7

Hydrodynamics with radiation: waves and stability

The goal of this chapter is to explore the effect on the hydrodynamic equations of the terms representing exchange of energy and momentum between matter and radiation. The present discussion will be rather general; most of the specific examples will be taken up after discussing velocity effects on transport and numerical methods. Background information on the properties of thermal equilibrium radiation is presented by, for example, Cox and Giuli (1968).

7.1 Imprisoned equilibrium radiation

We note one suspicious thing about the overall energy equation (4.47), which is that the flux term includes an enthalpy flux for matter, but there is no enthalpy flux for the radiation. Since our claim is that there is no intrinsic difference between material particles and photons, why is there this apparent difference that would persist even when the opacity is so great that the flux vanishes? The explanation was found in Chapter 6. In brief, it is that the flux depends on the reference frame. The same is true for the energy density and the radiation pressure, but for these variables the corrections are small. We have seen that

$$\mathbf{F} = \mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u}\cdot\ \ {}_0 \tag{7.1}$$

to first order in $\mathbf{u}/c$, where E_0, $\mathbf{F}_0$, and ${}_0$ are evaluated in the comoving frame. In the limit of vanishing mean free path it is $\mathbf{F}_0$ that goes to zero, not $\mathbf{F}$. What $\mathbf{F}$ tends to in that limit is the convective radiation enthalpy flux. If we make use of this in (4.47), we see that the symmetry between matter and radiation is restored.

This is the stellar interiors model of radiation. The intensity equals the Planck function at the local temperature, which implies that the energy density and radiation pressure are given by

$$E = aT^4 \tag{7.2}$$

and

$$P = \frac{1}{3}aT^4. \tag{7.3}$$

The radiation enthalpy whose flux is added to the material enthalpy flux in the total energy equation is $(4/3)aT^4$. The energy and enthalpy here are per unit volume; the specific internal energy and enthalpy contributions are obtained by dividing by ρ. The specific radiation entropy s_r comes from the first law of thermodynamics:

$$T\,ds_{\mathrm{r}} = d\left(\frac{aT^4}{\rho}\right) + \frac{aT^4}{3}d\left(\frac{1}{\rho}\right) = 4a\frac{T^3dT}{\rho} - \frac{4}{3}aT^4\frac{d\rho}{\rho^2}, \tag{7.4}$$

so

$$s_r = \frac{4}{3}\frac{aT^3}{\rho}. \tag{7.5}$$

All these radiation contributions can just be added to their material counterparts. The radiation makes a difference to the equation of state. The γ for radiation is $4/3$, less than that for the non-relativistic monatomic ideal gas, which is $5/3$. Thus radiation softens the equation of state. In massive stars Eddington showed years ago that a large part of the weight of the stellar material is supported by radiation pressure, and the effective spring constant of the star for radial oscillations, which is proportional to $\gamma - 4/3$, becomes smaller and smaller as the star's mass increases. Above some mass the stellar oscillations are so easily excited by the modulation of nuclear energy generation in the interior that the star becomes unstable to pulsations and is either disrupted or becomes an unusual kind of object instead of a quiescent main sequence star. Radiation pressure is also a several percent effect in the pulsating cepheid variables. The reduction of $\gamma - 4/3$ due to radiation appreciably lengthens their pulsation periods, which are very well studied and one of the bases of the extragalactic distance scale.

7.2 Nonadiabatic waves

We will consider some examples of waves that couple to a radiation field that is neither optically thick nor optically thin. The first example will be the cooling mode that was discussed earlier using a Newton's cooling model. For now we will omit the fluid motion part of that problem and just consider the zero sound speed limit. This discussion draws on Mihalas and Mihalas (1984), Sections 97, 101 and Castor (1972). What we are interested in is the thermal response, i.e., what is the decay rate of temperature fluctuations of different wavelengths. Earlier on we found the decay rate $1/\tau$ for the cooling mode, where $1/\tau$ was the coefficient in the cooling law (equation (2.52)). We write the energy coupling term $-g^0$ as

$$-g^0 = k_P(cE - acT^4), \tag{7.6}$$

where the emissivity has been expressed in terms of the thermal equilibrium energy density, and the frequency integration has been performed using an appropriate average value (Planck mean) of the absorptivity, k_P. If E is held fixed while T is perturbed, then the perturbation of $-g^0$ is $-k_P 4acT^3 \Delta T$. Dividing the coefficient of ΔT, namely $k_P 4acT^3$, by ρC_v, where C_v is the specific heat at constant volume, gives the damping constant $1/\tau$. Thus we identify τ with $\rho C_v/(k_P 4acT^3)$.

In the general case in which E responds to the temperature fluctuations the linearized coupling term is

$$-\delta g^0 = k_P(c\delta E - 4acT^3\delta T). \tag{7.7}$$

We use the frequency integrated form of the combined moment equation (4.42), and take for the unperturbed state an infinite homogeneous medium in which $E = aT^4$. The right-hand side of the integrated moment equation is just g^0, thus

$$\frac{\partial E}{\partial t} - \boldsymbol{\nabla} \cdot \left(\frac{c}{3k_R} \boldsymbol{\nabla} E \right) = g^0. \tag{7.8}$$

The frequency integral of the flux divergence term is approximated here using the Rosseland mean (q.v. Section 6.7) k_R of k_ν. We drop the time derivative term, which is to say we neglect the photon time of flight. When the equation is linearized and when spatial dependence $\exp(i\mathbf{k} \cdot \mathbf{r})$ is assumed it gives

$$\frac{k^2 c}{3k_R}\delta E = -k_P(c\delta E - 4acT^3\delta T). \tag{7.9}$$

The response of the radiation field to the temperature fluctuations is therefore given by

$$\delta E = \frac{4aT^3\delta T}{1 + k^2/(3k_R k_P)}. \tag{7.10}$$

This says that if the wavelength is very long, the energy density tracks the temperature perfectly, but that if the wavelength is short the energy density hardly varies. The roll-over occurs where the wavenumber is comparable to the geometric mean of the two different mean absorption coefficients, in other words, where the wavelength is about one mean free path. The value of $-\delta g^0$ that results when δE is substituted into (7.9) is

$$-\delta g^0 = -k_P 4acT^3\delta T \frac{k^2/(3k_R k_P)}{1 + k^2/(3k_R k_P)}. \tag{7.11}$$

Thus the cooling time is modified to

$$\tau \to \tau \left(1 + \frac{3k_R k_P}{k^2}\right) = \frac{\rho C_v}{4acT^3}\left(\frac{1}{k_P} + \frac{3k_R}{k^2}\right) \tag{7.12}$$

by the response of the radiation field to the temperature fluctuations.

It is interesting to consider the radiative cooling time in different parts of a star in response to a fixed wavelength. In the tenuous upper atmosphere the Planck mean absorption coefficient will become very small, and the cooling time will be long; the coupling between the matter and radiation is simply weak. In the deep interior the Rosseland mean absorption coefficient becomes very large and the cooling time again becomes long, but this time because the leakage of radiation through the opaque material is sluggish. The minimum cooling time is attained if the wavenumber is comparable with the geometric mean, as just mentioned. The minimum value of it is approximately

$$\tau_{\min} = \frac{2\rho C_v \lambda}{4acT^3}, \tag{7.13}$$

where λ is the spatial wavelength. This is comparable to the time it would take the radiation flux σT^4 to radiate the internal energy content of a wavelength-thick slab of material since $\sigma = ac/4$.

7.3 Atmospheric oscillations with radiation pressure

An extension of the wave propagation ideas of the previous section is the problem of the oscillations of a hydrostatic isothermal atmosphere, for which the equilibrium condition is an exponential stratification of the density, $\rho \propto \exp(-x/H)$. This problem is treated, with respect to the vertical oscillations, by Lamb (1945). It becomes more interesting when the horizontal motions are included, and more interesting still when the effect of the radiation pressure is included as a momentum coupling term in the optically thin approximation. This idea for atmospheric instability in luminous stars was advanced by Hearn (1972, 1973) under the name "radiation-driven sound waves." It provides an opportunity to look critically at what is meant by instability of an unbounded system, such as a stellar atmosphere.

Consider first the case without the radiation pressure term. The discussion of sound waves in Section 2.7 needs to be modified to account for the stratification. In this discussion the notation will also be modified: the variable ρ' will denote the perturbation of ρ divided by ρ, i.e., the logarithmic perturbation; likewise for p'.

The continuity equation (2.2) when linearized now becomes

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla \ln(\rho) = 0$$

or

$$\frac{\partial \rho'}{\partial t} + \nabla \cdot \mathbf{u} + \frac{\gamma}{a^2}\mathbf{g} \cdot \mathbf{u} = 0. \tag{7.14}$$

Here $\mathbf{g}$ is the downward-directed gravity vector, a is the adiabatic speed of sound, and γ is the ratio of specific heats, so $a/\sqrt{\gamma}$ is the isothermal speed of sound, and $a^2/(\gamma g)$ is H, the scale height of the static atmosphere. The perturbed momentum equation (2.3) becomes

$$\frac{\partial \mathbf{u}}{\partial t} - \rho'\mathbf{g} + p'\mathbf{g} + \frac{a^2}{\gamma}\nabla p' = 0, \tag{7.15}$$

and the linearization of the internal energy equation, including Newton's cooling as in Section 2.7 with a time constant τ, is

$$\frac{\partial p'}{\partial t} + \frac{\gamma}{a^2}\mathbf{g} \cdot \mathbf{u} - \gamma\frac{\partial \rho'}{\partial t} - \frac{\gamma^2}{a^2}\mathbf{g} \cdot \mathbf{u} = -\frac{p' - \rho'}{\tau}. \tag{7.16}$$

The system of equations (7.14)–(7.16) for the unknowns ρ', $\mathbf{u}$, and p' has constant coefficients, and therefore it is relatively simple to eliminate two of the unknowns to obtain a PDE for the remaining one, say ρ'. This is

$$\begin{aligned}\Bigg\{\left[\partial_t\left(\partial_t + \frac{1}{\gamma\tau}\right) + \frac{(\gamma-1)a^2}{\gamma^2 H^2}\right](\partial_x^2 + \partial_y^2) + \partial_t\left(\partial_t + \frac{1}{\gamma\tau}\right)\partial_z^2 \\ -\frac{1}{H}\partial_t\left(\partial_t + \frac{1}{\gamma\tau}\right)\partial_z - \frac{1}{a^2}\partial_t^3\left(\partial_t + \frac{1}{\tau}\right)\Bigg\}\rho' = 0.\end{aligned} \tag{7.17}$$

The approach to solving this equation is to first take the Laplace transform with respect to time, which gives an equation of this form:

$$(\nabla^{\mathrm{T}} \quad \nabla + 2\mathbf{b}^{\mathrm{T}}\nabla + c)\tilde{\rho}' = S, \tag{7.18}$$

in which S is a certain cubic polynomial in the transform variable s that contains the initial conditions for ρ', $\mathbf{u}$, and p' – cubic because (7.17) is fourth order in time

– and the coefficients , **b**, and c are defined by

$$= \begin{pmatrix} s\left(s+\frac{1}{\gamma\tau}\right)+\frac{(\gamma-1)a^2}{\gamma^2H^2} & 0 & 0 \\ 0 & s\left(s+\frac{1}{\gamma\tau}\right)+\frac{(\gamma-1)a^2}{\gamma^2H^2} & 0 \\ 0 & 0 & s\left(s+\frac{1}{\gamma\tau}\right) \end{pmatrix}, \tag{7.19}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2H}s\left(s+\frac{1}{\gamma\tau}\right) \end{pmatrix}, \tag{7.20}$$

and

$$c = -\frac{s^3(s+1/\tau)}{a^2}. \tag{7.21}$$

The variable $\tilde{\rho}'$ is the transform of ρ', the operator ∇ represents the column vector $(\partial_x, \partial_y, \partial_z)^{\mathrm{T}}$, and $^{\mathrm{T}}$ denotes the transpose. The same equation would be found for $\mathbf{u}$ and p'; only the initial condition function S would differ.

Some general conditions that can be imposed on the coefficients in (7.18) are: (1) is symmetric; (2) $\sim s^n$ for $s \to \infty$ for some n, where is positive definite; (3) $c = O(s^2|| \quad ||)$ for $s \to \infty$; (4) $||\mathbf{b}||^2/(|| \quad ||c) = o(s)$ for $s \to \infty$. These conditions are sufficient to ensure that the original system of PDEs is hyperbolic, and are met in this example.

If a solution for $\tilde{\rho}'$ is in hand, then the solution of (7.17) for ρ' is given by the Laplace inversion formula

$$\rho'(x, y, z, t) = \frac{1}{2\pi i}\int_{d-i\infty}^{d+i\infty} e^{st}\tilde{\rho}'\,ds, \tag{7.22}$$

where d is a positive real constant sufficiently large to ensure that the contour passes to the right of all the singularities of the integrand in the complex s plane. For large, real, positive s, is positive definite and can be factored:

$$= \quad ^{\mathrm{T}} \quad , \tag{7.23}$$

where is a nonsingular real matrix. The coordinate vector $\mathbf{r} = (x, y, z)^{\mathrm{T}}$ can be transformed using to a new set of coordinates ξ defined by $\mathbf{r} = \quad ^{\mathrm{T}}\xi$. In this way (7.18) may be turned into Helmholtz's equation in ξ-space, for which the Green's function is known. The steps of the transformation will be omitted; the result for the solution $\tilde{\rho}'$ is found to be

$$\tilde{\rho}' = \iiint\limits_V G(\mathbf{r} - \mathbf{r}')S(\mathbf{r}')\, dV, \tag{7.24}$$

where the integration volume includes that part of 3-D space where the initial conditions are nonvanishing, and the Green's function is

$$G(\mathbf{r}) = -\frac{1}{4\pi}\frac{\exp\left\{-\mathbf{b}^{\mathrm{T}}\ \ {}^{-1}\mathbf{r} - \left[(\mathbf{b}^{\mathrm{T}}\ \ {}^{-1}\mathbf{b} - c)\mathbf{r}^{\mathrm{T}}\ \ {}^{-1}\mathbf{r}\right]^{1/2}\right\}}{\left[\det(\quad)\mathbf{r}^{\mathrm{T}}\ \ {}^{-1}\mathbf{r}\right]^{1/2}}. \tag{7.25}$$

We now argue as follows about the behavior of the solution ρ' obtained from applying (7.22) and (7.25). The Green's function given by (7.25) is analytically continued from large, real, positive s in the negative real direction. At some value of $\Re(s)$ singularities will be encountered. The constant d in the Laplace inversion can be set to any number larger than the largest $\Re(s)$ of any of the singularities. If d turns out to be negative, then ρ' tends to zero for $t \to \infty$ if the initial conditions vanish outside a compact volume. If d is positive, then ρ' will become exponentially large for $t \to \infty$. If the singularities of G lie on the real axis a more careful analysis is required to decide whether ρ' grows or not.

This notion of stability is called *absolute stability.* A dynamical problem is absolutely stable if the response to an initial disturbance at a point in space eventually dies away at any other *fixed* point in space. It may still be true that no matter how large t may be, there is a point in space where the response at this time is large, and that as t increases the maximum response becomes larger and larger. This is consistent with absolute stability if the location of the maximum response moves further and further from the site of the initial disturbance. *Convective instability* is the term applied to this situation.[1] A dynamical problem is convectively stable or unstable depending on whether the maximum response over all space to an initial disturbance decays or grows in time. Convective stability guarantees absolute stability, but the reverse is not true. A third kind of stability, called global stability, pertains to dynamical systems in a bounded domain, with specific boundary conditions. Imposing the boundary conditions turns (7.18) into an eigenvalue problem. The signs of the real parts of all the discrete eigenvalues determine stability and instability in this case.

In other words, *the possible singularities of G determine whether the problem is absolutely stable or unstable*. Convective stability can be diagnosed by

[1] The term "convective instability" should not be confused with the instability associated with thermal convection. In fact, the latter could perhaps be called an absolute instability of the gravity wave modes.

finding solutions of the dispersion relation for all real propagation vectors, $\tilde{\rho}' \sim \exp(i\mathbf{k}\cdot\mathbf{r})$. It may be possible to relate global stability to absolute stability.

By inspection of (7.25) we can derive three conditions when G would be singular:

$$\text{(a)} \qquad \det(\quad) = 0; \tag{7.26}$$

$$\text{(b)} \qquad \mathbf{b}^{\mathrm{T}} \quad {}^{-1}\mathbf{b} - c = 0; \tag{7.27}$$

$$\text{(c)} \qquad \mathbf{r}^{\mathrm{T}} \quad {}^{-1}\mathbf{r} = 0. \tag{7.28}$$

We can establish absolute stability or instability by examining the real parts of all the values of s that obey any of (7.26)–(7.28).

We return to our specific problem, with the definitions (7.19)–(7.21). Condition (a) above gives

$$s\left(s + \frac{1}{\gamma\tau}\right)\left[s\left(s + \frac{1}{\gamma\tau}\right) + \frac{(\gamma-1)a^2}{\gamma^2 H^2}\right]^2 = 0, \tag{7.29}$$

condition (b) gives

$$s^3 + \frac{1}{\tau}s + \frac{a^2}{4H^2}s + \frac{a^2}{4\gamma H^2\tau} = 0 \tag{7.30}$$

and condition (c)

$$\frac{x^2+y^2}{s[s+1/(\gamma\tau)] + (\gamma-1)a^2/(\gamma^2H^2)} + \frac{z^2}{s[s+1/(\gamma\tau)]} = 0. \tag{7.31}$$

Condition (c) determines roots s that depend on the vector $\mathbf{r} = (x, y, z)$. This is interpreted as follows. If one of these roots has a positive real part, then $G(\mathbf{r})$ will grow exponentially for that $\mathbf{r}$. Absolute stability requires G to decay at *all* $\mathbf{r}$, and therefore one of the conditions is that *all* the roots of condition (c) should have a nonpositive real part, whatever the values of x, y, and z.

The roots of (7.29) are $s = 0$, $s = -1/(\gamma\tau)$ and the roots of the quadratic $s[s + 1/(\gamma\tau)] + (\gamma - 1)a^2/(\gamma^2 H^2) = 0$. Since there are no sign changes in the coefficients of the latter, those roots have negative real parts. Thus all the roots for condition (a) indicate stability except for the marginally stable trivial root $s = 0$.

The cubic equation (7.30) also has only positive coefficients, and therefore its roots have negative real parts, and condition (b) also indicates absolute stability. It is interesting to examine the roots of the cubic in the realistic limit that τ is small compared with H/a. (In the atmosphere of a typical hot star the dynamic time H/a is of order 10^3 s while the cooling time is of order 10 s.) A helpful tool for doing this is to sketch the overall power of τ that each term in the equation represents versus the exponent n in a hypothetical relation $s = \tau^n$, as in Figure 7.1. A possible

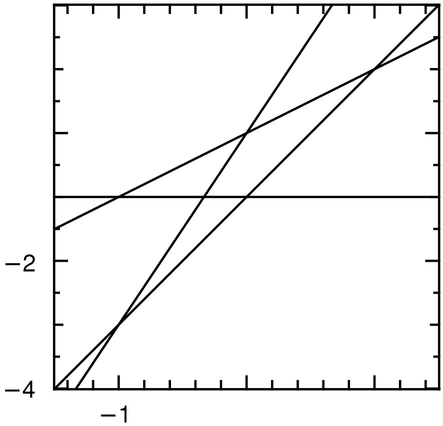

Fig. 7.1 Overall power of τ of each term in (7.30) if $s \propto \tau^n$ vs n.

value of n must be where two or more terms in the cubic have the same overall power of τ, since otherwise the terms cannot balance each other. Furthermore, since τ is a *small* parameter, the overall power these terms have must be smaller than the powers of the remaining terms, otherwise the terms that match will not be the dominant ones. Figure 7.1 shows us that the crossing points at $n = 0$, where the quadratic term balances the constant term, and at $n = -1$, where the cubic term balances the quadratic term, must describe the roots. The first crossing point gives two roots and the second one gives a single root, thus accounting for the three roots. When the dominant part of a given root has been determined in this way, the remaining terms can be evaluated to provide a first order correction to the dominant one, if desired. The roots are thus found to be

$$s \approx \pm \frac{ia}{2\sqrt{\gamma} H} - \frac{(\gamma - 1)a^2 \tau}{8\gamma H^2}, \qquad s \approx -\frac{1}{\tau}. \tag{7.32}$$

These roots can be physically identified with endpoints of branches of the wave spectrum. The complex conjugate pair are the endpoints of the acoustic branches, and the real root is an endpoint of the thermal mode branch. Since $\tau \ll a/H$, the damping of the thermal mode is very large, while the damping of the acoustic mode is slight.

Condition (c) in (7.31) can be reexpressed as this quadratic equation

$$s\left(s + \frac{1}{\gamma\tau}\right) + \frac{(\gamma - 1)a^2 \cos^2\theta}{\gamma^2 H^2} = 0, \tag{7.33}$$

where θ is the angle between $\mathbf{r}$ and the vertical. The roots all have a negative real part, except that $s = 0$ is a root if $\theta = \pi/2$. As θ ranges from 0 to

$\pi/2$ these roots fill in the space between the roots derived from condition (a). They are physically associated with the gravity mode waves, but, like the acoustic mode, they are mixed with the thermal mode. The roots are complex only if $\tau > H/(2a\sqrt{\gamma - 1})|\sec\theta|$. If the cooling is too efficient the buoyancy force that provides the "spring" for gravity waves is suppressed and the waves do not oscillate. In the adiabatic limit, $\tau \gg a/H$, the $\theta = 0$ endpoints of the gravity branches are at $s = \pm i\sqrt{\gamma - 1}a/(\gamma H)$, of which the imaginary part is the Brunt–Väisälä frequency. These are similar to, but just slightly smaller in magnitude than, the endpoints $\pm ia/(2H)$ of the acoustic branches in the adiabatic limit.

In summary, the oscillations of the exponential atmosphere are absolutely stable. The atmosphere is certainly not convectively stable, since the pulse produced by the initial disturbance increases in amplitude as $\exp[z/(2H)]$ as it rises through the atmosphere at the speed a.

We turn now to the case that the body force on the matter due to absorption or scattering of radiation is included in the material momentum balance. We saw earlier that this force should be the negative of $\mathbf{g}$ given by (4.44). When expressed per unit mass of material it is

$$\mathbf{g}_\mathrm{R} = \frac{1}{\rho c}\int d\nu \int_{4\pi} d\Omega\, \mathbf{n}(k_\nu I_\nu - j_\nu). \tag{7.34}$$

If the emissivity j_ν is isotropic that term in $\mathbf{g}_\mathrm{R}$ vanishes, and if k_ν is isotropic it can be taken out of the integral over angle, which becomes the total monochromatic flux, $\mathbf{F}_\nu$. For a stellar atmosphere with slab symmetry, the flux is a vector in the $+z$ direction. For the present purpose we assume that the frequency-dependent absorption coefficient can be replaced by the flux-weighted mean, $\kappa_\mathrm{F}\rho$. Thus we will use

$$\mathbf{g}_\mathrm{R} = \frac{\kappa_\mathrm{F} F}{c}\mathbf{e}_z, \tag{7.35}$$

where κ_F is the flux-mean opacity and F is the total radiative flux, which we will assume to be constant. The ratio of $g_\mathrm{R} = |\mathbf{g}_\mathrm{R}|$ to the normal gravity g will be denoted by a new variable Γ:[2]

$$\Gamma = \frac{\kappa_\mathrm{F} F}{gc}, \tag{7.36}$$

and therefore the momentum equation can be written

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} + \frac{1}{\rho}\nabla p = (1 - \Gamma)\mathbf{g}. \tag{7.37}$$

[2] The Eddington luminosity for a star is that value for which the radiation force balances gravity; it is $L_\mathrm{Edd} \equiv 4\pi G\mathcal{M}c/\kappa_\mathrm{F}$, and so $\Gamma = L/L_\mathrm{Edd}$. Normal stars approach but do not exceed $L = L_\mathrm{Edd}$.

The flux-mean opacity consists of a constant Thomson scattering part, denoted by σ_e, and an absorption part due to processes such as bound–free and free–free absorption, as well as line absorption. The absorption term varies with temperature and density, so it will be approximated by $\kappa_1 \rho^n T^{-q}$, where n and q are constant exponents, and therefore Γ can be written

$$\Gamma = \frac{F}{gc}\left(\sigma_e + \kappa_1 \rho^n T^{-q}\right). \tag{7.38}$$

The variation of the opacity with height, due to its ρ dependence, should be included, but that will be neglected here. However, the effect of density perturbations on the opacity *will* be included. The linearized form of the momentum coupling term $(1-\Gamma)\mathbf{g}$ becomes

$$-\Gamma'\mathbf{g} = -\frac{\kappa_1 F}{gc}\rho^n T^{-q}[(n+q)\rho' - qp']\mathbf{g} = -\Gamma_{\rm e}[(n+q)\rho' - gp']\mathbf{g}_{\rm eff}, \tag{7.39}$$

where $\Gamma_{\rm e}$ is defined by

$$\Gamma_{\rm e} = \frac{1}{1-\Gamma}\frac{\kappa_1 F}{gc}\rho^n T^{-q}, \tag{7.40}$$

and $\mathbf{g}_{\rm eff} = (1-\Gamma)\mathbf{g}$, in terms of Γ in the unperturbed atmosphere. $\Gamma_{\rm e}$ will be treated as a constant in the problem, in a Boussinesq-like approximation. The perturbed momentum equation, replacing (7.15), is now

$$\frac{\partial \mathbf{u}}{\partial t} - \rho'\mathbf{g}_{\rm eff} + p'\mathbf{g}_{\rm eff} + \frac{a^2}{\gamma}\nabla p' = -\Gamma_{\rm e}[(n+q)\rho' - qp']\mathbf{g}_{\rm eff}. \tag{7.41}$$

The occurrences of $\mathbf{g}$ in (7.14) and (7.16) will also be replaced by $\mathbf{g}_{\rm eff}$, since the scale height of the static atmosphere is now $H = a^2/(\gamma g_{\rm eff})$.

With these changes the equation satisfied by $\tilde{\rho}'$ remains of the form (7.18), but now , $\mathbf{b}$, and c are given by

$$= \begin{pmatrix} s\left(s+\dfrac{1}{\gamma\tau}\right) + \dfrac{(\gamma-1)a^2}{\gamma^2 H^2}[1-(n+q)\Gamma_{\rm e}] & 0 & 0 \\ 0 & s\left(s+\dfrac{1}{\gamma\tau}\right) + \dfrac{(\gamma-1)a^2}{\gamma^2 H^2}[1-(n+q)\Gamma_{\rm e}] & 0 \\ 0 & 0 & s\left(s+\dfrac{1}{\gamma\tau}\right) \end{pmatrix}, \tag{7.42}$$

$$\mathbf{b} = \begin{pmatrix} 0 \\ 0 \\ -\frac{1}{2H} s \left\{ s \left[1 + \frac{\Gamma_e}{\gamma} [n - (\gamma - 1) q] \right] + \frac{1 + n\Gamma_e}{\gamma\tau} \right\} \end{pmatrix}, \tag{7.43}$$

and

$$c = -\frac{s}{a^2}\left(s + \frac{1}{\tau}\right)\left(s^2 - \frac{n\Gamma_e a^2}{\gamma H^2}\right). \tag{7.44}$$

Condition (a) now leads to

$$s\left(s + \frac{1}{\gamma\tau}\right)\left\{s\left(s + \frac{1}{\gamma\tau}\right) + \frac{(\gamma - 1)a^2}{\gamma^2 H^2}[1 - (n + q)\Gamma_e]\right\}^2 = 0. \tag{7.45}$$

Condition (b) becomes

$$\frac{s}{4H^2[s + 1/(\gamma\tau)]}\left(s\left\{1 + \frac{\Gamma_e}{\gamma}[n - (\gamma - 1)q]\right\} + \frac{1 + n\Gamma_e}{\gamma\tau}\right)^2 + \frac{s}{a^2}\left(s + \frac{1}{\tau}\right)\left(s^2 - \frac{n\Gamma_e a^2}{\gamma H^2}\right) = 0. \tag{7.46}$$

Condition (c) becomes

$$s\left(s + \frac{1}{\gamma\tau}\right) + \frac{(\gamma - 1)a^2 \cos^2\theta}{\gamma^2 H^2}[1 - (n + q)\Gamma_e] = 0, \tag{7.47}$$

where θ, as before, is the angle between $\mathbf{r}$ and the z axis.

The change in conditions (a) and (c) is that the Brunt–Väisälä frequency is modified, and becomes

$$\omega_{BV}^2 = \frac{(\gamma - 1)a^2}{\gamma^2 H^2}[1 - (n + q)\Gamma_e]. \tag{7.48}$$

This has the important implication that if Γ_e exceeds the critical value $\Gamma_e = 1/(n + q)$, then ω_{BV} becomes imaginary. In this case one of the endpoints of a gravity mode branch will produce absolute instability. It is entirely possible for this condition to be met in the atmospheres of the most luminous stars, in which Γ approaches unity.

Condition (b) is more complicated with the addition of the radiation force term. Equation (7.46) can be rearranged as a quartic equation

$$s^4 + a_1 s^3 + a_2 s^2 + a_3 s + a_4 = 0 \tag{7.49}$$

after discarding the factor s, with the coefficients

$$
\begin{aligned}
a_1 &= \frac{\gamma+1}{\gamma\tau}, \\
a_2 &= \frac{a^2}{H^2}\left\{\frac{1}{\gamma}\left(\frac{H}{a\tau}\right)^2 + \frac{1}{4} - \frac{n+(\gamma-1)q}{2\gamma}\Gamma_e + \frac{[n-(\gamma-1)q]^2}{4\gamma^2}\Gamma_e^2\right\}, \\
a_3 &= \frac{a^2}{H^2\tau^2}\frac{1}{2\gamma^2}\left\{\gamma - [(\gamma+1)n + (\gamma-1)q]\,\Gamma_e + n[n-(\gamma-1)q]\Gamma_e^2\right\}, \\
a_4 &= \frac{a^2}{4\gamma^2H^2\tau^2}(1-n\Gamma_e)^2.
\end{aligned}
\tag{7.50}
$$

The Hurwitz–Routh criterion applied to this quartic equation leads to these conditions for stability:[3]

$$a_1 \geq 0, \tag{7.51}$$

$$a_1a_2 - a_3 \geq 0, \tag{7.52}$$

$$a_1a_2a_3 - a_1^2a_4 - a_3^2 \geq 0, \tag{7.53}$$

$$a_4(a_1a_2a_3 - a_1^2a_4 - a_3^2 \geq 0. \tag{7.54}$$

The inequalities (7.52) and (7.54) can be replaced by

$$a_3 \geq 0, \tag{7.55}$$

$$a_4 \geq 0 \tag{7.56}$$

without changing the results. Inspecting the coefficients shows that conditions (7.51) and (7.56) are always satisfied. Condition (7.55) may be violated for some Γ_e, depending on n and q. If $n < (\gamma-1)q$, the statement that the absorption opacity decreases in an adiabatic compression, the condition is certainly violated for sufficiently large Γ_e. That leaves condition (7.53) to discuss. It can be rearranged in this way:

$$2(\gamma+1)E\left(\frac{H}{a\tau}\right)^2 \geq -FG, \tag{7.57}$$

with three new coefficients that are polynomials in Γ_e:

$$E = (1+n\Gamma_e)[1-(n+2q)\Gamma_e], \tag{7.58}$$

$$F = \gamma - [(\gamma+1)n + (\gamma-1)q]\,\Gamma_e + n[n-(\gamma-1)q]\Gamma_e^2, \tag{7.59}$$

$$G = \gamma\left[1 + \frac{n-(\gamma-1)q}{\gamma}\Gamma_e\right]\left[1 - \frac{n+(\gamma+1)q}{\gamma}\Gamma_e\right]. \tag{7.60}$$

We note that $a_3 \propto F$ and that condition (7.55) is the same as $F \geq 0$.

[3] The roots of an algebraic equation are the same as the eigenvalues of a certain matrix derived from the coefficients, and the real parts of the eigenvalues will all lie in the left half-plane provided the principal minors of the matrix are nonnegative. See Ralston (1965).

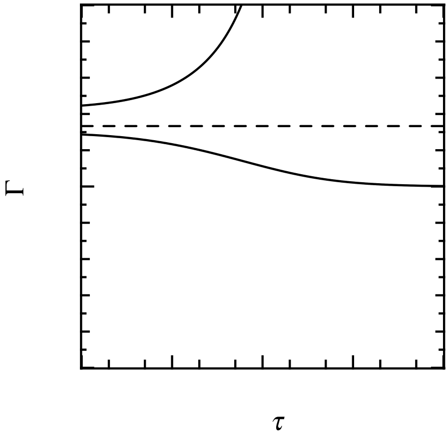

Fig. 7.2 Stability domain in radiation force vs adiabacy parameter. Acoustic modes are unstable between curves A and C. Gravity modes are unstable above dashed line B. Curves A and C and line B are described in the text.

The summary of the absolute stability for this problem is that there is stability in the isothermal limit $H/(a\tau) \to \infty$ for $\Gamma_e < 1/(n+2q)$. There is stability in the adiabatic limit $H/(a\tau) \to 0$ for Γ_e less than the smaller of the smallest positive zeroes of F and G. These limits turn out to be tighter than the limit $\Gamma_e < 1/(n+q)$ derived from the gravity modes. Depending on whether $n < (\gamma - 1)q$ or not, either F has one positive zero and G has two or none, or the reverse. The product FG has either one or three positive zeroes. With specific values of n, q, and γ it is a simple matter to map the stability domain in Γ_e as a function of $H/(a\tau)$. Figure 7.2 shows the domain for the choices $\gamma = 5/3$, $n = 1$, and $q = 1/2$. All pairs $(H/(a\tau), \Gamma_e)$ that produce equality in (7.57) define the locus in the $\Gamma_e, H/(a\tau)$ diagram that separates regions of stability and instability. For this case the locus has two disconnected branches, curves A and C in the diagram. Stability according to (7.57) occurs below curve A and above curve C. But the gravity wave criterion indicates instability above the line B, $\Gamma_e = 1/(n+q)$. Thus the region that is stable for both acoustic and gravity modes is the one below curve A, which goes smoothly between the isothermal limit $\Gamma_e = 1/(n+2q) = 1/2$ and the adiabatic limit, which is $\Gamma_e \approx 0.6492$. The implication is that the atmospheres of the most luminous stars may be unstable on this account, as suggested by Hearn (1972, 1973).