Module 20

Properties of Z-Transform

Objective: To understand the properties of Z-Transform and associating the knowledge of properties of ROC in response to different operations on discrete signals.

Introduction :

We are aware that the z transform of a discrete signal *x(n)* is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

And inverse z transform is given by

$$x(n) = \frac{1}{2\pi j}\oint X(z)z^{n-1}dz$$

The Properties of z-transform simplifies the work of finding the z-domain equivalent of a time domain function when different operations are performed on discrete signal like time shifting, time scaling, time reversal etc. These properties also signify the change in ROC because of these operations.

These properties are also used in applying z- transform to the analysis and characterization of Discrete Time LTI systems.

Description :

1. **Linearity**

   Statement:

   If $x_1(n) \overset{Z}{\leftrightarrow} X_1(z)$ with ROC = R_1

   and $x_2(n) \overset{Z}{\leftrightarrow} X_2(z)$ with ROC = R_2

   then $ax_1(n) + bx_2(n) \overset{Z}{\leftrightarrow} aX_1(z) + bX_2(z)$, with ROC containing $R_1 \cap R_2$

   Proof:

   Taking the z-transform

$$Z\{ax_1(n) + bx_2(n)\} = \sum_{n=-\infty}^{\infty} \{ax_1(n) + bx_2(n)\}z^{-n}$$

$$= a\textstyle\sum_{n=-\infty}^{\infty} x_1(n)z^{-n} + b\sum_{n=-\infty}^{\infty} x_2(n)z^{-n}$$

$$= aX_1(z) + bX_2(z)$$

   The ROC of the Linear combination is at least the intersection of R_1 and R_2. For sequences with rational z-transforms, if the poles of $aX_1(z) + bX_2(z)$ consist of all the poles of $X_1(z)$ and $X_2(z)$, indicating no pole-zero cancellation, then the ROC will be exactly equal to the overlap of the individual regions of convergence.

   If the Linear combination is such that some zeros are introduced that cancel poles, then the ROC may be larger.

Illustration:

A simple example of this occurs when $x_1(n)$ and $x_2(n)$ are both of infinite duration, but the linear combination is of finite duration. In this case the ROC of the linear combination is the entire z-plane, except for zero and / or infinity.

For example, the sequences $a^n u(n)$ and $a^n u(n-1)$ both have an ROC defined by $|z|>|a|$, but the sequence corresponding to the difference $\{a^n u(n) - a^n u(n-1)\}=\delta(n)$ has a region of convergence that is the entire z-plane.

2. **Time Shifting**

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$with ROC= R

then $x(n-m) \overset{Z}{\leftrightarrow} z^{-m}X(z)$with ROC= R, except for the possible addition or deletion of the origin or infinity

Proof:

$$Z\{x(n-m)\} = \sum_{n=-\infty}^{\infty} x(n-m)z^{-n}$$

Let n-m=p

$$= \sum_{p=-\infty}^{\infty} x(p)z^{-(p+m)}$$

$$= z^{-m} \sum_{p=-\infty}^{\infty} x(p)z^{-p}$$

$$= z^{-m}X(z)$$

Illustration:

Because of the multiplication by z^{-m}, for m>0 poles will be introduced at z=0, which may cancel corresponding zeros of X(z) at z=0. Consequently, z=0 may be a pole of $z^{-m}X(z)$ while it may not be a pole of X(z).In this case the ROC for $z^{-m}X(z)$ equals the ROC of X(z) but with the origin deleted.

Similarly, if m<0, zeros will be introduced at z=0, which may cancel corresponding poles of X(z) at z=0. Consequently, z=0 may be a zero of $z^{-m}X(z)$ while it may not be a pole of X(z). In this case z=∞ is a pole of $z^{-m}X(z)$, and thus the ROC for $z^{-m}X(z)$ equals the ROC of X(z) but with z=∞ deleted.

3. **Scaling in the z-Domain**

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then ${z_o}^n x(n) \overset{Z}{\leftrightarrow} X\left(\frac{z}{z_o}\right)$ with ROC= $|z_o|$R where, $|z_o|$R is the scaled version of R.

Proof:

$$Z\{z_o{}^n x(n)\} = \sum_{n=-\infty}^{\infty} z_o{}^n x(n) z^{-n} = \sum_{n=-\infty}^{\infty} x(n)\left(\frac{z}{z_o}\right)^{-n} = X\left(\frac{z}{z_o}\right)$$

Illustration:

If z is a point in the ROC of X(z), then the point $|z_o|z$is in the ROC of $X\left(\frac{z}{z_o}\right)$.

Also, if X(z) has a pole (or zero) at $z=a$, then $X\left(\frac{z}{z_o}\right)$has a pole (or zero) at $z=z_o a$.

An important special case of the property is when $z_o=e^{j\omega o}$. In this case, $|z_o|R=R$ and

$$e^{j\omega_o n}x(n) \overset{Z}{\leftrightarrow} X\left(e^{-j\omega_o}z\right)$$

The left-hand side of the above equation corresponds to multiplication by a complex exponential sequence. The right-hand side can be interpreted as a rotation in the z-plane; i.e., all pole-zero locations rotate in the z-plane by an angle of ω_o, as illustrated in the figure below.

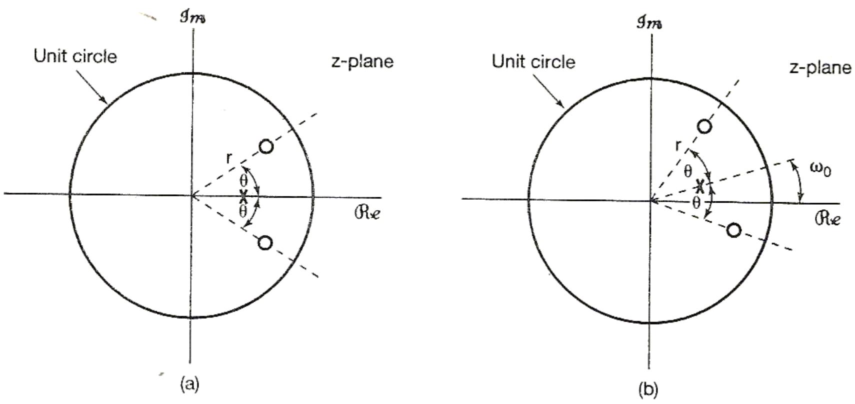

Fig (a) is the pole-zero pattern for the z-transform for a signal x(n)

Fig (b) is the pole-zero pattern for the z-transform of $e^{j\omega_o n}x(n)$

4. **Time Reversal**

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x(-n) \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ with ROC= $\frac{1}{R}$

Proof:

$$Z\{x(-n)\} = \sum_{n=-\infty}^{\infty} x(-n) z^{-n}$$

Let -n=p

$$= \sum_{p=-\infty}^{\infty} x(p)(z)^p = \sum_{p=-\infty}^{\infty} x(p)(z^{-1})^{-p} = X\left(\frac{1}{z}\right)$$

Illustration

If z_o is in the ROC for x(n), then $1/z_o$ is in the ROC for x(-n)

5. Conjugation

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x^*(n) \overset{Z}{\leftrightarrow} X^*(z^*)$ with ROC= R

Proof:

$$Z\{x^*(n)\} = \sum_{n=-\infty}^{\infty} x^*(n) z^{-n}$$

as we know that $z = re^{j\omega}$

$$= \sum_{n=-\infty}^{\infty} x^*(n) r^{-n}\, e^{-j\omega n}$$

$$= \left(\sum_{n=-\infty}^{\infty} x(n) r^{-n}\, e^{+j\omega n} \right)^*$$

$$= \left(\sum_{n=-\infty}^{\infty} x(n)\, (z^*)^{-n} \right)^*$$

$$= \left(X(z^*)\right)^* = X^*(z^*)$$

Also $X(z) = X^*(z^*)$ when x(n) is real.

Illustration:

If X(z) has a pole (or zero) at $z=z_o$, it must also have a pole (or zero) at the complex conjugate point $z=z_o^*$.

6. The Convolution Property

Statement:

If $x_1(n) \overset{Z}{\leftrightarrow} X_1(z)$ with ROC = R_1

and $x_2(n) \overset{Z}{\leftrightarrow} X_2(z)$ with ROC = R_2

then $x_1(n) * x_2(n) \overset{Z}{\leftrightarrow} X_1(z).X_2(z)$, with ROC containing $R_1 \cap R_2$

Proof:

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{\infty} \{x_1(n) * x_2(n)\} z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \{\sum_{m=-\infty}^{\infty} x_1(m)\, x_2(n-m)\} z^{-n}$$

Interchanging the order of summations

$$Z\{x_1(n) * x_2(n)\} = \sum_{m=-\infty}^{\infty} x_1(m) \left\{ \sum_{n=-\infty}^{\infty} x(n-m)\, z^{-n} \right\}$$

$$= \sum_{m=-\infty}^{\infty} x_1(m)\{z^{-m} X_2(z)\}$$

(Since from Time shifting property)

$$= X_2(z)\left\{\sum_{m=-\infty}^{\infty} x_1(m) z^{-m}\right\}$$

$$= X_1(z).X_2(z)$$

Illustration:

Just as with the convolution property for the Laplace transform, the ROC of $X_1(z).X_2(z)$ includes the intersection of R_1 and R_2 and may be larger if pole-zero cancellation occurs in the product.

Note: This property plays an important role in the analysis of Discrete Time LTI systems.

For example consider an LTI system for which y(n)=h(n)*x(n), where $h(n) = \delta(n) - \delta(n-1)$.

Note that $\delta(n) - \delta(n-1) \overset{Z}{\leftrightarrow} 1 - z^{-1}$, with ROC equal to the entire z-plane except the origin. Also, the z-transform has a zero at z=1.

Applying the property

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC = R, then
$y(n) \overset{Z}{\leftrightarrow} (1 - z^{-1})\, X(z)$ with ROC = R, with the possible deletion of z=0 and/or addition of z=1.

7. Accumulation

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC = R
then
$\sum_{k=-\infty}^{n} x(k) \overset{Z}{\leftrightarrow} X(z).\frac{1}{1-z^{-1}}$, with ROC containing $R \cap \{|z| > 1\}$

Proof:

$$\sum_{k=-\infty}^{n} x(k) = x(n) * u(n)$$

$$Z\left\{\sum_{k=-\infty}^{n} x(k)\right\} = Z\{x(n) * u(n)\}$$

Applying convolution property

$$Z\left\{\sum_{k=-\infty}^{n} x(k)\right\} = X(z).\frac{1}{1-z^{-1}}$$

8. **Time Expansion**

The continuous –time concept of time scaling does not directly extend to discrete time, since the discrete time index is defined only for integer values. However, the discrete time concept of time expansion can be defined and does play an important role in discrete time signal and system analysis. Let m be a positive integer, and define the signal

$$x_{(m)}(n) = \begin{cases} x\left(\frac{n}{m}\right), & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$$

$x_{(m)}(n)$ can be obtained from x(n) by placing m-1 zeros between successive values of the original signal. Intuitively, we can think of $x_{(m)}(n)$ as a slowed down version of x(n). Now,

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC = R

then $x_{(m)}(n) \overset{Z}{\leftrightarrow} X(z^m)$ with ROC= $R^{1/m}$

That is , if R is $a<$ $|z|$ $<b$, then the new ROC is $a<$ $|z^m|$ $<b$, or $a^{1/m}<$ $|z|$ $<b^{1/m}$. Also, if X(z) has a pole (or zero) at $z=a$, thenX(z^m)has apole (or zero) at $z^{1/m}$.

Proof:

The z transform of $x_{(m)}(n)$ is given by

$$Z\{x_{(m)}(n)\} = \sum_{n=-\infty}^{\infty} x_{(m)}(n)z^{-n} = \sum_{n=-\infty}^{\infty} x\left(\frac{n}{m}\right)z^{-n}$$

Changing the variables is performed by letting $r = n/m$, which also yields r = -∞ as n=-∞ and r=∞ as n=∞. Therefore,

$$Z\{x_{(m)}(n)\} = \sum_{r=-\infty}^{\infty} x(r)z^{-mr} = \sum_{n=-\infty}^{\infty} x(r)(z^m)^{-r} = X(z^m)$$

9. **Differentiation in the z-Domain**

Statement:

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $nx(n) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz}$ with ROC = R

Proof:

z transform is given by

$$Z\{x(n)\} = X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n}$$

Differentiating above on both sides with respect to ‘z’

$$\frac{dX(z)}{dz} = \frac{d}{dz}\left\{\sum_{n=-\infty}^{\infty} x(n)z^{-n}\right\} = \sum_{n=-\infty}^{\infty} x(n)\frac{d}{dz}\{z^{-n}\} = \sum_{n=-\infty}^{\infty} -nx(n)z^{-n-1}$$

$$-z\frac{dX(z)}{dz} = \sum_{n=-\infty}^{\infty} nx(n)z^{-n}$$

Comparing both equations $-z\frac{dX(z)}{dz}$ is the z transform of $nx(n)$.

ROC remains the same R because differentiating X(z) will increase the order of the poles present at the same location as earlier.

Illustration:

Consider

$$x(n) = a^n u(n)$$

$$X(z) = \sum_{n=-\infty}^{\infty} a^n u(n)z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

For convergence of X(z), we require that $\sum_{n=0}^{\infty}|az^{-1}|^n < \infty$. Consequently, the region of convergence is that range of values of z for which $|az^{-1}| < 1$, or equivalently, $|z| > |a|$

Then $X(z) = \sum_{n=0}^{\infty}(az^{-1})^n = \frac{1}{1-az^{-1}}$

$na^n u(n) \overset{Z}{\leftrightarrow} -z\frac{dX(z)}{dz} = \frac{az^{-1}}{(1-az^{-1})^2}$ with ROC: |z| > |a|

From the above result, it is observed that differentiating rational form z-domain function will result in multiple order poles. Therefore, ROC remains same as X(z).

10. The Initial Value Theorems

Statement:

If x(n)=0, for n < 0 then initial value of x(n) i.e., $x(0) = \lim_{z\to\infty} X(z)$

Proof:

We know that Z{x(n)}= $X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$ as x(n) is causal.

Expanding the summation

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots$$

Applying the $\lim_{z\to\infty}$ *on both sides*

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty}\{x(0) + x(1)z^{-1} + x(2)z^{-2} + \cdots\}$$

i.e.,

$$\lim_{z\to\infty} X(z) = x(0)$$

11. The Final Value Theorem

Statement:

If x(n) is causal and X(z) is the z-transform of x(n) and if all the poles of X(z) lie strictly inside the unit circle except possibly for a first order pole at z=1 then

$$\lim_{N\to\infty} x(n) = \lim_{z\to 1}(1 - z^{-1})X(z)$$

Proof:

Consider the z-transform of x(n)-x(n-1)

$$\mathrm{x(n)} - \mathrm{x(n-1)} \overset{Z}{\leftrightarrow} (1 - z^{-1})X(z)$$

$$\boldsymbol{Z}\{\, x(n) - x(n-1)\} = \sum_{n=0}^{\infty} \{x(n) - x(n-1)\} z^{-n} = \; (1 - z^{-1})X(z)$$

Also, the above can be written as

$$\lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\} z^{-n} = \; (1 - z^{-1})X(z)$$

Applying the limit z→1 on both sides

$$\lim_{z\to 1}\left\{\lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\} z^{-n}\right\} = \; \lim_{z\to 1}(1 - z^{-1})X(z)$$

LHS after applying the limit z→1 becomes

$$\left\{\lim_{N\to\infty} \sum_{n=0}^{N} \{x(n) - x(n-1)\}\right\} = \lim_{N\to\infty} \left\{\begin{matrix} x(0) - x(-1) + x(1) - x(0) + x(2) - x(1) + \cdots \\ +x(N-1) - x(N-2) + x(N) - x(N-1) \end{matrix}\right\}$$

All terms cancel except x(N). Therefore,

$$\lim_{N\to\infty} x(n) = \lim_{z\to 1}(1 - z^{-1})X(z)$$

Summary:

Property	Signal	z-Transform	ROC
	$x[n]$	$X(z)$	R
	$x_1[n]$	$X_1(z)$	R_1
	$x_2[n]$	$X_2(z)$	R_2
Linearity	$ax_1[n] + bx_2[n]$	$aX_1(z) + bX_2(z)$	At least the intersection of R_1 and R_2
Time shifting	$x[n - n_0]$	$z^{-n_0}X(z)$	R, except for the possible addition or deletion of the origin
Scaling in the z-domain	$e^{j\omega_0 n}x[n]$	$X(e^{-j\omega_0}z)$	R
	$z_0^n x[n]$	$X\left(\frac{z}{z_0}\right)$	z_0R
	$a^n x[n]$	$X(a^{-1}z)$	Scaled version of R (i.e., $\|a\|R$ = the set of points $\{\|a\|z\}$ for z in R)
Time reversal	$x[-n]$	$X(z^{-1})$	Inverted R (i.e., R^{-1} = the set of points z^{-1}, where z is in R)
Time expansion	$x_{(k)}[n] = \begin{cases} x[r], & n = rk \\ 0, & n \neq rk \end{cases}$ for some integer r	$X(z^k)$	$R^{1/k}$ (i.e., the set of points $z^{1/k}$, where z is in R)
Conjugation	$x^*[n]$	$X^*(z^*)$	R
Convolution	$x_1[n] * x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of R_1 and R_2
First difference	$x[n] - x[n-1]$	$(1 - z^{-1})X(z)$	At least the intersection of R and $\|z\| > 0$
Accumulation	$\sum_{k=-\infty}^{n} x[k]$	$\frac{1}{1 - z^{-1}}X(z)$	At least the intersection of R and $\|z\| > 1$
Differentiation in the z-domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	R

Initial Value Theorem

If $x[n] = 0$ for $n < 0$, then

$$x[0] = \lim_{z \to \infty} X(z)$$

Examples:

Solved Problems:

Problem 1: Compute the convolution y(n) of the signals

$$x(n) = \{1,2,1\}$$

$$h(n) = \begin{cases} 1 & 0 \le n \le 5 \\ 0 & elsewhere \end{cases}$$

Solution:

By definition

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=0}^{2} x(n)z^{-n} = x(0) + x(1)z^{-1} + x(2)z^{-2}$$
$$= 1 - 2z^{-1} + z^{-2} = (1 - z^{-1})^2$$

ROC is the entire z-plane expect z=0 because x(z) becomes unbounded for z=0.

Similarly, we have

$$H(z) = \sum_{n=-\infty}^{\infty} h(n)z^{-n} = \sum_{n=0}^{5} h(n)z^{-n} = \sum_{n=0}^{5} z^{-n} = \frac{1 - z^{-6}}{1 - z^{-1}}$$

ROC is the entire z-plane expect z=0 because x(z) becomes unbounded for z=0.

Now, consider $y(n) = x(n) * h(n)$

Using the convolution property, we obtain

$$Y(z) = X(z)H(z) = (1 - z^{-1})^2 \left(\frac{1 - z^{-6}}{1 - z^{-1}}\right) = (1 - z^{-1})(1 - z^{-6}) = 1 - z^{-1} - z^{-6} + z^{-7}$$

Then y(n)={1,-1,0,0,0,0,-1,1}

Problem 2: Determine the z-transform and ROC of the signal $x(n) = u(n) - u(n - 10)$

Solution:

$$Z\{x(n)\} = Z\{u(n)\} - z\{u(n - 10)\} = \frac{1}{1 - z^{-1}} - \frac{z^{-10}}{1 - z^{-1}} = \frac{1 - z^{-10}}{1 - z^{-1}} = \frac{1}{z^{10}}\left\{\frac{z^{10} - 1}{z - 1}\right\}$$

ROC is the entire z-plane except z=0

Problem 3: Find the z-transform of the signal $g(n)=|n|a^{|n|}$

Solution:

Consider the given signal $g(n)=|n|a^{|n|}$

$$g(n) = \begin{cases} na^n & n \geq 0 \\ -na^{-n} & n \leq 0 \end{cases}$$

Using the differentiation property in z-domain

$$Z\{x(n)\} = Z\{na^n u(n)\} = \frac{az^{-1}}{(1 - az^{-1})^2}, |z| > |a|$$

From the time reversal property in z-domain

If $x(n) \overset{Z}{\leftrightarrow} X(z)$ with ROC= R

then $x(-n) \overset{Z}{\leftrightarrow} X\left(\frac{1}{z}\right)$ with ROC= $\frac{1}{R}$

$$Z\{x(-n)\} = X\left(\frac{1}{z}\right) = \frac{a\left(\frac{1}{z}\right)^{-1}}{\left(1 - a\left(\frac{1}{z}\right)^{-1}\right)^2}, \left(\frac{1}{z}\right) > |a| = \frac{az}{(1 - az)^2}, |z| < \frac{1}{|a|}$$

Considering *g(n)=x(n)+x(-n)*

Taking z-transform yields $G(z) = X(z) + X\left(\frac{1}{z}\right) = \frac{a(1+a^2)(z+z^{-1})-4a^2}{(1-az)^2(1-az)^2}, |a| < |z| < \frac{1}{|a|}$

Problem 4: Find the z-transform and ROC of the signal $g(n) = a^{\frac{n}{3}}u\left(\frac{n}{3}\right) = \begin{cases} a^{\frac{n}{3}}, & n = 0,3,6,\dots \\ 0, & elsewhere \end{cases}$ where $|a|< 1$

Solution:

Consider the signal $g(n) = a^{\frac{n}{3}}u\left(\frac{n}{3}\right) = x\left(\frac{n}{3}\right)$

where $x(n)=a^n u(n)$

$x(n) = a^n u(n) \overset{Z}{\leftrightarrow} X(z) = \frac{1}{1-az^{-1}}$ with ROC:$|z| > |a|$

Using Time expansion property

$x\left(\frac{n}{3}\right) = a^{\frac{n}{3}}u\left(\frac{n}{3}\right) \overset{Z}{\leftrightarrow} X(z^3) = \frac{1}{1-az^{-3}}$ with ROC:$|z| > |a|^{1/3}$

Problem 5: Use the convolution property to show that $u(n) * u(n-1) = nu(n)$

Solution:

Let$x(n) = u(n) * u(n-1) = nu(n)$

Taking the z-transform of x(n) and using the convolution property, we get

$$Z\{x(n)\} = Z\{u(n) * u(n-1)\} = \left(\frac{1}{1 - z^{-1}}\right)\left(\frac{z^{-1}}{1 - z^{-1}}\right)$$

$$X(z) = \frac{z^{-1}}{(1 - z^{-1})^2}$$

Also from differentiation in z-domain property $n.u(n) \overset{Z}{\leftrightarrow} -z\frac{d}{dz}\left(\frac{1}{1-z^{-1}}\right) = \frac{z^{-1}}{(1-z^{-1})^2}$

Hence $u(n) * u(n-1) = nu(n)$

Problem 6: Apply the final value theorem to determine x(∞) for the signal

$$x(n) = \begin{cases} 1, & if\ n\ is\ even \\ 0, & otherwise \end{cases}$$

Solution:

Given that

$$x(n) = \begin{cases} 1, & if\ n\ is\ even \\ 0, & otherwise \end{cases}$$

From the definition of the unilateral z-transform, we have

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n} = \sum_{\substack{n=0 \\ n\ even}}^{\infty} (1)z^{-n}$$

Substituting n=2r,

$$X(z) = \sum_{r=0}^{\infty} z^{-2r} = \sum_{r=0}^{\infty} (z^{-2})^r = \frac{1}{1-z^{-2}}; ROC: |z^{-2}| < 1 \rightarrow |z| > 1$$

From the final value theorem, we have

$$x(\infty) = \lim_{z\to 1}(1-z^{-1})X(z) = \lim_{z\to 1}(1-z^{-1})\frac{1}{1-z^{-2}} = \lim_{z\to 1}\frac{1}{(1+z^{-1})} = \frac{1}{2}$$

Problem 7: Findthe transfer function of the system whose impulse response is given by $h(n) = \left(\frac{2}{3}\right)^n u(n) - 2\left(\frac{2}{3}\right)^{n-1} u(n-1)$.Check whether the system is stable.

Solution:

Given the impulse response of the system $h(n) = \left(\frac{2}{3}\right)^n u(n) - 2\left(\frac{2}{3}\right)^{n-1} u(n-1)$

Applying z-transform and applying time shifting property for the second term

$$H(z) = \left\{\frac{1}{1-\frac{2}{3}z^{-1}}\right\} - 2z^{-1}\left\{\frac{1}{1-\frac{2}{3}z^{-1}}\right\} = \frac{2z^{-1}}{1-\frac{2}{3}z^{-1}}$$

ROC: |z| > 2/3

System have only one pole at z=2/3 which is inside the unit circle. Hence the system is stable.

Problem 8: Find the z-transform and ROC of $x(n) = \delta(n+1) - 2\delta(n) + \delta(n-1)$

Solution:

Given signal$x(n) = \delta(n+1) - 2\delta(n) + \delta(n-1)$

Applying z-transform on both sides

$$X(z) = z - 2 + z^{-1}$$

ROC is entire z-plane except z=0 and z=∞

Problem 9: Find the initial value of the signal $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$

Solution:

Given signal $x(n) = 7\left(\frac{1}{3}\right)^n u(n) - 6\left(\frac{1}{2}\right)^n u(n)$

Applying z-transform

$$X(z) = \frac{7}{1-\frac{1}{3}z^{-1}} - \frac{6}{1-\frac{1}{2}z^{-1}} = \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)}$$

Applying initial value theorem

$$\lim_{z\to\infty} X(z) = \lim_{z\to\infty} \frac{1-\frac{3}{2}z^{-1}}{\left(1-\frac{1}{3}z^{-1}\right)\left(1-\frac{1}{3}z^{-1}\right)} = 1$$

Problem 10: Consider the rectangular signal

$$x(n) = \begin{cases} 1, & 0 \le n \le 5 \\ 0, & otherwise \end{cases}$$

Let $g(n) = x(n) - x(n-1)$

a) Find the signal g(n) and directly evaluate its z-transform
b) Noting that

$$x(n) = \sum_{k=-\infty}^{n} g(k)$$

Determine the z-transform of x(n)

Solution:

a) The signal g(n) is simplified as $g(n) = \delta(n) - \delta(n-6)$
Using the definition of z-transform $G(z) = 1 - z^{-6}; ROC: |z| > 0$
b) From accumulation property
$x(n) = \sum_{k=-\infty}^{n} g(k) \overset{z}{\leftrightarrow} X(z) = \frac{1}{1-z^{-1}} G(z) = \frac{1-z^{-6}}{1-z^{-1}}, ROC: |z| > 0$

Assignment:

Problem 1: Determine the signal x(n) whose z-transform is given by $X(z)=\log(1+az^{-1})$, $|z|>|a|$
Problem 2: Find the z-transform of a signal $x(n) = u\left(\frac{n-1}{2}\right)$

Problem 3: Determine the input to the system, using z-transform given output $y(n)=\delta(n-2)$ and impulse response $h(n)=(1/2)^n u(n)$

Problem 4: Determine the system function for the causal LTI system with difference equation$y(n) - \frac{1}{2}y(n-1) + \frac{1}{4}y(n-2) = x(n)$

Problem 5: Considera signal y(n) which is related to two signals x(n) and h(n) by $y(n) = x(n) * h(n)$ where $x(n) = \left(\frac{1}{2}\right)^n u(n)$ and $h(n) = \left(\frac{1}{3}\right)^n u(n)$. Use the properties of the z-transform to determine Y(z).

Problem 6: Consider a sequence $x_1(n)$ with z-transform $X_1(z)$ and a sequence $x_2(n)$ with z-transform $X_2(z)$, where $x_2(n) = x_1(-n)$.

Show that $X_2(z) = X_1(1/z)$, and from this, show that if $X_1(z)$ has a pole (or zero) at $z=z_o$, then $X_2(z)$ has a pole(or zero) at $z=1/z_o$

Problem 7. Find the z-transform and ROC of $x(n) = \left(\frac{1}{4}\right)^n u(3-n)$

Problem 8: Find the z-transform of t^2e^{-at} after performing sampling at T=1 sec

Problem 9: Find the z-transform as well as ROC for the sequence

$$x(n) = \left(\frac{1}{3}\right)^n [u(n) - u(n-8)]$$

Problem 10: Find the two-sided z-transform of the signal

$$x(n) = \begin{cases} \left(\frac{1}{3}\right)^n & n \geq 0 \\ (-2)^n & n \leq -1 \end{cases}$$

Simulation:

Z-transform using MATLAB is performed with the help of the function ***ztrans***

Example: Finding the z transform of $x(n) = \left(\frac{1}{4}\right)^{n-1} u(n-1)$

```
>>syms z
>>syms n
>>ztrans(((1/4)^(n-1)*heaviside(n-1)))

ans =

(1/(4*z - 1) + 1/2)/z

>> pretty(ans)

     1
  ------- + 1/2
  4 z - 1
  -------------
        z
```

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