1. Fourier 变换可以看作 Z 变换的一个特例.

2) Fourier 离散变换

$$F(u,v) = \frac{1}{MN} \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

Z 变换

$$Z(u,v) = \frac{1}{MN} \sum_{x=-\infty}^{x=+\infty} \sum_{y=-\infty}^{y=+\infty} f(x,y)\, z^{-\left(\frac{ux}{M}+\frac{vy}{N}\right)}$$

比较可知. 若 $z = e^{j2\pi}$, Z变换为 Fourier 变换

证明 Z 域上的卷积定理:

$$x_1(n) \to X_1(z)$$

$$x_2(n) \to X_2(z)$$

$$x_1(n) * x_2(n) \to X_1(z)\, X_2(z)$$

$$Z\{x_1(n) * x_2(n)\} = \sum_{n=-\infty}^{+\infty} \left(x_1(n) * x_2(n)\right) z^{-n}$$

$$= \sum_{n=-\infty}^{+\infty} \left[\sum_{m=-\infty}^{+\infty} x_1(m)\, x_2(n-m)\right] z^{-n}$$

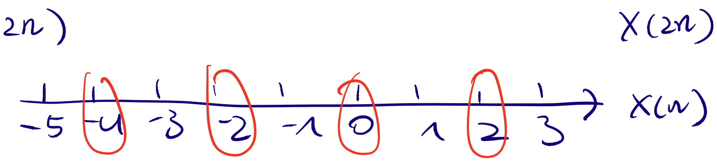
$$= \sum_{m=-\infty}^{+\infty} x_1(m) \underbrace{\left[\sum_{n=-\infty}^{\infty} x_2(n-m)\, z^{-n}\right]}$$

下面第3题证明了 $Z(x(n-m)) = z^{-m} X(z)$

$$= \sum_{m=-\infty}^{+\infty} x_1(m)\, z^{-m} X_2(z)$$

$$= X_1(z)\, X_2(z)$$

2. $X_{down}(n) = x(2n)$



对于 $x(n)$

$$X(z) = \sum_{n=-\infty}^{+\infty} X(n) z^{-n}$$

$$= X(0) + x(1) z^{-1} + \cdots + x(+\infty) z^{-\infty}$$
$$+ x(-1) z + \cdots + x(-\infty) z^{+\infty}$$

$$= X(0) + X(1) z^{-1} + X(2) z^{-2} + X(3) z^{-3} + X(4) z^{-4} + \cdots$$
$$+ X(-1) z + X(-2) z^{2} + X(-3) z^{3} + X(-4) z^{4} + \cdots$$

对于 $X(2n)$ 这个信号 只能拿到 上面绿色部分

即 $X(z) = \sum_{n=-\infty}^{+\infty} X(2n) z^{-n}$

$$= X(0) + X(2) z^{-1} + X(4) z^{-2} + \cdots$$
$$+ X(-2) z^{1} + X(-4) z^{2} + \cdots$$

若此时，令 $z = z^{\frac{1}{2}}$

$$X(z^{\frac{1}{2}}) = X(0) + \cancel{X(1) z^{-\frac{1}{2}}} + X(2) z^{-1} + \cancel{X(3) z^{-\frac{3}{2}}} + X(4) z^{-2} + \cdots$$
$$+ \cancel{X(-1) z^{\frac{1}{2}}} + X(-2) z^{1} + \cancel{X(-3) z^{\frac{3}{2}}} + X(-4) z^{2} + \cdots \quad 1)$$

令 $z = -z^{\frac{1}{2}}$

$$X(z^{-\frac{1}{2}}) = X(0) - X(1) z^{-\frac{1}{2}} + X(2) z^{-1} - X(3) z^{-\frac{3}{2}} + X(4) z^{-2} + \cdots$$
$$- X(-1) z^{\frac{1}{2}} + X(-2) z^{1} - X(-3) z^{\frac{3}{2}} + X(-4) z^{2} + \cdots \quad 2)$$

$$\frac{1) + 2)}{2} = X(0) + X(2)z^{-1} + X(z)z^{-2} + \cdots$$

可以 $X_{down}(z) = \frac{1}{2}X(z^{\frac{1}{2}}) + \frac{1}{2}X(z^{-\frac{1}{2}})$

- $X^{up}(n) = \begin{cases} X(n/2) & n=0,2,4,\cdots \\ 0 & \text{othos} \end{cases}$

$$X^{up}(z) = \sum_{n=-\infty}^{+\infty} X\left(\frac{n}{2}\right) z^{-n} \qquad \text{令 } n=2m$$

$$= \sum_{2m=-\infty}^{+\infty} X(m) z^{-2m}$$

$$= \sum_{2m=-\infty}^{+\infty} X(m) \left(z^{2}\right)^{-m} = X(z^{2})$$

3. ① $X(z) = \sum_{n=-\infty}^{n=+\infty} x(n) z^{-n}$

$X(-z) = \sum_{n=-\infty}^{n=+\infty} x(n) (-z)^{-n}$

当 n 为奇数 $(-z)^{-n} = \frac{1}{(-z)^n} = -z^{-n}$

当 n 为偶数 $(-z)^{-n} = \frac{1}{(-z)^n} = z^{-n}$

若让二式成立 则 奇数次乘 -1. 可用 $(-1)^n$ 控制

所以 $X(-z) = (-1)^n \sum_{n=-\infty}^{+\infty} x(n) z^{-n} = (-1)^n X(z)$

$Z^{-1}(X(-z)) = (-1)^n x(n)$

② $X(z) = \sum_{n=-\infty}^{n=+\infty} x(n) z^{-n} = x(0) z^0 + x(1) z^{-1} + x(2) z^{-2} \cdots$

$+ x(-1) z^1 + x(-2) z^2 \cdots$

对于 $x(-n)$ 信号

$Z\{x(-n)\} = \sum_{n=-\infty}^{+\infty} x(-n) z^{-n}$

若 $z = z^{-1}$ $= \sum_{n=-\infty}^{+\infty} x(-n) z^{n}$ 因为 n 从 $-\infty \to +\infty$

所以 index 左右对称.

$x(-n) \Leftrightarrow X(z^{-1})$

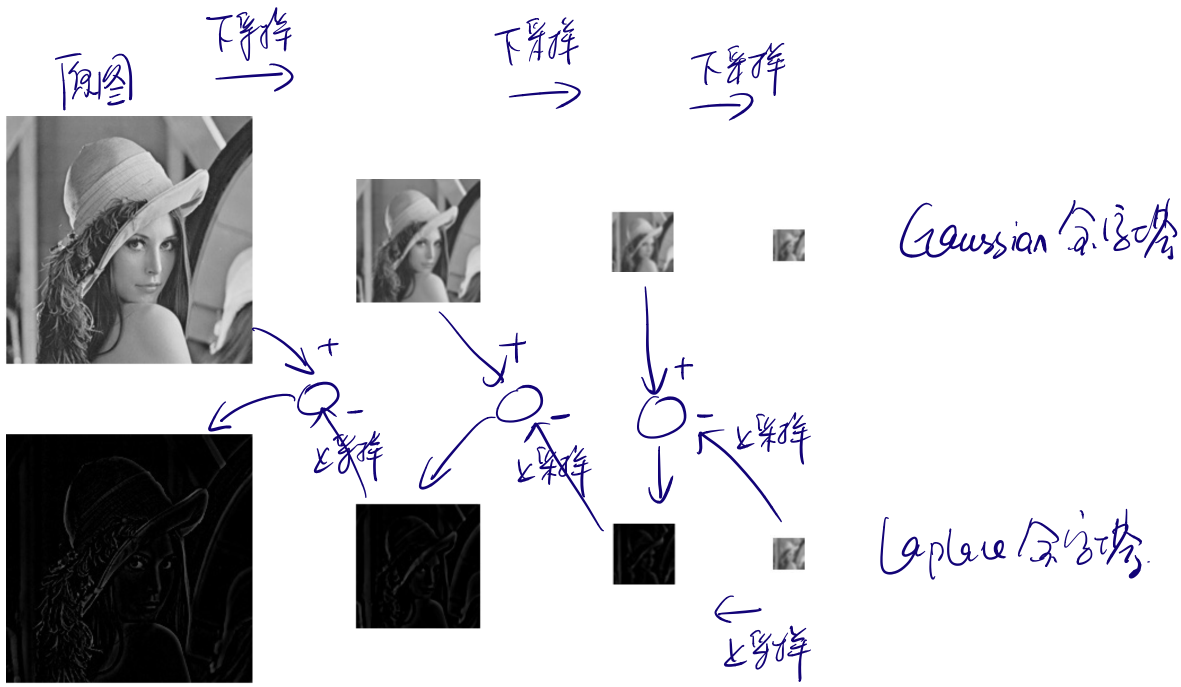
③ $Z\{x(n-k)\} = \sum_{n=-\infty}^{+\infty} x(n-k) z^{-n}$

令 $n-k=p$ $\quad = \sum_{p=-\infty}^{+\infty} x(p) z^{-k} z^{-p}$

$$= z^{-k} \sum_{p=-\infty}^{+\infty} x(p) z^{-p} = z^{-k} X(z)$$

4. Matlab Code：(附上).

下面是结果图.



5. $G_1(z) = -z^{-2k+1}G_0(-z^{-1})$

利用的性质 1. $\mathcal{Z}^{-1}[G_0(z^{-1})] = g_0(-n)$

利用的性质 2. $\mathcal{Z}^{-1}[G_0(-z)] = (-1)^n g_0(n)$

利用的性质 3. $\mathcal{Z}[z^{-k}G_0(z)] = g_0(n-k)$

$$G_0(z) \xrightarrow{①} \underbrace{-z^{-2k+1}G_0(z)}_{T(z)}$$

$$T(z) \xrightarrow{②} T(-z^{-1})$$

$$G_0(z) \bullet\!\!-\!\!-\!\!\circ\ g_0(n)$$

$$z^{-2k+1}G_0(z) \bullet\!\!-\!\!-\!\!\circ\ g_0(n-2k+1) \qquad n = n-2k+1$$

$$T(z) \bullet\!\!-\!\!-\!\!\circ\ t(n)$$

$$T(-z^{-1}) \bullet\!\!-\!\!-\!\!\circ\ (-1)^n t(-n)$$

$$\Rightarrow G_1(z) = (-1)(-1)^n g_0(2k-n-1)$$

$$= (-1)^{n+1} g_0(2k-n-1)$$ 与题目不符.

6. $h_0(n) = g_0(2k-1-n)$ $\qquad h_0(n) = (-1)^n g_1(n)$

$h_1(n) = g_1(2k-1-n)$ $\quad$ 2)

$g_1(n) = (-1)^n g_0(2k-1-n)$

2): 令 $2k-1-n = n$ $\qquad n = 2k-1-n.$

$\Rightarrow h_1(2k-1-n) = g_1(n)$

$$h_1(2k-1-n) = (-1)^n g_0(2k-1-n) = (-1)^n h_0(n)$$

7. $N=16$ $\quad \frac{1}{\sqrt{N}} = \frac{1}{4}$ $\quad$ 16阶Haar变换矩阵

$$\frac{1}{4}\begin{bmatrix}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 & -1 \\
\sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{2} & \sqrt{2} & \sqrt{2} & \sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} & -\sqrt{2} \\
2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & -2 & -2 \\
2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2} & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2\sqrt{2} & -2\sqrt{2}
\end{bmatrix}$$