Operations Research, Spring 2024 (112-2)
Homework 1

Ling-Chieh Kung*

March 4, 2024

1 Rules

- This homework is due at ***23:59, March 16***. Submissions late by no more than twelve hours get 10 points off; those late by no more than 24 hours get 20 points off; those late by more than 24 hours get no point.
- For this homework, students should work individually. While discussions are encouraged, copying is prohibited.
- Please submit a ***PDF file*** through NTU COOL and make sure that the submitted work contains the student ID and name. Those who fail to do these will get 10 points off.
- You are ***required*** to ***type*** your work with LaTeX (***strongly suggested***) or a text processor with a formula editor. Hand-written works are not accepted. You are responsible to make your work professional in mathematical writing by following at least the following rules:[1]

  1. When there is a symbol denoted by an English letter, make it italic. For example, write $a + b = 3$ rather than a + b = 3.
  2. An operator (e.g., +) should not be italic. A function with a well-known name (e.g., log, max and sin) is considered as an operator.
  3. A number should not be italic. For example, it should be $a + b = 3$ rather than $a + b = \mathit{3}$.
  4. Superscripts or subscripts should be put in the right positions. For example, a_1 and $a1$ are completely different: The former is a variable called a_1 while the latter is actually $a \times 1$.
  5. When there is a subtraction, write $-$ rather than -. For example, write $a - b = 3$ rather than a - $b = 3$. The same thing applies to the negation operator. For example, write $a = -3$ rather than $a =$ -3.
  6. If you want to write down the multiplication operator, write $\times$ rather than *.
  7. For an exponent, write it as a superscript rather than using ^. For example, write 10^2 rather than 10^2.
  8. There should be proper space beside a binary operator. For example, it should be $a + b = 3$ rather than $a{+}b{=}3$.

  Those who fail to follow these rules will be points deducted.

- As we may see, there are many students, many problems, but only a few TAs. Therefore, when the TAs grade this homework, it is possible for only some problems to be randomly selected and graded. For all problems, detailed suggested solutions will be provided.

*Department of Information Management, National Taiwan University. E-mail: lckung@ntu.edu.tw.

[1] A more complete list of formatting rules is on NTU COOL.

2 Problems

Here are some technical notes before you start. When we say a "linear program", it cannot contain nonlinear functions and integer variables; when we say an "integer program", it cannot contain nonlinear functions; when we say a "nonlinear program", it cannot contain integer variables. Moreover, nonlinear functions include maximum functions, minimum functions, multiplication of decision variables, division of decision variables, among others.

1. (30 points; 10 points each) During the next n months, the IEDO company has demand D_t for air conditioners in month t, $t = 1, ..., n$. Air conditioners can be produced in m sites. It takes L_i hours of skilled labor to produce an air conditioner in site i, $i = 1, ..., m$. It costs C_i^P to produce an air conditioner in site i, $i = 1, ..., m$. During each month, each city has K hours of skilled labor available. It costs C^H to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has I air conditioners in stock.

   (a) Suppose that IEDO must meet all the demands on time. Formulate a linear program whose solution will tell IEDO how to minimize the cost of meeting air conditioner demands for the next six months.

   (b) Suppose that those demands are the maximum number that IEDO may sell, and IEDO can decide the sales quantity in each month. Each air conditioner can be sold at \$600. Formulate an LP whose solution maximizes the profit of selling air conditioners for the next six months. Your formulation must be a compact one.

   (c) Continue from Part (a) and solve the linear program to obtain an optimal solution according to the following information. During the next six months, the IEDO company has following demands for air conditioners: month 1, 2500; month 2, 4000; month 3, 4500; month 4, 4200; month 5, 3800; month 6, 4400. Air conditioners can be produced in either Hsinchu or Taoyuan. It takes 2 hours of skilled labor to produce an air conditioner in Hsinchu, and 2.5 hours in Taoyuan. It costs \$400 to produce an air conditioner in Hsinchu, and \$350 in Taoyuan. During each month, each city has 4000 hours of skilled labor available. It costs \$80 to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has 2000 air conditioners in stock.

   Note. You may use Microsoft Excel Solver (introduced in the first Coursera course) or any other tool you like to solve this problem. However, you are suggested to preview the TA lectures in the second Coursera course to learn Gurobi Optimizer with Python and use it to solve this problem. The reason to preview Gurobi Optimizer is to save you some valuable time later. After all, later materials will be more challenging, a typical student gets more and more busy when semester continues, and Gurobi Optimizer will be used a lot by most of the students in this course eventually. In any case, please write down an optimal solution and its objective value; do not include a screenshot, computer program, or anything related to your solution process.

2. (30 points; 10 points each) A city is divided into n districts. The time (in minutes) it takes an ambulance to travel from district i to j is D_{ij}. The population of district i (in thousands) is H_i. The city has p ambulances and wants to locate them. Formulate an integer program that can achieve each of the following goals.

   (a) To maximize the number of people who live within (no greater than) B minutes of at least one ambulance.

   (b) To maximize the number of people who live within (no greater than) B minutes of at least two ambulances.

   (c) To maximize the minimum of the following two quantities: (1) the number of people who live within (no greater than) B_1 minutes of at least two ambulances, and (2) the number of people who live within (no greater than) B_2 minutes of at least one ambulance or within (no greater than) B_3 minutes of at least three ambulances.

3. (30 points; 10 points each) Michelle is traveling from Germany to Taiwan. She bought two bags, each can carry up to K kg of items. There are several items in a set $I = \{1, 2, ..., n\}$ that she considers to carry. The weights and values of item i is W_i and V_i, $i \in I$. Michelle wants to maximize the total values of the items she carry while satisfying the capacity constraint, i.e., each bag cannot carry more than K kg. Do each of the following problems independently.

   (a) Formulate an integer program that solves Michelle's problem.

   (b) Suppose that items 2 and 3 cannot be put in the same bag, items 4, 5, and 6 cannot be all put in the same bag (but two of them can be put in the same bag), at least two of items 8 to 12 must be carried, and at least one of items 1 and 2 must be carried if item 3 is not carried. Formulate an integer program that solves Michelle's problem.

   (c) Suppose that if items i and $i + 1$ are both brought in the trip, $i = 1, ..., n - 1$, an additional value A_i will be created for Michelle. Moreover, suppose that if items i and $i+1$ are both put in the same bag, an additional value B_i will be created for Michelle. For example, if Michelle put items 1 and 2 in bag 1 and items 3 and 4 in bag 2, her total value is $V_1 + V_2 + V_3 + V_4 + A_1 + A_2 + A_3 + B_1 + B_3$. Formulate an integer program that solves Michelle's problem.

4. (10 points) Consider the following n-item ordering problem. You purchase n items from a supplier to sell to a market. The demands for item i is D_i per day (i.e., for all items the demand rates are constants), the ordering cost is K per order, and the holding cost of item i is h_i per unit per day. You are allowed to choose an ordering cycle and n order quantities for the n products, but the supplier forces you to order all items with the same cycle. For example, suppose that $n = 2$, you are allowed order 5 and 10 units of items 1 and 2 every six days, but you are not allowed to order 5 units of item 1 every six days and 5 units of item 2 every seven days. Formulate a nonlinear program that minimizes the average daily total cost, which includes ordering and holding costs.

   Hint. We know the problem is to minimize the average cost in the long run. Suppose that in a cycle we order "too much" for a product so that at the end of the cycle there are some leftover of that product, because this will repeat again and again, in the long run the amount of leftover will go to infinity, and then the average cost will be infinite. This implies that there is a "constraint imposed by ourselves" (rather than by a supplier, a government, or whoever): the order quantity of a product must be equal to it demand quantity in a cycle. In other words, you may have a formulation with only one decision variable, the ordering cycle. Once the ordering cycle is determined, the ordering quantities will be determined (if we want to avoid infinitely large cost). We say this is a "constraint imposed by ourselves" because this restriction does not come from the environment; it is created due to our understanding about an optimal solution. In your answer, you may define T as the cycle time and q_i as the order quantities and set equality constraints to relate them. Alternatively, you may define T as your only decision variable and express anything else with T. Of course you may also choose the notations you prefer.