

## Refraction and polarized light

### 10.1 Refraction

Our picture of the radiation has to be modified when the atoms, ions, and electrons respond coherently to the oscillating electric field. This gives the medium a dielectric constant and therefore an index of refraction, and the transport of radiation is modified. The discussion we give here of refraction effects in radiation transport is derived from Cox and Giuli (1968).

The material in a volume element  $dV$  can have a mean electric dipole moment  $\mathbf{p}$ , induced by the local electric field  $\mathcal{E}$ , given by

$$\mathbf{p} = \chi \mathcal{E} dV, \quad (10.1)$$

where  $\chi$  is the AC electric susceptibility of the material. This coherent oscillating dipole modifies the propagation of the electromagnetic waves, since now there is a dielectric constant

$$\epsilon = 1 + 4\pi \chi, \quad (10.2)$$

and the phase velocity of electromagnetic waves will be

$$v_p = \frac{c}{n}, \quad (10.3)$$

where  $n = \sqrt{\epsilon}$  is the refractive index. (We will continue to use  $\mathbf{n}$  for the direction vector of photon propagation. The scalar symbol will always be the refractive index, and the vector will refer to the photon direction.) The susceptibility, dielectric constant, and refractive index may all be complex, since the atomic dipoles do not necessarily oscillate just in phase with the local electric field.

This continuous-medium picture that leads to the dielectric constant is the result of another application of coarse graining. It does not make sense unless the wavelength of the electromagnetic wave is large compared with the particle spacing. When this is done more carefully, the result is found to depend on the two-particle

correlation functions for the plasma species at the values of  $\mathbf{k}$  and  $\omega$  appropriate for the electromagnetic wave. We cannot go into that level of detail here.

The very simplest example of the plasma susceptibility is provided by the free electrons. The susceptibility per electron is found by solving an equation of motion which we write

$$m\ddot{\mathbf{r}} + m\nu_c\dot{\mathbf{r}} = -e\mathcal{E}, \quad (10.4)$$

in which  $\mathcal{E}$  is presumed to oscillate as  $\exp(-i\omega t)$ . We have included a “drag” force  $-m\nu_c\dot{\mathbf{r}}$  that represents heuristically the electron–ion collisions that produce relaxation of the electron velocities to the mean. This term would be the right-hand side  $(\partial f/\partial t)_{\text{coll}}$  term in the Boltzmann equation for the electron distribution function. The solution for the electron displacement is

$$\mathbf{r} = \frac{e\mathcal{E}}{m(\omega^2 + i\omega\nu_c)}, \quad (10.5)$$

so the susceptibility per electron is

$$\alpha = -\frac{e\mathbf{r}}{\mathcal{E}} = -\frac{e^2}{m} \frac{1}{\omega^2 + i\omega\nu_c}. \quad (10.6)$$

The complex dielectric constant becomes

$$\epsilon = 1 - \frac{\omega_{pe}^2}{\omega^2 + i\omega\nu_c}, \quad (10.7)$$

where  $\omega_{pe}$  is the electron plasma frequency  $\sqrt{4\pi N_e e^2/m}$ . The collision frequency  $\nu_c$  is generally quite small, and if it is neglected the dielectric constant is real and the refractive index is  $n = \sqrt{1 - (\omega_{pe}/\omega)^2}$  provided that  $\omega$  is above the plasma frequency. For frequencies below the plasma frequency electromagnetic waves cannot propagate in this picture.

Let us return to the case that  $\epsilon$  and  $n$  have a small imaginary part and see what that does to the propagation of the wave. The wave vector is  $\omega/v_p \hat{\mathbf{k}} = n\omega \hat{\mathbf{k}}/c$ . The exponential  $\exp(i\mathbf{k} \cdot \mathbf{r})$  contains a factor  $\exp[-\Im(n)\omega \hat{\mathbf{k}} \cdot \mathbf{r}/c]$ , which corresponds to an absorptivity, i.e., attenuation coefficient of the energy flux, of

$$k_v = \frac{2\omega}{c} \Im(n). \quad (10.8)$$

We can also express this as

$$k_v = \frac{\Im(\epsilon)\omega}{\Re(n)c} = \frac{\omega_{pe}^2 \nu_c}{\Re(n)c(\omega^2 + \nu_c^2)}. \quad (10.9)$$

The collision frequency, for Coulomb collisions with ions  $Z$ , is a number of order  $N_i(Ze^2/kT)^2\sqrt{kT/m}$ , and when this and the definition of the plasma frequency are inserted, the absorptivity for  $\omega \gg \nu_c$  is found to be of order

$$k_\nu \approx \frac{4\pi Z^2 e^6}{c(mkT)^{3/2} \omega^2 \Re(n)} N_e N_i. \quad (10.10)$$

This is nothing more or less than the free-free absorptivity, *after correction for stimulated emission*, that we discussed earlier (cf. (8.79)), apart from the numerical factors and the factor of the real refractive index in the denominator. We see that free-free absorption emerges from a completely classical plasma physics discussion, in which, if need be, the correlations of the plasma species can be included.

If the oscillating dipoles represented by all the electrons are added incoherently instead of coherently, then each one produces scattered radiation according to the Rayleigh-scattering formula  $\sigma = (8\pi/3)k^4|\alpha|^2$ , which becomes

$$\sigma = \frac{8\pi}{3} \left( \frac{e^2}{mc^2} \right)^2 \frac{\omega^2}{\omega^2 + \nu_c^2}. \quad (10.11)$$

This is exactly Thomson scattering except for the roll-off at very low frequency. Thus plasma dispersion and Thomson scattering are alternatives, and which occurs under a particular set of plasma conditions depends on the correlation functions. All three processes, Thomson scattering, free-free absorption, and plasma dispersion, must be treated in a unified way for dense plasmas.

Returning to the case of a real index of refraction, we inquire next what happens when we apply quantization to the electromagnetic field defined using the macroscopic fields that account for the dielectric contribution. We must understand that quanta of these fields do not represent just a disturbance of  $\mathcal{E}$  and  $\mathcal{H}$  traveling through space, but include the sympathetic response of the plasma as well. The energy in such waves is really partly in field energy proper, and partly in material energy. If we decline to use the macroscopic fields for our photon states, then we will find that the transport equation, which itself results from a coarse-graining procedure, will transport radiation at the wrong rate or in the wrong direction.

Our “dressed” photons, the quanta of the macroscopic electromagnetic field, still have the energy  $\hbar\omega$ , but their momentum is  $\hbar\mathbf{k} = n\omega\hat{\mathbf{k}}/c$ . The number of photon states of a particular (transverse) polarization in a volume  $V$  in a range  $dk$  of  $k$  and a solid angle  $d\Omega$  is still

$$\frac{V k^2 dk d\Omega}{(2\pi)^3}, \quad (10.12)$$

but when  $k$  is replaced by its value in terms of  $\omega$  this becomes

$$\frac{n^2 \omega^2 d\Omega d\omega}{(2\pi)^3 c^2 v_g}, \quad (10.13)$$

where  $v_g$  is the group velocity

$$v_g \equiv \frac{d\omega}{dk}. \quad (10.14)$$

The thermodynamic equilibrium intensity can be calculated by multiplying the number of photons per mode, the energy per photon, the number of modes per unit volume per unit frequency per unit solid angle and the group velocity of the photon. The Bose–Einstein value for the number of photons per mode in thermodynamic equilibrium at temperature  $T$  is unchanged, as is the energy of the photon, while the number of modes is multiplied by  $n^2 c/v_g$  and the group velocity is multiplied by  $v_g/c$ ; we conclude that the equilibrium intensity is altered by the dielectric according to

$$B_\nu \rightarrow n^2 B_\nu. \quad (10.15)$$

The modification of the expansion of the vector potential  $\mathbf{A}$  in the creation and annihilation operators requires a careful calculation of the energy density including the material contribution for a dispersive medium, see Landau and Lifshitz (1960). This leads to a modification of the multiplicative factor  $(2\pi\hbar c^2/\omega V)^{1/2}$  to  $(2\pi\hbar c v_g/n\omega V)^{1/2}$ . As a result the cross section for emission processes is modified by a factor  $n^2 c/v_g$  from the density of states times a factor  $v_g/cn$  from the  $\mathbf{A}$  expansion, or a factor  $n$  overall. The cross sections for photoabsorption are modified by a factor  $v_g/cn$  from the  $\mathbf{A}$  expansion and a factor  $c/v_g$  from the division by the photon flux, or a factor  $1/n$  overall. This gives the results quoted by Cox and Giuli (1968),

$$k_\nu \rightarrow \frac{k_\nu^0}{n}, \quad (10.16)$$

$$j_\nu \rightarrow n j_\nu^0, \quad (10.17)$$

where the quantities with superscript  $^0$  are the ones given by the atomic physics in the absence of refraction. See also Mercier (1964) and Dawson and Oberman (1962). It is an interesting and unsettled question how general the relation is between the refracting absorptivity and a hypothetical nonrefracting absorptivity.

The transport equation including refraction is modified in three respects. First, the rays are bent, so the transport operator must correspond to differentiation along a curved path. Second, the absorption and emission must use the modified values

of absorptivity and emissivity. The third correction is that owing to the refraction, which can produce focusing or defocusing of a beam, the intensity is not itself constant, even in the absence of absorption and emission. It is  $I_\nu/n^2$  that is constant. This can be shown either by calculating how a beam expands in solid angle following a cluster of rays, or by invoking the second law of thermodynamics and the relation (10.15). The transport equation as modified is

$$\frac{1}{c} \frac{\partial(I_\nu/n^2)}{\partial t} + \mathbf{n} \cdot \nabla \left( \frac{I_\nu}{n^2} \right) + \frac{d\mathbf{n}}{ds} \cdot \nabla \left( \frac{I_\nu}{n^2} \right) = \frac{1}{n^2} (j_\nu - k_\nu I_\nu). \quad (10.18)$$

The vector  $d\mathbf{n}/ds$  is the rate of bending of the ray direction per unit length measured along the ray. This comes from Snell's law, which takes this form for our problem:

$$\frac{d\mathbf{n}}{ds} = (-\mathbf{nn}) \cdot \nabla \log n. \quad (10.19)$$

As discussed by Cox and Giuli (1968), the definitions of energy density and radiation pressure are modified from those for the nonrefracting medium. The three moments are now given by

$$E_\nu = \frac{1}{v_g} \int_{4\pi} I_\nu d\Omega, \quad (10.20)$$

$$\mathbf{F}_\nu = \int_{4\pi} \mathbf{n} I_\nu d\Omega, \quad (10.21)$$

and

$$P_\nu = \frac{n}{c} \int_{4\pi} \mathbf{nn} I_\nu d\Omega. \quad (10.22)$$

The energy density has a division by the group velocity replacing the division by  $c$ , while the pressure has a division by the phase velocity  $c/n$  replacing the division by  $c$ .

The angle moments of (10.18) are complicated by the presence of the ray-bending term in the transport operator. Cox and Giuli (1968, Vol. 1, pp. 135–136, 173) carry out the calculations, from which we find the following:

$$\frac{\partial}{\partial t} \left( \frac{v_g}{c} E_\nu \right) + \nabla \cdot \mathbf{F}_\nu = 4\pi j_\nu - k_\nu v_g E_\nu, \quad (10.23)$$

$$\frac{\partial}{\partial t} \left( \frac{1}{n^2 c} \mathbf{F}_\nu \right) + \nabla \cdot \left( \frac{c}{n^3} \mathbf{F}_\nu \right) = -\frac{k_\nu}{n^2} \mathbf{F}_\nu \quad (10.24)$$

for the case of isotropic absorption, emission, and refraction. Equation (10.23) shows that our notions of energy and momentum conservation are modified by

refraction. Equation (10.24) can also be written in steady state, as

$$\nabla \cdot \mathbf{v} = -\frac{n}{c} k_v \mathbf{F}_v + 3 \mathbf{v} \cdot \nabla \log n. \quad (10.25)$$

The left-hand side is the actual rate at which the radiation field is gaining momentum; the first term on the right-hand side is the momentum imparted through absorption/emission processes, since the momentum flux is  $k/\omega = n/c$  times the energy flux. The second term on the right is a ponderomotive force term.

In the diffusion limit the intensity goes to  $n^2$  times the ordinary Planck function, and therefore  $E_v$  goes to  $n^2 c/v_g$  times  $4\pi B_v/c$  and the pressure  $P_v$  goes to  $n^3$  times  $4\pi B_v/3c$ . As a result the diffusion expression for the flux is

$$\mathbf{F}_v = -\frac{4\pi n^2}{3k_v} \nabla B_v, \quad (10.26)$$

and if the absorptivity is put in terms of the nonrefracting value  $k_v^0$  this becomes

$$\mathbf{F}_v = -\frac{4\pi n^3}{3k_v^0} \nabla B_v. \quad (10.27)$$

We take the frequency integral of this and express the total flux in terms of a new Rosseland mean opacity:

$$\mathbf{F} = -\frac{16\sigma T^3}{3\kappa_R \rho} \nabla T, \quad (10.28)$$

with

$$\frac{1}{\kappa_R} = \frac{\int_0^\infty dv \frac{n^3}{\kappa_v^0} \frac{dB_v}{dT}}{\int_0^\infty dv \frac{dB_v}{dT}}, \quad (10.29)$$

Notice that the index of refraction is included (cubed) in the numerator, but not in the denominator. That is because we want the denominator to continue to be exactly  $4\sigma T^3/\pi$ , since this is used in obtaining (10.28). In the simple model that  $n = [1 - (\omega_{pe}/\omega)^2]^{1/2}$  the numerator integral in (10.29) must be cut off below the plasma frequency  $\omega_{pe}$ , and we can see that the integrand drops to zero as this limit is approached, so the contribution to  $1/\kappa_R$  from frequencies between  $\omega_{pe}$  and, say,  $2\omega_{pe}$  is much reduced by the effect of refraction. If the density is high enough that  $\hbar\omega_{pe} \approx kT$  there will be a significant increase in  $\kappa_R$ . This modification to the Rosseland mean opacity is also discussed by Cox and Giuli (1968).

The importance of refraction in most astrophysical problems is not great since the plasma frequency is given by

$$\hbar\omega_{pe} = 0.37133 \left( \frac{N_e}{10^{20} \text{ cm}^{-3}} \right)^{1/2} \text{ eV}, \quad (10.30)$$

which puts it well below the optical frequency range at the densities likely to be encountered. It remains a significant issue at radio frequencies since, for example, even at the typical coronal density of  $10^8 \text{ cm}^{-3}$  the plasma frequency is 90 MHz. The correction to the Rosseland mean opacity indicated in (10.29) is often omitted even in the large opacity tabulations because in the conditions where this correction is significant there are uncertainties in the opacity calculation that are even larger.

## 10.2 Description of polarized light

We have so far assumed that the radiation field is equally strong in the two modes of polarization, and we have summed or averaged over the polarizations as appropriate, to get results for unpolarized light. This is really not correct in a number of circumstances, and in this section we will show what might be done about that. In truth, for most purposes we continue to use the results for unpolarized light, at the cost of some loss of accuracy, since the number of unknowns goes up by 2–4 times when polarization is considered, and therefore it is an issue of computational expense.

What is the mathematical description of polarized light? Here is the classical way of defining it, derived from Chandrasekhar (1960). Consider propagation in the  $z$  direction. The  $x$  component of the electric field is one possible random function of time and the  $y$  component is another. The mean square value of  $\mathcal{E}_x$  is  $4\pi/c$  times the intensity if all the radiation is propagating in the  $z$  direction and there is no  $y$  component, so we call  $c\langle\mathcal{E}_x\rangle^2/4\pi$   $I_x$ , the  $x$  intensity. Likewise for  $I_y$ . The sum  $I_x + I_y$  is indeed the total intensity  $I$ . This would be the end of the story, and in fact it sometimes is, except for the possibility of correlation between  $\mathcal{E}_x$  and  $\mathcal{E}_y$ . And it turns out that there are pieces of optical apparatus that can detect correlations between  $\mathcal{E}_x$  and  $\mathcal{E}_y$  when one of the two components has been shifted in phase by some controlled amount. Such an apparatus is the quarter-wave plate, for example, which introduces a shift of  $\pi/2$ . A mathematical approach that brings out the way that  $\mathcal{E}_x$  and  $\mathcal{E}_y$  may be correlated with a shift in phase is the following. We pick an interval of time  $\mathcal{T}$  that is long enough for  $1/\mathcal{T}$  to be smaller than the frequency resolution  $\Delta\nu$  we are interested in. We expand  $\mathcal{E}_x$  and  $\mathcal{E}_y$  over this

interval in complex exponential series:

$$\mathcal{E}_x(t) = \sum_k \left( a_{x,k} e^{-i\omega_k t} + a_{x,k}^* e^{+i\omega_k t} \right), \quad (10.31)$$

$$\mathcal{E}_y(t) = \sum_k \left( a_{y,k} e^{-i\omega_k t} + a_{y,k}^* e^{+i\omega_k t} \right). \quad (10.32)$$

If we now pick out one frequency of interest, say  $\omega_k$ , the Fourier coefficients for the two field components are represented by two complex numbers,  $a_{x,k}$  and  $a_{y,k}$ . From these we build a  $2 \times 2$  Hermitian matrix  $\rho_k$  as follows:

$$\rho_k = \begin{pmatrix} a_{x,k} \\ a_{y,k} \end{pmatrix} \begin{pmatrix} a_{x,k}^* & a_{y,k}^* \end{pmatrix} = \begin{pmatrix} |a_{x,k}|^2 & a_{x,k} a_{y,k}^* \\ a_{x,k}^* a_{y,k} & |a_{y,k}|^2 \end{pmatrix}. \quad (10.33)$$

The value of  $\rho_{k,11}$  is the power spectrum of  $\mathcal{E}_x$ , and if this is summed over the frequencies then, by Parseval's theorem, we recover  $I_x$ . Thus this is proportional to the spectral intensity for the  $x$  polarization. Likewise for  $\rho_{k,22}$ , which gives the spectral intensity for the  $y$  polarization. The off-diagonal components come in when we introduce a funny polarizer that responds to

$$\left| \cos \alpha a_{x,k} + \sin \alpha e^{i\beta} a_{y,k} \right|^2, \quad (10.34)$$

where  $\alpha$  and  $\beta$  are arbitrarily adjustable angles. The response is seen to be given by the expression

$$\begin{aligned} & (\cos \alpha \quad \sin \alpha e^{i\beta}) \begin{pmatrix} |a_{x,k}|^2 & a_{x,k} a_{y,k}^* \\ a_{x,k}^* a_{y,k} & |a_{y,k}|^2 \end{pmatrix} \begin{pmatrix} \cos \alpha \\ \sin \alpha e^{-i\beta} \end{pmatrix} \\ &= \cos^2 \alpha \rho_{k,11} + \sin^2 \alpha \rho_{k,22} + \cos \alpha \sin \alpha (e^{i\beta} \rho_{k,21} + e^{-i\beta} \rho_{k,12}). \end{aligned} \quad (10.35)$$

The full matrix is needed to determine this response for all possible values of  $\alpha$  and  $\beta$ .

The matrix  $\rho_k$  is often called the *coherency* matrix (cf., Born and Wolf (1989), Section 10.8.1). In quantum mechanical language it is a density matrix with respect to the polarization modes. The components have conventional designations as shown here:

$$\rho_k = \frac{1}{2} \begin{pmatrix} I + Q & U + iV \\ U - iV & I - Q \end{pmatrix}, \quad (10.36)$$

where the quantities  $I$ ,  $Q$ ,  $U$ , and  $V$  are the *Stokes parameters*. The matrix is positive semi-definite, so while  $Q$ ,  $U$  and  $V$  can each have either sign,  $I$  must be positive, and  $I \geq \sqrt{Q^2 + U^2 + V^2}$ .



The extension of our treatment to the quantized radiation field is fairly easy since expansions (10.31) and (10.32) are just like the representation of the fields in terms of creation and annihilation operators. The quantum equivalent to (10.33) is

$$\rho_k = \text{Tr} \left[ \rho \begin{pmatrix} a_{\mathbf{k}x}^\dagger a_{\mathbf{k}x} & a_{\mathbf{k}x} a_{\mathbf{k}y}^\dagger \\ a_{\mathbf{k}x}^\dagger a_{\mathbf{k}y} & a_{\mathbf{k}y}^\dagger a_{\mathbf{k}y} \end{pmatrix} \right], \quad (10.37)$$

in which  $\rho$  without a subscript is the overall density matrix for the radiation field, and the trace is taken over all the radiation states. This entails a summation over all occupation vectors  $|n_{k\varpi}\rangle$  in which the  $n$ s take all possible values. In the case that the radiation field is relatively weak, the only nonvanishing density matrix values connect states with either no photons at all, or just one photon total in all the modes. In this case  $\rho_k$  reduces to just the part of  $\rho$  connecting the states that have one photon. That is, we can regard  $\rho_k$  itself as the radiation density matrix. The density matrix for situations with multiple photon occupancies is another interesting subject, but it takes us beyond what can be discussed in the intensity picture. For example, some measurements can distinguish between different probability distributions for the photon number that correspond to the same intensity. The intensity interferometer of Hanbury Brown and Twiss (Hanbury Brown and Twiss, 1954, 1956a,b; see also Baym, 1997) relies on such an effect, since photon statistics are not Poisson owing to the photons' boson nature.

So far, since we have chosen just one time interval  $\mathcal{T}$ , the  $\rho_k$  matrix is related precisely to one complex vector  $(a_{x,k} \ a_{y,k})$ , therefore it is rank 1, and the Stokes parameters obey the relation  $I = \sqrt{Q^2 + U^2 + V^2}$  precisely. This is not the general case. When the electric field contains noise, which is true when emission from thermal sources is considered, the time-dependent field components are stochastic processes and the Fourier coefficients over any finite time interval are random variables. In this situation the correct quantity to use to describe the radiation is the average of  $\rho_k$  over an infinite number of realizations of the time-dependent fields, or over an infinite number of different time windows  $\mathcal{T}$ . After averaging,  $\rho_k$  will still have the Hermiticity property, and it will still be semi-definite. In fact, it is exceedingly unlikely that  $I = \sqrt{Q^2 + U^2 + V^2}$  will remain true, and therefore the matrix is highly likely to be positive definite. This means that every funny polarizer as described above will give some nonzero response.

Some general properties of the coherency matrix or the Stokes parameters are the following. When two beams of light are merged, supposing them to come from distinct sources so they are completely uncorrelated, then the coherency matrices or Stokes parameters can be added component by component. The case that  $I = \sqrt{Q^2 + U^2 + V^2}$  is called elliptically polarized light; it contains just one, possibly complex, polarization mode. It is called elliptical because the points  $(\mathcal{E}_x, \mathcal{E}_y)$  trace

out an ellipse in the  $x$ - $y$  plane. If  $U = 0$  the semi-major and semi-minor axes of the ellipse are aligned with  $x$  and  $y$ . If  $V = 0$  then the light is linearly polarized in general, and the plane of polarization depends on  $Q$  and  $U$ , the position angle with respect to the  $x$ -axis being  $(1/2) \tan^{-1} U/Q$ . If  $U = 0$  then the light is  $x$ -polarized if  $Q = I$  and  $y$ -polarized if  $Q = -I$ . In the  $U = 0$  case, if also  $Q = 0$ , the light is circularly polarized. Circularly polarized light with  $V = I$  has the relation that  $a_{y,k} = \exp(-i\pi/2)a_{x,k}$  and therefore the  $y$  component of the field *leads* the  $x$  component of the field by a quarter period. Thus the point representing  $(\mathcal{E}_x, \mathcal{E}_y)$  moves clockwise around a circle in the  $x$ - $y$  diagram as  $t$  increases at a fixed  $z$  (assuming that the  $xyz$  coordinate system is right-handed). This is right-circular polarization. If a quarter-wave plate is used to *delay*  $\mathcal{E}_y$  relative to  $\mathcal{E}_x$  by  $\pi/2$  for right-circularly polarized light, the result is light that is linearly polarized at  $\theta = \pi/4$ . In the case that  $V = -I$  the motion is counter-clockwise around the circle, this is left-circular polarization, and the effect of the quarter-wave plate is to produce linearly polarized light at  $\theta = 3\pi/4$ .

Every Hermitian matrix can be diagonalized by a unitary transformation, which means that  $\rho_k$  can be expressed in terms of rotated, possibly complex, orthogonal polarization modes, with respect to which the two field components are completely uncorrelated. There are two physical pictures of a beam of arbitrarily polarized light that go with this mathematical statement. The first picture is to regard the beam as the sum of two parts, of which the first part is a mixture of equal amounts of the two modes, and is therefore unpolarized, and the second part is a residual amount of just one of the two modes, which is thus elliptically polarized. Thus any kind of light can be formed by the mixture of some unpolarized light with a beam of elliptically polarized light. The alternative physical picture is that any kind of light is the sum of suitable amounts of two particular orthogonal elliptical polarizations. Orthogonal elliptical polarizations, by the way, differ from each other by interchanging the major and minor axes, and reversing the sense of rotation. So linear  $x$  polarization is orthogonal to linear  $y$  polarization; linear polarization along  $y = 2x$  is orthogonal to linear polarization along  $x = -2y$ . Right-circular polarization is orthogonal to left-circular polarization.

Elliptically polarized light that is normalized to unit intensity, so  $Q^2 + U^2 + V^2 = 1$ , corresponds to a point on the unit sphere in  $QUV$  space. This is called the Poincaré sphere. It gives an easy way of visualizing types of polarization, and the actions of various polarizers. (See Huard (1997).) The “north” pole of the Poincaré sphere is right-circularly polarized light and the “south” pole is left-circularly polarized. The equator contains the types of linear polarization. The longitude is twice the position angle of the major axis of the ellipse, and the latitude determines the ellipticity according to  $\text{latitude} = 2\epsilon$  with  $\tan \epsilon = \text{minor axis}/\text{major axis}$ . The northern hemisphere contains the right-handed polarizations, and the southern

hemisphere the left-handed ones. Diametrically opposite points on the Poincaré sphere represent orthogonal polarizations. The action of a phase plate or compensator is visualized as a rotation of the sphere about a certain direction. For example, the quarter-wave plate produces a  $90^\circ$  rotation about the  $Q$  axis, so that  $U$  becomes  $V$  and  $V$  becomes  $-U$ . In general, the action of polarizers and compensators is represented in matrix language by  $\mathbf{I}' = \mathbf{I}$ , a transformation mapping the old Stokes vector  $\mathbf{I}$  into a new one  $\mathbf{I}'$ . The  $4 \times 4$  matrix is called the Mueller matrix for the device.

If we change from one set of polarization basis vectors, for example the vectors  $\mathbf{e}_x$  and  $\mathbf{e}_y$  we have been discussing, to some other pair  $\mathbf{e}_1$  and  $\mathbf{e}_2$ , where we will require that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are normalized and orthogonal:  $\mathbf{e}_1^* \cdot \mathbf{e}_1 = \mathbf{e}_2^* \cdot \mathbf{e}_2 = 1$  and  $\mathbf{e}_1^* \cdot \mathbf{e}_2 = 0$ , what happens to the coherency matrix? Suppose that

$$(\mathbf{e}_x \quad \mathbf{e}_y) = (\mathbf{e}_1 \quad \mathbf{e}_2) M, \quad (10.38)$$

in terms of a  $2 \times 2$  unitary matrix  $M$ . Then the vector field Fourier amplitude can be expressed in two equivalent ways,

$$(\mathbf{e}_x \quad \mathbf{e}_y) \begin{pmatrix} a_{x,k} \\ a_{y,k} \end{pmatrix} \quad (10.39)$$

or

$$(\mathbf{e}_1 \quad \mathbf{e}_2) \begin{pmatrix} a_{1,k} \\ a_{2,k} \end{pmatrix}. \quad (10.40)$$

Therefore the amplitudes are related by

$$\begin{pmatrix} a_{1,k} \\ a_{2,k} \end{pmatrix} = M \begin{pmatrix} a_{x,k} \\ a_{y,k} \end{pmatrix}. \quad (10.41)$$

When these amplitudes are used to construct the coherency matrix  $\rho'_k$  with respect to these new polarization modes, it turns out to be

$$\rho'_k = M \rho_k M^\dagger. \quad (10.42)$$

So the unitary transformations that diagonalize the coherency matrix do indeed just correspond to picking different polarization modes for the basis, as suggested above.

As an example of this transformation let us take the modes

$$\mathbf{e}_1 = \frac{-\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}, \quad (10.43)$$

$$\mathbf{e}_2 = \frac{\mathbf{e}_x + i\mathbf{e}_y}{\sqrt{2}}. \quad (10.44)$$

These are the circular polarization modes. The matrix  $M$  in this case is

$$M = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & -i \\ 1 & -i \end{pmatrix}. \quad (10.45)$$

Applying this to find the transformed coherency matrix in terms of the Stokes parameters gives

$$\rho'_k = \frac{1}{2} \begin{pmatrix} I + V & -Q - iU \\ -Q + iU & I - V \end{pmatrix}. \quad (10.46)$$

So as a result of the transformation  $V$  now plays the role that  $Q$  did before,  $-Q$  plays the role that  $U$  did before, and  $-U$  plays the role that  $V$  did before. We recognize  $\mathbf{e}_1$  as the basis vector for the right-circular polarization mode, and  $\mathbf{e}_2$  as the basis vector for the left-circular polarization.

It must be added here that, sadly, the conventions related to the Stokes parameters and the coherency matrix are not too well established. Some authors assume that the time-dependent exponential factor is  $\exp(i\omega t)$  instead of  $\exp(-i\omega t)$ . This reverses the sign of  $V$ . Some authors define a positive  $V$  as referring to counter-clockwise rotation of the electric vector, which does the same thing. And there are other variations. In this presentation  $\exp(-i\omega t)$  and clockwise rotation have been assumed, and in this and other respects the conventions described by Rees (1987) have been followed.

### 10.3 Transport equation for an isotropic medium

The transport equation for polarized light we shall consider first will include only the effect of scattering on the polarization components, and refraction and any dependence of the absorptivity on polarization mode will be put aside for now. We will deal with a vector-valued intensity based on the four Stokes components,

$$\mathbf{I} = \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix}. \quad (10.47)$$

The absorptivity is (in our approximation) a scalar, but the emissivity is also a vector, since it is potentially different for the four components. Thus the transport equation is

$$\frac{1}{c} \frac{\partial \mathbf{I}_v}{\partial t} + \mathbf{n} \cdot \nabla \mathbf{I}_v = \mathbf{j}_v - k_v \mathbf{I}_v. \quad (10.48)$$

Incidentally, the additivity of Stokes vectors is what allows us to write a linear transport equation. There is an implicit assumption that the gains or losses of radiation in a volume element have a random phase compared with the radiation field itself. That will not be the case if the “scatterers” are spatially ordered, for example.

The discussion hinges on the form of the Stokes vector emissivity  $\mathbf{j}_\nu$ . Thermal emission is unpolarized, and therefore it corresponds to

$$\mathbf{j}_\nu = \begin{pmatrix} k_\nu B_\nu \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (10.49)$$

If this is the only kind of emission, if any radiation incident at the boundaries is unpolarized, and if any radiation present at the initial time is unpolarized, then the transport equation shows that the radiation remains everywhere unpolarized at all time, a common-sense result. However, we shall see that scattering processes produce polarization when there is none present to begin with.

There is a simple result for the Stokes vector emissivity for scattering processes like Thomson and Rayleigh scattering. To see how to get there from the quantum mechanics, we need to begin with a density-matrix form of Fermi’s Golden Rule, which is

$$\dot{\rho}_{f'f} = \frac{2\pi}{\hbar} \sum_{ii'} \langle f' | H_{\text{int}} | i' \rangle \rho_{i'i} \langle i | H_{\text{int}} | f \rangle \delta(E_f - E_i), \quad (10.50)$$

where the initial states  $i$  and  $i'$  are one degenerate set in energy, with populations and possible correlations as given by  $\rho_{i'i}$ , and the final set of states  $f$  and  $f'$  are another degenerate set, and furthermore there is overall conservation of energy. In the case of Rayleigh and Thomson scattering the indices on the initial density (polarization) matrix are  $\varpi'_1$  and  $\varpi_1$ , and we want to find the contribution to the final state density matrix with indices  $\varpi'_2$  and  $\varpi_2$ . The first matrix element of  $H_{\text{int}}$  contains a factor  $\mathbf{e}_{\mathbf{k}_2 \varpi'_2}^* \cdot \mathbf{e}_{\mathbf{k}_1 \varpi'_1}$  and the second matrix element contains a factor  $\mathbf{e}_{\mathbf{k}_1 \varpi_1}^* \cdot \mathbf{e}_{\mathbf{k}_2 \varpi_2}$ . The product of these two factors is the dependence of the differential cross section for coherency matrix components on the polarization modes. If we define the direction cosine matrix  $C$  by  $C_{\varpi_1 \varpi_2} = \mathbf{e}_{\mathbf{k}_1 \varpi_1}^* \cdot \mathbf{e}_{\mathbf{k}_2 \varpi_2}$ , then the polarization-matrix rate of emission due to scattering of photons in the solid angle  $d\Omega_1$  can be written as the matrix product

$$N_e \left( \frac{e^2}{mc^2} \right)^2 C^\dagger \boldsymbol{\rho}_k C d\Omega_1. \quad (10.51)$$

Obtaining the direction cosine matrix explicitly is an exercise in angular algebra; indeed the cleanest way of representing angular quantities such as  $\rho_k$  and the related emissivity function is by means of irreducible tensors, for which all the methods of Racah angular algebra may be invoked. We are not able to delve into that here.

The direction cosine matrix  $C$  is easy to give if the polarizations are referred to the plane containing the initial and final photon directions. Let polarization 1 be perpendicular to that plane and polarization 2 be parallel to it. Then the direction cosine matrix is

$$C = (\mathbf{e}_{\mathbf{k}_1\varpi_1}^* \cdot \mathbf{e}_{\mathbf{k}_2\varpi_2}) = \begin{pmatrix} 1 & 0 \\ 0 & \cos \Theta \end{pmatrix}, \quad (10.52)$$

where  $\Theta$  is the scattering angle. Working out the matrix products in (10.51) using the definition (10.36) of  $\rho_k$  and then rearranging the polarization components as Stokes vector components gives

$$\frac{1}{2}N_e \left( \frac{e^2}{mc^2} \right)^2 \begin{pmatrix} 1 + \cos^2 \Theta & \sin^2 \Theta & 0 & 0 \\ \sin^2 \Theta & 1 + \cos^2 \Theta & 0 & 0 \\ 0 & 0 & 2 \cos \Theta & 0 \\ 0 & 0 & 0 & 2 \cos \Theta \end{pmatrix} \begin{pmatrix} I \\ Q \\ U \\ V \end{pmatrix} d\Omega_1 \quad (10.53)$$

for the contribution to the Stokes emissivity vector from this particular initial photon direction. However, before being used to actually solve the equation of transfer the matrix appearing here must still be transformed to account for the fact that the polarization modes must be referred to a consistent azimuthal angle of reference, since the scattering plane rotates around as the photon directions vary. Equation (10.53) as it stands is sufficient to show us that unpolarized radiation scattered through  $\Theta = 90^\circ$  becomes 100% linearly polarized perpendicular to the scattering plane.

### 10.4 Polarized light in an anisotropic medium

The equation of transfer in the preceding subsection assumes that the absorption coefficient is a scalar quantity; i.e., it is the same for the two modes of polarization. Another implicit assumption is that the index of refraction is the same for the two modes. Neither of these things is true for an anisotropic medium. This is a medium in which the dielectric tensor is not a scalar tensor, and in which, therefore, the speed of light is different in different directions. This is encountered in birefringent crystals, and in astrophysics in describing light propagation through magnetized plasmas. Four examples of the latter are: (1) the formation of Zeeman

split or broadened spectral lines in the solar photosphere (Beckers, 1969), (2) remote probing of the earth's ionosphere using microwave emission of  $O_2$  (Lenoir, 1968), (3) formation of the cyclotron line in the accretion column of a neutron star (Mészáros and Nagel, 1985), and (4) Faraday rotation of radio signals from distant pulsars by the interstellar medium.

The last of these examples points out the need to consider in the transfer equation not only the absorption of the Stokes components, but also the rotation – the exchange of energy *between* components. The theory of Faraday rotation has long included the rotation effect based on classical optics without incorporating it in the transfer equation, while the theory of spectral line formation in the magnetized solar photosphere initially ignored the rotation (Unno, 1956). A unified treatment including both effects has now been given for the radio astronomy case by Lenoir (1967) and for the solar Zeeman effect by Rachkovsky (1962), E. Landi Degl'Innocenti and M. Landi Degl'Innocenti (1972). A general formulation for the Zeeman effect problem based on QED is found in E. Landi Degl'Innocenti (1983). E. Landi Degl'Innocenti (1987) reviews the application of his methods to the formation of polarized spectral lines in the solar atmosphere.

Lenoir (1967) uses a semi-classical development based on Maxwell's equations to arrive at this form of the transfer equation expressed in terms of the coherency matrix  $\rho$

$$\frac{d\rho}{ds} + \mathbf{G}\rho + \rho\mathbf{G}^\dagger = 2B_\nu, \quad (10.54)$$

in which  $\mathbf{G}$  is, in effect,  $-2\pi i/\lambda$  times the complex tensor index of refraction,  $\mathbf{G}^\dagger$  is its Hermitian adjoint, and  $\mathbf{G} + \mathbf{G}^\dagger$  is its Hermitian part,  $(\mathbf{G} + \mathbf{G}^\dagger)/2$ . (Lenoir's phases have to be corrected for time dependence  $\exp(-i\omega t)$ .) In Lenoir's formulation he assumes the permittivity  $\epsilon_0$  of vacuum but includes a permeability tensor  $\boldsymbol{\mu} = \mu_0(\mathbf{1} + \boldsymbol{\chi})$  that incorporates the plasma Zeeman effect, and it is then found to be sufficient to take  $\mathbf{G} = -(i\pi/\lambda)\boldsymbol{\chi}_\perp$ , where  $\boldsymbol{\chi}_\perp$  stands for the  $2 \times 2$  projection of  $\boldsymbol{\chi}$  into the plane perpendicular to the direction of propagation. Alternatively, the ideal permeability  $\mu_0$  may be used, and the permittivity given by the general dielectric tensor  $\boldsymbol{\epsilon}$ . The result is the same, with  $\boldsymbol{\epsilon}/\epsilon_0 - \mathbf{1}$  replacing  $\boldsymbol{\chi}$ .

Lenoir (1967) specifically considers LTE radiative transfer in the microwave region of the spectrum, thus his absorption coefficient implicitly contains a correction factor  $1 - \exp(-h\nu/kT) \approx h\nu/kT$  for stimulated emission, and he may use the Rayleigh–Jeans approximation  $B_\nu \approx 2kT\nu^2/c^2$ . He therefore refers to brightness temperature rather than to the specific intensity and its coherence components. However, nothing in the formulation prevents using it, as quoted above, for the coherency matrix in intensity units, provided it is understood that  $\mathbf{G}$  has been corrected for stimulated emission.

In the foregoing equations all the tensors are  $2 \times 2$ . If the polarization basis vectors are  $\mathbf{e}_x$  and  $\mathbf{e}_y$ , corresponding to the relation (10.36) between the coherency matrix and the Stokes parameters, then the tensor  $\chi_{\perp}$ , for example, is just

$$\chi_{\perp} = \begin{pmatrix} \chi_{xx} & \chi_{xy} \\ \chi_{yx} & \chi_{yy} \end{pmatrix} \quad (10.55)$$

in the case that the radiation direction in question is the  $+z$  direction. If, as above, a different set of polarization vectors is used, related to these by a unitary transformation with a matrix  $M$ , then the transformed coherency matrix  $\rho'$  of (10.42) will obey the transfer equation

$$\frac{d\rho'}{ds} + M\mathbf{G}M^{\dagger}\rho' + \rho'M\mathbf{G}^{\dagger}M^{\dagger} = 2B_v M^{-1}M^{\dagger}, \quad (10.56)$$

as we can see by multiplying (10.54) on the left by  $M$  and on the right by  $M^{\dagger}$ ; the matrices pass through the derivative with respect to  $s$  if the polarization modes are space-independent. The form of the transfer equation is unchanged in this basis, and it is necessary just to transform the  $\mathbf{G}$  and  $\mathbf{I}$  tensors to the new basis.

The transfer equation for the coherency matrix  $\rho$  can be made into a useful transfer equation for the Stokes vector  $\mathbf{I}$ . If the  $\mathbf{G}$  matrix is written as

$$\mathbf{G} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix}, \quad (10.57)$$

then working out the matrix products in (10.54) using (10.36) leads to

$$\frac{d\mathbf{I}}{ds} + \mathbf{K}\mathbf{I} = \mathbf{j} = \mathbf{K}\mathbf{S}, \quad (10.58)$$

with the absorption matrix  $\mathbf{K}$  and the emissivity and source function vectors  $\mathbf{j}$  and  $\mathbf{S}$  defined by

$$\mathbf{K} = \begin{pmatrix} \phi_I & \phi_Q & \phi_U & \phi_V \\ \phi_Q & \phi_I & \phi'_V & -\phi'_U \\ \phi_U & -\phi'_V & \phi_I & \phi'_Q \\ \phi_V & \phi'_U & -\phi'_Q & \phi_I \end{pmatrix}, \quad \mathbf{j} = B_v \begin{pmatrix} \phi_I \\ \phi_Q \\ \phi_U \\ \phi_V \end{pmatrix}, \quad \mathbf{S} = B_v \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \quad (10.59)$$



The seven parameters that appear in (10.59) are related to the  $\mathbf{G}$  elements by

$$\begin{aligned}\phi_I &= \Re(G_{11}) + \Re(G_{22}), \\ \phi_Q &= \Re(G_{11}) - \Re(G_{22}), \\ \phi_U &= \Re(G_{12}) + \Re(G_{21}), \\ \phi_V &= -\Im(G_{12}) + \Im(G_{21}), \\ \phi'_Q &= \Im(G_{11}) - \Im(G_{22}), \\ \phi'_U &= \Im(G_{12}) + \Im(G_{21}), \\ \phi'_V &= \Re(G_{12}) - \Re(G_{21}).\end{aligned}\tag{10.60}$$

This notation follows Jefferies, Lites, and Skumanich (1989); the  $\phi$ s should not be confused either with angles or with values of line profile functions, although it turns out that for polarized line transfer the unprimed  $\phi$ s are various combinations of profile functions of the Zeeman components times geometrical factors, and the primed parameters contain the corresponding dispersion functions, which are similar to the derivatives of the profiles; see below.

The action of the  $\phi'$  elements of  $\mathbf{K}$  in (10.58) is worthy of note. These elements, by themselves, cause the Stokes parameters  $Q, U, V$  to rotate as one progresses along the ray. The axis of rotation is  $(\phi'_Q, \phi'_U, \phi'_V)$ , and the magnitude of this vector gives the angle of rotation per unit of path length. The effect can be visualized by considering a counter-clockwise rotation of the Poincaré sphere about this direction.

The diagonal  $\phi_I$  elements of  $\mathbf{K}$  produce uniform absorption of all the components of  $\mathbf{I}$ , so that the percentage and type of polarization are unchanged as  $I$  is reduced. But the effect of  $\phi_Q, \phi_U$ , and  $\phi_V$  is to selectively absorb one polarization and to absorb less the orthogonal one. Specifically, the polarization component along  $(Q, U, V) = (\phi_Q, \phi_U, \phi_V)$  is absorbed at the increased rate  $\phi_I + (\phi_Q^2 + \phi_U^2 + \phi_V^2)^{1/2}$ , while the orthogonal polarization  $(Q, U, V) = (-\phi_Q, -\phi_U, -\phi_V)$  has the decreased absorption coefficient  $\phi_I - (\phi_Q^2 + \phi_U^2 + \phi_V^2)^{1/2}$ . (The large and small absorption coefficients are the same as twice the eigenvalues of  $\mathbf{K}$ , the Hermitian part of  $\mathbf{G}$ . A condition of physical reasonableness is that  $\mathbf{K}$  should be positive definite.) If the path length through this material is sufficiently large, then the attenuated emerging radiation is 100% polarized along  $(Q, U, V) = (-\phi_Q, -\phi_U, -\phi_V)$ , regardless of its initial polarization. So the recipe for making a polarizer that passes only linear polarization in the  $x$  direction ( $Q > 0, U = V = 0$ ) is to provide a slab of material with large, negative  $\phi_Q, \phi_I \approx |\phi_Q|$ , and negligible values of the other parameters.

A simple example of the transfer equation for anisotropic polarized light is provided by Faraday rotation of cosmic radio waves due to magnetized interstellar

plasma. The dielectric tensor  $\epsilon$  for this case is a generalization of (10.7) discussed earlier. Following Allis, Buchsbaum, and Bers (1963) and correcting the phases for time dependence  $\exp(-i\omega t)$  leads to the dielectric tensor, for the case of a uniform static field  $\mathbf{B}$  along the  $z$  axis, given by

$$\frac{\epsilon}{\epsilon_0} = -\frac{\omega_{pe}^2}{2\omega} \begin{pmatrix} \ell + r & -i(\ell - r) & 0 \\ i(\ell - r) & \ell + r & 0 \\ 0 & 0 & 2p \end{pmatrix}, \quad (10.61)$$

with the definitions

$$r = \frac{1}{\omega + \omega_B + i\nu_c}, \quad \ell = \frac{1}{\omega - \omega_B + i\nu_c}, \quad p = \frac{1}{\omega + i\nu_c}, \quad (10.62)$$

in which  $\omega_B$  is the electron cyclotron frequency  $eB/mc$  and  $\omega_{pe}$  and  $\nu_c$  are, as before, the electron plasma frequency  $\sqrt{4\pi N_e e^2/m}$  and the electron-ion collision frequency. Making the assumption that both  $\omega_B$  and  $\nu_c$  are much smaller than  $\omega$  leads to

$$\frac{\epsilon}{\epsilon_0} \approx \left(1 - \frac{\omega_{pe}^2}{\omega^2} + i\frac{\omega_{pe}^2 \nu_c}{\omega^3}\right) + i\frac{\omega_{pe}^2 \omega_B}{\omega^3} \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (10.63)$$

The isotropic part of this tensor is what we found before; the second term is new. The components as written are for a coordinate system with the  $z$  axis along  $\mathbf{B}$ . But the matrix in the second term can also be written in tensor language as  $b_{ij} = e_{ijk} B_k/B$  in terms of the anti-symmetric tensor  $e_{ijk}$  which is  $+1$  if  $ijk$  is an even permutation,  $-1$  if  $ijk$  is an odd permutation, and is otherwise 0. Since  $e_{ijk}$  and  $B_k$  transform like a good tensor and vector under rotations, the same formula can be used to express this matrix for a general orientation of  $\mathbf{B}$ , which gives

$$\frac{\epsilon}{\epsilon_0} \approx \left(1 - \frac{\omega_{pe}^2}{\omega^2} + i\frac{\omega_{pe}^2 \nu_c}{\omega^3}\right) + i\frac{\omega_{pe}^2 \omega_B}{\omega^3 B} \begin{pmatrix} 0 & B_z & -B_y \\ -B_z & 0 & B_x \\ B_y & -B_x & 0 \end{pmatrix}, \quad (10.64)$$

and therefore

$$\mathbf{G} = \left(\frac{i\omega_{pe}^2}{2\omega c} + \frac{\omega_{pe}^2 \nu_c}{2\omega^2 c}\right) + \frac{\omega_{pe}^2 \omega_{Bz}}{2\omega^2 c} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad (10.65)$$

where the identity matrix is now  $2 \times 2$ , and  $\omega_{Bz}$  is a new variable,  $\omega_{Bz} = \omega_B B_z/B = eB_z/mc$ . In contrast to  $\omega_B$ ,  $\omega_{Bz}$  can have either sign, depending on the angle between  $\mathbf{B}$  and the  $+z$  axis. The Faraday rotation will apparently depend on only that component of  $\mathbf{B}$  that is along the direction of propagation of the wave.

The scalar imaginary (dispersive) part of  $\mathbf{G}$  disappears in the transfer equation, since it has no effect on the intensities. It is, however, responsible for pulsar

dispersion measures. The real scalar part of  $\mathbf{G}$  represents absorption due to inverse bremsstrahlung, as discussed before. Given this result for  $\mathbf{G}$ , the elements of  $\mathbf{K}$  can be read off:  $\phi_I = \omega_{pe}^2 v_c / (\omega^2 c)$ ,  $\phi_Q = 0$ ,  $\phi_U = 0$ ,  $\phi_V = 0$ ,  $\phi'_Q = 0$ ,  $\phi'_U = 0$  and  $\phi'_V = \omega_{pe}^2 \omega_{Bz} / (\omega^2 c)$ . Neglecting the emission by the cold interstellar material, the Stokes transfer equations become

$$\frac{dI}{ds} = -\frac{\omega_{pe}^2 v_c}{\omega^2 c} I, \quad (10.66)$$

$$\frac{dQ}{ds} = -\frac{\omega_{pe}^2 v_c}{\omega^2 c} Q - \frac{\omega_{pe}^2 \omega_{Bz}}{\omega^2 c} U, \quad (10.67)$$

$$\frac{dU}{ds} = -\frac{\omega_{pe}^2 v_c}{\omega^2 c} U + \frac{\omega_{pe}^2 \omega_{Bz}}{\omega^2 c} Q, \quad (10.68)$$

$$\frac{dV}{ds} = -\frac{\omega_{pe}^2 v_c}{\omega^2 c} V. \quad (10.69)$$

If the light from the distant source is partially linearly polarized, then at the source  $V = 0$ , and it remains so. We see that  $I$ ,  $Q$ , and  $U$  are attenuated at the same rate by the inverse bremsstrahlung. But the coupling of  $Q$  and  $U$  produces rotation of the plane of polarization. The rate of rotation can be calculated by

$$\frac{d\theta}{ds} = \frac{d}{ds} \left( \frac{1}{2} \tan^{-1} \frac{U}{Q} \right) = \frac{1}{2} \frac{1}{Q^2 + U^2} \left( Q \frac{dU}{ds} - U \frac{dQ}{ds} \right) = \frac{\omega_{pe}^2 \omega_{Bz}}{2\omega^2 c}. \quad (10.70)$$

The rotation per unit path length is proportional to the electron density, the projection of  $\mathbf{B}$  on the line of sight and inversely to the square of the frequency. If the component of  $\mathbf{B}$  in the direction of propagation is positive, the sense of the rotation is counter-clockwise. It often happens that the total rotation angle as observed on the earth for a certain radio source is many times  $2\pi$  at typical frequencies. What is observed in these cases is a linear variation of the position angle of linear polarization with  $\lambda^2$  over a small range of  $\lambda$ . The coefficient is proportional to  $\int N_e B_z ds$  over the line of sight to the source, called the rotation measure.

E. Landi Degl'Innocenti's (1983) form of the transfer equation for  $\rho$  is

$$\frac{d\rho}{ds} = \frac{1}{2} \left( \begin{smallmatrix} & + \\ & \end{smallmatrix} \right) - \left( \begin{smallmatrix} \rho & + \\ \rho & \end{smallmatrix} \right) + \left( \begin{smallmatrix} \rho & + \\ \rho & \end{smallmatrix} \right). \quad (10.71)$$

The three new matrices that appear here:  $\begin{smallmatrix} & + \\ & \end{smallmatrix}$ ,  $\begin{smallmatrix} \rho & + \\ \rho & \end{smallmatrix}$ , and  $\begin{smallmatrix} \rho & + \\ \rho & \end{smallmatrix}$ , take the places of  $\mathbf{G}$  and  $\mathbf{K}$ , and are expressed by E. Landi Degl'Innocenti as specific sums of atomic density matrix elements multiplied by two factors of quantities like  $\langle a | \mathbf{p} \cdot \mathbf{e} | b \rangle$  and by the complex line absorption profile function, about which more below. They represent spontaneous emission, absorption, and stimulated emission, respectively. The

stimulated emission term, with  $\rho_{11}$  enters the transfer equation as a subtraction from the absorption term, as expected. The spontaneous emission matrix  $\mathbf{B}_{10}$  is the stimulated emission matrix multiplied by the factor  $2h\nu^3/c^2$ , also as expected. When the atomic density matrices reduce to simple populations, and furthermore are in LTE, then the  $\rho_{11}$  matrix becomes  $\exp(-h\nu/kT)$  times the  $\rho_{00}$  matrix and we can denote  $\rho_{11}$  by  $\mathbf{G}$ , with the result that  $\mathbf{B}_{10}$  is  $\mathbf{G}\mathbf{B}_{10}$ . This reduces E. Landi Degl'Innocenti's equation to Lenoir's form.

E. Landi Degl'Innocenti's result for the  $\mathbf{B}_{10}$  matrix can be looked at in more detail in the simple case of absorption for a normal Zeeman triplet, for example in a  $^1S_0$ – $^1P_1$  transition. The  $^1P_1$  level is split by the Zeeman effect into the  $M = -1$ ,  $M = 0$ , and  $M = 1$  sublevels, and the line is split into the  $0 \rightarrow -1$   $\sigma_-$  component, the  $0 \rightarrow 0$   $\pi$  component and the  $0 \rightarrow 1$   $\sigma_+$  component. The frequencies of the components turn out to be  $\nu = \nu_0 - \nu_B$ ,  $\nu_0$ , and  $\nu_0 + \nu_B$ , respectively, where  $\nu_0$  is the unperturbed frequency and  $\nu_B = eB/4\pi mc$  is the normal Zeeman splitting. The expression for the components of the  $\mathbf{B}_{10}$  matrix at frequency  $\nu$  becomes

$$\begin{aligned} g_{\alpha\beta} = C [ & \langle u, -1 | \mathbf{p} \cdot \mathbf{e}_\alpha | \ell, 0 \rangle \langle u, -1 | \mathbf{p} \cdot \mathbf{e}_\beta | \ell, 0 \rangle^* \Phi(\nu_0 - \nu_B - \nu) \\ & + \langle u, 0 | \mathbf{p} \cdot \mathbf{e}_\alpha | \ell, 0 \rangle \langle u, 0 | \mathbf{p} \cdot \mathbf{e}_\beta | \ell, 0 \rangle^* \Phi(\nu_0 - \nu) \\ & + \langle u, +1 | \mathbf{p} \cdot \mathbf{e}_\alpha | \ell, 0 \rangle \langle u, +1 | \mathbf{p} \cdot \mathbf{e}_\beta | \ell, 0 \rangle^* \Phi(\nu_0 + \nu_B - \nu) ], \quad (10.72) \end{aligned}$$

in which  $C$  is a certain combination of atomic constants with the population density of atoms in the lower level. The complex function  $\Phi(\Delta\nu)$  has a real part  $H$ , which is the Voigt profile function for a Doppler width  $\Delta\nu_D$  and a Lorentzian width  $\Gamma = a\Delta\nu_D$ , and the imaginary part is  $2F$ , the dispersion function that can be derived from  $H$  using the Kramers–Kronig relation. If we define  $v = \Delta\nu/\Delta\nu_D$ , then the functions are given by

$$\begin{aligned} \Delta\nu_D \Phi(\Delta\nu) &= \frac{i}{\pi^{3/2}} \int_{-\infty}^{\infty} \frac{\exp(-t^2) dt}{v + ia - t} \\ &= \frac{1}{\sqrt{\pi}} H(a, v) + \frac{2i}{\sqrt{\pi}} F(a, v) \end{aligned} \quad (10.73)$$

with

$$(10.74)$$

$$H(a, v) = \frac{a}{\pi} \int_{-\infty}^{\infty} \frac{\exp(-t^2) dt}{(v - t)^2 + a^2} \quad (10.75)$$

$$F(a, v) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{(v - t) \exp(-t^2) dt}{(v - t)^2 + a^2}, \quad (10.76)$$

also

$$\Delta v_D \Phi(\Delta v) = \frac{1}{\sqrt{\pi}} w(v + ia) \quad (10.77)$$

$$= \frac{1}{\sqrt{\pi}} \exp\left(-(v + ia)^2\right) \operatorname{erfc}(a - iv). \quad (10.78)$$

The last relation connects the complex profile shape with the complex error function (Abramowitz and Stegun, 1964).

The factors like  $\langle u, -1 | \mathbf{p} \cdot \mathbf{e}_\alpha | \ell, 0 \rangle$  evaluate to physical constants times the reduced matrix element  $\langle u || \mathbf{p} || \ell \rangle$  times the trigonometrical factor relating the direction of  $\mathbf{e}_\alpha$  to the  $-1$  spherical component of a unit vector along the  $\mathbf{B}$  field. In fact, the vector  $\mathbf{p}$  can be expressed as

$$\mathbf{p} = -p'_1 \mathbf{e}'_{-1} + p'_0 \mathbf{e}'_0 - p'_{-1} \mathbf{e}'_1 \quad (10.79)$$

in terms of its spherical components in an  $x'y'z'$  coordinate system with the  $z'$  axis along  $\mathbf{B}$ , and the spherical basis vectors in this coordinate system. An application of the Wigner–Eckart theorem to  $\langle u, -1 | \mathbf{p} | \ell, 0 \rangle$  gives

$$\begin{aligned} \langle u, -1 | \mathbf{p} | \ell, 0 \rangle &= -\langle u, -1 | p'_{-1} | \ell, 0 \rangle \mathbf{e}'_1 \\ &= -\langle u || \mathbf{p} || \ell \rangle \begin{pmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \end{pmatrix} \mathbf{e}'_1, \end{aligned} \quad (10.80)$$

where the  $3j$  symbol has the value  $1/\sqrt{3}$ . Expressing  $\mathbf{e}'_1$  in  $xyz$  coordinates requires selecting a direction for the  $x'$  axis in the plane perpendicular to  $\mathbf{B}$ ; a different choice of this direction will multiply this matrix element by a complex phase factor  $\exp(i\delta)$ . Fortunately this will cancel out when  $\langle u, -1 | \mathbf{p} \cdot \mathbf{e}_\alpha | \ell, 0 \rangle$  is combined with  $\langle u, -1 | \mathbf{p} \cdot \mathbf{e}_\beta | \ell, 0 \rangle^*$ . Arbitrarily selecting  $\mathbf{e}'_x$  to lie in the  $x$ – $y$  plane leads to

$$\begin{aligned} \mathbf{e}'_x &= -\sin \phi_B \mathbf{e}_x + \cos \phi_B \mathbf{e}_y \\ \mathbf{e}'_y &= -\cos \theta_B \cos \phi_B \mathbf{e}_x - \cos \theta_B \sin \phi_B \mathbf{e}_y + \sin \theta_B \mathbf{e}_z \\ \mathbf{e}'_z &= \sin \theta_B \cos \phi_B \mathbf{e}_x + \sin \theta_B \sin \phi_B \mathbf{e}_y + \cos \theta_B \mathbf{e}_z, \end{aligned} \quad (10.81)$$

where  $\theta_B$  and  $\phi_B$  are the polar and azimuthal angles, respectively, of  $\mathbf{B}$  in the  $xyz$  coordinate system. The result is

$$\begin{aligned} \mathbf{e}'_1 &= \frac{1}{\sqrt{2}} [(-\sin \phi_B - i \cos \theta_B \cos \phi_B) \mathbf{e}_x \\ &\quad + (\cos \phi_B - i \cos \theta_B \sin \phi_B) \mathbf{e}_y + i \sin \theta_B \mathbf{e}_z], \end{aligned} \quad (10.82)$$

from which the contributions of just the  $\sigma_-$  Zeeman component to  $\mathbf{G}$  are found to be

$$G_{11}(\sigma_-) = C' \left( \sin^2 \phi_B + \cos^2 \theta_B \cos^2 \phi_B \right) \Phi(v_0 - v_B - v), \quad (10.83)$$

$$G_{12}(\sigma_-) = C' \left( -\sin^2 \theta_B \sin \phi_B \cos \phi_B - i \cos \theta_B \right) \Phi(v_0 - v_B - v), \quad (10.84)$$

$$G_{21}(\sigma_-) = C' \left( -\sin^2 \theta_B \sin \phi_B \cos \phi_B + i \cos \theta_B \right) \Phi(v_0 - v_B - v), \quad (10.85)$$

$$G_{22}(\sigma_-) = C' \left( \cos^2 \phi_B + \cos^2 \theta_B \sin^2 \phi_B \right) \Phi(v_0 - v_B - v), \quad (10.86)$$

in which  $C'$  is the constant  $C' = C[1 - \exp(-h\nu_0/kT)]|\langle u||\mathbf{p}||\ell\rangle|^2/6$ .

The contributions of the  $\sigma_-$  component to the  $\mathbf{K}$  matrix follow from (10.59) with

$$\begin{aligned} \phi_I(\sigma_-) &= C''(1 + \cos^2 \theta_B) H_r, \\ \phi_Q(\sigma_-) &= -C'' \sin^2 \theta_B \cos 2\phi_B H_r, \\ \phi_U(\sigma_-) &= -C'' \sin^2 \theta_B \sin 2\phi_B H_r, \\ \phi_V(\sigma_-) &= C'' 2 \cos \theta_B H_r, \\ \phi'_Q(\sigma_-) &= -C'' \sin^2 \theta_B \cos 2\phi_B 2F_r, \\ \phi'_U(\sigma_-) &= -C'' \sin^2 \theta_B \sin 2\phi_B 2F_r, \\ \phi'_V(\sigma_-) &= C'' 2 \cos \theta_B 2F_r. \end{aligned} \quad (10.87)$$

In these relations  $C'' = C'/(\sqrt{\pi} \Delta\nu_D)$  and  $H_r$  stands for the *red* component (the  $\sigma_-$  component) profile function  $H(a, v)$  with  $v = (v_0 - v_B - v)/\Delta\nu_D$ , and likewise for  $F_r$ .

In a similar way the contributions of the blue ( $\sigma_+$ ) and unshifted ( $\pi$ ) components, denoted by subscripts  $b$  and  $p$ , respectively, can be evaluated. The final result is represented by these relations:

$$\begin{aligned} \phi_I &= C'' \left[ 2 \sin^2 \theta_B H_p + (1 + \cos^2 \theta_B) (H_b + H_r) \right], \\ \phi_Q &= C'' \sin^2 \theta_B \cos 2\phi_B (2H_p - H_b - H_r), \\ \phi_U &= C'' \sin^2 \theta_B \sin 2\phi_B (2H_p - H_b - H_r), \\ \phi_V &= C'' 2 \cos \theta_B (H_r - H_b), \\ \phi'_Q &= C'' \sin^2 \theta_B \cos 2\phi_B (4F_p - 2F_b - 2F_r), \\ \phi'_U &= C'' \sin^2 \theta_B \sin 2\phi_B (4F_p - 2F_b - 2F_r), \\ \phi'_V &= C'' 2 \cos \theta_B (2F_r - 2F_b). \end{aligned} \quad (10.88)$$

This transfer equation is illustrated with a calculation, following Rees (1987), of the Stokes line profiles of a generic line formed in the magnetized solar atmosphere. The line formation model is the Milne–Eddington one, with the assumption

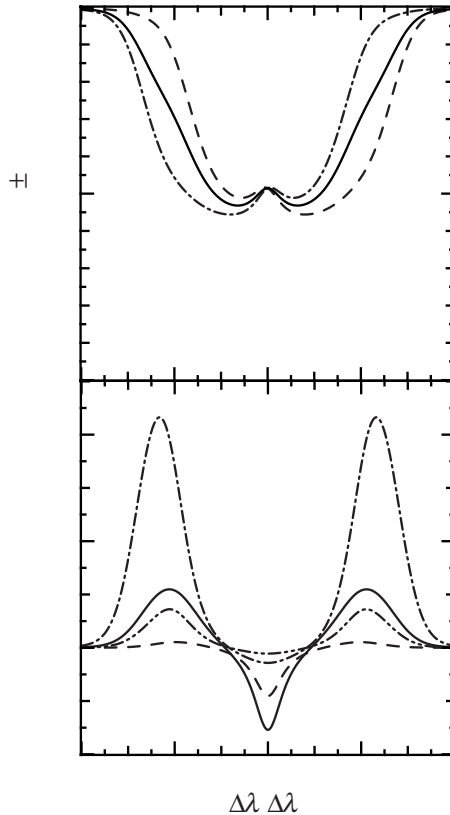


Fig. 10.1 Stokes parameter profiles are shown as functions of wavelength for a line formed in a sunspot. (a) The curves show the total intensity  $I$  (solid) and the circularly polarized components  $I + V$  (dashed) and  $I - V$  (dash-dotted). (b) The linear polarization parameters  $Q$  and  $U$  are shown both with and without Faraday depolarization. Solid curve:  $U$  with depolarization; dashed curve:  $Q$  with depolarization; dash-dotted curve:  $U$  without depolarization; dash-double-dotted curve:  $Q$  without depolarization.

of LTE, a ratio of line opacity to continuous opacity that is independent of depth, a linear variation of the Planck function with continuous optical depth, and a constant geometry of the magnetic field. For this model the line-center ratio of line to continuous opacity is 10, the Planck function is  $B_\nu \propto 1 + (3/2)\tau_c$ , the line wavelength is  $4500 \text{ \AA}$ , the Voigt parameter is  $a = 0.01$ , the Zeeman shift is 1 Doppler width unit, the atomic mass is that of iron, 56, the temperature used to calculate the Doppler width is taken to be 5000 K and the magnetic field is inclined  $60^\circ$  to the vertical direction at an azimuth of  $30^\circ$  with respect to the  $x$  axis. The radiation is viewed along the vertical. The magnetic field implied by these numbers is close to 2000 G, which is reasonable for a sunspot.

The profiles of the Stokes components are illustrated in Figure 10.1. The upper portion shows the total intensity profile, and also shows the profiles for

right-circular and left-circular polarization, i.e., of  $I + V$  and  $I - V$ . At this high value of the magnetic field there is a very sizeable shift of the profile between the two circular modes. Indeed, the two  $\sigma$  components are almost resolved in the total intensity. The magnetometer, which looks for modulation of the intensity in the wing of the profile as the polarization is flipped between left and right circular, easily measures this large shift. The lower part of the figure shows the profiles of  $Q$  and  $U$  by themselves. Because the magnetic field is viewed obliquely, linear polarization is seen with a similar magnitude to circular polarization, except for one effect. That is, that the Faraday rotation produces depolarization of the linear components. This is indicated in the figure. The linear polarization is much larger when the Faraday rotation is absent. The degree of linear polarization peaks in the line wings near where the circular polarization is largest. By using models such as this the observers can infer the magnitude and direction of the transverse magnetic field component, albeit with some inaccuracy. At lower values of the field the contrast between the two circular polarizations becomes very slight, and careful signal processing is needed to extract the longitudinal field. The linear polarization effect is even smaller, so the derived transverse fields can never be as accurate as the longitudinal ones. The reason why the linear polarization is so small, besides Faraday depolarization, can be seen in (10.88):  $\phi_Q$  and  $\phi_U$  have the form of a second difference of the profile function  $H$ , and will be proportional to  $\omega_B^2$  for small  $\omega_B$ , while  $\phi_V$  is a first difference and will be proportional to  $\omega_B$  to the first power.

This example has assumed LTE and a normal Zeeman triplet. The majority of the lines we might want to study have so-called anomalous Zeeman patterns, and the level populations may well be non-LTE. The complete formulation by E. Landi Degl'Innocenti (1983) encompasses these complications.