text{ min}^{-1}, \qquad k_2 = 0 \text{ min}^{-1}, \qquad k_3 = 2 \text{ min}^{-1}, \qquad k_4 = 3 \text{ min}^{-1}$$

Time

$$\begin{aligned} &\mathbf{c}_0 \\ &\mathbf{c}_1 = e^{\mathbf{K}\Delta t}\mathbf{c}_0 \\ &\mathbf{c}_2 = e^{\mathbf{K}\Delta t}\mathbf{c}_1 \\ &\quad \dots \\ &\quad \dots \\ &\mathbf{c}_{N-1} = e^{\mathbf{K}\Delta t}\mathbf{c}_{N-2} \end{aligned}$$

We only need to compute one exponential of the matrix $e^{\mathbf{K}\Delta t}$.

Live coding: Analytical solution

- Open Colab: Analytical solution

- Find more in the Github repository of the course: https://github.com/process-intelligence-research/computational_practicum_lecture_coding/tree/main

What if we do not know the analytical solution?

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Non-Linear ODEs: Initial value problem

- General form

$$\frac{dy}{dt} = f(t, y)$$

- Initial condition

$$y(t_0) = y_0$$

HINT

For every n-th order ODE, we need n initial conditions.

- Example

Second order elementary reaction

$$2A \xrightarrow{k} B$$

Non-linear term

$$\frac{dC_A}{dt} = -kC_A^2$$
$$\frac{dC_B}{dt} = kC_A^2$$

- Initial condition

$$C_A(0) = 1$$
$$C_B(0) = 0$$

Numerical solution through discretization

- Discretization

$$t = [t_0, t_1, \dots, t_i, \dots, t_{N-1}] \Rightarrow y = [y_0, y_1, \dots, y_i, \dots, y_{N-1}]$$

- How can we get from an initial value at $y_i = y(t_i)$ to the next value at $y_{i+1} = y(t_{i+1})$?

Non-Linear ODEs: Initial value problem

$$\frac{dy}{dt} = f(t, y)$$

- Find the solution by integrating from an initial value y_i to the subsequent value y_{i+1}.

$$\int_{y_i}^{y_{i+1}} dy = \int_{t_i}^{t_{i+1}} f(t, y) dt$$

$$y(t_{i+1}) - y(t_i) = \int_{t_i}^{t_{i+1}} f\big(t, y(t)\big) dt$$

$$y(t_{i+1}) = y(t_i) + \underbrace{\int_{t_i}^{t_{i+1}} f\big(t, y(t)\big) dt}$$

Numerical approximation
for definite integrals!

As taught
in lecture 4

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Non-Linear ODEs: Initial value problem
The forward Euler method

- Rewriting the ODE

$$\frac{dy}{dt} = f(t,y) \qquad \Big|\ \text{Integrate over } \int_{t_i}^{t_{i+1}} dt$$

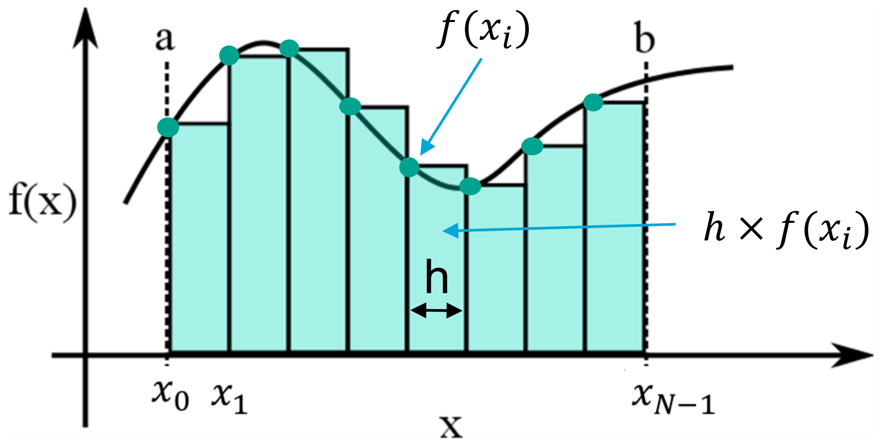
$$\int_{t_i}^{t_{i+1}} \frac{dy}{dt} dt = \int_{t_i}^{t_{i+1}} f(t,y)dt$$

$$y_{i+1} - y_i = \int_{t_i}^{t_{i+1}} f(t,y)dt \qquad \Big|\ +y_i$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)dt \qquad \Big|\ \text{Substitute}$$

$$y_{i+1} \approx y_i + hf(t_i, y_i)$$

Rectangle approach for integration
x_i^* in left corner (c.f. lecture 4)



$$\int_{t_i}^{t_{i+1}} f(t,y)dt \approx hf(t_i, y_i)$$

Non-Linear ODEs: Initial value problem
The Forward Euler method

- Using Euler for ODEs

Definition

$$\frac{dy}{dt} = f(t, y)$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t, y)dt$$

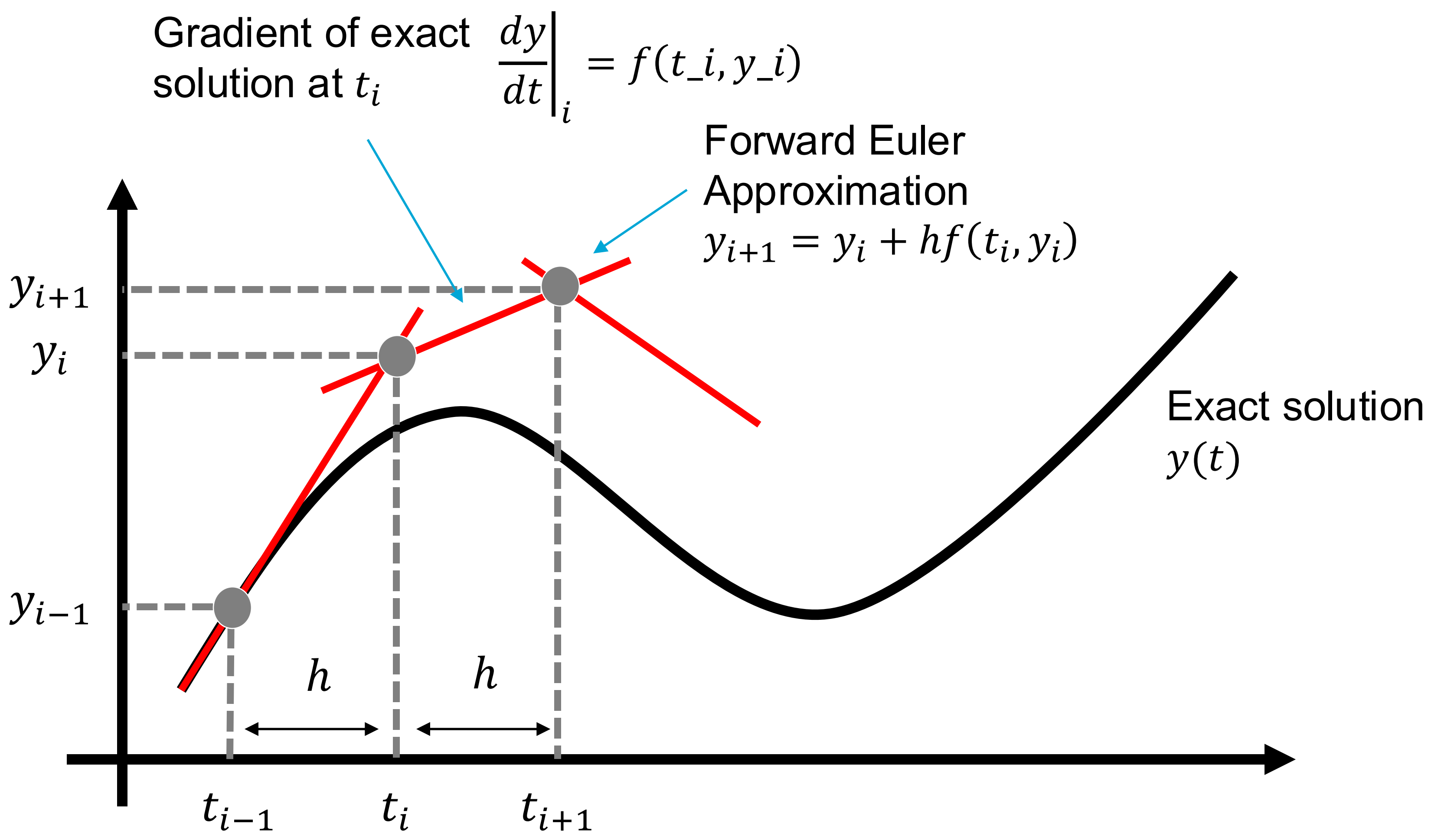
$$y_{i+1} = y_i + hf(t_i, y_i)$$

HINT

Note that step size h can be taken to be negative.

- Forward Euler is also called the **explicit** Euler method, because it gives an explicit expression for y_{i+1} ($y_{i+1} = g(t_i, y_i)$).

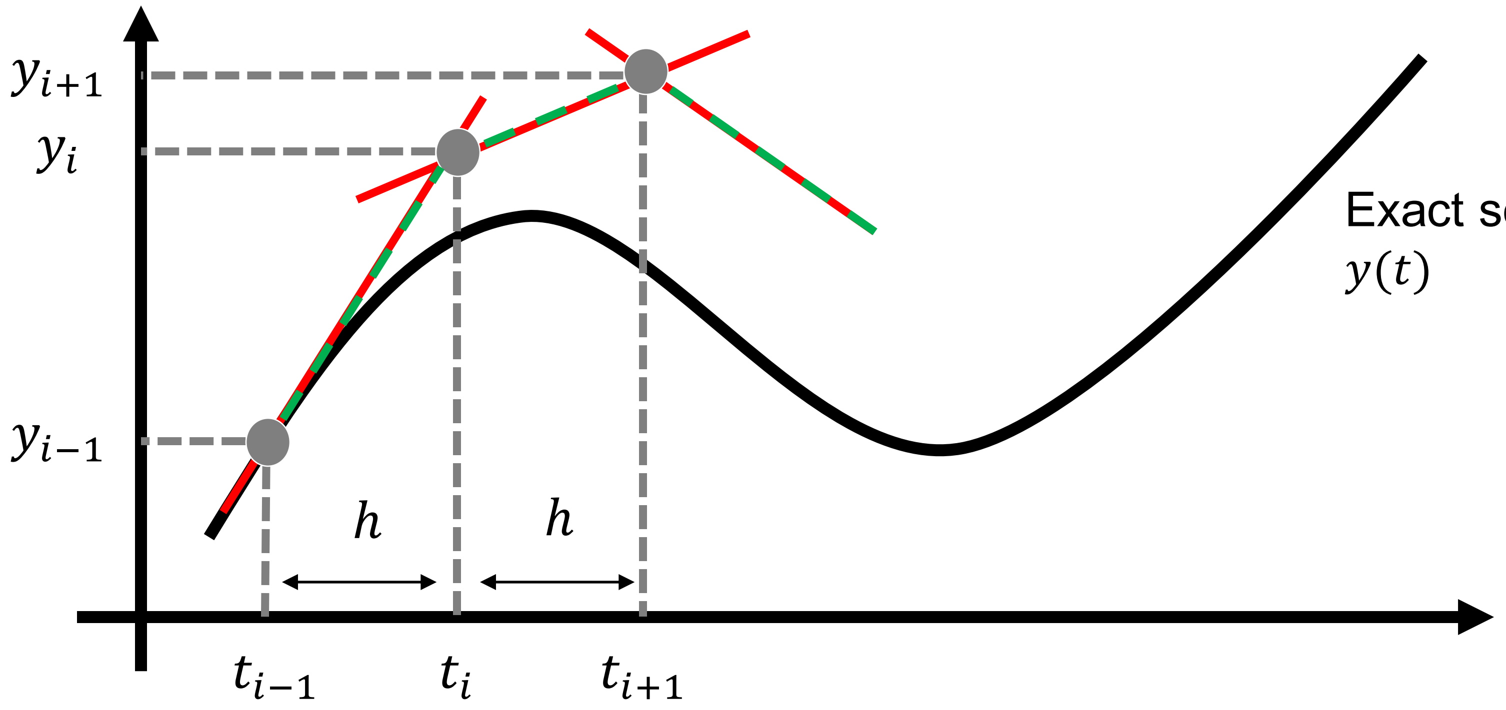
Non-Linear ODEs: Initial value problem
The forward Euler method



Non-Linear ODEs: Initial value problem
Grid points vs. evaluation points

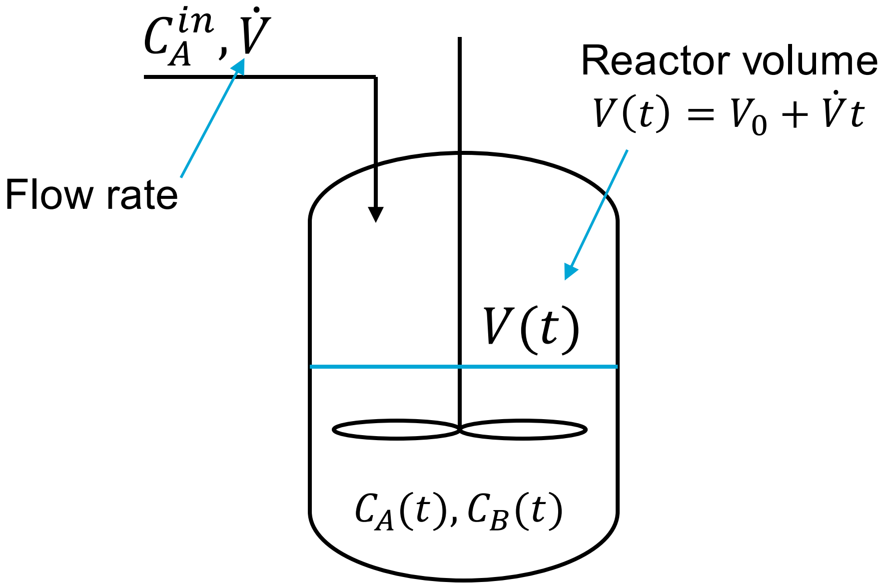
Grid points:
- Used to obtain numerical solution
- Chosen a priori

Evaluation points:
- Evaluate numerical solution
- Chosen a posteriori



Case study for lecture coding: Semi batch example

- 1st order reaction scheme $A \xrightarrow{k} B$.
- Semi-batch reactor model, assuming constant density.



$$\begin{cases} \dfrac{dC_A}{dt} = \dfrac{\dot{V}}{V_0 + \dot{V}t}\left(C_A^{in} - C_A\right) - kC_A \\ \dfrac{dC_B}{dt} = kC_A - \dfrac{\dot{V}}{V_0 + \dot{V}t} C_B \end{cases}$$

$$with:\ C_A(0) = 1, C_B(0) = 0$$

Non-autonomous term which contains the independent variable t explicitly.

Try at home:
1. Derive the ODE system equations by performing component balances.
2. Transform the semi batch ODE system to the 1st order, linear, autonomous form.

Live coding: Forward Euler

- Open Colab: Forward Euler

- Find more in the Github repository of the course: https://github.com/process-intelligence-research/computational_practicum_lecture_coding/tree/main

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Non-Linear ODEs: Initial value problem
The backward Euler method

- Using Euler for ODEs

Definition

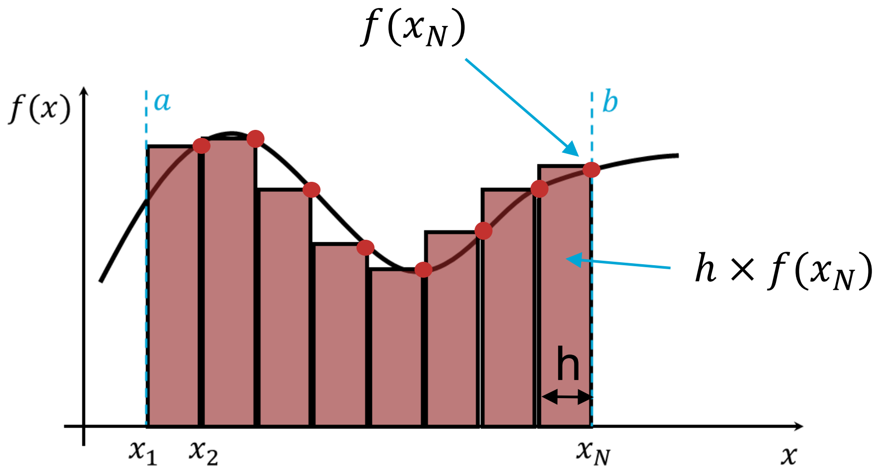
$$\frac{dy}{dt} = f(t,y)$$

$$y_{i+1} = y_i + \int_{t_i}^{t_{i+1}} f(t,y)dt$$

$$y_{i+1} \approx y_i + hf(t_{i+1}, y_{i+1})$$

Substitute

Rectangle approach for integration
x_i^* in right corner (c.f. lecture 4)



$$\int_{t_i}^{t_{i+1}} f(t,y)dt \approx hf(t_{i+1}, y_{i+1})$$

- Backward Euler is also called the **implicit** Euler method, because it gives an implicit expression for y_{i+1} ($y_{i+1} = g(t_i, t_{i+1}, y_i, y_{i+1})$).

Non-Linear ODEs: Initial value problem
The backward Euler method

$$y_{i+1} = y_i + hf(t_{i+1}, y_{i+1})$$

- The term y_{i+1} appears on both sides of the equation (*implicit*).
  - We can formulate this as a root finding problem.

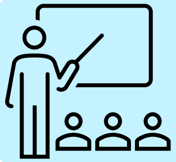
$$y_{i+1} = g(y_{i+1})$$

$$\text{e.g.}\ \frac{dy}{dt} = y^2 \rightarrow y_{i+1} = y_i + hy_{i+1}^2$$

  - We can exactly reformulate the problem to be explicit (not possible for all equations).

$$y_{i+1} = f^*(t_i, y_i, h)$$

$$\text{e.g.}\ \frac{dy}{dt} = y \rightarrow y_{i+1} = y_i + hy_{i+1} \rightarrow y_{i+1} = \frac{y_i}{1-h}$$



As taught in lecture 2

Root-finding problem for the backward Euler method

$$y_{i+1} = g(y_{i+1})$$

- Root finding methods: Fixed-point iteration, Newton-Raphson method.
- At every integration step, we need to solve the root finding problem.

Root finding step

$$y_{i+1}^{[0]} = y_i, \qquad y_{i+1}^{[k+1]} = y_i + hf\left(t_{i+1}, y_{i+1}^{[k]}\right)$$

Integration step

- → expansive numerical method for ODEs
- … but there are advantages, stay tuned!

As taught in lecture 2

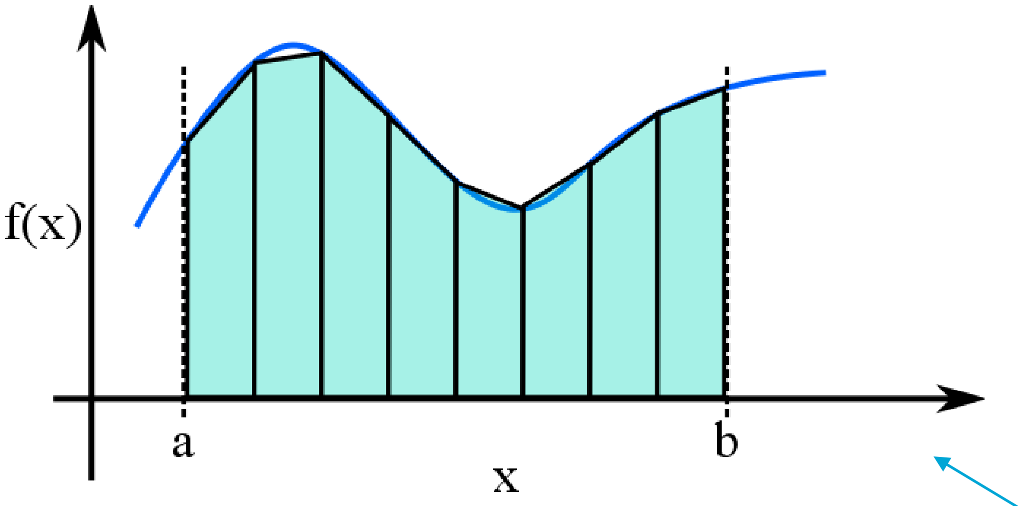
Live coding: Backward Euler

- Open Colab: Backward Euler

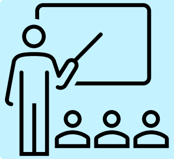
- Find more in the Github repository of the course: https://github.com/process-intelligence-research/computational_practicum_lecture_coding/tree/main

Non-Linear ODEs: Initial value problem
The Heun's method

- In addition, other numerical integration methods, such as the trapezoid method, can be used to solve the IVP.
  → Heun's method (or modified Euler method, improved Euler method).



Trapezoid method



Heun's method in Assignment 5

Non-Linear ODEs: Initial value problem Runge-Kutta method

See PDC course lecture

- There exist more advanced methods like Runge-Kutta (RK4) which will be covered in the Process Dynamics and Control (PDC) lecture.
- Runge-Kutta is the most widely used method in practice.
- Explicit method that uses multiple intermediate points to estimate the slope, e.g. RK4:

$$
\begin{aligned}
k_1 &= f(t_n, y_n) \\
k_2 &= f(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_1) \\
k_3 &= f\left(t_n + \frac{h}{2}, y_n + \frac{h}{2} k_2\right) \\
k_4 &= f(t_n + h, y_n + h k_3) \\
y_{n+1} &= y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)
\end{aligned}
$$

- Greater accuracy per step than forward or backward Euler.

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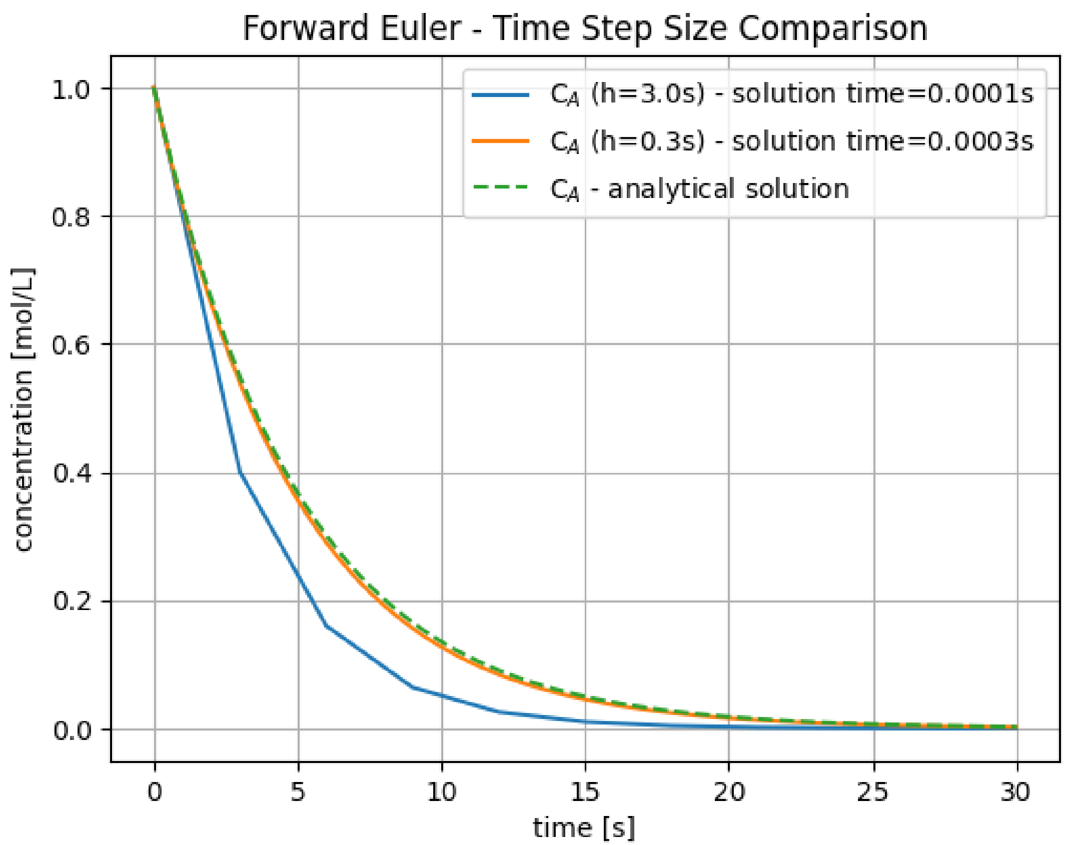
Influence of the time step size

$$A \xrightarrow{k} B$$
$$C_A(0) = 1$$

- Solution

$$C_A(t) = e^{-kt}$$
$$C_B(t) = 1 - e^{-kt}$$

There is a trade-off between the solution time (size of the problem, N) and the accuracy of the solution (error).



Live coding: Numerical error

- Open Colab: Numerical error

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Errors in numerical solution of ODEs

- Rounding error
  - Due to finite precision of floating-point arithmetic (cf. lecture 1)
- Truncation error
  - Due to the approximate nature of the method
  - Global truncation error $e_i = y_i - y(t_i)$
  - Local truncation error $\ell_i = y_i - u_{i-1}(t_i)$

Numerical solution y at point i

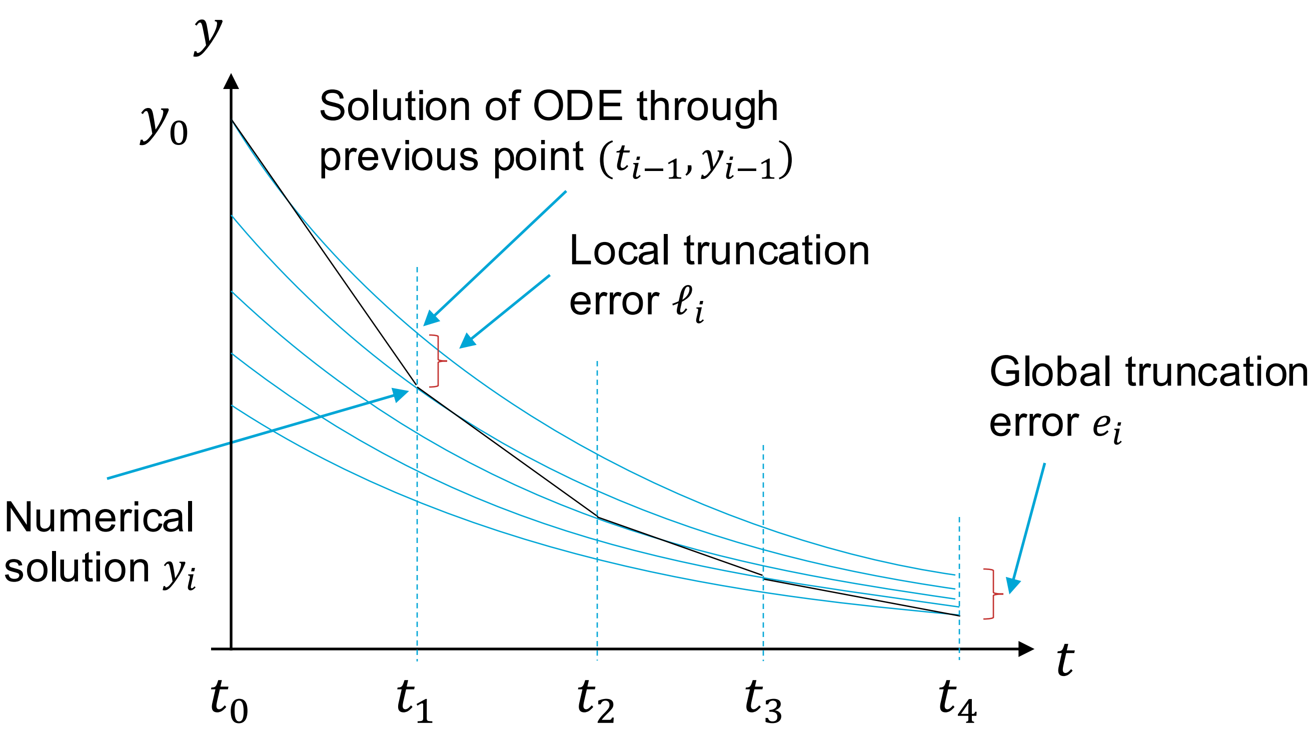
True solution y at point i

True solution of ODE through previous point (t_{i-1}, y_{i-1})

Heath M.T - Scientific computing: an introductory survey, SIAM (2018)

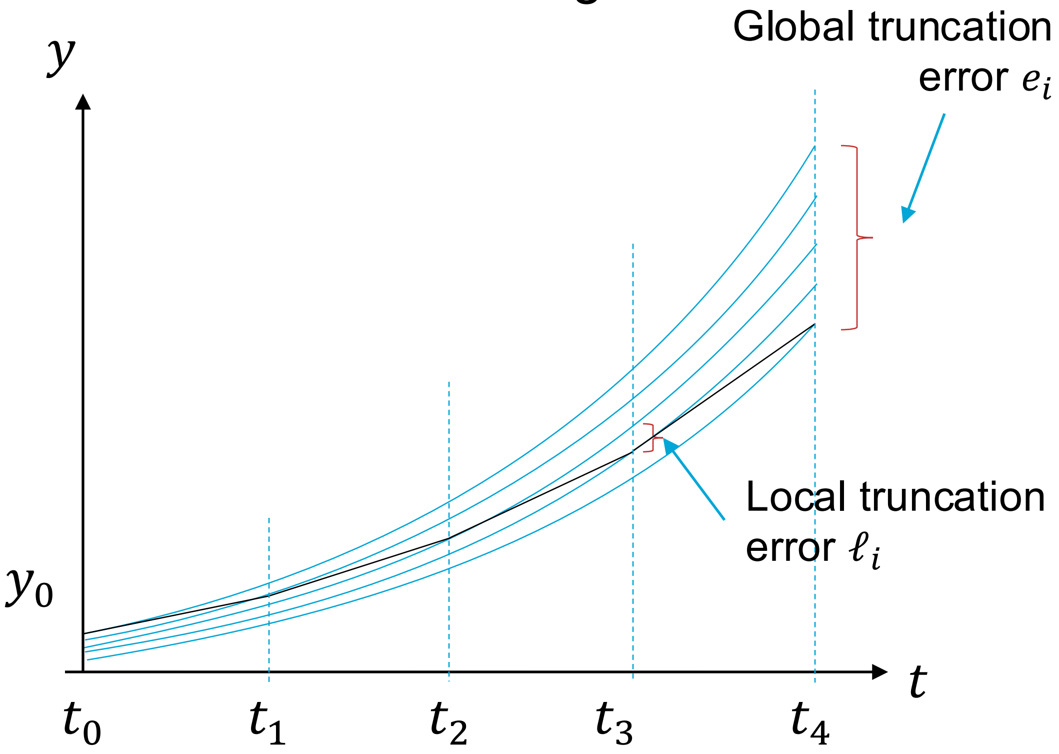
Errors in numerical solution of ODEs

- Stable solution,
  errors in numerical solution may diminish



$$\frac{dy}{dt} = -y$$

- Unstable solution,
  errors in numerical solution grow



$$\frac{dy}{dt} = y$$

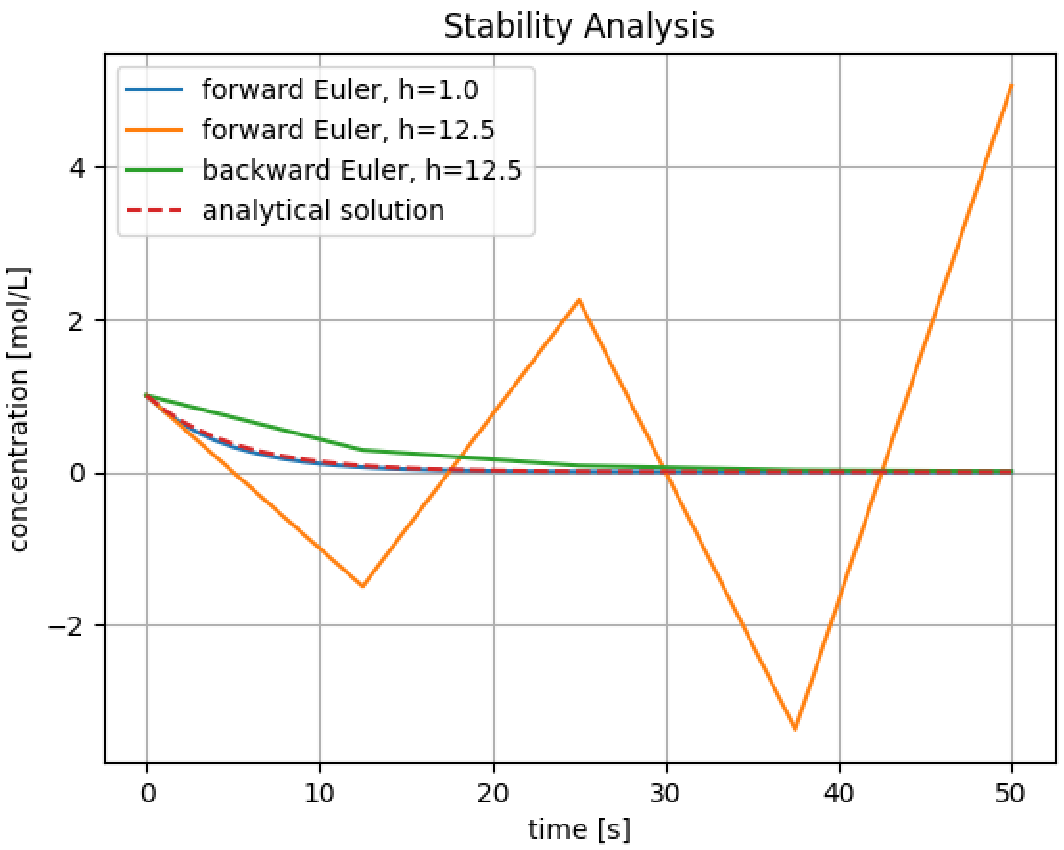
Heath M.T - Scientific computing: an introductory survey, SIAM (2018)

Stability analysis

- Backward (implicit) Euler *unconditionally* stable
- Forward (explicit) Euler and Runge-Kutta *conditionally* stable
  - Depending on step size h and stiffness of the problem

Definition

Stiff ODEs often involve competing physical phenomena with **widely varying time and/or spatial scales**. There is no precise definition in literature for stiffness. In general, an ODE is stiff if the eigenvalues of the Jacobian differ greatly in magnitude.



Amos Gilat, Vish Subramaniam - Numerical Methods for Engineers and Scientists: An Introduction with Applications using MATLAB, Wiley (2013)

Live coding: Stability

- Open Colab: Stability

- Find more in the Github repository of the course: https://github.com/process-intelligence-research/computational_practicum_lecture_coding/tree/main

Agenda

- **Ordinary differential equations (ODEs)**
  - Classification of ODEs
  - System of linear ODEs
- **Numerical solution methods for Initial value problems (IVPs)**
  - Forward Euler
  - Backward Euler
  - Errors in numerical solution of ODEs and stability
  - ODE solver in scipy

Solving IVPs using scipy's *solve_ivp*

- scipy provides a built-in IVP solver called *solve_ivp*
- Function arguments:
  - *fun*: Function containing the ODE system.
  - *t_span*: Border of integration interval.
  - *y0*: Initial values.
  - *method*: Integration method. There are explicit (e.g., "RK45") and implicit methods (e.g., "Radau") available. Choose wisely!
  - *t_eval:* Grid points for which solution is returned. Solver uses dynamic grid points to calculate solution.

Read the complete description of *solve_ivp* (link) at home.

scipy.integrate.solve_ivp

`scipy.integrate.solve_ivp(fun, t_span, y0, method='RK45', t_eval=None, dense_output=False, events=None, vectorized=False, args=None, **options)` [source]

Solve an initial value problem for a system of ODEs.

This function numerically integrates a system of ordinary differential equations given an initial value:

```
dy / dt = f(t, y)
y(t0) = y0
```

Here t is a 1-D independent variable (time), y(t) is an N-D vector-valued function (state), and an N-D vector-valued function f(t, y) determines the differential equations. The goal is to find y(t) approximately satisfying the differential equations, given an initial value y(t0)=y0.

Some of the solvers support integration in the complex domain, but note that for stiff ODE solvers, the right-hand side must be complex-differentiable (satisfy Cauchy-Riemann equations [11]). To solve a problem in the complex domain, pass y0 with a complex data type. Another option always available is to rewrite your problem for real and imaginary parts separately.

Parameters:

fun : ***callable***

Right-hand side of the system: the time derivative of the state `y` at time `t`. The calling signature is `fun(t, y)`, where `t` is a scalar and `y` is an ndarray with `len(y) = len(y0)`. Additional arguments need to be passed if `args` is used (see documentation of `args` argument). `fun` must return an array of the same shape as `y`. See *vectorized* for more information.

t_span : ***2-member sequence***

Interval of integration (t0, tf). The solver starts with t=t0 and integrates until it reaches t=tf. Both t0 and tf must be floats or values interpretable by the float conversion function.

y0 : ***array_like, shape (n,)***

Initial state. For problems in the complex domain, pass *y0* with a complex data type (even if the initial value is purely real).

method : ***string or*** `OdeSolver`***, optional***

Live coding: *solve_ivp*

- Open Colab: [solve_ivp]

- Find more in the Github repository of the course: https://github.com/process-intelligence-research/computational_practicum_lecture_coding/tree/main

Learning goals of this lecture

After successfully completing this lecture, you are able to…

- categorize ordinary differential equations (ODEs).
- derive the derive the linear, 1^{st} order, autonomous form of an ODE.
- implement different numerical solution approaches to ODEs from scratch, namely,
  - Backward Euler.
  - Forward Euler.
- use Python libraries' built-in functions for numerical solution approaches to ODEs.
- discuss numerical errors and stability of numerical solution approaches to ODEs.

Thank you very much for your attention!