TU/e EINDHOVEN UNIVERSITY OF TECHNOLOGY

Dynamics and Control of Processes

Topic 2: Introduction to frequency domain and Laplace transformation

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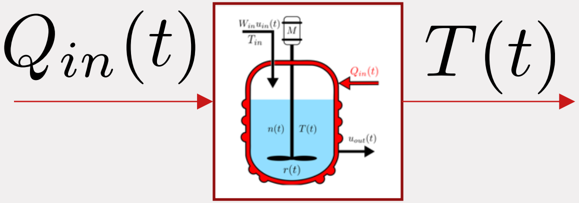
Course 6E8X0

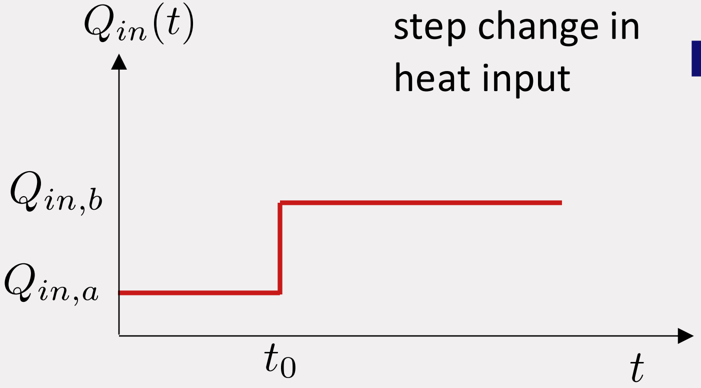
Outline

- Analysis of Dynamics
- Frequency domain and Introduction of Laplace Transformation
- Final Value Theorem

Analysis of Dynamics

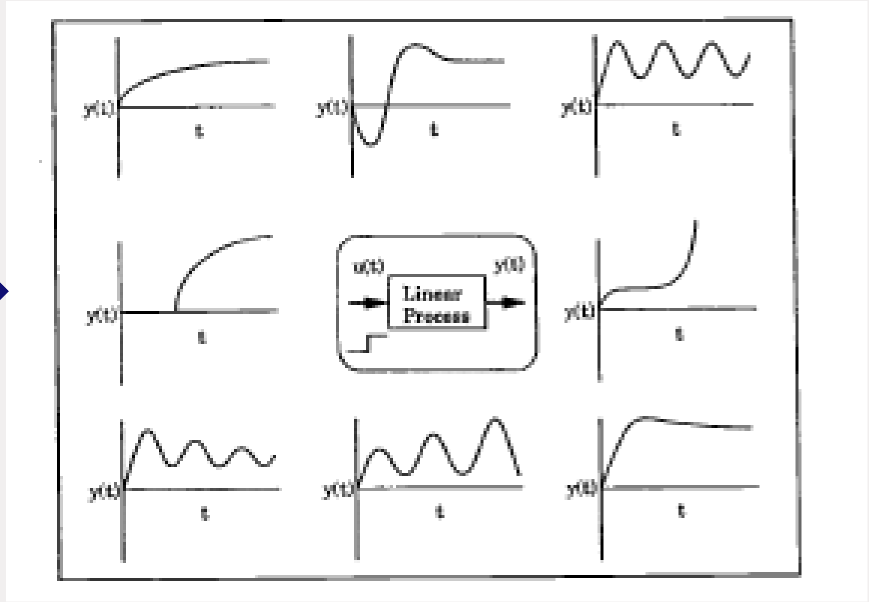
Linear Process





step change in heat input

Typical responses of a linear process to a step change in the input



Analysis of Dynamics

Differential equations are the mathematical tools describing the dynamic behavior of processes.

$$\partial_t y(x,t) = D\left(y(x,t)\right)$$

D : partial differential operator

- Process dynamics can be very well approximated by finite order, linear, time-invariant differential equations when each of the assumptions apply:
  - the content of a relevant reactor volume can be considered homogeneous
  - the conditions and properties don’t change as function of time
  - the process is operated in a well defined operating point

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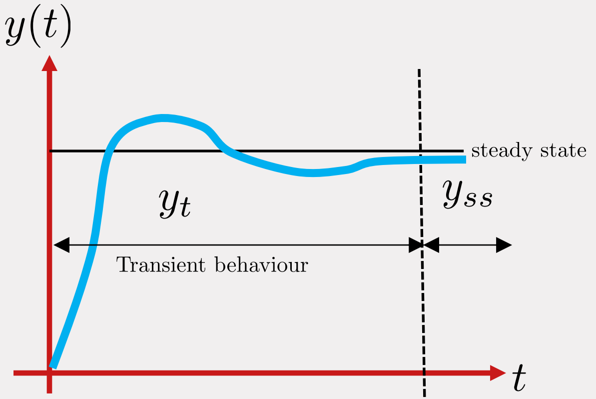
Analysis of Dynamics

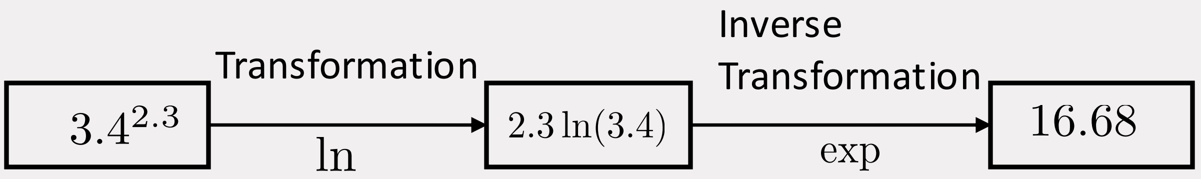
Time Domain Analysis:



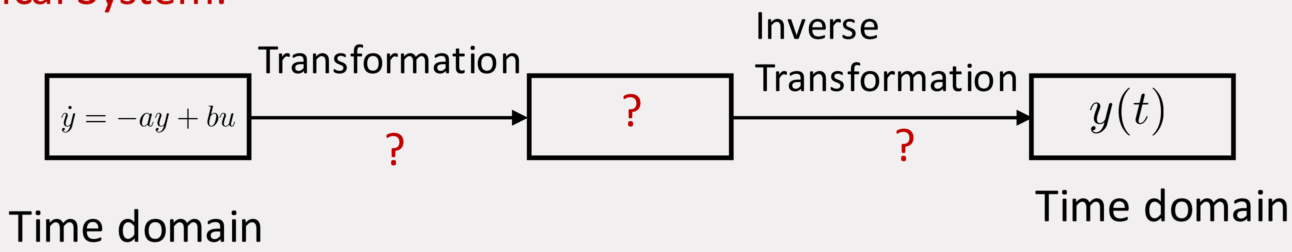
Is there a convenient way (Transformation) of analyzing linear systems?

Analysis of Dynamics

Example: Calculate $3.4^{2.3}$



Dynamical System:



Laplace transform

from

ordinary, linear, time-invariant, finite order

differential equations

to

algebraic equations

Pierre-Simon Laplace 1749 -1827

Laplace Transform

Definition: Given a linear function $f(t)$, the Laplace transform is

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

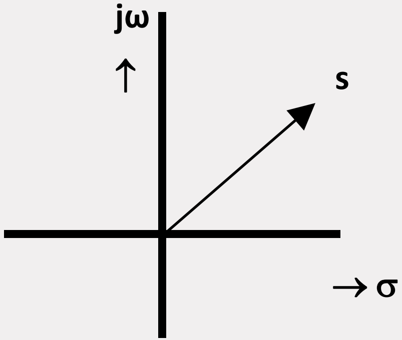
Unilateral transformation

$f(t)$: function in time domain

$F(s)$: function in s-domain

$s = \sigma + j\omega$: complex frequency

$Re(s) = \sigma \qquad Im(s) = \omega$



Assumption: $f(t) = 0 \quad \forall t \leq 0$

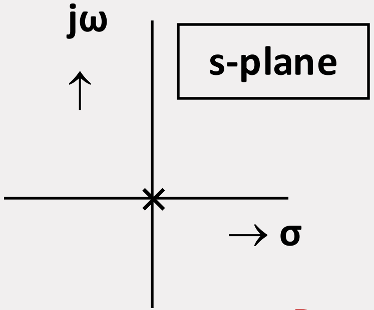
Laplace transform

Example: Find the laplace transform of unit step function $x(t) = unitstep(t=0)$

Unit step:

$$x(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \quad \xrightarrow{\mathcal{L}} \quad X(s) = \int_0^\infty e^{-st} x(t) dt = \int_0^\infty 1.e^{-st} dt = \frac{1}{s}$$

Time Domain



exists for $s > 0$ ROC[1)]

s- Domain

([1)]ROC:=Region of Convergence)

Laplace transform

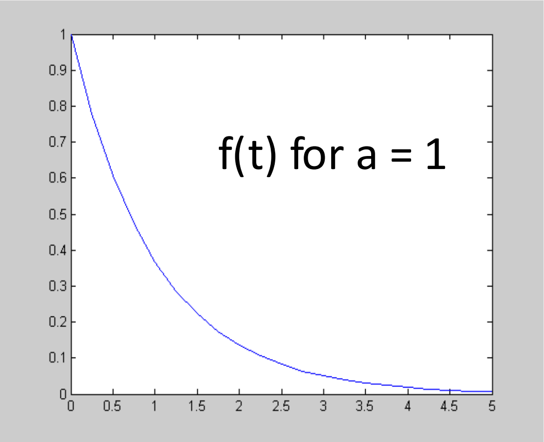
Example: Find the laplace transform of e^{-at}

$$f(t) = e^{-at} \quad t \geq 0$$
$$f(t) = 0 \quad t < 0$$

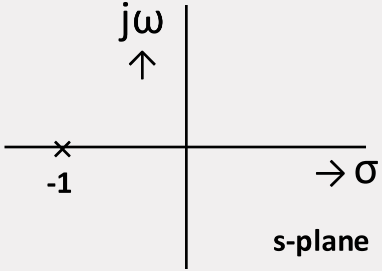
$\mathcal{L}$

$$F(s) = \int_0^\infty e^{-at} e^{st} dt = \int_0^\infty e^{-(s+a)t} dt$$

$$F(s) = -\frac{1}{s+a} e^{-(s+a)t} \Big|_0^\infty = 0 - \left(-\frac{1}{s+a}\right) = \frac{1}{s+a}$$



$\Leftrightarrow$



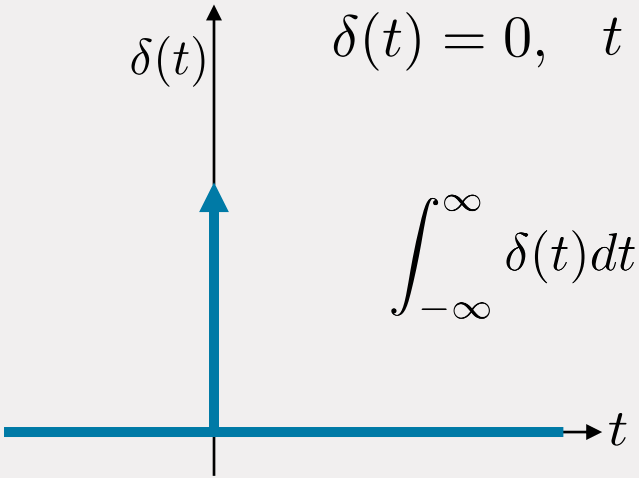
exists for $s > -a$

singularity in $s = -a$

*)ROC:=Region of Convergence

Laplace transform

Example: Find the laplace transform of impulse (dirac) function



$$\delta(t) = 0, \quad t \neq 0$$

$$\int_{-\infty}^{\infty} \delta(t)dt = 1$$

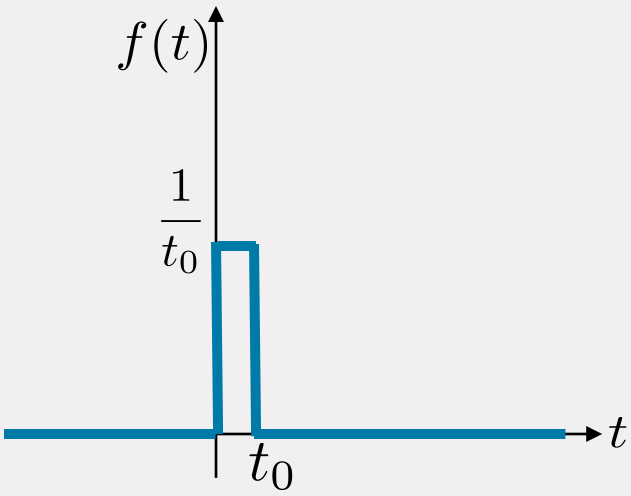
Tall and narrow pulse

Laplace transform

Example: Find the laplace transform of impulse (dirac) function

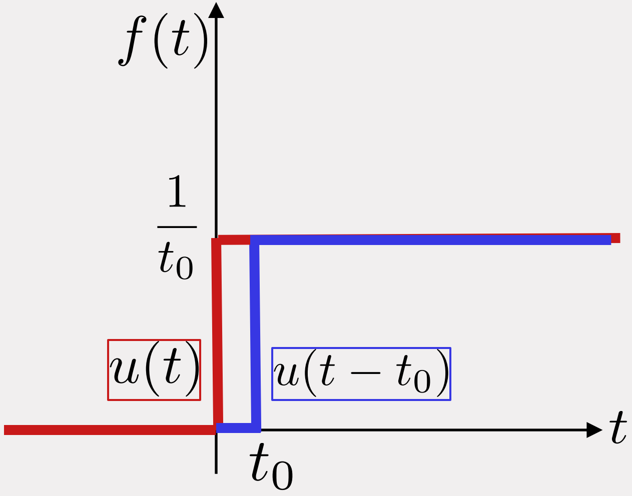
Consider

$$f(t) = \begin{cases} \frac{1}{t_0}, & 0 \leq t \leq t_0 \\ 0, & t < 0 \text{ or } t > t_0 \end{cases}$$



Sum of two step functions

$$f(t) = u(t) - u(t - t_0)$$



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Laplace transform

Example: Find the laplace transform of impulse (dirac) function

$$f(t) = \begin{cases} \frac{1}{t_0}, & 0 \leq t \leq t_0 \\ 0, & t < 0 \text{ or } t > t_0 \end{cases}$$

$$f(t) = u(t) - u(t - t_0)$$

$$\mathcal{L}\left(f(t)\right) = \mathcal{L}\left(u(t) - u(t - t_0)\right) = \mathcal{L}\left(u(t)\right) - \mathcal{L}\left(u(t - t_0)\right)$$

$$\mathcal{L}\left(f(t)\right) = \mathcal{L}\left\{\frac{1}{t_0}1(t)\right\} - \mathcal{L}\left\{\frac{1}{t_0}1(t - t_0)\right\}$$

$$\mathcal{L}\left(f(t)\right) = \frac{1}{t_0 s} - \frac{e^{-t_0 s}}{t_0 s}$$

We use the time shifting property of Laplace transformation

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Laplace transform

Example: Find the laplace transform of impulse (dirac) function

In the limit as $t_0 \to 0$

$$\mathcal{L}\left\{\delta(t)\right\} = \lim_{t_0 \to 0} \mathcal{L}\left\{f(t)\right\} = \lim_{t_0 \to 0} \left(\frac{1 - e^{-t_0 s}}{t_0 s}\right)$$

Applying l'Hopital rule

$$\mathcal{L}\left\{\delta(t)\right\} = \lim_{t_0 \to 0} \frac{\frac{d}{dt_0}\left(1 - e^{-t_0 s}\right)}{\frac{d}{dt_0}\left(t_0 s\right)} = \lim_{t_0 \to 0} \frac{s e^{-t_0 s}}{s} = \frac{s}{s} = 1$$

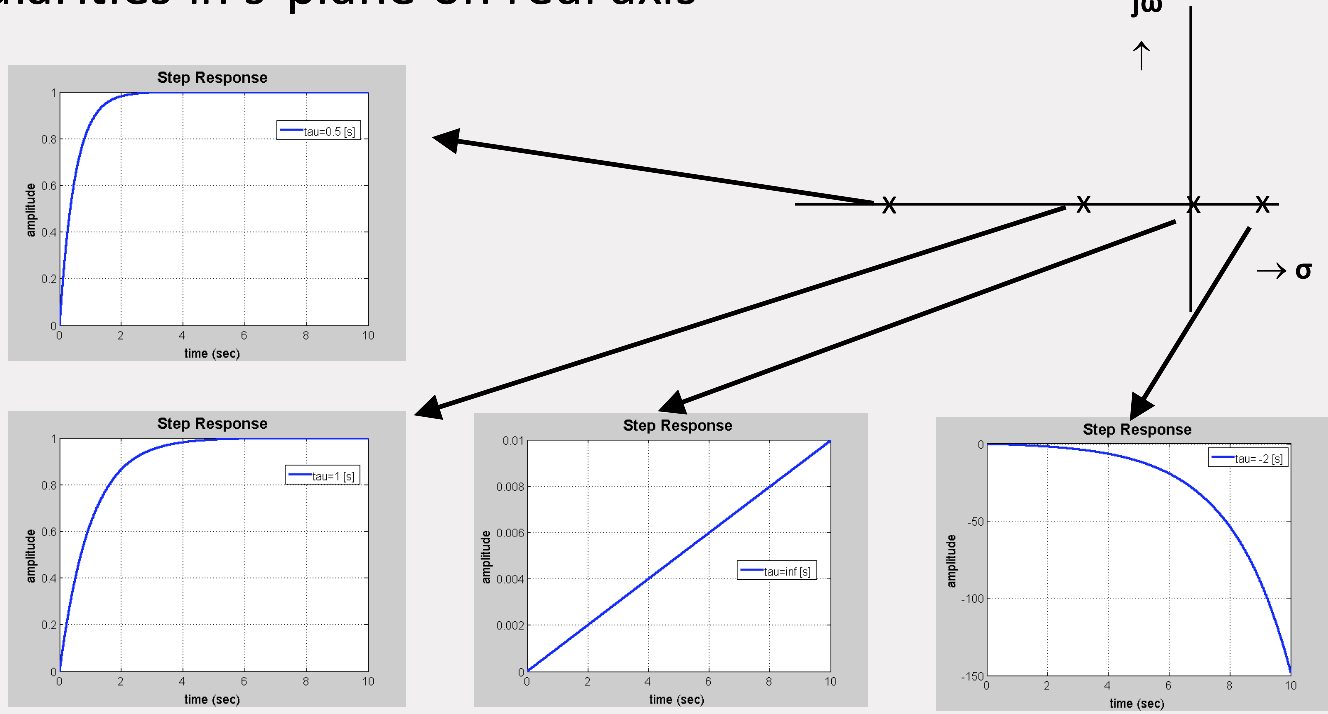
Laplace transform

Construct table of basic transforms (see page 721 in the course textbook):

time domain	s-domain
$U(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
$\delta(t)$	1
$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$

Laplace transform

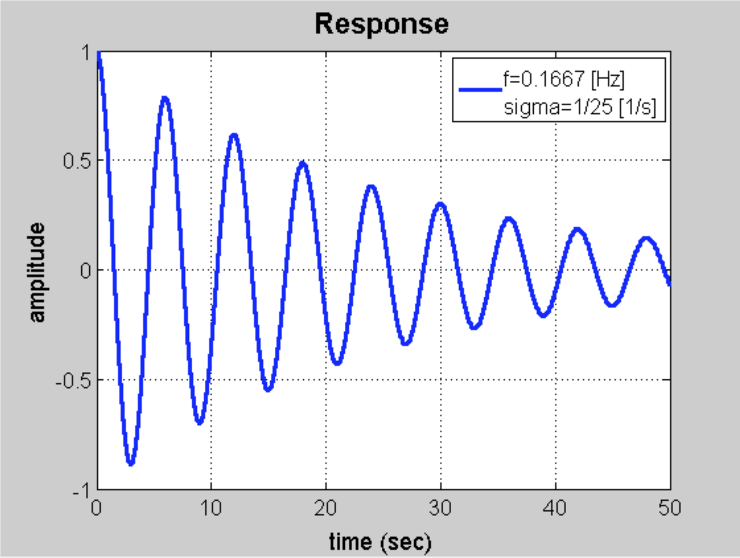
Singularities in s-plane on real axis

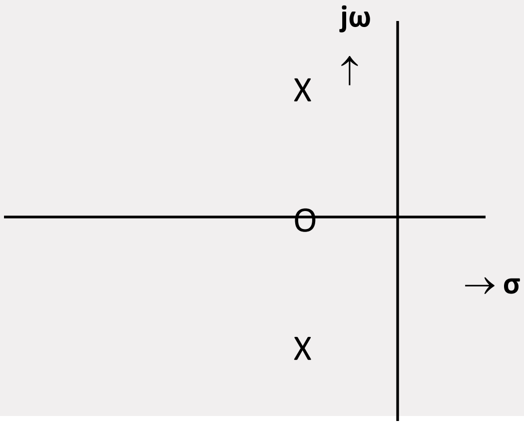


Laplace transform

singularities in s-plane

$$f(t) = e^{-\sigma t}\cos(\omega t) \xrightarrow{\mathcal{L}} F(s) = \frac{s+\sigma}{(s+\sigma)^2+\omega^2} = \frac{s+\sigma}{(s+\sigma+j\omega)(s+\sigma-j\omega)}$$





Laplace transform

Properties and Rules:

- linearity $$\mathcal{L}\left\{af_1 + bf_2(t)\right\} = aF_1(s) + bF_2(s)$$
- time shift $$\mathcal{L}\left\{f(t-\tau)\right\} = e^{-s\tau}F(s)$$
- differentiation $$\mathcal{L}\left\{\frac{df(t)}{dt}\right\} = sF(s) - f(0)$$
- integration $$\mathcal{L}\left\{\int_0^t f(\tau)d\tau\right\} = \frac{1}{s}F(s)$$
- modulation $$\mathcal{L}\left\{e^{-at}f(t)\right\} = F(s+a)$$

Laplace transform

Higher Order Differentiation

$$\frac{df(t)}{dt} = \dot{f}(t) \qquad \mathcal{L}\left\{\dot{f}(t)\right\} = sF(s) - f(0)$$

$$\frac{d^2 f(t)}{dt^2} = \ddot{f}(t)$$

Let us define $f_1(t) = \dot{f}(t) \qquad F_1(s) = sF(s) - f(0)$

$$\mathcal{L}\left\{\frac{d^2 f(t)}{dt^2}\right\} = \mathcal{L}\left\{\frac{df_1(t)}{dt}\right\} = sF_1(s) - f_1(0) = s\left[sF(s) - f(0)\right] - f_1(0)$$
$$= s^2 F(s) - \dot{f}(0) - sf(0)$$

Laplace transform

Differential Equation:

$$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + 3x(t)$$

Apply Laplace Transform to both sides

$$\mathcal{L}\left\{\ddot{y}(t) + 3\dot{y} + 2y(t)\right\} = \mathcal{L}\left\{\dot{x}(t) + 3x(t)\right\}$$

$$s^2Y(s) - \dot{y}(0) - sy(0) + 3\left(sY(s) - y(0)\right) + 2Y(s) = sX(s) - x(0) + 3X(s)$$

Collect similar terms together

$$\left(s^2 + 3s + 2\right)Y(s) - \dot{y}(0) - (s+3)\,y(0) = (s+3)\,X(s) - x(0)$$

Laplace transform

$$\left(s^2 + 3s + 2\right) Y(s) - \dot{y}(0) - (s+3)\, y(0) = (s+3)\, X(s) - x(0)$$

We can express $Y(s)$ in s-domain

$$Y(s) = \frac{(s+3)}{s^2+3s+2} X(s) - \frac{1}{(s^2+3s+2)} x(0) + \frac{1}{(s^2+3s+2)} \dot{y}(0) + \frac{(s+3)}{(s^2+3s+2)} y(0)$$

due to input $x(t)$

due to initial conditions

$x(0), \dot{y}(0), y(0)$

Laplace transform

Assume

$$\dot{y}(0) = y(0) = 0$$
$$x(0) = 0$$

$$\Rightarrow \quad Y(s) = \frac{s+3}{(s^2+3s+2)} X(s)$$

Input (Forcing function) $x(t) = U(t)$ $\Rightarrow$ $X(s) = \frac{1}{s}$

$$Y(s) = \frac{s+3}{(s^2+3s+2)} \frac{1}{s} = \frac{s+3}{s(s+1)(s+2)} = \frac{1.5}{s} + \frac{-2}{s+1} + \frac{0.5}{s+2}$$

$$\Downarrow \mathcal{L}^{-1}$$

$$U(t) \Leftrightarrow \frac{1}{s}$$

$$e^{-at} \Leftrightarrow \frac{1}{s+a}$$

$$y(t) = 1.5U(t) - 2e^{-t} + 0.5e^{-2t}$$

Laplace transform

How did we obtain the coefficients 1.5 , -2, 0.5?

$$\frac{s+3}{s(s+1)(s+2)} = \frac{a}{s} + \frac{b}{s+1} + \frac{c}{s+2} = \underset{(s+1)(s+1)}{\frac{a}{s}} + \underset{s(s+2)}{\frac{b}{s+1}} + \underset{s(s+1)}{\frac{c}{s+2}}$$

We compare the numerators

$$s+3 = a(s+1)(s+2) + bs(s+2) + cs(s+1) = (a+b+c)s^2 + (3a+2b+c)s + 2a$$

$$a+b+c=0 \qquad 3a+2b+c=1 \qquad 2a=3$$

$$a=1.5 \qquad b=-2 \qquad c=0.5$$

Laplace transform

Non zero initial conditions:

Example: $x(0) = 0,\ \dot{x}(0) = 0,\ y(0) = 0,\ \dot{y}(0) = 1$

$$Y(s) = \frac{s+3}{s^2+3s+2}X(s) + \frac{s+3}{s^2+3s+2}y(0) + \frac{1}{s^2+3s+2}\dot{y}(0)$$

$$Y(s) = \frac{s+3}{s^2+3s+2}\frac{1}{s} + \frac{1}{s^2+3s+2} = \frac{s+3}{s(s+1)(s+2)} + \frac{1}{s+1} + \frac{-1}{s+2}$$

$$\Big\Downarrow \mathcal{L}^{-1}$$

$$y(t) = 1.5U(t) - 2e^{-t} + 0.5e^{-2t} + e^{-t} - e^{-2t} = 1.5U(t) - e^{-t} - 0.5e^{-2t}$$

Laplace transform

Inverse Laplace transform:

Definition:

$$f(t) = \mathcal{L}^{-1}\left\{F(s)\right\} = \lim_{\omega\to\infty} \frac{1}{2\pi j}\int_{\sigma-j\omega}^{\sigma+j\omega} F(s)e^{st}ds$$

Integration path within ROC

Practical Inverse Laplace Transform

- re-arrange $F(s)$ into sum of simple functions $F_i(s)$
- Use tables for $\mathcal{L}^{-1}\left\{F_i\right\}$

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Laplace transform

Partial Fractions:

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a_1)(s+a_2)\cdots(s+a_n)} = \frac{c_1}{s+a_1} + \frac{c_2}{s+a_2} + \cdots + \frac{c_n}{s+a_n}$$

Find c_i

$$(s+a_1)F(s) = c_1 + \frac{c_2(s+a_1)}{s+a_2} + \cdots + \frac{c_n(s+a_1)}{s+a_n}$$

If we evaluate

$$(s+a_1)F(s)\big|_{s=-a_1} = c_1$$

Laplace transform

Partial Fractions with repeated roots:

$$F(s) = \frac{N(s)}{D(s)} = \frac{N(s)}{(s+a_1)^k} = \frac{c_k}{(s+a_1)^k} + \frac{c_{k-1}}{(s+a_1)^{k-1}} + \cdots + \frac{c_1}{(s+a_1)}$$

Find c_i

$$(s+a_1)^k F(s) = c_k + c_{k-1}(s+a_1) + \cdots + c_1(s+a_1)^{k-1}$$

$$(s+a_1)^k F(s)\Big|_{s=-a_1} = c_k$$

$$\lim_{s \to a_1} \frac{d(s+a_1)^k F(s)}{ds} = \lim_{s \to a_1} \left(\frac{dN(s)}{ds} \right) = c_{k-1}$$

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Laplace transform

$$\lim_{s\to a_1} \frac{d^2(s+a_1)^k F(s)}{ds^2} = \lim_{s\to a_1}\left(\frac{d^2 N(s)}{ds^2}\right) = 2c_{k-2}$$

$$\lim_{s\to a_1} \frac{d^3(s+a_1)^k F(s)}{ds^3} = \lim_{s\to a_1}\left(\frac{d^3 N(s)}{ds^3}\right) = 3*2*c_{k-3}$$

$$\vdots$$

$$\lim_{s\to a_1} \frac{d^i(s+a_1)^k F(s)}{ds^i} = \lim_{s\to a_1}\left(\frac{d^i N(s)}{ds^i}\right) = i*\cdots*3*2*c_{k-i}$$

Laplace transform

Example:

$$\tau \frac{df(t)}{dt} + f(t) = Ku(t) \qquad u(t) = 1.0, f(0) = 0$$

$$\mathcal{L}\left\{\tau \frac{df(t)}{dt} + f(t)\right\} = \mathcal{L}\left\{Ku(t)\right\}$$

$$\tau\left[sF(s) - f(0)\right] + F(s) = \frac{K}{s}$$

$$F(s) = \frac{K}{s(\tau s + 1)} = \frac{\frac{K}{\tau}}{s\left(s + \frac{1}{\tau}\right)} = \frac{c_1}{s} + \frac{c_2}{s + \frac{1}{\tau}}$$

Laplace transform

Example:

$$c_1 = sF(s)\big|_{s=0} = s\frac{\frac{K}{\tau}}{s\left(s+\frac{1}{\tau}\right)}\Bigg|_{s=0} = K$$

$$c_2 = \left(s+\frac{1}{\tau}\right)F(s)\Bigg|_{s=-\frac{1}{\tau}} = \left(s+\frac{1}{\tau}\right)\frac{\frac{K}{\tau}}{s\left(s+\frac{1}{\tau}\right)}\Bigg|_{s=-\frac{1}{\tau}} = -K$$

$$F(s) = \frac{K}{s} + \frac{-K}{s+\frac{1}{\tau}}$$

$$f(t) = \mathcal{L}^{-1}\left(F(s)\right) = KU(t) - Ke^{-\frac{1}{\tau}t} = K\left(1 - e^{-\frac{1}{\tau}t}\right)$$

Laplace transform

Final Value Theorem and Its Derivation

$$\lim_{s\to 0}\left[\int_0^{\infty}\frac{d(f(t))}{dt}e^{-st}dt\right]=\lim_{s\to 0}\left[sF(s)-f(0)\right]$$

Bringing the $\lim$ inside the integral

$$\lim_{s\to 0}\left[\int_0^{\infty}\frac{d(f(t))}{dt}e^{-st}dt\right]=\int_0^{\infty}\frac{df(t)}{dt}\left(\lim_{s\to 0}\left[e^{-st}\right]\right)dt=\int_0^{\infty}1\frac{df(t)}{dt}dt=f(\infty)-f(0)$$

Combining the results gives

$$f(\infty)=\lim_{s\to 0}\left[sF(s)\right]$$

Final Value Theorem

Laplace transform

Conclusions:

- transformation from time domain to complex frequency domain and vise versa
- existence of transformation: Region Of Convergence (ROC)
- transformation is linear
- Transforms any Ordinary Linear Differential Equation (time domain) to a unique corresponding algebraic equation (Laplace or s-domain)
- useful for calculation of system output
- final value theorem enables direct calculation of the steady state response