Pre-lecture Problems for Lecture 12: Lagrange Duality and the KKT Condition

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1. (10 points; 2 points each) Consider the following nonlinear program

$$\begin{aligned} \min \quad & (x_1 - 3)^2 + (x_2 - 2)^2 \\ \text{s.t.} \quad & x_1 + 2x_2 \geq 10. \end{aligned}$$

   (a) Prove or disprove that the NLP is a convex program.

   Ans. The Hessain matrix of the objective function is

$$\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

   which is positive semi-definite. Therefore, the NLP is convex.

   (b) Find the Lagrangian of this NLP. What is the sign constraint for your Lagrangian mulitplier?

   Ans.

$$\mathcal{L}(x|\lambda) = (x_1 - 3)^2 + (x_2 - 2)^2 + \lambda(10 - x_1 - 2x_2)$$

   Since it is a minimization, so the sign constraint for the Lagrangian multiplier is $\lambda \geq 0$.

   (c) According to the FOC of the Lagrangian, find a necessary condition for any optimal solution.

   Ans.

   The FOC of the Lagrangian is

$$\begin{cases} \frac{\partial L}{\partial x_1} = 2(x_1 - 3) + \lambda = 0 \\ \frac{\partial L}{\partial x_2} = 2(x_2 - 2) + 2\lambda = 0 \end{cases}$$

   if $\lambda = 0$, then $x_1 = 3, x_2 = 2$, but it is not feasible; if $\lambda > 0$, then $x_1 = \frac{18}{5}, x_2 = \frac{16}{5}$ and $\lambda = \frac{6}{5}$, which is feasible, so this is a necessary condition.

(d) Find an optimal solution for the NLP.

Ans. From (c), we have an optimal solution $(x_1, x_2) = (\frac{18}{5}, \frac{16}{5})$, and the optimal value is 1.8.