Operations Research, Spring 2024 (112-2)
Homework 0

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February 25, 2024

1 Rules

- This homework is due at ***23:59, March 2***. Those who submit their works late but are late by less than one hour get 10 points off. Works that are late more than one hour get no point.
- For this homework, students should work individually. While discussions are encouraged, copying is prohibited.
- Please submit a ***PDF file*** through NTU COOL and make sure that the submitted work contains the student ID and name. Those who fail to do these will get 10 points off.
- You are ***required*** to ***type*** your work with a text processor with a formula editor or LaTeX (***strongly suggested***). Hand-written works are not accepted. You need to make your work professional in mathematical writing by following at least the following rules:[1]
  1. When there is a symbol denoted by an English letter, make it italic. For example, write $a + b = 3$ rather than a + b = 3.
  2. An operator (e.g., +) should not be italic. A function with a well-known name (e.g., log, max and sin) is considered as an operator.
  3. A number should not be italic. For example, it should be $a + b = 3$ rather than $a + b = \mathit{3}$.
  4. Superscripts or subscripts should be put in the right positions. For example, a_1 and $a1$ are completely different: The former is a variable called a_1 while the latter is actually $a \times 1$.
  5. When there is a subtraction, write $-$ rather than -. For example, write $a - b = 3$ rather than a - $b = 3$. The same thing applies to the negation operator. For example, write $a = -3$ rather than a = -3.
  6. If you want to write down the multiplication operator, write $\times$ rather than *.
  7. For an exponent, write it as a superscript rather than using ^. For example, write 10^2 rather than 10^2.
  8. There should be proper space beside a binary operator. For example, it should be $a + b = 3$ rather than a+b=3.

  Those who fail to follow these rules may get at most 10 points off.
- As we may see, there are many students, many problems, but only a few TAs. Therefore, when the TAs grade this homework, it is possible for only some problems to be randomly selected and graded. For all problems, detailed suggested solutions will be provided.

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[1] A more complete list of formatting rules and useful LaTeX templates are on NTU COOL.

2 Special notes regarding ChatGPT

- This assignment is to test your background knowledge and to make you warm up for typesetting.
- You may see that you may input some problems into ChatGPT to obtain answers. In fact, we encourage you to try ChatGPT and make your own judgment about whether an answer is correct and clear enough. If you believe that it is correct, you may utilize the answer made by ChatGPT to make your own answer. This is not considered as plagiarism in this assignment.
- Of course, we suggest you to do all the problems by yourself to improve yourself. It will be much better if you do so and use ChatGPT only to check the correctness of your answers (again, you need to make your own judgment regarding the correctness of the answers given by ChatGPT). After all, if you use ChatGPT to solve easy problems, and ChatGPT cannot solve a hard problem, you will have no way to solve hard problems.
- Just to let you know: When the instructor tried to use ChatGPT to solve the subproblems, ChatGPT gave wrong answers to some of them. Please be careful in using ChatGPT.

3 Special notes regarding typesetting

- For this assignment, the TAs will spend huge efforts to let you know where you did not do well in typesetting. This is to help you know the universal standard of professional writing and typesetting.
- Starting from the next assignment, we will deduct points if you do not do well in typesetting. However, as many of you are not familiar with such a standard, for this assignment we do not deduct points due to imperfect typesetting. Please take this chance to get familiar with typesetting tools (LaTeX or any other appropriate ones) and the standard.

4 Problems

1. (30 points; 10 points each) Solve the following problems.
   (a) Let $f(x) = ax^2 + 8x + 6$, where $a \in \mathbb{R}$. Find all values of a such that $f(x)$ is maximized at $x = 2$.
   (b) Let $f(x) = ax^2 + 8x + 6$, where $a \in \mathbb{R}$. Find $F(t) = \int_0^t f(x)dx$ as a function of a and t for all $t > 0$.
   (c) Find all values of $a \in \mathbb{R}$ such that the inverse of
   $$\begin{bmatrix} 1 & 0 & 1 \\ a & 1 & 2 \\ 3 & 1 & 4 \end{bmatrix}$$
   does not exist. If for all values of a the inverse exists, prove it.
2. (20 points; 10 points each) Consider the problem of determining whether a given integer n is a prime number.
   (a) Write down a pseudocode of an algorithm that solves the problem for any given positive integer n. If you do not know what a pseudocode is, your may choose to write a real program in C++, Python, Java, or any modern language you like. Please indicate the language of your program. Still, as this term will show up again in this course, please teach yourself what a pseudocode is.
   (b) What is the time complexity of your algorithm? Please use the big-O notation to express your solution.

3. (30 points; 10 points each) For each subproblem, use the graphical approach to solve

$$\begin{aligned} \max \quad & 2x_1 + Ax_2 \\ \text{s.t.} \quad & x_1 + x_2 \leq 8 \\ & 2x_1 - x_2 \geq 12 \\ & x_2 \leq B \\ & x_1 \geq 0, x_2 \geq 0, \end{aligned}$$

for the given values of A and B. If there are multiple optimal solutions, please list just one of them. As long as there is at least one optimal solution, write one down and also list all constraints binding at that optimal solution. If there is no optimal solution, graphically demonstrate it.

(a) $A = 1$, $B = -4$.

(b) $A = 5$, $B = 4$.

(c) $A = -1$, $B = 1$.

4. (20 points; 10 points each) For each of the following subproblems, formulate a linear program that maximizes IEDO's profits for the next year.

(a) IEDO Oil has refineries in Kaohsiung and Taipei. Currently, the Kaohsiung refinery can refine up to K_1 million barrels of oil per year, and the Taipei refinery up to K_2 million. Once refined, oil is shipped to two distribution points: Hsinchu and Taichung. IEDO Oil estimates that each distribution point can sell up to D million barrels per year. Because of differences in shipping and refining costs, the profit earned per million barrels of oil shipped depends on where the oil was refined and on the point of distribution. In particular, the profit per million barrels is P_{11} from Kaohsiung to Hsinchu, P_{12} from Kaohsiung to Taichung, P_{21} from Taipei to Hsinchu, and P_{22} from Taipei to Taichung.

(b) IEDO Oil has refineries in n cities. Currently, the refinery in city i can refine up to K_i million barrels of oil per year. Once refined, oil is shipped to m distribution points. IEDO Oil estimates that each distribution point can sell up to D million barrels per year. Because of differences in shipping and refining costs, the profit earned per million barrels of oil shipped depends on where the oil was refined and on the point of distribution. In particular, the profit per million barrels is P_{ij} from refinery in city i to distribution point j.