Operations Research Homework 1

B10705034 資管三 許文鑫

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Here are some technical notes before you start. When we say a "linear program", it cannot contain nonlinear functions and integer variables; when we say an "integer program", it cannot contain nonlinear functions; when we say a "nonlinear program", it cannot contain integer variables. Moreover, nonlinear functions include maximum functions, minimum functions, multiplication of decision variables, division of decision variables, among others.

1. (30 points; 10 points each) During the next n months, the IEDO company has demand D_t for air conditioners in month t, $t = 1, ..., n$. Air conditioners can be produced in m sites. It takes L_i hours of skilled labor to produce an air conditioner in site i, $i = 1, ..., m$. It costs C_i^P to produce an air conditioner in site i, $i = 1, ..., m$. During each month, each city has K hours of skilled labor available. It costs C^H to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has I air conditioners in stock.

   (a) Suppose that IEDO must meet all the demands on time. Formulate a linear program whose solution will tell IEDO how to minimize the cost of meeting air conditioner demands for the next six months.

   Ans.

   Let x_{it} be the number of air conditioners produced in site i in month t, and h_t be the number of air conditioners in stock at the end of month t. Then the linear program is

$$
\begin{aligned}
\min\ & \sum_{t=1}^{6}(\sum_{i=1}^{m} C_i^P x_{it}) + C^H h_t \\
\text{s.t.}\ & h_0 = I \\
& h_t = h_{t-1} + \sum_{i=1}^{m} x_{it} - D_t, \quad \forall t = 1,2,3,4,5,6 \\
& x_{it} L_i \leq K, \quad \forall i = 1,2,...,m, \quad \forall t = 1,2,3,4,5,6 \\
& x_{it} \geq 0, \quad \forall i = 1,2,...,m, \quad \forall t = 1,2,3,4,5,6 \\
& h_t \geq 0, \quad \forall t = 1,2,3,4,5,6
\end{aligned}
$$

   (b) Suppose that those demands are the maximum number that IEDO may sell, and IEDO can decide the sales quantity in each month. Each air conditioner can be sold at \$600. Formulate an LP whose solution maximizes the profit of selling air conditioners for the next six months. Your formulation must be a compact one.

   Ans.

Let y_t be the number of air conditioners sold in month t. Then the linear program is

$$\begin{aligned}
\max \ & \sum_{t=1}^{6} 600 \cdot y_t - \sum_{t=1}^{6} \sum_{i=1}^{m} C_i^P x_{it} - C^H h_t \\
\text{s.t.} \ & h_0 = I \\
& h_t = h_{t-1} + \sum_{i=1}^{m} x_{it} - y_t, \quad \forall t = 1,2,3,4,5,6 \\
& y_t \leq D_t, \quad \forall t = 1,2,3,4,5,6 \\
& x_{it} L_i \leq K, \quad \forall i = 1,2,...,m, \quad \forall t = 1,2,3,4,5,6 \\
& x_{it} \geq 0, \quad \forall i = 1,2,...,m, \quad \forall t = 1,2,3,4,5,6 \\
& h_t \geq 0, \quad \forall t = 1,2,3,4,5,6 \\
& y_t \geq 0, \quad \forall t = 1,2,3,4,5,6
\end{aligned}$$

(c) Continue from Part (a) and solve the linear program to obtain an optimal solution according to the following information. During the next six months, the IEDO company has following demands for air conditioners: month 1, 2500; month 2, 4000; month 3, 4500; month 4, 4200; month 5, 3800; month 6, 4400. Air conditioners can be produced in either Hsinchu or Taoyuan. It takes 2 hours of skilled labor to produce an air conditioner in Hsinchu, and 2.5 hours in Taoyuan. It costs \$400 to produce an air conditioner in Hsinchu, and \$350 in Taoyuan. During each month, each city has 4000 hours of skilled labor available. It costs \$80 to hold an air conditioner in inventory for a month. At the beginning of month 1, IEDO has 2000 air conditioners in stock.

Ans.

Let Hsinchu be site $t = 0$, and Taoyuan be site $t = 1$. Then the linear program is The optimal solution is:

$x_{01} = 1800, x_{11} = 1600,$
$x_{02} = 2000, x_{12} = 1600,$
$x_{03} = 2000, x_{13} = 1600,$
$x_{04} = 2000, x_{14} = 1600,$
$x_{05} = 2000, x_{15} = 1600,$
$x_{06} = 2000, x_{16} = 1600,$
$h_1 = 2900,$
$h_2 = 2500,$
$h_3 = 1600,$
$h_4 = 1000,$
$h_5 = 800,$
$h_6 = 0$
and the minimum cost is \$8944000.

2. (30 points; 10 points each) A city is divided into n districts. The time (in minutes) it takes an ambulance to travel from district i to j is D_{ij}. The population of district i (in thousands) is H_i. The city has p ambulances and wants to locate them. Formulate an integer program that can achieve each of the following goals.

(a) To maximize the number of people who live within (no greater than) B minutes of at least one ambulance.

Ans.

Let x_i be the binary variable that indicates whether an ambulance is located in district i.

Let y_{ij} be the binary variable that indicates $D_{ij} \leq B$.

Let z_i be the binary variable that indicates whether district z is within B minutes of at least one ambulance.

Let M be the maximum value of $D_{ij} + B$.

Then the integer program is:

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} H_i z_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq p \\
& D_{ij} - B \geq -M y_{ij} \\
& D_{ij} - B \leq M(1 - y_{ij}) \\
& z_i \leq \sum_{j=1}^{n} x_j y_{ij}, \quad \forall i = 1, 2, ..., n \\
& x_i \geq 0, \quad \forall i = 1, 2, ..., n \\
& y_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n \\
& z_i \in \{0, 1\}, \quad \forall i = 1, 2, ..., n
\end{aligned}$$

(b) To maximize the number of people who live within (no greater than) B minutes of at least two ambulances.

Ans.

$$\begin{aligned}
\max \quad & \sum_{i=1}^{n} H_i z_i \\
\text{s.t.} \quad & \sum_{i=1}^{n} x_i \leq p \\
& D_{ij} - B \geq -M y_{ij} \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n \\
& D_{ij} - B \leq M(1 - y_{ij}) \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n \\
& \sum_{j=1}^{n} x_j y_{ij} \geq 2 z_i, \quad \forall i = 1, 2, ..., n \\
& x_i \geq 0, \quad \forall i = 1, 2, ..., n \\
& y_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n \\
& z_i \in \{0, 1\}, \quad \forall i = 1, 2, ..., n
\end{aligned}$$

(c) To maximize the minimum of the following two quantities: (1) the number of people who live within (no greater than) B_1 minutes of at least two ambulances, and (2) the number of people who live within (no greater than) B_2 minutes of at least one ambulance or within (no greater than) B_3 minutes of at least three ambulances.

Ans.

Let z_{ai} be the binary variable that indicates whether district i is within B_k minutes of at least a

ambulances. $a = \{1, 2, 3\}, k = \{2, 1, 3\}$
Let y_{ijk} be the binary variable that indicates $D_{ij} \leq B_k$.
Let v_i be the binary that indicates $z_{1i} = 1$ or $z_{3i} = 1$.
Then the integer program is:

$$
\begin{aligned}
\max\ & \min\{\sum_{i=1}^{n} H_i z_{2i}, \sum_{i=1}^{n} H_i v_i\} \\
\text{s.t. } & \sum_{i=1}^{n} x_i \leq p \\
& D_{ij} - B_k \geq -M y_{ijk} \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n, \quad \forall k = 1, 2, 3 \\
& D_{ij} - B_k \leq M(1 - y_{ijk}) \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n, \quad \forall k = 1, 2, 3 \\
& \sum_{j=1}^{n} x_j y_{ij2} \geq z_{1i}, \quad \forall i = 1, 2, ..., n \\
& \sum_{j=1}^{n} x_j y_{ij1} \geq 2 z_{2i}, \quad \forall i = 1, 2, ..., n \\
& \sum_{j=1}^{n} x_j y_{ij3} \geq 3 z_{3i}, \quad \forall i = 1, 2, ..., n \\
& v_i \leq z_{1i} + z_{3i}, \quad \forall i = 1, 2, ..., n \\
& v_i \in \{0, 1\}, \quad \forall i = 1, 2, ..., n \\
& x_i \geq 0, \quad \forall i = 1, 2, ..., n \\
& y_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2, ..., n \\
& z_{ai} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall a = 1, 2, 3
\end{aligned}
$$

Then we can let $w = \min\{\sum_{i=1}^{n} H_i z_{2i}, \sum_{i=1}^{n} H_i v_i\}$, and add the following constraints to the integer program:

$$
\begin{aligned}
\max\ & w \\
\text{s.t. }\ & w \le \sum_{i=1}^{n} H_i z_{2i} \\
& w \le \sum_{i=1}^{n} H_i v_i \\
& \sum_{i=1}^{n} x_i \le p \\
& D_{ij} - B_k \ge -M y_{ijk} \quad \forall i = 1,2,...,n, \quad \forall j = 1,2,...,n, \quad \forall k = 1,2,3 \\
& D_{ij} - B_k \le M(1 - y_{ijk}) \quad \forall i = 1,2,...,n, \quad \forall j = 1,2,...,n, \quad \forall k = 1,2,3 \\
& \sum_{j=1}^{n} x_j y_{ij2} \ge z_{1i}, \quad \forall i = 1,2,...,n \\
& \sum_{j=1}^{n} x_j y_{ij1} \ge 2 z_{2i}, \quad \forall i = 1,2,...,n \\
& \sum_{j=1}^{n} x_j y_{ij3} \ge 3 z_{3i}, \quad \forall i = 1,2,...,n \\
& v_i \le z_{1i} + z_{3i}, \quad \forall i = 1,2,...,n \\
& v_i \in \{0,1\}, \quad \forall i = 1,2,...,n \\
& x_i \ge 0, \quad \forall i = 1,2,...,n \\
& y_{ij} \in \{0,1\}, \quad \forall i = 1,2,...,n, \quad \forall j = 1,2,...,n \\
& z_{ai} \in \{0,1\}, \quad \forall i = 1,2,...,n, \quad \forall a = 1,2,3
\end{aligned}
$$

3. (30 points; 10 points each) Michelle is traveling from Germany to Taiwan. She bought two bags, each can carry up to K kg of items. There are several items in a set $I = 1,2,...,n$ that she considers to carry. The weights and values of item i is W_i and V_i, $i \in I$. Michelle wants to maximize the total values of the items she carry while satisfying the capacity constraint, i.e., each bag cannot carry more than K kg. Do each of the following problems independently.

   (a) Formulate an integer program that solves Michelle's problem.

   Ans.

   Let x_{ij} be the binary variable that indicates whether item i is carried in bag j. Then the integer program is

$$
\begin{aligned}
\max\ & \sum_{i=1}^{n} \sum_{j=1}^{2} V_i x_{ij} \\
\text{s.t. }\ & \sum_{j=1}^{2} x_{ij} \le 1, \quad \forall i = 1,2,...,n \\
& \sum_{i=1}^{n} W_i x_{ij} \le K, \quad \forall j = 1,2 \\
& x_{ij} \in \{0,1\}, \quad \forall i = 1,2,...,n, \quad \forall j = 1,2
\end{aligned}
$$

(b) Suppose that items 2 and 3 cannot be put in the same bag, items 4, 5, and 6 cannot be all put in the same bag (but two of them can be put in the same bag), at least two of items 8 to 12 must be carried, and at least one of items 1 and 2 must be carried if item 3 is not carried. Formulate an integer program that solves Michelle's problem.

Ans.

$$\begin{aligned}
\max \; & \sum_{i=1}^{n} \sum_{j=1}^{2} V_i x_{ij} \\
\text{s.t.} \; & \sum_{j=1}^{2} x_{ij} \leq 1, \quad \forall i = 1, 2, ..., n \\
& \sum_{i=1}^{n} W_i x_{ij} \leq K, \quad \forall j = 1, 2 \\
& x_{2j} + x_{3j} \leq 1, \quad \forall j = 1, 2 \\
& x_{4j} + x_{5j} + x_{6j} \leq 2, \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{8j} + x_{9j} + x_{10j} + x_{11j} + x_{12j} \geq 2 \\
& \sum_{j=1}^{2} x_{1j} + x_{2j} \geq 1 - \sum_{j=1}^{2} x_{3j} \\
& x_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2
\end{aligned}$$

(c) Suppose that if items i and $i+1$ are both brought in the trip, $i = 1, ..., n-1$, an additional value A_i will be created for Michelle. Moreover, suppose that if items i and $i+1$ are both put in the same bag, an additional value B_i will be created for Michelle. For example, if Michelle put items 1 and 2 in bag 1 and items 3 and 4 in bag 2, her total value is $V1+V2+V3+V4+A1+A2+A3+B1+B3$. Formulate an integer program that solves Michelle's problem.

Ans.

Let y_i be the binary variable that indicates whether item i and $i+1$ are both brought in the trip. Let z_{iJ} be the binary variable that indicates whether item i and $i+1$ are both put in the same bag j.

Then the integer program is

$$
\begin{aligned}
\max \; & \sum_{i=1}^{n}\sum_{j=1}^{2} V_i x_{ij} + \sum_{i=1}^{n-1} A_i y_i + \sum_{i=1}^{n-1}\sum_{j=1}^{2} B_i z_{ij} \\
\text{s.t.} \; & \sum_{j=1}^{2} x_{ij} \le 1, \quad \forall i = 1, 2, ..., n \\
& \sum_{i=1}^{n} W_i x_{ij} \le K, \quad \forall j = 1, 2 \\
& \sum_{j=1}^{2} x_{ij} + x_{i+1j} \ge 2y_i, \quad \forall i = 1, 2, ..., n-1 \\
& x_{ij} + x_{i+1j} \ge 2z_{ij}, \quad \forall i = 1, 2, ..., n-1, \quad \forall j = 1, 2 \\
& x_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n, \quad \forall j = 1, 2 \\
& y_i \in \{0, 1\}, \quad \forall i = 1, 2, ..., n-1 \\
& z_{ij} \in \{0, 1\}, \quad \forall i = 1, 2, ..., n-1, \quad \forall j = 1, 2
\end{aligned}
$$

4. (10 points) Consider the following n-item ordering problem. You purchase n items from a supplier to sell to a market. The demands for item i is D_i per day (i.e., for all items the demand rates are constants), the ordering cost is K per order, and the holding cost of item i is h_i per unit per day. You are allowed to choose an ordering cycle and n order quantities for the n products, but the supplier forces you to order all items with the same cycle. For example, suppose that $n = 2$, you are allowed order 5 and 10 units of items 1 and 2 every six days, but you are not allowed to order 5 units of item 1 every six days and 5 units of item 2 every seven days. Formulate a nonlinear program that minimizes the average daily total cost, which includes ordering and holding costs.

   Ans.

   Let T be the ordering cycle. Then the nonlinear program is

$$
\begin{aligned}
\min \quad & \frac{K}{T} + \frac{1}{T}\sum_{t=1}^{T}\sum_{i=1}^{n} h_i * D_i * (T - t) \\
\text{s.t.} \;\; & T \ge 1
\end{aligned}
$$