Pre-lecture Problems for Lecture 8: Linear Programming Duality

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1. (10 points; 2 points each) Consider a primal LP

   Ans.

$$
\begin{aligned}
\max \quad & 5x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 \leq 8 \\
& x_1 + 2x_2 \leq 10 \\
& x_1 \geq 0, x_2 \geq 0
\end{aligned}
$$

   (a) Find a primal optimal solution $\bar{x}$ by the simplex method.

   Ans. The standard form of the primal LP is

$$
\begin{aligned}
\max \quad & 5x_1 + 3x_2 \\
\text{s.t.} \quad & x_1 + x_2 + x_3 = 8 \\
& x_1 + 2x_2 + x_4 = 10 \\
& x_i \geq 0 \quad i = 1, 2, 3, 4
\end{aligned}
$$

$$
\begin{array}{cccc|c}
-5 & -3 & 0 & 0 & 0 \\
\hline
1 & 1 & 1 & 0 & x_3 = 8 \\
1 & 2 & 0 & 1 & x_4 = 10
\end{array}
\rightarrow
\begin{array}{cccc|c}
0 & 2 & 5 & 0 & 40 \\
\hline
1 & 1 & 1 & 0 & x_1 = 8 \\
0 & 1 & -1 & 1 & x_4 = 2
\end{array}
$$

   The optimal solution is $(8, 0, 0, 2)$ with the optimal value 40.

   (b) Formulate the dual LP.

Ans.

$$\begin{aligned} \min \quad & 8y_1 + 10y_2 \\ \text{s.t.} \quad & y_1 + y_2 \geq 5 \\ & y_1 + 2y_2 \geq 3 \\ & y_1 \geq 0, y_2 \geq 0 \end{aligned}$$

(c) Solve the dual LP in any way you like to get a dual optimal solution $\bar{y}$. Show that $c^T\bar{x} = \bar{y}^T b$, where c and b represent the primal and dual objective coefficients.

Ans.

Phase-I:

$$\begin{aligned} \min \quad & y_5 + y_6 \\ \text{s.t.} \quad & y_1 + y_2 - y_3 + y_5 = 5 \\ & y_1 + 2y_2 - y_4 + y_6 = 3 \\ & y_i \geq 0 \quad i = 1, 2, 3, 4 \end{aligned}$$

$$\begin{array}{cccccc|c} 0 & 0 & 0 & 0 & -1 & -1 & 0 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & y_5 = 5 \\ 1 & 2 & 0 & -1 & 0 & 1 & y_6 = 3 \end{array} \rightarrow \begin{array}{cccccc|c} 2 & 3 & -1 & -1 & 0 & 0 & 8 \\ \hline 1 & 1 & -1 & 0 & 1 & 0 & y_5 = 5 \\ 1 & 2 & 0 & -1 & 0 & 1 & y_6 = 3 \end{array}$$

$$\rightarrow \begin{array}{ccccc|c} 0 & -1 & -1 & 1 & 0 & 2 \\ \hline 0 & -1 & -1 & 1 & 1 & y_5 = 2 \\ 1 & 2 & 0 & -1 & 0 & y_1 = 3 \end{array} \rightarrow \begin{array}{cccc|c} 0 & 0 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array}$$

Phase-II:

$$\begin{array}{cccc|c} -8 & -10 & 0 & 0 & 0 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array} \rightarrow \begin{array}{cccc|c} 0 & -2 & -8 & 0 & 40 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array}$$

$$\rightarrow \begin{array}{cccc|c} 0 & -2 & -8 & 0 & 40 \\ \hline 0 & -1 & -1 & 1 & y_4 = 2 \\ 1 & 1 & -1 & 0 & y_1 = 5 \end{array}$$

We get $\bar{y} = \begin{bmatrix} 5 \\ 0 \end{bmatrix}$, and the optimal is value 40.

$$c^T\bar{x} = \begin{bmatrix} 5 & 3 \end{bmatrix} \begin{bmatrix} 8 \\ 0 \end{bmatrix} = 40 = \bar{y}^T b \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 8 \\ 10 \end{bmatrix}$$

(d) Use the primal optimal basis B you found in Part (a) to verify that $c_B^T A_B^{-1} = \bar{y}^T$.

Ans.

$$x_B = \begin{bmatrix} x_1 \\ x_4 \end{bmatrix}, c_B = \begin{bmatrix} 5 \\ 0 \end{bmatrix}, A_B = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, A_B^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$$

then we can verify that

$$c_B^T A_B^{-1} = \begin{bmatrix} 5 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 5 & 0 \end{bmatrix} = \bar{y}^T$$

(e) Find the shadow prices for the two primal constraints.

Ans.

Shadow prices are the dual optimal solution, so

$$x_1 + x_2 \leq 8 \rightarrow \text{shadow price} = 5$$
$$x_1 + 2x_2 \leq 10 \rightarrow \text{shadow price} = 0$$