1 Go to Cartesian coordinates

velocity in Cartesian coordinates:

$$u^{(cart)^\alpha} = u^{(boy)^\mu} \frac{\partial x^{(cart)^\alpha}}{x^{(boyer)^\mu}} \tag{1}$$

Christoffel connection in Cartesian coordinates:

$$\Gamma^{(Cart)^\alpha}{}_{\beta\gamma} = \Gamma^{(Boyer)^\mu}{}_{\sigma\rho} \frac{\partial x^{(cart)^\alpha}}{x^{(boyer)^\mu}} \frac{\partial x^{(boyer)^\sigma}}{x^{(cart)^\beta}} \frac{\partial x^{(boyer)^\rho}}{x^{(cart)^\gamma}} \tag{2}$$

To evaluate $\partial x^{(cart)^\alpha}/x^{(boyer)^\mu}$:

$$\begin{aligned} x =& \sqrt{r^2+a^2}\sin\theta\cos\phi \\ y =& \sqrt{r^2+a^2}\sin\theta\sin\phi \\ z =& r\cos\theta \end{aligned} \tag{3}$$

To evaluate $\partial x^{(boyer)^\sigma}/x^{(cart)^\beta}$:

$$\begin{aligned} & r^2 + a^2 - z^2a^2/r^2 = x^2+y^2+z^2 \\ & z^2\tan^2\theta + a^2\sin^2\theta = x^2+y^2 \\ & \tan\phi = y/x \end{aligned} \tag{4}$$

2 compute up to 7th derivatives

geodesic eq.s with respect to coordinate time t:

$$\frac{d^2x^i}{dt^2} = ... = -\Gamma^i_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt} + \Gamma^0_{\mu\nu}\frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{dx^i}{dt} \tag{5}$$

third, fourth, fifth, sixth, seventh derivatives:

$$\frac{d^3x^i}{dt^3} = -\Gamma^i_{\mu\nu}(\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt} + \frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}) + \Gamma^0_{\mu\nu}(\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d^2x^i}{dt^2}) \tag{6}$$

$$\begin{aligned} \frac{d^4x^i}{dt^4} =& -\Gamma^i_{\mu\nu}(\frac{d^3x^\mu}{dt^3}\frac{dx^\nu}{dt} + 2\frac{d^2x^\mu}{dt^2}\frac{d^2x^\nu}{dt^2} + \frac{dx^\mu}{dt}\frac{d^3x^\nu}{dt^3}) \\ &+ \Gamma^0_{\mu\nu}(\frac{d^3x^\mu}{dt^3}\frac{dx^\nu}{dt}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{d^3x^\nu}{dt^3}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d^3x^i}{dt^3} \\ &+ 2\frac{d^2x^\mu}{dt^2}\frac{d^2x^\nu}{dt^2}\frac{dx^i}{dt} + 2\frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}\frac{d^2x^i}{dt^2} + 2\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt}\frac{d^2x^i}{dt^2}) \end{aligned} \tag{7}$$

$$\begin{aligned} \frac{d^5x^i}{dt^5} =& -\Gamma^i_{\mu\nu}(\frac{d^4x^\mu}{dt^4}\frac{dx^\nu}{dt} + 3\frac{d^3x^\mu}{dt^3}\frac{d^2x^\nu}{dt^2} + 3\frac{d^2x^\mu}{dt^2}\frac{d^3x^\nu}{dt^3} + \frac{dx^\mu}{dt}\frac{d^4x^\nu}{dt^4}) \\ &+ \Gamma^0_{\mu\nu}(\frac{d^4x^\mu}{dt^4}\frac{dx^\nu}{dt}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{d^4x^\nu}{dt^4}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d^4x^i}{dt^4} \\ &+ 3\frac{d^3x^\mu}{dt^3}\frac{d^2x^\nu}{dt^2}\frac{dx^i}{dt} + 3\frac{d^2x^\mu}{dt^2}\frac{d^3x^\nu}{dt^3}\frac{dx^i}{dt} + 3\frac{dx^\mu}{dt}\frac{d^3x^\nu}{dt^3}\frac{d^2x^i}{dt^2} + 3\frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}\frac{d^3x^i}{dt^3} \\ &+ 3\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt}\frac{d^3x^i}{dt^3} + 3\frac{d^3x^\mu}{dt^3}\frac{dx^\nu}{dt}\frac{d^2x^i}{dt^2} + 6\frac{d^2x^\mu}{dt^2}\frac{d^2x^\nu}{dt^2}\frac{d^2x^i}{dt^2}) \end{aligned} \tag{8}$$

$$
\begin{aligned}
\frac{d^6x^i}{dt^6} =& -\Gamma^i_{\mu\nu}(\frac{d^5x^\mu}{dt^5}\frac{dx^\nu}{dt} + 4\frac{d^4x^\mu}{dt^4}\frac{d^2x^\nu}{dt^2} + 6\frac{d^3x^\mu}{dt^3}\frac{d^3x^\nu}{dt^3} + 4\frac{d^2x^\mu}{dt^2}\frac{d^4x^\nu}{dt^4} + \frac{dx^\mu}{dt}\frac{d^5x^\nu}{dt^5}) \\
&+ \Gamma^0_{\mu\nu}(\frac{d^5x^\mu}{dt^5}\frac{dx^\nu}{dt}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{d^5x^\nu}{dt^5}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d^5x^i}{dt^5} \\
&+ 4\frac{d^4x^\mu}{dt^4}\frac{d^2x^\nu}{dt^2}\frac{dx^i}{dt} + 4\frac{d^2x^\mu}{dt^2}\frac{d^4x^\nu}{dt^4}\frac{dx^i}{dt} + 4\frac{dx^\mu}{dt}\frac{d^4x^\nu}{dt^4}\frac{d^2x^i}{dt^2} \\
&+ 4\frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}\frac{d^4x^i}{dt^4} + 4\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt}\frac{d^4x^i}{dt^4} + 4\frac{d^4x^\mu}{dt^4}\frac{dx^\nu}{dt}\frac{d^2x^i}{dt^2} \\
&+ 6\frac{d^3x^\mu}{dt^3}\frac{d^3x^\nu}{dt^3}\frac{dx^i}{dt} + 6\frac{d^3x^\mu}{dt^3}\frac{dx^\nu}{dt}\frac{d^3x^i}{dt^3} + 6\frac{dx^\mu}{dt}\frac{d^3x^\nu}{dt^3}\frac{d^3x^i}{dt^3} \\
&+ 12\frac{d^2x^\mu}{dt^2}\frac{d^2x^\nu}{dt^2}\frac{d^3x^i}{dt^3} + 12\frac{d^2x^\mu}{dt^2}\frac{d^3x^\nu}{dt^3}\frac{d^2x^i}{dt^2} + 12\frac{d^3x^\mu}{dt^3}\frac{d^2x^\nu}{dt^2}\frac{d^2x^i}{dt^2})
\end{aligned}
\tag{9}
$$

$$
\begin{aligned}
\frac{d^7x^i}{dt^7} =& -\Gamma^i_{\mu\nu}(\frac{d^6x^\mu}{dt^6}\frac{dx^\nu}{dt} + 5\frac{d^5x^\mu}{dt^5}\frac{d^2x^\nu}{dt^2} + 10\frac{d^4x^\mu}{dt^4}\frac{d^3x^\nu}{dt^3} + 10\frac{d^3x^\mu}{dt^3}\frac{d^4x^\nu}{dt^4} + 5\frac{d^2x^\mu}{dt^2}\frac{d^5x^\nu}{dt^5} + \frac{dx^\mu}{dt}\frac{d^6x^\nu}{dt^6}) \\
&+ \Gamma^0_{\mu\nu}(\frac{d^6x^\mu}{dt^6}\frac{dx^\nu}{dt}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{d^6x^\nu}{dt^6}\frac{dx^i}{dt} + \frac{dx^\mu}{dt}\frac{dx^\nu}{dt}\frac{d^6x^i}{dt^6} \\
&+ 5\frac{d^5x^\mu}{dt^5}\frac{d^2x^\nu}{dt^2}\frac{dx^i}{dt} + 5\frac{d^2x^\mu}{dt^2}\frac{d^5x^\nu}{dt^5}\frac{dx^i}{dt} + 5\frac{dx^\mu}{dt}\frac{d^5x^\nu}{dt^5}\frac{d^2x^i}{dt^2} \\
&+ 5\frac{dx^\mu}{dt}\frac{d^2x^\nu}{dt^2}\frac{d^5x^i}{dt^5} + 5\frac{d^2x^\mu}{dt^2}\frac{dx^\nu}{dt}\frac{d^5x^i}{dt^5} + 5\frac{d^5x^\mu}{dt^5}\frac{dx^\nu}{dt}\frac{d^2x^i}{dt^2} \\
&+ 10\frac{d^4x^\mu}{dt^4}\frac{d^3x^\nu}{dt^3}\frac{dx^i}{dt} + 10\frac{d^3x^\mu}{dt^3}\frac{d^4x^\nu}{dt^4}\frac{dx^i}{dt} + 10\frac{dx^\mu}{dt}\frac{d^4x^\nu}{dt^4}\frac{d^3x^i}{dt^3} \\
&+ 10\frac{dx^\mu}{dt}\frac{d^3x^\nu}{dt^3}\frac{d^4x^i}{dt^4} + 10\frac{d^3x^\mu}{dt^3}\frac{dx^\nu}{dt}\frac{d^4x^i}{dt^4} + 10\frac{d^4x^\mu}{dt^4}\frac{dx^\nu}{dt}\frac{d^3x^i}{dt^3} \\
&+ 20\frac{d^2x^\mu}{dt^2}\frac{d^2x^\nu}{dt^2}\frac{d^4x^i}{dt^4} + 20\frac{d^2x^\mu}{dt^2}\frac{d^4x^\nu}{dt^4}\frac{d^2x^i}{dt^2} + 20\frac{d^4x^\mu}{dt^4}\frac{d^2x^\nu}{dt^2}\frac{d^2x^i}{dt^2} \\
&+ 30\frac{d^3x^\mu}{dt^3}\frac{d^3x^\nu}{dt^3}\frac{d^2x^i}{dt^2} + 30\frac{d^3x^\mu}{dt^3}\frac{d^2x^\nu}{dt^2}\frac{d^3x^i}{dt^3} + 30\frac{d^2x^\mu}{dt^2}\frac{d^3x^\nu}{dt^3}\frac{d^3x^i}{dt^3})
\end{aligned}
\tag{10}
$$

3 compute 5th, 6th derivatives of Quadrupole

ATTENTION: in the text below, all indexes are in subscripts, consistent with PRD 52 R3159. They are not in covariant form.

Quadrupole mass moment and Quadrupole current moment:

$$
\begin{aligned}
I_{ij}/\mu =& x_i x_j \\
J_{ij}/\mu =& x_i \epsilon_{jkm} x_k \dot{x}_m - \frac{3}{2} a x_i \delta_{j3}
\end{aligned}
\tag{11}
$$

5th and 6th derivatives (superscripts ${}^{(s)}$ means sth order derivatives to t):

$$
I^{(5)}_{ij}/\mu = x^{(5)}_i x_j + 5x^{(4)}_i x^{(1)}_j + 10x^{(3)}_i x^{(2)} x_j + 10x^{(2)}_i x^{(3)}_j + 5x^{(1)}_i x^{(4)}_j + x_i x^{(5)}_j \tag{12}
$$

$$
\begin{aligned}
J_{ij}^{(5)}/\mu = \epsilon_{jkm}(& x_i^{(5)} x_k x_m^{(1)} + x_i x_k^{(5)} x_m^{(1)} + x_i x_k x_m^{(6)} \\
&+ 5x_i^{(4)} x_k^{(1)} x_m^{(1)} + 5x_i^{(1)} x_k^{(4)} x_m^{(1)} + 5x_i x_k^{(4)} x_m^{(2)} + 5x_i^{(4)} x_k x_m^{(2)} + 5x_i^{(1)} x_k x_m^{(5)} + 5x_i x_k^{(1)} x_m^{(5)} \\
&+ 10x_i^{(3)} x_k^{(2)} x_m^{(1)} + 10x_i^{(2)} x_k^{(3)} x_m^{(1)} + 10x_i^{(3)} x_k x_m^{(3)} + 10x_i x_k^{(3)} x_m^{(3)} + 10x_i^{(2)} x_k x_m^{(4)} + 10x_i x_k^{(2)} x_m^{(4)} \\
&+ 20x_i^{(3)} x_k^{(1)} x_m^{(2)} + 20x_i^{(1)} x_k^{(3)} x_m^{(2)} + 20x_i^{(1)} x_k^{(1)} x_m^{(4)} \\
&+ 30x_i^{(2)} x_k^{(2)} x_m^{(2)} + 30x_i^{(2)} x_k^{(1)} x_m^{(3)} + 30x_i^{(1)} x_k^{(2)} x_m^{(3)}) - \frac{3}{2} a x_i^{(5)} \delta_{j3}
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
J_{ij}^{(6)}/\mu = \epsilon_{jkm}(& x_i^{(6)} x_k x_m^{(1)} + x_i x_k^{(6)} x_m^{(1)} + x_i x_k x_m^{(7)} \\
&+ 6x_i^{(5)} x_k^{(1)} x_m^{(1)} + 6x_i^{(1)} x_k^{(5)} x_m^{(1)} + 6x_i^{(5)} x_k x_m^{(2)} + 6x_i x_k^{(5)} x_m^{(2)} + 6x_i^{(1)} x_k x_m^{(6)} + 6x_i x_k^{(1)} x_m^{(6)} \\
&+ 15x_i^{(4)} x_k^{(2)} x_m^{(1)} + 15x_i^{(2)} x_k^{(4)} x_m^{(1)} + 15x_i^{(4)} x_k x_m^{(3)} + 15x_i x_k^{(4)} x_m^{(3)} + 15x_i^{(2)} x_k x_m^{(5)} + 15x_i x_k^{(2)} x_m^{(5)} \\
&+ 20x_i^{(3)} x_k^{(3)} x_m^{(1)} + 20x_i^{(3)} x_k x_m^{(4)} + 20x_i x_k^{(3)} x_m^{(4)} \\
&+ 30x_i^{(4)} x_k^{(1)} x_m^{(2)} + 30x_i^{(1)} x_k^{(4)} x_m^{(2)} + 30x_i^{(1)} x_k^{(1)} x_m^{(5)} \\
&+ 60x_i^{(3)} x_k^{(2)} x_m^{(2)} + 60x_i^{(2)} x_k^{(3)} x_m^{(2)} + 60x_i^{(3)} x_k^{(1)} x_m^{(3)} + 60x_i^{(1)} x_k^{(3)} x_m^{(3)} + 60x_i^{(2)} x_k^{(1)} x_m^{(4)} + 60x_i^{(1)} x_k^{(2)} x_m^{(4)} \\
&+ 90x_i^{(2)} x_k^{(2)} x_m^{(3)}) - \frac{3}{2} a x_i^{(6)} \delta_{j3}
\end{aligned}
\tag{14}
$$

4 Compute radiative acceleration

Acceleration(Ref: PRD 52 R3159):

$$
\begin{aligned}
a_j = &- \frac{2}{5} I_{jk}^{(5)} x_k + \frac{16}{45} \epsilon_{jpq} J_{pk}^{(6)} x_q x_k + \frac{32}{45} \epsilon_{jpq} J_{pk}^{(5)} x_k \dot{x}_q \\
&+ \frac{16}{45} \epsilon_{pqj} J_{kp}^{(5)} x_q \dot{x}_k - \frac{16}{45} \epsilon_{pqk} J_{jp}^{(5)} x_q \dot{x}_k + \frac{8}{15} a J_{3j}^{(5)}
\end{aligned}
\tag{15}
$$

ATTENTION: in the text above, all indexes are in subscripts, consistent with PRD 52 R3159. They are not in covariant form.

Transform to derivative with respect to τ ($x_\dagger$ means the contribution from radiative form to coordinate deviation):

$$
a = \frac{d^2 x_\dagger}{dt^2} = \frac{d\tau}{dt} \frac{d}{d\tau} \left(\frac{dx_\dagger}{d\tau} / \frac{dt}{d\tau} \right) \tag{16}
$$

$$
\frac{d^2 x_\dagger}{d\tau^2} = a \left(\frac{dt}{d\tau} \right)^2 - \frac{dx_\dagger}{d\tau} \frac{d^2 t}{d\tau^2} / \frac{dt}{d\tau} \tag{17}
$$

Transform to Boyer-Lindquist coordinates:

$$
\frac{d^2 x_\dagger^\mu}{d\tau^2} \Big|_{boy} = \frac{d^2 x_\dagger^\alpha}{d\tau^2} \Big|_{car} \frac{dx^{(boy)^\mu}}{dx^{(car)^\alpha}} \tag{18}
$$