Note on the spin angular momentum of scalar field in classical field theory

It is known that, for a system defined by the Lagrangian $\mathcal{L}(\phi, \partial_\mu\phi, x_\mu)$, the scalar field does not has spin angular momentum (SAM). However, this conclusion is dependent on a strong condition that the Lagrangian density $\mathcal{L}$ is only dependent on ϕ and its first order derivative. When there are higher order terms in the Lagrangian, e.g. elastic system with only longitudinal wave $\vec{u} = \vec{\nabla}\phi$, the SAM does not vanish, as shown in the following text. This property is purely classical, i.e. does not involve relativistic or quantum effects. In the following we use Greek letters $\mu, \nu, ...$ to denote spacetime indexes with 0 being time and Latin letters $i, j, k, ...$ to denote spatial indexes

1 Lagrangian formulation of classical field theory

1.1 scalar field

For a scalar field ϕ and a Lagrangian $\mathcal{L}$ dependent up to first order derivative of ϕ, we define the action $S = \int d^4x\mathcal{L}(\phi, \partial_\mu\phi, x_\mu)$. The evolution of ϕ over spacetime is determined by imposing the variation of action δS induce by variation of the field $\delta\phi$ is zero. This gives us the Euler-Lagrangian equation:

$$\frac{\partial\mathcal{L}}{\partial\phi} = \partial_\mu \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} \tag{1}$$

Using Noether's theorem, we are able to define the energy-momentum tensor and angular momentum endowed by this field.

(1) First, consider the translational transformation, the symmetry of which will lead to energy-momentum tensor. Consider an infinitesimal spacetime translation of the whole system by a_μ (an infinitesimal vector), the coordinate will transform as:

$$x_\mu \to x_\mu - a_\mu \tag{2}$$

i.e. the variation of x_μ is $\delta x_\mu = -a_\mu$. Thus induce the variation of the field $\delta\phi = -\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$. The variation of the Lagrangian is $\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi}\delta\phi + \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\delta\partial_\nu\phi + \frac{\partial\mathcal{L}}{\partial x^\mu}\delta x^\mu$. Inserting the Euler-Lagrangian equation and after some algebra, we have:

$$\delta\mathcal{L} = \partial_\nu \left(\frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)} \frac{\partial\phi}{\partial x_\mu} - \delta_{\mu\nu}\mathcal{L} \right) a_\mu \tag{3}$$

The translational symmetry means that the Lagrangian density is invariant under this transformation, i.e. $\delta\mathcal{L} = 0$. Therefore we have the energy-momentum tensor $T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\nu\phi)}\frac{\partial\phi}{\partial x_\mu} - \delta_{\mu\nu}\mathcal{L}$, which satisfies $\partial_\nu T_{\mu\nu} = 0$. The conservation of energy and momentum is that $\int_V d^3x\partial_t T_{\mu 0} = -\int_V d^3x\partial_i T_{\mu i} =$ boundary terms at infinity $\to 0$.

(2) Then, consider the rotational transformation, the symmetry of which will lead to angular momentum tensor. Consider a infinitesimal rotational transformation of the whole system by an infinitesimal angle along $\vec{\theta}$. The coordinate transforms as:

$$x_j \to x_j - \theta_i \epsilon_{jik} x_k \tag{4}$$

i.e. the variation of x_j is $-\theta_i \epsilon_{jik} x_k$. Thus induce the variation of the field $\delta\phi = -\frac{\partial \phi}{\partial x_\nu} \delta x_\nu$ (the field itself does not rotate since the scalar field does not have orientation, but this is not the case for vector field). Again, the variation of the Lagrangian is $\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial \phi}\delta\phi + \frac{\partial \mathcal{L}}{\partial(\partial_\nu \phi)}\delta\partial_\nu\phi + \frac{\partial \mathcal{L}}{\partial x^\mu}\delta x^\mu$. Inserting all above and after some algebra, we have:

$$\delta\mathcal{L} = \theta_i \partial_\mu \left(T_{j\mu} \epsilon_{jik} x_k\right) \tag{5}$$

Similarly, the rotational symmetry leads to conservation of angular momentum $L_i = \epsilon_{ikj} x_k T_{j0}$ in that $\int_V d^3x \partial_t L_i = \int_V d^3x \partial_l T_{jl} \epsilon_{jik} x_k =$ boundary terms at infinity $\to 0$. We can see that all the angular momentum is dependent on choice of reference point x_μ (orbital angular momentum). This is the classical conclusion that scalar field has no SAM.

1.2 vector field

For a vector field u_i, we have similar Lagrangian description for the dynamics of the field with $\mathcal{L} = \mathcal{L}(u_i, \partial_\mu u_i, x_\mu)$. The Euler-Lagrangian equation is:

$$\frac{\partial \mathcal{L}}{\partial u_i} = \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu u_i)} \tag{6}$$

We similarly use the Noether's theorem to construct energy-momentum tensor and angular momentum

(1) consider an infinitesimal translation by a_μ:

$$x_\mu \to x_\mu - a_\mu \tag{7}$$

i.e. $\delta x_\mu = -a_\mu$. The vector field itself does not change but change with the spatial coordinate.

$$\delta u_i = -\frac{\partial u_i}{\partial x_\nu} \delta x_\nu \tag{8}$$

The variation of Lagrangian will be

$$\delta\mathcal{L} = \partial_\nu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\nu u_i)} \frac{\partial u_i}{\partial x_\mu} - \delta_{\mu\nu}\mathcal{L} \right) a_\mu \tag{9}$$

Thus we have the energy-momentum tensor $T_{\mu\nu} = \frac{\partial \mathcal{L}}{\partial(\partial_\nu u_i)} \frac{\partial u_i}{\partial x_\mu} - \delta_{\mu\nu}\mathcal{L}$. $\delta\mathcal{L} = 0$ means $\partial_\nu T_{\mu\nu} = 0$. Then $\int d^3x T_{\mu 0}$ is conserved. $\int d^3x T_{00}$ is the total energy endowed by the field and $\int d^3x T_{i0}$ is the momentum endowed by the field.

(2) Consider a infinitesimal rotational transformation of the whole system by an infinitesimal angle along $\vec{\theta}$. The coordinate transforms as:

$$x_j \to x_j - \theta_i \epsilon_{jik} x_k \tag{10}$$

i.e. the variation of x_j is $-\theta_i \epsilon_{jik} x_k$.

In the vector case, the field not only changes with respect to the change of coordinate, but also changes itself due to rotation, which is absent in scalar case. The total variation of the vector field is:

$$\delta u_i = \theta_i \epsilon_{jik} u_k - \frac{\partial u_i}{\partial x_\nu} \delta x_\nu \tag{11}$$

The variation of the Lagrangian is:

$$\delta\mathcal{L} = \theta_i \partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u_j)} \epsilon_{jik} u_k + T_{j\mu} \epsilon_{jik} x_k \right] \tag{12}$$

$\delta\mathcal{L} = 0$ means $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu u_j)} \epsilon_{jik} u_k + T_{j\mu} \epsilon_{jik} x_k \right] = 0$. Then we have the spin angular momentum and the orbital angular momentum:

$$s_i = \frac{\partial \mathcal{L}}{\partial(\partial_t u_j)} \epsilon_{jik} u_k \; : \; \vec{s} = \vec{u} \times \frac{\partial \mathcal{L}}{\partial(\partial_t \vec{u})} \tag{13}$$

$$j_i = T_{j0} \epsilon_{jik} x_k \; : \; \vec{j} = \vec{x} \times \vec{T}_{\cdot 0} \tag{14}$$

The total angular momentum is conserved in that:

$$\partial_t \left(s_i + j_i \right) + \nabla \cdot (\ldots) = 0 \Rightarrow \partial_t \int d^3 (\vec{s} + \vec{j}) = 0 \tag{15}$$

It is worth noting that the SAM is dependent on specific physical system (i.e. the Lagrangian $\mathcal{L}$) and, generally, has nothing to do with its curl $\vec{\nabla} \times \vec{u}$.

2 Elastic Wave

Then we apply the classical field theory to elastic waves. The Lagrangian for elastic field u_i is (see Eq. (7) of [1]):

$$\mathcal{L} = \frac{1}{2} \left(\rho \dot{u}_i \dot{u}_i - c_{ijkl} u_{i,j} u_{k,l} \right) \tag{16}$$

The Euler-Lagrangian equation leads to the general equation for elastic fields:

$$\rho \ddot{u}_i = \frac{\partial}{\partial x_j} \left(c_{ijkl} u_{k,l} \right) \tag{17}$$

where c_{ijkl} specifies the mechanical properties of the medium. Using the result above, the SAM density of elastic wave is

$$\vec{s} = \vec{u} \times \frac{\partial \mathcal{L}}{\partial \dot{\vec{u}}} = \rho \vec{u} \times \dot{\vec{u}} \tag{18}$$

Here comes the myth: If the wave is longitudinal, i.e. $\vec{u} = \vec{\nabla}\phi$ where ϕ is a scalar, still we should have non-vanishing SAM: $\rho \vec{\nabla}\phi \times \vec{\nabla}\dot{\phi}$. On the other hand, if we forget the vector field u_i, the scalar field obey an Euler-Lagrangian equation of itself. But why does this scalar field have SAM? Because its corresponding Lagrangian have higher order derivatives of ϕ, e.g. inserting $u_i = \partial_i \phi$ into Eq. 16 will leads to second order derivatives. Then the discussion in the previous section is not valid. The clarification is shown in the following.

3 Classical field theory with higher order derivatives

We restrict our scope to the Lagrangian of elastic field. The Lagrangian density for the scalar field ϕ is simply inserting $u_i = \partial_i \phi$ into Eq. 16:

$$\mathcal{L}(\partial_\mu \partial_j \phi, x_\mu) = \frac{1}{2} \left(\rho \partial_i \partial_t \phi \partial_i \partial_t \phi - c_{ijkl} \partial_i \partial_j \phi \partial_k \partial_l \phi \right) \tag{19}$$

We first check that the Euler Lagrangian equation is still the elastic wave equation. Consider a variation $\partial_i\phi \to \partial_i\phi + \delta\partial_i\phi$ (ϕ can be infinitely large at infinity since only $\vec{\nabla}\phi$ is physical). The variation of action is

$$\begin{aligned}\delta S &= \int d^4x \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_i\phi)}\partial_\mu\partial_i\delta\phi \\ &= -\int d^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_i\phi)}\right)\delta\partial_i\phi + \int d^4x\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_i\phi)}\delta\partial_i\phi\right)\end{aligned} \tag{20}$$

The second is the boundary term at infinity and vanishes. Note that we cannot do an additional integration-by-part since ϕ can be large at infinity. Then the Euler Lagrangian equation should be:

$$\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_i\phi)}\right) = 0 \tag{21}$$

This is the same as elastic field equation:

$$\begin{aligned}(\mu = t) : \partial_t\left(\frac{\partial\mathcal{L}}{\partial(\partial_t\partial_i\phi)}\right) &= \quad\partial_t\left(\rho\partial_i\partial_t\phi\right) = \rho\partial_t\partial_t u_i \\ (\mu = j) : \partial_j\left(\frac{\partial\mathcal{L}}{\partial(\partial_j\partial_i\phi)}\right) &= \quad -\partial_j\left(c_{ijkl}\partial_k\partial_l\phi\right) = -\partial_j\left(c_{ijkl}\partial_l u_k\right)\end{aligned} \tag{22}$$

$$(\text{in total})\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_i\phi)}\right) = \rho\partial_t\partial_t u_i - \partial_j\left(c_{ijkl}\partial_l u_k\right) \tag{23}$$

which is the same as Eq. 17.

Then we use the Noether's Theorem to give the energy-momentum tensor and the angular momentum endowed by ϕ.

(1) Consider a spacetime translation by infinitesimal a_μ ($\delta x_\mu = -a_\mu$)

$$x_\mu \to x_\mu - a_\mu \tag{24}$$

Variation of Lagrangian $\mathcal{L}(\partial_\mu\partial_j\phi, x_\mu)$ is:

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_j\phi)}\delta\partial_\mu\partial_j\phi + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu \tag{25}$$

Do an integration-by-part

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_j\phi)}\delta\partial_j\phi\right) - \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_j\phi)}\right)\delta\partial_j\phi + \frac{\partial\mathcal{L}}{\partial x_\mu}\delta x_\mu \tag{26}$$

By Euler Lagrangian equation, the second term vanishes. Then we give the $\delta\partial_j\phi$ under spacetime translation. The variation of ϕ is $\delta\phi = -\frac{\partial\phi}{\partial x_\nu}\delta x_\nu$. Then $\delta\partial_j\phi = -\partial_j\partial_\mu\phi\delta x_\mu$ (Since the variation is constant, $\partial_j\delta x_\mu = 0$). Then the variation of Lagrangian is:

$$\delta\mathcal{L} = \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_j\phi)}\partial_j\partial_\nu\phi - \delta_{\mu\nu}\mathcal{L}\right)a_\nu \tag{27}$$

So the energy-momentum tensor is

$$T_{\mu\nu} = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\partial_j\phi)}\partial_j\partial_\nu\phi - \delta_{\mu\nu}\mathcal{L} \tag{28}$$

(2) Consider an infinitesimal rotation by $\vec{\theta}$

$$x_j \to x_j - \theta_i \epsilon_{jik} x_k \tag{29}$$

i.e. the variation of x_j is $-\theta_i \epsilon_{jik} x_k$.

Still, the variation of Lagrangian is:

$$\delta\mathcal{L} = \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_i \phi)} \delta \partial_i \phi \right) + \frac{\partial \mathcal{L}}{\partial x_\mu} \delta x_\mu \tag{30}$$

The variation of ϕ is $\delta\phi = -\frac{\partial \phi}{\partial x_j} \delta x_j$. Then the variation of $\partial_i \phi$ is

$$\delta \partial_i \phi = -\partial_i \partial_j \phi \delta x_j - \partial_j \phi \partial_i (\delta x_j) \tag{31}$$

Note that this time the second term does not vanish and it will contribute to the SAM. Since $\delta x_j = -\theta_l \epsilon_{jlk} x_k$, $\partial_i(\delta x_j) = -\theta_i \epsilon_{jlk} \delta_{ki}$, so we have

$$\delta \partial_i \phi = -\partial_i \partial_j \phi \delta x_j + \partial_j \phi \theta_l \epsilon_{jlk} \delta_{ki} \tag{32}$$

Inserting into the variation of Lagrangian:

$$\begin{aligned} \delta\mathcal{L} =& \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_i \phi)} \partial_j \phi \theta_l \epsilon_{jli} \right) \\ & - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_i \phi)} \partial_j \partial_i \phi - \delta_{\mu j} \right) \delta x_j \\ =& \theta_i \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_k \phi)} \partial_j \phi \epsilon_{jik} \right) + \theta_i \partial_\mu \left(\epsilon_{jik} T_{\mu j} x_k \right) \end{aligned} \tag{33}$$

$\delta\mathcal{L} = 0$ means $\partial_\mu \left[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_k \phi)} \epsilon_{ijk} \partial_j \phi + T_{\mu j} \epsilon_{jik} x_k \right] = 0$. Then we have the spin angular momentum and the orbital angular momentum:

$$s_i = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_k \phi)} \partial_j \phi \epsilon_{jik} \ : \ \vec{s} = \vec{\nabla} \phi \times \frac{\partial \mathcal{L}}{\partial(\partial_t \vec{\nabla} \phi)} \tag{34}$$

$$j_i = T_{j0} \epsilon_{jik} x_k \ : \ \vec{j} = \vec{x} \times \vec{T}_{\cdot\, 0} \tag{35}$$

References

[1] Vincent Laude and Jean-Charles Beugnot. Lagrangian description of brillouin scattering and electrostriction in nanoscale optical waveguides. *New Journal of Physics*, 17(12):125003, 2015.