5

Steady-state transfer

In this chapter we will review some of the standard results of radiative transport/transfer theory. A large part of the theoretical development has been done for the steady-state case and for slab geometry. Within these approximations the results can be made quite refined in the sense of avoiding approximations to the angle dependence of the radiation field such as diffusion or Eddington's approximation. These results lead to concepts that are helpful in understanding more complicated cases.

5.1 Formal solution in three dimensions

Since the transport equation (4.23) says in effect that

$$\frac{dI}{ds} = j - kI \tag{5.1}$$

along the ray path, where j and k are the local in space and time emissivity and absorptivity, the solution is just

$$I = I_{\mathrm{B}} \exp\left(-\int_{s_{\mathrm{B}}}^{s} ds'\, k(s')\right) + \int_{s_{\mathrm{B}}}^{s} ds' \exp\left(-\int_{s'}^{s} ds''\, k(s'')\right) j(s'); \tag{5.2}$$

see, for example, Pomraning (1973). The notations ν and $\mathbf{n}$ have been omitted here, since these parameters are constant in this fixed-frame picture. The suffix "B" denotes a point on the spatial boundary of the problem, or a point at the initial time, where the boundary or initial data prescribe I_{B}. The values of j and k inside the integrals over s' or s'' are at the displaced location and retarded time given, for example, by $\mathbf{r}' = \mathbf{r} - \mathbf{n}(s - s')$ and $t' = t - (s - s')/c$. Thus a more notationally complete but harder to read version of the same formal solution is

$$
\begin{aligned}
I(\mathbf{r}, t) = I(\mathbf{r} - (s - s_{\mathrm{B}})\mathbf{n}, t - (s - s_{\mathrm{B}})/c) \\
\times \exp\left[- \int_{s_{\mathrm{B}}}^{s} ds'\, k(\mathbf{r} - (s - s')\mathbf{n}, t - (s - s')/c) \right] \\
+ \int_{s_{\mathrm{B}}}^{s} ds' \exp\left[- \int_{s'}^{s} ds''\, k(\mathbf{r} - (s - s'')\mathbf{n}, t - (s - s'')/c) \right] \\
\times j(\mathbf{r} - (s - s')\mathbf{n}, t - (s - s')/c).
\end{aligned}
\tag{5.3}
$$

This form of the transport equation is not too useful as it stands, but after making the simplifications of time independence and slab symmetry we will get a form that lends itself to the theoretical elaborations that were promised. Equation (5.3) as it stands corresponds well with the Monte Carlo approach to radiation transport that will be touched on later.

5.2 Time-independent slab geometry

We assume now that our problem is in a steady state and that there is translational symmetry in x and y, so the z coordinate is the only nontrivial one. This also results in the problem being axially symmetric about the z axis. Radiative transfer in this kind of geometry – slab geometry – has an extensive literature, including the older books by Chandrasekhar (1960) and Kourganoff (1963), and more general stellar atmospheres texts like Mihalas (1978). The notation in these works is standard, and has been followed here. As a result of the symmetry the intensity can depend only on the component of $\mathbf{n}$ in the z direction. In astrophysics it is traditional to include a minus sign here, so a positive direction cosine will refer to propagation toward $-z$. The reason is that z is thought of as a coordinate measured from the outside inward toward the center of a star, but a positive direction cosine is used for the radiation the external observer views. So we take

$$n_z \equiv -\mu, \tag{5.4}$$

and use μ as our label for angles.

We also introduce the optical depth variable τ at this point. There is a powerful justification for this, which is that a variety of transfer problems which differ from each other just in how the opacity is distributed in the z coordinate become the same problem when viewed in τ space. The notational problem of carrying factors of $k(z)$ vanishes. At the end it is very simple to transform back to physical space. The variable τ is reckoned from the outside of the star inward, in the same sense as z, but in the direction of negative μ. Thus

$$\tau = \int_{-\infty}^{z} dz' \, k(z'). \tag{5.5}$$

The assumption here is that the star (cloud, ...) has an "outside" beyond which there is vacuum, and the zero point of τ is located there. If the star trails off gradually to infinite radius, it is assumed that the optical depth integral converges in that limit. If this is not the right picture, if instead a "wall" of some kind is encountered on the outside, then this must be accounted for in the boundary condition. We mention again that all quantities are in fact frequency-dependent, which will not be shown explicitly unless it is necessary to do so.

In τ space the equation of transfer, which is what we call transport in the steady-state case, becomes

$$\mu \frac{dI}{d\tau} = I - \frac{j}{k}. \tag{5.6}$$

The mapping to τ space allows us to largely ignore the spatial variation of $k(z)$; we make this complete by introducing the *source function* S defined by

$$S \equiv \frac{j}{k}. \tag{5.7}$$

The disadvantage of having to remember the meaning of yet another symbol is compensated by the greater simplicity of the equations, besides which it has been found by old hands of radiative transfer that the source function concept is actually an aid to understanding.

In thermodynamic equilibrium the radiation field is in equilibrium with its sources, and therefore the right-hand side of (5.6) must be zero. Also in this case the intensity is the Planck function, and therefore the source function becomes the Planck function in thermodynamic equilibrium. Since j and k are properties of the matter – scattering aside, which we discuss next – the source function is the Planck function whenever the state of the matter is the same as if it actually were in thermodynamic equilibrium, whether the intensity I is the Planck function or not. This is the concept of LTE.

Scattering complicates this picture. The atomic basis of scattering will be discussed later, but here we can simply say that a scattering process is one in which a photon that is removed by the interaction is returned to the radiation field *instantly* instead of causing an excitation of the atoms. The emissivity due to this process depends directly on the actual radiation field rather than on the state of the matter, and therefore the corresponding source function need not be the Planck function even when the matter is in LTE.

The emissivity due to scattering can often be treated as isotropic, and for pure scattering every photon removed from the radiation field by a scattering process comes back at some other angle. The consequence of these two conditions is that the scattering source function is given by

$$S = \frac{1}{2}\int_{-1}^{1} d\mu\, I(\mu) = J = \frac{cE}{4\pi}. \tag{5.8}$$

One comment that must be made here concerns the frequencies. Energy is conserved overall in a scattering process, and we are considering processes that leave the scatterer (electron or atom) in the same state after the event as before. The only way that energy can be lost by the photon (or conceivably gained) is in the recoil kinetic energy of the scatterer. This effect is exceedingly small except for x-rays and gamma-rays being scattered by electrons, and for Raman scattering. For electron scattering processes in the IR, optical, and UV it is a very good assumption that there is no frequency change in the scattering. That means that the monochromatic source function in (5.8) depends on I, J or E at exactly the same frequency. The relaxation of this assumption for Compton scattering of x-rays by electrons will be discussed below.

When the absorptivity and emissivity include both scattering processes and what is called "true" absorption, i.e., everything that is *not* scattering, then the emissivity in the LTE case can be expressed as

$$j = kS = k_a B + k_s J, \tag{5.9}$$

where k_a is the absorptivity for "true" absorption and k_s is the absorptivity for scattering. Thus the total source function is

$$S = \frac{k_a}{k_a + k_s} B + \frac{k_s}{k_a + k_s} J = \frac{k_a}{k_a + k_s} B + \frac{k_s}{k_a + k_s}\frac{1}{2}\int_{-1}^{1} d\mu\, I(\mu). \tag{5.10}$$

The ratio $k_s/(k_a + k_s)$ might be called the *single-scattering albedo*, although sometimes that term is reserved for things like scattering of light by dirty water droplets. If we denote the albedo by ϖ then we can write

$$S = (1 - \varpi)B + \frac{\varpi}{2}\int_{-1}^{1} d\mu\, I(\mu). \tag{5.11}$$

The formal integral (5.3) turns into the following form with our simplified geometry:

$$I = \begin{cases} I_{\mathrm{B}} \exp(\tau/\mu) + \displaystyle\int_0^{\tau} \frac{d\tau'}{|\mu|} \exp\left[(\tau - \tau')/\mu\right] S(\tau') & \mu < 0 \\ \displaystyle\int_{\tau}^{\infty} \frac{d\tau'}{\mu} \exp[(\tau - \tau')/\mu] S(\tau') & \mu > 0 \end{cases}. \tag{5.12}$$

If the source function is already known, e.g., it is the Planck function, the job is done at this point. Otherwise, as for instance in the case that S is given by a relation like (5.11), we proceed further. Since S depends in such cases on B, which is known, and on J, which is unknown, we will want the expression for J that can be derived from (5.12), and while we are at it, we will get the flux and pressure moments as well. The integrals over μ of the exponential functions like $\exp(-x/\mu)$ are defined in terms of the special functions called exponential integral functions. The generic exponential integral function is $E_n(x)$ defined by

$$E_n(x) = \int_1^\infty \frac{e^{-xy}}{y^n}\, dy. \tag{5.13}$$

The properties of the exponential-integral functions are described at length in Abramowitz and Stegun (1964). Another form that is useful here is

$$E_n(x) = \int_0^1 t^{n-2} e^{-x/t}\, dt. \tag{5.14}$$

These functions obey the recursion relations

$$E_n'(x) = -E_{n-1}(x), \tag{5.15}$$

and

$$E_n(x) = \frac{1}{n-1}\left[e^{-x} - xE_{n-1}(x)\right]. \tag{5.16}$$

It is sometimes useful to know that the E_n functions are related to the incomplete gamma function.

Forming the moments of $I(\mu)$ given by (5.12) leads to

$$J(\tau) = \frac{cE}{4\pi} = \frac{1}{2}E_2(\tau)I_{\mathrm{B}} + \frac{1}{2}\int_0^\infty d\tau'\, E_1(|\tau' - \tau|)S(\tau'), \tag{5.17}$$

$$H(\tau) = \frac{F}{4\pi} = -\frac{1}{2}E_3(\tau)I_{\mathrm{B}} + \frac{1}{2}\int_0^\infty d\tau'\, \mathrm{sgn}(\tau' - \tau)E_2(|\tau' - \tau|)S(\tau'), \tag{5.18}$$

and

$$K(\tau) = \frac{cP}{4\pi} = \frac{1}{2}E_4(\tau)I_{\mathrm{B}} + \frac{1}{2}\int_0^\infty d\tau'\, E_3(|\tau' - \tau|)S(\tau'), \tag{5.19}$$

which also identically obey the moment equations

$$\frac{dH}{d\tau} = J - S \tag{5.20}$$

and

$$\frac{dK}{d\tau} = H. \tag{5.21}$$

All the functions $E_n(x)$ behave at large x as $\exp(-x)/x$. Therefore when τ is significantly larger than 1, say $\tau > 4$, the boundary terms in J, H, and K are negligible, and the lower limits of the integrals in (5.17)–(5.19) can be extended to $-\infty$ without affecting the answer. Thus it is as if the medium were infinitely extended. These are the hallmarks of the interior part of the problem. If S is slowly varying there, as it should be, then the integrals in the expressions for J and K can be approximated by taking S outside the integral and evaluating it at τ. Since also

$$\int_{-\infty}^{\infty} E_n(|x|)\, dx = \frac{2}{n}, \tag{5.22}$$

the conclusion is that $K \approx J/3$, which is Eddington's approximation. In other words, the radiation becomes isotropic once $\tau \gg 1$.

Another insight into the behavior of the transfer equation is obtained by taking S out of the integral in (5.17) whether τ is large or not. The resulting relation

$$\begin{aligned} J &\approx \frac{1}{2}E_2(\tau)I_\mathrm{B} + S\frac{1}{2}\int_0^\infty E_1(|\tau' - \tau|)\, d\tau' \\ &= \frac{1}{2}E_2(\tau)I_\mathrm{B} + \left[1 - \frac{1}{2}E_2(\tau)\right]S \end{aligned} \tag{5.23}$$

is called the *escape-probability* approximation. The expression

$$p_\mathrm{esc} = \frac{1}{2}E_2(\tau) \tag{5.24}$$

is called the escape probability, or the two-sided single-flight escape probability, to be more precise. We can regard this as the average of the one-sided escape probability going toward larger τ, namely zero, and the one-sided escape probability going toward smaller τ, which is $E_2(\tau)$. The E_2 function goes between the limits of 1 and 0 as τ goes from 0 to ∞, so it is reasonable to think of it as a probability. We will return to escape probabilities in Sections 6.8 and 11.8.

5.3 Milne's equation

Some of the standard problems in radiative transfer theory involve solving for J in the case of a semi-infinite atmosphere with no externally-incident radiation, when there is a relation of the kind in (5.11) between S and J. Thus $I_\mathrm{B} = 0$ and there is a vacuum boundary at $\tau = 0$ but the problem extends to infinity in the positive τ

direction. The simplest of these is the homogeneous Milne problem:

$$S = J = \frac{1}{2}\int_0^\infty S(\tau')E_1(|\tau' - \tau|)\,d\tau. \tag{5.25}$$

This arises in conservative scattering, i.e., when there is no absorption at all. It also is a model for an atmosphere in radiative energy balance, for which the absorption and emission rates are just equal, with a frequency-independent (gray) opacity that permits integrating all the radiation quantities over frequency. This problem is described at length by Chandrasekhar (1960) and Kourganoff (1963), and the latter reference gives the exact solution obtained using the Wiener–Hopf method.

Sometimes we need a notation for the kernel in (5.25), and for this we use $K_1(\tau)$:

$$K_1(\tau) = \frac{1}{2}E_1(|\tau|). \tag{5.26}$$

If the medium were infinite the lower limit of integration in (5.25) would be $-\infty$ not 0, and then we see that the equation would have a displacement kernel, i.e., be of convolution type. We would want to know the Fourier transform of the kernel. We can readily calculate that from the definition of the E_1 function, as follows:

$$\begin{aligned}\tilde{K}_1(k) &= \frac{1}{2}\int_{-\infty}^{\infty} d\tau\ \exp(-ik\tau)\int_0^1 \frac{du}{u}\exp(-|\tau|/u)\\ &= \Re\left[\int_0^1 \frac{du}{u}\left(\frac{1}{u} + ik\right)^{-1}\right]\\ &= \int_0^1 \frac{du}{1 + k^2u^2} = \frac{\tan^{-1} k}{k}.\end{aligned} \tag{5.27}$$

Equation (5.25) is a Fredholm equation of the second kind on a half-space. This is the kind of equation for which the Wiener–Hopf method was designed, and the solution has been obtained in terms of definite integrals. Since $J = S$, (5.20) says that the total flux H is constant, in which case (5.21) can be integrated to give

$$K = H(\tau + \text{constant}). \tag{5.28}$$

The Eddington approximation should become accurate at large τ, so the solution for J can be written

$$J = 3H(\tau + q(\tau)), \tag{5.29}$$

where $q(\tau)$, called the Hopf function, tends to a constant at large τ.

The constant value of the flux $F = 4\pi H$ defines the effective temperature of a star according to

$$F = \sigma T_{\text{eff}}^4. \tag{5.30}$$

Furthermore, the frequency-integrated mean intensity J and the source function S are both the same, in the gray atmosphere problem,[1] as the frequency-integrated Planck function B, which is

$$B = \frac{\sigma}{\pi} T^4. \tag{5.31}$$

Therefore (5.29) gives the temperature distribution in the gray atmosphere

$$T^4 = \frac{3}{4} T_{\text{eff}}^4 (\tau + q(\tau)). \tag{5.32}$$

The Hopf function is the main result of Milne's problem. As we see, in Eddington's approximation it is replaced by a constant. The value of the constant is derived from the relation imposed at $\tau = 0$, of the kind

$$\frac{H(0)}{J(0)} = \frac{1}{3q(0)} = \frac{\int_0^1 \mu I(0, \mu)\, d\mu}{\int_0^1 I(0, \mu)\, d\mu} = \langle \mu \rangle. \tag{5.33}$$

Different estimates of $\langle \mu \rangle$ have been proposed, each with its corresponding value of $q(\tau)$. If I is approximately constant on the range $[0, 1]$ then $\langle \mu \rangle$ turns out to be $1/2$, and so the constant value of $q(\tau)$ should be $2/3$. A quadrature for the integrals based on two-point Gaussian quadrature (abscissae at $\pm 1/\sqrt{3}$) leads to $\langle \mu \rangle = 1/\sqrt{3}$ and to $q(\tau) = 1/\sqrt{3}$. Neither estimate of $q(\tau)$ is perfect. The exact function is shown in Figure 5.1. The value of $q(\tau)$ for $\tau \to \infty$ is $0.710\,446\,09\ldots$. As it happens, the exact value of $q(0)$ is $1/\sqrt{3} = 0.577\,35\ldots$, but the estimate $q(\tau) = 2/3$ is closer to the exact solution in the average. The exact function is very nearly constant for $\tau > 4$ as expected based on the earlier discussion. In fact, $q(\tau)$ is within 1% of q_∞ for $\tau > 1.3$, and within 0.1% of q_∞ for $\tau > 3$.

[1] When the opacity is *gray*, i.e., independent of frequency, the radiative equilibrium condition becomes $J = B$, and therefore $S = B$ in that case.

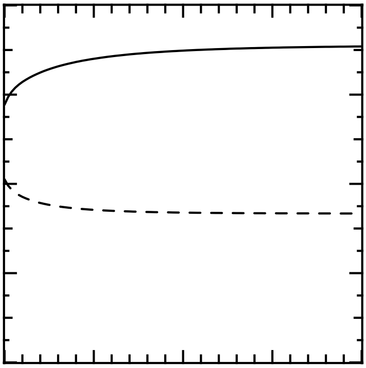

Fig. 5.1 Solid curve: exact Hopf function $q(\tau)$; dashed curve: corresponding Eddington factor, vs optical depth τ.

5.4 Eddington factor

The ratio of the zz component of the radiation pressure to the energy density, i.e., of K to J, defines the *Eddington factor*:

$$f \equiv \frac{P_{zz}}{E} = \frac{K}{J}. \tag{5.34}$$

It is a variable, not a constant, unless the Eddington approximation is being made, in which case it has the value $1/3$. If the Eddington factor is somehow already known, then the system of the first two moment equations is closed, just as in the Eddington approximation. Furthermore, if f is known then probably the mean cosine $f_H \equiv \langle\mu\rangle$ in (5.33) is also known. This allows us to write down the answer to the Milne problem immediately:

$$fJ = H\left[\tau + \frac{f(0)}{f_H}\right]. \tag{5.35}$$

This is the idea behind variable-Eddington-factor approximation methods, about which we will say more later. For the Milne problem there is a simple relation between the Eddington factor and the Hopf function, which is

$$3f(\tau) = \frac{\tau + q_\infty}{\tau + q(\tau)}. \tag{5.36}$$

This result for f based on the exact Hopf function is also shown in Figure 5.1.

5.5 Milne's second equation – thermalization

The next standard problem is the inhomogeneous version of the first. The model in this case is that there is a specified Planck function that may depend on position in the atmosphere and a scattering albedo that is smaller than unity, $\varpi < 1$. The objective is to solve for S in this case. The integral equation is

$$S(\tau) = (1-\varpi)B(\tau) + \frac{\varpi}{2}\int_0^\infty E_1(|\tau' - \tau|)S(\tau'), \tag{5.37}$$

a Fredholm equation of the first kind. The solution in terms of integrals has been derived using another application of the Wiener–Hopf method, and also, by Sobolev (1963), using "elementary" methods, which is to say, without using analytic function theory. The general solution is too complicated to derive here. It is given in the form of integrals for the resolvent function $R(\tau, \tau')$ in terms of which the solution to (5.37) is

$$S(\tau) = (1-\varpi)B(\tau) + \varpi\int_0^\infty R(\tau, \tau')B(\tau'). \tag{5.38}$$

The second Milne equation and the theory of the resolvent are discussed at length in Sobolev, and very similar material pertaining to spectral line transport is found in Ivanov (1973). The Wiener–Hopf theory is based on the factorization of a function $T(z)$ related to the Fourier transform of the kernel:

$$T(z) \equiv 1 - \varpi\tilde{K}_1(i/z) = 1 - \varpi z \coth^{-1} z, \tag{5.39}$$

where the last equality comes from (5.27). The nontrivial part is to factor $T(z)$ into parts that are analytic and nonvanishing in the left and right half-planes,

$$T(z) = \frac{1}{H(z)H(-z)}, \tag{5.40}$$

where the H function, introduced by Chandrasekhar,[2] is analytic and nonvanishing in $\Re z \geq 0$. Finding $H(z)$ can be done formally, by applying Cauchy's integral formula to $\log[T(z)]$, or computationally using the method of discrete ordinates or a numerical inversion of an integral equation satisfied by H. The result for the resolvent function is given in terms of its double Laplace transform, i.e., transformed with respect to both τ and τ':

$$\tilde{\tilde{R}}(p, q) = \frac{1-\varpi}{\varpi}\frac{H(1/p)H(1/q) - 1}{p+q}. \tag{5.41}$$

[2] Also attributed to V. A. Ambartsumian.

The integral formulae by which $R(\tau, \tau')$ can be obtained from its double transform are developed in the radiative transfer literature, e.g., Sobolev (1963) and Ivanov (1973).

The physical interpretation of (5.38) is the following. The first term in S is the contribution from the local source, i.e., the photons emitted for the first time at this location. The second term is the intensity at the location τ of the photons first emitted at another location τ' by the primary source, which then travel from τ' to τ in any number of flights with scattering events between successive flights. Now imagine we time-reverse this picture. A collection of photons at τ will contain some that are destroyed on the spot by absorption, and others that travel from τ to τ' in any number of flights before being destroyed at that location. This final destruction of a photon after perhaps a large number of scatterings is the process called thermalization of the radiation. The meaning of $R(\tau, \tau')$ is therefore the conditional probability distribution of thermalization positions τ' given that a photon originates at τ. It is not too hard to show that $R(\tau, \tau') = R(\tau', \tau)$, and therefore R is also the conditional probability of thermalization at τ for photons created at τ'.

When the albedo is small, R is nearly the same as $E_1(|\tau' - \tau|)/2$, in other words, it is sharply peaked at $\tau' = \tau$. As the albedo approaches 1 the distribution becomes broader and broader. The typical extent of R in $|\tau' - \tau|$ is referred to as the *thermalization length*, and it tends to infinity as $\varpi \to 1$.

This progression in R as the albedo approaches unity is easiest to illustrate for the infinite medium, for which all the integrals are taken over the range $-\infty$ to ∞. The integral equation is then a convolution integral and its solution is easily found using Fourier transforms; the inverse transformation to obtain $R(\tau' - \tau)$ can be done by the method of residues. Figure 5.2 shows how R progresses as the albedo

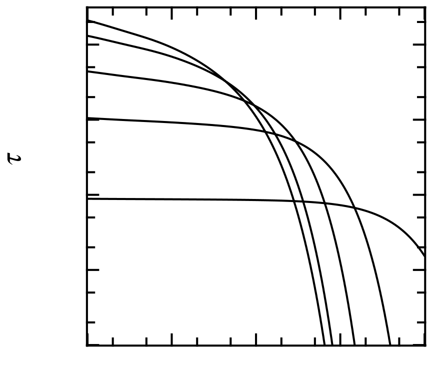

Fig. 5.2 Resolvent function for the infinite medium vs optical depth τ. The curves are ordered at the left, top to bottom, as $\epsilon = 1, 0.5, 0.1, 0.01, 0.0001$.

is raised from zero to 0.9999. When $\epsilon \equiv 1 - \varpi$ is small the entire contribution to R comes from a single pole in $\tilde{R}(k)$. This pole is located at $k = i\kappa_0$, where κ_0 is the root of

$$\frac{\tanh^{-1}\kappa_0}{\kappa_0} = \frac{1}{\varpi}, \tag{5.42}$$

which also means that $z = 1/\kappa_0$ is a zero of $T(z)$. This root appears in various guises in asymptotic diffusion theory. For small ϵ the root is approximately

$$\kappa_0 \approx \sqrt{3\epsilon}. \tag{5.43}$$

This is identical to what would be obtained using the Eddington approximation, and in fact the whole solution $R(\tau)$ tends to the Eddington approximation result when ϵ is small. In the half-space case only the long-range part of R is well approximated by Eddington, and the short-range part shows some effects of angle-dependent transfer.

The half-width of $R(\tau)$ at the $1/e$ point, in the small-ϵ case, is just

$$L = \frac{1}{\sqrt{3\epsilon}}, \tag{5.44}$$

and this is as good a definition of the thermalization length as any. This expression has a simple interpretation. Since ϵ is the probability that a photon will be absorbed on any single material interaction, then the mean number of times it will survive scattering is $1/\epsilon - 1$, and the mean number of total flights is $1/\epsilon$. The rms displacement in τ per flight is $1/\sqrt{3}$. Thus in a random walk with $1/\epsilon$ steps the net displacement in τ will be $1/\sqrt{3\epsilon}$.

In summary, the results for the inhomogeneous Milne problem are the following. The solution for the source function depends both locally and nonlocally on the primary source, which is the Planck function in this case. The nonlocal term is expressed by a resolvent function. When the albedo is close to unity the resolvent includes the effect of a large number of scatterings, and the typical extent of the resolvent in optical depth space is what you would expect for a random walk with a large number of steps. This extent varies in proportion to $1/\sqrt{\epsilon}$, where ϵ is the destruction probability per scattering, 1− the scattering albedo. In a large but finite volume filled with scatterers and with a smoothly distributed primary source of photons, the source function and the mean intensity will be depressed from the local equilibrium value for all points that are within a thermalization depth of the boundary. This may be quite a thick layer. But except within one or two mean free paths of the boundary the radiation will be nearly isotropic, albeit perturbed in magnitude owing to the presence of the boundary.

5.6 The Feautrier or even-parity equation

A simple manipulation of the equation of transfer in slab geometry turns out to be very useful in numerical solution methods. This is to define the even and odd combinations of the intensity $I(\mu)$ and $I(-\mu)$. For $0 \leq \mu \leq 1$ those two variables are replaced by the combinations

$$j(\mu) = \frac{1}{2}[I(\mu) + I(-\mu)], \qquad h(\mu) = \frac{1}{2}[I(\mu) - I(-\mu)]. \tag{5.45}$$

The use of j for the even-parity combination should not be confused with the emissivity function j_ν; the notation here is meant to suggest that $j(\mu)$ is like an angle-dependent J and $h(\mu)$ is like an angle-dependent H. In fact, they are related in this way

$$J = \int_0^1 d\mu\, j(\mu), \qquad H = \int_0^1 d\mu\, \mu\, h(\mu). \tag{5.46}$$

We now introduce the important assumption that the absorption coefficient and the source function are isotropic. Taking the even and odd combinations of (5.6) for $\pm\mu$, given this assumption, leads to the following:

$$\mu \frac{dh}{d\tau} = j - S, \qquad \mu \frac{dj}{d\tau} = h. \tag{5.47}$$

Now it is obvious that h can easily be eliminated to produce this second order equation:

$$\mu^2 \frac{d^2 j}{d\tau^2} = j - S. \tag{5.48}$$

Equation (5.48) is the heart of Feautrier's method (Feautrier, 1964). Sometimes this is called the second order form of the transfer equation while (5.6) is called the first order form. The great virtue of this equation is that it lends itself to this simple and accurate finite-difference form:

$$\mu_k^2 \frac{j_{i-1,k} - 2j_{i,k} + j_{i+1,k}}{\Delta\tau^2} = j_{i,k} - S_i, \tag{5.49}$$

for a uniformly-space grid in τ, in which i is the index of τ points, and k is the index for μ values, which all lie in the range $0 \rightarrow 1$. There is a simple extension of this to non-equally-spaced τ values. This discretization is second order accurate, and the solution $j_{i,k}$ is guaranteed to be positive for all i and k if the source function values S_i are all positive. As it turns out, these two properties are difficult to obtain simultaneously with finite-difference forms of the first order (5.6). The large majority of the work in astrophysical radiative transfer for slab geometry since 1964 has used Feautrier's equation (5.48). Many of the exceptions have been when fluid motion effects are included, which cause the absorption coefficient to depend on direction, or with nonisotropic scattering or angle-dependent frequency redistribution in lines, for which the source function is angle-dependent.

5.7 Eddington–Barbier relation

We now address the question: what intensity does an external observer see when he looks into an atmosphere with a certain temperature distribution? The rigorous result requires solving the general Milne equation for the source function, then doing an additional integration to get the emergent intensity. But a useful semi-quantitative formula can be found by supposing that the Planck function varies slowly with optical depth; this is the Eddington–Barbier relation.

We consider the case without scattering first. The emergent intensity is

$$I(0, \mu) = \int_0^\infty \frac{d\tau}{\mu} \exp(-\tau/\mu) B(\tau). \tag{5.50}$$

Now suppose we adopt a linear approximation for $B(\tau)$, namely

$$B(\tau) \approx a + b\tau. \tag{5.51}$$

Then it is easy to see that

$$I(0, \mu) \approx a + b\mu = B(\mu). \tag{5.52}$$

The intensity seen by the observer is the Planck function one mean free path into the atmosphere as measured along the ray. This is the basic form of the Eddington–Barbier relation. Another relation gives the value of the total flux leaving the atmosphere at the surface, which we get by multiplying the intensity by $2\pi\mu$ and integrating over μ:

$$F(0) \approx \pi \left(a + \frac{2}{3} b \right) = \pi B \left(\frac{2}{3} \right), \tag{5.53}$$

in other words, the flux is π times the Planck function at a location two-thirds of a mean free path into the atmosphere from the outside. Not coincidentally, the value of the temperature at optical depth 2/3 is just the effective temperature, if (5.32) is used and $q(\tau) \approx 2/3$.

The Eddington–Barbier relation is due to Barbier (1943), and a good discussion is given by Kourganoff (1963), Section 18. The two forms of the Eddington–Barbier relation are found to be very useful in understanding qualitatively what the spectrum should look like for atmospheres with complicated temperature structure, like the sun, when the opacity is very different at different frequencies.

When scattering is included the results are more complicated. The Planck function should be replaced by the source function $S(\tau)$ in (5.50), from which we see that the emergent intensity apart from a factor μ is the Laplace transform of the source function at $p = 1/\mu$. We can get this for the case that the Planck function is given by (5.51) from the double transform of the resolvent (5.41), by a suitable limiting procedure $q \to 0$. What we get is

$$\begin{aligned} I(0, \mu) &= \sqrt{\epsilon} H(\mu) \left[a + b \left(\mu + \frac{1-\epsilon}{2\sqrt{\epsilon}} \alpha_1 \right) \right] \\ &= \sqrt{\epsilon} H(\mu) B \left(\mu + \frac{1-\epsilon}{2\sqrt{\epsilon}} \alpha_1 \right). \end{aligned} \tag{5.54}$$

Here $H(\mu)$ is the Chandrasekhar H function introduced earlier, which depends on $\epsilon = 1 - \varpi$ in addition to μ, and α_1 is its first moment:

$$\alpha_1 = \int_0^1 d\mu\, \mu H(\mu). \tag{5.55}$$

The function $H(\mu)$ is identically 1 when there is no scattering, $\epsilon = 1$, and in that case we see that (5.54) reduces to (5.52). The scattering-dominated case, with an albedo that approaches 1 so ϵ is small, is more interesting. The H function tends to a limit, the function for conservative scattering, which is roughly equal to $1 + \sqrt{3}\mu$. The exact value of α_1 for this limiting H function is $2/\sqrt{3}$. We see that the emergent intensity depends in this limit on the Planck function at a large value of τ, which is approximately the thermalization depth $1/\sqrt{3\epsilon}$ since the added term μ is negligible. Thus for $\epsilon \to 0$ the emergent intensity is

$$I(0, \mu) \approx \sqrt{\epsilon} H(\mu) B \left(\frac{1}{\sqrt{3\epsilon}} \right). \tag{5.56}$$

We also note that the emergent intensity is a lot *smaller* than the Planck function at the thermalization depth. We can also obtain an expression for the emergent flux by doing the integration over $2\pi \mu d\mu$ as before. We note that the integral of H becomes α_1, for which we substitute $2/\sqrt{3}$. We find

$$F(0) \approx 4\pi \sqrt{\frac{\epsilon}{3}} B \left(\frac{1}{\sqrt{3\epsilon}} \right). \tag{5.57}$$

There is a cartoon-level way of understanding (5.56), which is the following. While in reality the primary source term ϵB acts throughout the whole problem, including in the thermalization layer, the cartoon version is to suppose that in the whole of this layer, down to the thermalization depth, there is only conservative scattering, but from the thermalization depth on down the radiation field is thermalized, i.e., is exactly Planckian. The radiation that emerges at the surface is therefore the Planckian emission at the thermalization depth, but attenuated by the effect of multiple scatterings between there and the surface, which reduce its intensity by the diffuse transmission factor for that layer. For a thick scattering layer we know that the fraction of radiation that is reflected is almost 100% and the transmitted fraction is small, of order the reciprocal of the optical depth. For our problem the optical thickness of the layer is the thermalization depth, $\approx 1/\sqrt{3\epsilon}$, and so the diffuse transmission fraction is about $\sqrt{3\epsilon}$. Except for the numerical factors, this gives (5.56).