Pre-lecture Problems for Lecture 9: Sensitivity Analysis and Dual Simplex Method

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2. (10 points) Consider an LP:

$$\begin{aligned}\max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 16 \\ & x_1 + 2x_2 \leq 20 \\ & x_1 \geq 0, x_2 \geq 0\end{aligned}$$

(a) Find an optimal solution by the simplex method and its optimal tableau.

Ans.

Transform the LP to standard form:

$$\begin{aligned}\max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 + x_3 = 16 \\ & x_1 + 2x_2 + x_4 = 20 \\ & x_i \geq 0, \forall i = 1, 2, 3, 4\end{aligned}$$

Solve the LP by simplex method:

$$\begin{array}{cccc|c} -3 & -5 & 0 & 0 & 0 \\ \hline 1 & 1 & 1 & 0 & x_3 = 16 \\ 1 & 2 & 0 & 1 & x_4 = 20 \end{array} \rightarrow \begin{array}{cccc|c} 0 & -2 & 3 & 0 & 48 \\ \hline 1 & 1 & 1 & 0 & x_1 = 16 \\ 0 & 1 & -1 & 1 & x_4 = 4 \end{array} \rightarrow \begin{array}{cccc|c} 0 & 0 & 1 & 2 & 56 \\ \hline 1 & 0 & 2 & -1 & x_1 = 12 \\ 0 & 1 & -1 & 1 & x_2 = 4 \end{array}$$

The optimal solution is $(x_1, , x_2, x_3, x_4) = (12, 4, 0, 0)$ and the optimal value is 56.

(b) Suppose that the LP becomes:

$$\begin{aligned} \max \quad & 3x_1 + 5x_2 \\ \text{s.t.} \quad & x_1 + x_2 \le 16 \\ & x_1 + 2x_2 \le 20 \\ & 3x_1 + 3x_2 \le 42 \\ & x_1 \ge 0, x_2 \ge 0 \end{aligned}$$

Ans.

Let $B = (x_1, x_2, s_3), N = (s_1, x_2)$. We get $c_B = \begin{bmatrix} 3 \\ 5 \\ 0 \end{bmatrix}, c_N = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, A_B = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 0 \\ 3 & 3 & 1 \end{bmatrix},$

$A_N = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, b = \begin{bmatrix} 16 \\ 20 \\ 42 \end{bmatrix}.$

Then $A_B^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix}, A_b^{-1} A_N = \begin{bmatrix} 2 & -1 \\ -1 & 1 \\ -3 & 0 \end{bmatrix}, A_B^{-1} b = \begin{bmatrix} 12 \\ 4 \\ -6 \end{bmatrix},$

$c_B^T A_B^{-1} A_N - c_N^T = \begin{bmatrix} 1 & 2 \end{bmatrix}, c_B^T A_B^{-1} b = 56$

We can get a tableau:

$$\begin{array}{ccccc|c} 0 & 0 & 1 & 2 & 0 & 56 \\ \hline 1 & 0 & 2 & -1 & 0 & 12 \\ 0 & 1 & -1 & 1 & 0 & 4 \\ 0 & 0 & -3 & 0 & 1 & -6 \end{array} \rightarrow \begin{array}{ccccc|c} 0 & 0 & 1 & 2 & 0 & 56 \\ \hline 1 & 0 & 2 & -1 & 0 & 12 \\ 0 & 1 & -1 & 1 & 0 & 4 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 2 \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & 0 & 0 & 2 & \frac{1}{3} & 54 \\ \hline 1 & 0 & 0 & -1 & \frac{2}{3} & 8 \\ 0 & 1 & 0 & 1 & -\frac{1}{3} & 6 \\ 0 & 0 & 1 & 0 & -\frac{1}{3} & 2 \end{array}$$

The optimal solution is $(x_1, x_2, s_1, s_2, s_3) = (8, 6, 2, 0, 0)$ and the optimal value is 54.