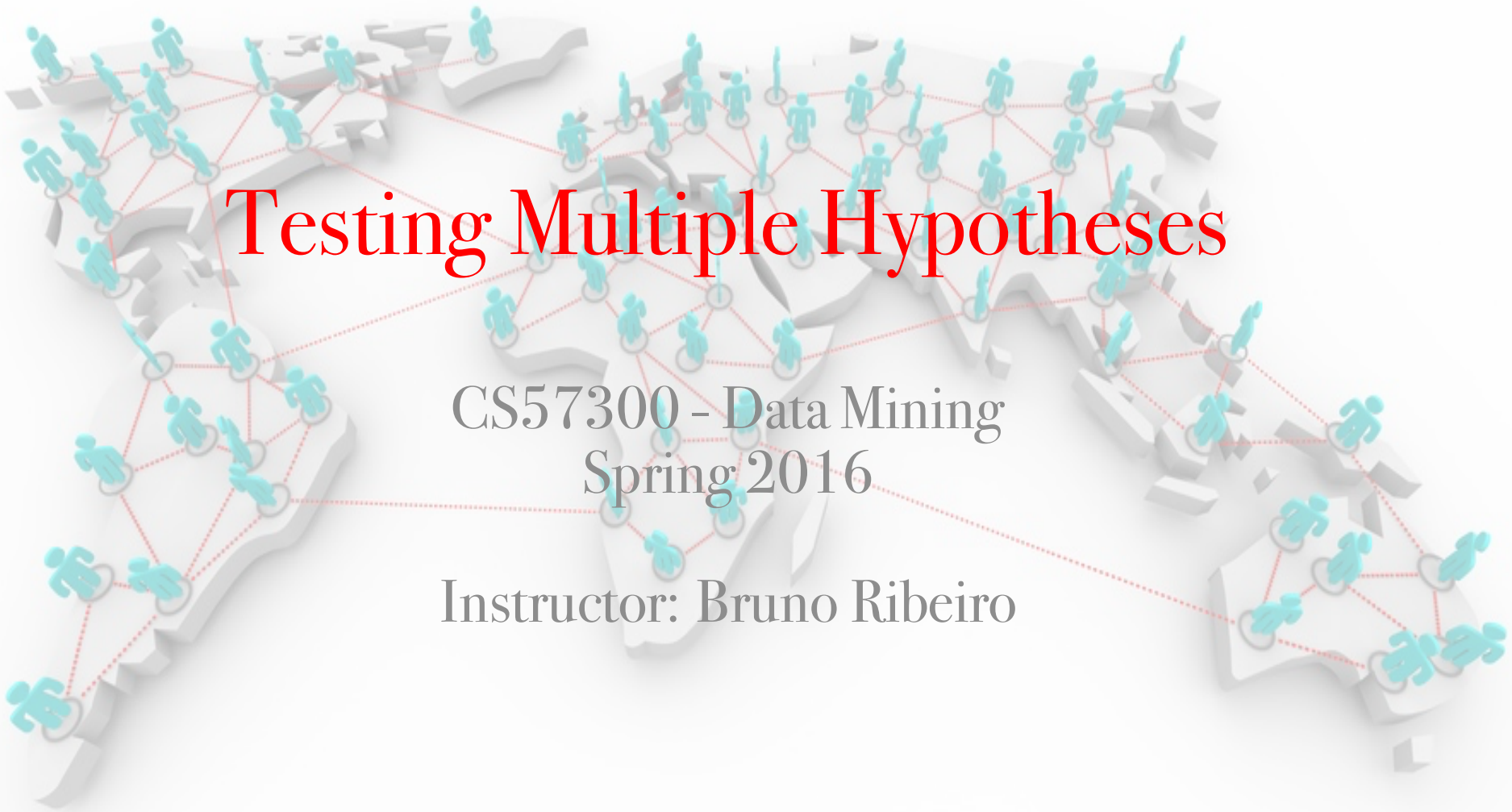

Testing Multiple Hypotheses

CS57300 - Data Mining
Spring 2016

Instructor: Bruno Ribeiro

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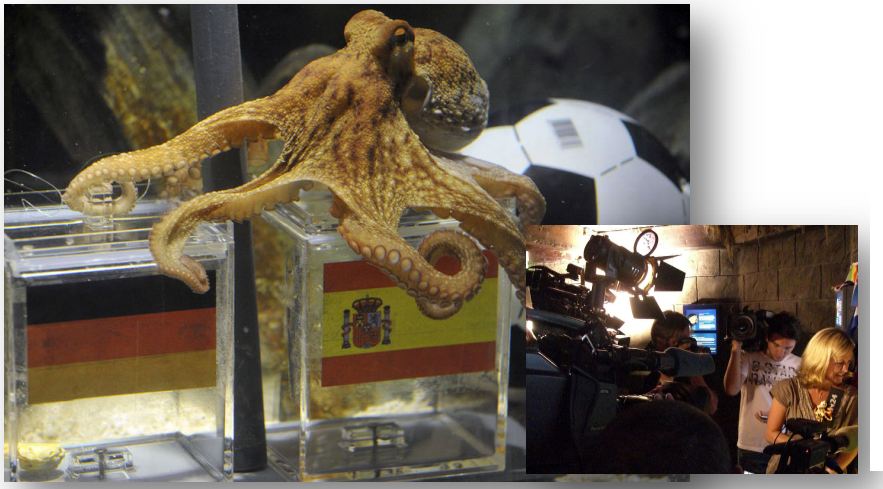
This Class:

- Testing Multiple Hypotheses
- Sequential Analysis
- Non-parametric Tests
- Independence Tests

Hypothesis Testing Example

Paul the Octopus (2008-2010)

- Paul was an animal oracle
- Paul's keepers would present him with two boxes containing food
- Whichever teams is in the box Paul chooses first is the predicted winner



Results involving Germany

Opponent	Tournament	Outcome
Poland	Euro 2008	Correct
Croatia	Euro 2008	Incorrect
Austria	Euro 2008	Correct
Portugal	Euro 2008	Correct
Turkey	Euro 2008	Correct
Spain	Euro 2008	Incorrect
Australia	World Cup 2010	Correct
Serbia	World Cup 2010	Correct
Ghana	World Cup 2010	Correct
England	World Cup 2010	Correct
Argentina	World Cup 2010	Correct
Spain	World Cup 2010	Correct
Uruguay	World Cup 2010	Correct

Hypothesis Testing Paul the Octopus as an Oracle

- Random variable (i.i.d.)

$$X_i = \begin{cases} 1 & \text{, if Paul predicts correct outcome} \\ 0 & \text{, otherwise} \end{cases}$$

- Variable of interest: $Y_{13} = \sum_{i=1}^{13} X_i$

- What is the Null Hypothesis?
  - Paul is not an animal oracle
  - Mathematical definition?
    - $H_0 := P[X_i = 1] = p = 0.5$

$$\Rightarrow P[Y_{13} = k|H_0] = \binom{13}{k} 0.5^k (1-0.5)^{n-k}$$

- Should we reject H_0 with significance level 0.05? (one-sided test)

$$P[Y_{13} \geq 11|H_0] = \sum_{k=11}^{13} \binom{13}{k} 0.5^k (1-0.5)^{n-k} = 0.0112 < 0.05$$

Anything Wrong in our Hypothesis Test?

Hypothesis Test Possible Outcomes

Actual Situation "Truth"

	H_0 True	H_0 False
Do Not Reject H_0	$P[H_0 \mid H_0]$ $1-\alpha$	Type II error (false negative) $P[H_0 \mid \neg H_0]$ β
Reject H_0	Type I error (false positive) $P[\neg H_0 \mid H_0]$ α	$P[\neg H_0 \mid \neg H_0]$ $1-\beta$

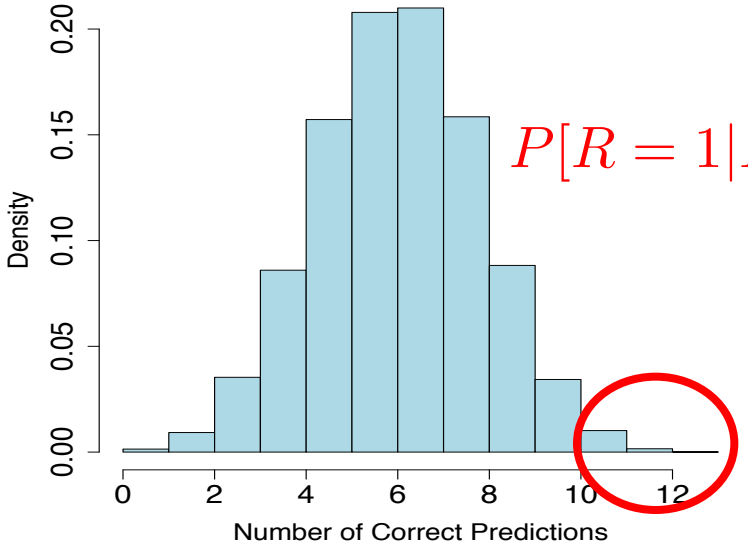
Hypothesis Test as Random Variable

$$X_i = \begin{cases} 1 & \text{, if Paul predicts correct outcome} \\ 0 & \text{, otherwise} \end{cases}$$

- R is random variable that defines if hypothesis is rejected
  if $P[Y_{13} \geq k | H_0] < 0.05$ then $R = 1;$ otherwise $R = 0$

k correct predictions by animal

Binomial Distribution (p=0.5)

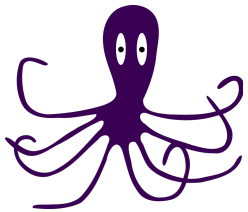


$$P[R = 1 | H_0] = P[Y_{13} \geq 10 | H_0] = 0.046$$

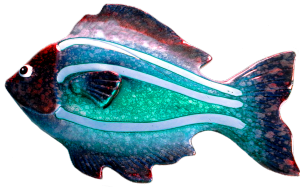
Testing Multiple Hypotheses

Familywise Error

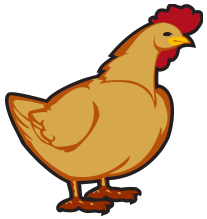
(probability of rejecting a true hypothesis in multiple hypotheses tests)



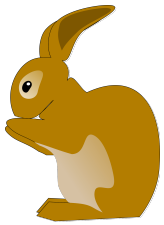
Paul



Peter



Paloma



Philis

- Probability we reject "not an oracle" hypothesis of Paul based on chance alone?

$$P[R = 1|H_0] = 0.046$$

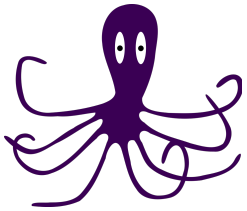
- Probability we reject "not an oracle" hypothesis of one or more animals (Paul, Peter, Paloma,Philis)

$$1 - \underbrace{(1 - P[R = 1|H_0])}^{4} = 0.17$$

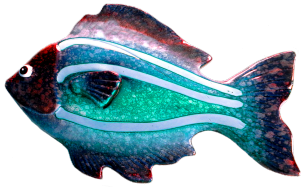
$P[R=0|H_0]^4$ = Probability we correctly reject all 4 hypotheses

Bonferoni's correction

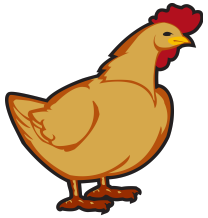
- Used when there aren't too many hypotheses
- Tends to be too conservative for large number of hypotheses



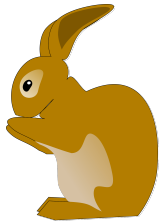
Paul



Peter



Paloma



Philis

- Per-hypothesis significance level of m hypotheses: α/m
- In our animal oracle example:
  - Old significance level α=0.05
  - Bonferoni's corrected significance level α'=0.05/4 = 0.0125
  - Hypothesis test: "Paul is not an animal oracle"

$$P[Y_{13} \geq 11 | H_0] = \sum_{k=11}^{13} \binom{13}{k} 0.5^k (1-0.5)^{n-k} = 0.0112 < 0.0125$$

- Effect on $P[H_0|\neg H_0]$?

False Discovery Rate

- Often used for large number of tests
- Bonferoni's correction seeks to ensure that no true hypotheses are rejected
  - Low statistical power for large number of hypotheses (rejects no hypotheses n >> 1)
- False Discovery Rate:
  - Controls: $mP[\neg H_0 | H_0]$
  - Greater statistical power at expense of more false positives
  - Order p-values of all m tests: $p_1 \leq p_2 \leq \cdots \leq p_m$
  - Holm's Method:
    - $\tilde{p}_i = \min((m - i + 1)p_i, 1)$
    - Reject if adjusted p-value < α
  - Benjamini-Hochberg method:
    - Reject j null hypothesis if $p_j \leq \alpha \frac{j}{m}$

Sequential Analysis

Motivation: *The New York Times* Dilemma

- Select 50% users to see headline A
  - Titanic Sinks
- Select 50% users to see headline B
  - Ship Sinks Killing Thousands

"All the News That's Fit to Print."

The New York Times.

THE WEATHER.

NEW YORK, TUESDAY, APRIL 16, 1912.—TWENTY-FOUR PAGES. ONE CENT

TITANIC SINKS FOUR HOURS AFTER HITTING ICEBERG; 866 RESCUED BY CARPATHIA, PROBABLY 1250 PERISH; ISMAY SAFE, MRS. ASTOR MAYBE, NOTED NAMES MISSING

Col. Astor and Bride, Isidor Straus and Wife, and Maj. Butt Aboard.

"RULE OF SEA" FOLLOWED

Women and Children Put Over in Lifeboats and Are Supposed to be Safe on Carpathia.

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FRANKLIN HOPEFUL ALL DAY

Manager of the Line Insisted Titanic Was Unsinkable Even After She Had Gone Down.

HEAD OF THE LINE ABOARD

J. Bruce Ismay Making First Trip on Gigantic Ship That Was to Surpass All Others.

Biggest Liner Plunges to the Bottom at 2:20 A. M.

RESCUERS THERE TOO LATE

Except to Pick Up the Few Hundreds Who Took to the Lifeboats.

WOMEN AND CHILDREN FIRST

Cunarder Carpathia Rushing to New York with the Survivors.

SEA SEARCH FOR OTHERS

The California Stands By on Chance of Picking Up Other Boats or Rafts.

OLYMPIC SENDS THE NEWS

Only Ship to Flash Wireless Messages to Shore After the Disaster.

- Hypothesis?
  - H_0: % page views A = % page views B
- Assign half the readers to headline A and half to headline B?

If A is **much** better than B then we could reject hypothesis H_0 quickly

Reject hypothesis before end of experiment

- Early hypothesis rejection
- We should not have to wait to declare hypothesis is rejected
- Problem:

Consider experiment order

Opponent	Tournament	Outcome
Uruguay	World Cup 2010	Correct
Spain	World Cup 2010	Correct
Argentina	World Cup 2010	Correct
England	World Cup 2010	Correct
Ghana	World Cup 2010	Correct
Serbia	World Cup 2010	Correct
Australia	World Cup 2010	Correct
Spain	Euro 2008	Incorrect
Croatia	Euro 2008	Incorrect
Poland	Euro 2008	Correct

Results involving Germany

Complete experiment:

$$P[Y_{13} \geq 11|H_0] = \sum_{k=11}^{13} \binom{13}{k} 0.5^k (1-0.5)^{n-k} = 0.0112 \not< 0.01$$

First 7 of Paul's predictions:

$$P[Y_7 = 7|H_0] = 0.5^7 = 0.0078 < 0.01$$

Sequential Analysis

Example Application: Jung et al. 2004

- Detect whether a remote host is a port scanner or a benign host
- Ground truth: based on percentage of local hosts which a remote host has a failed connection
- Example of parameters:
  - for a scanner, the probability of hitting inactive local host is 0.8
  - for a benign host, that probability is 0.1

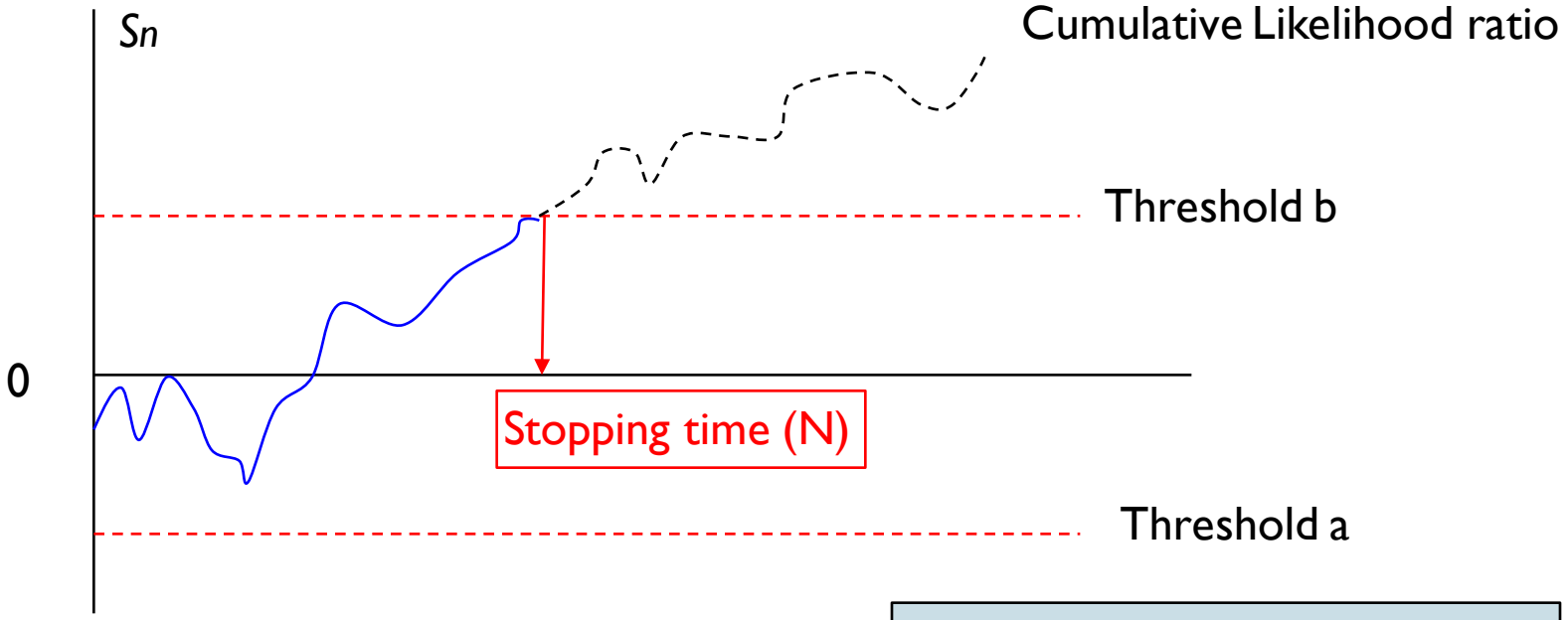
Ack: M. Jordan

Wald's Sequential Probability Ratio Test

- Starts with two hypotheses
- Example:
  - A remote host attempts to connect a local host at time *i*
    let $X_i = 0$ if the connection attempt is a success,
    1 if failed connection
  - As outcomes $X_1, X_2, \ldots$ are observed we wish to determine whether host is a scanner or not
  - Two competing hypotheses:
    - H0: remote host is benign: $P[X_i = 1|H_0] = 0.1$
    - H1: remote host is a scanner: $P[X_i = 1|H_1] = 0.8$

- Calculate the cumulative sum of the log-likelihood ratio
  - $S_0 = 0$
  - $S_i = S_{i-1} + \log P[X_i|H_0] - \log P[X_i|H_1]$

Thresholds vs. errors



- False alarm rate $\alpha = P[R = 1|H_0]$
- Misdetection rate $\beta = P[R = 0|H_1]$

Wald's approximation :

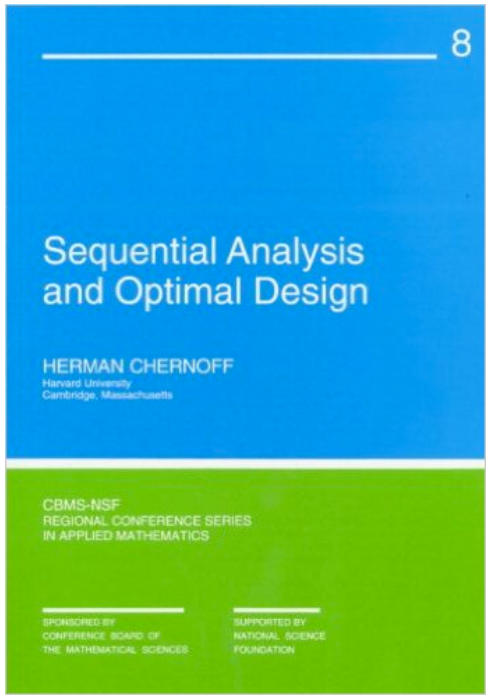
$$a \geq \log\frac{\beta}{1-\alpha} \quad \Rightarrow a \approx \log\frac{\beta}{1-\alpha}$$

$$b \leq \log\frac{1-\beta}{\alpha} \quad \Rightarrow \quad b \approx \log\frac{1-\beta}{\alpha}$$

$$\text{So, } \alpha \approx \frac{1-e^{a}}{e^{b}-e^{a}} \quad \text{and} \quad \beta \approx \frac{e^{-b}-1}{e^{-b}-e^{-a}}$$

Sequential Analysis (Sequential Hypothesis Test)

- Variety of techniques
- Random walk-based methods



Back to Single (Two-sample) Hypothesis Tests

Nonparametric Test Procedures

- Not Related to Population Parameters
  Example: Probability Distributions, Independence

- Data Values not Directly Used
  Uses Ordering of Data

Examples:
  Wilcoxon Rank Sum Test , Komogorov-Smirnov Test

Example of Nonparametric Test

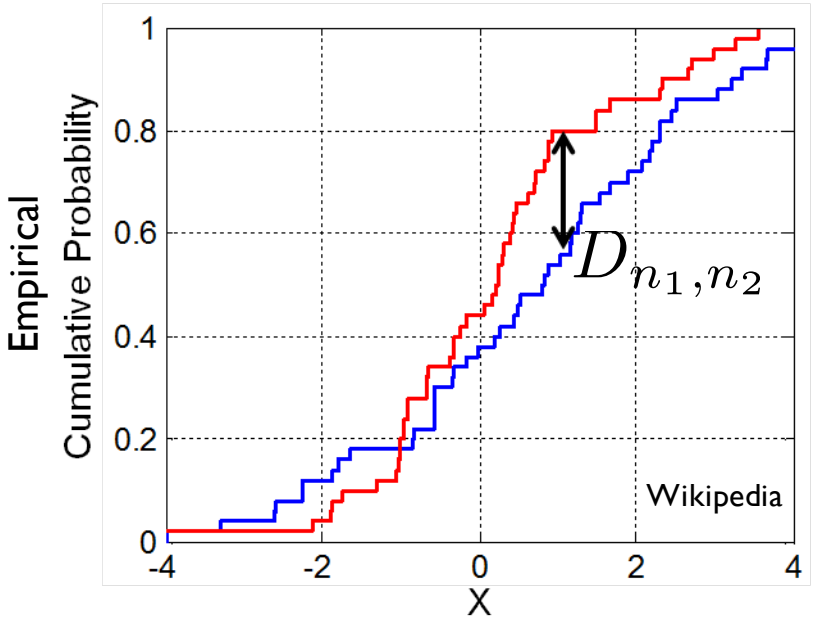
Nonparametric Testing of Distributions

- Two-sample Kolmogorov-Smirnov Test
  - Do $\mathbf{X}^{(0)}$ and $\mathbf{X}^{(1)}$ come from same underlying distribution?
  - Hypothesis (same distribution) rejected at level p if

$$D_{n_1,n_2} > c(p)\sqrt{\frac{n_1+n_2}{n_1 n_2}}$$

Sample size correction

Confidence interval factor



The K-S test is less sensitive when the differences between curves is greatest at the beginning or the end of the distributions. Works best when distributions differ at center.

Good reading:

M. Tygert, Statistical tests for whether a given set of independent, identically distributed draws comes from a specified probability density. *PNAS 2010*

Are Two User Features Independent?

Chi-Squared Test

- Twitter users have features gender and number of tweets.
- We want to determine whether gender is related to number of tweets.
- Use chi-square test for independence

When to use Chi-Squared test

- When to use chi-square test for independence:
  - Uniform sampling design
  - Categorical features
  - Population is significantly larger than sample

- State the hypotheses:
  - H_0 ?
  - H_1 ?

Example Chi-Squared Test

```
men = c(300, 100, 40)
women = c(350, 200, 90)

data = as.data.frame(rbind(men, women))

names(data) = c('low', 'med', 'large')

data

chisq.test(data)
```

Reject H_0 ($p<0.05$) means …

Next Class

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Only Ship to Flash Wireless Messages to Shore After the Disaster.

- If we just care about page visitors we never need to decide which one is best