Operations Research, Spring 2024 (112-2)

Pre-lecture Problems for Lecture 12: Lagrange Duality and the KKT Condition

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Note. The deadline of submitting the pre-lecture problem is ***9:30, May 6***. Please submit a hard copy of your work in class. Late submissions will not be accepted. Each student must submit her/his individual work. Submit ONLY the problem that counts for grades.

1. (0 point) Consider the following nonlinear program

$$\begin{array}{rl} \min & (x_1-4)^2+(x_2-2)^2 \\ \text{s.t.} & 2x_1+x_2 \leq 6. \end{array}$$

   (a) Prove or disprove that the NLP is a convex program.
   (b) Find the Lagrangian of this NLP. What is the sign constraint for your Lagrangian multiplier?
   (c) Formulate the Lagrangian relaxation.
   (d) According to the FOC of the Lagrangian, find a necessary condition for any optimal solution.
   (e) Find an optimal solution for the NLP.

   Answer. To help you solve the next problem, here we provide the answer for this problem directly.

   (a) Let $f(x_1,x_2)=(x_1-4)^2+(x_2-2)^2$. We then have

$$\nabla f(x_1,x_2)=\begin{bmatrix} 2x_1-8 \\ 2x_2-4 \end{bmatrix} \quad \text{and} \quad \nabla^2 f(x_1,x_2)=\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}.$$

   For the Hessian matrix, the leading principal minors are

$$|2|=2>0 \quad \text{and} \quad \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix}=4>0.$$

   Therefore, $f(x_1,x_2)$ is convex over $\mathbb{R}^2$. Because the feasible region is obviously convex, and the objective function is to minimize a convex function, the NLP is a convex program.

   (b) For a given $\lambda \leq 0$, the Lagrangian is

$$\mathcal{L}(x_1,x_2|\lambda)=(x_1-4)^2+(x_2-2)^2+\lambda(6-2x_1-x_2).$$

   (c) For a given $\lambda \leq 0$, the Lagrangian relaxation is

$$\begin{aligned} z^L(\lambda) &= \min_{x\in\mathbb{R}^2} \ \mathcal{L}(x_1,x_2|\lambda) \\ &= \min_{x\in\mathbb{R}^2} \ (x_1-4)^2+(x_2-2)^2+\lambda(6-2x_1-x_2). \end{aligned}$$

   (d) The FOC of the Lagrangian leads to

$$\nabla\mathcal{L}=0 \Rightarrow \begin{bmatrix} 2x_1-8-2\lambda \\ 2x_2-4-\lambda \end{bmatrix}=0 \Rightarrow x_1-4-\lambda=2x_2-4-\lambda \Rightarrow x_1-2x_2=0.$$

   In other words, if (x_1,x_2) is optimal, it must satisfy $x_1=2x_2$.

(e) Based on the complementary slackness condition $\lambda(6-2x_1-x_2)=0$, we know either $\lambda=0$ or $6-2x_1-x_2=0$ is true. If $\lambda=0$, the FOC of the Lagrangian gives a solution $(x_1,x_2)=(4,2)$. As this solution violates the primal feasibility condition $2x_1+x_2\leq 6$, it is not optimal. If $6-2x_1-x_2=0$, solving this equality with $x_1=2x_2$ leads to the solution $(x_1,x_2)=(\frac{12}{5},\frac{6}{5})$. As this is primal feasible, it is optimal. The objective value is $\frac{16}{5}$.

2. (10 points; 2 points each) Consider the following nonlinear program

$$\begin{array}{rl} \min & (x_1-3)^2+(x_2-2)^2 \\ \text{s.t.} & x_1+2x_2\geq 10. \end{array}$$

(a) Prove or disprove that the NLP is a convex program.

(b) Find the Lagrangian of this NLP. What is the sign constraint for your Lagrangian multiplier?

(c) Formulate the Lagrangian relaxation.

(d) According to the FOC of the Lagrangian, find a necessary condition for any optimal solution.

(e) Find an optimal solution for the NLP.