6

The comoving-frame picture

There is just one good reason for viewing the radiation field in the comoving frame of the fluid, and developing equations based on this picture, and it is an important one. It is that owing to the Doppler and aberration effects it is only in the comoving frame that the emissivity and absorptivity have the values specified by atomic physics. When the photorecombination process results in the emission of photons, only in the comoving frame will this emission be isotropic. Only in the comoving frame does the photoionization edge appear in the absorptivity at the same frequency for every angle, and is that frequency the one in the tables of atomic absorption energies. Isotropic emission and absorption in the comoving frame mean that the equilibrium intensity, (4.33), will be isotropic, and that therefore the flux will vanish. As mentioned before, the flux F in the fixed frame does not vanish no matter how opaque the medium may be. The plan of this chapter is to present the transformation relations for the various radiation quantities for going from the comoving frame to the fixed frame (or vice-versa), and to follow through some of the implications of these relations. The development will be carried out only to order u/c, not because the relativistic treatment is especially hard, but because we have no use for this when we are using Newtonian mechanics for the fluid equations. We note that all the complexities of comoving-frame transport arise from the space and time variation of the fluid velocity. Indeed, if the velocity is a uniform constant then the comoving frame is an inertial frame and all the earlier simple relations apply in it, just as in any other inertial frame.

This material is admittedly complicated. The reader is encouraged to find the other references, including the key paper of L. H. Thomas (1930) and the report by Fraser (1966). This topic is Chapter 7 in Mihalas and Mihalas (1984), and the present discussion also draws on Castor (1972).

6.1 The Doppler and aberration transformations

We have to dip into the regime of relativistic kinematics for a little while. We will talk about the position four-vector $(x^\mu) = (t, \mathbf{r})$. The Greek index μ runs from 0 to 3, with the 0 value designating the time component and 1, 2, and 3 designating the space components. If a particle moves from $(t, \mathbf{r})$ to $(t + dt, \mathbf{r} + d\mathbf{r})$ in the (relative) time dt, then the amount of proper time elapsed in its own frame is ds given by

$$ds^2 = dt^2 - \frac{d\mathbf{r}^2}{c^2}. \tag{6.1}$$

The particle's four-velocity is

$$(dx^\mu/ds) = \frac{dt}{ds}(1, \mathbf{v}) = (\gamma, \gamma\mathbf{v}), \tag{6.2}$$

where $\gamma = 1/\sqrt{1 - v^2/c^2}$ (not to be confused with the ratio of specific heats). Its four-momentum is

$$(p^\mu) = \left(m\frac{dx^\mu}{ds}\right) = (m\gamma, m\gamma\mathbf{v}). \tag{6.3}$$

We will use m only for the rest masses of particles. The time component of the four-momentum is E/c^2, where E is the relativistic energy of the particle. For photons, whose speed is c, $\mathbf{v}$ is $c\mathbf{n}$, but γ tends to infinity. However, the rest mass also vanishes, and $m\gamma$ tends to a finite limit such that E is $h\nu$, i.e., $m\gamma = h\nu/c^2$. Thus the photon four-momentum is

$$(p^\mu) = \frac{h\nu}{c^2}(1, \mathbf{n}c). \tag{6.4}$$

We consider the Lorentz transformation. We want the transformation from one set of coordinates $(x^\mu_{(0)}) = (t_0, \mathbf{r}_0)$ to the $(t, \mathbf{r})$ set such that a point with a fixed $\mathbf{r}_0$ is moving with the velocity $\mathbf{u}$ in the $(t, \mathbf{r})$ coordinates. We quote the result:

$$x^\mu = A^\mu_\lambda x^\lambda_{(0)} \tag{6.5}$$

(using the summation convention), where (A^μ_λ) is the 4×4 matrix given by

$$(A^\mu_\lambda) = \begin{pmatrix} \gamma_u & \gamma_u \mathbf{u}^{\mathrm{T}}/c^2 \\ \gamma_u \mathbf{u} & + (\gamma_u - 1)\mathbf{u}\mathbf{u}^{\mathrm{T}}/u^2 \end{pmatrix}. \tag{6.6}$$

The velocity $\mathbf{u}$ is represented by a column vector here, and its transpose is the row vector $\mathbf{u}^T$. The scalar γ_u is $1/\sqrt{1 - u^2/c^2}$. The 3×3 matrix in the lower right-hand corner is arranged to leave unchanged a vector it multiplies that is perpendicular to $\mathbf{u}$, and to multiply by γ_u one that is parallel to $\mathbf{u}$.

The very point of the Lorentz transformation is to leave the proper time element ds unchanged in the transformation, so the transformation relation for a four-velocity is the same as for the coordinates,

$$\frac{dx^\mu}{ds} = A^\mu_\lambda \frac{dx^\lambda_{(0)}}{ds}. \tag{6.7}$$

Therefore the same is true for the four-momentum:

$$p^\mu = A^\mu_\lambda p^\lambda_{(0)}. \tag{6.8}$$

Now we can apply this to photons and get the Doppler and aberration relations:

$$\nu = \nu_0 \gamma_u \left(1 + \frac{\mathbf{n}_{(0)} \cdot \mathbf{u}}{c}\right), \tag{6.9}$$

$$\mathbf{n} = \frac{\gamma_u \mathbf{u}/c + \mathbf{n}_{(0)} + (\gamma_u - 1)(\mathbf{n}_{(0)} \cdot \mathbf{u})\mathbf{u}/u^2}{\gamma_u (1 + \mathbf{n}_{(0)} \cdot \mathbf{u}/c)}. \tag{6.10}$$

For small values of the relative velocity $\mathbf{u}$ of the two frames these relations simplify considerably. In that case we can neglect all the u^2/c^2 terms, which means that γ_u can be replaced by 1. Therefore

$$\nu = \nu_{(0)} \left(1 + \frac{\mathbf{n}_{(0)} \cdot \mathbf{u}}{c}\right), \tag{6.11}$$

$$\mathbf{n} = \frac{\mathbf{n}_{(0)} + \mathbf{u}/c}{1 + \mathbf{n}_{(0)} \cdot \mathbf{u}/c}. \tag{6.12}$$

These relations will be adequate for most (all?) of the subsequent discussion.

6.2 Transforming I, k, j

Phase-space density functions for relativistic gases require a little more of our relativistic kinematics. The reason is that the momentum space volume element $d^3\mathbf{p}$ is not a Lorentz invariant. There are good discussions of this in Synge (1957), and in Mihalas and Mihalas (1984) . We will give the quick version here. All of the possible four-momentum values p^μ that are allowed for a given kind of particle do not fill up space-time, since they must all be consistent with the value of the proper mass, and therefore

$$(p^0)^2 - \mathbf{p}^2/c^2 = m^2. \tag{6.13}$$

Thus the allowed four-momentum values form a hypersurface (mass shell) in space-time. An infinitesimal piece of the mass shell is an oriented three-volume in space-time, just as a patch on an ordinary surface in space is an oriented 2-D element. Oriented means that it is associated with a particular vector perpendicular to it, namely the surface normal at that point. The oriented three-volume element for an infinitesimal piece of the mass shell is a good four-vector that points along the hypersurface normal for the mass shell at the momentum value in question. What direction is that? The answer is that it is along p^μ itself. The reason is that p^μ is a four-vector of constant length, as indicated by (6.13), and therefore the allowed displacements dp^μ consistent with staying on the mass shell are perpendicular to p^μ (in the sense of the Minkowski metric). The surface normal is the one vector perpendicular to all the allowed displacements in the surface, therefore it is along p^μ.

Thus the oriented three-volume element for the mass shell is one good four-vector, and p^μ is another good four-vector, and we now know that they are parallel. The constant of proportionality between them is therefore a Lorentz invariant, and this is the invariant mass shell volume element. In a particular frame, the time-like component of the oriented three-volume is just $d^3\mathbf{p}$ if we choose to use p^1, p^2, and p^3 as the coordinates on the hypersurface. Since the time-like component of p^μ is proportional to the relativistic energy E, this combination is a Lorentz invariant:

$$\frac{d^3\mathbf{p}}{E}. \tag{6.14}$$

For photons, the momentum-space volume element is proportional to $\nu^2 d\nu d\Omega$ (see (4.4)), and E is proportional to ν, so the invariant volume element is $\nu d\nu d\Omega$.

What about the ordinary spatial volume element $d^3\mathbf{r}$? This is not Lorentz invariant either. In a particular coordinate frame, three-space is a slice through space-time at a constant value of the appropriate time variable; it is another hypersurface. The hypersurface normal is in the direction of the four-velocity corresponding to the velocity of that frame. Let us call that four-velocity U^μ. In this one frame U^μ has the components $(1, 0, 0, 0)$. We conclude that the oriented 3-volume element for this time slice is $d^3\mathbf{r}U^\mu$. We next make use of the fact that dot products of four-vectors are Lorentz invariant. Since p^μ is one four-vector and $d^3\mathbf{r}U^\mu$ is another, we conclude that $Ed^3\mathbf{r}$ is invariant. Here E is the energy of one particular photon in the frame for which $d^3\mathbf{r}$ is the correct 3-space volume element.

Notice that we have to divide the momentum space volume element by E to get an invariant and we have to multiply the coordinate volume element by E to get an invariant. That means that the phase-space volume element $d^3\mathbf{r}d^3\mathbf{p}$ is an invariant by itself. We recall from the discussion above that the phase-space density of photons is

$$\frac{I_\nu}{h^4\nu^3/c^2}. \tag{6.15}$$

If we change the units here to measure the number per volume h^3 in phase space, and divide by 2 to find the average number per mode of polarization, we get

$$\mathcal{I}_\nu = \frac{I_\nu}{2h\nu^3/c^2} \tag{6.16}$$

for the Lorentz-invariant intensity. This is exactly the quantity referred to as the number of photons per mode. This is the central quantity in the general covariant formulation of radiation transport by Lindquist (1966). Since $\mathcal{I}_\nu$ is invariant, the rule for transforming the intensity when going between different frames is just

$$I_\nu = \left(\frac{\nu}{\nu_0}\right)^3 I_\nu^{(0)}. \tag{6.17}$$

We can get the rules for transforming the absorptivity and emissivity by considering two time slices separated by dt, with matching three-volumes $d^3\mathbf{r}$ on each, in addition to a momentum volume $d^3\mathbf{p}$. The number of photons added to $d^3\mathbf{r}$ during dt that lie in $d^3\mathbf{p}$ is equal to

$$\Delta N = \frac{j_\nu}{h\nu} d^3\mathbf{r}\,dt\,d\nu\,d\Omega = \frac{j_\nu}{h^4\nu^3/c^3} d^3\mathbf{r}\,dt\,d^3\mathbf{p}. \tag{6.18}$$

The product $d^3\mathbf{r}dt$ by itself is the four-volume element, and it is invariant. Thus $d^3\mathbf{r}dtd^3\mathbf{p}$ is not invariant, but $d^3\mathbf{r}dtd^3\mathbf{p}/E$ is, in view of the earlier result. So we rearrange (6.18) as

$$\Delta N = \frac{j_\nu}{h^3\nu^2/c^3} \frac{d^3\mathbf{r}\,dt\,d^3\mathbf{p}}{h\nu} \tag{6.19}$$

and since ΔN should be invariant, we conclude that

$$\mathcal{e}_\nu = \frac{c}{2}\frac{j_\nu}{\nu^2} \tag{6.20}$$

is Lorentz invariant. The same reasoning applies to $k_\nu I_\nu$, therefore

$$\mathcal{a}_\nu = \frac{h}{c}\nu k_\nu \tag{6.21}$$

is invariant. (The constant factors included in these definitions are for the purpose of making (6.24) consistent with our prior notation.) This gives the transformation relations

$$j_\nu = \left(\frac{\nu}{\nu_0}\right)^2 j_\nu^{(0)}, \tag{6.22}$$

$$k_\nu = \frac{\nu_0}{\nu} k_\nu^{(0)}. \tag{6.23}$$

More discussion of the covariant absorption and emission is found in Linquist (1966).

As a final note to the business of Lorentz transformations of the intensity, we quote the covariant form of the transport equation for Cartesian coordinate systems (see Lindquist (1966)):

$$p^{\mu}\mathcal{I}_{\nu,\mu} = \mathcal{e}_{\nu} - \mathcal{a}_{\nu}\mathcal{I}_{\nu}. \tag{6.24}$$

The comma subscript here indicates that the following subscripts denote the coordinates by which this quantity is differentiated. Thus $f_{,\mu}$ means $\partial f/\partial x^{\mu}$. Putting in the appropriate powers of ν shows that this is identical to the equation we usually use, (4.23).

6.3 Transforming E, F, and

The moments of the radiation field integrated over frequency have a natural physical interpretation as the parts of the stress-energy tensor. The tensor is defined by

$$T^{\lambda\mu} = \int \frac{d^3\mathbf{p}}{E}\, p^{\lambda} p^{\mu} \mathcal{I}. \tag{6.25}$$

This is a good contravariant second rank tensor since it is built from the product of two contravariant four-vectors multiplied by scalars. We evaluate the integral using the relations we just derived and find, after discarding some irrelevant factors of h and c,

$$(T^{\lambda\mu}) = c\int\int_{4\pi}\begin{pmatrix}1\\ \mathbf{n}c\end{pmatrix}\begin{pmatrix}1 & \mathbf{n}^{\mathrm{T}}c\end{pmatrix}\left(\frac{\nu}{c}\right)^2 \frac{I_\nu}{\nu^3}\frac{\nu^2 d\nu d\Omega}{\nu} \tag{6.26}$$

$$= \int d\nu \int_{4\pi} d\Omega\, I_\nu \begin{pmatrix}1/c & \mathbf{n}^{\mathrm{T}}\\ \mathbf{n} & \mathbf{n}\mathbf{n}^{\mathrm{T}}c\end{pmatrix} \tag{6.27}$$

$$= \begin{pmatrix}E & \mathbf{F}^{\mathrm{T}}\\ \mathbf{F} & c^2\end{pmatrix}. \tag{6.28}$$

Thus the time–time part of $T^{\lambda\mu}$ is the energy density, the time-space part is the flux vector, and the space–space part is the pressure tensor, apart from factors of c for unit conversion.

We are familiar with stress-energy tensors like this in other fields. The electromagnetic stress-energy tensor is of this kind (see Panofsky and Philips (1962)). Its time–time part is the electromagnetic energy density

$$\frac{1}{8\pi}(\boldsymbol{\mathcal{E}}^2 + \boldsymbol{\mathcal{H}}^2), \tag{6.29}$$

the space–time part is the Poynting vector, and the space–space part is the Maxwell stress tensor. As a matter of fact, the radiation stress-energy tensor is the electromagnetic stress-energy tensor after coarse-graining.

Since $T^{\lambda\mu}$ is a good tensor, we can use the Lorentz transformation matrix A to obtain its components in the fixed frame from those in the comoving frame. The rule is to multiply the T matrix by one factor of A from the left and by one factor of A from the right. We carry this out for the small $\mathbf{u}$ case and discard terms of order u^2/c^2 and higher. The result is

$$\begin{pmatrix} E & \mathbf{F}^{\mathrm{T}} \\ \mathbf{F} & c^2 \end{pmatrix} = \begin{pmatrix} E_0 + \dfrac{2\mathbf{u}}{c^2} \cdot \mathbf{F}_0 & (\mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u} \cdot {}_0)^{\mathrm{T}} \\ \mathbf{F}_0 + \mathbf{u}E_0 + \mathbf{u} \cdot {}_0 & c^2 {}_0 + \mathbf{F}_0\mathbf{u}^T + \mathbf{u}\mathbf{F}_0^{\mathrm{T}} \end{pmatrix}. \tag{6.30}$$

The transformation relation for the flux is the one that was used earlier in discussing the total energy equation for matter and radiation.

The quantities cE, $\mathbf{F}$, and c are potentially all the same order of magnitude. In diffusion regions the flux will be smaller than the other two, however. So the velocity corrections to the moments are formally of order u/c, but in a diffusion region the relative magnitude of the correction to the flux may be much larger than that, while the relative corrections to the energy density and pressure are even smaller than u/c. In other words, the correction to the flux is the important one to keep in mind.

The next interesting thing that we can do with the stress-energy tensor is to form its divergence,

$$T_{,\lambda}^{\lambda\mu} = g^{\mu}. \tag{6.31}$$

The four-vector g^{μ} is made up of the energy and momentum source rates for the radiation field. We can find an expression for g^{μ} by multiplying the invariant transport equation (6.24) by p^{μ} and then integrating with $d^3\mathbf{p}/E$. This leads to

$$(T_{,\lambda}^{\lambda\mu}) = \int\int_{4\pi} \begin{pmatrix} 1 \\ \mathbf{n}c \end{pmatrix} (j_\nu - k_\nu I_\nu) d\nu d\Omega, \tag{6.32}$$

and therefore

$$g^0 = \int\int_{4\pi} (j_\nu - k_\nu I_\nu) d\nu d\Omega \tag{6.33}$$

and

$$\mathbf{g} = (g^i) = \int\int_{4\pi} \mathbf{n}c(j_\nu - k_\nu I_\nu) d\nu d\Omega, \tag{6.34}$$

exactly as given earlier, in (4.43) and (4.44). Writing out the components of the tensor divergence in the fixed frame and substituting these results for g^{μ} into (6.31) recovers (4.27) and (4.28) discussed earlier. What is new is the realization that g^{μ} is a four-vector, and that therefore it can be evaluated by going to the comoving frame where the atomic properties are known, and then transforming back to the fixed frame; see Mihalas and Mihalas (1984), Section 91. The Lorentz transformation to first order in u/c is

$$g^0 = g^0_{(0)} + \frac{\mathbf{u}}{c^2} \cdot \mathbf{g}_{(0)}, \tag{6.35}$$

$$\mathbf{g} = \mathbf{g}_{(0)} + \mathbf{u} g^0_{(0)}. \tag{6.36}$$

The second term on the right-hand side of the $\mathbf{g}$ equation is another of those pesky ones that do not have an analog in nonrelativistic mechanics. It comes about because the addition of energy increases the relativistic mass density and therefore also the relativistic momentum density. When we treat the material fluid nonrelativistically there is no similar term in the material momentum equation to compensate this one. Fortunately it is small and we can discard it or hide it somewhere. The velocity term in the g^0 equation does have a nonrelativistic meaning. It is the rate of doing work by the force exerted on the radiation by the matter.

The components of g^{μ} in the fluid frame are easily found from the atomic properties of the matter, without the need for Doppler and aberration transformations. Furthermore, we can almost always assume isotropy of the absorptivity and emissivity in the fluid frame. With that assumption we obtain

$$g^0_{(0)} = \int d\nu \left(4\pi j^{(0)}_{\nu} - k^{(0)}_{\nu} c E^{(0)}_{\nu}\right), \tag{6.37}$$

$$\mathbf{g}_{(0)} = -c \int d\nu \, k^{(0)}_{\nu} \mathbf{F}^{(0)}_{\nu}. \tag{6.38}$$

These are the same expressions that we used for the right-hand sides of the fixed-frame moment equations earlier, only repeated here using the fluid-frame absorptivity and emissivity and the fluid-frame radiation moments. The important difference is that the earlier fixed-frame expressions are wrong if there is an appreciable velocity while the fluid-frame expressions are correct as long as the velocity is nonrelativistic. The components g^0 and $\mathbf{g}$ obtained by substituting (6.37) and (6.38) into (6.35) and (6.36) give the correct quantities to put on the right-hand sides of the fixed-frame moment equations.

We have not described transforming the monochromatic radiation moments from the comoving frame to the fixed frame. That is because the monochromatic moments are not space-time tensors, and therefore the Lorentz transformation rules do not apply. That is one mathematical reason. A second mathematical reason is that an angle moment at a constant value of the fixed-frame frequency is an integral over a different slice through photon momentum space than an angle moment at any constant fluid-frame frequency. In other words, there is no simple relation between the two kinds of moments. The relations that have been obtained involve substituting the transformation relations for the intensity and the direction vector into the integrals for the fixed-frame moments, then using Taylor expansions to first order of $I^{(0)}$ to perform the mappings $\nu_0 \to \nu$ and $\mathbf{n}_0 \to \mathbf{n}$. The results will not be given here because this Taylor expansion technique is a very poor idea. It makes the implicit assumption that $v/c \ll \Delta\nu/\nu$, where $\Delta\nu$ is the width of the narrowest feature in the spectrum. It is easy to think of important examples that violate this limit by a wide margin: supernovae and stellar winds, to name just two.

6.4 The comoving-frame transport equation

There are two basically different approaches to solving radiation transport problems involving time-dependent flows. The first is to solve partial differential equations for the radiation intensity or its moments as viewed in a fixed frame of reference. The second is to let the unknown be the radiation field in the comoving frame of the fluid. There are advantages and disadvantages of each approach. The fixed-frame method has the advantage of simplicity in the partial differential equation, and the disadvantage of complexity in the absorption and emission coefficients, and in the corresponding energy/momentum terms for the matter. For the treatment with comoving-frame radiation the advantages and disadvantages are exchanged. In this section we examine the comoving-frame view of the radiation.

There are also two approaches to deriving the comoving-frame transport equation. One is to substitute the transformation relations for the radiation variables into the fixed-frame transport equation and perform the necessary expansions (Buchler, 1979, 1983). The other is to treat the Lagrangian frame as a curvilinear coordinate system, work out the appropriate metric tensor, and write the transport equation using a curvilinear generalization of (6.24) (see Castor (1972) and Mihalas and Mihalas (1984), Section 95). In spherical symmetry the second approach is in fact somewhat simpler than the first, and is an aid to understanding the problem. This technique fails to generalize to higher-dimensional cases, however. The reason is interesting: the curvilinear coordinate system is very awkward to deal with unless the metric tensor is diagonal, so that locally it agrees with a Lorentz transformation from the fixed space. This imposes certain conditions on the mapping from fixed space to comoving space, and it turns out that these are impossible to satisfy if the fluid velocity is rotational. Rather than discuss this further, we will turn immediately to the technique of transforming the fixed-frame transport equation.

The objective is very simple: Substitute the relations (6.11), (6.12), (6.17), (6.22), and (6.23) into the transport equation (4.23) and expand, discarding all the terms that are $O(u^2/c^2)$ or higher. This result is obtained:

$$\left(1+\frac{\mathbf{n}_0}{c}\cdot\mathbf{u}\right)\left(\frac{1}{c}\frac{\partial I^{(0)}}{\partial t}+\frac{\mathbf{u}}{c}\cdot\nabla I^{(0)}\right)+\mathbf{n}_0\cdot\nabla I^{(0)}$$
$$-\frac{\nu_0}{c}\left(\frac{\mathbf{a}}{c}+\mathbf{n}_0\cdot\nabla\mathbf{u}\right)\cdot\nabla_{\nu_0\mathbf{n}_0}I^{(0)}$$
$$+\frac{3}{c}\left(\frac{\mathbf{n}_0\cdot\mathbf{a}}{c}+\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right)I^{(0)}=j^{(0)}-k^{(0)}I^{(0)}. \tag{6.39}$$

It was first given in this form by Buchler (1983). For simplicity here and below, the subscripts indicating that all the frequency-dependent quantities in the comoving frame are evaluated at ν_0 have been omitted. There are two features of this equation on which we should comment first. The term $\mathbf{n}_0\cdot\mathbf{u}/c$ in the coefficient of the first term would go away if the transformation from the fixed frame to the comoving frame had really been a Lorentz transformation, since a spatial derivative at constant time in the moving frame is not the same as a spatial derivative at constant time in the fixed frame owing to the relativity of time. But we are not altering the time coordinate in our procedure, so this term is left over. The terms involving the acceleration, $\mathbf{a}$, arise from $\partial\mathbf{u}/\partial t$. The actual acceleration is $\partial\mathbf{u}/\partial t+\mathbf{u}\cdot\nabla\mathbf{u}$, of course, so we might wonder about the other piece. But the extra parts would bring in terms that are of order u^2/c^2, and all such terms have been discarded. It is our opinion that the acceleration terms, as well as the $\mathbf{n}_0\cdot\mathbf{u}/c$ part of the coefficient of the first term, should be discarded. The reasoning is as follows. In a fluid flow problem the time derivative and the flow derivative $\mathbf{u}\cdot\nabla$ are generally of the same order of magnitude. If the acceleration terms above are ordered in this way then all of them are seen to be of order u^2/c^2 compared with the dominant terms, and may be discarded. The same is true of the $\mathbf{n}_0\cdot\mathbf{u}/c$ term multiplying $\partial I^{(0)}/\partial t$.

We suggest that (6.39) be retained for those problems involving nonrelativistic velocities that evolve on a light-transit time scale, and we simplify this equation by dropping the subject terms for problems with the fluid-flow time scale. Carrying out the simplification leads to

$$\frac{1}{c}\frac{DI^{(0)}}{Dt}+\mathbf{n}_0\cdot\nabla I^{(0)}-\frac{\nu_0}{c}\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\nabla_{\nu_0\mathbf{n}_0}I^{(0)}$$
$$+\frac{3}{c}\mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0 I^{(0)}=j^{(0)}-k^{(0)}I^{(0)}. \tag{6.40}$$

The operator D/Dt that appears here is the Lagrangian time derivative discussed in Section 2.2, $D/Dt\equiv\partial/\partial t+\mathbf{n}\cdot\nabla$.

We have so far not commented on the gradient with respect to momentum components. The reason for this term is simple. A photon with a fixed momentum travels along its ray and, as the local fluid velocity changes, so do its momentum components when referred to the local comoving frame. Thus the transport operator must account for this change with a gradient term multiplied by the rate of change of the comoving momentum along the ray. We normally use spherical coordinates, i.e., ν_0 and $\mathbf{n}_0$, rather than Cartesian coordinates for momentum space. So let's separate the vector $\mathbf{n}_0 \cdot \nabla\mathbf{u}$ into its radial and angular components in momentum space:

$$\mathbf{n}_0 \cdot \nabla\mathbf{u} = (\mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0)\mathbf{n}_0 + \mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot (\quad - \mathbf{n}_0\mathbf{n}_0). \tag{6.41}$$

When dotted with the momentum-space gradient, the first term picks up the radial derivative, i.e., $\partial/\partial\nu_0$, and the second one picks up $1/\nu_0$ times the angle gradient, the gradient on the unit sphere. We end up with a form like this:

$$\frac{1}{c}\frac{DI^{(0)}}{Dt} + \mathbf{n}_0 \cdot \nabla I^{(0)} - \frac{1}{c}\mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0 \nu_0 \frac{\partial I^{(0)}}{\partial \nu_0} - \frac{1}{c}\mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot (\quad - \mathbf{n}_0\mathbf{n}_0) \cdot \nabla_{\mathbf{n}_0} I^{(0)}$$
$$+ \frac{3}{c}\mathbf{n}_0 \cdot \nabla\mathbf{u} \cdot \mathbf{n}_0 I^{(0)} = j^{(0)} - k^{(0)} I^{(0)}. \tag{6.42}$$

The projection operator term here looks more complicated than it really is. The frequency-derivative term here is the Doppler correction, and the angle-derivative term is the abberation correction.

We will give the monochromatic and frequency-integrated moment equations that follow from this comoving-frame transport equation as they were derived by Buchler (1983). The only fussy part of the derivation is an integration by parts of the angle-derivative term. The frequency-dependent moment equations based on (6.39) are found to be

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot (\mathbf{u}E_\nu) + \frac{1}{c^2}\frac{D(\mathbf{u} \cdot \mathbf{F}_\nu)}{Dt} + \nabla \cdot \mathbf{F}_\nu + \left(\quad_\nu - \frac{\partial(\nu \quad_\nu)}{\partial \nu} \right) : \nabla\mathbf{u}$$
$$+ \frac{1}{c^2}\left(\mathbf{F}_\nu - \frac{\partial(\nu \mathbf{F}_\nu)}{\partial \nu} \right) \cdot \mathbf{a} = 4\pi j_\nu - k_\nu c E_\nu \tag{6.43}$$

and

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t} + \frac{1}{c}\nabla \cdot (\mathbf{u}\mathbf{F}_\nu) + \frac{1}{c}\frac{D(\mathbf{u} \cdot \quad_\nu)}{Dt} + c\nabla \cdot \quad_\nu + \frac{\mathbf{a}}{c}E_\nu + \frac{1}{c}\mathbf{F}_\nu \cdot \nabla\mathbf{u}$$
$$- \frac{1}{c}\frac{\partial(\nu \quad_\nu)}{\partial \nu} : \nabla\mathbf{u} - \frac{1}{c}\frac{\partial(\nu \quad_\nu)}{\partial \nu} \cdot \mathbf{a} = -k_\nu \mathbf{F}_\nu. \tag{6.44}$$

To avoid the dreadful profusion of superscripts and subscripts the designations $^{(0)}$ or $_{(0)}$ have been omitted from all the quantities. The object $\quad_\nu$ is the symmetric third rank tensor

$$ {}_{\nu} = \int_{4\pi} d\Omega\, \mathbf{nnn} I_{\nu}. \tag{6.45}$$

The colon ":" operator indicates summing the product of the tensor on the left with the tensor on the right over two indices, viz.,

$$: \quad \equiv R_{ij} S_{ij}. \tag{6.46}$$

The factors are symmetrical in their indices in such cases.

The frequency-integrated moment equations are

$$\begin{aligned} &\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u} E) + \frac{1}{c^2} \frac{D(\mathbf{u} \cdot \mathbf{F})}{Dt} + \nabla \cdot \mathbf{F} + \quad : \nabla \mathbf{u} + \frac{\mathbf{a}}{c^2} \cdot \mathbf{F} \\ &\quad = \int d\nu\, (4\pi j_{\nu} - k_{\nu} c E_{\nu}) = g^{0}_{(0)} \end{aligned} \tag{6.47}$$

and

$$\begin{aligned} &\frac{1}{c} \frac{\partial \mathbf{F}}{\partial t} + \frac{1}{c} \nabla \cdot (\mathbf{u} \mathbf{F}) + \frac{1}{c} \frac{D(\mathbf{u} \cdot \quad)}{Dt} + c \nabla \cdot \quad + \frac{\mathbf{a}}{c} E + \frac{1}{c} \mathbf{F} \cdot \nabla \mathbf{u} \\ &\quad = - \int d\nu\, k_{\nu} \mathbf{F}_{\nu} = \frac{\mathbf{g}_{(0)}}{c}. \end{aligned} \tag{6.48}$$

The term in third rank tensor vanishes in the frequency integration.

After considering the ordering of the terms in several different hydrodynamic regimes, Buchler (1983) suggests that certain of the terms can be dropped that would not be of relative magnitude u/c or larger in any regime. His suggestions are to drop all terms with c^2 in the denominator in (6.39) (thus reducing it to (6.40)), the terms with c^2 in the denominator in (6.43) and the terms in (6.44) with c in the denominator other than the $\partial \mathbf{F}_{\nu}/\partial t$ term. His suggested frequency-dependent moment equations are therefore

$$\frac{\partial E_{\nu}}{\partial t} + \nabla \cdot (\mathbf{u} E_{\nu}) + \nabla \cdot \mathbf{F}_{\nu} + \left({}_{\nu} - \frac{\partial (\nu \quad_{\nu})}{\partial \nu} \right) : \nabla \mathbf{u} = 4\pi j_{\nu} - k_{\nu} c E_{\nu} \tag{6.49}$$

and

$$\frac{1}{c} \frac{\partial \mathbf{F}_{\nu}}{\partial t} + \frac{1}{c} \nabla \cdot (\mathbf{u} \mathbf{F}_{\nu}) + c \nabla \cdot \quad_{\nu} = -k_{\nu} \mathbf{F}_{\nu}. \tag{6.50}$$

The zeroth moment equation (6.49) follows exactly from (6.40), but the first moment equation (6.50) does not. Terms of order $\mathbf{F}_{\nu} \nabla \cdot \mathbf{u}$ and $\quad_{\nu} : \nabla \mathbf{u}$ are neglected by Buchler in deriving (6.50). Buchler's simplified forms of the frequency-integrated equations are

$$\frac{\partial E}{\partial t} + \nabla \cdot (\mathbf{u} E) + \nabla \cdot \mathbf{F} + \quad : \nabla \mathbf{u} = \int d\nu\, (4\pi j_{\nu} - k_{\nu} c E_{\nu}) = g^{0}_{(0)} \tag{6.51}$$

and

$$\frac{1}{c}\frac{\partial \mathbf{F}}{\partial t}+\frac{1}{c}\nabla\cdot(\mathbf{uF})+c\nabla\cdot \quad =-\int d\nu\, k_\nu \mathbf{F}_\nu=\frac{\mathbf{g}_{(0)}}{c}. \tag{6.52}$$

6.5 A common-sense summary

After spending page after page of messy algebra deriving all the u/c corrections to radiation transport, it is easy to lose sight of the fact that some of the corrections really *are* significant while others are less so. Here are some of the main points:

- Ignoring the Doppler effect entirely and using the fixed-frame transport equation with the absorptivity and emissivity appropriate for matter at rest gives the wrong answer for the radiation when the velocity is supersonic and spectral lines dominate the opacity, and it also makes a significant error in the energy coupling rate of radiation and matter.
- The coupling terms are correctly given by the usual relations (6.33), (6.34) but those values are in the fluid frame. The transformations (6.35) and (6.36) have to be used to get the correct coupling terms for the fixed-frame equations.
- The coupling term in the material *internal* energy equation is indeed the fluid-frame energy term $g^0_{(0)}$ from (6.37), but in the material *total* energy equation it is combined with the work done by the radiation force, which turns it into the fixed-frame energy term equation (6.35).
- The diffusion limit gives a Fick's law form ($\mathbf{F} \propto -\nabla E$) for the flux in the fluid frame, not the fixed frame. In the fixed frame the convective flux of radiation enthalpy is added on.
- *Confusion of the correct frames for the coupling terms and the Fick's law flux is more serious than the other moderate corrections the u/c terms give.*
- The velocity terms in the comoving-frame energy equation (6.49) matter, especially the Doppler-shift frequency-derivative term; the ones in the comoving-frame momentum equation are less important. The aberration (angle-derivative) term in the transport equation does not survive in the energy moment and does not matter in the flux moment, so this can be dropped with little consequence.

The remaining part of this chapter is concerned with the problem of solving the comoving-frame equation if the choice is made to take $I_\nu^{(0)}$ or its moments as the basic variable(s).

6.6 The comoving-frame equation as a boundary-value problem

From a mathematical point of view the comoving-frame transport equation, say (6.40), is a partial differential equation for one scalar dependent variable in seven independent variables, namely x, y, z, ν, the two angles that describe $\mathbf{n}$, and t.

By contrast the fixed-frame equation has only four independent variables, x, y, z, and t, and the three photon momentum coordinates enter only as parameters, not as differentiation variables. The seven-dimensional problem has not been attacked yet, but efforts have been made to solve the comoving-frame equation in cases of lower dimensionality.

To illustrate the ideas in this case we will consider a problem with one spatial dimension, and for which photon flight time can be neglected, which allows us to drop the time derivatives. Spherical geometry is the interesting case, but we will discuss slab geometry for simplicity. Let the nontrivial space coordinate be x. If we particularize the simplified transport equation (6.40) to this case, and also drop the angle-derivative term, we end up with

$$\left(n_x + \frac{u_x}{c}\right)\frac{\partial I}{\partial x} - \frac{\nu}{c}n_x^2\frac{du_x}{dx}\frac{\partial I}{\partial \nu} = j_\nu - k_\nu I. \tag{6.53}$$

The seven independent variables have been reduced to two by the symmetry assumptions and by neglecting aberration. It is now quite feasible to apply a numerical technique for solving a hyperbolic equation in two dimensions to (6.53).

The concept of characteristics applies to (6.53). In fact, the fixed-frame transport equation is already in characteristic form, so the characteristic equation for (6.53) is just the equation for the Doppler effect:

$$\left(n_x + \frac{u_x}{c}\right)dI = (j_\nu - k_\nu I)dx \qquad \text{on} \quad \frac{d\nu}{dx} = -\frac{\nu}{cn_x + u_x}n_x^2\frac{du_x}{dx}. \tag{6.54}$$

The characteristic slope here can be simplified by dropping the u_x term in the denominator; the difference this makes is $O(u^2/c^2)$. The slope becomes

$$\frac{d\nu}{dx} \approx -\frac{\nu}{c}n_x\frac{du_x}{dx}. \tag{6.55}$$

We see that in (6.54) the direction in which the radiation flows is toward $+x$ if $n_x + u_x/c > 0$ and toward $-x$ if $n_x + u_x/c < 0$. In either case the sign of the change in $d\nu$ is the sign of $-du_x/dx$. That is, the comoving-frame frequency decreases along the ray if u_x increases, and increases if u_x decreases. Figure 6.1 illustrates three cases of characteristic morphology depending on whether du_x/dx is positive, negative or indefinite. The ordinate is $\log \nu$ in these figures, so the characteristics are parallel curves. Since u_x/c is most often relatively small the slopes of the characteristics are modest. This suggests solving the PDE as an initial-value problem with x as the time-like variable. If $n_x > 0$ we can sweep in the direction of increasing x, solving for one slice in the ν direction at each step. This procedure

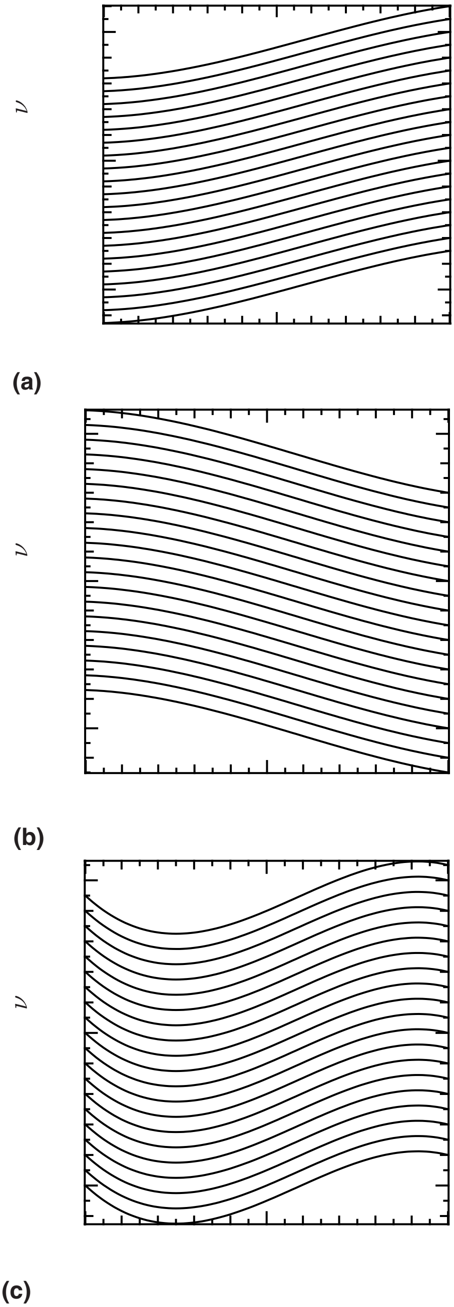

Fig. 6.1 Characteristics in ν vs x if $n_x u_x$: (a) decreases, (b) increases or (c) is nonmonotonic with x. Ordinate: log (ν) abscissa: .

will be subject to a Courant-like condition

$$\frac{\Delta x |du_x/dx|}{c\Delta \log \nu} < 1 \tag{6.56}$$

unless implicit x-differencing is used, which requires solving a system of equations at each ν slice. Normally this would not be a problem, since $|u_x| \ll c$. However, in order to solve transfer problems involving spectral lines we may want to use very fine frequency meshes, and if the flow is supersonic, (6.56) may be violated by a large factor. We can then turn to implicit differencing, which ensures stability, but accuracy may still be badly compromised.

The alternative to a spatial sweep is a frequency sweep. The question is, in which direction? If u_x is monotonically increasing with x, then a sweep from high frequency to low works. If u_x is decreasing, a sweep from low frequency to high works. The direction of the sweep also agrees with the frequency boundary on which we can give "initial" values that make the problem well posed. This boundary must be an "inflow" boundary. In order to have a well-posed problem, we should specify one and only one condition on each characteristic, at a place where it enters the problem on either a spatial or a frequency boundary. With $n_x > 0$ and $du_x/dx > 0$ that will be either the lower boundary in x or the upper boundary in ν, depending on which one the characteristic intersects. If $du_x/dx < 0$ the lower frequency boundary is used instead. Then what about the nonmonotonic velocity case? The frequency sweep is simply not feasible in that case, and we have to fall back on the spatial sweep, and use implicit x-differencing if necessary. In the nonmonotonic case the same characteristic can easily intersect the same frequency boundary two or more times, which makes the problem badly posed.

This discussion has been based on the transport equation, i.e., the equation for $I_\nu^{(0)}$. Very similar considerations arise if the system of comoving-frame moment equations is solved instead. The two equations for $E_\nu^{(0)}$ and $F_\nu^{(0)}$ (in one dimension) can be rearranged by addition and subtraction to give two equations mathematically similar to the transport equation, one with $n_x > 0$ and one with $n_x < 0$, to which the preceding remarks apply.

The successful applications of the comoving-frame transfer or moment equations have been to cases with a monotonic velocity.

6.7 Diffusion in the comoving frame

We have mentioned several times now that the diffusion approximation should be applied in the comoving frame if the intent is to obtain a flux that tends to zero as the mean free path becomes small, i.e., one which obeys Fick's law. We will substantiate that claim by rederiving the diffusion results beginning with the comoving-frame transport equation. The present discussion is new, and is intended as a complement to Section 97 of Mihalas and Mihalas (1984).

We begin with (6.39), which we arrange in the form

$$I^{(0)} = \frac{j^{(0)}}{k^{(0)}} - \frac{1}{k^{(0)}}\Bigg[\left(1 + \frac{\mathbf{n}_0}{c}\cdot\mathbf{u}\right)\left(\frac{1}{c}\frac{\partial I^{(0)}}{\partial t} + \frac{\mathbf{u}}{c}\cdot\nabla I^{(0)}\right) + \mathbf{n}_0\cdot\nabla I^{(0)} - \frac{\nu_0}{c}\left(\frac{\mathbf{a}}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\right)\cdot\nabla_{\nu_0\mathbf{n}_0} I^{(0)} + \frac{3}{c}\left(\frac{\mathbf{n}_0\cdot\mathbf{a}}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right) I^{(0)}\Bigg]. \tag{6.57}$$

The zeroth approximation is that $I^{(0)}$ is $j^{(0)}/k^{(0)}$ which we identify with the thermodynamic equilibrium value, the Planck function,

$$\frac{j^{(0)}}{k^{(0)}} = B_\nu(T). \tag{6.58}$$

Making this replacement for $I^{(0)}$ in (6.57) leads to

$$I^{(0)} = B_\nu(T) - \frac{1}{k^{(0)}}\Bigg[\left(1 + \frac{\mathbf{n}_0}{c}\cdot\mathbf{u}\right)\left(\frac{1}{c}\frac{\partial B_\nu}{\partial t} + \frac{\mathbf{u}}{c}\cdot\nabla B_\nu\right) + \mathbf{n}_0\cdot\nabla B_\nu - \frac{\nu_0}{c}\left(\frac{\mathbf{a}}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\right)\cdot\nabla_{\nu_0\mathbf{n}_0} B_\nu + \frac{3}{c}\left(\frac{\mathbf{n}_0\cdot\mathbf{a}}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right) B_\nu\Bigg]. \tag{6.59}$$

Now the task is to expand (6.59) and keep terms of first order in the velocity. The Planck function is isotropic, but it does have spatial and temporal gradients. Its momentum-space gradient is

$$\nabla_{\nu_0\mathbf{n}_0} B_\nu = \frac{\partial B_\nu}{\partial \nu}\mathbf{n}_0. \tag{6.60}$$

What results is

$$I^{(0)} = B_\nu - \frac{1}{k^{(0)}}\Bigg[\frac{dB_\nu}{dT}\left(\frac{1}{c}\frac{DT}{Dt} + \mathbf{n}_0\cdot\nabla T + \frac{\mathbf{n}_0\cdot\mathbf{u}}{c^2}\frac{DT}{Dt}\right) - \frac{1}{c}\left(\frac{\mathbf{a}\cdot\mathbf{n}_0}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right)\nu_0\frac{\partial B_\nu}{\partial \nu} + \frac{3}{c}\left(\frac{\mathbf{n}_0\cdot\mathbf{a}}{c} + \mathbf{n}_0\cdot\nabla\mathbf{u}\cdot\mathbf{n}_0\right) B_\nu\Bigg]. \tag{6.61}$$

Integrating (6.61) over angles leads to the diffusion formula for the comoving-frame monochromatic energy density

$$E_\nu^{(0)} = \frac{4\pi B_\nu}{c} - \frac{4\pi}{k_\nu^{(0)} c^2}\left[\frac{dB_\nu}{dT}\frac{DT}{Dt} + \frac{\nabla\cdot\mathbf{u}}{3}\left(3B_\nu - \nu\frac{\partial B_\nu}{\partial \nu}\right)\right]. \tag{6.62}$$

By virtue of Wien's displacement law, which is

$$B_\nu(T) = \nu^3 \times \text{function}(\nu/T), \tag{6.63}$$

it follows that

$$3B_\nu - \nu \frac{\partial B_\nu}{\partial \nu} = T \frac{dB_\nu}{dT}. \tag{6.64}$$

Making this replacement in (6.62) leads to the simple form

$$E_\nu^{(0)} = \frac{4\pi B_\nu}{c} - \frac{4\pi}{k_\nu^{(0)} c^2} \frac{dB_\nu}{dT}\left[\frac{DT}{Dt} + \frac{\nabla \cdot \mathbf{u}}{3} T\right]. \tag{6.65}$$

A frequency integration of (6.65) leads to the following expression for the total comoving-frame energy density

$$E = \frac{4\pi B}{c}\left[1 - \frac{4}{k_R c}\left(\frac{1}{T}\frac{DT}{Dt} + \frac{\nabla \cdot \mathbf{u}}{3}\right)\right], \tag{6.66}$$

where the Rosseland mean has been put in to replace its defining integral,

$$\frac{1}{k_R} = \frac{\int d\nu\, (1/k_\nu^{(0)}) dB_\nu/dT}{\int d\nu\, dB_\nu/dT}, \tag{6.67}$$

and where the notation $B = \int d\nu\, B_\nu$ has been used. In fact, $4\pi B/c$ is just the thermodynamic-equilibrium radiation energy density aT^4, and the fact that $dB/dT = 4B/T$ has been used in deriving (6.66). If the characteristic length scale for the problem is L and the characteristic time scale is τ, then the correction term in (6.66) is of order $\lambda_R/[c \min(\tau, L/u)]$, where λ_R denotes the Rosseland mean of the mean free path. So if the flow time L/u is longer than the characteristic time τ then the order of the correction is $\lambda_R/(c\tau)$, while if the flow time is shorter then the order is $(\lambda_R/L)(u/c)$. If the characteristic time is so short that it is comparable with the light-transit time c/L, then the order is just λ_R/L. When the flow time is shortest the size of the correction is the product of two small quantities, λ_R/L and u/c. This is why Mihalas and Mihalas (1984), Section 97, refer to this order of diffusion expansion as "second order diffusion."

The monochromatic flux derived from (6.61) is

$$\mathbf{F}_\nu^{(0)} = -\frac{4\pi}{3k_\nu^{(0)}} \frac{dB_\nu}{dT}\left(\nabla T + \frac{\mathbf{u}}{c^2}\frac{DT}{Dt} + \frac{\mathbf{a}}{c^2} T\right), \tag{6.68}$$

where we have again been able to use Wien's displacement law (6.64). We have the problem now of finding the physical explanations for the at-first-sight puzzling terms in the velocity and the acceleration. The velocity one is relatively easy. We need to recall that we are using the fixed-frame coordinates, and that in the Lorentz transformation to a locally comoving frame not only does the time derivative transform according to

$$\frac{\partial}{\partial t} \to \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{u} \cdot \nabla, \tag{6.69}$$

but the spatial derivative changes from one at constant t to one at constant comoving time t' according to the other Lorentz relation

$$\nabla \to \nabla' = \nabla + \frac{\mathbf{u}}{c^2}\frac{\partial}{\partial t}. \tag{6.70}$$

As we see, the $\mathbf{u}$ term in (6.68) is absorbed in changing ∇T to $\nabla' T$, apart from an error that is $O(u^2/c^2)$. In short, this velocity term is *real* and represents a relativistic effect on the diffusion flux. The acceleration term has a physical interpretation as well. The mean free path $1/k_\nu^{(0)}$ corresponds to a flight time $1/(k_\nu^{(0)}c)$ during which the local fluid velocity has increased by an amount $\mathbf{a}/(k_\nu^{(0)}c)$. The incremental boost by this velocity change makes a contribution to the flux of $-\mathbf{a}/(k_\nu^{(0)}c)$ times the sum of E and , which gives the term in question. To put it another way, if the flux is zero at the time when the photon flight begins, then by the end of the flight the fluid has accelerated away from the rest frame of that radiation, which produces a flux in the new fluid rest frame in the direction opposite to the acceleration that is proportional to the acceleration times the flight time.

The frequency-integrated flux is simply

$$\mathbf{F}^{(0)} = -\frac{4\pi}{3k_R}\frac{dB}{dT}\left(\nabla' T + \frac{\mathbf{a}}{c^2}T\right), \tag{6.71}$$

which, since $B = acT^4/(4\pi)$, can be written

$$\mathbf{F}^{(0)} = -K_R\left(\nabla' T + \frac{\mathbf{a}}{c^2}T\right), \tag{6.72}$$

in terms of the Rosseland-mean radiative conductivity

$$K_R = \frac{4\pi}{3k_R}\frac{dB}{dT} = \frac{4acT^3}{3k_R}. \tag{6.73}$$

The gradient operator here is the Lorentz-corrected one from (6.70).

Before writing the result for the pressure tensor we have to obtain the fully symmetric angle average of the product of four $\mathbf{n}$s. In other words, we want to evaluate

$$\frac{1}{4\pi}\int_{4\pi} d\Omega\, n_i n_j n_k n_l. \tag{6.74}$$

First we note that at least two of the four indices must be equal, since there are four indices and only three possible values they can take. If one pair out of the four are equal, say $i = j$, then it must be true that the other pair are equal as well, $k = l$, or otherwise the integral vanishes. Thus one contribution to the integral is proportional to $\delta_{ij}\delta_{kl}$. By choosing the first pair in the three possible ways and adding the results we get the required fully-symmetric form

$$\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}. \tag{6.75}$$

The integral must be proportional to this. We get the proportionality factor by taking the case $i = j = k = l = 1$, for which the angle average of n_x^4 is seen to be 1/5 by doing the integrations in that case, and for which the sum of delta functions is 3. Thus

$$\frac{1}{4\pi}\int_{4\pi} d\Omega\, n_i n_j n_k n_l = \frac{1}{15}\left(\delta_{ij}\delta_{kl} + \delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right). \tag{6.76}$$

Now we can do the integrations to find the monochromatic pressure tensor:

$$^{(0)}_{\nu} = \left(\frac{4\pi B_\nu}{c} - \frac{4\pi}{k_\nu^{(0)} c^2}\frac{dB_\nu}{dT}\frac{DT}{Dt}\right)\frac{}{3} - \frac{4\pi}{15 k_\nu^{(0)} c^2} T \frac{dB_\nu}{dT}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}} + \nabla\cdot\mathbf{u} \quad \right] \tag{6.77}$$

and the frequency-integrated form

$$^{(0)} = \frac{1}{3} a T^4 \left(1 - \frac{4}{k_R c}\frac{1}{T}\frac{DT}{Dt}\right) \quad - \frac{4aT^4}{15 k_R c}\left[\nabla\mathbf{u} + (\nabla\mathbf{u})^T + \nabla\cdot\mathbf{u} \quad \right]. \tag{6.78}$$

The form of this pressure tensor implies that radiation in the diffusion limit contributes normal and bulk viscosity coefficients (cf., (2.19))

$$\mu_R = \frac{4aT^4}{15 k_R c} \tag{6.79}$$

and

$$\zeta_R = \frac{5}{3}\mu_R = \frac{4aT^4}{9 k_R c} \tag{6.80}$$

to the mixed fluid. The bulk viscosity in this picture is not at all close to zero, which is a cause of consternation in view of some very general relativistic results. Mihalas and Mihalas (1984) discuss this subtle point and its explanation by Weinberg in terms of a slight renormalization of the definition of temperature.

The order of magnitude of the shear stress contribution to compared with the isotropic pressure is $O(\lambda_R u/(Lc))$ in the notation used above. In other words, isotropy of is a very good approximation in a diffusion regime owing to the smallness of both λ_R/L and u/c.

We want to compare the size of the radiative viscosity with the gas-kinetic viscosity of the matter, and also estimate the size of a Reynolds number based on radiative viscosity. For gas-kinetic viscosity we use an estimate $\mu = \rho\lambda_g v_{th}$, where λ_g is a gas-kinetic mean free path, and v_{th} is the mean thermal speed of a gas particle. In plasmas the electron and ion contributions must be summed, but it turns out that the ion contribution dominates by a factor $\approx \sqrt{m_H/m}$, where m_H is the hydrogen-atom mass. Thus the v_{th} that is appropriate is comparable with the sound speed. An estimate of $\rho\lambda_g$ is m_H/σ_C, where $\sigma_C \approx (e^2/kT)^2$ is the Coulomb scattering cross section. Working out the ratio of μ_R to μ_g gives this estimate

$$\frac{\mu_R}{\mu_g} \approx \frac{\sigma_C}{\sigma_R}\frac{aT^4}{p_g}\frac{c_s}{c}, \tag{6.81}$$

where σ_R is the Rosseland mean photo cross section, p_g is the gas pressure and c_s is the (gas) speed of sound. Radiative viscosity is only relevant in a diffusion region, which means inside a star as opposed to high in its atmosphere.[1] As a result the radiation pressure may be a few times larger than the gas pressure, but not orders of magnitude larger. The Coulomb cross section can be orders of magnitude larger than the photo cross section, however. For example, deep in a stellar envelope, say where kT is about 100 eV, σ_C is around 10^{-17} cm^2 while the Rosseland mean photo cross section is only 10^{-24}–10^{-23}. Thus the ratio of the cross sections can be of order 10^6. The ratio of the sound speed to the speed of light is the small factor, about 5×10^{-4} at 100 eV. The effect of the large cross section ratio more than makes up for the speed ratio, and we see that radiative viscosity dominates gas viscosity unless the radiation pressure is much smaller than the gas pressure.

The estimate of the Reynolds number based on radiative viscosity is

$$\mathrm{R_e} = \frac{\rho v L}{\mu_R} \approx k_R L \frac{p_g}{aT^4}\frac{v}{c_s}\frac{c}{c_s}. \tag{6.82}$$

We see that $\mathrm{R_e} \gg 1$ is quite likely whenever diffusion is valid, since then the optical depth $k_R L$ is large, the ratio of gas pressure to radiation pressure is not *very* small, the Mach number v/c_s is not too small, and c/c_s is large, perhaps $O(10^3)$.

6.8 Sobolev approximation

In a number of astrophysical environments, such as active galactic nuclei, molecular clouds, and some circumstellar outflows, as well as in laboratory plasmas that may be generated by high power lasers or magnetic pinches, a flow speed that is highly supersonic is combined with spectral line transport. This is the regime that Sobolev's approximation addresses. The original work on the high-velocity-gradient approximation is that of Sobolev (1960). The present discussion follows

[1] The momentum effects of radiation may be large for $\tau \ll 1$, but viscosity is not a useful way of treating them.

Castor (1970, 1974b). A special topic is needed in this case because when the flow velocity is supersonic the common approximation of forgetting about the Doppler shifts in calculating the opacity is quite wrong. An alternative approximation, of fully accounting for the fluid velocity but ignoring some spatial gradients, is better suited to these problems. This approximation, called after the first to study it, offers a relatively easy way to analyze a non-LTE problem with a myriad of lines and level populations to be computed. It has also led to a simple but useful way of approximating the force on the stellar material due to the large number of lines, and some simple but realistic stellar wind models (see Section 12.5).

The environment to which we apply the Sobolev approximation is a fluid flow with a velocity field $\mathbf{u}$ and a corresponding rate-of-strain tensor $\nabla\mathbf{u}$. A restriction that has to be imposed at the outset is that the principal strains all have the same sign in all parts of the flow. In other words, either the whole flow is expanding or it is contracting, but nowhere is there stretch in one direction and compression in another. (This case, which introduces nonlocal coupling, is treated by Rybicki and Hummer (1978).) We suppose that the velocity is not so terribly large, perhaps at most a few percent of c, so most of the $\mathbf{u}/c$ corrections to transport are relatively unimportant. The radiation–enthalpy–advection correction is probably important, but that is not the one we are going to talk about here. We will focus on line transport, and on the role the Doppler frequency-derivative term plays in line transport when the velocity is supersonic. If we let our imagination work on a typical ray cutting through the flow, and suppose that we follow a photon as it tracks along the ray, we see that the photon changes its local comoving-frame frequency as it goes, and the whole range it scans is typical of the flow velocity. Since the flow is supersonic, that range is larger than a typical line width. For some of the wind models, and for supernovae, the flow is actually around Mach 100, which means that not only does the range of comoving-frame frequencies cover one line, it can span the spacing between lines, and so the photon may even visit the frequencies of several lines as it moves along a single path.

We quote the comoving-frame transport equation again:

$$\frac{1}{c}\frac{DI_\nu}{Dt} + \mathbf{n}\cdot\nabla I_\nu - \frac{1}{c}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,\nu\frac{\partial I_\nu}{\partial\nu} - \frac{1}{c}\mathbf{n}\cdot\nabla\mathbf{u}\cdot(\quad - \mathbf{nn})\cdot\nabla_{\mathbf{n}}\,I_\nu$$
$$+ \frac{3}{c}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,I_\nu = j_\nu - k_\nu I_\nu, \tag{6.83}$$

where from now on we will understand that the comoving frame is used without explicitly adding superscripts to indicate this. In particular, ν will be the comoving-frame frequency.

As we just mentioned, the $1/c$ terms are relatively unimportant with the exception of the frequency-derivative term that expresses the Doppler effect. So we will first of all discard the time-derivative, angle-derivative, and dilation terms and keep only the spatial transport and the frequency-derivative terms on the left-hand side. The dimensional estimate of the ratio of these is

$$\frac{\text{frequency derivative}}{\text{spatial transport}} = \frac{u}{c}\frac{\nu}{\Delta\nu}, \tag{6.84}$$

where $\Delta\nu$ is the scale of the variations with frequency; we have assumed that the spatial scale lengths for the velocity and the comoving intensity are the same order. Thus in the flows being discussed, for which the velocity is larger than the line width converted to velocity units, the spatial transport term is dominated by the frequency-derivative term. Dropping the smaller spatial term leads to the *Sobolev equation*

$$-\frac{1}{c}\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}\,\nu\frac{\partial I_\nu}{\partial\nu} = j_\nu - k_\nu I_\nu. \tag{6.85}$$

This can be integrated immediately, but first we want to discuss what we should integrate over. We will handle individual lines singly using (6.85), which means we select a frequency band that contains that line, with the endpoints located in stretches of continuous spectrum just outside the line. Depending on the strain tensor $\nabla\mathbf{u}$, the comoving photon frequency either decreases as the photon travels its path (expansion) or increases (compression). We pick an initial value for the intensity on the high-frequency side in the first case and on the low-frequency side in the second case. We call that value I_c, where the $_c$ stands for "continuum". We assume that the opacity and emissivity are due to the line alone, because the line most often dominates the total opacity at frequencies within the line. The addition of a background continuum adds other terms that Hummer and Rybicki (1985) discuss. The expressions for absorptivity and emissivity of a line will be discussed in detail later, see (9.4) and (9.5). We write them here as

$$k_\nu = k_L\phi(\nu), \tag{6.86}$$

$$j_\nu = k_L S_L\phi(\nu). \tag{6.87}$$

The quantity k_L depends on the upper and lower level populations and the oscillator strength of the line:

$$k_L = \frac{\pi e^2}{mc}(gf)_{\ell u}\left(\frac{N_\ell}{g_\ell} - \frac{N_u}{g_u}\right) = \frac{h\nu}{4\pi}\left(N_\ell B_{\ell u} - N_u B_{u\ell}\right), \tag{6.88}$$

and S_L is the line source function which is either the Planck function or the result of a non-LTE kinetics model, as in the case of resonance line scattering. The line profile function $\phi(\nu)$ is normalized over frequency,

$$\int d\nu\, \phi(\nu) = 1, \tag{6.89}$$

and is given very often by the Doppler broadening formula, a Gaussian. That is what we will assume here. The Gaussian profile provides sharp edges to the line bandwidth, and avoids concerns about the effect of an extended tail of the profile function that complicates or even invalidates the integration over frequency.

We need a variable for the indefinite integral of the profile function. We let this be y defined by

$$y = \begin{cases} \int_{\nu}^{\infty} d\nu' \phi(\nu') & \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} > 0 \\ \int_{-\infty}^{\nu} d\nu' \phi(\nu') & \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n} < 0 \end{cases}, \tag{6.90}$$

i.e., the beginning point for y is on the side of the line where the initial value I_c is specified. The variation of y is from $y = 0$ on the incoming side of the line to $y = 1$ on the outgoing side. When integrating (6.85) over frequency we neglect the variation of the factor ν that multiplies the frequency derivative and replace it with the line-center frequency ν_0. Using the variable y in place of ν gives this result for the differential equation:

$$\frac{\nu_0|\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}|}{c}\frac{dI(y)}{dy} = k_L[S_L - I(y)], \tag{6.91}$$

which is immediately integrated using the boundary condition $I = I_c$ at $y = 0$:

$$I(y) = I_c \exp[-\tau(\mathbf{n})y] + S_L\{1 - \exp[-\tau(\mathbf{n})y]\}, \tag{6.92}$$

where $\tau(\mathbf{n})$, a direction-dependent quantity, is the *Sobolev optical depth*, defined by

$$\tau(\mathbf{n}) = \frac{k_L c}{\nu_0|\mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}|}. \tag{6.93}$$

This differs from the usual optical depth that increases from one spatial side of the problem to the other in that this is a strictly local variable. It is a measure of the optical thickness of the resonance zone along a given ray where a particular photon might be absorbed by the line. This resonance zone is quite limited in extent, because the velocity gradient is large, when the Sobolev approximation is appropriate.

We find the quantity $\bar{J}$ to which the photoabsorption rate is proportional by integrating $I(y)$ from $y = 0$ to $y = 1$, which is equivalent to first multiplying by $\phi(\nu)$ then integrating over ν, and then forming the average over direction. The first integration gives

$$\bar{I}(\mathbf{n}) = I_c(\mathbf{n})\beta(\mathbf{n}) + S_L[1 - \beta(\mathbf{n})], \tag{6.94}$$

in which $\beta(\mathbf{n})$ is what we shall call the *angle-dependent escape probability*,

$$\beta(\mathbf{n}) = \frac{1 - \exp[-\tau(\mathbf{n})]}{\tau(\mathbf{n})}. \tag{6.95}$$

The angle average gives

$$\bar{J} = \frac{1}{4\pi}\int_{4\pi} d\Omega\, \beta(\mathbf{n}) I_c(\mathbf{n}) + (1 - \beta) S_L, \tag{6.96}$$

and β, the *Sobolev escape probability*, is the angle average of $\beta(\mathbf{n})$:

$$\beta = \frac{1}{4\pi}\int_{4\pi} d\Omega\, \beta(\mathbf{n}) = \frac{1}{4\pi}\int_{4\pi} d\Omega\, \frac{1 - \exp[-\tau(\mathbf{n})]}{\tau(\mathbf{n})}. \tag{6.97}$$

We have allowed for the direction dependence of the continuum intensity $I_c(\mathbf{n})$.

The Sobolev escape probability result (6.96) was given in 3-D form by Rybicki and Hummer (1983), and earlier for spherical geometry by Castor (1970). Equation (6.96) has the same structure as the static escape probability approximation (5.23) and also the accelerated lambda iteration (ALI) *ansatz* (11.142), to be discussed later. In all the escape probability approximations, including ALI which uses such an approximation as the acceleration operator, $\bar{J}$ is replaced by a linear expression in terms of the local source function. Among these methods, the Sobolev approximation is unique in that the approximation is actually accurate in the circumstances for which it is intended – high Mach number flows. It has been applied extensively to the study of stellar winds and supernovae as part of non-LTE modeling of the spectra using multilevel model atoms.

A further application of the Sobolev approximation is to the calculation of the body force associated with the absorption of radiation by the lines. We recall that the body force (per unit mass) is given by (see (6.38))

$$\mathbf{g}_R = \frac{1}{\rho c}\int_0^\infty d\nu\, k_\nu \mathbf{F}_\nu. \tag{6.98}$$

The contribution to $\mathbf{g}_R$ from a single line treated in the Sobolev approximation can be evaluated by using the Sobolev expression for the intensity and forming the integral over frequency before integrating over direction. What results is

$$\mathbf{g}_R = \frac{k_L}{\rho c}\int_{4\pi} d\Omega\, \mathbf{n}\{I_c(\mathbf{n})\beta(\mathbf{n}) + S_L[1 - \beta(\mathbf{n})]\}. \tag{6.99}$$

We observe that $\beta(\mathbf{n})$ is an even function of angle, and that therefore the contribution of the local emissions to the flux, and thus to the net force, vanishes. This becomes a somewhat subtle point when we examine the diffusion-like corrections to the Sobolev approximation, because a moderate gradient in the source function produces another force contribution in the opposite direction. The size of the latter contribution compared with the Sobolev formula is proportional to the ratio v_{th}/u and is therefore small in a hypersonic flow. We drop the S_L term in the force and obtain

$$\mathbf{g}_R = \frac{k_L}{\rho c} \int_{4\pi} d\Omega \, \mathbf{n} I_c(\mathbf{n}) \beta(\mathbf{n}). \tag{6.100}$$

The Sobolev approximation has therefore given a formula for the force that requires no further calculations if $I_c(\mathbf{n})$ is just the free-streaming radiation of a stellar photosphere, for instance.

The applications of Sobolev theory have so far been almost exclusively to spherically symmetric problems. We recall that the strain tensor in spherical coordinates looks like

$$\begin{pmatrix} du/dr & 0 & 0 \\ 0 & u/r & 0 \\ 0 & 0 & u/r \end{pmatrix}, \tag{6.101}$$

and that therefore the strain rate projected on the ray direction is

$$\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} = \mu^2 \frac{du}{dr} + (1 - \mu^2) \frac{u}{r}. \tag{6.102}$$

We introduce an auxiliary quantity σ, *not* a cross section, by

$$\sigma = \frac{r}{u} \frac{du}{dr} - 1, \tag{6.103}$$

so we can write the projected strain rate as

$$\mathbf{n} \cdot \nabla \mathbf{u} \cdot \mathbf{n} = \frac{u}{r} (1 + \sigma \mu^2). \tag{6.104}$$

This means that the directional Sobolev optical depth is

$$\tau(\mathbf{n}) = \frac{\tau_0}{1 + \sigma \mu^2}, \tag{6.105}$$

in which the angle-independent optical depth τ_0 is

$$\tau_0 = \frac{k_L c r}{\nu_0 u}. \tag{6.106}$$

The formula for the escape probability then becomes

$$\beta = \int_0^1 d\mu \, \frac{1 + \sigma \mu^2}{\tau_0} \left[1 - \exp\left(-\frac{\tau_0}{1 + \sigma \mu^2} \right) \right]. \tag{6.107}$$

For the useful case that the continuum intensity is uniform within a cone $\mu > \mu_c$, which arises when the continuum radiation is supposed to come from a well-defined photosphere with no limb darkening, the integral for the I_c contribution to $\bar{J}$ can be expressed in terms of β:

$$\frac{1}{4\pi}\int_{4\pi} d\Omega\,\beta(\mathbf{n})I_c(\mathbf{n}) = \beta_c I_c \tag{6.108}$$

with

$$\beta_c(\tau_0,\sigma) = \frac{1}{2}[\beta(\tau_0,\sigma) - \mu_c\beta(\tau_0,\sigma\mu_c^2)]. \tag{6.109}$$

The behavior of $\beta(\tau_0,\sigma)$ is simple. When the optical depth is zero the escape probability is unity. For large optical depth the integral (6.107) becomes

$$\beta \sim \frac{1+\sigma/3}{\tau_0}. \tag{6.110}$$

The variation as the inverse of the optical depth is reminiscent of line scattering with complete redistribution over a Doppler profile, although the meaning of optical depth is different in the two cases. The static atmosphere gray escape probability decays exponentially at large optical depth, so that is quite unlike either case of line transfer.

The integral for $\beta(\tau_0,\sigma)$ cannot be evaluated in terms of elementary functions or the common special functions, but some excellent numerical approximations are available. Results for the 3-D case are given by Rybicki and Hummer (1983) as a single integral involving complete elliptic integrals of the first kind. Rybicki (private communication, August, 1978) suggested a very useful method of approximating $\beta(\tau_0,\sigma)$. It is based on a rational approximation to $[1-\exp(-x)]/x$:

$$\frac{1-\exp(-x)}{x} \approx \frac{P_{n-1}(x)}{Q_n(x)}, \tag{6.111}$$

in which P_{n-1} is a polynomial of the $(n-1)$th degree and Q_n is a polynomial of the nth degree. The zeroes of Q_n are complex in general. A partial-fraction expansion of P_{n-1}/Q_n leads to

$$\frac{1-\exp(-x)}{x} \approx \sum_{i=1}^{n}\frac{r_i}{x-z_i}. \tag{6.112}$$

If this approximation is inserted into (6.107), the integration over μ can be done analytically, with the result

$$\beta(\tau_0,\sigma) \approx -\sum_{i=1}^{n}\frac{r_i}{z_i}\left[1+\frac{\tau_0}{2t_i\sigma z_i}\log\left(\frac{t_i-1}{t_i+1}\right)\right], \tag{6.113}$$

in which

$$t_i = \sqrt{\frac{1}{\sigma}\left(\frac{\tau_0}{z_i} - 1\right)}. \tag{6.114}$$

The complex square root and logarithm functions are needed in these expressions. The $n = 2$ approximation of this kind, constrained to be accurate at $x = 0$ and $x \to \infty$, is

$$\frac{1 - \exp(-x)}{x} = \frac{1 + c_1 x}{1 + c_2 x + c_1 x^2}[1 + \epsilon(x)], \tag{6.115}$$

with $|\epsilon(x)| < 1.61 \times 10^{-2}$. The coefficients are $c_1 = 0.422\,26$ and $c_2 = 0.820\,47$. There is one complex-conjugate pair of roots z, given by

$$z = -0.971\,515 \mp 1.193\,464i, \qquad \frac{r}{z} = -0.5 \pm 0.011\,933\,91i. \tag{6.116}$$

Since the roots and residues are complex conjugates, it is sufficient to calculate just one of the terms in (6.113) and keep twice the real part. The approximation for $\beta(\tau_0, \sigma)$ has the same global relative accuracy as the rational approximation for $[1 - \exp(-x)]/x$, namely 1.61%. If more accuracy is needed, the next good approximation, for $n = 4$, may be used. The accuracy in that case is 1.5×10^{-4}, and there are two complex-conjugate pairs of roots and residues, $z = -1.915\,394 \pm 1.201\,751i$ with $r/z = -0.492\,975 \pm 0.216\,820i$, and $z = -0.048\,093 \pm 3.655\,564i$ with $r/z = -0.007\,025 \pm 0.050\,338i$.

Returning to the general discussion, we want to evaluate the radiation force. The reduction of the formula for $\mathbf{g}_R$ in spherical geometry gives

$$g_R = \frac{2\pi\nu_0}{\rho c^2}\frac{du}{dr}\int_{-1}^{1}\mu d\mu\, I_c(\mu)\frac{1 + \sigma\mu^2}{1 + \sigma}\left[1 - \exp\left(-\frac{\tau_0}{1 + \sigma\mu^2}\right)\right]. \tag{6.117}$$

When the continuum flux is confined to a narrow cone about the radial direction it is a good approximation to evaluate $1 + \sigma\mu^2$ as $1 + \sigma$ inside the integral, which then depends on

$$\tau_{\text{rad}} = \frac{\tau_0}{1 + \sigma} = \frac{k_L c}{\nu_0(du/dr)}. \tag{6.118}$$

In this radial-beaming approximation g_R is a function of the radial velocity gradient. We see that optically thin lines contribute an amount $(k_L F_{\nu_0}/c)$ that depends on the opacity and the continuum flux, but not on the velocity gradient. Optically thick lines contribute an amount that depends on the flux and the velocity gradient, but not on the opacity. The physical interpretation of this latter result is important. It says that a band $\Delta\nu$ of the continuous spectrum, which contains a momentum flux $\Delta\nu F_c/c$, gives its momentum to the matching spherical shell in the expanding envelope. The column thickness of that shell is $\rho\Delta r = \rho c\Delta\nu/(\nu du/dr)$, and dividing the momentum by the mass thickness gives the body force $(\nu F_c/\rho c^2)du/dr$.

6.9 Expansion opacity

Expansion opacity is a concept that grows out of Sobolev escape probability theory. It was introduced with that name by Karp, Lasher, Chan, and Salpeter (1977). The concept treats the situation in which the spectrum is filled with a forest of lines; there is a fluid velocity giving a significant strain-rate tensor $\nabla\mathbf{u}$ as discussed in the previous section; and the lines are spaced in $\log\nu$ by an amount that is comparable to or smaller than u/c. When these conditions occur, a photon may fly along until it hits resonance with one of the lines. If the strain rate in the medium corresponds to expansion, i.e., the eigenvalues of the strain-rate tensor are positive, then the photon's fluid-frame frequency steadily decreases as it goes along its path. When it hits resonance it is absorbed or scattered with a probability $1 - \exp[-\tau(\mathbf{n})]$, where $\tau(\mathbf{n})$ is the Sobolev optical depth discussed earlier, see (6.93). If this does occur, then the photon may be reemitted or scattered with a new direction, and it again steadily marches down in frequency. If we conceive of the distributions of lines in frequency as stochastic, the whole process looks like a random walk or diffusion. The expansion opacity is defined by identifying the mean free path in this random walk as $1/(\kappa_{\exp}\rho)$.

6.9.1 The Karp et al. model

The Karp *et al.* (1977) formulation is slightly different in flavor from the preceding paragraph. Karp *et al.* do not adopt a stochastic picture, and they also include the effect of continuous opacity (electron scattering). Their basic result is

$$\kappa_{\exp}(\nu) = \sigma_T \left\{ 1 - \sum_{j=J}^{N} \left[1 - \exp(-\tau_j)\right] (\nu_j/\nu)^s \exp\left(-\sum_{i=J}^{j-1} \tau_i \right) \right\}^{-1} . \quad (6.119)$$

The parameter s that appears as an exponent of ν_j/ν in (6.119) is defined as

$$s = \frac{\sigma_T \rho c}{du/d\ell}, \quad (6.120)$$

where σ_T is the Thomson scattering opacity and $du/d\ell = \mathbf{n}\cdot\nabla\mathbf{u}\cdot\mathbf{n}$. In terms of s the Sobolev optical depth (equation (6.93)) is $\tau(\mathbf{n}) = k_L s/\sigma_T \rho\nu_0$. The list of lines $\{\nu_j,\ j = 1, \ldots, N\}$ is arranged in descending order of frequency, and J is the index of the first line in the list with a frequency less than ν.

In order to gain a little more insight into (6.119), we can work out the simplified case introduced in their paper. We suppose that the lines all have the same strength, so their Sobolev optical depth is a constant τ, which we do *not* assume is small. We suppose the lines are equally spaced by Δ in frequency. We will assume $s\Delta/\nu \ll 1$. Equation (6.119) can be expressed as $\kappa_{\rm exp}(\nu) = \sigma_T/(1-\epsilon_\nu)$, in which ϵ_ν is the summation in the denominator of the right-hand side. In this simplified case the sum becomes a geometrical series, and leads to

$$\epsilon_\nu = \frac{[1-\exp(-\tau)](\nu_J/\nu)^s}{1-\exp(-s\Delta/\nu-\tau)}. \tag{6.121}$$

To first order in the small quantity $s\Delta/\nu$ this result is the same as

$$\epsilon_\nu = \left[1 + \frac{s\Delta}{\nu}\frac{\exp(-\tau)}{1-\exp(-\tau)} + \frac{(\nu-\nu_J)s}{\nu}\right]^{-1}. \tag{6.122}$$

This result leads to the following expression for the expansion opacity:

$$\kappa_{\rm exp} = \sigma_T + \frac{\sigma_T\nu}{s}\frac{\exp(\tau)-1}{\Delta+(\nu-\nu_J)[\exp(\tau)-1]}. \tag{6.123}$$

In the large-τ limit this is the result given by Karp *et al.* The expansion opacity in this limit is not dependent on τ, nor on the spacing of the lines, but solely on the frequency displacement from the next-lower line in the list: $\kappa_{\rm exp} \approx \sigma_T[1+\nu/(\nu-\nu_J)/s]$. If τ is finite but we set $\nu=\nu_J$ the results for the line contribution to $\kappa_{\rm exp}$ would be $\sigma_T\nu/(s\Delta)[\exp(\tau)-1]$.

Blinnikov (1996) gives a criticism of Karp *et al.* (1977), based on a Boltzmann equation solution. He finds that the proper mean free path calculation of the photon should average over its history in the *upwind* direction, not in the downwind direction as in Karp *et al.* This is pointed out by Pinto and Eastman (2000). But, as a matter of fact, Blinnikov's result (20) for the expansion opacity, when evaluated for the case of equally-spaced lines of uniform strength, and with $s\Delta/\nu \ll 1$ and $s \gg 1$ as before, leads to the same result for ϵ_ν as in Karp *et al.*'s model, except that $\nu-\nu_J$ is replaced by $\nu_L-\nu$; the index L is that of the closest line in the list with frequency larger than ν, just as J is the index of the closest line with frequency smaller than ν. Blinnikov's model and that of Karp *et al.* are equivalent apart from reversing the order of the line list.

The next step in the use of the monochromatic expansion opacity is to evaluate its Rosseland mean over a frequency band that is large compared with the line spacing; this is the quantity that enters diffusion calculations in the Sobolev regime. We find, following Karp *et al.*, that

$$\frac{\sigma_T}{\kappa_{R,\rm exp}} = 1 - \langle\epsilon_\nu\rangle, \tag{6.124}$$

where the brackets signify the average over the frequency band. Using (6.122) for equally-spaced constant-strength lines leads to

$$\langle \epsilon_\nu \rangle = \frac{\nu}{s\Delta} \log \left[\frac{1 - \exp(-\tau) + s\Delta/\nu}{1 - \exp(-\tau) + (s\Delta/\nu)\exp(-\tau)} \right]. \tag{6.125}$$

The corresponding value of the Rosseland mean depends on how large τ is:

$$\kappa_{R,\mathrm{exp}} \approx \begin{cases} 2\dfrac{\nu|du/d\ell|}{\rho c\Delta} & \tau \gg 1 \\ \dfrac{k_L}{\rho\Delta} & \dfrac{\sigma_T \rho c\Delta}{\nu|du/d\ell|} \ll \tau \ll 1 \\ \sigma_T & \tau \ll \dfrac{\sigma_T \rho c\Delta}{\nu|du/d\ell|} \end{cases}. \tag{6.126}$$

The first case here is when the lines are optically thick in the Sobolev sense. The Rosseland absorption coefficient, per unit length, becomes twice the probability of encountering a line per unit path length. This factor 2 is related to the choice of equally-spaced lines. We will comment on this factor below, in the discussion of Wehrse, Baschek, and van Waldenfels (2003). The second case applies if the lines are optically thin in the Sobolev sense, yet the smeared-out line opacity is still greater than the continuum absorption coefficient. This only applies if the continuum optical depth of the typical path length between line encounters, $\Delta c/(\nu du/d\ell)$, is small, the usual case (and assured if $s\Delta/\nu \ll 1$). In the third case the smeared-out line opacity is less than the continuum opacity, and the Rosseland mean is unaffected by lines.

6.9.2 Friend and Castor model

The stochastic approach to line transfer with a forest of lines treated in the Sobolev approximation is presented by Friend and Castor (1983). The statistical model of the line distribution is that the lines with various values of the line strength k_L have independent Poisson distributions in frequency, so that the mean number of lines in a frequency interval $[\nu, \nu + \Delta\nu]$ and in a strength interval $[k_L, k_L + \Delta k_L]$ is $\mu(k_L, \nu)\Delta k_L \Delta\nu$. If we now consider a photon of frequency ν traveling in the direction $\mathbf{n}$, the probability that it will encounter a line in this strength range in traveling a distance $d\ell$ is

$$\mu(k_L, \nu)\Delta k_L \frac{\nu d\ell}{c} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n}. \tag{6.127}$$

The probability that there will be an interaction, given that a line is encountered, is $1 - \exp[-\tau(\mathbf{n})]$, with the Sobolev optical depth given by (6.93) in terms of k_L and $\mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n}$. Summing over the line strength distribution gives the total probability of encountering a line and being absorbed or scattered in $d\ell$:

$$\mu(k_L, \nu) dk_L \frac{\nu}{c} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n} \{1 - \exp[-\tau(\mathbf{n})]\} d\ell, \tag{6.128}$$

from which the effective opacity, the same as we mean by the expansion opacity and including now the continuous opacity, is seen to be

$$\kappa_{\exp}(\nu) = \frac{\nu}{\rho c} \mathbf{n} \cdot \nabla\mathbf{u} \cdot \mathbf{n} \int_0^\infty \mu(k_L, \nu) \{1 - \exp[-\tau(\mathbf{n})]\}\, dk_L + \sigma_T, \tag{6.129}$$

where, in the integrand, $\tau(\mathbf{n})$ should be substituted using (6.93). The result (6.129) is expressed in terms of the hypothetical Poisson distribution of lines of different strengths. That turns out to be very useful when the line statistics have been expressed in analytic form, and the further development in Friend and Castor (1983) makes use of this to evaluate the overlapping line effect on radiatively-driven stellar winds (cf. Section 12.5). But we can also estimate the line densities by using the actual line list: the density is approximated by summing the lines within a bandwidth $\Delta\nu$ and dividing by the bandwidth. This gives equation (9) of Friend and Castor (1983), which takes this form in the present notation:

$$\kappa_{\exp}(\nu) = \frac{\nu}{\rho c \Delta\nu} \mathbf{n} \cdot \nabla\mathbf{u} \cdot s\mathbf{n} \sum_{\Delta\nu} \{1 - \exp[-\tau(\mathbf{n})]\} + \sigma_T. \tag{6.130}$$

6.9.3 Eastman and Pinto model; Wehrse, Baschek, and von Waldenfels model

In an ambitious study of non-LTE spectrum modeling of supernovae, Eastman and Pinto (1993) employ expansion opacity to treat, in an approximate way, the effect of the forest of weak lines for which a detailed transfer solution is not feasible. They independently derive the expansion opacity, formula (4.2) in their paper, which is identical with (6.130). The more general result, in which the Sobolev approximation has not been made, is their equation (23), in which the left-hand side should read κ_{R}^{-1}. This equation agrees with Blinnikov's equation (14).

Wehrse *et al.* (2003) discusses methodically the calculation of the expansion opacity and the general issue of the diffusion approximation for a flow with velocity gradients. They consider both a deterministic distribution of lines, and a stochastic distribution of lines, as well as both infinitely narrow lines (Sobolev limit) and lines with finite widths. Their results for the monochromatic opacity with infinitely narrow lines reproduce Blinnikov's expressions, and with finite line widths the Wehrse *et al.* expressions are essentially the same as Blinnikov's; both imply a local harmonic mean opacity

$$\kappa_{\rm eff}(\nu)^{-1} = \frac{s}{\sigma_T}\left\langle \int_0^\infty \exp\left[-\frac{s}{\sigma_T}\int_{\xi-\eta}^{\xi}\kappa(\zeta)\,d\zeta\right]d\eta\right\rangle, \tag{6.131}$$

in which $\xi = -\log\nu$, the variable η is defined in terms of the path length ℓ by $\eta = \sigma_T\rho\ell/s$, and the angle brackets signify an average over a moderate bandwidth centered at ν. In the inner integral $\kappa(\zeta)$ is the monochromatic opacity at $\nu = \exp(-\zeta)$ calculated without the fluid velocity but including all the other line broadening mechanisms.

For the stochastic, infinitely narrow line case, in which the lines form a Poisson point process as described above, Wehrse *et al.* give a result for the effective opacity that is equivalent to (6.129) or (6.130), and thus also in agreement with the Eastman and Pinto (1993) formula and Pinto and Eastman (2000), equation (9). Wehrse *et al.* give results using the stochastic model in the particular case that the line strengths follow a power-law model $\mu(k_L,\nu) \propto k_l^{-\alpha}$, just as in the Castor, Abbott, and Klein (1975) stellar wind model (cf. Section 12.5), and as used in Friend and Castor (1983). The exponent α in Wehrse *et al.* is $1-\alpha$ as used by Castor *et al.* Wehrse *et al.*, use high and low cutoffs in k_L in evaluating the effective opacity; this is unnecessary if $0 < \alpha < 1$.

Some clarification is needed on the differences between the monochromatic Karp *et al.* opacity and the stochastic model opacity. The former is a strong function of frequency in the spaces between lines. The latter, which is a constant in a given medium-scale frequency band,[2] actually corresponds to the *expectation value* of the mean free path with respect to realizations of the Poisson process that produces the line spectrum. Friend and Castor (1983) argue that the expectation of the monochromatic intensity can be calculated from the opacity as defined by (6.129). The calculation by Wehrse *et al.* of the diffusion flux for infinitely narrow Poisson lines gives a result consistent with (6.129), in confirmation of this argument. Thus the local Rosseland mean expansion opacity, in the infinitely narrow stochastic line case, is the same quantity given by (6.129) or (6.130), Eastman and Pinto (1993) and Pinto and Eastman (2000), equation (9).

We now return to the factor 2 difference between the Rosseland average of the expansion opacity for equally-spaced strong lines, and the Poisson average opacity for strong lines with the same mean density in frequency. In fact, this is the difference between the Poisson statistics and equally-spaced lines. A property of the Poisson process is that if a frequency is selected at random, and the frequency displacement is found to the next higher frequency in a realization of the Poisson

[2] As in (6.130); a band large enough to include many lines, but small in comparison with the frequency itself or the interval over which the line statistics change substantially.

distribution, then the mean value of that interval is just equal to the reciprocal of the line density. But if this same frequency is compared with the list of equally-spaced lines, then the mean displacement is one half of the line spacing in the list. This is the factor of 2. These results are verified by a simple Monte Carlo calculation, which also shows that (6.119) does lead to $\langle 1 - \epsilon_\nu \rangle = s\Delta/\nu$ for strong lines with a mean spacing in frequency of Δ, twice the result for equally spaced lines.

In summary, the stochastic model is a useful tool for simulations of spectra when there is a dense forest of lines and the Sobolev approximation is valid, such as in novae and supernovae, stellar winds, and high-velocity laser-plasma experiments.

6.9.4 Expansion opacity example: the iron spectrum

The ideas of expansion opacity are best illustrated using an idealized but realistic example. We have chosen the spectrum of the ions Fe II, Fe III, and Fe IV as calculated in LTE at a temperature $T = 3$ eV and electron density $N_e = 10^{16}\,\mathrm{cm}^{-3}$. Iron is assumed to have the number abundance 4×10^{-5} relative to hydrogen. For simplicity the three ions are assumed to have equal abundance. The data of Kurucz and Bell (1995) are used for the line frequencies, energy levels, oscillator strengths, and the radiative and Stark damping constants.[3] Each of the lines has a Voigt profile based on the thermal Doppler width and the damping constant. The total number of iron lines treated is about 37 000, and of these 836 lie within the range of Figure 6.2(a), which shows a portion of the synthesized opacity spectrum between 213 and 222 nm. In order to produce the expansion opacities a larger region, from 182 to 222 nm, had to be synthesized first, to provide the data for (6.124), (6.130) and (6.131). We calculate the harmonic mean opacity over the band illustrated in Figure 6.2(a), accounting for the velocity gradient, in three ways: (i) using the Karp *et al.*/Blinnikov formula (6.119) with (6.124); (ii) using the Friend–Castor/Eastman–Pinto formula (6.130); and (iii) using the Blinnikov/Wehrse *et al.* formula (6.131) for lines with finite widths. These results for k_{eff} in this band are shown as functions of s in Figure 6.2(b).

It is apparent that the calculations of k_{eff} that omit the intrinsic line widths, i.e., in the Sobolev approximation, considerably underestimate the effective opacity unless the velocity gradient is quite large, i.e., s is less than some amount. In this case that value of s is about 10^3. This is the s for which about half the lines in the frequency band in question have Sobolev optical depths less than unity. The two Sobolev calculations shown in Figure 6.2(b) agree well with each other. Since one is a deterministic formula that makes no assumption about the line statistics,

[3] We are grateful to R. L. Kurucz and B. Bell and the Smithsonian Astrophysical Observatory for making these data available.

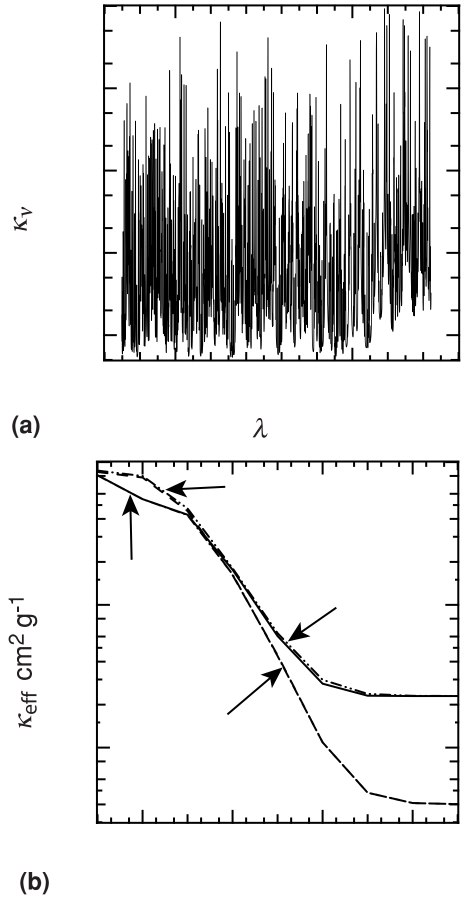

Fig. 6.2 (a) Illustration of the opacity in a synthetic spectrum of Fe II, Fe III, and Fe IV vs wavelength. (b) Mean expansion opacity over the band in (a) using: A (6.119) (dash-dotted line); B (6.130) (dashed line); C (6.131) (solid line); and D (6.130) with the replacement of σ_T with the static Rosseland mean (dash-dot-dot line). The four curves overlie each other in pairs, but differently on the left- and right-hand sides, as indicated in the figure by the labels A, B, C, and D.

and the other assumes that the lines have a Poisson distribution, the agreement indicates that the stochastic approach gives a quite usable answer.

The large-s and small-s limits of $k_{\rm eff}$ have natural interpretations. The large-s limit is the harmonic mean or Rosseland mean for this band of the $du/d\ell = 0$ spectral opacity, or at least it should be if the approximation were accurate. The small-s limit is the arithmetic or Planck mean for this band. As $s \to 0$, the region over which the average is taken becomes larger and larger for (6.124) and (6.131), which accounts for differences that are seen in the different models at small s, owing also to the fact that the line density varies somewhat with frequency. The values of the harmonic and arithmetic means are 2.23 and 86 $\mathrm{cm}^2\,\mathrm{g}^{-1}$ for this band.

An *ad hoc* modification of the Sobolev calculations that makes them substantially more usable is to replace the added quantity σ_T in (6.130) with the actual harmonic mean opacity calculated without the velocity gradient. The Sobolev result for the line opacity alone tends to zero as the velocity gradient becomes small, while we know that with the intrinsic line widths taken into account the harmonic mean can be significantly larger than σ_T owing to line blanketing, i.e., bandwidth constriction. The fourth curve plotted in Figure 6.2(b) shows the result using (6.130) with σ_T replaced by $\kappa_{\mathrm{eff}}(du/d\ell = 0)$. The agreement with (6.131) is good over the entire range.