Pre-lecture Problems for Lecture 3: Integer Programming

B10705034 資管三 許文鑫

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1. (0 point) Consider the following integer program

$$
\begin{aligned}
\max\ & x_1 + x_2 \\
\text{s.t.}\ & 2x_1 + x_2 - 6 \leq M_1 z \\
& x_1 + 2x_2 - 8 \leq M_2(1 - z) \\
& x_1 \leq 10 \\
& x_2 \leq 10 \\
& x_i \geq 0 \quad \forall i = 1, 2 \\
& z \in \{0, 1\},
\end{aligned}
$$

where M_1 and M_2 are parameters and x_1, x_2, and z are variables. The binary variable z is to select at least one constraint to be satisfied.

(a) What values of M_1 and M_2 can enable z to do the "at-least-one" selection?

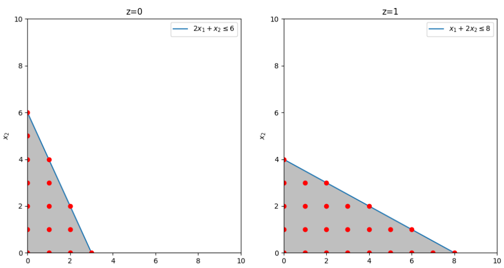
Ans. $M_1 = 24, M_2 = 22$

(b) Depict the feasible region on the (x_1, x_2) plane. Then graphically solve the IP.

Hint. The feasible region is nonconvex on the (x_1, x_2) plane. To solve this problem graphically, no branch-and-bound is needed.

Ans.

$x_1 = 8, x_2 = 0$



2. (0 point) Six towns locate at the following six points on a Cartesian plane: $(0, 60)$, $(20, 50)$, $(30, 20)$, $(40, 80)$, $(50, 50)$, and $(90, 60)$ (in km). Currently a company owns six retail stores, one in each town. The weekly sales of a product in these stores are 10000, 15000, 12000, 8000, 20000, and 3000. The company currently has one distribution center (DC) in town 3. It plans to build some more DCs for the retail stores to replenish from. The construction costs of building a DC in these towns are \$200000, \$180000, \$160000, \$190000, \$150000, and \$200000. A truck can carry 500 units of this product. The shipping cost for a truck to move 1 km is \$1. The existing DC and any newly built DC can be used for 5 years. There is no capacity limit for a DC. Transportation between two locations can be done by traveling through the straight line connecting them.

   (a) Formulate the problem of minimizing the 5-year total construction and shipping costs as an integer program if DCs can only be built in towns.

   Ans. Let x_{ij} be the amount of product shipped from town i to town j, and y_i be the binary variable indicating whether a DC is built in town i, c_{ij} be the distance between town i and j, c_i be the costs of building a DC in town i, s_j be the weekly sales of a product in the store of town j. Then the integer program is

$$\begin{aligned} \min \quad & \sum_{i=1}^{6} \sum_{j=1}^{6} c_{ij} \times \lceil \frac{x_{ij}}{500} \rceil + \sum_{i=1}^{6} y_i c_i \\ \text{s.t.} \quad & \sum_{i=1}^{6} x_{ij} \geq s_j \quad \forall j = 1, 2, 3, 4, 5, 6 \\ & y_i \in \{0, 1\} \quad \forall i = 1, 2, 3, 4, 5, 6 \end{aligned}$$

   (b) Suppose DCs may also be built in the following locations: $(0, 20)$, $(20, 40)$, $(40, 30)$, and $(60, 40)$. Do Part (a) again.

   (c)

3. (10 point; 5 points each) Ten jobs should be scheduled on one single machine. The processing times for these jobs are $7, 4, 3, 9, 10, 6, 8, 9, 7,$ and 10 (in hours). The due times for these jobs are $40, 43, 45, 46, 49, 50, 52, 57, 58,$ and 60 (in hours).

   (a) Suppose that we want to minimize the total ***tardiness*** of all jobs. For a job, its tardiness is its completion time minus its due time if this quantity is positive or zero otherwise. For the schedule (1,2,3,4,5,6,7,8,9,10), which means first processing job 1, then job 2 right after job 1 is done, then job 3 right after job 2 is done, etc., calculate the total tardiness.

   Ans.

The completion time of these job is $7, 11, 14, 23, 33, 39, 47, 56, 63, 73$, and the tardiness of these job is $0, 0, 0, 0, 0, 0, 0, 0, 5, 13$, so the total tardiness is 18.

(b) Formulate an integer program that finds a schedule to minimize the total tardiness of all jobs.

Ans.

Let w_j be the amount of tardiness of job j, p_j be the processing time of job j, d_j be the due time of job j, x_j be the ocmpletion time of job j. Then the integer program is

$$
\begin{aligned}
\min\ & \sum_{j=1}^{10} w_j \\
\text{s.t. } & w_j \geq 0 \quad \forall j = 1, 2, ..., 10 \\
& w_j \geq x_j - d_j \quad \forall j = 1, 2, ..., 10 \\
& x_j \geq p_j \quad \forall j = 1, 2, ..., 10 \\
& x_j \geq 0 \quad \forall j = 1, 2, ..., 10 \\
& x_i + p_j - x_j \leq M z_{ij} \quad \forall j = 1, 2, ...10, \forall i = 1, 2, ..., 10, i < j \\
& x_j + p_i - x_i \leq M(1 - z_{ij}) \quad \forall j = 1, 2, ...10, \forall i = 1, 2, ..., 10, i < j \\
& z_{ij} \in \{0, 1\} \quad \forall j = 1, 2, ...10, \forall i = 1, 2, ..., 10, i < j
\end{aligned}
$$