CHAPTER TWELVE

Incompressible Viscous Flows via Finite Element Methods

As noted in Chapter 5, the condition of incompressibility for incompressible flows is difficult to satisfy. The consequence of this difficulty results in a checkerboard type pressure oscillation which occurs when the primitive variables (pressure and velocity) are calculated directly from the governing equations of continuity and momentum. Various methods are used to overcome this difficulty. Among them are: mixed methods, penalty methods, pressure correction methods, generalized Petrov-Galerkin (GPG) methods, operator splitting (fractional) methods, and semi-implicit pressure correction methods. Another approach is to use the vortex methods in which stream functions and vorticity are calculated, thus avoiding the pressure term. Some of the earlier and recent contributions to the finite element analyses of incompressible flows are found in [Hughes, Liu, and Brooks, 1979; Carey and Oden, 1986; Zienkiewicz and Taylor, 1991; Gunzburger and Nicholaides, 1993; Gresho and Sani, 1999], among many others.

Instead of being limited to incompressible flows, we may begin with the conservation form of the Navier-Stokes system of equations for compressible flows, in which special steps can be devised to obtain solutions near incompressible limits ($M_\infty \cong 0$). This allows us to use a single formulation to handle both compressible and incompressible flows. We shall address this subject in Section 13.6. For this reason, treatments of incompressible flows in this chapter will be brief.

12.1 PRIMITIVE VARIABLE METHODS

12.1.1 MIXED METHODS

Consider the governing equations of continuity and momentum for incompressible flow in the form:

Continuity

$$\mathrm{v}_{i,i} = 0 \tag{12.1.1a}$$

Momentum

$$\rho \mathrm{v}_{i,j}\mathrm{v}_j + p_{,i} - \mu \mathrm{v}_{i,jj} = 0 \tag{12.1.1b}$$

It is well known that the standard Galerkin formulation of the simultaneous system of equations for continuity and momentum (12.1.1a,b) becomes ill-conditioned, known as the LBB condition [Ladyszhenskaya, 1969; Babuska, 1973; Brezzi, 1974] as pointed out in Section 10.1.4. In order to circumvent the numerical instability, trial functions for pressure are chosen one order lower than those for the velocity, defined as shown in Figure 10.1.3. We may write the standard Galerkin integrals in nondimensional form as follows:

$$\int_\Omega \Phi_\alpha \left(\mathrm{v}_{i,j}\mathrm{v}_j + p_{,i} - \frac{1}{\mathrm{Re}}\mathrm{v}_{i,jj} \right) d\Omega = 0 \tag{12.1.2a}$$

$$\int_\Omega \hat{\Phi}_\alpha \mathrm{v}_{i,i} d\Omega = 0 \tag{12.1.2b}$$

where the pressure approximation is of one order lower than the velocity approximation so that the incompressibility condition may be satisfied as discussed in Section 10.1.4. Combining (12.1.2a,b) yields

$$\begin{bmatrix} D_{\alpha\beta ij} & C_{\alpha\beta i} \\ C_{\alpha\beta j} & 0 \end{bmatrix} \begin{bmatrix} \mathrm{v}_{\beta j} \\ p_\beta \end{bmatrix} = \begin{bmatrix} G_{\alpha i} \\ 0 \end{bmatrix} \tag{12.1.3}$$

with

$$D_{\alpha\beta ij} = \int_\Omega \left(\Phi_\alpha \Phi_{\beta,k}\mathrm{v}_k \delta_{ij} + \frac{1}{\mathrm{Re}}\Phi_{\alpha,k}\Phi_{\beta,k}\delta_{ij} \right) d\Omega$$

$$C_{\alpha\beta i} = \int_\Omega \Phi_\alpha \hat{\Phi}_{\beta,j}\delta_{ij} d\Omega, \qquad C_{\alpha\beta j} = \int_\Omega \hat{\Phi}_\alpha \Phi_{\beta,j} d\Omega,$$

$$G_{\alpha i} = \int_\Gamma \frac{1}{\mathrm{Re}} \overset{*}{\Phi}_\alpha \mathrm{v}_{i,j} n_j d\Gamma$$

where the test function $\hat{\Phi}_\alpha$ for continuity is the same as the pressure trial function.

As mentioned in Section 10.1.4, if pressure is interpolated as constant (pressure node at the center of an element) and velocity as a linear function (velocity defined at corner node, Figure 10.1.3a), then such element becomes overconstrained (known as locking element). This situation can be alleviated by using linear pressure and quadratic velocity approximations (Figure 10.1.3b). In this process of unequal order approximations for pressure, we seek to achieve the computational stability. Many other available options are discussed below.

12.1.2 PENALTY METHODS

As seen in Section 10.1.4, the incompressibility condition can be satisfied by means of the penalty function λ such that

$$p = -\lambda \mathrm{v}_{i,i} \tag{12.1.4a}$$

$$p_{,i} = -\lambda \mathrm{v}_{j,ji} \tag{12.1.4b}$$

which is designed to replace the pressure gradient term in (12.1.2a). The reduced Gaussian quadrature integration for the penalty term is still required to avoid being over-constrained, as discussed in Section 10.1.4. In this way, we obtain the solution of

(12.1.2a) without (12.1.2b), but the mass conservation is maintained through the penalty constraint.

Another approach is to combine the penalty formulation with the mixed method of (12.1.2a,b). This can be achieved by replacing the continuity equation with the Galerkin integral of (12.1.4a),

$$\int_\Omega \Phi_\alpha \left(\mathrm{v}_{i,i} + \frac{p}{\lambda} \right) d\Omega = 0 \tag{12.1.5}$$

This will then revise (12.1.3) in the form

$$\begin{bmatrix} D_{\alpha\beta ij} & C_{\alpha\beta i} \\ C_{\alpha\beta j} & E_{\alpha\beta} \end{bmatrix} \begin{bmatrix} \mathrm{v}_{\beta j} \\ p_\beta \end{bmatrix} = \begin{bmatrix} G_{\alpha i} \\ 0 \end{bmatrix} \tag{12.1.6}$$

with

$$E_{\alpha\beta} = \int_\Omega \frac{1}{\lambda} \Phi_\alpha \Phi_\beta d\Omega$$

which provides an additional computational stability in comparison with (12.1.3).

12.1.3 PRESSURE CORRECTION METHODS

The basic idea of the pressure correction methods is to split the pressure and velocity in the form [Patankar and Spalding, 1972]

$$p^{n+1} = p^n + p' \tag{12.1.7a}$$

$$\mathrm{v}_i^{n+1} = \mathrm{v}_i^* + \mathrm{v}_i' \tag{12.1.7b}$$

where v_i^* denotes the intermediate step velocity. Using (12.1.7) in (12.1.1b) we obtain, for the case of unsteady flow,

$$\left(\frac{\partial \mathrm{v}_i}{\partial t} \right)^* + \left(\frac{\partial \mathrm{v}_i}{\partial t} \right)' \cong \left(\frac{1}{\mathrm{Re}} \mathrm{v}_{i,jj}^* - \mathrm{v}_{i,j}^* \mathrm{v}_j^n - (p_{,i})^n - (p_{,i})' \right)$$

which may be split into

$$\left(\frac{\partial \mathrm{v}_i}{\partial t} \right)^* = \frac{1}{\mathrm{Re}} \mathrm{v}_{i,jj}^* - \mathrm{v}_{i,j}^* \mathrm{v}_j^n - (p_{,i})^n \tag{12.1.8a}$$

$$\left(\frac{\partial \mathrm{v}_i}{\partial t} \right)' = -(p_{,i})' \tag{12.1.8b}$$

where the asterisk and prime indicate intermediate and correction values. The solution of (12.1.8a) is not expected, in general, to satisfy the conservation of mass. In order to rectify this situation, we take a divergence of (12.1.8b) and write

$$p'_{,ii} = -\frac{\partial}{\partial t} (\mathrm{v}_{i,i})' \tag{12.1.9a}$$

which may be recast in a difference form

$$p'_{,ii} \cong -\frac{1}{\Delta \mathrm{t}} (\mathrm{v}_{i,i}^{n+1} - \mathrm{v}_{i,i}^*) \tag{12.1.9b}$$

Here we intend to force $v_{i,i}^{n+1}$ to vanish for mass conservation so that

$$p'_{,ii} = \frac{1}{\Delta t}(v^*_{i,i}) \tag{12.1.10}$$

Thus, the solution procedure consists of

(1) Solve (12.1.8a) for v^*_i with initial and boundary conditions and assumed pressure.
(2) Solve (12.1.10) for pressure corrections, p', with the boundary conditions $p' = 0$ on Γ_D and $p'_{,i}n_i$ on Γ_N.
(3) Determine v'_i from (12.1.8b).
(4) Determine

$$p^{n+1} = p^n + p'$$

$$v_i^{n+1} = v^*_i + v'_i$$

(5) Repeat steps (1) through (4) until convergence has been achieved.

The generalized Galerkin formulations may be used for (12.1.8a), (12.1.10), and (12.1.8b). Mixed interpolations (between velocity and pressure) are not required. Although the mass conservation is achieved through the pressure correction methods, the convective terms may still contribute to nonconvergence if convection dominates the flowfield. Toward this end, the generalized Galerkin formulation can be replaced by GPG methods.

12.1.4 GENERALIZED PETROV-GALERKIN METHODS

The mixed method may be modified so that both pressure and velocity can be interpolated in a same order. The convection and pressure gradient terms are treated with generalized Petrov-Galerkin (GPG), and the pressure is updated using the standard pressure Poisson equation.

$$\int_0^1 \hat{W}(\xi)\left\{\int_\Omega\left[\Phi_\alpha\left(\frac{\partial v_i}{\partial t} + v_{i,j}v_j - \frac{1}{\text{Re}}v_{i,jj}\right) + \Psi_\alpha(v_{i,j}v_j + p_{,i})\right]d\Omega\right\}d\xi = 0 \tag{12.1.11}$$

$$\int_\Omega \Phi_\alpha[p_{,ii} + (v_{i,j}v_j)_{,i}]d\Omega = 0 \tag{12.1.12}$$

Integrating (12.1.11) by parts leads to

$$\left[A_{\alpha\beta} + \frac{\Delta t}{2}(B_{\alpha\beta} + C_{\alpha\beta} + K_{\alpha\beta})\right]v_{\beta i}^{n+1} = \left[A_{\alpha\beta} - \frac{\Delta t}{2}(B_{\alpha\beta} + C_{\alpha\beta} + K_{\alpha\beta})\right]v_{\beta i}^{n} + \Delta t\,(F_{\alpha i} + G_{\alpha i}) \tag{12.1.13}$$

where

$$F_{\alpha i} = -\int_\Omega \tau v_k \Phi_{\alpha,k}\Phi_{\beta,i}\,d\Omega p_\beta \tag{12.1.14}$$

with all other quantities being the same as in (11.4.5) except for the Reynolds number.

The nodal pressure p_β will be updated from (12.1.12), which assumes the form

$$E_{\alpha\beta} p_\beta = H_\alpha + Q_\alpha \tag{12.1.15}$$

with

$$E_{\alpha\beta} = \int_\Omega \Phi_{\alpha,i}\Phi_{\beta,i}\, d\Omega$$

$$H_\alpha = \int_\Omega \Phi_\alpha (\overline{\mathrm{v}}_{i,j}\overline{\mathrm{v}}_j)_{,i}\, d\Omega$$

$$Q_\alpha = \int_\Gamma \overset{*}{\Phi}_\alpha p_{,i} n_i\, d\Gamma$$

Note that pressure oscillations are suppressed not only from (12.1.15) but also the damping effect built into (12.1.14).

Remarks: We note that GPG methods can be applied to the incompressible Navier-Stokes system of equations in which the special treatment for pressure is no longer required. In this case, the conservation form of the Navier-Stokes system of equations can be utilized and it is possible to formulate various schemes which can handle both compressible and incompressible flows. Furthermore, the conservation variables can be transformed into primitive variables in order to accommodate the incompressible nature of the flow. In this case, details of derivations of GPG schemes for incompressible flows are the same as in the case of compressible flows, which will be presented in Section 13.3.

12.1.5 OPERATOR SPLITTING METHODS

The pressure correction methods may be solved with fractional steps, often called operator splitting methods or fractional step methods [Yanenko, 1971], such that equations of hyperbolic, parabolic, and elliptic types are solved separately [Chorin, 1967]. To this end, we consider the standard Galerkin finite element equations of momentum and continuity in the form

$$A_{\alpha\beta}\dot{\mathrm{v}}_{\beta i} + E_{\alpha\beta j\gamma}\mathrm{v}_{\beta i}\mathrm{v}_{\gamma j} - C_{\alpha\beta i}\, p_\beta + K_{\alpha j\beta j}\mathrm{v}_{\beta i} - G_{\alpha i} = 0 \tag{12.1.16}$$

$$C_{\alpha\beta}\mathrm{v}_{\beta i} = 0 \tag{12.1.17}$$

(1) Hyperbolic Fractional Step Operator for Convective Terms

$$A_{\alpha\beta}\dot{\mathrm{v}}_{\beta i} = -E_{\alpha\beta j\gamma}\mathrm{v}_{\beta i}\mathrm{v}_{\gamma j} + G_{\alpha i} \tag{12.1.18}$$

where

$$E_{\alpha\beta j\gamma} = B_{\alpha\beta j\gamma} + C_{\alpha\beta j\gamma}$$

with $C_{\alpha\beta j\gamma}$ indicating the term constructed from the numerical diffusion test functions.

The solution of (12.1.18) is obtained from the GPG formulation,

$$\left(A_{\alpha\beta} + \frac{\Delta t}{2} E_{\alpha\beta j\gamma} \mathrm{v}_{\gamma j}^n\right)\hat{\mathrm{v}}_{\beta i}^{n+1} = A_{\alpha\beta}\mathrm{v}_{\beta i}^n - \frac{\Delta t}{2} E_{\alpha\beta j\gamma}\mathrm{v}_{\beta i}^n \mathrm{v}_{\gamma j}^n + \Delta t\, G_{\alpha i} \tag{12.1.19}$$

(2) Parabolic Fractional Step Operator for Dissipation Term

$$A_{\alpha\beta}\dot{\mathrm{v}}_{\beta i} = -K_{\alpha j\beta j}\mathrm{v}_{\beta i} \tag{12.1.20}$$

$$\begin{cases} \mathrm{v}_i = \bar{\mathrm{v}}_i & \text{on } \Gamma_D \\ \mathrm{v}_{i,j} n_j = g_i & \text{on } \Gamma_N \end{cases}$$

We solve (12.1.20) with TGM formulation so that

$$\left(A_{\alpha\beta} + \frac{\Delta t}{2} E_{\alpha\beta ij}\right)\tilde{\mathrm{v}}_{\beta i}^{n+1} = A_{\alpha\beta}\hat{\mathrm{v}}_{\beta i}^{n+1} - \frac{\Delta t}{2} K_{\alpha j\beta j}\hat{\mathrm{v}}_{\beta i}^{n+1} + \Delta t\, G_{\alpha i} \tag{12.1.21}$$

(3) Elliptic Fractional Step Operator for Pressure Term

$$A_{\alpha\beta}\frac{\mathrm{v}_{\beta i}^{n+1} - \tilde{\mathrm{v}}_{\beta i}^{n+1}}{\Delta t} = C_{\alpha\beta i}\, p_\beta^{n+1} \tag{12.1.22}$$

$$C_{\alpha\beta}\mathrm{v}_{\beta i}^{n+1} = 0 \tag{12.1.23}$$

$$\begin{cases} p = p_0 & \text{on } \Gamma_D \\ p_{,i} n_i = g_i & \text{on } \Gamma_N \end{cases}$$

Here the enforcement of incompressibility is achieved by substituting the first term on the right-hand side of (12.1.22) by (12.1.23).

$$D_{\alpha\beta i}\, p_\beta^{n+1} = -\frac{1}{\Delta t} C_{\alpha\beta}\mathrm{v}_{\beta i}^{n+1} \tag{12.1.24}$$

where

$$D_{\alpha\beta i} = C_{\alpha\gamma} A_{\gamma\delta}^{-1} C_{\delta\beta i} \tag{12.1.25}$$

We calculate p_β^{n+1} from (12.1.25) and determine the final velocity from (12.1.22),

$$\mathrm{v}_{\alpha i}^{n+1} = \tilde{\mathrm{v}}_{\alpha i}^{n+1} + \Delta t\, A_{\alpha\gamma}^{-1} C_{\gamma\beta i}\, p_\beta^{n+1} \tag{12.1.26}$$

Note that the fractional step methods are similar to the pressure correction methods, although there are two distinctly different aspects:

(1) The solution involved in (12.1.8a) is split into two steps: hyperbolic step and parabolic step.
(2) The processes (12.1.8b) and (12.1.10) of pressure correction methods are combined into an elliptic step of the fractional step methods. The pressure Poisson equation is not used here.

It should be noted that (12.1.22) may be differentiated spatially to obtain the pressure Poisson equation as in the pressure correction method, expediting convergence to a certain extent.

12.1.6 SEMI-IMPLICIT PRESSURE CORRECTION

In this scheme, the GPG method is used for convection dominated flows, but we resort to the pressure correction method to maintain conservation of mass and to suppress pressure oscillations.

With the continuity equation written in the form

$$\frac{1}{c^2}\frac{\partial p}{\partial t} + (\rho \mathrm{v}_i)_{,i} = 0 \tag{12.1.27}$$

we obtain the finite element equations as follows:

Continuity

$$D_{\alpha\beta}\dot{p}_\beta + C_{\alpha\beta i}\mathrm{v}_{\beta i} = 0 \tag{12.1.28}$$

Momentum

$$A_{\alpha\beta}\dot{\mathrm{v}}_{\beta i} + (B_{\alpha\beta jj} + K_{\alpha j\beta j})\mathrm{v}_{\beta i} + C_{\alpha\beta i}p_\beta = 0 \tag{12.1.29}$$

where $B_{\alpha\beta jj}$ contains the GPG terms.

Denote the following:

$$\Delta p_\alpha^n = p_\alpha^{n+1} - p_\alpha^n \tag{12.1.30}$$

$$\Delta \mathrm{v}_{\alpha i}^n = \mathrm{v}_{\alpha i}^{n+1} - \mathrm{v}_{\alpha i}^n = \Delta \mathrm{v}_{\alpha i}^{n(1)} - \Delta \mathrm{v}_{\alpha i}^{n(2)} \tag{12.1.31}$$

and

$$\begin{aligned} p &= (1-\theta)p^n + \theta p^{n+1} \\ &= \theta(p^{n+1} - p^n) + p^n \\ &= \theta(\Delta p^n) + p^n \end{aligned} \tag{12.1.32}$$

$$\begin{aligned} \mathrm{v}_i &= (1-\theta)\mathrm{v}_i^n + \theta \mathrm{v}_i^{n+1} \\ &= \theta\left(\Delta \mathrm{v}_i^{n(1)} - \Delta \mathrm{v}_i^{n(2)}\right) + \mathrm{v}_i^n \\ &= \theta\left(\Delta \mathrm{v}_i^n\right) + \mathrm{v}_i^n \end{aligned} \tag{12.1.33}$$

Substituting (12.1.32) into (12.1.29) and taking a temporal approximation, we obtain

$$\Delta \mathrm{v}_{\beta i}^n = -\Delta t\left[(B_{\alpha\beta jj} + K_{\alpha j\beta j})\left(\theta\Delta \mathrm{v}_{\beta i}^n + \mathrm{v}_{\beta i}^n\right) + C_{\alpha\beta i}\left(\theta\Delta p_\beta^n + p_\beta^n\right)\right] \tag{12.1.34}$$

Combining (12.1.32) into (12.1.34) and separating the resulting equation into two parts leads to

$$[\Delta t\theta(B_{\alpha\beta jj} + K_{\alpha j\beta j})]\Delta \mathrm{v}_{\alpha i}^{(1)} = \Delta t\left[(B_{\gamma\beta jj} + K_{\gamma j\beta j})\mathrm{v}_{\beta i}^n + C_{\gamma\beta i}p_\beta^n\right] \tag{12.1.35a}$$

$$[A_{\alpha\beta} + \Delta t\theta(B_{\alpha\beta jj} + K_{\alpha j\beta j})]\Delta \mathrm{v}_{\alpha i}^{(2)} = \Delta t\theta C_{\gamma\beta i}\Delta p_\beta^n \tag{12.1.35b}$$

Substituting (12.1.32) into (12.1.28) and using (12.1.33) and (12.1.35), we obtain

$$\left(C_{\alpha\gamma i}Q_{\gamma\delta}^{-1}C_{\delta\beta i}\right)\Delta p_\beta^n = -C_{\alpha\beta i}\Delta t\left(\mathrm{v}_{\beta i}^n + \theta\Delta \mathrm{v}_{\beta i}^{n(1)}\right) \tag{12.1.36}$$

where

$$D_{\alpha\beta} = \int_\Omega \frac{1}{\alpha^2}\Phi_\alpha\Phi_\beta d\Omega = \int_\Omega \frac{M^2}{q^2}\Phi_\alpha\Phi_\beta d\Omega, \qquad Q_{\alpha\beta} = A_{\alpha\beta} + \Delta t\theta(B_{\alpha\beta jj} + K_{\alpha j\beta j}) \tag{12.1.37}$$

For incompressible flows, we have $D_{\alpha\beta} = 0$. This gives

$$\Delta t^2\theta^2 C_{\alpha\gamma i} Q^{-1}_{\gamma\delta} C_{\delta\beta i}\Delta p^n_\beta = C_{\alpha\beta i}\Delta t\left(\mathrm{v}^n_{\beta i} + \theta\Delta \mathrm{v}^{n(1)}_{\beta i}\right) \tag{12.1.38}$$

The von Neumann analysis shows that, for stable solutions, Δt must be limited by

$$\Delta t \leq \frac{h}{|\mathrm{v}|}\sqrt{\frac{1}{\mathrm{Re}} + 1} - \frac{1}{\mathrm{Re}} \tag{12.1.39}$$

Upon solution of the pressure equation (12.1.38), we return to (12.1.34) for the corrected velocity components.

A simplified version of the previous approach arises in the absence of viscosity terms:

$$\frac{1}{a^2}\frac{\partial p}{\partial t} + \mathrm{v}_{i,i} = 0 \tag{12.1.40}$$

$$\frac{\partial \mathrm{v}_i}{\partial t} + p_{,i} = 0 \tag{12.1.41}$$

Rewriting (12.1.40) and (12.1.41) yields

$$\frac{1}{a^2}\Delta p^n + \Delta t\left(\mathrm{v}^n_i + \theta\Delta \mathrm{v}^{n(2)}_i - \theta\Delta \mathrm{v}^{n(1)}_i\right)_{,i} = 0 \tag{12.1.42}$$

$$\Delta \mathrm{v}^{n(2)}_i + \theta\Delta t\Delta p^n_{,i} = 0 \tag{12.1.43}$$

Substituting (12.1.43) into (12.1.42), we obtain

$$\frac{1}{a^2}\Delta p^n + \Delta t\left(\mathrm{v}^n_i + \theta\Delta \mathrm{v}^{n(1)}_i\right)_{,i} - (\theta\Delta t)^2\Delta p^n_{,ii} = 0 \tag{12.1.44}$$

With the finite element approximation,

$$\mathrm{v}_i = \Phi_\alpha \mathrm{v}_{\alpha i}, \qquad p = \hat{\Phi}_\alpha p_\alpha$$

we have

$$(D_{\alpha\beta} - \Delta t^2\theta^2 E_{\alpha i\beta i})\Delta p^n_\beta = -G_{\alpha\beta i}\Delta t\left(\Delta \mathrm{v}^n_{\beta i} + \theta\Delta \mathrm{v}^{n(1)}_{\beta i}\right) \tag{12.1.45}$$

The pressure correction as obtained from (12.1.45) can be used to solve (12.1.44) in which the viscosity term is now restored.

12.2 VORTEX METHODS

Recall that the vortex methods as examined in Section 5.4 utilize the vortex transport equation in which the terms with pressure gradients vanish upon satisfaction of the conservation of mass. Thus, the pressure oscillation is not expected to occur in the solution of the vortex transport equation.

In many engineering problems, it is not feasible to make two-dimensional simplifications because the flowfield is physically three-dimensional, such as in high-speed rotational flows and high-Reynolds number turbulent flows. Thus, we begin with three-dimensional formulations and demonstrate that the two-dimensional analysis can be derived easily as a simplification of the three-dimensional process if permitted by the special physical situations.

12.2.1 THREE-DIMENSIONAL ANALYSIS

Three-Dimensional Vorticity Transport Equations

The system of three-dimensional vorticity transport equation takes the form

$$\frac{\partial \boldsymbol{\omega}}{\partial t} + (\mathbf{v} \cdot \nabla)\, \boldsymbol{\omega} - (\boldsymbol{\omega} \cdot \nabla)\, \mathbf{v} = \nu \nabla^2 \boldsymbol{\omega} \tag{12.2.1}$$

with

$$\boldsymbol{\omega} = \nabla \times \mathbf{v} \tag{12.2.2}$$

$$\nabla^2 p = \rho \nabla \cdot [(\mathbf{v} \cdot \nabla)\, \mathbf{v}] \tag{12.2.3}$$

The above system provides seven unknowns (ω, $\mathbf{v}$, p) and seven equations in three dimensions. We may use GGM , TGM, or GPG to solve the system of equations (12.2.1–12.2.3).

Three-Dimensional Biharmonic Equation with Stream Function

It is also possible to write (12.2.1) in terms of the stream function vector Ψ as defined in (5.4.15),

$$\frac{\partial}{\partial t}(\nabla^2 \Psi) + (\nabla \times \Psi \cdot \nabla)\nabla^2 \Psi - (\nabla^2 \Psi \cdot \nabla)(\nabla \times \Psi) = \nu \nabla^4 \Psi \tag{12.2.4a}$$

or

$$\frac{\partial}{\partial t}(\Psi_{i,jj}) + \varepsilon_{rjk}\Psi_{k,j}\Psi_{i,mmr} - \varepsilon_{isk}\Psi_{r,jj}\Psi_{k,sr} = \nu \Psi_{i,jjkk} \tag{12.2.4b}$$

with

$$\omega = \nabla(\nabla \cdot \Psi) - \nabla^2 \Psi = -\nabla^2 \Psi$$

To obtain the TGM equation for (12.2.4b), we proceed as follows:

$$\int_0^1 \hat{W}(\xi) \int_\Omega \Phi_\alpha \left(\frac{\partial}{\partial t}(\Psi_{i,jj}) + \varepsilon_{rjk}\Psi_{k,j}\Psi_{i,mmr} - \varepsilon_{isk}\Psi_{r,jj}\Psi_{k,sr} - \nu \Psi_{i,jjkk} \right) d\Omega d\xi = 0 \tag{12.2.5}$$

Integrate (12.2.5) twice to obtain

$$A_{\alpha\beta}\delta_{ij}\frac{\partial \Psi_{\beta j}}{\partial t} - B_{\alpha\beta\gamma k}\Psi_{\beta k}\Psi_{\gamma i} + C_{\alpha\beta\gamma imk}\Psi_{\beta m}\Psi_{\gamma k} + K_{\alpha\beta}\delta_{ij}\Psi_{\beta j} = -G_{\alpha i} \tag{12.2.6}$$

with

$$A_{\alpha\beta} = \int_\Omega \Phi_{\alpha,k}\Phi_{\beta,k}\, d\Omega$$

$$B_{\alpha\beta\gamma k} = \int_\Omega \varepsilon_{njk}\Phi_\alpha\Phi_{\beta,j}\Phi_{\gamma,mmn}\, d\Omega$$

$$C_{\alpha\beta\gamma imk} = \int_\Omega \varepsilon_{isk}\Phi_\alpha\Phi_{\beta,jj}\Phi_{\gamma,sm}\, d\Omega$$

$$K_{\alpha\beta} = \int_\Omega \nu\Phi_{\alpha,kk}\Phi_{\beta,mm}\, d\Omega$$

$$G_{\alpha i} = \int_\Gamma \nu\overset{*}{\Phi}_\alpha\Psi_{i,jjk}n_k\, d\Gamma - \int_\Gamma \nu\overset{*}{\Phi}_{\alpha,k}\Psi_{i,jj}n_k\, d\Gamma$$

Here, there are nine variables (Ψ_1, Ψ_2, Ψ_3, $\Psi_{1,2}$, $\Psi_{1,3}$, $\Psi_{2,1}$, $\Psi_{2,3}$, $\Psi_{3,1}$, and $\Psi_{3,2}$) to be specified and calculated at each of the eight nodes of the 3-D cubic isoparametric element. To this end, we require thirty-two constants to be determined with four of them (Ψ_i, $\Psi_{i,1}$, $\Psi_{i,2}$, $\Psi_{i,3}$) at each of the eight nodes as follows (Figure 12.2.1):

$$1, \xi, \eta, \zeta, \xi\eta, \xi\zeta, \eta\zeta, \xi\eta\zeta, \xi^2, \eta^2, \zeta^2, \xi^2\eta, \xi^2\zeta, \eta^2\xi, \eta^2\zeta, \zeta^2\xi, \zeta^2\eta, \xi^2\eta\zeta, \xi\eta^2\xi, \xi\eta\zeta^2, \xi^3,$$
$$\eta^3, \zeta^3, \xi^3\eta, \xi^3\zeta, \eta^3\xi, \eta^3\zeta, \zeta^3\xi, \zeta^3\eta, \xi^3\zeta\eta, \xi\eta^3\zeta, \xi\eta\zeta^3$$

The TGM Newton-Raphson formulation of (12.2.6) takes the form

$$J^{(r)}_{\alpha\beta ij}\Delta\Psi^{(n+1)(r+1)}_{\beta j} = -R^{(n+1)(r)}_{\alpha i} \tag{12.2.7}$$

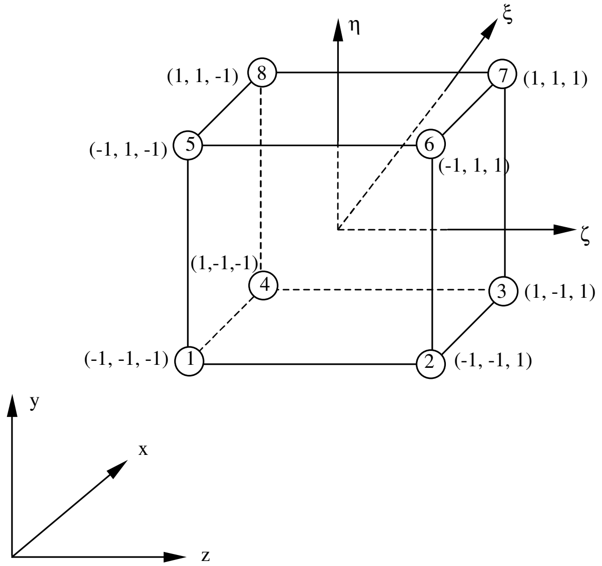

Figure 12.2.1 Hexahedral element for 3-D vortex transport analysis.

where

$$
\begin{aligned}
R_{\alpha i}^{(n+1)(r)} = {} & A_{\alpha\beta}\delta_{ij}\Psi_{\beta j}^{(n+1)} + \frac{\Delta t}{2}\Big(B_{\alpha\beta\gamma k}\Psi_{\beta k}^{(n+1)}\Psi_{\gamma i}^{(n+1)} - C_{\alpha\beta\gamma imk}\Psi_{\beta m}^{(n+1)}\Psi_{\gamma k}^{(n+1)} \\
& - K_{\alpha\beta}\delta_{ij}\Psi_{\beta j}^{(n+1)}\Big) - A_{\alpha\beta}\delta_{ij}\Psi_{\beta j}^{(n)} + \frac{\Delta t}{2}\Big(B_{\alpha\beta\gamma k}\Psi_{\beta k}^{(n)}\Psi_{\gamma i}^{(n)} - C_{\alpha\beta\gamma imk}\Psi_{\beta m}^{(n)}\Psi_{\gamma k}^{(n)} \\
& - K_{\alpha\beta}\delta_{ij}\Psi_{\beta j}^{(n)}\Big) - \Delta t\, G_{\alpha i}
\end{aligned}
\tag{12.2.8}
$$

$$
J_{\alpha\beta ij}^{(r)} = \frac{\partial R_{\alpha i}^{(n+1)(r)}}{\partial \Psi_{\beta j}^{(n+1)(r)}} = A_{\alpha\beta}\delta_{ij} + \frac{\Delta t}{2}(B_{\alpha\beta\gamma j}\Psi_{\gamma i} + B_{\alpha\beta\gamma k}\delta_{ij}\Psi_{\gamma k} - 2C_{\alpha\beta\gamma ijk}\Psi_{\gamma k} - K_{\alpha\beta}\delta_{ij})
\tag{12.2.9}
$$

First of all, the local element interpolation functions must be polynomials of at least third degree which will allow the stream function to be linear. The total number of element unknowns are thirty-two with four at each node (Figure 12.2.1). Explicit interpolation functions have been described in Elshabka and Chung [1999].

Typical Neumann and Dirichlet boundary conditions associated with the 3-D stream function vector components are shown in Figure 12.2.2. The Newton-Raphson solution of (12.2.7) is expected to be free of numerical oscillations because of the Jacobian matrix which is well-conditioned.

Computations of (12.2.7) based on the definition of the three-dimensional stream function vector components as given in (5.4.15) have been carried out in Elshabka [1995]. Some of the highlights are given in Section 12.3.

The Curl of Three-Dimensional Vorticity Transport Equations

The vorticity transport equations (12.2.1) are derived by taking a curl of the momentum equations. In this process, the pressure gradient terms of the momentum equations are eliminated, resulting in computationally more stable formulations. However, both vorticity and velocity are coupled together in the vorticity transport equations. The vorticity transport equations are written in a modified form,

$$
\frac{\partial \omega_i}{\partial t} + \varepsilon_{ijk}S_{k,j} - \nu\omega_{i,jj} = 0 \tag{12.2.10}
$$

with

$$
S_i = (\mathrm{v}_i \mathrm{v}_j)_{,j}
$$

To arrive at a single variable, say velocity alone, we take a curl of (12.2.10) and obtain

$$
\frac{\partial}{\partial t}(\mathrm{v}_{i,jj}) + S_{i,jj} - (S_j)_{,ji} - \nu \mathrm{v}_{i,jjkk} = 0 \tag{12.2.11}
$$

or

$$
\frac{\partial}{\partial t}(\mathrm{v}_{i,jj}) + (\mathrm{v}_i \mathrm{v}_k)_{,kjj} - (\mathrm{v}_j \mathrm{v}_k)_{,kji} - \nu \mathrm{v}_{i,jjkk} = 0 \tag{12.2.12}
$$

This will allow calculations of velocity by solving (12.2.12) alone. Other options include solving (12.2.10) and (12.2.11) simultaneously with $\omega = \nabla \times \mathbf{v}$.

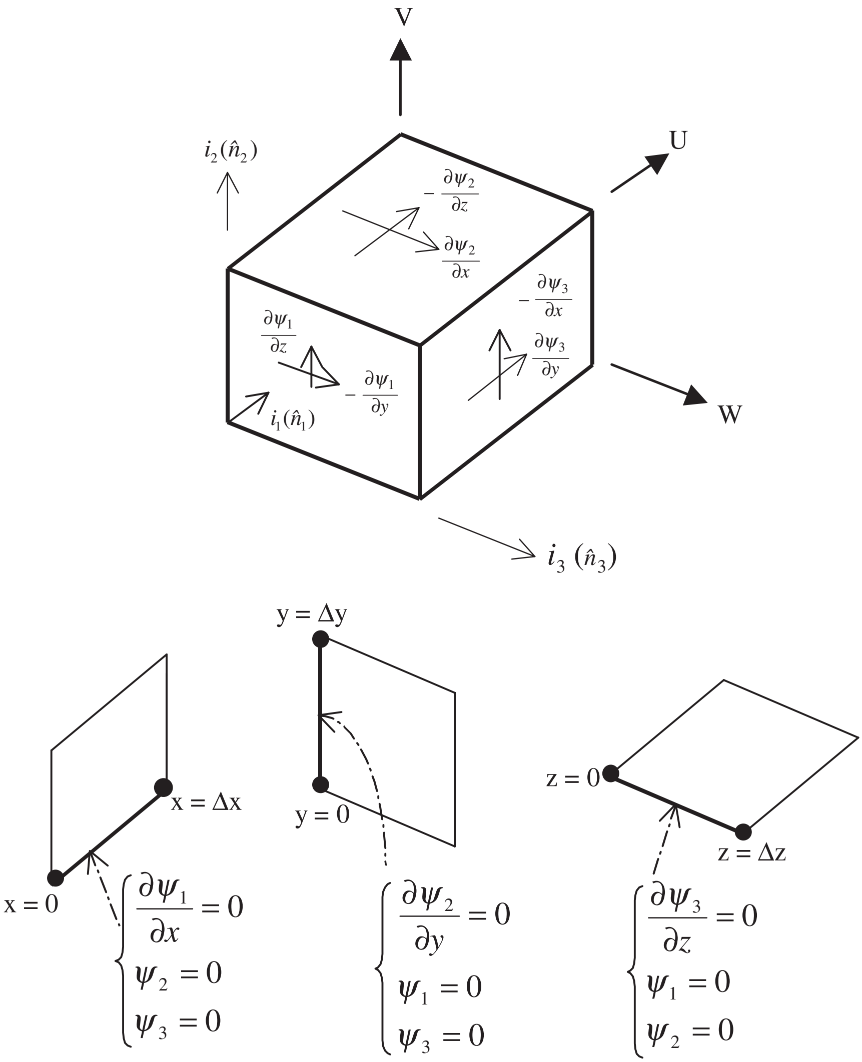

Figure 12.2.2 Three-dimensional boundary conditions.

12.2.2 TWO-DIMENSIONAL ANALYSIS

The two-dimensional vorticity transport equation is simplified to

$$\frac{\partial \omega}{\partial t} + (\mathbf{v} \cdot \nabla)\,\omega = \nu \nabla^2 \omega \tag{12.2.13a}$$

with

$$\omega = \frac{\partial \mathrm{v}}{\partial x} - \frac{\partial u}{\partial y} \tag{12.2.13b}$$

$$\frac{\partial \mathrm{v}}{\partial y} + \frac{\partial u}{\partial x} = 0 \tag{12.2.13c}$$

Here, there are three unknowns (u, v, ω) in the system of three equations (12.2.13a,b,c). The pressure is then calculated from the Poisson equation.

$$\nabla^2 p = 2\rho\left(\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y} - \frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial y}\right) \tag{12.2.14}$$

We may rewrite (12.2.13a) in terms of a scalar stream function, ψ,

$$\frac{\partial}{\partial t}(\psi_{,jj}) + \varepsilon_{ik}\psi_{,k}\psi_{,jji} - \nu\psi_{,iijj} = 0 \tag{12.2.15}$$

The TGM equation for (12.2.15) becomes

$$A_{\alpha\beta}\frac{\partial \psi_\beta}{\partial t} + B_{\alpha\beta\gamma}\psi_\beta\psi_\gamma - K_{\alpha\beta}\psi_\beta = G_\alpha \tag{12.2.16}$$

where

$$A_{\alpha\beta} = \int_\Omega \Phi_{\alpha,i}\Phi_{\beta,i}\,d\Omega$$

$$B_{\alpha\beta\gamma} = \int_\Omega \varepsilon_{ik}\Phi_\alpha\Phi_{\beta,k}\Phi_{\gamma,jji}\,d\Omega$$

$$K_{\alpha\beta} = \int_\Omega \nu\Phi_{\alpha,ii}\Phi_{\beta,jj}\,d\Omega$$

$$G_{\alpha i} = \int_\Gamma \nu\overset{*}{\Phi}_\alpha\psi_{,iij}n_j\,d\Gamma - \int_\Gamma \nu\overset{*}{\Phi}_{\alpha,j}\psi_{,ii}n_j\,d\Gamma$$

Here, there are three variables (Ψ, $\Psi_{,1}$, $\Psi_{,2}$) which are to be specified and calculated at each of the four nodes of the 2-D isoparametric element. To this end, we require twelve constants to be determined, with three of them (Ψ, $\Psi_{,1}$, $\Psi_{,2}$) at each of the four nodes:

$$1, \xi, \eta, \xi\eta, \xi^2, \eta^2, \xi^2\eta, \eta^2\xi, \xi^3, \eta^3, \xi\eta^3, \xi^3\eta$$

The 2-D TGM Newton-Raphson formulation of (12.2.16) can be constructed similarly as in (12.2.7) for the 3-D case with the boundary conditions reduced to the two-dimensional geometry from Figure 12.2.2 and Table 12.2.1.

12.2.3 PHYSICAL INSTABILITY IN TWO-DIMENSIONAL INCOMPRESSIBLE FLOWS

Unstable motions occur during the transition from laminar to turbulent flows. To examine such motions, the so-called Orr-Sommerfeld equation is solved. Here we may begin with the 2-D velocity and vorticity as a sum of the mean and fluctuation components,

$$\mathrm{v}_i = \overline{\mathrm{v}}_i + \mathrm{v}_i^* \quad (i = 1, 2) \tag{12.2.17a}$$

$$\omega_i = \overline{\omega}_i + \omega_i^* \quad (i = 3) \tag{12.2.17b}$$

where ($^-$) and (*) denote mean and fluctuation quantities, respectively.

Table 12.2.1 Boundary Conditions (3-D cavity)

$At\ x = 0, 1$	$\psi_{1,1} = \psi_2 = \psi_{2,2} = \psi_{2,3} = \psi_3 = \psi_{3,2} = \psi_{3,3}$ $\psi_{1,3} - \psi_{3,1} = 0$ $\psi_{2,1} - \psi_{1,2} = 0$
$At\ y = 0$	$\psi_1 = \psi_{1,1} = \psi_{1,3} = \psi_{2,2} = \psi_3 = \psi_{3,1} = \psi_{3,3}$ $\psi_{3,2} - \psi_{2,3} = 0$ $\psi_{2,1} - \psi_{1,2} = 0$
$At\ y = 1$	$\psi_1 = \psi_{1,1} = \psi_{1,3} = \psi_{2,2} = \psi_3 = \psi_{3,1} = \psi_{3,3}$ $\psi_{3,2} - \psi_{2,3} = U_{\max}$ $\psi_{2,1} - \psi_{1,2} = 0$
$At\ z = 0, 1$	$\psi_1 = \psi_{1,1} = \psi_{1,2} = \psi_2 = \psi_{2,1} = \psi_{2,2} = \psi_{3,3}$ $\psi_{3,2} - \psi_{2,3} = 0$ $\psi_{1,3} - \psi_{3,1} = 0$
$At\ z = 0.5$	$\psi_1 = \psi_{1,1} = \psi_{1,2} = \psi_2 = \psi_{2,1} = \psi_{2,2} = \psi_{3,3}$

For two-dimensional flows with v_i $(i = 1, 2)$, ω_i $(i = 3)$, the vorticity transport equation takes the form

$$\frac{\partial}{\partial t}(-\psi^*_{,ii}) + \varepsilon_{ik}\overline{v}_{k,ij}\overline{v}_j + \varepsilon_{ik}\overline{v}_{k,ij}\varepsilon_{jr}\psi^*_{,r} - \psi^*_{,iij}\overline{v}_j - \psi^*_{,iij}\varepsilon_{jr}\psi^*_{,r} - \frac{1}{\mathrm{Re}}(\varepsilon_{ik}\overline{v}_{k,ijj} - \psi^*_{,iijj}) = 0 \tag{12.2.18}$$

where we have used the following relationship:

$$\overline{\omega} = \varepsilon_{ik}\overline{v}_{k,i}$$

$$\omega^* = \varepsilon_{ik}v^*_{k,i} = \varepsilon_{ik}\varepsilon_{kr}\psi^*_{,ri} = -\psi^*_{,ii}$$

Denote

$$\psi^*(x, y, t) = q(x, y)e^{-i\beta t} = Q(y)e^{ikx}e^{-i\beta t} \tag{12.2.19a}$$

$$\beta = \beta^{(R)} + i\beta^{(I)} \tag{12.2.19b}$$

where $\beta^{(R)}$ is the circular frequency and $\beta^{(I)}$ is the amplification factor, related as

$$\beta = kc, \quad c = c^{(R)} + ic^{(I)} \tag{12.2.20}$$

with $k =$ wave number and c is the velocity of propagation, (R) and (I) indicating the real and imaginary parts, respectively. In view of (12.2.18) and (12.2.19) and neglecting higher order terms $(\varepsilon_{ik}\overline{v}_{k,ij}\overline{v}_j, \psi^*_{,iij}\varepsilon_{jr}\psi^*_{,r},$ and $\varepsilon_{ik}\overline{v}_{k,ijj})$, we obtain

$$-i\beta q_{,ii} + \varepsilon_{ik}\overline{v}_{k,ij}\varepsilon_{jr}q_{,r} + q_{,iij}\overline{v}_j - \frac{1}{\mathrm{Re}}q_{,iijj} = 0 \tag{12.2.21}$$

We further denote that

$$\overline{v}_1 = \mathbf{U}(y) \quad \text{and} \quad \overline{v}_2 = 0 \tag{12.2.22a}$$

and

$$q(x, y) = Q(y)e^{ikx} \tag{12.2.22b}$$

Combine (12.2.22) with (12.2.21) to obtain

$$-i\beta Q(ik)^2 - i\beta Q_{,22} + U_{,22}Q(ik) + UQ(ik)^3 + UQ_{,22}(ik)$$
$$-\frac{1}{\mathrm{Re}}[Q_{,2222} + Q(ik)^4 - 2Q_{,22}(ik)^2] = 0 \qquad (12.2.23)$$

Dividing (12.2.23) by ik, we arrive at the Orr-Sommerfeld equation

$$c\left(k^2 Q - Q_{,22}\right) - QU_{,22} - k^2 QU + UQ_{,22} + \frac{i}{k\mathrm{Re}}(Q_{,2222} + k^4 Q - 2k^2 Q_{,22}) = 0 \qquad (12.2.24)$$

or

$$(U-c)\left(\frac{d^2Q}{dy^2} - k^2 Q\right) - Q\frac{d^2U}{dy^2} = -\frac{i}{k\mathrm{Re}}\left(\frac{d^4Q}{dy^4} - 2k^2\frac{d^2Q}{dy^2} + k^4 Q\right) = 0 \qquad (12.2.25)$$

Since (12.2.25) represents variations only in the lateral direction y, the trial functions are constructed in one dimension. The finite element formulations of (12.2.25) can be carried out in a standard manner, resulting in the form,

$$(K_{\alpha\beta} - cM_{\alpha\beta})Q_\beta = 0 \qquad (12.2.26)$$

with the boundary conditions

$$Q_\alpha = 0 \quad \text{and} \quad \partial Q_\alpha / \partial y = 0 \qquad (12.2.27)$$

The expression (12.2.26) is a standard eigenvalue problem,

$$|K_{\alpha\beta} - cM_{\alpha\beta}|Q_\beta = 0 \qquad (12.2.28)$$

Eigenvalues are the phase velocity (c) with real and imaginary parts as defined in (16.6.20),

$$c^{(I)} < 0 \quad \text{stable} \qquad (12.2.29a)$$

$$c^{(I)} = 0 \quad \text{neutral stability} \qquad (12.2.29b)$$

$$c^{(I)} > 0 \quad \text{unstable} \qquad (12.2.29c)$$

Eigenvectors Q_β represent fluctuation parts of stream function, which provide fluctuation parts of velocity $v_i^* = \varepsilon_{ij}\psi^*_{,j}$. The eigenvalue problem involved in a complex number may be solved using the so-called QR algorithm [Wilkinson, 1965].

12.3 EXAMPLE PROBLEMS

Three-Dimensional Vorticity Transport Equations

A convenient benchmark problem is the lid-driven cubic cavity flow as shown in Figure 12.3.1. The corresponding boundary conditions are shown in Table 12.2.1.

In Figure 12.3.2, we show comparisons between the TGM solution of the 3-D vorticity transport equations (12.2.4) and the results of other approaches reported by Takami and Kuwahara [1974] with the $20 \times 10 \times 20$ FDM velocity-pressure formulation, Goda [1979] with the $20 \times 10 \times 20$ FDM velocity-pressure formulation, and Mahallati and

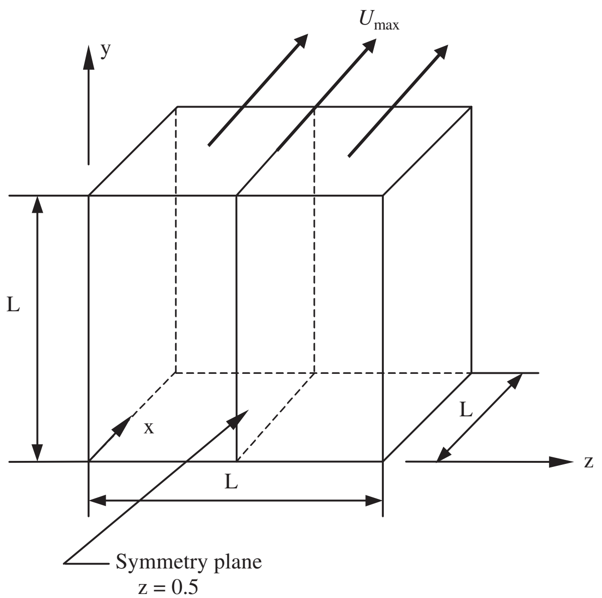

Figure 12.3.1 Geometry of the cubic cavity flow.

Militzer [1993] with the $21 \times 11 \times 21$ vorticity vector potential formulation using the finite analytic method (FAM).

In Figure 12.3.3, the x-velocity profiles for Re $= 100$ at different x-planes are shown. Note that the effect of boundary layers is clearly evident in the y–z-planes, indicating that the velocity profiles in planes closer to the wall are less developed due to the boundary layer effects than in the symmetry planes.

The 3-D cavity streamline distributions for Re $= 10$ and Re $= 100$ at different planes are as shown in Figure 12.3.4. It is noted that, for higher Reynolds number (Re $= 100$), the vortex center moves toward downstream.

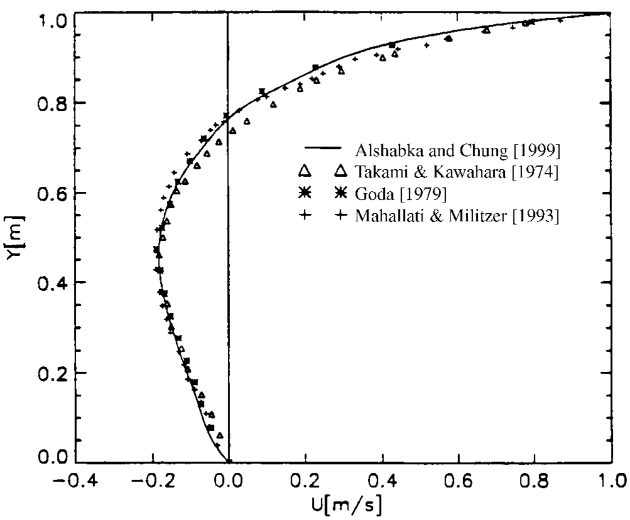

Figure 12.3.2 Velocity profiles on vertical centerline of the 3-D cavity for Re $= 100$.

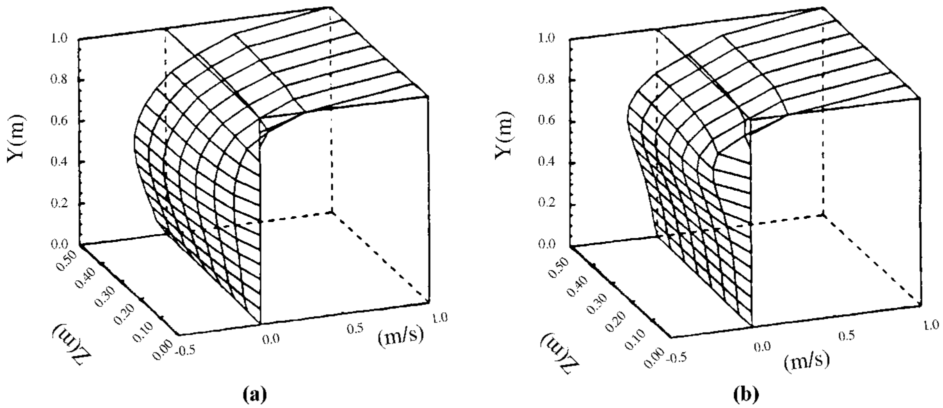

Figure 12.3.3 Profiles of the x-component of the velocity of the 3-D cavity flow at Re = 100. (**a**) The $X = 0.5$ plane. (**b**) The $x = 0.786$ plane.

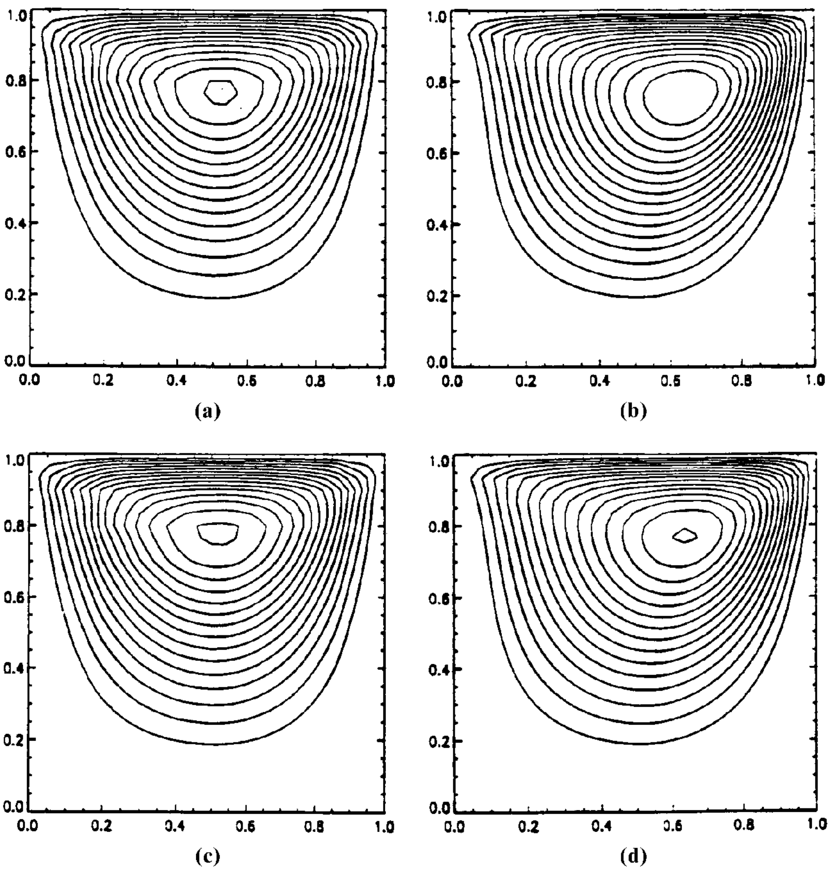

Figure 12.3.4 The 3-D cavity streamlines (ψ_3). (**a**) The symmetry plane ($z = 0.5$) for Re = 10. (**b**) The symmetry plane ($z = 0.5$) for Re = 100. (**c**) The $Z = 0.2$ plane for Re = 10. (**d**) The $Z = 0.2$ plane for Re = 100.

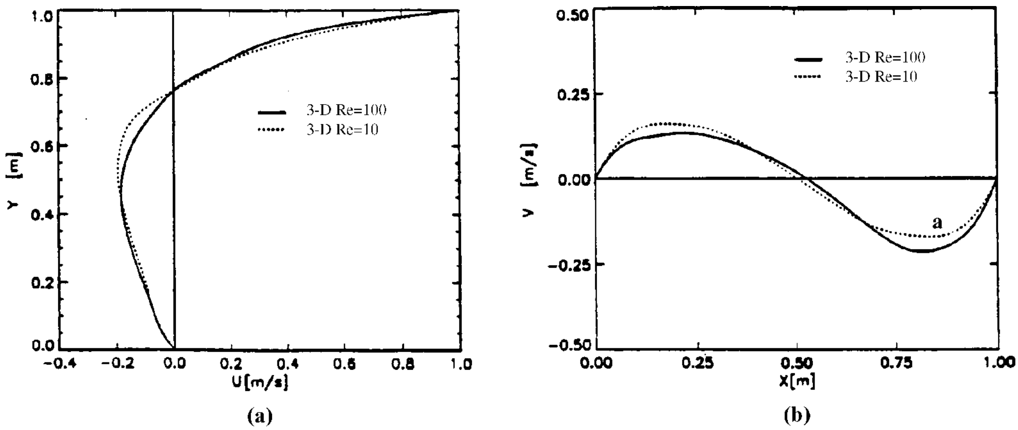

Figure 12.3.5 Velocity profile on the 3-D cavity. (**a**) Vertical centerline. (**b**) X-horizontal centerline.

Figure 12.3.5 shows the velocity profiles along the vertical and horizontal centerlines of the symmetry plane of the 3-D cavity. It is seen in Figure 12.3.5a that an increase in Reynolds number tends to reduce negative x-velocity in the region around $y = 0.6$, with the point of maximum negative x-velocity moving downward. At the same time, the y-velocity becomes less positive upstream and more negative downstream as the Reynolds number increases, with the position of zero velocity shifted toward downstream as shown in Figure 12.3.5b.

Overall, the fourth order partial differential equations of vorticity transport in terms of the three dimensional stream function vector components lead to an accurate solution, in which the pressure oscillations are eliminated from the governing equations.

12.4 SUMMARY

Difficulties involved in the satisfaction of mass conservation and prevention of pressure oscillations discussed in Chapter 5 for FDM are the focus of attention also in this chapter for FEM. Traditional approaches in FEM include mixed methods, penalty methods, pressure correction methods, operator splitting methods, and vortex methods. These methods can be formulated by finite elements using GGM, TGM, or GPG.

Although the incompressible flows occur in many engineering problems and their accurate solution methods are important, recent trends appear to be an emphasis in developing computational schemes capable of treating all speed regimes for both incompressible and compressible flows and, in particular, interactions between incompressible and compressible flows. Recall that this was the case for the incompressible flows using FDM. Toward this end, two approaches were introduced: the preconditioning of compressible flow equations and the FDV methods. Similar treatments are available for FEM applications. These and other topics will be discussed in the next chapter on compressible flows.

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