1. Using the hint, we can create a bipartite map with all team vertices on one side and all day vertices on the other side. If a team wins on a certain day, we set up an edge between that team vertex and the day vertex. Now, we are trying to find a matching that satisfies all the day vertices, but not necessarily all the team vertices. Suppose the converse is true, that there does not exist such matching. Then for a subset A of day vertices, according to Hall's theorem, must have been connected to fewer team vertices. See below for graph illustration:

Figure 1. an example of a subset of day vertices, where more day vertices are connected to fewer team vertices. In this figure, over a period of four days, only three teams are the winning candidates.

Now that there are fewer wining candidates than the day vertices, but the total number of team vertices is greater than the total number of day vertices, we are guaranteed to find a team vertex p that lost every match during the period of the day vertices subset. This implies that p has lost to x many teams, such that the value of x equals the size of the day vertices subset. This means that there are at least x many different teams that are winning candidates over the period of day vertices. However, our assumption is that there are fewer than x winning candidates over the period of day vertices subset. Thus there is a contradiction. And this concludes our assumption is false. Namely, the claim that such a matching does not exist is false. Thus, there exists such a matching, such that all the day vertices are matched to some team vertices. This concludes that we can select a winning team for each day without having to select the same team twice.

2. To show that independent set is NP complete, we first show that we can verify it in polynomial time, namely, independent set belongs to NP. Suppose we are given a set of vertices V and a graph G, we can simply go through all the edges to check if each edge incident on at most one vertex in V, this will cost at most O(E) time, which is polynomial. Thus independent set belongs to NP.

Now we show the reduction of clique problem to independent set. Given a graph G and set of vertices V. we claim that there is a clique V in G if and only if there is an independent set V in the complement graph of G, namely, $\bar{G}$.

If V is a clique in G, then all the vertices in V are connected in G. Thus in the complement graph $\bar{G}$, all these edges are removed and there will be no edge between any vertices in V. Thus any edge can have at most one endpoint in set V. And thus V forms an independent set in $\bar{G}$.

If V is an independent set in $\bar{G}$, then any edge in $\bar{G}$ can only have one endpoint in V, namely, there cannot be an edge that connects two vertices in V. Thus in graph G, all edges between any two vertices of V will be added, and V will form a clique in G.

Thus we have shown that independent set belongs to NP, and can be reduced from the clique problem, which is NP. We can then conclude that independent set is NP.

3. The hint outlined two methods to solve this question. Since the second method is available online, we will use the first method to solve it.

1) set cover is reducible to hitting set

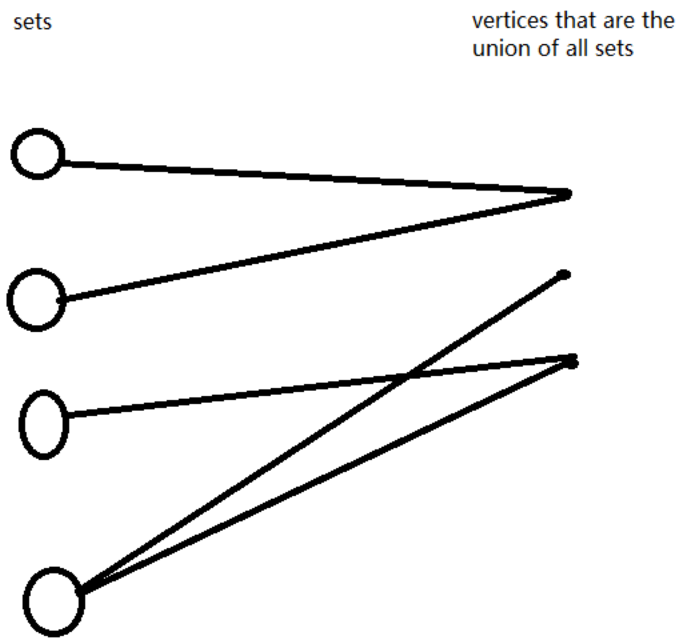

Figure 2. A sample set cover in this figure will be choosing the first and the fourth set on the left. This is reducible to the hitting set problem if we let each vertex x on the right become a set that contains sets: namely, a new and bigger set, which contains all the sets that x belongs to in the original graph. Figure 2 corresponds to graph G in the explanation below.

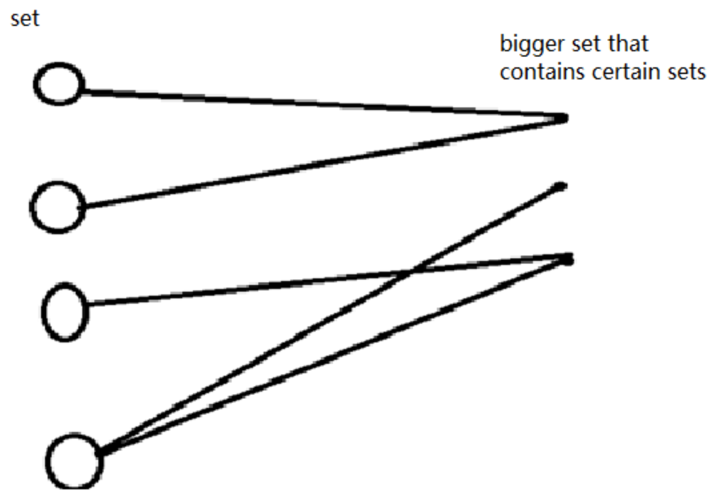

Figure 3. a hitting set problem reduced from figure 2. Figure 3 corresponds to G' in the explanation below.

We create a bipartite map G that has all sets on the left, and all vertices that belong to the union of all sets on the right. We set up an edge if the vertex on the right belongs to certain sets on the left. Note that all the sets must have at least one edge connected to it, because it is smaller than the union of all sets. On the other hand, all vertices on the right must connect to some sets, because they are generated by the union of all sets so each of them has to belong to some set. The problem of set cover wants to find a number k of sets on the left, such that the vertices these sets connect to on the right are saturated, namely, all vertices on the right are connected by some edges. Now we can create a new bipartite graph G', by replacing each vertex x on the right side of G with a new set y that contains all the sets x belong to in G. This will require polynomial time as we just go through all the nodes on the right side. Now we can view the left hand side of G' as elements of the union of all bigger sets on the right hand side, and view the right hand side nodes as all sets. We claim that G has a set cover of at most k if and only if G' has a hitting set of at most k.

If G has a set cover, let us call all right hand nodes of G the set of vertices V. Then there will be a way of connecting V to a set of left nodes of G, (which we will call it V'), such that V' is of size at most k. From G to G', all the edges will stay the same because the bigger sets are connected to the exact same sets. This implies that there will be a mapping from the bigger sets such that all the bigger sets are connected to some sets, and the total number of sets is at most k. This means that the sets on the left will form a hitting set of all the bigger sets on the right hand side of G' of size at most k.

If G' has a hitting set, this implies that there is a way of mapping such that all the bigger sets on the right side of G' is connected to some sets on the left side of G' of size at most k. From G' to G, since all the edges stay the same, this implies that there is a mapping in G such that all the vertices on the right side of G is connected to some sets on the left of G of size at most k. This means that all elements in the union of all sets are connected to some sets of at most k. Thus there is a set cover of size at most k in G.

Thus we have proved that set cover is reducible to hitting set.

(2)hitting set is reducible to dominating set

The graph illustration is on next page. We can create a bipartite map G that has all elements in the union of all sets on the left, and all sets on the right. We set up an edge if the element belongs to certain sets. And we connect all the elements on the left side of G by edges. This transformation can be done in polynomial time because it corresponds to the number of vertices. A hitting set in G means that there is mapping such that all nodes on the right side is connected to some nodes on the left of G. Now we claim that there is a hitting set in G if and only if there is a dominating set in G.

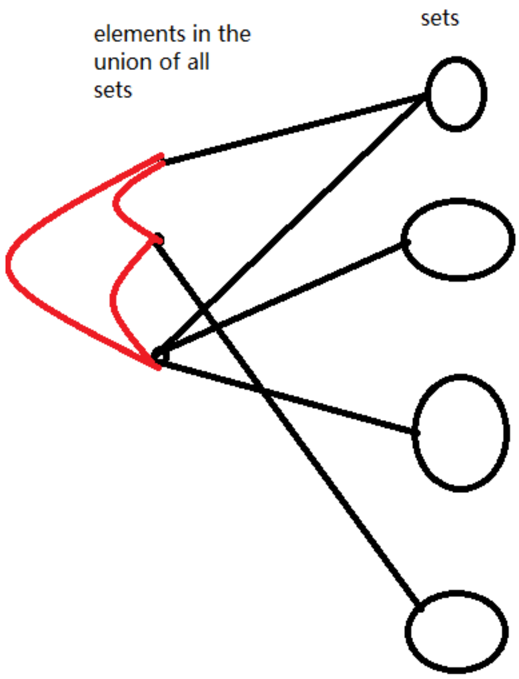

Figure 4. graph G such that all elements in the union of all sets are on the left, all sets are on the right, and all nodes on the left are connected by edges.

If G has a hitting set of size at most k, then there is a mapping such that all the right nodes are connected to some left nodes. Now the hitting set is itself a dominating set, because all the right nodes are connected to hitting sets and are thus dominated, and all left nodes are dominated because they are all connected. Thus the dominating set is at most of size k.

If G has a dominating set of size at most k, then we can move all the vertices in the dominating set that are on the right side of G to a node it connects to on the left side of G. After this operation, the set remains a dominating set because the right side nodes that are in the original dominating set can only dominate left side nodes, and thus if we move it to a left node it connects to, the original left nodes that it used to dominate will still be dominated because all the left nodes are all connected. This moving will not increase the size of k, and only possibly reduce k. After this operation, the dominating set will be composed of purely left side nodes. This implies that this set of left nodes are connected to all the right side nodes. Thus this set is also a hitting set of size at most k.

Thus we proved that hitting set is reducible to dominating set.

(3)dominating set is reducible to set cover

For each vertex v in graph G, we find all the points it connects to, and create a corresponding set s that contains vertex v and all the vertices it connects to. This can be done in polynomial time because we just iterate through the vertices and find its neighbors.

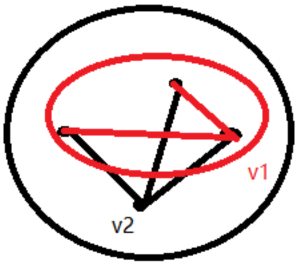

Figure 5. the red point v1 when considered, is reduced to the set of vertices in the red circle. The black point v2 when considered, is reduced to the set of vertices in the black circle.

Now we claim that the set V of vertices is a dominating set if and only if the corresponding set of sets S is a set cover of all vertices in G.

If V is a dominating set of size at most k, then all the vertices that are not in V is connected by an edge to V. This means that all vertices that are not in V will be added to certain set s when the vertex v in the dominating set it connects to is considered. Thus all the sets s that contains dominating set v and its neighboring vertices will contain all the vertices in G. Thus set S is a set cover of all vertices in G of size at most k.

If S is a set cover of size at most k, then we can find all center points v of s. Because S covers all the vertices, any vertex on the graph will appear in certain set s, and will be either the center point of s, or connected by an edge to the center point of s. Thus each vertex will either be in V or be connected by an edge to V. Thus V forms a dominating set of size at most k.

Thus we have proved that dominating set is reducible to set cover.

This concludes our proof that Set-Cover, Hitting-Set and Dominating-Set are polynomial-time reducible to each other.

4. We first show that Hamiltonian cycle is reducible to Hamiltonian path.
Let G be the graph we want to find if there exist a Hamiltonian cycle. For each edge x_i in G, we take off the edge and generate a graph G_i. We call the set of all E graphs from G_1 to G_E the graph G'. This will cost polynomial time because we will only run E iterations. Now we claim that there is a hamiltonian cycle in G if and only if there is a hamiltonian path in any sub-graph G_i of G'.

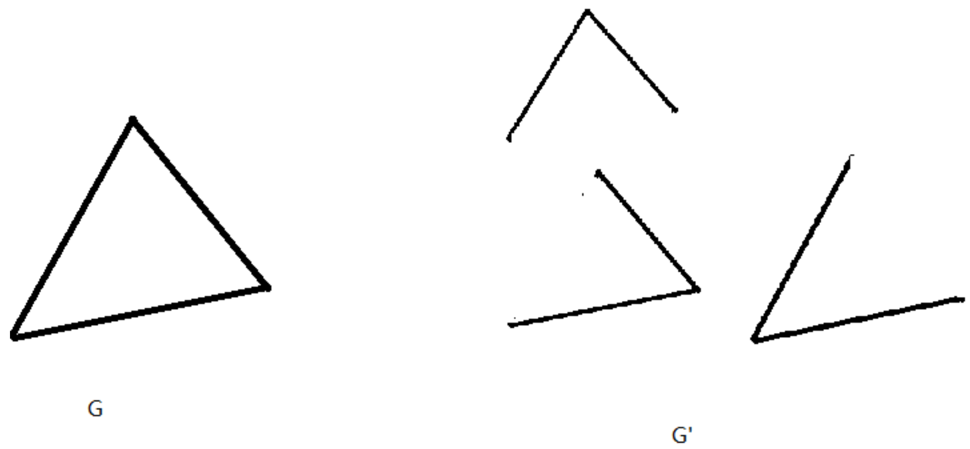

Figure 6. A sample graph G of a triangle that is reduced to G'. G' contains three sub-graphs, each contains two edges, corresponding to removing one edge from G.

If G has a Hamiltonian cycle, then removing one of the edges on the Hamiltonian cycle will generate a Hamiltonian path in G'.

If G' has a Hamiltonian path, then connecting the two endpoints of the Hamiltonian path will generate a Hamiltonian cycle in G.

Thus we have proved that Hamiltonian cycle can be reduced to Hamiltonian path.

Secondly we show that Hamiltonian path can be reduced to Hamiltonian cycle. Let the Hamiltonian path be from s to t, and the whole path be named G. We create a new graph G' that adds a node u in between s and t, and connect (s,u) and (u,t). This can be done in polynomial time because we only add a node and two edges. Now we claim that G has a Hamiltonian path if and only if G' has a Hamiltonian cycle.

If G has a Hamiltonian path, then adding edges (s,u) and (u,t) will generate a Hamiltonian cycle in G'.

If G' has a Hamiltonian cycle, the cycle must have visited node u because Hamiltonian cycle travels through each vertex exactly once. Bur since node u is only connected to s and t, the Hamiltonian cycle must have edges (s,u) and (u,t). Thus if we take off edges (s,u) and (u,t) from the Hamiltonian cycle in G', we will generate a Hamiltonian path between s and t in G.

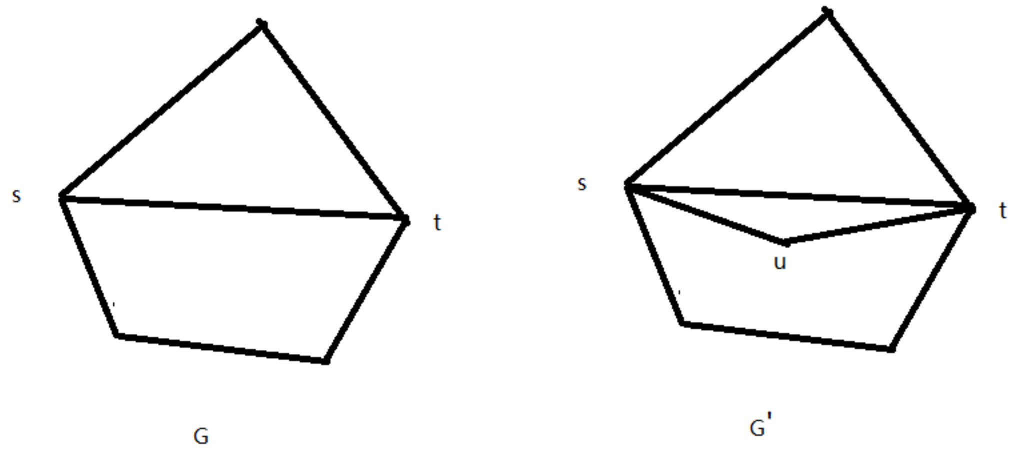

Figure 7. we transform G to G' by adding a node u between s and t, and connect edge (s,u) and (u,t).

Thus we have proved that Hamiltonian path is reducible to Hamiltonian cycle.

This concludes our proof that Hamiltonian path and Hamiltonian cycle problems are polynomial-time reducible to each other.