CHAPTER TWO

Governing Equations

2.1 CLASSIFICATION OF PARTIAL DIFFERENTIAL EQUATIONS

Partial differential equations (PDEs) in general, or the governing equations in fluid dynamics in particular, are classified into three categories: (1) elliptic, (2) parabolic, and (3) hyperbolic. The physical situations these types of equations represent can be illustrated by the flow velocity relative to the speed of sound as shown in Figure 2.1.1. Consider that the flow velocity u is the velocity of a body moving in the quiescent fluid. The movement of this body disturbs the fluid particles ahead of the body, setting off the propagation velocity equal to the speed of sound a. The ratio of these two competing speeds is defined as Mach number

$$M = \frac{u}{a}$$

For subsonic speed, M $<$ 1, as time t increases, the body moves a distance, ut, which is always shorter than the distance at of the sound wave (Figure 2.1.1a). The sound wave reaches the observer, prior to the arrival of the body, thus warning the observer that an object is approaching. The zones outside and inside of the circles are known as the zone of silence and zone of action, respectively.

If, on the other hand, the body travels at the speed of sound, $M = 1$, then the observer does not hear the body approaching him prior to the arrival of the body, as these two actions are simultaneous (Figure 2.1.1b). All circles representing the distance traveled by the sound wave are tangent to the vertical line at the position of the observer. For supersonic speed, $M > 1$, the velocity of the body is faster than the speed of sound (Figure 2.1.1c). The line tangent to the circles of the speed of sound, known as a Mach wave, forms the boundary between the zones of silence (outside) and action (inside). Only after the body has passed by does the observer become aware of it.

The governing equations for subsonic flow, transonic flow, and supersonic flow are classified as elliptic, parabolic, and hyperbolic, respectively. We shall elaborate on these equations below. Most of the governing equations in fluid dynamics are second order partial differential equations. For generality, let us consider the partial differential equation of the form [Sneddon, 1957] in a two-dimensional domain

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G = 0 \tag{2.1.1}$$

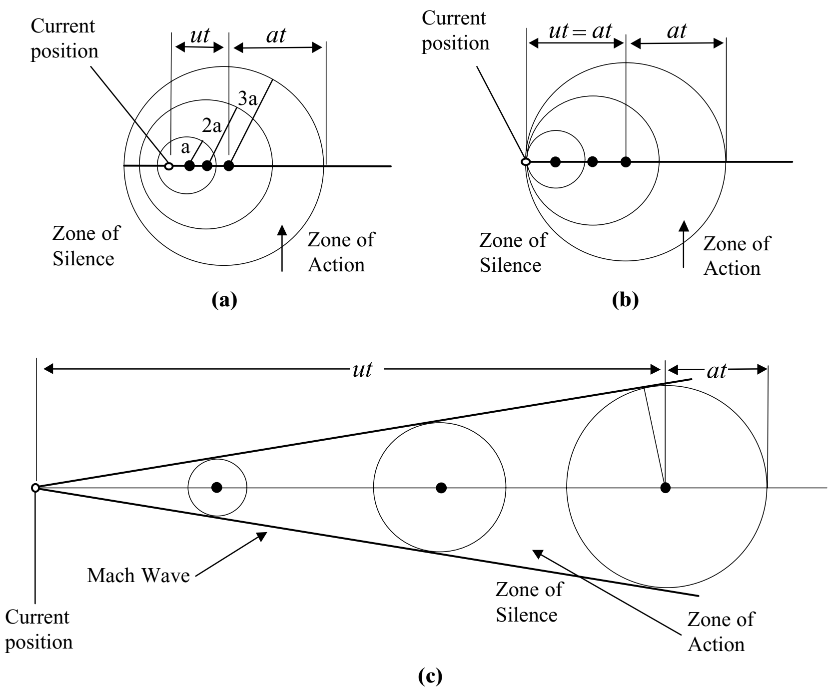


Figure 2.1.1 Subsonic, sonic, and supersonic flows. **(a)** Subsonic ($u < a$, $M < 1$). **(b)** Sonic ($u = a$, $M = 1$). **(c)** Supersonic ($u > a$, $M > 1$).

where the coefficients A, B, C, D, E, and F are constants or may be functions of both independent and/or dependent variables. To assure the continuity of the first derivative of u, $u_x \equiv \partial u/\partial x$ and $u_y \equiv \partial u/\partial y$, we write

$$du_x = \frac{\partial u_x}{\partial x}dx + \frac{\partial u_x}{\partial y}dy = \frac{\partial^2 u}{\partial x^2}dx + \frac{\partial^2 u}{\partial x \partial y}dy \tag{2.1.2a}$$

$$du_y = \frac{\partial u_y}{\partial x}dx + \frac{\partial u_y}{\partial y}dy = \frac{\partial^2 u}{\partial x \partial y}dx + \frac{\partial^2 u}{\partial y^2}dy \tag{2.1.2b}$$

Here u forms a solution surface above or below the $x - y$ plane and the slope dy/dx representing the solution surface is defined as the characteristic curve.

Equations (2.1.1), (2.1.2a), and (2.1.2b) can be combined to form a matrix equation

$$\begin{bmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{bmatrix} \begin{bmatrix} u_{xx} \\ u_{xy} \\ u_{yy} \end{bmatrix} = \begin{bmatrix} H \\ du_x \\ du_y \end{bmatrix} \tag{2.1.3}$$

where

$$H = -\left(D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu + G \right) \tag{2.1.4}$$

Since it is possible to have discontinuities in the second order derivatives of the dependent variable along the characteristics, these derivatives are indeterminate. This

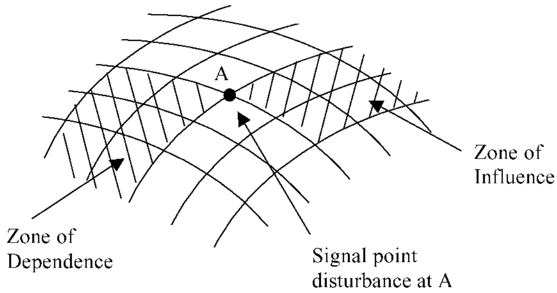


Figure 2.1.2 Propagation of disturbance and characteristics.

happens when the determinant of the coefficient matrix in (2.1.3) is equal to zero.

$$\begin{vmatrix} A & B & C \\ dx & dy & 0 \\ 0 & dx & dy \end{vmatrix} = 0 \tag{2.1.5}$$

which yields

$$A\left(\frac{dy}{dx}\right)^2 - B\left(\frac{dy}{dx}\right) + C = 0 \tag{2.1.6}$$

Solving this quadratic equation yields the equation of the characteristics in physical space,

$$\frac{dy}{dx} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A} \tag{2.1.7}$$

Depending on the value of $B^2 - 4AC$, characteristic curves can be real or imaginary. For problems in which real characteristics exist, a disturbance propagates only over a finite region (Figure 2.1.2). The downstream region affected by this disturbance at point A is called the zone of influence. A signal at point A will be felt only if it originates from a finite region called the zone of dependence of point A.

The second order PDE is classified according to the sign of the expression $(B^2 - 4AC)$.

(a) Elliptic if $B^2 - 4AC < 0$
In this case, the characteristics do not exist.
(b) Parabolic if $B^2 - 4AC = 0$
In this case, one set of characteristics exists.
(c) Hyperbolic if $B^2 - 4AC > 0$
In this case, two sets of characteristics exist.

Note that (2.1.1) resembles the general expression of a conic section,

$$AX^2 + BXY + CY^2 + DX + EY + F = 0 \tag{2.1.8}$$

in which one can identify the following geometrical properties:

$B^2 - 4AC < 0$	ellipse
$B^2 - 4AC = 0$	parabola
$B^2 - 4AC > 0$	hyperbola

This is the origin of terms used for classification of partial differential equations.

Examples

(a) Elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1.9}$$

$$A = 1, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4 < 0$$

(b) Parabolic equation

$$\frac{\partial u}{\partial t} - \alpha \frac{\partial^2 u}{\partial x^2} = 0 \quad (\alpha > 0) \tag{2.1.10}$$

$$A = -\alpha, \quad B = 0, \quad C = 0$$

$$B^2 - 4AC = 0$$

(c) Hyperbolic equation

1-D First Order Wave Equation

$$\frac{\partial u}{\partial t} + a \frac{\partial u}{\partial x} = 0 \quad (a > 0) \tag{2.1.11}$$

1-D Second Order Wave Equation

Differentiating (2.1.11) with respect to x and t,

$$\frac{\partial^2 u}{\partial t \partial x} + a \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1.12a}$$

$$\frac{\partial^2 u}{\partial t^2} + a \frac{\partial^2 u}{\partial t \partial x} = 0 \tag{2.1.12b}$$

Combining (2.1.12a) and (2.1.12b) yields

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \tag{2.1.13}$$

where

$$A = 1, \quad B = 0, \quad C = -a^2$$

$$B^2 - 4AC = 4a^2 > 0$$

(d) Tricomi equation

$$y \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \tag{2.1.14}$$

$$A = y, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4y$$

elliptic $\quad y > 0$

parabolic $\quad y = 0$

hyperbolic $\quad y < 0$

(e) 2-D small disturbance potential equation

$$(1 - M^2)\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \tag{2.1.15}$$

$$A = 1 - M^2, \quad B = 0, \quad C = 1$$

$$B^2 - 4AC = -4(1 - M^2)$$

$$\begin{array}{ll} \text{elliptic} & M < 1 \\ \text{parabolic} & M = 1 \\ \text{hyperbolic} & M > 1 \end{array}$$

In CFD applications, computational schemes and specification of boundary conditions depend on the types of PDEs. In many cases, the governing equations in fluids and heat transfer are of mixed types. For this reason, selections of computational schemes and methods to apply boundary conditions are important subjects in CFD. We shall examine them in detail for the remainder of this book.

2.2 NAVIER-STOKES SYSTEM OF EQUATIONS

Physics of fluids and heat transfer as a part of continuum mechanics has now been well established. The nonconservation form of the governing equations in fluids can be derived from the first law of thermodynamics, written as [Truesdell and Toupin, 1960; Chung, 1996]

$$\frac{DK}{Dt} + \frac{DU}{Dt} = M + Q \tag{2.2.1}$$

where K, U, M, and Q denote the kinetic energy, internal energy, mechanical power, and heat energy, respectively,

$$K = \int_\Omega \frac{1}{2}\rho \mathrm{v}_i \mathrm{v}_i \, d\Omega \tag{2.2.2}$$

$$U = \int_\Omega \rho\varepsilon \, d\Omega \tag{2.2.3}$$

$$M = \int_\Omega \rho F_i \mathrm{v}_i \, d\Omega + \int_\Gamma \sigma_{ij}\mathrm{v}_j n_i \, d\Gamma \tag{2.2.4}$$

$$Q = \int_\Omega \rho r \, d\Omega \pm \int_\Gamma q_i n_i \, d\Gamma \tag{2.2.5}$$

with

$$\varepsilon = c_p T - \frac{p}{\rho} \tag{2.2.6a}$$

$$\sigma_{ij} = -p\delta_{ij} + \tau_{ij} \tag{2.2.6b}$$

$$\tau_{ij} = \mu(\mathrm{v}_{i,j} + \mathrm{v}_{j,i}) - \frac{2}{3}\mu \mathrm{v}_{k,k}\delta_{ij} \tag{2.2.6c}$$

$$q_i = \pm kT_{,i} \tag{2.2.6d}$$

where the repeated indices imply summing and the comma denotes partial derivatives with respect to the independent variables x_i, Ω represents the domain of the flowfield with n_i being the components of a vector normal to the boundary surface Γ, with $\rho =$ density per unit mass, $\mathrm{v}_i =$ components of the velocity vector, $\varepsilon =$ internal energy per unit mass, $F_i =$ components of body force vector, $c_p =$ specific heat at constant pressure, $\sigma_{ij} =$ total stress tensor, $\tau_{ij} =$ viscous stress tensor, $\mu =$ coefficient of dynamic viscosity, $p =$ pressure, $q_i =$ heat flux, $T =$ temperature, $k =$ coefficient of thermal conductivity, and $r =$ heat supply per unit mass. Note that δ_{ij} denotes the Kronecker delta with $\delta_{ij} = 1$ for $i = j$ and $\delta_{ij} = 0$ for $i \neq j$.

The dynamic viscosity and thermal conductivity coefficients are functions of temperature as given by Sutherland's law,

$$\mu = \frac{C_1 T^{3/2}}{T + C_2} \tag{2.2.7}$$

$$k = \frac{C_3 T^{3/2}}{T + C_4} \tag{2.2.8}$$

with $C_1, C_2, C_3,$ and C_4 being the constants for a given gas. For air at moderate temperatures, we may use $C_1 = 1.458 \times 10^{-6}$ kg/(m s K$^{1/2}$), $C_2 = 110.4$ K, $C_3 = 2.495 \times 10^{-3}$ kg m/(s^3 K$^{3/2}$), and $C_4 = 194$ K.

Substituting (2.2.2) through (2.2.5) into (2.2.1) and using the Green-Gauss theorem, we obtain the governing equations of continuity, momentum, and energy,

Continuity

$$\frac{\partial \rho}{\partial t} + (\rho \mathrm{v}_i)_{,i} = 0 \tag{2.2.9a}$$

Momentum

$$\rho \frac{\partial \mathrm{v}_j}{\partial t} + \rho \mathrm{v}_{j,i} \mathrm{v}_i + p_{,j} - \tau_{ij,i} - \rho F_j = 0 \tag{2.2.9b}$$

Energy

$$\rho \frac{\partial \varepsilon}{\partial t} + \rho \varepsilon_{,i} \mathrm{v}_i + p \mathrm{v}_{i,i} - \tau_{ij} \mathrm{v}_{j,i} + q_{i,i} - \rho r = 0 \tag{2.2.9c}$$

with the equation of state

$$p = \rho R T \tag{2.2.10}$$

where R is the specific gas constant. Note that equations (2.2.9a) through (2.2.9c) are known as the nonconservation form of the Navier-Stokes system of equations for compressible viscous flows.

The above equations may be recast in the so-called conservation form of the Navier-Stokes system of equations,

$$\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} = \mathbf{B} \tag{2.2.11}$$

where $\mathbf{U}$, $\mathbf{F}_i$, $\mathbf{G}_i$, and $\mathbf{B}$ are the conservation flow variables, convection flux variables, diffusion flux variables, and source terms, respectively

$$\mathbf{U} = \begin{bmatrix} \rho \\ \rho \mathrm{v}_j \\ \rho E \end{bmatrix}, \quad \mathbf{F}_i = \begin{bmatrix} \rho \mathrm{v}_i \\ \rho \mathrm{v}_i \mathrm{v}_j + p\delta_{ij} \\ \rho E \mathrm{v}_i + p\mathrm{v}_i \end{bmatrix}, \quad \mathbf{G}_i = \begin{bmatrix} 0 \\ -\tau_{ij} \\ -\tau_{ij}\mathrm{v}_j + q_i \end{bmatrix}, \quad \mathbf{B} = \begin{bmatrix} 0 \\ \rho F_j \\ \rho r + \rho F_j \mathrm{v}_j \end{bmatrix}$$

with E being the total (stagnation) energy,

$$E = \varepsilon + \frac{1}{2}\mathrm{v}_j \mathrm{v}_j \tag{2.2.12a}$$

which is related by the pressure and temperature as

$$p = (\gamma - 1)\rho\left(E - \frac{1}{2}\mathrm{v}_j\mathrm{v}_j\right) \tag{2.2.12b}$$

$$T = \frac{1}{c_\mathrm{v}}\left(E - \frac{1}{2}\mathrm{v}_j\mathrm{v}_j\right) \tag{2.2.12c}$$

with c_v being the specific heat at constant volume. The Navier-Stokes system of equations is simplified to the Euler equations if the diffusion flux variables $\mathbf{G}_i$ are neglected.

It should be noted that, upon differentiation as implied in (2.2.11), we recover the nonconservation form of the Navier-Stokes system of equations given by (2.2.9).

On the other hand, integrating (2.2.11) spatially over the volume of the domain,

$$\int_\Omega \left(\frac{\partial \mathbf{U}}{\partial t} + \frac{\partial \mathbf{F}_i}{\partial x_i} + \frac{\partial \mathbf{G}_i}{\partial x_i} - \mathbf{B}\right) d\Omega = 0 \tag{2.2.13}$$

we obtain another form of governing equations,

$$\int_\Omega \left(\frac{\partial \mathbf{U}}{\partial t} - \mathbf{B}\right) d\Omega + \int_\Gamma (\mathbf{F}_i + \mathbf{G}_i) n_i \, d\Gamma = 0 \tag{2.2.14}$$

Note that the surface integral in (2.2.14) represents the convection and diffusion fluxes through the control surfaces, which are in balance with $\partial \mathbf{U}/\partial t$ and $\mathbf{B}$ inside the control volume. The surface integral in (2.2.14) has two important roles. First, it lays the foundation for the finite volume methods (FVM). Second, it provides appropriate numerical treatments for high gradient flows or discontinuities such as shock waves. Conservation properties across the discrete element boundary surfaces are satisfied if the surface integral components in (2.2.14) are properly implemented in the numerical solution.

Various types of fluid flows emerge from the Navier-Stokes system of equations in nonconservation and conservation forms. In general, computational schemes are dictated from the physics of flows characterized by special forms of the governing equations.

We have written the governing equations in fluid dynamics in three different ways. Equations (2.2.9a) through (2.2.9c) derived from the **F**irst **L**aw of **T**hermodynamics (FLT) are the nonconservation form of the Navier-Stokes system of equations in terms of the primitive variables ρ, v_i, p, T, whereas the **C**onservation form of **N**avier-**S**tokes system (CNS) of (2.2.11) are written in terms of the conservation variables $\mathbf{U}$, $\mathbf{F}_i$, and $\mathbf{G}_i$. In contrast, the **C**ontrol **V**olume-**S**urface (CVS) equations (2.2.14) are expressed in volume and surface integral forms, but still in terms of the conservation variables $\mathbf{U}$, $\mathbf{F}_i$, and $\mathbf{G}_i$. All of these three different forms of the governing equations represent certain types of numerical schemes to be developed, each playing special roles in CFD.

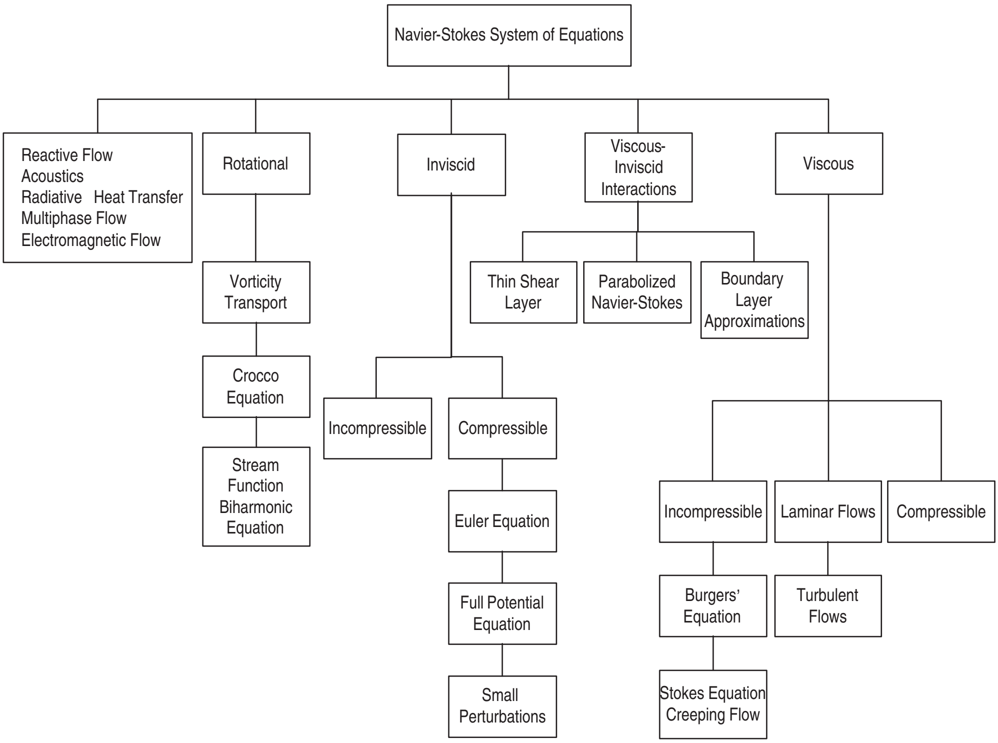
The FLT equations are convenient when the primitive variables ρ, v_i, p, T are to be solved directly, whereas this is not possible if CNS or CVS equations are used. It is seen that the conservation variables must be solved first with primitive variables extracted indirectly. Despite this inconvenience, the CNS or CVS equations are preferred in many CFD problems. For example, when the solution of density ρ is discontinuous, such as in shock waves, the solution through FLT is difficult. On the other hand, the mass flow $\rho\mathrm{v}_i$ is a smooth function and so are all other conservation variables, whereby the solution of CNS or CVS equations makes it possible to obtain discontinuous solution of primitive variables (indirectly). So, the conclusion here is that we can use FLT if the solution does not contain discontinuities such as in incompressible flows (no shock waves). This is known as the *pressure-based* formulation. Otherwise, CNS or CVS equations can be chosen, in which satisfactory results are assured in general, when the solution may contain discontinuities such as in compressible flows. This is known as the *density-based* formulation.

The Navier-Stokes system of equations as given by (2.2.11) may be simplified by disregarding one or more equations and/or some of the terms of each equation. For example, the momentum equations (2.2.9b) alone are often called the Navier-Stokes equations, thus distinguished from the Navier-Stokes *system* of equations which includes all equations (2.2.9a) through (2.2.9c). If all viscous terms are eliminated from the Navier-Stokes system of equations, then the resulting equations are known as Euler equations. The momentum equations without the pressure gradients are called the Burgers' equation. The Burgers' equation can be inviscid linear (no viscosity terms with convection terms being linearized), inviscid nonlinear, linear viscous, and nonlinear viscous. Simpler forms of these equations will be treated in Chapter 4. The governing equations for incompressible and compressible flows are discussed in Chapters 5 and 6 for FDM and Chapters 12 and 13 for FEM. More complicated governing equations are the subjects of Chapters 21 through 27.

The Navier-Stokes system of equations can be modified into various different forms, corresponding to particular physical phenomena, with the following subject areas included: compressible viscous flow (Navier-Stokes system of equations), compressible inviscid flow (diffusion terms are neglected), incompressible viscous flow (temporal and spatial variations of density are neglected), incompressible inviscid flow (both diffusion and density variations are neglected), vortex flow in terms of vorticity and stream function, compressible inviscid flow in terms of velocity potential function, turbulence, chemically reacting flows and combustion, acoustics, combined mode radiative heat transfer, and two-phase flows, as summarized in Table 2.2.1.

The governing equations in fluids and heat transfer in general are of mixed types: elliptic, parabolic, and hyperbolic partial differential equations. The presence or absence of each of the terms in these equations will determine their specific classifications. It will be shown throughout the book that numerical schemes depend on the types of partial

Table 2.2.1 Various types of flows



differential equations. In general, physical phenomena dictate the types of equations to be used, which are then accommodated by appropriate numerical schemes for solutions of the equations.

The Navier-Stokes system of equations presented above is cast in the Eulerian coordinates in which the current flowfield is fixed at the reference coordinates. In dealing with multiphase flows, however, it is convenient to work with the Lagrangian coordinates in which displacements of fluid or solid particles are tracked relative to the initial reference coordinates. Both Eulerian and Lagrangian coordinates may be coupled in dealing with certain physical phenomena. These and other topics of coordinate systems are discussed in Section 16.4 and Chapter 25. Detailed mathematics of Eulerian and Lagrangian coordinates are given in Chung [1996].

For flows coupled with magnetic and electric forces, it is necessary to solve the Maxwell's equations together with the modified Navier-Stokes system of equations. Applications of these equations to coronal mass ejection and semiconductor plasma processing are presented in Chapter 26.

The Navier-Stokes system of equations discussed in this section is based on the macroscopic nonrelativistic continuum view. In dealing with extremely high velocities such as occur in supernova explosions, the cosmic expansion, and cosmic singularity, however, the relativity principles based on the microscopic kinetic theory must be used. The governing equations for the relativistic astrophysical flows and their numerical solutions are discussed in Chapter 27.

2.3 BOUNDARY CONDITIONS

In Section 1.2 we dealt with boundary conditions for the second order differential equation: Dirichlet boundary conditions (values of variables specified at boundaries) and Neumann boundary conditions (derivatives of variables specified at boundaries).

In general, the boundary conditions are identified by constructing the inner product of the residual of the given differential equation with an arbitrary function. For example, consider the biharmonic fourth order partial differential equation of the stream function ψ

$$\nu\nabla^4\psi - f = 0 \tag{2.3.1}$$

which is obtained from the curl of the vector form of the two-dimensional momentum equation (2.2.9b), with $\nu = \mu/\rho$ and f being the nonlinear function of velocity gradients. We shall demonstrate which boundary conditions are required for this equation. To determine them, we construct an inner product of (2.3.1) with an arbitrary function ϕ [Chung, 1996]:

$$(\phi, \nu\nabla^4\psi - f) = \int_\Omega \phi(\nu\psi_{,iijj} - f)d\Omega = 0 \tag{2.3.2}$$

Integrate (2.3.2) by parts four times, successively,

$$\int_\Gamma \phi\nu\psi_{,iij}n_j d\Gamma - \int_\Omega \phi_{,j}\nu\psi_{,iij}d\Omega - \int_\Omega \phi f d\Omega = 0$$

$$\int_\Gamma \phi\nu\psi_{,iij}n_j d\Gamma - \int_\Gamma \phi_{,j}\nu\psi_{,ii}n_j d\Gamma + \int_\Omega \phi_{,jj}\nu\psi_{,ii}d\Omega - \int_\Omega \phi f d\Omega = 0$$

$$\int_\Gamma \phi\nu\psi_{,iij}n_j d\Gamma - \int_\Gamma \phi_{,j}\nu\psi_{,ii}n_j d\Gamma + \int_\Gamma \phi_{,jj}\nu\psi_{,i}n_i d\Gamma - \int_\Omega \phi_{,jji}\nu\psi_{,i}d\Omega$$
$$- \int_\Omega \phi f d\Omega = 0$$

Finally,

$$\int_\Gamma (\phi\nu\psi_{,iij}n_j - \phi_{,j}\nu\psi_{,ii}n_j + \phi_{,jj}\nu\psi_{,i}n_i - \phi_{,jji}\nu\psi\, n_i)d\Gamma$$
$$+ \int_\Omega \phi_{,jjii}\nu\psi\, d\Omega - \int_\Omega \phi f d\Omega = 0 \tag{2.3.3}$$

where the boundary conditions consist of two Neumann and two Dirichlet conditions:

Neumann Boundary Conditions

$\psi_{,iij}n_j$ *normal stress gradient*
$\psi_{,ii}n_j$ *normal velocity gradient* (2.3.4a)

Dirichlet Boundary Conditions

$$\begin{array}{ll} \psi_{,i} n_i & \textit{normal velocity} \\ \psi & \textit{stream function} \end{array} \tag{2.3.4b}$$

It is seen that, for the $2m$th order differential equation, the Neumann boundary conditions are of the order $2m-1, 2m-2, \ldots m$ and the Dirichlet boundary conditions are of the order $m-1, m-2, \ldots 0$. These boundary conditions are to be prescribed on the boundary surfaces. Similarly, for the second order equation ($\nabla^2\psi = 0$), there is one Neumann boundary condition ($\psi_{,i}\, n_i$) and one Dirichlet boundary condition (ψ). It was seen in Chapter 1 that the implementation of the Neumann boundary conditions "naturally" arises in the formulation process of FEM, whereas in FDM they must be carried out "manually" with appropriate forms of the difference equations.

Often, mixed Dirichlet and Neumann conditions (called Cauchy or Robin conditions) are used. For example, for the second order differential equation such as in combined conductive and convective heat transfer boundary conditions, we may write

$$\alpha T + \beta \frac{\partial T}{\partial n} = \gamma \tag{2.3.5}$$

with

$$\frac{\partial T}{\partial n} = (\boldsymbol{n} \cdot \boldsymbol{\nabla})T = T_{,i} n_i = \frac{\partial T}{\partial x} n_1 + \frac{\partial T}{\partial y} n_2 + \frac{\partial T}{\partial z} n_3 \tag{2.3.6}$$

$$\begin{array}{ll} \beta = 0 & \text{Dirichlet} \\ \alpha = 0 & \text{Neumann} \\ \alpha \neq 0,\ \beta \neq 0 & \text{Cauchy/Robin} \end{array}$$

Note that the notation $\partial T/\partial n$ is misleading since n in this derivative is neither the unit normal vector $\boldsymbol{n}$, nor its components n_i. However, this unfortunate notation has been generally accepted in the literature.

For time dependent problems, we must provide initial conditions as well as boundary conditions. Let us consider the case of hyperbolic, parabolic, and elliptic equations as shown in Figure 2.3.1.

(1) Hyperbolic equations associated with Cauchy conditions in an open region (Figure 2.3.1a).

Second Order Equation

$$\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < 1 \tag{2.3.7}$$

$$\text{Two initial conditions given} \quad \left\{ u(x,0) \quad \text{and} \quad \frac{\partial u}{\partial t}(x,0) \right.$$

$$\text{Two boundary conditions given} \quad \left\{ u(0,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0,t) \right.$$

$$\left\{ u(1,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(1,t) \right.$$

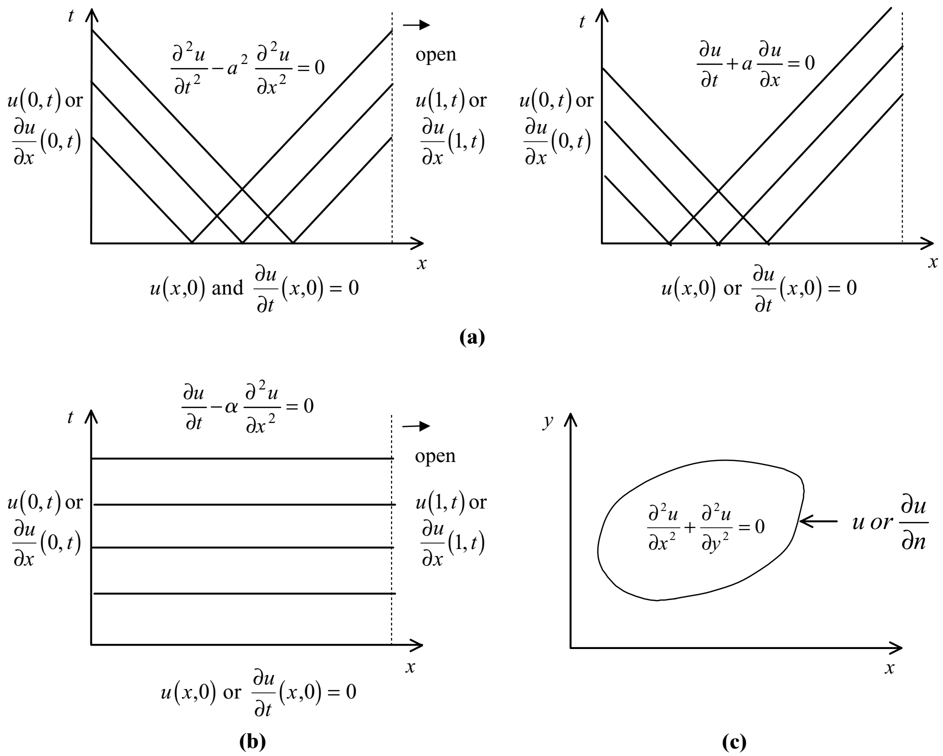


Figure 2.3.1 Initial and boundary conditions for hyperbolic, parabolic, and elliptic equations. **(a)** Hyperbolic equations (two sets of characteristics), Cauchy conditions in open region for second order equation. **(b)** Parabolic equations (one set of characteristics), Dirichlet or Neumann boundary conditions in an open region. **(c)** Elliptic equations (no real characteristics), Dirichlet or Neumann boundary conditions in closed region.

First Order Equation

$$\frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \quad 0 < x < 1 \tag{2.3.8}$$

One initial condition given $\left\{ u(x,0) \quad \text{or} \quad \frac{\partial u}{\partial t}(x,0) \right.$

One boundary condition given at $x = 0$ $\left\{ u(0,t) \quad \text{or} \quad \frac{\partial u}{\partial x}(0,t) \right.$

(2) Parabolic equations associated with Dirichlet or Neumann conditions in an open region (Figure 2.3.1b).

$$\frac{\partial u}{\partial t} - \nu\frac{\partial^2 u}{\partial x^2} = 0 \quad 0 < x < 1 \tag{2.3.9}$$

One initial condition given $\left\{ u(x,0) \quad \text{or} \quad \frac{\partial u}{\partial t}(x,0) \right.$

$$\text{Two boundary conditions given} \quad \begin{cases} u(0,t) \quad \text{or} \quad \dfrac{\partial u}{\partial x}(0,t) \end{cases}$$
$$\begin{cases} u(1,t) \quad \text{or} \quad \dfrac{\partial u}{\partial x}(1,t) \end{cases}$$

(3) Elliptic equations associated with Dirichlet or Neumann conditions in a closed region (Figure 2.3.1c).

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad \text{in} \quad \Omega \tag{2.3.10}$$

Two boundary conditions given

$$u \quad \text{on} \quad \Gamma_D$$
$$\frac{\partial u}{\partial n} \quad \text{on} \quad \Gamma_N$$

where Γ_D and Γ_N denote the Dirichlet and Neumann boundaries, respectively.

In general, more complicated boundary and initial conditions are required for CFD. Discussions on detailed boundary conditions for the Euler equations and the Navier-Stokes system of equations in FDM will be presented in Section 6.7, various aspects of boundary conditions associated with FEM in Sections 10.1.2, 11.1, and 13.6.6, and special boundary conditions for multiphase flows in Section 22.2.6.

2.4 SUMMARY

The basic properties of partial differential equations have been described and classified as elliptic, parabolic, and hyperbolic equations. The Navier-Stokes system of equations which represents mixed elliptic, parabolic, and hyperbolic partial differential equations can be written in three different forms: first law of thermodynamics (FLT) nonconservation form, conservation form of Navier-Stokes system (CNS), and control volume-surface integral form (CVS). The nonconservation form of the Navier-Stokes system of equations is derived from the first law of thermodynamics (FLT) written in terms of primitive variables, suitable for low-speed incompressible flows in which the solution surfaces are relatively smooth and not discontinuous. The conservation form of the Euler equations or Navier-Stokes (CNS) system of equations, on the other hand, is convenient for discontinuities such as in shock waves, thus suitable for high-speed compressible flows. Another conservation form is the control volume-surface (CVS) integral equations, applicable for the finite volume methods in which conservation requirements through discrete interior boundary surfaces as well as the exterior boundary surfaces are self-enforced. Relationships of these three forms of the Navier-Stokes system of equations have been mathematically linked together, traced back to the first law of thermodynamics [Chung, 1996].

The governing equations presented in this chapter are based on the Eulerian coordinates, which are fixed on the reference coordinates in which velocity components of fluid particles are calculated at any fixed point rather than tracing the particles downstream. In some problems, however, it is convenient to use the Lagrangian coordinates where the coordinate points are allowed to move together with fluid particles such as in multiphase flows. This subject will be discussed in Section 16.4.2 and Chapter 25.

In this chapter, we also discussed the boundary conditions for simple geometries and simple physics. The general method of identifying the existence of Neumann and Dirichlet boundary conditions of higher order partial differential equations was demonstrated. However, in reality, determination of boundary conditions is a difficult task in multidimensional, complex geometrical configurations with complex physical phenomena. Applications of boundary conditions will be the subject of discussion throughout the remainder of this book.

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