A brief introduction to Computibility

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Hilbert's 10th Problem: Decidability

David Hilbert's tenth problem was to devise an algorithm (procedure) that tests whether a polynomial has an integral root (*Diophantine eqations*).

The set of such polynomials is easily recognizable but was proved to be not decidable in 1970.

Turing Machines

A Turing Machine (TM) is a theoretical model of computation that consists of:

- A finite set of **states** Q.
- A finite **input alphabet** Σ not containing the blank symbol $\sqcup$.
- A **tape alphabet** Γ such that $\Sigma \subseteq \Gamma$ and $\sqcup \in \Gamma$.
- A **transition function** $\delta : Q \times \Gamma \to Q \times \Gamma \times \{L, R\}$.
- A **start state** $q_0 \in Q$.
- A set of **accept states** $F \subseteq Q$.

The machine operates on an infinite tape, reading and writing symbols, and moving left or right, one cell at a time.

Recognizable & Decidable

The collection of strings M accepts is the ***language*** M, or the ***language*** recognized by M (written $L(M)$). Call a language ***Turing-recognizable*** if some TM recognizes it.

TM M is a ***decider*** if M halts on all inputs. Say that M decides A if $A = L(M)$ and M is a decider. Call a language ***Turing-decidable*** or simply ***decidable*** if some TM decides it.

Enumerators

An ***enumerator*** is a TM with a printer. It starts on a blank tape and can print strings $w_1, w_2, w_3, \ldots,$ possibly forever. For enumerator E say $L(E) = \{w | E \text{ prints } w\}$.

Thm A is T-recognizable iff $A = L(E)$ for some enumerator E.

Recognizable iff Enumerable: Proof Idea

($\Leftarrow$) Simulate E with TM M. For input w, whenever E prints x, test if $x = w$. Accept or continue accordingly.
($\Rightarrow$) Simulate TM M with E on *each* possible input w_i. Let E print accordingly whenever M accepts.

We can do this in a "time-sharing" scheme, for example let M go through 1 transitions on input w_1, then 2 transitions on input w_1 and w_2 respectively, then 3 transitions on input w_1, w_2 and w_3 respectively and so forth. When switching the input that M is currently working on, keep track of the state and the place of the head on the tape and the input together on a block of the tape.

Church-Turing thesis

The definition of ***algorithms*** came in the 1936 papers of Alonzo Church and Alan Turing.Church used a notational system called the λ-calculus to define algorithms. Turing did it with his "machines". These two definitions were shown to be equivalent.

This equivalence relation (in some sense) between the informal notion of algorithm and the precise definition (TM) has come to be called the ***Church–Turing thesis***.

Note that it is about the equivalence between the intuitive and the formal, thus *not* provable. Instead it's a philosophical postulate.

$\mathbb{R}$ is not countable.

- Suppose $\mathbb{R}$ is countable. Then so is the interval $(0,1)$.
- So we can list all real numbers in $(0,1)$ as a sequence:

$$r_1 = 0.\underline{d_{11}}d_{12}d_{13}\ldots \quad r_2 = 0.d_{21}\underline{d_{22}}d_{23}\ldots \quad \ldots$$

- Construct a new number $r = 0.c_1c_2c_3\ldots$ where:

$$c_n \neq d_{nn}, \quad c_n \in \{1,2,\ldots,9\} \setminus \{d_{nn}\}$$

- Then $r \in (0,1)$ but $r \neq r_n$ for all n, since it differs at the n-th digit from r_n.
- Contradiction. So $\mathbb{R}$ is uncountable.

This proof uses *Diagonalization*.

A_{TM} is undecidable.

$A_{\mathrm{TM}} = \{\langle M, w\rangle | M$ is a TM that accepts $w\}$ is undecidable.

Proof.

Assume some TM H decides A_{TM}. We use H to construct TM

$D =$ "On input $\langle M\rangle$

1. Simulate H on input $\langle M, \langle M\rangle\rangle$.
2. Accept if H rejects and reject if H accepts."

Then D accepts $\langle D\rangle$ iff D does not accept $\langle D\rangle$. Hence contradiction. □

This proof uses diagonalization as well. But How?

A_{TM} is undecidable: Diagonalization

TMs	$\langle M_1 \rangle$	$\langle M_2 \rangle$	$\langle M_3 \rangle$	$\langle M_4 \rangle$	$\cdots$	$\langle D \rangle$
M_1	acc	rej	acc	acc	$\cdots$	acc
M_2	rej	rej	rej	rej	$\cdots$	rej
M_3	acc	acc	acc	acc	$\cdots$	acc
M_4	rej	rej	acc	acc	$\cdots$	rej
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
D	rej	acc	rej	rej	$\cdots$	?

Figure: Undecidability hinges on diagonalization.

An unrecognizable language

Thm A language is decidable iff both it and its complement are recognizable.

Observe that A_{TM} is recognizable.

Cor $\overline{A_{\mathrm{TM}}}$ is unrecognizable.

Halting Problem

$H_{\mathrm{TM}} = \{\langle M, w\rangle | M$ halts on input $w\}$ is undecidable.

Proof.

Assume that R decides H_{TM}. Construct TM

$S =$ "On input $\langle M, w\rangle$

1. Simulate R on input $\langle M, w\rangle$. If M does not halt reject.
2. Simulate M on w.
3. Accept if M accepts and reject if M rejects."

Then S decides A_{TM}. Hence contradiction. $\square$

Q&A

Questions?