Overview of lecture topics

- Topic 1: Course introduction
- Topic 2: Introduction to frequency domain and Laplace transformation
- Topic 3: Dynamic Behavior of Linear systems
- Topic 4: Frequency Response Analysis and Bode plots
- Topic 5: Mathematical Description of Chemical Systems
- Topic 6: Nonlinear ODE’s, Linearity, Linearization, Feedback, Stability, Root Locus
- Topic 7: Feedback Controller Design and Bode stability
- Topic 8: Advanced (Enhanced) Process Control

EINDHOVEN UNIVERSITY OF TECHNOLOGY

Process Dynamics and Process Control

Lecture 6: Nonlinear ODE's, Linearity, Linearization, Feedback, Stability, Root Locus

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Course 6E8X0

Outline of lecture 6

- Derivation of linear approximate models
- How to check whether a system is linear or not?
- Derivation of linear approximate models
  - Taylor series expansion
- Stability
- Root Locus

Derivation of linear approximate models

Definition of System Linearity

$$x_1(t) \rightarrow \boxed{\text{System}} \rightarrow y_1(t) \qquad a x_1(t) \rightarrow \boxed{\text{System}} \rightarrow a y_1(t)$$

$$x_2(t) \rightarrow \boxed{\text{System}} \rightarrow y_2(t) \qquad b x_2(t) \rightarrow \boxed{\text{System}} \rightarrow b y_2(t)$$

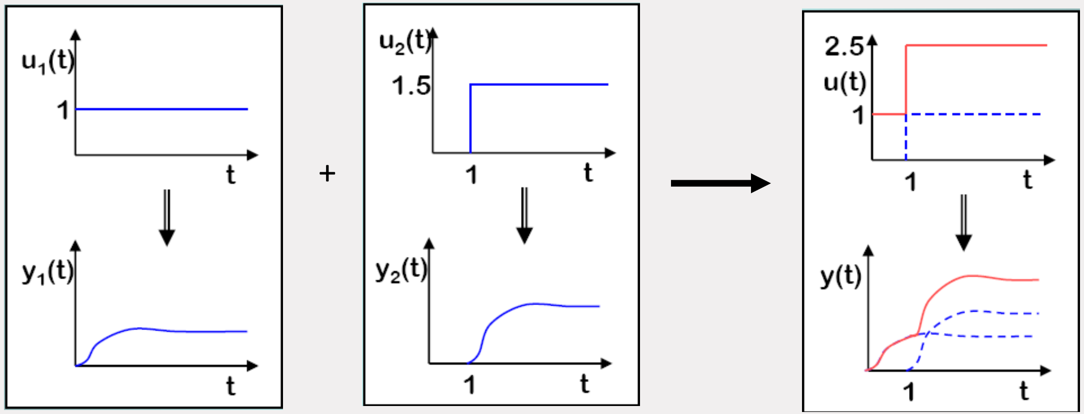
$$a x_1(t) + b x_2(t) \rightarrow \boxed{\text{System}} \rightarrow a y_1(t) + b y_2(t)$$

Superposition Principle:

$$F(ax_1(t) + bx_2(t)) = aF(x_1(t)) + bF(x_2(t))$$

F is a linear operator

Illustration of Superposition Principle



Derivation of linear approximate models

An approximate linear model can be obtained by local linearization of the process behavior in the operating point; each non-linear term is approximated by a Taylor series expansion

- Taylor series for a function of one variable:

$$F(x) \approx= F(x_s) + \left.\frac{\partial F(x)}{\partial x}\right|_{x_s} (x - x_s) + \frac{1}{2!} \left.\frac{\partial^2 F(x)}{\partial x^2}\right|_{x_s} (x - x_s)^2 + \mathrm{H.O.T}$$

Derivation of linear approximate models

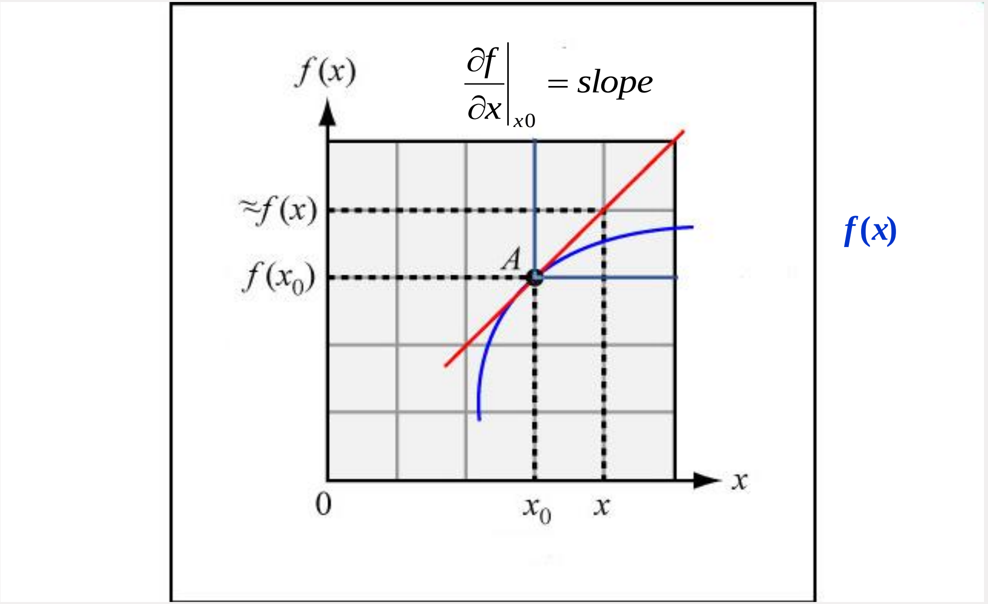
Taylor series for a function of two variables:

$$F(x_1, x_2) \cong F(x_{1s}, x_{2s}) + \frac{\partial F}{\partial x_1} \cdot \Big|_{x_1 = _{1s}, x_2 = x_{2s}} (x_1 - x_{1s}) + \frac{\partial F}{\partial x_2} \cdot \Big|_{x_1 = _{1s}, x_2 = x_{2s}} (x_2 - x_{2s}) +$$

$$\frac{1}{2!} \frac{\partial^2 F}{\partial x_1^2} \cdot \Big|_{x_1 = _{1s}, x_2 = x_{2s}} (x_1 - x_{1s})^2 + \frac{1}{2!} \frac{\partial^2 F}{\partial x_2} \cdot \Big|_{x_1 = _{1s}, x_2 = x_{2s}} (x_2 - x_{2s})^2 +$$

$$\frac{1}{2!} \frac{\partial^2 F}{\partial x_1 \partial x_2} \cdot \Big|_{x_1 = _{1s}, x_2 = x_{2s}} (x_1 - x_{1s})(x_2 - x_{2s}) +$$

Illustration of linearization up to 1st order



$f(x)$

$$\frac{f(x) - f(x_0)}{x - x_0} = \left.\frac{\partial f(x)}{\partial x}\right|_{x_0}$$

$$f(x) = f(x_0) + \left.\frac{\partial f(x)}{\partial x}\right|_{x_0} \cdot (x - x_0)$$

Linearization

Example: Linearize $y = xz$ around the operating point/reference point (x_s, z_s)

$$y(x,z) \cong y(x_s,z_s) + \left.\frac{\partial y}{\partial x}\right|_{x_s,z_s} \cdot (x - x_s) + \left.\frac{\partial y}{\partial z}\right|_{x_s,z_s} \cdot (z - z_s)$$

$$y(x,z) \cong y(x_s,z_s) + z_s(x - x_s) + x_s(z - z_s)$$

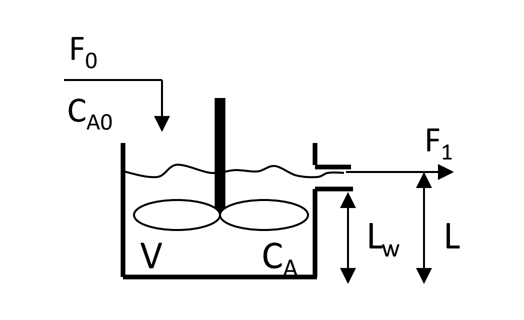
Deviation Variables: represent perturbations from normal operating point/reference point

$$\Delta y \cong z_s \Delta x + x_s \Delta z, \quad \Delta y = y - y(x_s,z_s), \quad \Delta x = x - x_s, \ \Delta z - z_s$$

Some modelling examples

Isothermal CSTR with 2nd order chemical reaction

- Goal: Determine the transient step response of tank concentration CA to the inlet concentration CA_0 both for the non-linear model and for the linear approximate model
- Information: See figure
- Assumptions:
  - Well mixed vessel
  - Density solvent and A are equal
  - Flow in is constant
- Data:
  - F_0=0.085 m^3/min; V=2.1 m^3; $(CA_0)_{init}$=0.925 mole/m3;
  - ΔCA_0=0.925 mole/m^3; $(CA)_{init}$=0.236 mole/m^3;
    - $r_A=-kC_A^2$; k=0.5 [(mole/m^3)min]$^{-1}$
  - The reactor is isothermal



TU/e

Some modelling examples

Formulation of the model:

- Apply overall and component balances
  - Overall mass balance:

$$\frac{d(\rho \cdot V)}{dt} = \rho \cdot \frac{dV}{dt} + V \cdot \frac{d\rho}{dt} = \rho \cdot \frac{dV}{dt} = F_0 \cdot \rho - F_1 \cdot \rho$$

  - F_0 and V are both variables in this equation; an additional equation -linking F_1 and L- is applied to cover the two degrees of freedom:

$$F_1 = k_F \cdot \sqrt{L - L_w} \qquad \text{for} \qquad L > L_w$$

  - If we assume $(L-L_w) << L$, the liquid level in the tank may be assumed to be constant and $F_0=F_1=F$
  - As a consequence:

$$\rho \cdot \frac{dV}{dt} = \rho \cdot (F_0 - F_1) = 0 \qquad \rightarrow \qquad \therefore V = \text{constant} \qquad (1)$$

Some modelling examples

- Component balance for component A:

$$Mw_A \cdot V \cdot \frac{dC_A}{dt} = Mw_A \cdot F \cdot (C_{A_0} - C_A) - Mw_A \cdot V \cdot k \cdot C_A^2 \qquad (2)$$

with: Mw_A molecular weight of A

- Degree of freedom analysis:
  - Variables: C_A, F_1
  - External or disturbance variables: C_{A0}, F_0
  - Equations: (1), (2)
- Rewriting and linearization of the non-linear term in (2):

$$V \cdot \frac{dC_A}{dt} = F \cdot (C_{A_0} - C_A) - V \cdot k \cdot (C_{As}^2 + 2 \cdot C_{As} \cdot (C_A - C_{As})) \qquad (3)$$

Some modelling examples

- At steady state condition the following holds:

$$V \cdot \frac{dC_{As}}{dt} = 0 = F \cdot (C_{A_0} - C_{As}) - V \cdot k \cdot C_{As}^2$$

- Subtracting this result from (3) gives the model in deviation variables (indicated by x') from the steady state:

$$V \cdot \frac{dC_A^{'}}{dt} = F \cdot (C_{A_0}^{'} - C_A^{'}) - 2 \cdot V \cdot k \cdot C_{As} \cdot C_A^{'} \qquad (4)$$

- This equation can be rewritten as a standard first order linear ordinary differential equation:

$$\frac{dC_A^{'}}{dt} + \frac{1}{\tau} \cdot C_A^{'} = \frac{F}{V} \cdot C_{A_0}^{'} \qquad \text{With:} \quad \tau = \frac{V}{F + 2 \cdot V \cdot k \cdot C_{As}}$$

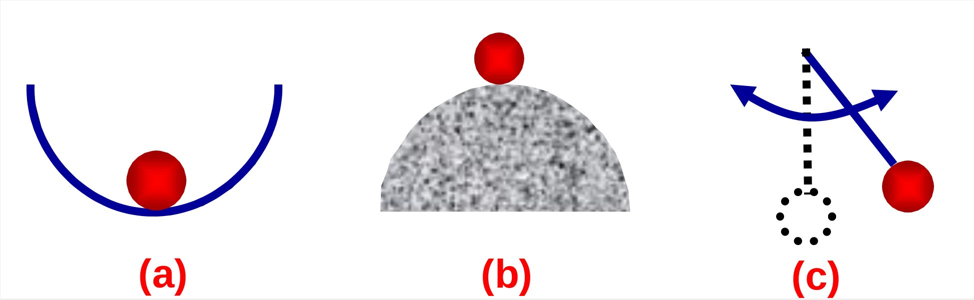
Take Home Message

How to obtain transfer functions from non-linear process models:

- Find steady-state of process
- Linearize around the steady-state
- Express in terms of deviations variables around the steady-state
- Take Laplace transform

Stability

- A dynamic system is stable if the system output response is bounded for all bounded inputs (Bounded Input Bounded Output Stability).
- A stable system will tend to return to its equilibrium point following a disturbance.
- An unstable system will have the tendency to move away from its equilibrium point following a disturbance.



Stability

Question: Can you give an example of a chemical system that can be unstable?

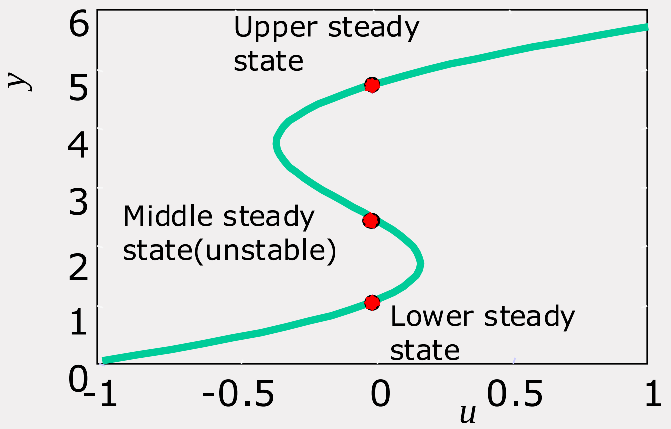
CSTR with an irreversible reaction[*)]

A ⇨ B

$$\dot{x}_1 = -\phi x_1 \kappa(x_2) + q(x_{1f} - x_1)$$

$$\dot{x}_2 = \beta\phi x_1 \kappa(x_2) - (q+\delta)x_2 + \delta u + q x_{2f}$$

$$y = x_2 \qquad \kappa(x_2) = \exp\left(\frac{x_2}{1+\frac{x_2}{\gamma}}\right)$$



Steady state input-output behavior

*) Chemical Reactor Analysis and Design, Chapter 10.4.1, Froment & Bischoff

Stability

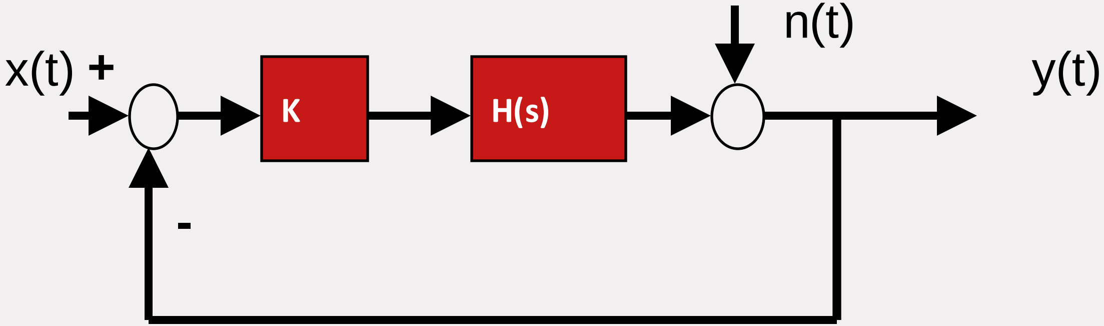
How to determine whether a linear system is stable?

- Roots of the characteristic equation
- Simulation
- Bode stability

Closed loop Stability

Roots of the characteristic equation

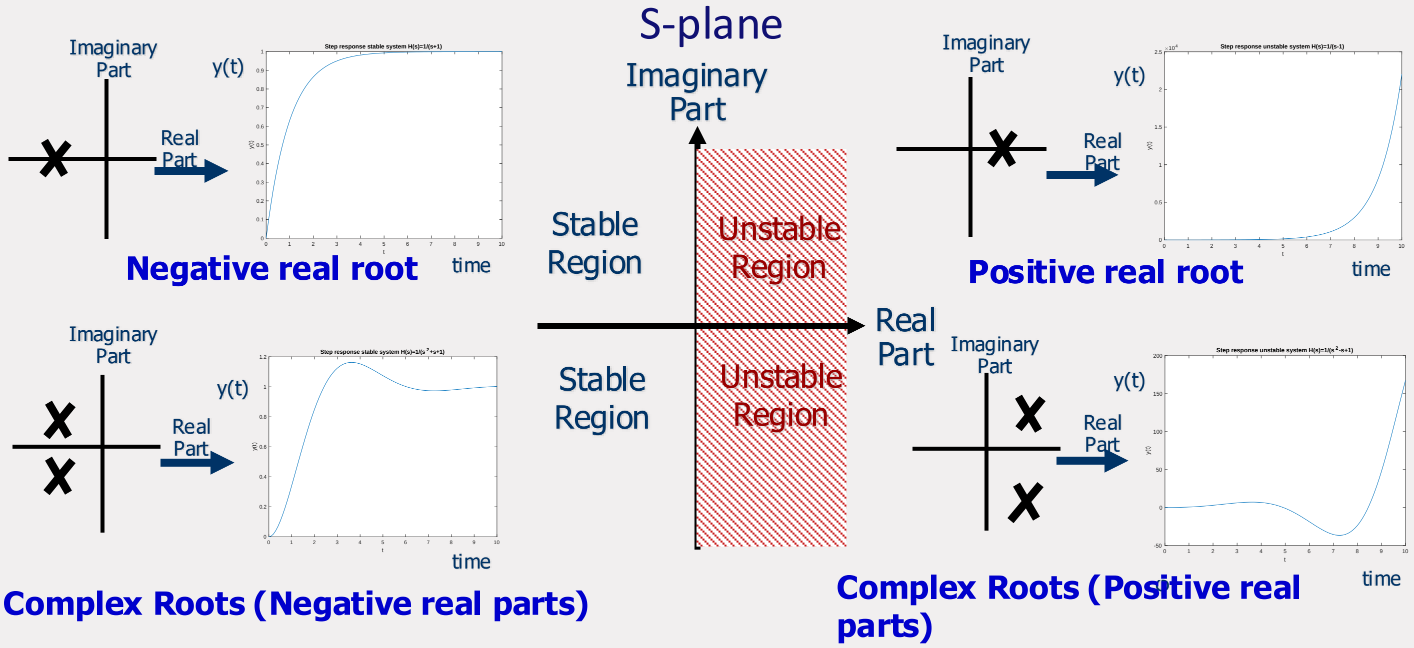
The location of the roots in the s-plane determines if a system is stable or not



$$Y(s) = H_{xy,cl}(s)X(s) + H_{ny,cl}N(s) = \frac{KH(s)}{1+KH(s)}X(s) + \frac{1}{1+KH(s)}N(s)$$

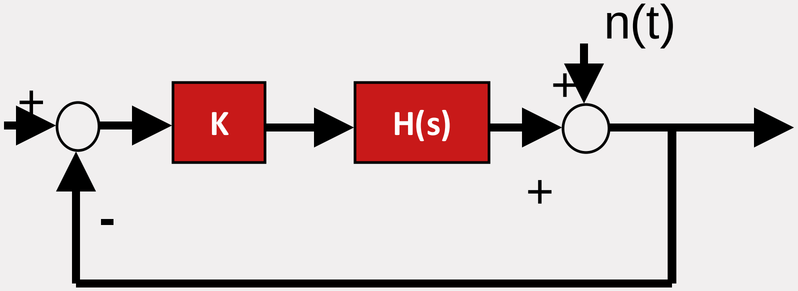
$$1 + KH(s) = 0$$

Stability


S-plane
Imaginary Part
Real Part
Stable Region
Stable Region
Unstable Region
Unstable Region
Imaginary Part
Real Part
y(t)
Step response stable system H(s)=1/(s+1)
time
Negative real root
Imaginary Part
Real Part
y(t)
Step response unstable system H(s)=1/(s-1)
time
Positive real root
Imaginary Part
Real Part
y(t)
Step response stable system H(s)=1/(s^2+s+1)
time
Complex Roots (Negative real parts)
Imaginary Part
Real Part
y(t)
Step response unstable system H(s)=1/(s^2-s+1)
time
Complex Roots (Positive real parts)

Rootlocus

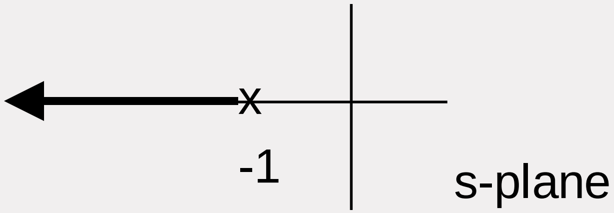
Effect of gain in loop



Example 1: $$H(s) = \frac{1}{s+1} \qquad H_{cl}(s) = \frac{KH(s)}{1+KH(s)} \qquad H_{cl}(s) = \frac{K}{s+1+K}$$

- pole position: s_1 =-(1+K)
- if K varies from 0 to ∞

  pole s_1 of $H_{cl}(s)$ moves from -1 to - ∞
- The track followed is called the rootlocus of H(s)



Rootlocus

The path that poles follow when gain K in the loop is varied

Example 2: $H(s) = \dfrac{s+b}{s+a}$ $\qquad H_{cl}(s) = \dfrac{K(s+b)}{(1+K)s+(a+Kb)}$

closed loop system has pole in:

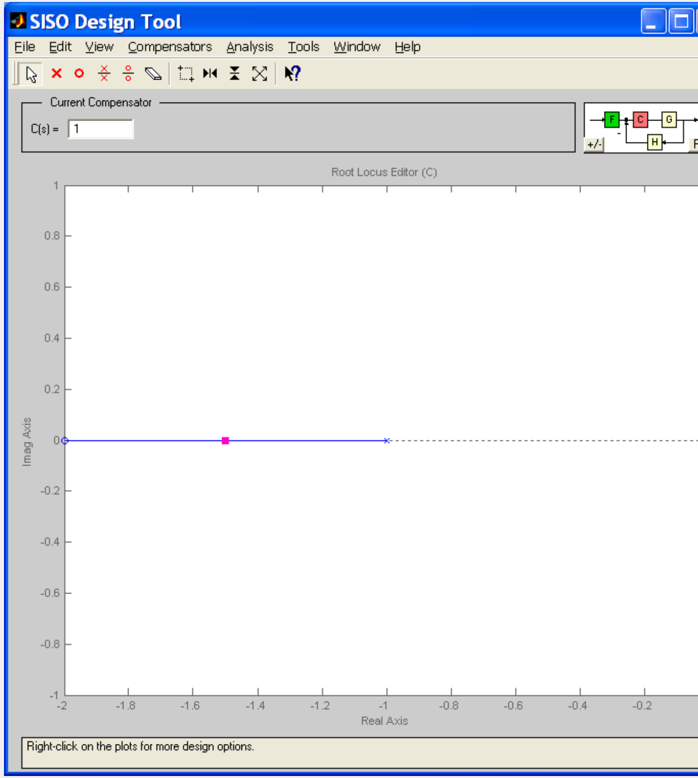
$$s_1 = -\frac{a+Kb}{1+K}$$

for K = 0 pole is located in s_1 = -a

for K = ∞ is pole is located in s_1 = -b

<u>rootlocus</u>

moves from pole to zero of H(s)!



Rootlocus

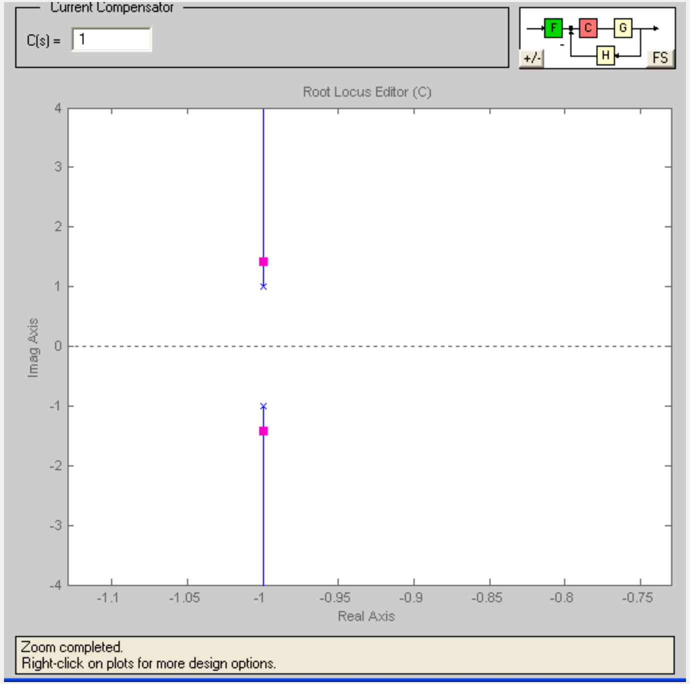
Example 3:

$$H(s) = \frac{1}{s^2 + 2s + 2} \qquad H_{cl}(s) = \frac{K}{s^2 + 2s + 2 + K}$$

Open loop Poles: $$s_{1,2} = -1 \pm j$$

Closed loop Poles:

$$s_{1,2} = -1 \pm j\sqrt{1+K}$$



Rootlocus

Example 4: $$H(s) = \frac{1}{(s+1)^3}$$

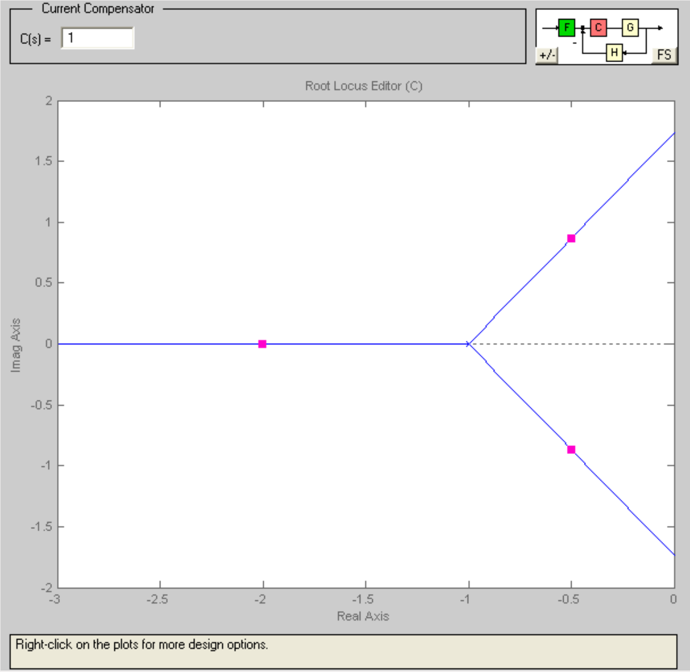
$$H_{cl}(s) = \frac{K}{(s+1)^3 + K} = \frac{K}{s^3 + 3s^2 + 3s + 1 + K}$$

open loop system H(s) has 3 poles in $s_i = -1$

closed loop system $H_{cl}(s)$ has 3 poles:

- one pole is real
- two poles are complex conjugate

The rootlocus of $H_{cl}(s)$ is shown in the figure



Rootlocus

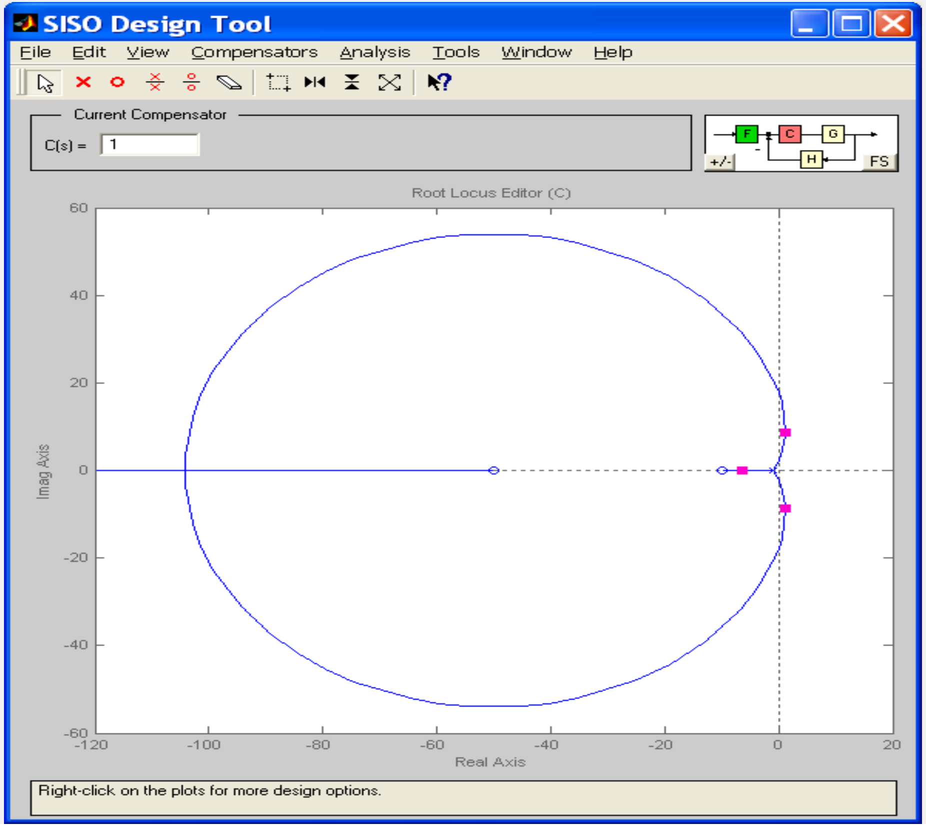
rules for rootloci:

- suppose that H(s) has N poles and M zeros $N \geq M$
- the zeros of $H_{cl}(s)$ are the zeros of H(s)
- the poles of $H_{cl}(s)$ can be calculated from $1 + KH(s) = 0$
- the loci depart from the N poles of H(s) for K = 0;
  - the locus has N branches
  - M branches of the loci arrive, for $K = \infty$, in the M zeros of H(s)
  - N-M branches disappear, for $K \rightarrow \infty$, to ∞ along asymptotes
  - the angles of the asymptotes are given by:
    - N - M = 1: -180°
    - N - M = 2: +90° and -90°
    - N – M = 3: +60°, -180° and -60°
    - etc.

Rootlocus

Example 5:

$$H(s) = \frac{(s+10)(s+50)}{(s+1)^3}$$

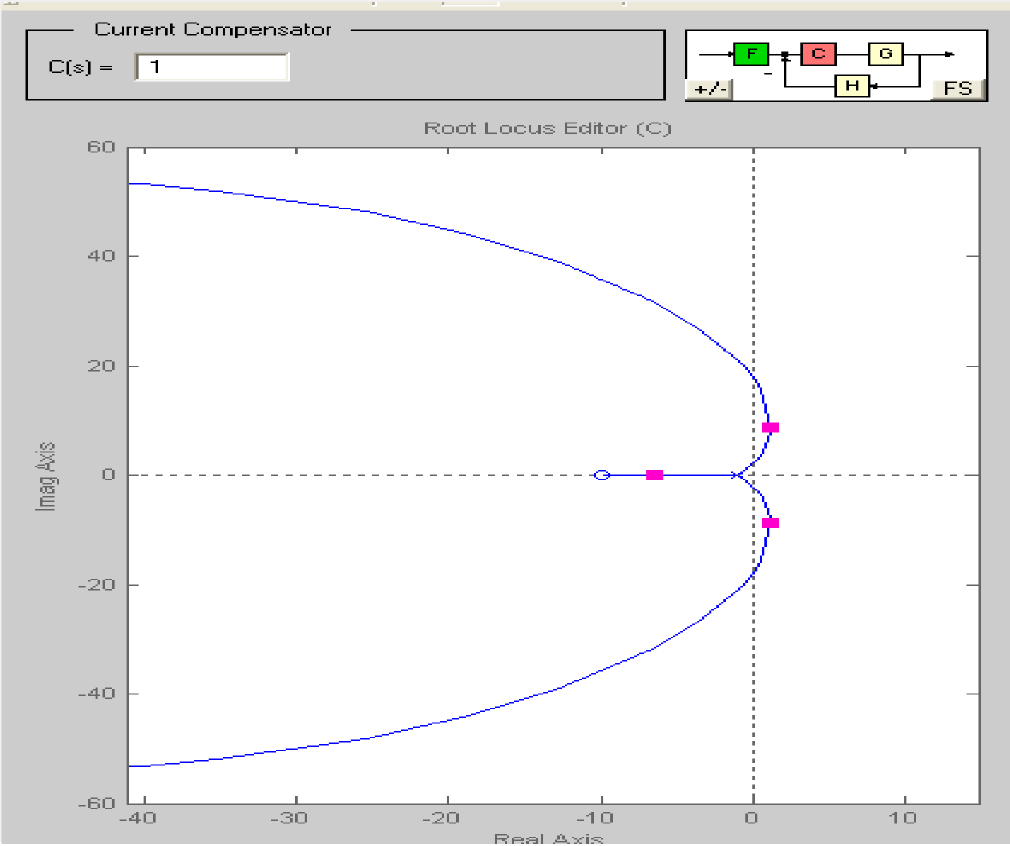


Rootlocus

Example 5:

$$H(s)=\frac{(s+10)(s+50)}{(s+1)^3}$$

part of the rootlocus close to the origin of the s-plane



Rootlocus

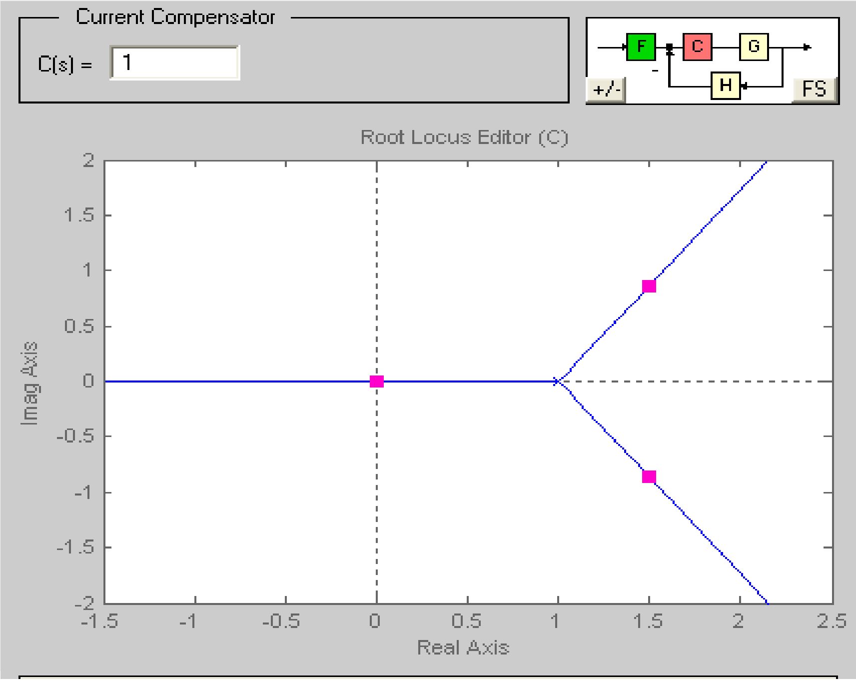
Rootlocus as designtool

Change behavior of systems in closed loop

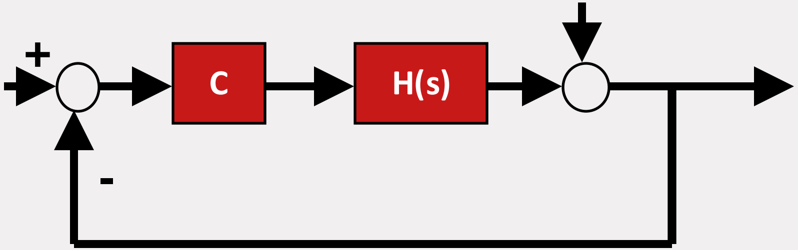
to make unstable processes stable

Example 6: $$H(s) = \frac{1}{(s-1)^3}$$

H(s) is the open loop transfer function of an unstable system with 3 poles in s_i=+1



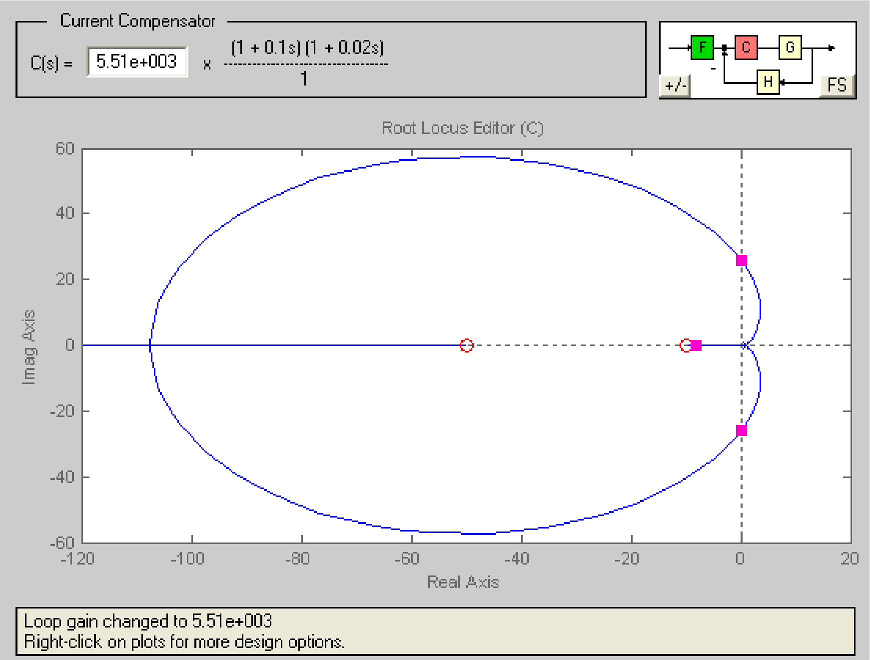
Rootlocus



Rootlocus as designtool

Open loop system transfer function:

$$H(s) = \frac{1}{(s-1)^3}$$



Now choose the controller:

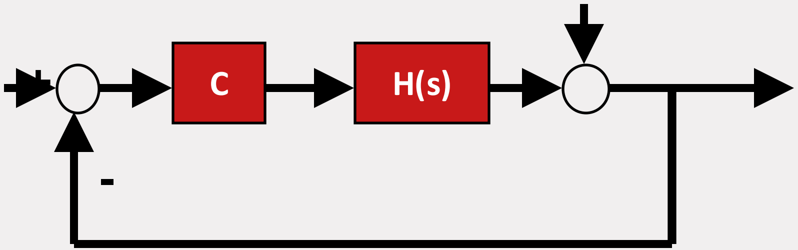
$$C(s) = K(s+10)(s+50)$$

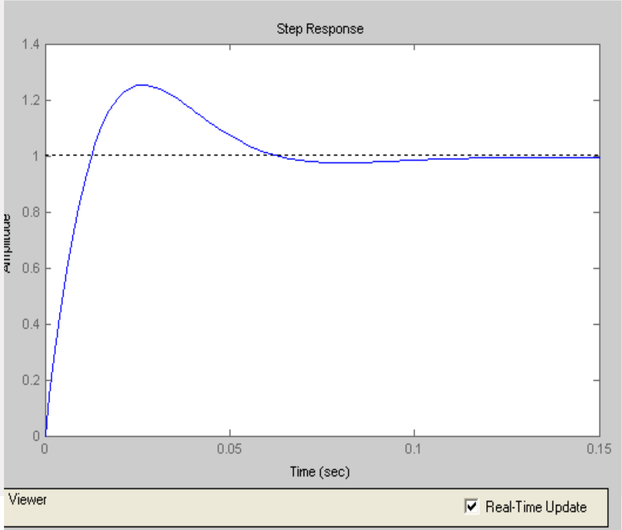
Rootlocus

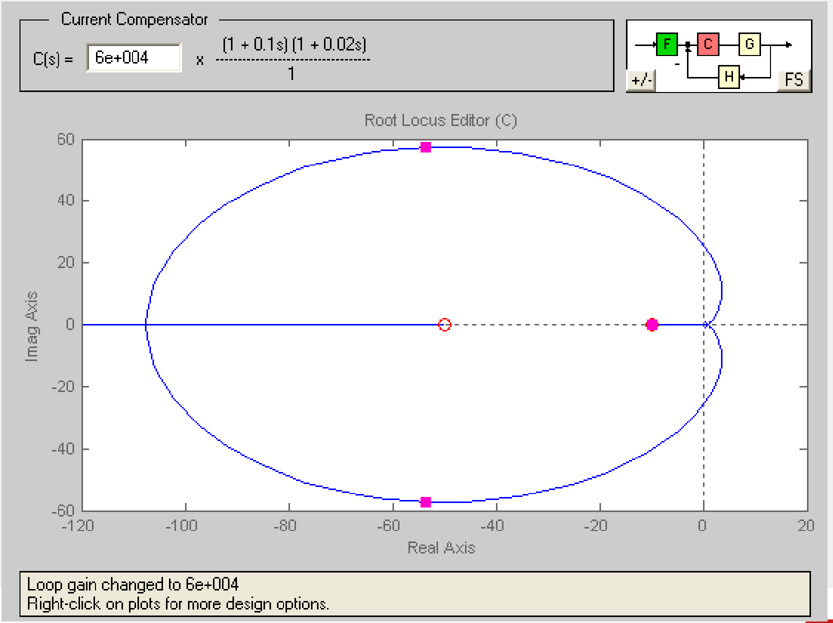
Rootlocus as designtool

$$H(s) = \frac{1}{(s-1)^3}$$

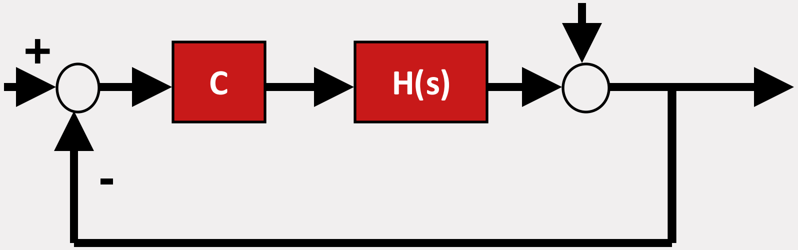
$$C(s) = 120(s+10)(s+50)$$





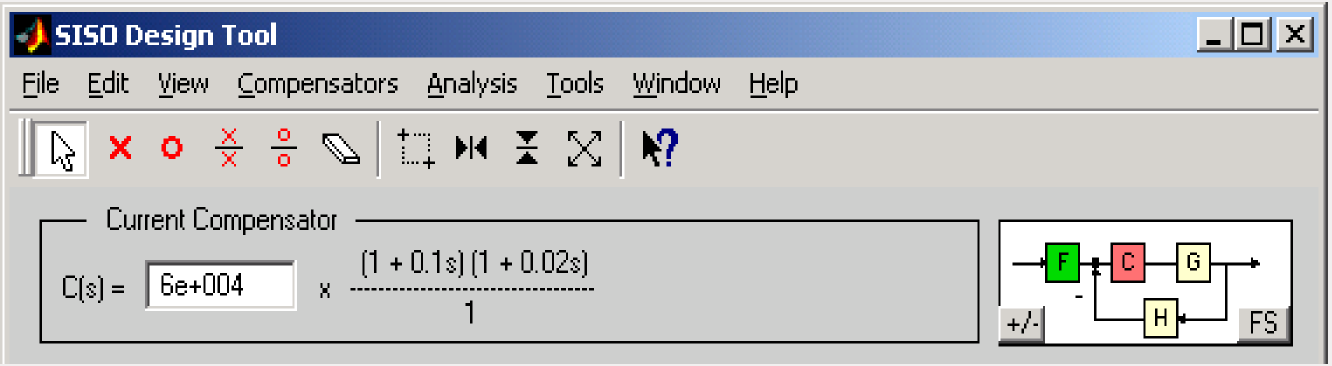


Rootlocus



$$H(s) = \frac{1}{(s-1)^3}$$

$$C(s) = 120(s+10)(s+50)$$



Summary

- Non-linear system behavior can be approximated by a linear system description:
  - apply Taylor series expansion to each of the non-linear terms of the non-linear differential equation
  - describe the true system by just retaining the linear terms

- To go to deviation variables steady state behavior needs to be subtracted from the resulting linear differential equation

- Rootloci (Latin for: “path’s followed by poles”) describe the positions poles will take in the s-plane as function of change in the controller gain

- Pole locations of the closed loop transfer function move from original open-loop pole locations of H(s) for K=0 to open-loop zero locations or to infinity