CHAPTER NINE

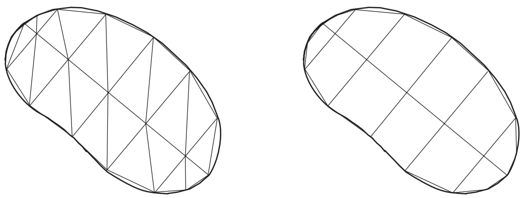
Finite Element Interpolation Functions

9.1 GENERAL

We saw in Section 1.3 that finite element equations are obtained by the classical approximation theories such as variational or weighted residual methods. However, there are some basic differences in philosophy between the classical approximation theories and finite element methods. In the finite element methods, the global functional representations of a variable consist of an assembly of local functional representations so that the global boundary conditions can be implemented in local elements by modification of the assembled algebraic equations. The local interpolation (shape, basis, or trial) functions are chosen in such a manner that continuity between adjacent elements is maintained.

The finite element interpolations are characterized by the shape of the finite element and the order of the approximations. In general, the choice of a finite element depends on the geometry of the global domain, the degree of accuracy desired in the solution, the ease of integration over the domain, etc.

In Figure 9.1.1, a two-dimensional domain is discretized by a series of triangular elements and quadrilateral elements. It is seen that the global domain consists of many subdomains (the finite elements). The global domain may be one-, two-, or three-dimensional. The corresponding geometries of the finite elements are shown in Figure 9.1.2. A one-dimensional element (as we have studied in Chapters 1 and 8) is simply a straight line, a two-dimensional element may be triangular, rectangular, or quadrilateral, and a three-dimensional element can be a tetrahedron, a regular hexahedron, an irregular hexahedron, etc. The three-dimensional domain with axisymmetric geometry and axisymmetric physical behavior can be represented by a two-dimensional element generated into a three-dimensional ring by integration around the circumference. In general, the interpolation functions are the polynomials of various degrees, but often they may be given by transcendental or special functions. If polynomial expansions are used, the linear variation of a variable within an element can be expressed by the data provided at the corner nodes. For quadratic variations, we add a side node located midway between the corner nodes (Figure 9.1.3). Cubic variations of a variable are represented by two side nodes in addition to the corner nodes. Sometimes a complete expansion of certain degree polynomials may require installation of nodes at various points within the element (interior nodes). Thus, there are three different types of nodes: vertex nodes in which only corner nodes are installed at vertices, side nodes



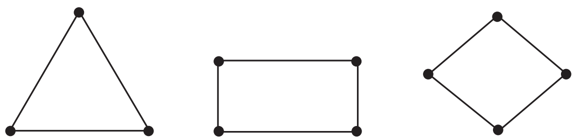
(a) Discretization by triangular elements **(b)** Discretization by quardrilateral elements

Figure 9.1.1 Finite element discretization of a two-dimensional domain.

in which one or more nodes are installed along the element sides, and internal nodes in which one or more interior nodes are provided inside of an element.

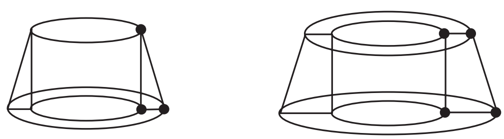
Nodal configurations and corresponding polynomials may be selected from the so-called Pascal triangle, Pascal tetrahedron, two-dimensional hypercube, or three-dimensional hypercube, as shown in Figure 9.1.4. Various combinations between the number of nodes and degrees of polynomials for two-dimensional geometries can be selected as illustrated in Figures 9.1.5 and 9.1.6. Similar approaches may be used for three-dimensional geometries. In choosing a suitable element, the number of nodes

(a)



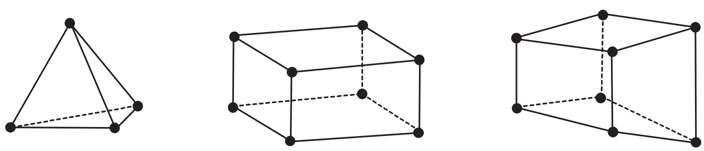
Triangular Rectangular Quadrilateral

(b)



Triangular ring Quadrilateral ring

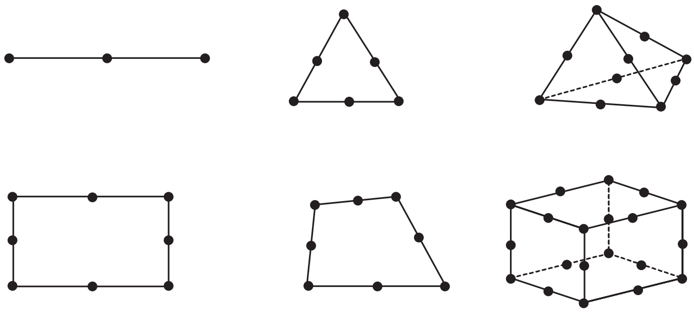
(c)



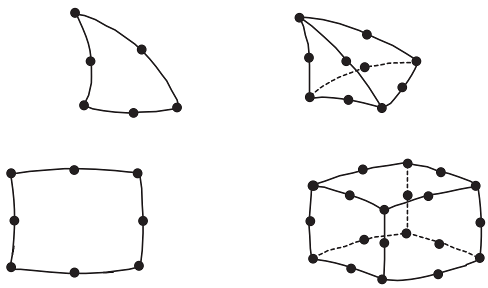
Tetrahedral Regular hexahedral Irregular hexahedral

(d)

Figure 9.1.2 Various shapes of finite elements with corner nodes: **(a)** One-dimensional element; **(b)** two-dimensional elements; **(c)** two-dimensional element generated into three-dimensional ring element for axisymmetric geometry; and **(d)** three-dimensional elements.



(a)



(b)

Figure 9.1.3 Quadratic elements. (**a**) Quadratic elements with straight edges. (**b**) Quadratic elements with curved edges.

must match the number of terms in the polynomials, and must be symmetrically arranged. They must also be as complete as possible so that all possible degrees of freedom are allowed to be present within the highest polynomial degrees chosen.

Types of finite elements may be distinguished by: (1) geometries (one-, two-, and three dimensional); (2) choices of interpolation functions (polynomials, Lagrange or Hermite polynomials), etc.; (3) choices of element coordinates (cartesian or natural coordinates); and (4) choices of specified variables and gradients of the variables at nodes (Lagrange family with variables alone or Hermite families with gradients included). Earlier developments of finite element interpolation functions include Argyris [1963] and Zienkiewicz and Cheung [1965], among many others. These and other topics will be presented in the following sections.

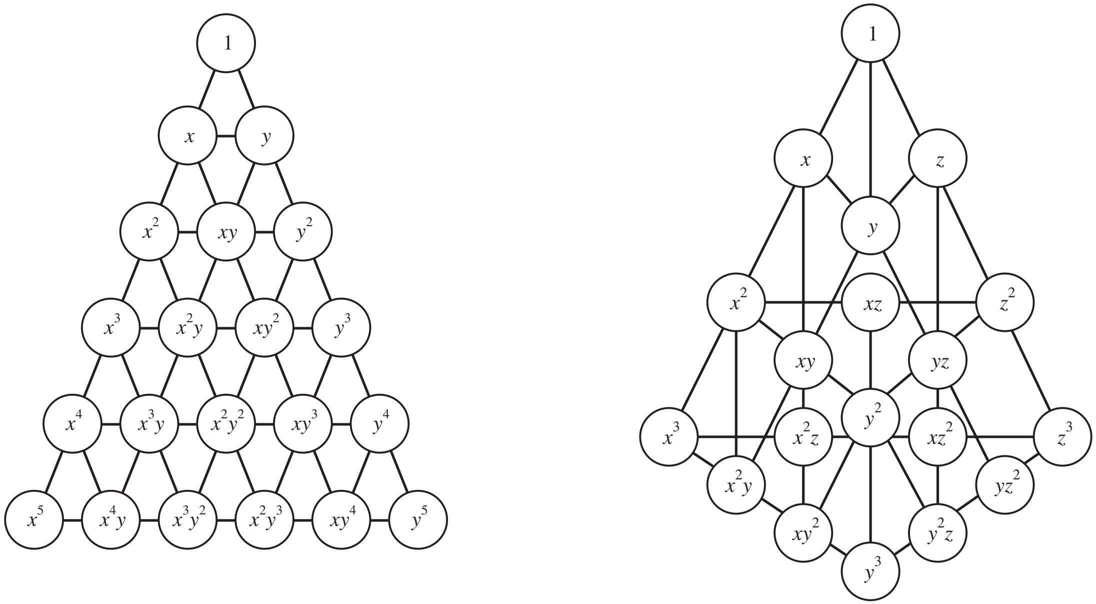
9.2 ONE-DIMENSIONAL ELEMENTS

9.2.1 CONVENTIONAL ELEMENTS

The polynomial expansion for a variable u to be approximated in a one-dimensional element may be written as

$$u = \alpha_1 + \alpha_2 x + \alpha_3 x^2 + \alpha_4 x^3 + \cdots \tag{9.2.1}$$

For a linear variation of u, we need a two-node system with one node at each



(a) (b)

Figure 9.1.4 Polynomial expansions in finite elements. **(a)** Pascal triangle. **(b)** Pascal tetrahedron. **(c)** Two-dimensional hypercube. **(d)** Three-dimensional hypercube.

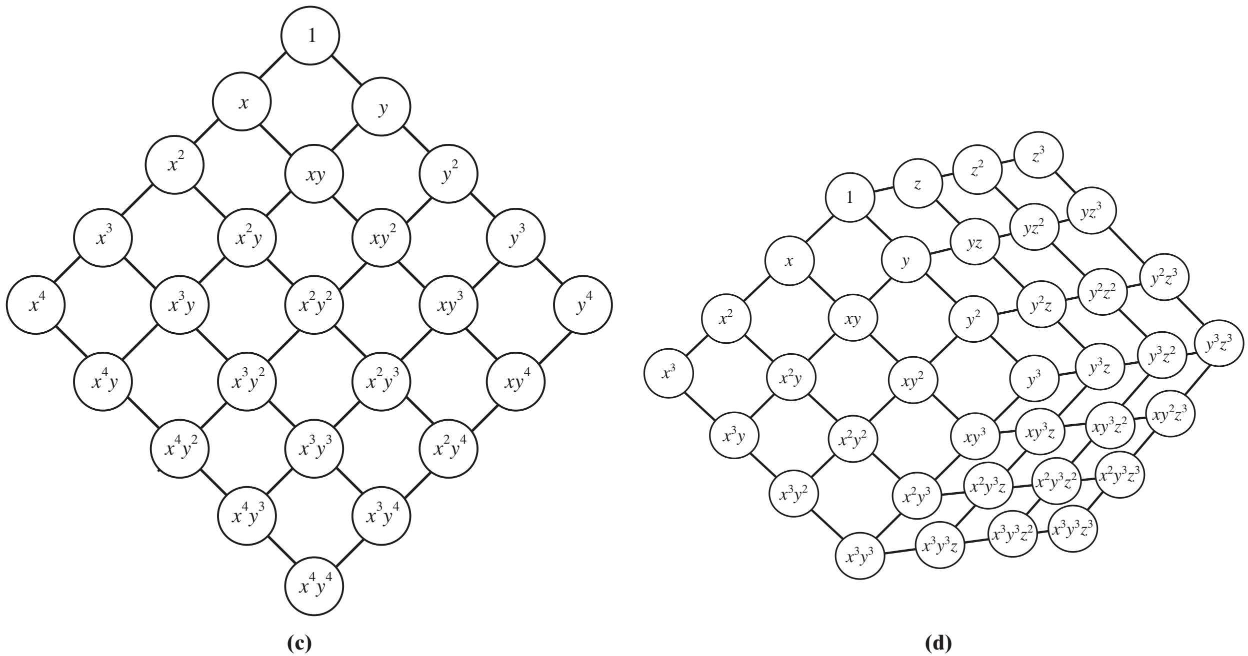


Figure 9.1.4 *(continued)*

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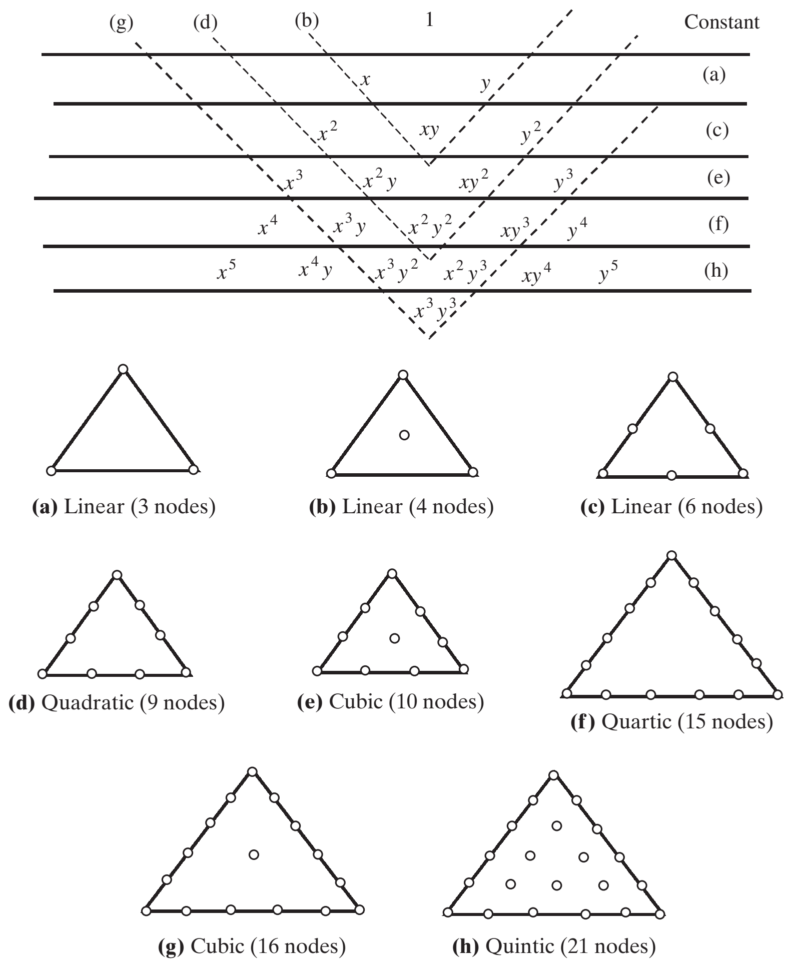


Figure 9.1.5 Various combinations between the number of nodes and degrees of polynomials for two-dimensional triangular geometries.

end. The interpolation functions for this case were derived in Section 1.3, based on Figure 1.3.1. An alternative method, perhaps the more general approach, is to use the natural (nondimensional) coordinate, ξ, with the origin set as in Figure 9.2.1a ($\xi = x/h$) or Figure 9.2.1b ($\xi = x/(h/2)$). Then (9.2.1) becomes

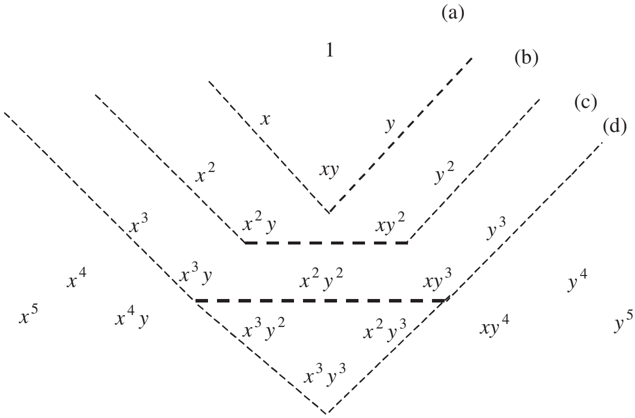
$$u = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3 + \cdots \tag{9.2.2}$$

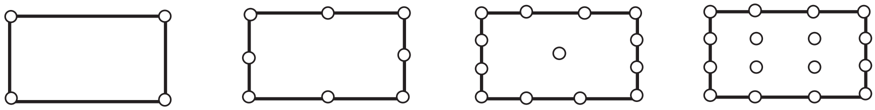
For a linear element (two-node system), we have

$$u = \alpha_1 + \alpha_2\xi \tag{9.2.3}$$

Writing (9.2.3) at each node, solving for the constants, and substituting them into (9.2.3) for an element, we obtain $u^{(e)}$ [u in element (e)]:

$$u^{(e)} = \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} = \Phi_N^{(e)} u_N^{(e)}, \quad (N = 1, 2)$$





(a) Linear (4 nodes) **(b)** Quadratic (8 nodes) **(c)** Cubic (13 nodes) **(d)** Cubic (16 nodes)

Figure 9.1.6 Various combinations between the number of nodes and degrees of polynomials for two-dimensional rectangular or quadrilateral geometries.

where the interpolation functions are

$$\Phi_1^{(e)} = 1 - \xi, \qquad \Phi_2^{(e)} = \xi \qquad \text{for Figure 9.2.1a} \tag{9.2.4a}$$

$$\Phi_1^{(e)} = \frac{1}{2}(1 - \xi), \qquad \Phi_2^{(e)} = \frac{1}{2}(1 + \xi) \qquad \text{for Figure 9.2.1b} \tag{9.2.4b}$$

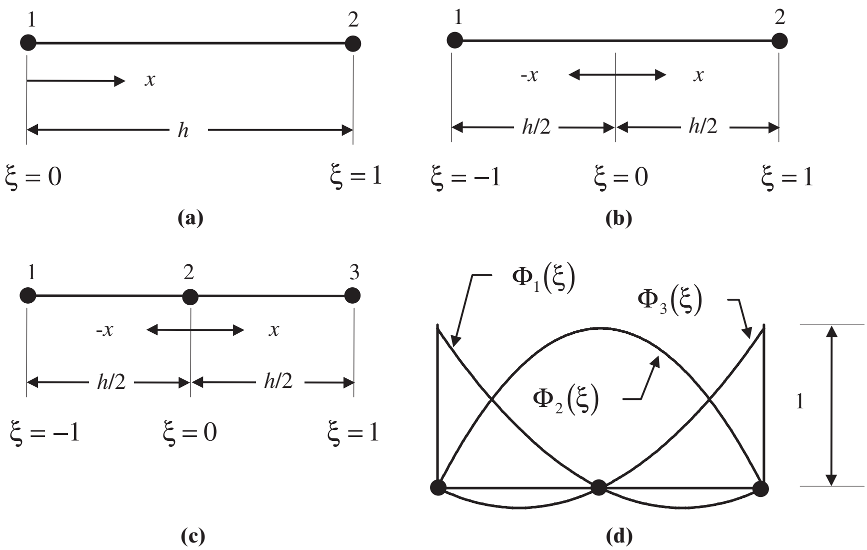


Figure 9.2.1 One-dimensional element. **(a)** Origin at end node (linear variation). **(b)** Origin at center (linear variation). **(c)** Origin at center node (quadratic variation). **(d)** Quadratic variation.

Likewise, for quadratic approximations in which we require an additional node, preferably at the midside (Figure 9.2.1c), we have

$$u = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 \tag{9.2.5}$$

and writing (9.2.5) at each node yields

$$u_1 = \alpha_1 - \alpha_2 + \alpha_3, \qquad u_2 = \alpha_1, \qquad u_3 = \alpha_1 + \alpha_2 + \alpha_3 \tag{9.2.6}$$

Evaluating the constants, we obtain

$$u^{(e)} = \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} + \Phi_3^{(e)} u_3^{(e)} = \Phi_N^{(e)} u_N^{(e)}, \quad (N = 1, 2, 3) \tag{9.2.7}$$

where the interpolation functions are (see Figure 9.2.1d)

$$\Phi_1^{(e)} = \frac{1}{2}\xi(\xi - 1), \qquad \Phi_2^{(e)} = 1 - \xi^2, \qquad \Phi_3^{(e)} = \frac{1}{2}\xi(\xi + 1) \tag{9.2.8}$$

It is easily seen that the limits of integration of the interpolation functions should be changed such that

$$\int_{-h/2}^{h/2} f(x)dx = \int_{-1}^{1} f(\xi)\frac{\partial x}{\partial \xi}d\xi = \frac{h}{2}\int_{-1}^{1} f(\xi)d\xi \tag{9.2.9}$$

where $x = (h/2)\xi$. If the interpolation functions are derived in terms of nondimensionalized spatial variables, then such a normalized system is called a natural coordinate. Note that the basic properties of interpolation functions as given by (8.2.12) are satisfied for both (9.2.4) and (9.2.8).

9.2.2 LAGRANGE POLYNOMIAL ELEMENTS

To avoid the inversion of the coefficient matrix for higher order approximations, we may use the Lagrange interpolation function L_N, which can be obtained as follows. Let $u(x)$ be given by (Figure 9.2.2)

$$u(x) = L_1(x)u_1 + L_2(x)u_2 + \cdots L_n(x)u_n$$

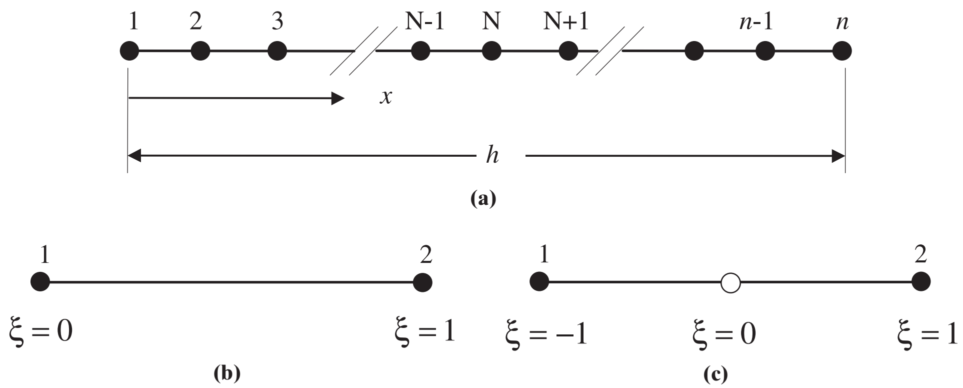


Figure 9.2.2 Lagrange element with natural coordinates. **(a)** Lagrange element of the n-1th degree approximation. **(b)** Linear approximation with origin at the left node. **(c)** Linear variation with origin at the center.

where $L_N(x)$ is chosen such that

$$L_N(x_M) = \delta_{NM}$$

$L_N(x)$ may be expanded in the form

$$L_N(x) = c_N(x - x_1)(x - x_2)\cdots(x - x_{N-1})(x - x_{N+1})\cdots(x - x_n)$$

where

$$L_N(x_M) = \begin{cases} 0 & M \neq N \\ 1 = c_N \displaystyle\prod_{M=1,\, M\neq N}^{n} (x_N - x_M) & M = N \end{cases}$$

Solving for the coefficient c_N and substituting it to the expression for $L_N(x)$, we obtain

$$\Phi_N^{(e)}(x) = L_N(x) = \prod_{M=1,\, M\neq N}^{n} \frac{x - x_M}{x_N - x_M} \tag{9.2.10}$$

$$= \frac{(x - x_1)(x - x_2)\cdots(x - x_{N-1})(x - x_{N+1})\cdots(x - x_n)}{(x_N - x_1)(x_N - x_2)\cdots(x_N - x_{N-1})(x_N - x_{N+1})\cdots(x_N - x_n)}$$

with the symbol $\prod$ denoting a product of binomials over the range $M = 1, 2, \ldots, n$ (see Figure 9.2.2). Here the element is divided into equal length segments by the $n = m + 1$ nodes, with m and n equal to the order of approximations and the number of nodes in an element, respectively. Let us consider a first order approximation of a variable u such that

$$u^{(e)} = L_N u_N^{(e)} \quad (N = 1, 2)$$

with

$$L_1 = \frac{x - x_2}{x_1 - x_2} = \frac{x - h}{-h} = 1 - \frac{x}{h}$$

$$L_2 = \frac{x - x_1}{x_2 - x_1} = \frac{x}{h}$$

with $x_1 = 0$ and $x_2 = h$. If the nondimensionalized form $\xi = x/h$ is used, we have

$$L_N = \prod_{M=1,\, M\neq N}^{n} \frac{\xi - \xi_M}{\xi_N - \xi_M} \tag{9.2.11}$$

and

$$L_1 = \frac{\xi - \xi_2}{\xi_1 - \xi_2} = 1 - \xi, \qquad L_2 = \frac{\xi - \xi_1}{\xi_2 - \xi_1} = \xi$$

If the origin is taken as shown, at the center of the element (Figure 9.2.2c) using the natural coordinate system, we note that

$$L_1 = \frac{1}{2}(1 - \xi), \qquad L_2 = \frac{1}{2}(1 + \xi)$$

These functions are the same as in (9.2.4b).

For quadratic approximations, we have $n = m + 1 = 3$ and

$$L_1 = \frac{(\xi - \xi_2)(\xi - \xi_3)}{(\xi_1 - \xi_2)(\xi_1 - \xi_3)} = 2\left(\xi - \frac{1}{2}\right)(\xi - 1)$$

$$L_2 = \frac{(\xi - \xi_1)(\xi - \xi_3)}{(\xi_2 - \xi_1)(\xi_2 - \xi_3)} = -4\xi(\xi - 1)$$

$$L_3 = \frac{(\xi - \xi_1)(\xi - \xi_2)}{(\xi_3 - \xi_1)(\xi_3 - \xi_2)} = 2\xi\left(\xi - \frac{1}{2}\right)$$

For the natural coordinate system with the origin at the center, we obtain

$$L_1 = \frac{1}{2}\xi(\xi - 1), \qquad L_2 = 1 - \xi^2, \qquad L_3 = \frac{1}{2}\xi(\xi + 1)$$

which are identical to (9.2.8), the results one would expect to obtain.

The interpolation functions derived using the natural coordinates are convenient to generate multidimensional element interpolation functions by means of tensor products as shown in Section 9.3.2.

9.2.3 HERMITE POLYNOMIAL ELEMENTS

If continuity of the derivative of a variable at common nodes is desired, one efficient way of assuring this continuity is to use the Hermite polynomials. For a one-dimensional element with two end nodes, the development of Hermite polynomials for a variable u begins with

$$u = \alpha_1 + \alpha_2\xi + \alpha_3\xi^2 + \alpha_4\xi^3$$

We write the nodal equations for $u(\xi)$ and $du(\xi)/d\xi$ at two end nodes and evaluate the constants to obtain

$$u^{(e)}(\xi) = H_N^0(\xi)u_N^{(e)} + H_N^1(\xi)\left(\frac{\partial u}{\partial \xi}\right)_N^{(e)} \quad (N = 1, 2) \tag{9.2.12a}$$

or

$$u^{(e)}(\xi) = \Phi_r^{(e)} Q_r \quad (r = 1, 2, 3, 4) \tag{9.2.12b}$$

where the Hermite polynomials have the properties [see Hildebrand, 1956]

$$H_N^0(\xi_M) = \delta_{NM}, \qquad \frac{d}{d\xi}H_N^1(\xi_M) = \delta_{NM}$$

Here $H_N^0(\xi)$ and $H_N^1(\xi)$, which are now used as the finite element interpolation functions,

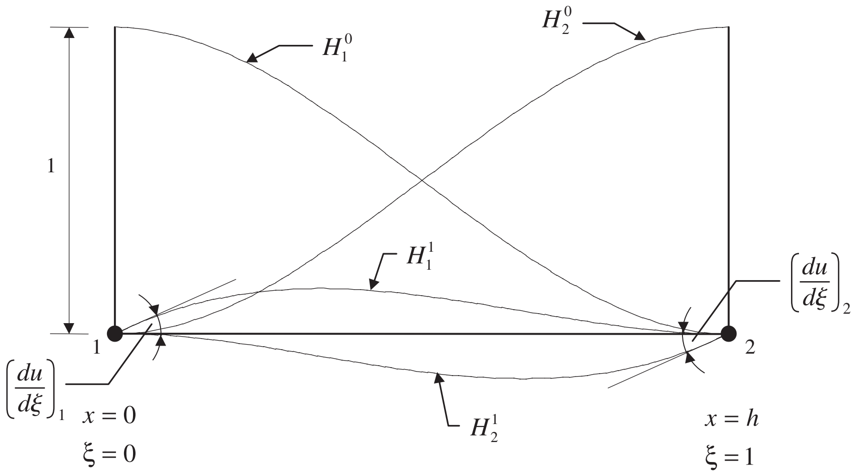


Figure 9.2.3 Hermite interpolation functions.

are the cubic polynomials of the form (see Figure 9.2.3)

$$
\begin{aligned}
\Phi_1^{(e)} &= H_1^0 = 1 - 3\xi^2 + 2\xi^3 & Q_1 &= u_1^{(e)} \\
\Phi_2^{(e)} &= H_2^0 = 3\xi^2 - 2\xi^3 & Q_2 &= u_2^{(e)} \\
\Phi_3^{(e)} &= H_1^1 = \xi - 2\xi^2 + \xi^3 & Q_3 &= \left(\frac{\partial u}{\partial \xi}\right)_1^{(e)} \\
\Phi_4^{(e)} &= H_2^1 = \xi^3 - \xi^2 & Q_4 &= \left(\frac{\partial u}{\partial \xi}\right)_2^{(e)}
\end{aligned}
\tag{9.2.13}
$$

with $\xi = x/h$, h being the length of the element. Note that $\Phi_3^{(e)}$ and $\Phi_4^{(e)}$ must be multiplied by h, if the nodal values of derivatives are given by $\partial u/\partial x$.

If the second as well as the first derivative is to be specified at the end nodes, we require a fifth degree Hermite polynomial such that

$$
u^{(e)}(\xi) = H_N^0(\xi)u_N^{(e)} + H_N^1(\xi)\left(\frac{\partial u}{\partial \xi}\right)_N^{(e)} + H_N^2(\xi)\left(\frac{\partial^2 u}{\partial \xi^2}\right)_N^{(e)}
\tag{9.2.14}
$$

with

$$
\begin{aligned}
\Phi_1^{(e)} &= H_1^0 = 1 - 10\xi^3 + 15\xi^4 - 6\xi^5 & Q_1 &= u_1^{(e)} \\
\Phi_2^{(e)} &= H_2^0 = 10\xi^3 - 15\xi^4 + 6\xi^5 & Q_2 &= u_2^{(e)} \\
\Phi_3^{(e)} &= H_1^1 = \xi - 6\xi^3 + 8\xi^4 - 3\xi^5 & Q_3 &= \left(\frac{\partial u}{\partial \xi}\right)_1^{(e)} \\
\Phi_4^{(e)} &= H_2^1 = -4\xi^3 + 7\xi^4 - 3\xi^5 & Q_4 &= \left(\frac{\partial u}{\partial \xi}\right)_2^{(e)}
\end{aligned}
$$

$$\Phi_5^{(e)} = H_2^2 = \frac{1}{2}(\xi^2 - 3\xi^3 + 3\xi^4 - \xi^5) \qquad Q_5 = \left(\frac{\partial^2 u}{\partial \xi^2}\right)_1^{(e)}$$

$$\Phi_6^{(e)} = H_2^2 = \frac{1}{2}(\xi^3 - 2\xi^4 + \xi^5) \qquad Q_6 = \left(\frac{\partial^2 u}{\partial \xi^2}\right)_2^{(e)}$$

Note that $\Phi_3^{(e)}$, $\Phi_4^{(e)}$, and $\Phi_5^{(e)}$, $\Phi_6^{(e)}$ must be multiplied by h and h^2, respectively, if the nodal values of derivatives are given by $\partial u/\partial x$, and $\partial^2 u/\partial x^2$.

Additional discussions of Hermite polynomials can be found in Birkoff, Schultz, and Varga [1968].

9.3 TWO-DIMENSIONAL ELEMENTS

Among the two-dimensional elements, the triangular element was the first investigated in the early days of development. In recent years, however, the four-sided isoparametric element has become equally popular, or more convenient in some applications. Various features of these elements are described below.

9.3.1 TRIANGULAR ELEMENTS

As noted in the one-dimensional element, we may use the standard rectangular cartesian coordinates or the natural coordinates (nondimensionalized) to obtain the interpolation functions. It will be seen that the choice of a particular coordinate system influences the amount of algebra required in the formulation of finite element equations. For higher order approximations (with higher order polynomials), an evaluation of constants is particularly easy if natural coordinates are used.

Cartesian Coordinate Triangular Elements

In this element, the properties of the element are determined in terms of the local rectangular cartesian coordinates (x_i) with their origin at the centroid of the triangle (Figure 9.3.1).

$$x_1 + x_2 + x_3 = 0 \quad \text{and} \quad y_1 + y_2 + y_3 = 0$$

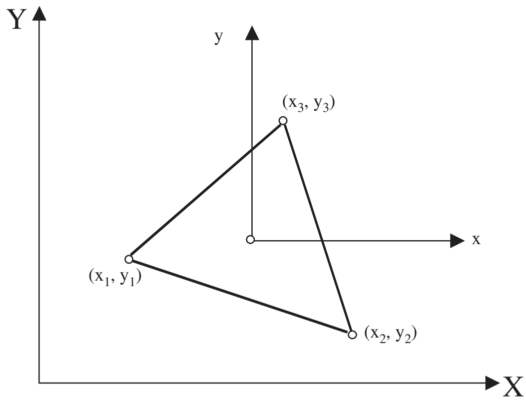


Figure 9.3.1 Cartesian coordinate triangular element.

or

$$\sum_{N=1}^{3} x_{Ni} = 0 \quad (N = 1, 2, 3, i = 1, 2)$$

with $x_{N1} = x_N$ and $x_{N2} = y_N$. If this triangle is identified from the global rectangular cartesian coordinates (X_i) with their origin outside the triangle, we note that the following relationships hold:

$$x_1 = X_1 - \frac{1}{3}(X_1 + X_2 + X_3)$$

$$x_2 = X_2 - \frac{1}{3}(X_1 + X_2 + X_3)$$

$$\vdots$$

$$y_3 = Y_3 - \frac{1}{3}(Y_1 + Y_2 + Y_3)$$

Or, combining these equations, we write

$$x_{Ni} = X_{Ni} - \frac{1}{3}\sum_{N=1}^{3} X_{Ni} \quad (N = 1, 2, 3,\ \ i = 1, 2) \tag{9.3.1}$$

Now consider the polynomial expansion of a variable $u^{(e)}$ in the form

$$u^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y \tag{9.3.2}$$

This represents a linear variation of u in both x and y directions within the triangular element. To evaluate the three constants α_1, α_2, and α_3, we must provide three equations in terms of the known values of u, x, and y at each of the three nodes.

$$u_1^{(e)} = \alpha_1 + \alpha_2 x_1 + \alpha_3 y_1$$

$$u_2^{(e)} = \alpha_1 + \alpha_2 x_2 + \alpha_3 y_2$$

$$u_3^{(e)} = \alpha_1 + \alpha_2 x_3 + \alpha_3 y_3$$

Writing in a matrix form, we obtain

$$\begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} = \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{bmatrix} \tag{9.3.3}$$

Solving for the constants and substituting them into (9.3.2) gives

$$\begin{aligned} u^{(e)} &= \begin{bmatrix} 1 & x & y \end{bmatrix} \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}^{-1} \begin{bmatrix} u_1^{(e)} \\ u_2^{(e)} \\ u_3^{(e)} \end{bmatrix} \\ &= (a_1 + b_1 x + c_1 y)u_1^{(e)} + (a_2 + b_2 x + c_2 y)u_2^{(e)} + (a_3 + b_3 x + c_3 y)u_3^{(e)} \\ &= \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} + \Phi_3^{(e)} u_3^{(e)} \end{aligned}$$

or

$$u^{(e)} = \Phi_N^{(e)} u_N^{(e)} \quad (N = 1, 2, 3)$$

where the interpolation function $\Phi_N^{(e)}$ is given by

$$\Phi_N^{(e)} = a_N + b_N x + c_N y \tag{9.3.4}$$

$$a_1 = \frac{1}{|D|}(x_2 y_3 - x_3 y_2) \qquad a_2 = \frac{1}{|D|}(x_3 y_1 - x_1 y_3) \qquad a_3 = \frac{1}{|D|}(x_1 y_2 - x_2 y_1) \tag{9.3.4a}$$

$$b_1 = \frac{1}{|D|}(y_2 - y_3) \qquad b_2 = \frac{1}{|D|}(y_3 - y_1) \qquad b_3 = \frac{1}{|D|}(y_1 - y_2) \tag{9.3.4b}$$

$$c_1 = \frac{1}{|D|}(x_3 - x_2) \qquad c_2 = \frac{1}{|D|}(x_1 - x_3) \qquad c_3 = \frac{1}{|D|}(x_2 - x_1) \tag{9.3.4c}$$

with

$$|D| = \det \begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix} = 2A$$

where A denotes the area of triangle.

Note that the node numbers 1, 2, 3, are assigned counterclockwise in Figure 9.3.1. If assigned clockwise, however, it is seen that the determinant |D| yields $-2A$, twice the negative area. Observe that the fundamental requirements of the interpolation functions for one dimension,

$$\sum_{N=1}^{3} \Phi_N^{(e)} = 1, \qquad 0 \leq \Phi_N^{(e)} \leq 1, \qquad \Phi_N^{(e)}(z_M) = \delta_{NM}$$

are also established in this case in two dimensions.

In view of (9.3.1) and (9.3.4a), we note that

$$\begin{aligned} a_1 &= \frac{1}{2A}(x_2 y_3 - x_3 y_2) \\ &= \frac{1}{2A}\left\{\left(X_2 - \frac{1}{3}\sum_{N=1}^{3} X_N\right)\left(Y_3 - \frac{1}{3}\sum_{N=1}^{3} Y_N\right) - \left(X_3 - \frac{1}{3}\sum_{N=1}^{3} X_N\right)\right. \\ &\quad \left. \times \left(Y_2 - \frac{1}{3}\sum_{N=1}^{3} Y_N\right)\right\} \\ &= \left(\frac{1}{2A}\right)\frac{1}{3}\begin{vmatrix} 1 & X_1 & Y_1 \\ 1 & X_2 & Y_2 \\ 1 & X_3 & Y_3 \end{vmatrix} = \left(\frac{1}{2A}\right)\frac{2A}{3} = \frac{1}{3} \end{aligned}$$

Similarly, we may prove that $a_1 = a_2 = a_3 = 1/3$.

If the variable u is assumed to vary quadratically or cubically, then we require additional nodes along the sides and possibly at the interior. The evaluation of constants would require an inversion of a matrix of the size corresponding to the total number of

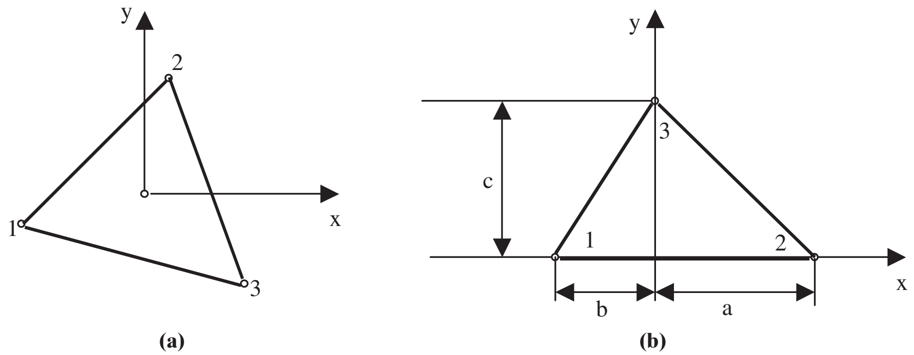


Figure 9.3.2 Integration over the triangular element using cartesian coordinates. (**a**) Integration with origin at centroid; (**b**) Integration with x-axis along one side of triangle.

nodes. An explicit inversion of a large size matrix in terms of nodal coordinate values is difficult, but such complications are avoided if natural coordinates are used.

With the interpolation functions constructed for various degrees of approximations, one generally encounters integration over the spatial domain of the form

$$\iint f\left(\Phi_N^{(e)}\right)dxdy = \iint f(x, y)dxdy$$

If the functions $f(x, y)$ are of higher order, the explicit integration becomes extremely cumbersome. Let us consider an integral

$$P_{rs} = \iint x^r y^s dxdy$$

The limits of this integral must be calculated from the slope of each side of the triangle oriented from the reference cartesian coordinates. The final form of the integral consists of the sum of the integrals performed along all three sides of the triangle. With the origin of the cartesian coordinates at the centroid (Figure 9.3.2a), the following results are obtained:

$$\begin{aligned}
&n = r + s \\
&n = 1 \qquad P_{rs} = \iint xdxdy = \iint ydxdy = 0 \\
&n = 2 \qquad P_{rs} = \frac{A}{12}\left(x_1^r y_1^s + x_2^r y_2^s + x_3^r y_3^s\right) \\
&n = 3 \qquad P_{rs} = \frac{A}{30}\left(x_1^r y_1^s + x_2^r y_2^s + x_3^r y_3^s\right) \\
&n = 4 \qquad P_{rs} = \frac{A}{60}\left(x_1^r y_1^s + x_2^r y_2^s + x_3^r y_3^s\right) \\
&n = 5 \qquad P_{rs} = \frac{2A}{105}\left(x_1^r y_1^s + x_2^r y_2^s + x_3^r y_3^s\right)
\end{aligned} \tag{9.3.5}$$

Triangular Element with Origin on One Side

Integration formulas for $r + s > 5$ are difficult to obtain for the triangular element with the origin at the centroid. An easier, more compact integration formula can be derived from a triangle with the origin on the side between nodes 1 and 2 designated as the x-axis with the y-axis passing through node 3 as shown in Figure 9.3.2b. In this triangle, we obtain the integration formula as follows:

$$
\begin{aligned}
\iint x^r y^s \, dx dy &= \int_0^c \int_{-\frac{b}{c}(c-y)}^{\frac{a}{c}(c-y)} x^r y^s \, dx dy \\
&= \int_0^c \frac{1}{r+1} [x^{r+1}]_{-\frac{b}{c}(c-y)}^{\frac{a}{c}(c-y)} y^s \, dy \\
&= \frac{1}{r+1} \frac{a^{r+1} - (-b)^{r+1}}{c^{r+1}} \int_0^c (c-y)^{r+1} y^s \, dy \\
&\;\vdots \\
&= \frac{r!s!}{(s+r+2)!} \left[a^{r+1} - (-b)^{r+1}\right] c^{s+1}
\end{aligned}
\tag{9.3.6}
$$

The triangular element characterized by (9.3.6) is effective in the solution of fourth order differential equations [Cowper, et al., 1969].

Example 9.3.1 Local Element Stiffness Matrix

Given: Consider the local element stiffness matrix which arises from the two-dimensional Laplace equation $\nabla^2 u = 0$ in the form

$$
K_{NM}^{(e)} = \iint \left(\frac{\partial \Phi_N^{(e)}}{\partial x} \frac{\partial \Phi_M^{(e)}}{\partial x} + \frac{\partial \Phi_N^{(e)}}{\partial y} \frac{\partial \Phi_M^{(e)}}{\partial y} \right) dx dy
$$

Required: Determine the explicit form of the above expression in a linear triangular element using the interpolation functions given by (9.3.4).

Solution: Using the formula given by (9.3.3), we obtain

$$
\frac{\partial \Phi_N^{(e)}}{\partial x} \frac{\partial \Phi_M^{(e)}}{\partial x} = b_N b_M, \qquad \frac{\partial \Phi_N^{(e)}}{\partial y} \frac{\partial \Phi_M^{(e)}}{\partial y} = c_N c_M
$$

Since the area of the triangle is given by

$$
\iint dx dy = A
$$

the local element stiffness matrix becomes

$$
K_{NM}^{(e)} = A(b_N b_M + c_N c_M) = A \begin{bmatrix} b_1^2 + c_1^2 & b_1 b_2 + c_1 c_2 & b_1 b_3 + c_1 c_3 \\ b_2 b_1 + c_2 c_1 & b_2^2 + c_2^2 & b_2 b_3 + c_2 c_3 \\ b_3 b_1 + c_3 c_1 & b_3 b_2 + c_3 c_2 & b_3^2 + c_3^2 \end{bmatrix}
$$

where b_N and c_N are explicitly shown by (9.3.4b) and (9.3.4c), respectively. The cartesian coordinate triangular element is simple to use as long as the interpolation function is linear. It is cumbersome for nonlinear interpolation functions with $n = r + s > 5$ in (9.3.5). Notice that the element characterized by the integration formula (9.3.6) is free from this restriction.

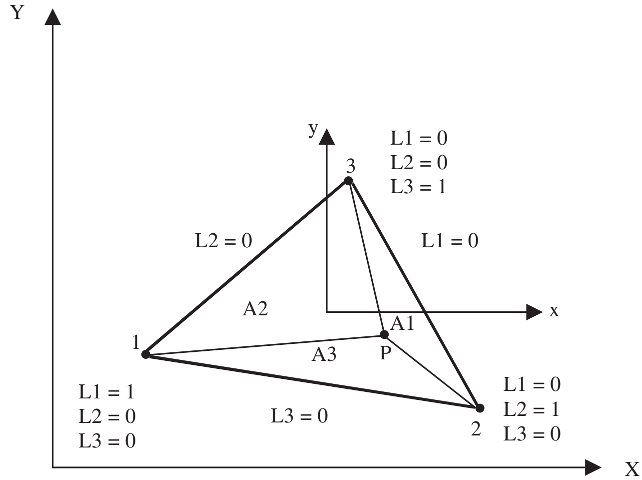


Figure 9.3.3 Natural coordinate triangular element (linear variation).

Natural Coordinate Triangular Element

Consider a triangle with the natural coordinates L_N whose values are zero along the sides and unity on the vertices with a linear variation in between, as shown in Figure 9.3.3. These coordinates are defined as

$$L_i = \frac{A_i}{A}$$

Here $L_1 = A_1/A$, $L_2 = A_2/A$, and $L_3 = A_3/A$ with A_1, A_2, and A_3 being the areas obtained by connecting the three vertices from any point within the triangle such that the total area A is

$$A = A_1 + A_2 + A_3 \tag{9.3.7}$$

$$1 = L_1 + L_2 + L_3 \tag{9.3.8}$$

with A_1 = area (P23), A_2 = area (P31), and A_3 = area (P12). It is now possible to establish a relationship between the cartesian coordinates x and the natural coordinates L_N in the form

$$x = L_1x_1 + L_2x_2 + L_3x_3 \tag{9.3.9a}$$

$$y = L_1y_1 + L_2y_2 + L_3y_3 \tag{9.3.9b}$$

Writing (9.3.8) and (9.3.9) in matrix form, we obtain

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} L_1 \\ L_2 \\ L_3 \end{bmatrix} \tag{9.3.10}$$

We note that the 3×3 matrix on the right-hand side of (9.3.10) is the transpose of the matrix appearing on the right-hand side of (9.3.3). Solving for the natural coordinates, we obtain the interesting result

$$L_1 = \Phi_1^{(e)} \qquad L_2 = \Phi_2^{(e)} \qquad L_3 = \Phi_3^{(e)} \text{ or } L_N = \Phi_N^{(e)} \tag{9.3.11}$$

with $\Phi_N^{(e)}$ identical to (9.3.4).

The natural coordinates as used here for the triangular element are often called the area coordinates or triangular coordinates. Any variable u may now be written as

$$u^{(e)} = L_N u_N^{(e)}$$

It is possible to write (9.3.9a) in the form

$$x = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 \tag{9.3.12}$$

Writing for each node, we obtain

$$x_1 = \alpha_1, \qquad x_2 = \alpha_2, \qquad x_3 = \alpha_3$$

Substituting these into (9.3.12) yields the same expression as (9.3.9a).

Advantages of the natural coordinates can be demonstrated for higher order elements. Notice that, if cartesian coordinates are used for quadratic elements, we require an inversion of the 6×6 matrix corresponding to the three corner nodes plus three side nodes. This difficulty can be avoided in the natural coordinate system. For example, for quadratic approximations, we may write

$$x = \alpha_1 L_1 + \alpha_2 L_2 + \alpha_3 L_3 + \alpha_4 L_1 L_2 + \alpha_5 L_2 L_3 + \alpha_6 L_3 L_1 \tag{9.3.13}$$

Referring to Figure 9.3.4 with three additional nodes installed at midsides of the triangle, we may write (9.3.13) at each corner and midside node,

$$x_1 = \alpha_1 \qquad x_2 = \alpha_2 \qquad x_3 = \alpha_3$$

$$x_4 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_2 + \frac{1}{4}\alpha_4 \qquad x_5 = \frac{1}{2}\alpha_2 + \frac{1}{2}\alpha_3 + \frac{1}{4}\alpha_5 \qquad x_6 = \frac{1}{2}\alpha_1 + \frac{1}{2}\alpha_3 + \frac{1}{4}\alpha_6$$

Solving for the constants and substituting them into (9.3.13) yields

$$x = \Phi_r^{(e)} x_r \quad (r = 1, 2, \ldots, 6) \tag{9.3.14}$$

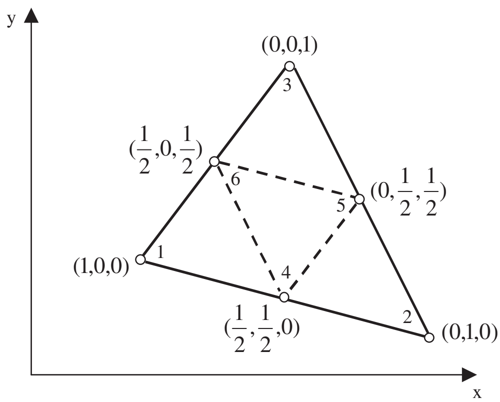


Figure 9.3.4 Natural coordinate triangular element (quadratic variation).

with

$$\Phi_1^{(e)} = (2L_1 - 1)L_1 \qquad \Phi_2^{(e)} = (2L_2 - 1)L_2 \qquad \Phi_3^{(e)} = (2L_3 - 1)L_3$$
$$\Phi_4^{(e)} = 4L_1L_2 \qquad \Phi_5^{(e)} = 4L_2L_3 \qquad \Phi_6^{(e)} = 4L_3L_1 \tag{9.3.15}$$

Similarly, we write

$$y = \Phi_r^{(e)} y_r$$

and consequently, for any variable u

$$u^{(e)} = \Phi_r^{(e)} u_r^{(e)}$$

Using the index notations for a cubic variation, we may proceed similarly as follows (see Figure 9.3.5):

$$x = a_N L_N + b_{NM} L_N L_M + c_{NMQ} L_N L_M L_Q \tag{9.3.16}$$

with N, M, $Q = 1, 2, 3$ and $b_{NM} = 0$ for $N = M$ and $c_{NMQ} = 0$ for $N = M = Q$. Writing (9.3.16) for the three corner nodes, six side nodes (equally spaced), and the interior node, we evaluate the ten constants. Returning to (9.3.16) with these constants, we can now write

$$x = \Phi_{\mathrm{r}}^{(\mathrm{e})} x_r \quad (r = 1, 2, \ldots, 10) \tag{9.3.17}$$

Here, for corner nodes:

$$\Phi_N^{(e)} = \frac{1}{2}(3L_N - 1)(3L_N - 2)L_N \quad (N = 1, 2, 3)$$

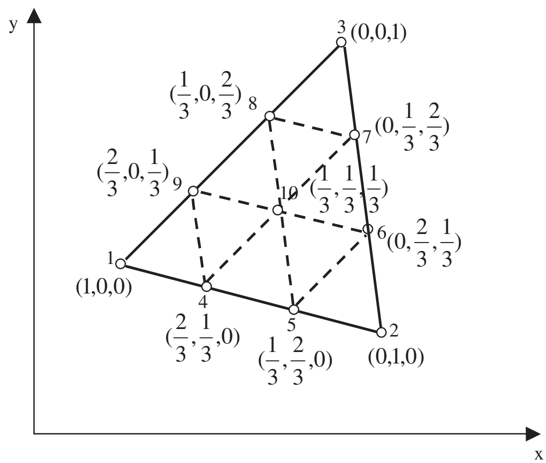


Figure 9.3.5 Natural coordinate triangular element (cubic variation).

for side nodes:

$$\Phi_4^{(e)} = \frac{9}{2} L_1 L_2 (3L_1 - 1) \qquad \Phi_7^{(e)} = \frac{9}{2} L_2 L_3 (3L_3 - 1)$$

$$\Phi_5^{(e)} = \frac{9}{2} L_1 L_2 (3L_2 - 1) \qquad \Phi_8^{(e)} = \frac{9}{2} L_3 L_1 (3L_3 - 1)$$

$$\Phi_6^{(e)} = \frac{9}{2} L_2 L_3 (3L_2 - 1) \qquad \Phi_9^{(e)} = \frac{9}{2} L_3 L_1 (3L_1 - 1)$$

for interior node:

$$\Phi_{10}^{(e)} = 27 L_1 L_2 L_3 \tag{9.3.18}$$

It has been shown that the determination of the interpolation functions for the natural coordinate triangular element can be accomplished quite easily by noting the special geometrical features that make it possible to avoid the inversion.

An additional feature, which should be noted, is the fact that the Lagrange interpolation formula can be used to generalize the procedure. Consider the higher order elements as depicted in Figure 9.3.6. The Lagrange interpolation formula may be

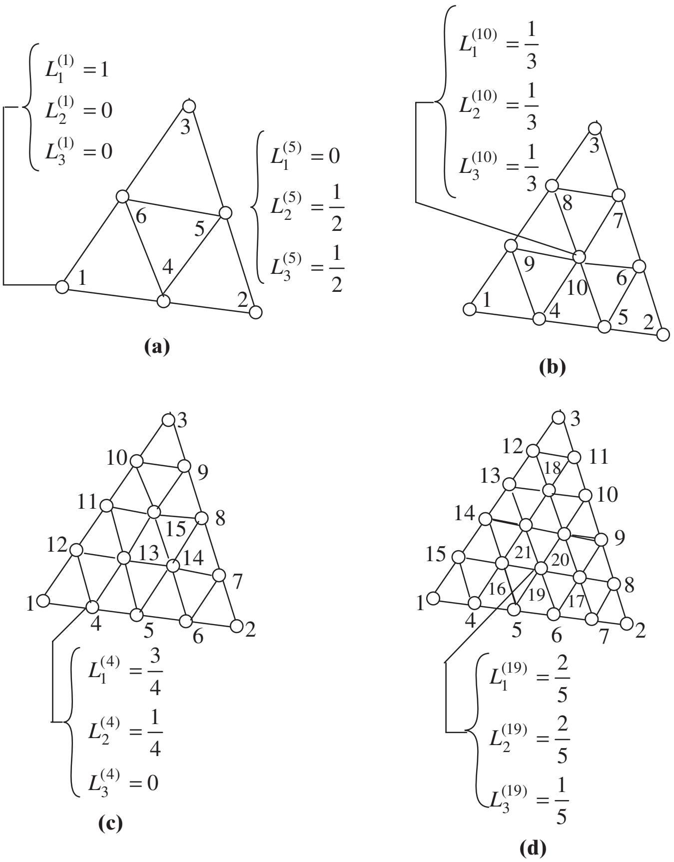


Figure 9.3.6 High order natural coordinate elements. **(a)** Quadratic (m = 2); **(b)** cubic (m = 3); **(c)** quartric (m = 4); **(d)** quintic (m = 5).

transformed to natural coordinates by

$$B^{(r)}(L_N) = \begin{cases} \displaystyle\prod_{s=1}^{s=d} \frac{1}{s}(mL_N - s + 1) & \text{for } d \geq 1 \\ 1 & \text{for } d = 0 \end{cases} \tag{9.3.19}$$

with $d = mL_N^{(r)}$. Here m denotes the degree of approximations and $L_N^{(r)}$ $(N = 1, 2, 3, r = 1, 2, \ldots, n, n =$ total number of nodes) represents the values of area coordinates at each node. The interpolation functions are given by

$$\Phi_r^{(e)} = B^{(r)}(L_1)B^{(r)}(L_2)B^{(r)}(L_3) \tag{9.3.20}$$

To determine $\Phi_1^{(e)}$, we write (for $m = 2$)

$$\Phi_1^{(e)} = B^{(1)}(L_1)B^{(1)}(L_2)B^{(1)}(L_3)$$

$$B^{(1)}(L_1) = (2L_1 - 1 + 1)\frac{1}{2}(2L_1 - 2 + 1)$$

$$B^{(1)}(L_2) = 1$$

$$B^{(1)}(L_3) = 1$$

Thus,

$$\Phi_1^{(e)} = L_1(2L_1 - 1)$$

The interpolation functions corresponding to other nodes may be obtained similarly, and we note that the results are identical to those derived from the polynomial expansions.

The finite element application of the triangular natural coordinates involves integration of a typical form

$$I = \int_A f(L_1, L_2, L_3)dA \tag{9.3.21}$$

Referring to Figure 9.3.7, the differential area dA is given by

$$dA = \frac{(dh)(dH)}{\sin\alpha} = \frac{(hdL_2)(HdL_1)}{\sin\alpha} = 2AdL_1dL_2$$

The limits of integration for L_1 and L_2 are 0 to 1 and 0 to $1 - L_1$, respectively. Thus,

$$I = 2A\int_0^1 \int_0^{1-L_1} f(L_1, L_2, L_3)dL_1dL_2 \tag{9.3.22}$$

where the function f may occur in the form

$$f(L_1, L_2, L_3) = L_1^m L_2^n L_3^p \tag{9.3.23}$$

with m, n, p being the arbitrary powers. In view of (9.3.22) and (9.3.23), we have

$$I = 2A\int_0^1 \int_0^{1-L_1} L_1^m L_2^n L_3^p dL_1dL_2$$

or

$$I = 2A\int_0^1 JL_1^m dL_1 \tag{9.3.24}$$

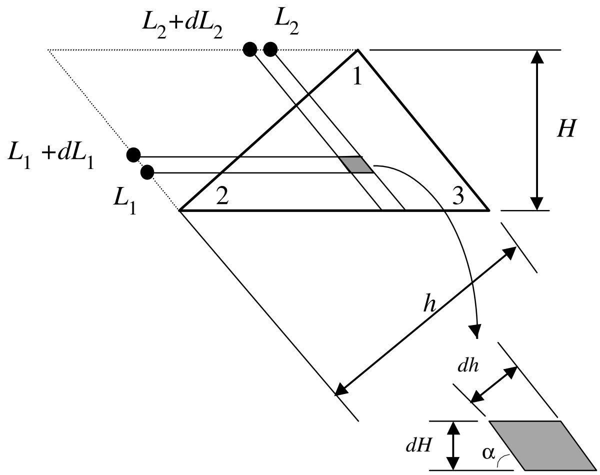


Figure 9.3.7 Geometry for area coordinate integration.

where

$$J = \int_0^{1-L_1} L_2^n L_3^p dL_2 = \int_0^{1-L_1} L_2^n (1 - L_1 - L_2)^p dL_2 \tag{9.3.25}$$

Integrating (9.3.25) by parts gives

$$\begin{aligned} J &= \left[\frac{L_2^{n+1}}{n+1} (1 - L_1 - L_2)^p \right]_0^{1-L_1} + \int_0^{1-L_1} \frac{p L_2^{n+1}}{n+1} (1 - L_1 - L_2)^{p-1} dL_2 \\ &= \frac{p}{n+1} \int_0^{1-L_1} L_2^{n+1} (1 - L_1 - L_2)^{p-1} dL_2 \\ &= \frac{p(p-1)}{(n+1)(n+2)} \int_0^{1-L_1} L_2^{n+2} (1 - L_1 - L_2)^{p-2} dL_2 \end{aligned}$$

or

$$J = \frac{p!n!}{(n+p)!} \int_0^{1-L_1} L_2^{n+p} dL_2 = \frac{p!n!(1 - L_1)^{n+p+1}}{(n+p+1)!} \tag{9.3.26}$$

Substituting (9.3.26) into (9.3.24) and integrating by parts again, we obtain

$$I = \frac{2Am!p!n!}{(n+m+p+2)!} \tag{9.3.27}$$

For example, if $m = 2$, $n = 0$, and $p = 3$, we obtain

$$\int_A L_1^2 L_3^3 dx dy = \frac{2A(2!)(0!)(3!)}{(2+0+3+2)!} = \frac{A}{210}$$

It is clear that the advantage of this element is that higher order elements are generated easily and a simple integration formula is available without limitation to the order of polynomial degrees.

9.3.2 RECTANGULAR ELEMENTS

If the entire domain of study is rectangular, it is more efficient to use rectangular elements rather than triangular elements. Consider a domain with a rectangular mesh. The mesh can also be generated using triangular elements with sides forming diagonals passed through each rectangle. This, of course, results in twice as many elements. That such a system of refined meshes with triangles does not necessarily provide more accurate results is well known. A simple explanation is that the additional node in the rectangular element leads to additional degrees of freedom or constants that may be specified at all nodes of an element, which contributes to more precise or adequate representation of a variable across the element than in the triangular element having an area equal to the rectangular element.

Cartesian Coordinate Elements

To construct interpolation functions for a rectangular element, one might be tempted to use a polynomial expansion in terms of the standard cartesian coordinates.

$$u^{(e)} = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy + \dots \tag{9.3.28}$$

The necessary terms of polynomials corresponding to the side and interior nodes, as well as the corner nodes as related to the degrees of approximations of a variable, must be chosen wisely. Polynomials are often incomplete for the desired inclusion of side and interior nodes. Furthermore, the inverses of coefficient matrices may not exist in some cases. The natural coordinates, on the other hand, usually provide an efficient means of obtaining acceptable forms of the interpolation functions. Lagrange and Hermite polynomials, as discussed in the one-dimensional case, are also frequently used for the rectangular elements. A special element popularly known as an isoparametric element is perhaps the most widely adopted. Among the many desirable features of the isoparametric element is the fact that it may be used not only for the rectangular geometry but also for irregular quadrilateral geometries.

Lagrange and Hermite Elements

The advantage of using Lagrange or Hermite elements for a rectangular element is that desired interpolation functions are constructed simply by a tensor product of the one-dimensional counterparts for the x and y directions, respectively.

Consider the Lagrange interpolations in two dimensions, as shown in Figure 9.3.8. For a linear variation of u (Figure 9.3.8a), we write

$$u^{(e)} = \Phi_N^{(e)} u_N^{(e)} \quad (N = 1, 2, 3, 4) \tag{9.3.29}$$

with

$$\Phi_1^{(e)} = L_1^{(x)} L_1^{(y)}, \qquad \Phi_2^{(e)} = L_2^{(x)} L_1^{(y)}, \qquad \Phi_3^{(e)} = L_2^{(x)} L_2^{(y)} \quad \text{and} \quad \Phi_4^{(e)} = L_1^{(x)} L_2^{(y)}$$

where

$$L_1^{(x)} = \frac{1}{2}(1-\xi), \qquad L_2^{(x)} = \frac{1}{2}(1+\xi), \qquad L_1^{(y)} = \frac{1}{2}(1-\eta),$$
$$L_2^{(y)} = \frac{1}{2}(1+\eta), \qquad \xi = \frac{2x}{a}, \qquad \eta = \frac{2y}{b}$$

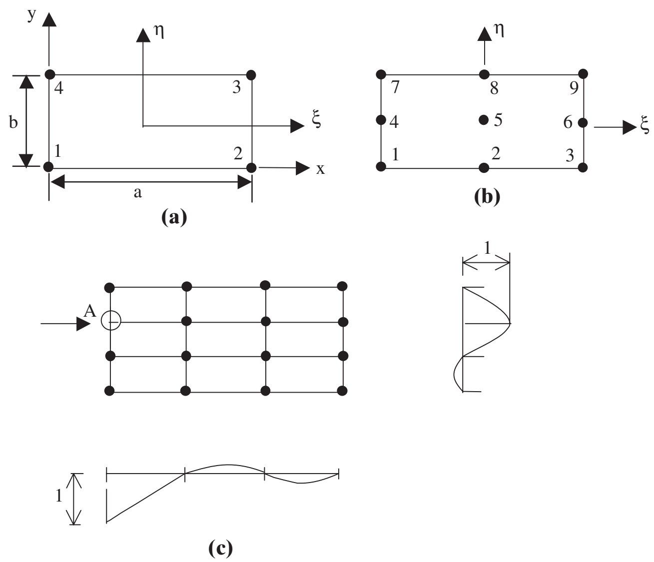


Figure 9.3.8 Lagrange interpolation functions: **(a)** linear, **(b)** quadratic, **(c)** cubic (variations of functions along the line through A).

Interpolations of quadratic and cubic variations can be constructed in the same way (see Figure 9.3.8 b,c).

The Hermite polynomials may be applied similarly to the rectangular element as the Lagrange polynomials. For bicubic Hermite polynomials, we have (Figure 9.3.9):

$$u^{(e)} = \Phi_r^{(e)} Q_r \quad (r = 1, 2, \ldots 16) \tag{9.3.30a}$$

with

$$\begin{array}{llll}
\Phi_1^{(e)} = H^0_{1(x)} H^0_{1(y)} & \Phi_5^{(e)} = H^0_{2(x)} H^0_{1(y)} & \Phi_9^{(e)} = H^0_{1(x)} H^0_{2(y)} & \Phi_{13}^{(e)} = H^0_{2(x)} H^0_{2(y)} \\
\Phi_2^{(e)} = H^1_{1(x)} H^0_{1(y)} & \Phi_6^{(e)} = H^1_{2(x)} H^0_{1(y)} & \Phi_{10}^{(e)} = H^1_{1(x)} H^0_{2(y)} & \Phi_{14}^{(e)} = H^1_{2(x)} H^0_{2(y)} \\
\Phi_3^{(e)} = H^0_{1(x)} H^1_{1(y)} & \Phi_7^{(e)} = H^0_{2(x)} H^1_{1(y)} & \Phi_{11}^{(e)} = H^0_{1(x)} H^1_{2(y)} & \Phi_{15}^{(e)} = H^0_{2(x)} H^1_{2(y)} \\
\Phi_4^{(e)} = H^1_{1(x)} H^1_{1(y)} & \Phi_8^{(e)} = H^1_{2(x)} H^1_{1(y)} & \Phi_{12}^{(e)} = H^1_{1(x)} H^1_{2(y)} & \Phi_{16}^{(e)} = H^1_{2(x)} H^1_{2(y)}
\end{array} \tag{9.3.30b}$$

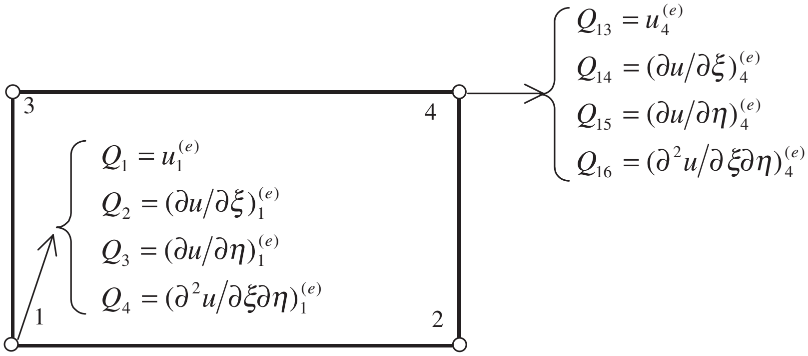


Figure 9.3.9 Hermite bicubic rectangular element.

and

$$
\begin{array}{llll}
Q_1 = u_1^{(e)} & Q_5 = u_2^{(e)} & Q_9 = u_3^{(e)} & Q_{13} = u_4^{(e)} \\
Q_2 = \left(\dfrac{\partial u}{\partial \xi}\right)_1^{(e)} & Q_6 = \left(\dfrac{\partial u}{\partial \xi}\right)_2^{(e)} & Q_{10} = \left(\dfrac{\partial u}{\partial \xi}\right)_3^{(e)} & Q_{14} = \left(\dfrac{\partial u}{\partial \xi}\right)_4^{(e)} \\
Q_3 = \left(\dfrac{\partial u}{\partial \eta}\right)_1^{(e)} & Q_7 = \left(\dfrac{\partial u}{\partial \eta}\right)_2^{(e)} & Q_{11} = \left(\dfrac{\partial u}{\partial \eta}\right)_3^{(e)} & Q_{15} = \left(\dfrac{\partial u}{\partial \eta}\right)_4^{(e)} \\
Q_4 = \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\right)_1^{(e)} & Q_8 = \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\right)_2^{(e)} & Q_{12} = \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\right)_3^{(e)} & Q_{16} = \left(\dfrac{\partial^2 u}{\partial \eta \partial \xi}\right)_4^{(e)}
\end{array}
\tag{9.3.30c}
$$

$$
\begin{array}{ll}
H^0_{1(x)} = 1 - 3\xi^2 + 2\xi^3 & H^0_{1(y)} = 1 - 3\eta^2 + 2\eta^3 \\
H^0_{2(x)} = 3\xi^2 - 2\xi^3 & H^0_{2(y)} = 3\eta^2 - 2\eta^3 \\
H^1_{1(x)} = \xi - 2\xi^2 + \xi^3 & H^1_{1(y)} = \eta - 2\eta^2 + \eta^3 \\
H^1_{2(x)} = \xi^3 - \xi^2 & H^1_{2(y)} = \eta^3 - \eta^2
\end{array}
\tag{9.3.30d}
$$

Note that, because of the combinations of the Hermite polynomials for both x and y directions, the mixed second derivatives must be included as nodal generalized coordinates. Higher order Hermite polynomials may be constructed similarly using (9.2.14).

A similar approach can be used to generate three-dimensional elements $\Phi_1^{(e)} = L_1^{(x)} L_2^{(y)} L_3^{(z)}$, etc. for Lagrange elements and similarly for Hermite elements. However, it should be noted that for nonorthogonal elements (arbitrary quadrilateral and hexahedral), appropriate coordinate transformation (geometrical Jacobian) will be required as discussed in the following section.

9.3.3 QUADRILATERAL ISOPARAMETRIC ELEMENTS

The isoparametric element was first studied by Zienkiewicz and his associates [see Zienkiewicz, 1971]. The name "isoparametric" derives from the fact that the "same" parametric function which describes the geometry may be used for interpolating spatial variations of a variable within an element. The isoparametric element utilizes a nondimensionalized coordinate and therefore is one of the natural coordinate elements.

Consider an arbitrarily shaped quadrilateral element as shown in Figure 9.3.10. The isoparametric coordinates (ξ, η) whose values range from 0 to ± 1 are established at the centroid of the element. The reference cartesian coordinates (x, y) are related to

$$x, y = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta \tag{9.3.31}$$

for the two-dimensional linear element in Figure 9.3.10. A linear variation of a variable u may also be written as

$$u^{(e)} = \alpha_1 + \alpha_2 \xi + \alpha_3 \eta + \alpha_4 \xi \eta \tag{9.3.32}$$

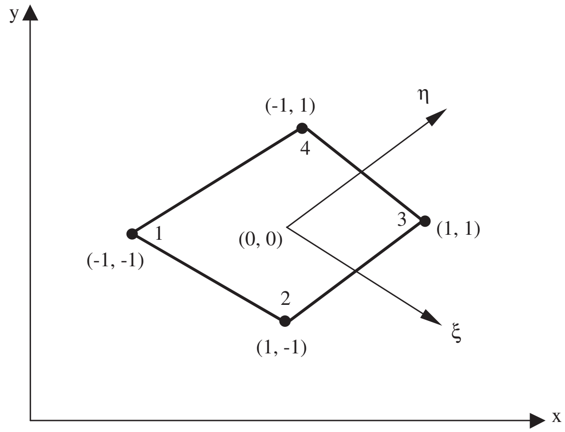


Figure 9.3.10 Quadrilateral isoparametric element (linear variation).

Writing (9.3.31) in terms of nodal values yields

$$
\begin{aligned}
x_1 &= \alpha_1 + \alpha_2(-1) + \alpha_3(-1) + \alpha_4(-1)(-1) \\
x_2 &= \alpha_1 + \alpha_2(1) + \alpha_3(-1) + \alpha_4(1)(-1) \\
x_3 &= \alpha_1 + \alpha_2(1) + \alpha_3(1) + \alpha_4(1)(1) \\
x_4 &= \alpha_1 + \alpha_2(-1) + \alpha_3(1) + \alpha_4(-1)(1)
\end{aligned}
\tag{9.3.33a}
$$

In a matrix form, we may rewrite (9.3.33a) as

$$[x] = [C][\alpha] \tag{9.3.33b}$$

Here the coefficient matrix [C] is given by

$$[C] = \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Thus,

$$[\alpha] = [C]^{-1}[x] \tag{9.3.34}$$

with

$$[C]^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

Substituting (9.3.34) into (9.3.31) yields

$$x_i = \Phi_N^{(e)} x_{Ni} \tag{9.3.35}$$

Here $\Phi_N^{(e)}$ is called the isoparametric function and has the form

$$\Phi_N^{(e)} = \frac{1}{4}(1 + \xi_{N1}\xi_1)(1 + \xi_{N2}\xi_2) \tag{9.3.36}$$

Substituting the nodal values of ξ_{N1} and ξ_{N2} into (9.3.36) gives, with $\xi_1 = \xi$, $\xi_2 = \eta$,

$$\begin{aligned}
\Phi_1^{(e)} &= \frac{1}{4}(1-\xi)(1-\eta) \\
\Phi_2^{(e)} &= \frac{1}{4}(1+\xi)(1-\eta) \\
\Phi_3^{(e)} &= \frac{1}{4}(1+\xi)(1+\eta) \\
\Phi_4^{(e)} &= \frac{1}{4}(1-\xi)(1+\eta)
\end{aligned} \tag{9.3.37}$$

It is interesting to note that the interpolation functions derived in (9.3.37) can be obtained by tensor products of the Lagrange polynomials with the origin at the centroid from (9.3.29) for the case in Figure (9.3.8a).

The quadratic element requires midside nodes as shown in Figure 9.3.11. Thus, we may approximate x or y in the form (see Figure 9.1.6)

$$x, y = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta + \alpha_5\xi^2 + \alpha_6\eta^2 + \alpha_7\xi^2\eta + \alpha_8\xi\eta^2 \tag{9.3.38}$$

A similar procedure as in the linear element may be used to determine $[C]$ and $[C]^{-1}$, and we obtain

at corner nodes:

$$\Phi_N^{(e)}(\xi_i) = \frac{1}{4}(1+\xi_{N1}\xi_1)(1+\xi_{N2}\xi_2)(\xi_{N1}\xi_1 + \xi_{N2}\xi_2 - 1) \tag{9.3.39}$$

at midside nodes:

$$\begin{aligned}
\Phi_N^{(e)}(\xi_i) &= \frac{1}{2}(1-\xi_1^2)(1+\xi_{N2}\xi_2) \quad \text{for } \xi_{N1} = 0 \\
\Phi_N^{(e)}(\xi_i) &= \frac{1}{2}(1+\xi_{N1}\xi_1)(1-\xi_2^2) \quad \text{for } \xi_{N2} = 0
\end{aligned} \tag{9.3.40}$$

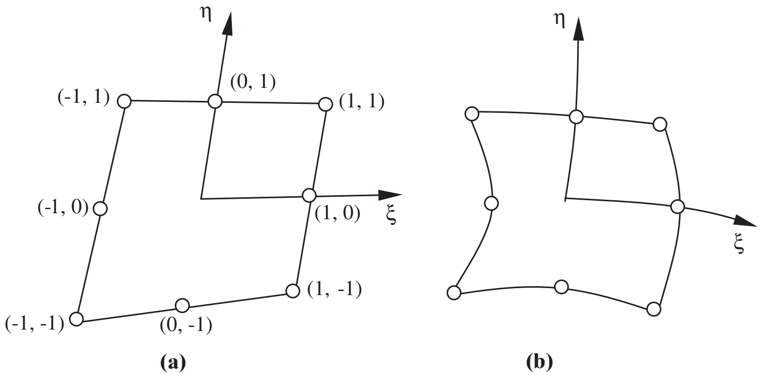


Figure 9.3.11 Quadrilateral isoparametric element (quadratic variation): **(a)** straight edges, **(b)** curved edges.

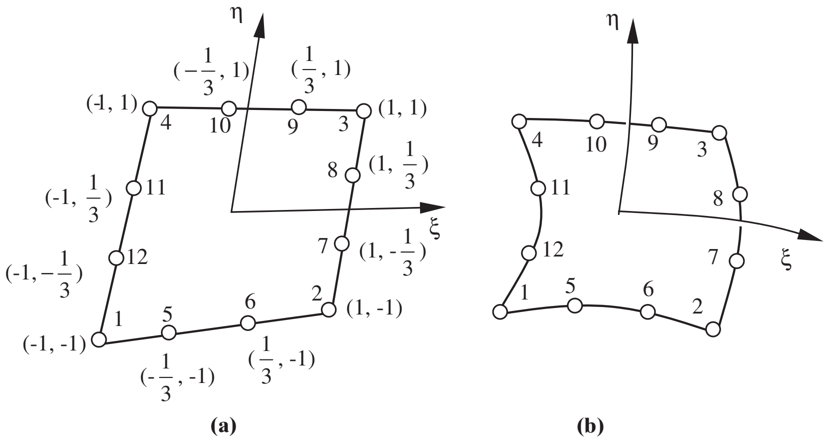


Figure 9.3.12 Quadrilateral isoparametric element (cubic variation): (**a**) straight edges; (**b**) curved edges.

For a cubic element as shown in Figure 9.3.12, we have (see Figure 9.1.6)

$$x, y = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\xi\eta + \alpha_5\xi^2 + \alpha_6\eta^2 + \alpha_7\xi^2\eta + \alpha_8\xi\eta^2 + \alpha_9\xi^3 + \alpha_{10}\eta^3 + \alpha_{11}\xi^3\eta + \alpha_{12}\xi\eta^3 \tag{9.3.41}$$

where we notice that the $x^2y^2(\xi^2\eta^2$ here) term is omitted from the complete cubic expansion (Figure 9.1.6) in order to match the number of nodes chosen here (12 terms instead of 13 terms)

at corner nodes:

$$\Phi_N^{(e)}(\xi_i) = \frac{1}{32}(1 + \xi_{N1}\xi_1)(1 + \xi_{N2}\xi_2)[9(\xi_1^2 + \xi_2^2) - 10] \tag{9.3.42a}$$

at side nodes:

$$\Phi_N^{(e)}(\xi_i) = \frac{9}{32}(1 + \xi_{N1}\xi_1)(1 - \xi_2^2)(1 + 9\xi_{N2}\xi_2) \quad \text{for } \xi_{N1} = \pm 1 \text{ and } \xi_{N2} = \pm\tfrac{1}{3}$$
$$\Phi_N^{(e)}(\xi_i) = \frac{9}{32}(1 + \xi_{N2}\xi_2)(1 - \xi_1^2)(1 + 9\xi_{N1}\xi_1) \quad \text{for } \xi_{N2} = \pm 1 \text{ and } \xi_{N1} = \pm\tfrac{1}{3} \tag{9.3.42b}$$

It should be remarked that for higher order isoparametric elements, Lagrange polynomials can still be used without interior nodes but with side constraints imposed.

In engineering applications, we are concerned with a derivative and the integration of quantity associated with a variable with respect to the cartesian reference coordinates. Since the variable is represented in terms of the nondimensionalized isoparametric coordinates, we require a transformation between the two coordinate systems. Consider a quantity given by

$$\iint \frac{\partial}{\partial x_i} f(\xi, \eta)\, dx\, dy \tag{9.3.43}$$

with $\xi = \xi_1, \eta = \xi_2, x = x_1$, and $y = x_2$. From the chain rule of calculus, we write

$$\begin{aligned}\frac{\partial f}{\partial \xi} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \xi}\\ \frac{\partial f}{\partial \eta} &= \frac{\partial f}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial \eta}\end{aligned} \tag{9.3.44}$$

or in a matrix form

$$\begin{bmatrix}\dfrac{\partial f}{\partial \xi}\\[2ex] \dfrac{\partial f}{\partial \eta}\end{bmatrix} = \begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix}\begin{bmatrix}\dfrac{\partial f}{\partial x}\\[2ex] \dfrac{\partial f}{\partial y}\end{bmatrix}$$

Thus,

$$\begin{bmatrix}\dfrac{\partial f}{\partial x}\\[2ex] \dfrac{\partial f}{\partial y}\end{bmatrix} = [J]^{-1}\begin{bmatrix}\dfrac{\partial f}{\partial \xi}\\[2ex] \dfrac{\partial f}{\partial \eta}\end{bmatrix} \tag{9.3.45}$$

where J is called the Jacobian given by

$$[J] = \begin{bmatrix}\dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi}\\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta}\end{bmatrix} \tag{9.3.46}$$

Here the derivatives $\partial f/\partial x$ or $\partial f/\partial y$ are determined from the inverse of the Jacobian and the derivatives $\partial f/\partial \xi$ and $\partial f/\partial \eta$. The integration over the domain referenced to the cartesian coordinates must be changed to the domain now referenced to the isoparametric coordinates

$$\iint dxdy = \int_{-1}^{1}\int_{-1}^{1} |J| d\xi d\eta \tag{9.3.47}$$

To prove (9.3.47), we consider the two coordinate systems shown in Figure 9.3.13. The directions of the cartesian coordinates and the arbitrary nonorthogonal (possibly curvilinear) isoparametric coordinates are given by the unit vectors $\mathbf{i}_1, \mathbf{i}_2$, and the tangent vectors $\mathbf{g}_1, \mathbf{g}_2$, respectively, related by

$$\begin{aligned}\mathbf{g}_1 &= \frac{\partial x}{\partial \xi}\mathbf{i}_1 + \frac{\partial y}{\partial \xi}\mathbf{i}_2\\ \mathbf{g}_2 &= \frac{\partial x}{\partial \eta}\mathbf{i}_1 + \frac{\partial y}{\partial \eta}\mathbf{i}_2\end{aligned}$$

The differential area (shaded) is

$$dx\,\mathbf{i}_1 \times dy\,\mathbf{i}_2 = dxdy\,\mathbf{i}_3 = \mathbf{g}_1 d\xi \times \mathbf{g}_2 d\eta = \begin{vmatrix}\mathbf{i}_1 & \mathbf{i}_2 & \mathbf{i}_3\\[1ex] \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} & 0\\[2ex] \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} & 0\end{vmatrix} d\xi d\eta$$

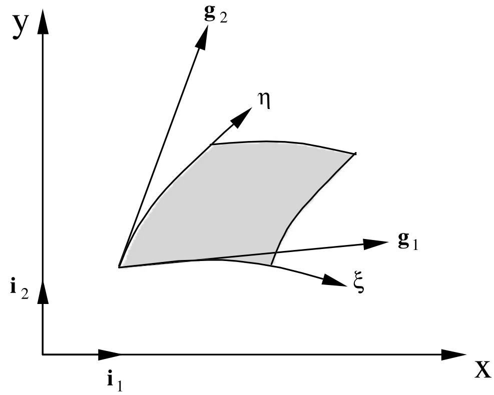


Figure 9.3.13 Coordinate transformation.

or

$$dxdy\,\mathbf{i}_3 = |J|d\xi d\eta\,\mathbf{i}_3$$

with

$$|J| = \begin{vmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{vmatrix}$$

Thus, we obtain the relations

$$dxdy = |J|d\xi d\eta \tag{9.3.48}$$

and

$$\iint \frac{\partial f}{\partial x}dxdy = \int_{-1}^{1}\int_{-1}^{1}\left(\overline{J}_{11}\frac{\partial f}{\partial \xi} + \overline{J}_{12}\frac{\partial f}{\partial \eta}\right)|J|d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1} g_x(\xi,\eta)d\xi d\eta \tag{9.3.49}$$

$$\iint \frac{\partial f}{\partial y}dxdy = \int_{-1}^{1}\int_{-1}^{1}\left(\overline{J}_{21}\frac{\partial f}{\partial \xi} + \overline{J}_{22}\frac{\partial f}{\partial \eta}\right)|J|d\xi d\eta = \int_{-1}^{1}\int_{-1}^{1} g_y(\xi,\eta)d\xi d\eta \tag{9.3.50}$$

where $\overline{J}_{11}$, $\overline{J}_{12}$, $\overline{J}_{21}$, and $\overline{J}_{22}$ are the components of the inverted Jacobian matrix (9.3.46).

The integration (9.3.49) may be performed most efficiently by means of the Gaussian quadrature [see Hildebrand, 1956]. For a one-dimensional case, we may write

$$\int_{-1}^{1} f(\xi)d\xi = \sum_{j=1}^{n} w_j f(\xi_j)$$

or, when extended to a tensor product in two dimensions, we write

$$\int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)d\xi d\eta = \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k f(\xi_j,\eta_k)$$

where w_j and w_k are the weight coefficients, and $f(\xi_k)$ and $f(\xi_j,\eta_k)$ denote the abscissae representing the values of the functions $f(\xi)$ and $f(\xi,\eta)$ corresponding to the n Gaussian points. The weight coefficients and abscissae for the first ten Gaussian points

Table 9.3.1 Abscissae and Weight Coefficients of the Gaussian Quadrature Formula

	Weight Coefficient	Abscissae
N	W_k	$\pm\xi_k, \pm\eta_k$
2	1.00000 00000	0.57735 02691
3	0.55555 55555	0.77459 66692
	0.88888 88888	0.00000 00000
4	0.34785 48451	0.86113 63115
	0.65214 51548	0.33998 10435
5	0.23692 68850	0.90617 98459
	0.47862 86704	0.53846 93101
	0.56888 88888	0.00000 00000
6	0.17132 44923	0.93246 95142
	0.36076 15730	0.66120 93864
	0.46791 39345	0.23861 91860
7	0.12948 49661	0.94910 79123
	0.27970 53914	0.74153 11855
	0.38183 00505	0.40584 51513
	0.41795 91836	0.00000 00000
8	0.10122 85362	0.96028 98564
	0.22238 10344	0.79666 64774
	0.31370 66458	0.52553 24099
	0.36268 37833	0.18343 46424
9	0.08127 43883	0.96816 02395
	0.18064 81606	0.83603 11073
	0.26061 06964	0.61336 14327
	0.31234 70770	0.32425 34234
	0.33023 93550	0.00000 00000
10	0.06667 13443	0.97390 65285
	0.14945 13491	0.86506 33666
	0.21908 63625	0.67940 95682
	0.26926 67193	0.43339 53941
	0.29552 42247	0.14887 43389

are shown in Table 9.3.1. In general, accuracy of integration increases with an increase of Gaussian points, but it can be shown that only a very few Gaussian points may lead to an acceptable accuracy. The basic idea of Gaussian quadrature is shown in Appendix B.

The Gaussian quadrature numerical integration may be easily extended to the three-dimensional element. Extension of the Gaussian quadrature integration to the triangular or tetrahedral elements are also possible with some modification of the procedure.

Example 9.3.2 Stiffness Matrix of an Isoparametric Element

Given:

$$K_{NM}^{(e)} = \iint \left(\frac{\partial \Phi_N^{(e)}}{\partial x} \frac{\partial \Phi_M^{(e)}}{\partial x} + \frac{\partial \Phi_N^{(e)}}{\partial y} \frac{\partial \Phi_M^{(e)}}{\partial y} \right) dx dy$$

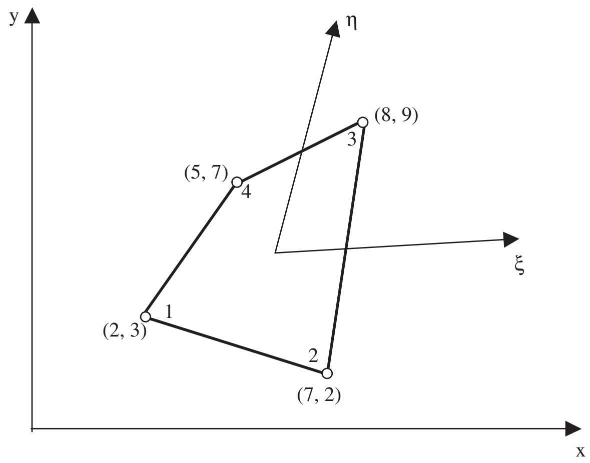


Figure E9.3.2 Geometry for Example 9.3.2.

Required: Work out the detailed algebra necessary for computer integration using the Gaussian quadrature. Compute the integral with 3, 4, 5 Gaussian points (Figure E9.3.2).

Solution:

$$
\begin{aligned}
K_{NM}^{(e)} &= \iint \left(\frac{\partial \Phi_N^{(e)}}{\partial x} \frac{\partial \Phi_M^{(e)}}{\partial x} + \frac{\partial \Phi_N^{(e)}}{\partial y} \frac{\partial \Phi_M^{(e)}}{\partial y} \right) dxdy \\
&= \int_{-1}^{1} \int_{-1}^{1} \left(\frac{\partial \Phi_N^{(e)}}{\partial x} \frac{\partial \Phi_M^{(e)}}{\partial x} + \frac{\partial \Phi_N^{(e)}}{\partial y} \frac{\partial \Phi_M^{(e)}}{\partial y} \right) |J| d\xi d\eta \\
&= \int_{-1}^{1} \int_{-1}^{1} k_{NM}(\xi, \eta) d\xi d\eta \\
&= \sum_{j=1}^{n} \sum_{k=1}^{n} w_j w_k k_{NM}(\xi_j, \eta_k)
\end{aligned}
$$

where

$$
\begin{aligned}
k_{NM}(\xi, \eta) = \Bigg[& \left(\bar{J}_{11} \frac{\partial \Phi_N^{(e)}}{\partial \xi} + \bar{J}_{12} \frac{\partial \Phi_N^{(e)}}{\partial \eta} \right) \left(\bar{J}_{11} \frac{\partial \Phi_M^{(e)}}{\partial \xi} + \bar{J}_{12} \frac{\partial \Phi_M^{(e)}}{\partial \eta} \right) \\
& + \left(\bar{J}_{21} \frac{\partial \Phi_N^{(e)}}{\partial \xi} + \bar{J}_{22} \frac{\partial \Phi_N^{(e)}}{\partial \eta} \right) \left(\bar{J}_{21} \frac{\partial \Phi_M^{(e)}}{\partial \xi} + \bar{J}_{22} \frac{\partial \Phi_M^{(e)}}{\partial \eta} \right) \Bigg] |J|
\end{aligned}
$$

with

$$
\bar{J}_{11} = \frac{1}{|J|} \frac{\partial y}{\partial \eta}, \qquad \bar{J}_{12} = -\frac{1}{|J|} \frac{\partial y}{\partial \xi}, \qquad \bar{J}_{21} = -\frac{1}{|J|} \frac{\partial x}{\partial \eta}, \qquad \bar{J}_{22} = \frac{1}{|J|} \frac{\partial x}{\partial \xi}
$$

$$
|J| = \frac{\partial x}{\partial \xi} \frac{\partial y}{\partial \eta} - \frac{\partial y}{\partial \xi} \frac{\partial x}{\partial \eta}
$$

with

$$\Phi_N^{(e)} = \frac{1}{4}(1 + \xi_{N1}\xi_1)(1 + \xi_{N2}\xi_2)$$

$$x_i = \Phi_N^{(e)} x_{Ni} = \frac{1}{4}(a_i + b_i\xi_1 + c_i\xi_2 + d_i\xi_1\xi_2)$$

$$a_i = x_{1i} + x_{2i} + x_{3i} + x_{4i}, \qquad b_i = -x_{1i} + x_{2i} + x_{3i} - x_{4i}$$

$$c_i = -x_{1i} - x_{2i} + x_{3i} + x_{4i}, \qquad d_i = x_{1i} - x_{2i} + x_{3i} - x_{4i}$$

$$\frac{\partial \Phi_N^{(e)}}{\partial x_i} = (J_{ik})^{-1} \frac{\partial \Phi_N^{(e)}}{\partial \xi_k} = \frac{1}{8|J|}\left(A_{Ni} + B_{Ni}^k \xi_k\right), \quad (i, k = 1, 2)$$

with

$$A_{11} = x_{22} - x_{42}, \qquad B_{11}^1 = x_{42} - x_{32}, \qquad B_{11}^2 = x_{32} - x_{22}$$

$$A_{21} = x_{32} - x_{12}, \qquad B_{21}^1 = x_{32} - x_{42}, \qquad B_{21}^2 = x_{12} - x_{42}$$

$$A_{31} = x_{42} - x_{22}, \qquad B_{31}^1 = x_{12} - x_{22}, \qquad B_{31}^2 = x_{42} - x_{12}$$

$$A_{41} = x_{12} - x_{32}, \qquad B_{41}^1 = x_{22} - x_{12}, \qquad B_{41}^2 = x_{22} - x_{32}$$

$$A_{12} = x_{41} - x_{21}, \qquad B_{12}^1 = x_{31} - x_{41}, \qquad B_{12}^2 = x_{21} - x_{31}$$

$$A_{22} = x_{11} - x_{31}, \qquad B_{22}^1 = x_{41} - x_{31}, \qquad B_{22}^2 = x_{41} - x_{11}$$

$$A_{32} = x_{21} - x_{41}, \qquad B_{32}^1 = x_{21} - x_{11}, \qquad B_{32}^2 = x_{11} - x_{41}$$

$$A_{42} = x_{31} - x_{11}, \qquad B_{42}^1 = x_{11} - x_{21}, \qquad B_{42}^2 = x_{31} - x_{21}$$

$$|J| = \frac{\partial x_1}{\partial \xi_1}\frac{\partial x_2}{\partial \xi_2} - \frac{\partial x_2}{\partial \xi_1}\frac{\partial x_1}{\partial \xi_2} = \frac{1}{8}(\alpha_0 + \alpha_1\xi_1 + \alpha_2\xi_2)$$

$$\alpha_0 = (x_{41} - x_{21})(x_{12} - x_{32}) - (x_{11} - x_{31})(x_{42} - x_{22})$$

$$\alpha_1 = (x_{31} - x_{41})(x_{12} - x_{22}) - (x_{11} - x_{21})(x_{32} - x_{42})$$

$$\alpha_2 = (x_{41} - x_{11})(x_{22} - x_{32}) - (x_{21} - x_{31})(x_{42} - x_{12})$$

where

$$x_{22} - x_{42} = y_2 - y_4, \qquad x_{11} - x_{31} = x_1 - x_3, \text{ etc.}$$

$$\frac{\partial \Phi_N^{(e)}}{\partial x_1} = \frac{1}{8|J|}\left(A_{N1} + B_{N1}^k \xi_k\right) = C_{N1}, \qquad \frac{\partial \Phi_N^{(e)}}{\partial x_2} = \frac{1}{8|J|}\left(A_{N2} + B_{N2}^k \xi_k\right) = C_{N2}$$

If we chose $n = 3$, then from Table 9.3.1 we have

$$w_1 = 0.55555555, \qquad w_2 = 0.88888888, \quad w_3 = 0.55555555$$

$$(\xi_1, \eta_1) = -0.77459666, \qquad (\xi_2, \eta_2) = 0.0, \qquad (\xi_3, \eta_3) = 0.77459666$$

We are now prepared to calculate

$$K_{NM}^{(e)} = \sum_{i=1}^{n}\sum_{j=1}^{n} w_i w_j k_{NM}(\xi_i, \eta_j)$$

where

$$k_{NM}(\xi_i, \eta_j) = (C_{N1}C_{M1} + C_{N2}C_{M2})|J|$$

Thus,

$$K_{NM}^{(e)} = \begin{bmatrix} 0.5449 & -0.2773 & -0.1035 & -0.1640 \\ -0.2773 & 0.8771 & 0.1380 & -0.7377 \\ -0.1035 & 0.1380 & 0.6378 & -0.6723 \\ -0.1640 & -0.7377 & -0.6723 & 1.5740 \end{bmatrix}$$

Similarly,
for $n = 4$

$$K_{NM}^{(e)} = \begin{bmatrix} 0.5457 & -0.2776 & -0.1026 & -0.1655 \\ -0.2776 & 0.8771 & 0.1377 & -0.7372 \\ -0.1026 & 0.1377 & 0.6390 & -0.6741 \\ -0.1655 & -0.7372 & -0.6741 & 1.5768 \end{bmatrix}$$

for $n = 5$

$$K_{NM}^{(e)} = \begin{bmatrix} 0.5457 & -0.2776 & -0.1025 & -0.1656 \\ -0.2776 & 0.8771 & 0.1376 & -0.7372 \\ -0.1025 & 0.1376 & 0.6391 & -0.6742 \\ -0.1656 & -0.7372 & -0.6742 & 1.5770 \end{bmatrix}$$

We notice that an asymptotic convergence is evident as the Gaussian integration point n increases from 3 to 5.

Example 9.3.3 Transition from Linear to Quadratic Element

Figure E9.3.3 presents irregular elements with transition from a linear element to a quadratic element. In this case, side (1-5-2) is quadratic for the element ($e = 1$). Element 2 is fully quadratic, whereas element 1 is partially linear and partially quadratic. Interpolation functions for element 1 can be derived by constructing tensor products as follows:

$$\Phi_1^{(e)} = L_1^{(2)}(\xi)L_1^{(1)}(\eta) = \frac{1}{4}\xi(\xi - 1)(1 - \eta)$$

$$\Phi_2^{(e)} = L_3^{(2)}(\xi)L_1^{(1)}(\eta) = \frac{1}{4}\xi(\xi + 1)(1 - \eta)$$

$$\Phi_3^{(e)} = L_2^{(1)}(\xi)L_2^{(1)}(\eta) = \frac{1}{4}(1 + \xi)(1 + \eta)$$

$$\Phi_4^{(e)} = L_1^{(1)}(\xi)L_2^{(1)}(\eta) = \frac{1}{4}(1 - \xi)(1 + \eta)$$

$$\Phi_5^{(e)} = L_2^{(2)}(\xi)L_1^{(1)}(\eta) = \frac{1}{2}(1 - \xi^2)(1 - \eta)$$

where the superscripts (1) and (2) for Lagrange polynomials denote linear and quadratic functions, respectively.

Example 9.3.4 Irregular Elements with an Irregular Node

Consider the irregular elements that may occur in the process of refinements as seen in Figure E9.3.4. All elements are to be approximated linearly. Interpolation functions

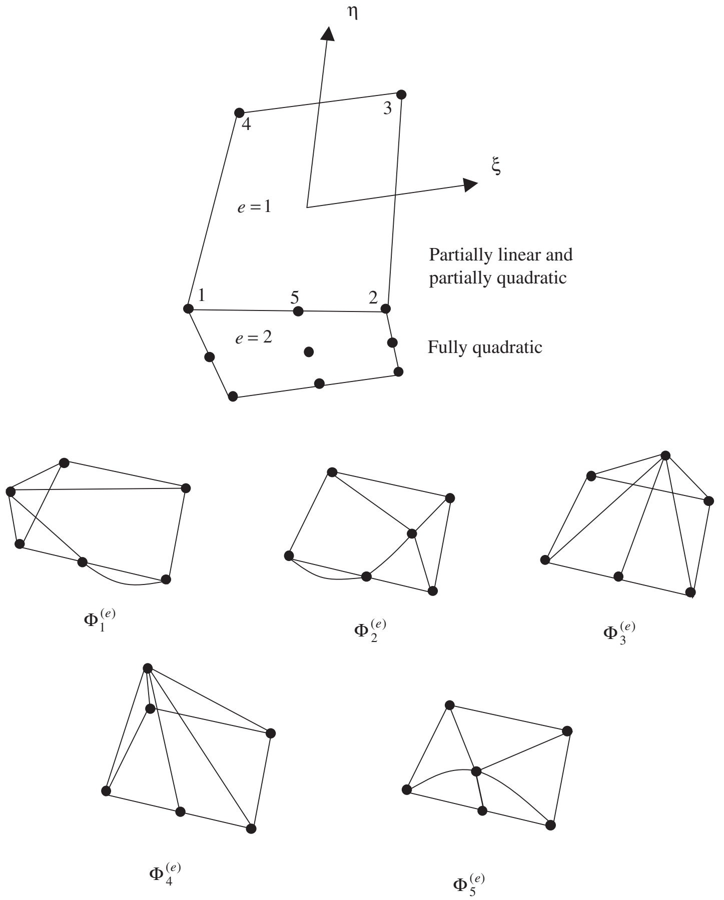


Figure E9.3.3 Five-node quadrilateral element, transition from linear to quadratic element.

are as follows:

$$\Phi_1^{(e)} = \begin{cases} \frac{1}{4}(1-\xi)(1-\eta) & \eta > -1 \\ -\xi & \eta = -1, \quad -1 \le \xi \le 0 \\ 0 & \eta = -1, \quad 0 \le \xi \le 1 \end{cases}$$

$$\Phi_2^{(e)} = \begin{cases} \tfrac{1}{4}(1+\xi)(1-\eta) & \eta > -1 \\ \xi & \eta = -1, \quad 0 \le \xi \le 1 \\ 0 & \eta = -1, \quad -1 \le \xi \le 0 \end{cases}$$

$$\Phi_3^{(e)} = \frac{1}{4}(1+\xi)(1+\eta)$$

$$\Phi_4^{(e)} = \frac{1}{4}(1-\xi)(1+\eta)$$

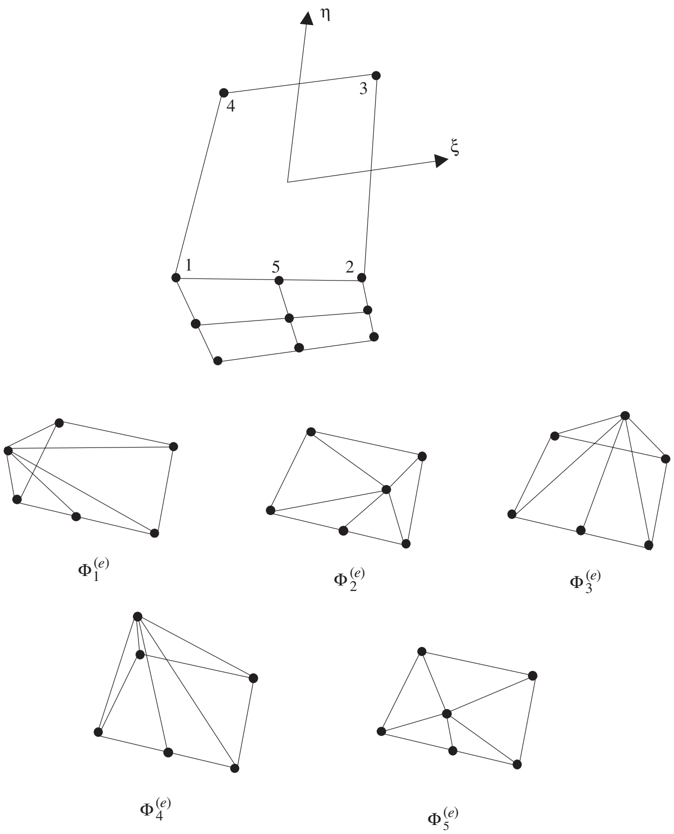


Figure E9.3.4 Irregular elements with irregular node which may occur in the refinement process, all elements are linear.

$$\Phi_5^{(e)} = \begin{cases} \dfrac{1}{2}(1-\xi)(1-\eta) & \xi > 0 \\ \dfrac{1}{2}(1+\xi)(1-\eta) & \xi \leq 0 \end{cases}$$

Here $\Phi_5^{(e)}$ for the midside node (hanging node) may be eliminated by readjusting the corner node functions, as is usually the case in adaptive mesh refinement methods (see Chapter 19).

Example 9.3.5 Collapse of Quadrilateral to Triangle

A quadrilateral element may be collapsed into a triangle by combining two of the quadrilateral nodes into one (Figure E9.3.5), as follows:

$$u^{(e)} = \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} + \Phi_3^{(e)} u_3^{(e)} + \Phi_4^{(e)} u_4^{(e)}$$

Equating $u_4^{(e)} = u_3^{(e)}$ we have for the triangle

$$u^{(e)} = \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} + \left(\Phi_3^{(e)} + \Phi_4^{(e)}\right) u_3^{(e)} = \Phi_1^{(e)} u_1^{(e)} + \Phi_2^{(e)} u_2^{(e)} + \overline{\Phi}_3^{(e)} u_3^{(e)}$$

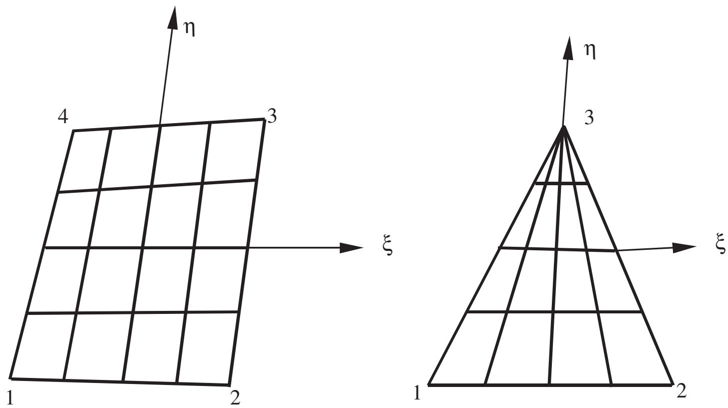


Figure E9.3.5 Collapsing a quadrilateral element into a triangle. Gaussian quadrature integration can be performed on the triangle as modified from the quadrilateral.

in which the modified interpolation for node 3 of the triangle is given by

$$\overline{\Phi}_3^{(e)} = \Phi_3^{(e)} + \Phi_4^{(e)} = \frac{1}{4}(1+\xi)(1+\eta) + \frac{1}{4}(1-\xi)(1+\eta)$$

Thus

$$\overline{\Phi}_3^{(e)} = \frac{1}{2}(1+\eta)$$

Gaussian quadrature integration may be used for this triangle in accordance with Table 9.3.1 with appropriate abscissae values.

9.4 THREE-DIMENSIONAL ELEMENTS

Three-dimensional elements are required when one- or two-dimensional idealization is not possible. Basic ingredients for three-dimensional elements have already been presented in earlier sections and no special conceptual developments are required. The three-dimensional elements may be constructed quite easily by direct extension of the ideas used for two-dimensional elements.

9.4.1 TETRAHEDRAL ELEMENTS

Consider the tetrahedral elements as shown in Figure 9.4.1. For linear variation of a variable (Figure 9.4.1a), we write

$$u^{(e)} = \alpha_0 + \alpha_1 x + \alpha_2 y + \alpha_3 z \tag{9.4.1}$$

It is a simple matter to write the above equation at each node, which yields a total of four equations. Evaluating the constants from these equations, we obtain

$$u^{(e)} = \Phi_N^{(e)} u_N^{(e)} \quad (N = 1, 2, 3, 4) \tag{9.4.2}$$

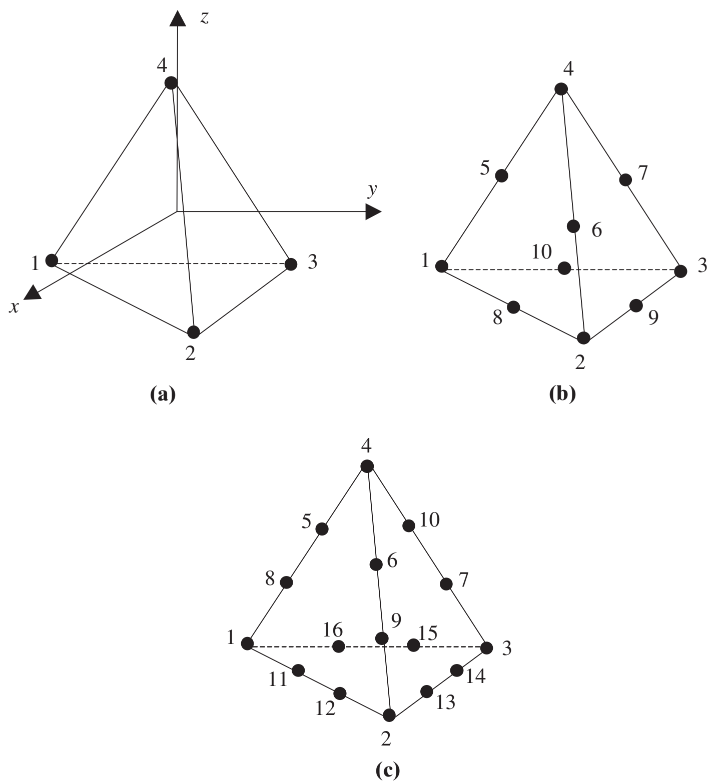


Figure 9.4.1 Tetrahedral element (cartesian coordinate): **(a)** linear variation, **(b)** quadratic variation, **(c)** cubic variation.

where

$$\Phi_N^{(e)} = a_N + b_N x + c_N y + d_N z \tag{9.4.3}$$

For $N = 1$, the coefficients a_1, b_1, c_1, d_1 are of the form

$$a_1 = \begin{vmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{vmatrix} \frac{1}{|D|}, \qquad b_1 = -\begin{vmatrix} 1 & y_2 & z_2 \\ 1 & y_3 & z_3 \\ 1 & y_4 & z_4 \end{vmatrix} \frac{1}{|D|}$$

$$c_1 = \begin{vmatrix} 1 & x_2 & z_2 \\ 1 & x_3 & z_3 \\ 1 & x_4 & z_4 \end{vmatrix} \frac{1}{|D|}, \qquad d_1 = -\begin{vmatrix} 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \\ 1 & x_4 & y_4 \end{vmatrix} \frac{1}{|D|} \tag{9.4.4}$$

$$|D| = \begin{vmatrix} 1 & x_{11} & x_{12} & x_{13} \\ 1 & x_{21} & x_{22} & x_{23} \\ 1 & x_{31} & x_{32} & x_{33} \\ 1 & x_{41} & x_{42} & x_{43} \end{vmatrix} = \begin{vmatrix} 1 & x_1 & y_1 & z_1 \\ 1 & x_2 & y_2 & z_2 \\ 1 & x_3 & y_3 & z_3 \\ 1 & x_4 & y_4 & z_4 \end{vmatrix} = 6V \tag{9.4.5}$$

where V is the volume of the tetrahedron. The rest of the coefficients can be determined similarly.

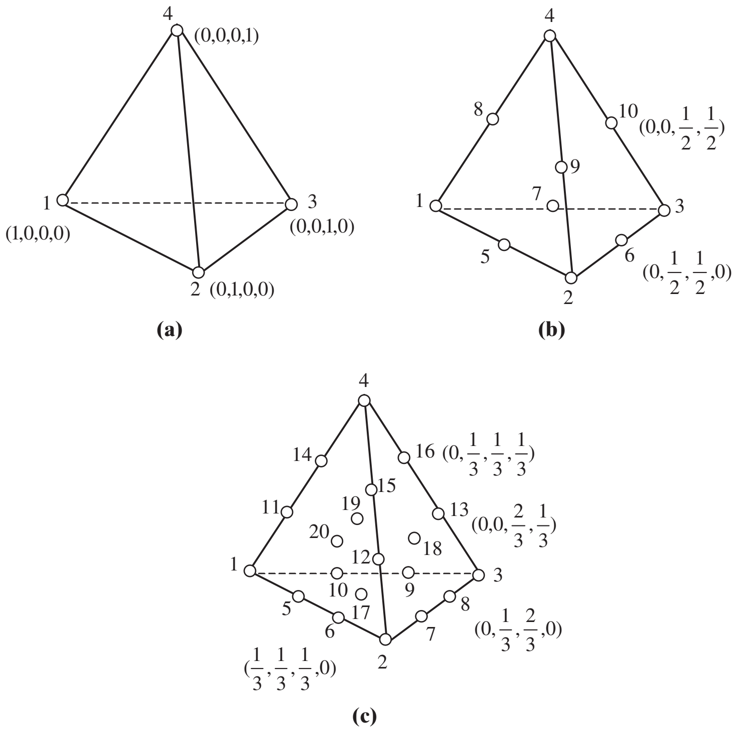


Figure 9.4.2 Tetrahedral element (natural volume, or tetrahedral coordinates): **(a)** linear variation; **(b)** quadratic variation; **(c)** cubic variation.

For higher order approximations, the coefficient matrix becomes very large in size and a resort to natural coordinates is inevitable. The most suitable choice is the volume coordinate system extended from the area coordinates for a two-dimensional triangular element.

If the three-dimensional natural coordinates (tetrahedral or volume coordinates) are used, a node having the coordinate of one decreases to zero as it moves to the opposite triangular surface formed by the rest of the nodes (Figure 9.4.2). For the linear element (Figure 9.4.2a), the interpolation functions are

$$\Phi_N^{(e)} = L_N \quad (N = 1, 2, 3, 4) \tag{9.4.6}$$

For higher order interpolations (Figure 9.4.2b,c), we invoke a formula similar to (9.3.20),

$$\Phi_r^{(e)} = B^{(r)}(L_1)B^{(r)}(L_2)B^{(r)}(L_3)B^{(r)}(L_4) \tag{9.4.7}$$

where $B^{(r)}(L_N)$ is given by (9.3.19). This provides the following results:
For quadratic variation (Figure 9.4.2b):
at corner nodes:

$$\Phi_N^{(e)} = (2L_N - 1)L_N$$

at midside nodes:

$$\Phi_6^{(e)} = 4L_2L_3, \qquad \Phi_{10}^{(e)} = 4L_3L_4, \text{ etc.}$$

For cubic variation (Figure 9.4.2c):
at corner nodes:

$$\Phi_N^{(e)} = \frac{1}{2}(3L_N - 1)(3L_N - 2)L_N$$

at side nodes:

$$\Phi_8^{(e)} = \frac{9}{2}L_2L_3(3L_3 - 1), \qquad \Phi_{13}^{(e)} = \frac{9}{2}L_3L_4(3L_3 - 1), \text{ etc.}$$

at midside nodes:

$$\Phi_{17}^{(e)} = 27L_1L_2L_3, \qquad \Phi_{18}^{(e)} = 27L_2L_3L_4, \text{ etc.}$$

The spatial integration of the tetrahedral coordinates may be derived similarly as in the triangular coordinates. This results in

$$I = \int_{\mathrm{v}} L_1^m L_2^n L_3^p L_4^q dv$$

or

$$I = \frac{6Vm!n!p!q!}{(m+n+p+q+3)!} \tag{9.4.8}$$

We may use a hexahedral element to generate five tetrahedral elements as shown in Figure 9.4.3. This approach is desirable in some applications where both hexahedral

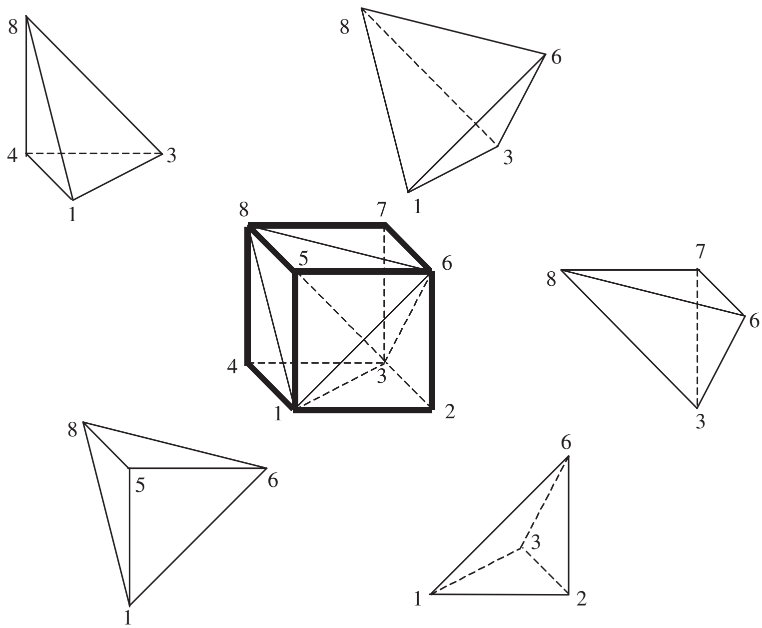


Figure 9.4.3 Five tetrahedral elements subdivided from a hexahedral element.

and tetrahedral elements are used. It is also convenient for the structured automatic grid generation.

9.4.2 TRIANGULAR PRISM ELEMENTS

It is possible to extend the tetrahedral element into triangular prism elements as shown in Figure 9.4.4. Note that triangular shapes may be completely arbitrary with the curvilinear coordinates ξ, η, ζ being distorted. Interpolation functions for linear and quadratic approximations are given as follows:

Linear (6 nodes)

$$\Phi_1^{(e)} = \frac{L_1(1+\eta)}{2}, \qquad \Phi_2^{(e)} = \frac{L_2(1+\eta)}{2}, \qquad \Phi_3^{(e)} = \frac{L_3(1+\eta)}{2} \tag{9.4.9a,b,c}$$

$$\Phi_4^{(e)} = \frac{L_1(1-\eta)}{2}, \qquad \Phi_5^{(e)} = \frac{L_2(1-\eta)}{2}, \qquad \Phi_6^{(e)} = \frac{L_3(1-\eta)}{2} \tag{9.4.9d,e,f}$$

Quadratic (15 nodes)

Corner nodes

$$\Phi_1^{(e)} = \frac{1}{2}L_1(2L_1-1)\eta(\eta+1)$$

$$\Phi_2^{(e)} = \frac{1}{2}L_2(2L_2-1)\eta(\eta+1)$$

$$\Phi_3^{(e)} = \frac{1}{2}L_3(2L_3-1)\eta(\eta+1) \tag{9.4.10a,b,c}$$

$$\Phi_4^{(e)} = \frac{1}{2}L_1(2L_1-1)\eta(\eta-1)$$

$$\Phi_5^{(e)} = \frac{1}{2}L_2(2L_2-1)\eta(\eta-1)$$

$$\Phi_6^{(e)} = \frac{1}{2}L_3(2L_3-1)\eta(\eta-1) \tag{9.4.10d,e,f}$$

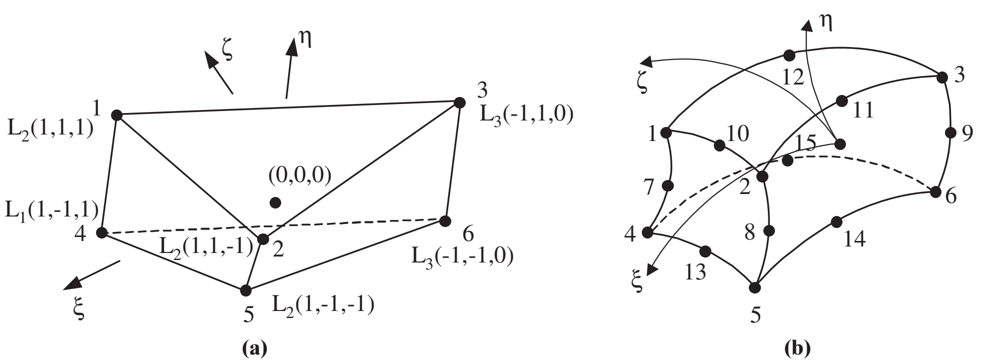


Figure 9.4.4 Triangular prism elements: **(a)** linear (6 nodes), **(b)** quadratic (15 nodes).

Midsides of Triangle

$$\Phi_{10}^{(e)} = 2L_1L_2\eta(\eta+1), \quad \Phi_{11}^{(e)} = 2L_2L_3\eta(\eta+1), \quad \Phi_{12}^{(e)} = 2L_1L_3\eta(\eta+1) \qquad (9.4.11a,b,c)$$

$$\Phi_{13}^{(e)} = 2L_1L_2\eta(\eta-1), \quad \Phi_{14}^{(e)} = 2L_2L_3\eta(\eta-1), \quad \Phi_{15}^{(e)} = 2L_1L_3\eta(\eta-1) \qquad (9.4.11d,e,f)$$

Midsides of Quadrilateral

$$\Phi_7^{(e)} = L_1(1-\eta^2), \qquad \Phi_8^{(e)} = L_2(1-\eta^2), \qquad \Phi_9^{(e)} = L_3(1-\eta^2) \qquad (9.4.12a,b,c)$$

9.4.3 HEXAHEDRAL ISOPARAMETRIC ELEMENTS

The four-sided two-dimensional elements may be extended to three-dimensional elements (Figure 9.4.5). The rectangular and arbitrary quadrilateral elements are developed into a regular hexahedron (brick) and irregular hexahedron. For a regular hexahedron, we may use either the Lagrange or Hermite element, but this becomes cumbersome as higher order approximations must include interior and surface nodes as well as corner and side nodes. Besides, neither may be applicable for irregular hexahedrons. An element which is free from these disadvantages is the isoparametric element.

In the isoparametric element for a linear variation of the geometry and variable, we write (see Figure 9.4.5a)

$$x, y, z = \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\zeta + \alpha_5\xi\eta\zeta + \alpha_6\xi\eta + \alpha_7\eta\zeta + \alpha_8\xi\zeta \qquad (9.4.13)$$

Using the same procedure as in the two-dimensional element, we obtain

$$\Phi_N^{(e)} = \frac{1}{8}(1+\xi_{N1}\xi_1)(1+\xi_{N2}\xi_2)(1+\xi_{N3}\xi_3) \qquad (9.4.14)$$

For a quadratic variation (Figure 9.4.4b), we have

$$\begin{aligned} x, y, z = {} & \alpha_1 + \alpha_2\xi + \alpha_3\eta + \alpha_4\zeta + \alpha_5\xi\eta\zeta + \alpha_6\xi\eta + \alpha_7\eta\zeta + \alpha_8\xi\zeta \\ & + \alpha_9\xi^2 + \alpha_{10}\eta^2 + \alpha_{11}\zeta^2 + \alpha_{12}\xi^2\eta + \alpha_{13}\xi\eta^2 + \alpha_{14}\eta^2\zeta + \alpha_{15}\eta\zeta^2 \\ & + \alpha_{16}\xi^2\zeta + \alpha_{17}\xi\zeta^2 + \alpha_{18}\xi^2\eta\varsigma + \alpha_{19}\xi\eta^2\varsigma + \alpha_{20}\xi\eta\varsigma^2 \end{aligned} \qquad (9.4.15)$$

The interpolation functions are:
at corner nodes :

$$\Phi_N^{(e)} = \frac{1}{8}(1+\xi_{N1}\xi_1)(1+\xi_{N2}\xi_2)(1+\xi_{N3}\xi_3)(\xi_{N1}\xi_1+\xi_{N2}\xi_2+\xi_{N3}\xi_3-2) \qquad (9.4.16a)$$

at midside nodes :

$$\Phi_N^{(e)} = \frac{1}{4}(1-\xi_1^2)(1+\xi_{N2}\xi_2)(1+\xi_{N3}\xi_3) \qquad (9.4.16b)$$

for

$$\xi_{N1} = 0, \qquad \xi_{N2} = \pm 1, \qquad \xi_{N3} = \pm 1, \text{ etc.}$$

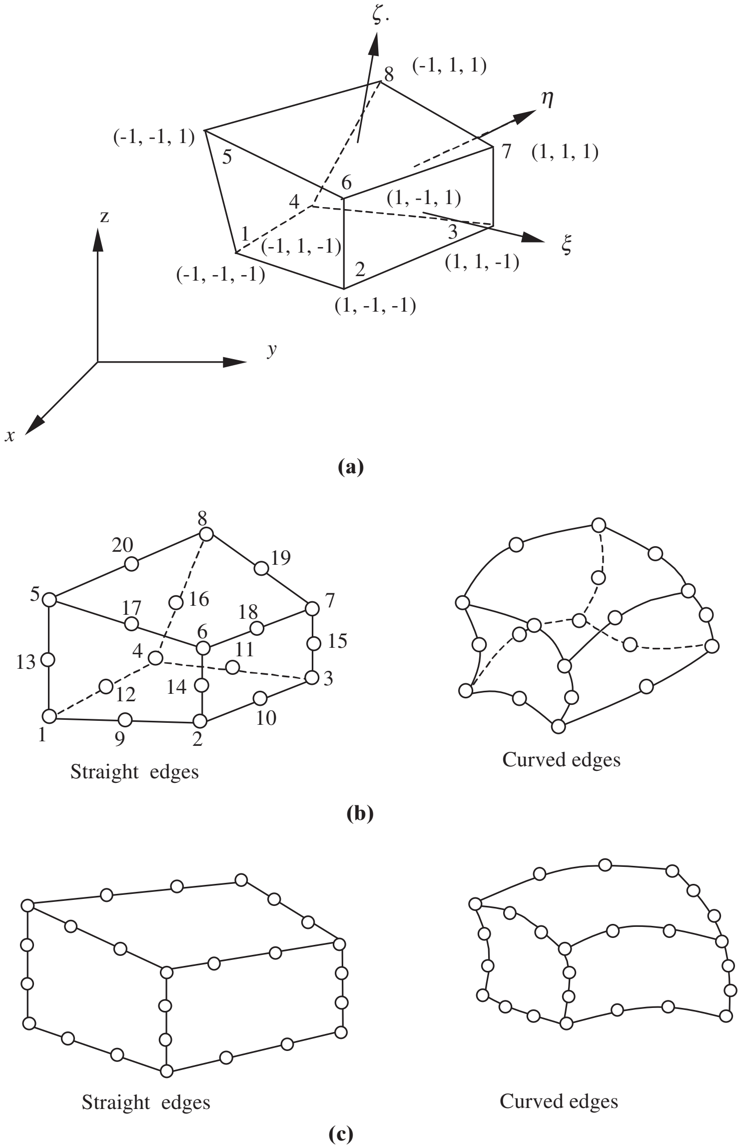


Figure 9.4.5 Hexahedral isoparametric element: **(a)** linear variation (8 nodes); **(b)** quadratic variation (20 nodes); **(c)** cubic variation (32 nodes).

Once again, Lagrange polynomials may be used to determine three-dimensional interpolation functions without interior nodes, but with side node constraint conditions. We now require integration of the form

$$\iiint \frac{\partial}{\partial x} f(\xi, \eta, \zeta)\, dx\, dy\, dz \tag{9.4.17}$$

with $\xi = \xi_1$, $\eta = \xi_2$, $\zeta = \xi_3$, $x = x_1$, $y = x_2$, and $z = x_3$. Proceeding similarly as in the two-dimensional case, we obtain

$$\iiint \frac{\partial f}{\partial x} dx dy dz = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}\left(\bar{J}_{11}\frac{\partial f}{\partial \xi} + \bar{J}_{12}\frac{\partial f}{\partial \eta} + \bar{J}_{13}\frac{\partial f}{\partial \zeta}\right)|J| d\xi d\eta d\zeta$$
$$= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g(\xi, \eta, \zeta) d\xi d\eta d\zeta \tag{9.4.18}$$

where $\bar{J}_{11}, \bar{J}_{12},$ and $\bar{J}_{13}$ are the first row of the 3×3 inverted Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\ \frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta} \end{bmatrix}$$

We may carry out differentiations of f with respect to y and z similarly, and write the general form of integration as follows:

$$\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} g(\xi, \eta, \zeta) d\xi d\eta d\zeta = \sum_{i=1}^{n}\sum_{j=1}^{n}\sum_{k=1}^{n} w_i w_j w_k g(\xi_i, \eta_j, \zeta_k) \tag{9.4.19}$$

The weight coefficients w_i, w_j, w_k, and the abscissae $g(\xi_i, \eta_j, \zeta_k)$ are obtained from Table 9.3.1 as a tensor product in three directions. A procedure similar to Example 9.3.1 may be followed for three dimensions to perform Gaussian quadrature integrations.

9.5 AXISYMMETRIC RING ELEMENTS

If the three-dimensional domain of study is axisymmetric, then any two-dimensional element may be used with the spatial integral replaced by

$$\iiint f(x, y, z) dx dy dz = \int_{0}^{2\pi}\iint f(r, z) r\, d\theta\, dr\, dz \tag{9.5.1}$$

where $dx = dr$, $dy = r d\theta$, and $dz = dz$ (see Figure 9.5.1). For quadrilateral isoparametric elements, we have

$$\int_{0}^{2\pi}\int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta) r\, d\theta |J| d\xi d\eta = 2\pi \int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta) r(\xi, \eta) |J| d\xi d\eta$$

or

$$2\pi \int_{-1}^{1}\int_{-1}^{1} g(\xi, \eta) d\xi d\eta = 2\pi \sum_{j=1}^{n}\sum_{k=1}^{n} w_j w_k g(\xi_j, \eta_k) \tag{9.5.2}$$

This represents a three-dimensional ring element generated by a two-dimensional element.

Note that the applications arise in the flowfields of missiles and rockets at zero angle of attack. For a nonzero angle of attack, the flowfields become asymmetric. In this case, the axisymmetric ring element can no longer be used and three-dimensional elements must be invoked instead. Another alternative is to keep the ring element and use Fourier

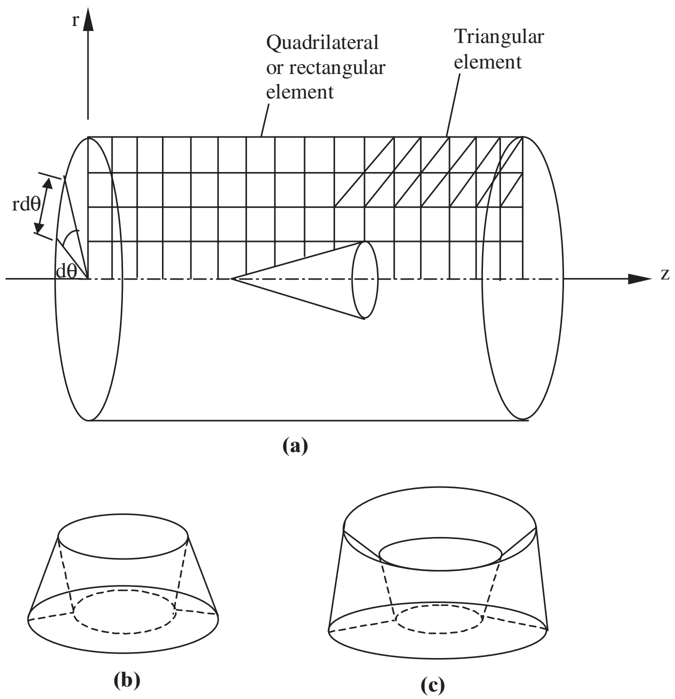


Figure 9.5.1 Axisymmetric ring elements. (**a**) Discretized geometry of a cylinder. (**b**) Triangular ring. (**c**) Quadrilateral ring.

series expansions around the circumference in order to accommodate nonaxisymmetric flowfields at every few degrees apart. This may result in a process just as expensive as three-dimensional elements.

9.6 LAGRANGE AND HERMITE FAMILIES AND CONVERGENCE CRITERIA

All finite elements, regardless of their geometrical shapes, may be grouped into two categories: Lagrange and Hermite families. The Lagrange family consists of finite elements in which the values of a variable are specified at nodes, whereas the Hermite family includes derivatives of the variable as well as its values defined at nodes.

Both Lagrange and Hermite families may be represented by polynomials derived from the Pascal triangle (Figure 9.1.4a) and Pascal tetrahedron (Figure 9.1.4b), or from two-dimensional hypercube (Figure 9.1.4c) and three-dimensional hypercube (Figure 9.1.4d).

In the Lagrange family, the polynomial terms contained in the circles represent the corresponding number of nodes required. However, in general, interior nodes lead to cumbersome bookkeeping and subsequent removal of some of the polynomial terms, resulting in an incomplete polynomial.

For a Hermite family, the number of nodes and polynomial terms required increases since derivatives in addition to the variable itself are to be specified. However, a reasonable compromise can be met by eliminating the values of a variable and specifying only the normal derivatives at side nodes. Let us consider twenty-one terms of a quintic

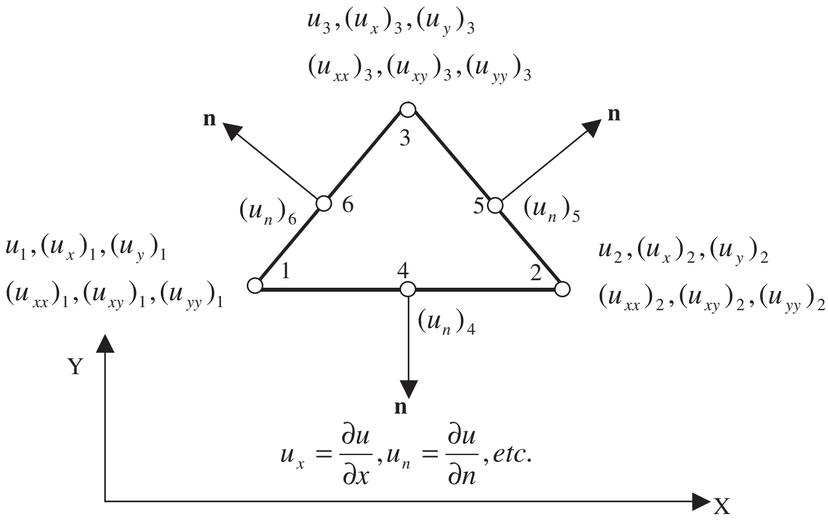


Figure 9.6.1 Hermite triangle, 21 constants to be determined.

polynomial for a triangular element given by

$$
\begin{aligned}
u = {} & \alpha_1 \\
& + \alpha_2 x + \alpha_3 y \\
& + \alpha_4 x^2 + \alpha_5 xy + \alpha_6 y^2 \\
& + \alpha_7 x^3 + \alpha_8 x^2 y + \alpha_9 xy^2 + \alpha_{10} y^3 \\
& + \alpha_{11} x^4 + \alpha_{12} x^3 y + \alpha_{13} x^2 y^2 + \alpha_{14} xy^3 + \alpha_{15} xy^4 \\
& + \alpha_{15} x^5 + \alpha_{17} x^4 y + \alpha_{18} x^3 y^2 + \alpha_{19} x^2 y^3 + \alpha_{20} xy^4 + \alpha_{21} y^5
\end{aligned}
\tag{9.6.1}
$$

The nodal values of u and derivatives of u to be specified are shown in Figure 9.6.1. Notice that we can write only eighteen equations with six equations at each node. We require three more equations which are furnished by writing normal derivatives at midsides. In this way, all twenty-one constants can be evaluated.

The $2m$th order differential equations associated with many of the engineering problems are in the form

$$\nabla^2 u = f \quad (m = 1) \tag{9.6.2}$$

$$\nabla^4 u = f \quad (m = 2) \tag{9.6.3}$$

Thus, the weak derivatives that appear in the Galerkin finite element formulations have $m = 1$ for (9.6.2) and $m = 2$ for (9.6.3). The choice of interpolation functions must ensure the convergence of solutions of the given differential equations. Toward this end, the following criteria should be satisfied.

(1) Smooth within the interior domain
(2) Continuity across each element
(3) Completeness

To satisfy (1), the degree of polynomial, p, should be $p \geq m$ so that the integrand of the finite element equation does not vanish (remaining at least a constant). For the stiffness integrands with derivatives of order m, we require C^m continuity within the domain (Ω) and C^{m-1} continuity across the boundary (Γ) in order to satisfy the convergence criteria of (1) and (2), respectively.

Interpolation functions associated only with the variable(s) of the differential equation such as in Lagrange polynomials are known as the C° element, whereas those with derivatives m are called the C^m elements. The Hermite polynomial interpolation functions of (9.2.12a) are referred to as the C^1 element.

The elements that satisfy both criteria (1) and (2) are known as conforming (compatible) elements. If these criteria are not satisfied, they are called nonconforming (incompatible) elements. Nonconforming elements, however, are useful in fourth order differential equations in which normal derivatives along the boundaries of C^1 triangle are specified.

The criterion (3) implies that complete polynomials as shown in Figures 9.1.4 through 9.1.6 be used, which cannot be met in many cases as the number of nodes to be provided does not match the number of complete polynomials of a given degree. As long as the symmetry of the polynomials is maintained, however, the convergence is, in general, not affected.

9.7 SUMMARY

Although the standard textbooks on finite elements provide information presented in this chapter, it was intended that a complete summary of finite element interpolation functions serve as a counterpart of Chapter 3, Derivation of Finite Difference Equations, as well as this text being self-contained and adequately balanced between FEM and FDM.

It is clear now that, instead of writing finite difference approximations using as many nodal points as necessary for desired order accuracy in FDM, we achieve similar objectives in FEM through interpolation functions. Instead of Taylor series expansions or Pade approximations used in finite difference equations, we resort to polynomial expansion in finite element interpolation functions. Although not covered in this chapter, special functions such as Chebyshev polynomials, Legendre polynomials, or Laguerre polynomials have been used in association with spectral elements. This subject will be discussed in Section 14.1.

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