Incompressible Viscous Flows via Finite Difference Methods

5.1 GENERAL

The basic concepts in FDM and applications to simple partial differential equations have been presented in the previous chapters. This chapter will focus on incompressible viscous flows in which the physical property of the fluid, *incompressibility*, requires substantial modifications of computational schemes discussed in Chapter 4.

In general, a flow becomes incompressible for low speeds, that is, $M < 0.3$ for air, and compressible for higher speeds, that is, $M \geq 0.3$, although the effect of compressibility may appear at the Mach number as low as 0.1, depending on pressure and density changes relative to the local speed of sound. Computational schemes are then dictated by various physical conditions: viscosity, incompressibility, and compressibility of the flow. The so-called pressure-based formulation is used for incompressible flows to keep the pressure field from oscillating, which may arise due to difficulties in preserving the conservation of mass or *incompressibility condition* as the sound speed becomes so much higher than convection velocity components. The pressure-based formulation for incompressible flows uses the primitive variables (p, v_i, T), whereas the density-based formulation applicable for compressible flows utilizes the conservation variables $(\rho, \rho\mathrm{v}_i, \rho E)$.

Incompressible viscous flows are usually computed by means of the continuity and momentum equations. If temperature changes in natural and/or forced convection heat transfer are considered, then the energy equation is also added. For simplicity in demonstrating the computational strategies for incompressible flows in general, we shall consider only the isothermal case in this chapter. In Chapter 6, it will be shown that computational schemes for incompressible flows can also be developed from preconditioning processes of the density-based formulation which is originally intended for compressible flows. This process leads to implementations of an algorithm applicable for both compressible and incompressible flows [Merkle et al., 1998].

In dealing with incompressible flows, there are two approaches: primitive variable methods and vortex methods. The primitive variable approach includes the artificial compressibility method (ACM) [Chorin, 1967], and the pressure correction methods (PCM) including the marker and cell (MAC) method [Harlow and Welch, 1965], the semi-implicit method for pressure linked equations (SIMPLE) [Patankar and Spalding, 1972], and the pressure implicit with splitting of operators (PISO) [Issa, 1985]. The main difficulty in incompressible flows is the accurate solution for pressure. Thus, the purpose of the vortex methods is to remove the pressure terms from the momentum equations, which can be achieved by solving the vorticity transport equation(s) (one scalar equation for 2-D and three vector component equations for 3-D).

In view of the fact that the transition between incompressible and compressible flows involves a complex process of interactions between inviscid and viscous properties, it is reasonable to seek a unified approach in which both incompressible and compressible flows can be accommodated. This subject will be discussed in Section 6.4, Preconditioning Process for Compressible Flows and Viscous Flows, and in Section 6.5 on the flowfield-dependent variation (FDV) methods. For this reason, treatments of incompressible flows in this chapter will be brief.

5.2 ARTIFICIAL COMPRESSIBILITY METHOD

The governing equations for incompressible viscous flows, known as the incompressible Navier-Stokes system of equations, are written in nondimensionalized form as

Continuity

$$\mathrm{v}_{i,i} = 0 \tag{5.2.1}$$

Momentum

$$\frac{\partial \mathrm{v}_i}{\partial t} + \mathrm{v}_{i,j}\mathrm{v}_j = -p_{,i} + \frac{1}{Re}\mathrm{v}_{i,jj} \tag{5.2.2}$$

where the following nondimensional quantities are used:

$$\mathrm{v}_i = \frac{\mathrm{v}_i^*}{\mathrm{v}_\infty}, \qquad x_i = \frac{x_i^*}{L}, \qquad p = \frac{p^*}{\rho \mathrm{v}_\infty^2}, \qquad t = \frac{t^*\mathrm{v}_\infty}{L}, \qquad Re = \frac{\mathrm{v}_\infty L}{\nu}$$

with asterisks implying the physical variable and Re being the Reynolds number.

In the artificial compressibility method (ACM), the continuity equation is modified to include an artificial compressibility term which vanishes when the steady state is reached [Chorin, 1967]:

$$\frac{\partial \tilde{\rho}}{\partial \tilde{t}} + \mathrm{v}_{i,i} = 0 \tag{5.2.3}$$

where $\tilde{\rho}$ is an artificial density, equated to the product of artificial compressibility factor β and pressure,

$$\tilde{\rho} = \beta^{-1} p \tag{5.2.4}$$

Here $\frac{\partial \tilde{\rho}}{\partial \tilde{t}} \to 0$ at the steady state and $\tilde{t}$ is a fictitious time.

With these definitions and combining (5.2.1–5.2.4), we may write the incompressible Navier-Stokes system of equations in the form

$$\frac{\partial \mathbf{W}}{\partial t} + \mathbf{A}_i \frac{\partial \mathbf{W}}{\partial x_i} = \frac{1}{Re}\frac{\partial}{\partial x_i}\left(\mathbf{B}_{ij}\frac{\partial \mathbf{W}}{\partial x_j}\right) \tag{5.2.5}$$

with

$$\mathbf{W} = \begin{bmatrix} p \\ \mathrm{v}_j \end{bmatrix}, \qquad \mathbf{A}_i = \frac{\partial \mathbf{D}_i}{\partial \mathbf{W}}, \qquad \mathbf{D}_i = \begin{bmatrix} \beta \mathrm{v}_i \\ \mathrm{v}_i \mathrm{v}_j + p\delta_{ij} \end{bmatrix}, \qquad \mathbf{B}_{ij} = \begin{bmatrix} 0 & \\ & \delta_{ij} \end{bmatrix}$$

$$\mathbf{A}_1 = \frac{\partial \mathbf{D}_1}{\partial \mathbf{W}} = \begin{bmatrix} 0 & \beta & 0 & 0 \\ 1 & 2u & 0 & 0 \\ 0 & \mathrm{v} & u & 0 \\ 0 & w & 0 & u \end{bmatrix} \qquad \mathbf{A}_2 = \frac{\partial \mathbf{D}_2}{\partial \mathbf{W}} = \begin{bmatrix} 0 & 0 & \beta & 0 \\ 0 & \mathrm{v} & u & 0 \\ 1 & 0 & 2\mathrm{v} & 0 \\ 0 & 0 & w & \mathrm{v} \end{bmatrix}$$

$$\mathbf{A}_3 = \frac{\partial \mathbf{D}_3}{\partial \mathbf{W}} = \begin{bmatrix} 0 & 0 & 0 & \beta \\ 0 & w & 0 & u \\ 0 & 0 & w & \mathrm{v} \\ 1 & 0 & 0 & 2w \end{bmatrix}$$

Let us now investigate the eigenvalues of $\mathbf{A}_i$,

$$|\mathbf{A}_i - \lambda_i \mathbf{I}| = 0$$

where the eigenvalues of $\mathbf{A}_i$ $(i = 1, 2, 3)$ are, respectively,

$$(u, u, u \pm \sqrt{u^2 + \beta}), \qquad (\mathrm{v}, \mathrm{v}, \mathrm{v} \pm \sqrt{\mathrm{v}^2 + \beta}), \qquad (w, w, w \pm \sqrt{w^2 + \beta}) \tag{5.2.6}$$

in which $\sqrt{\beta}$ is the artificial speed of sound (often called the artificial compressibility factor) with β being chosen adequately (between 0.1 and 10 as suggested by Kwak et al. [1986]). The idea is to maintain low enough β (close to the convective velocity) to overcome stiffness associated with a disparity in the magnitudes of the eigenvalues, but high enough such that pressure waves (moving with infinite speed at incompressible limit) be allowed to travel far enough to balance viscous effects. As a result, the conservation of mass or incompressibility condition is assured by means of an artificial compressibility. In this process, it is possible to obtain the correct pressure distributions. The solution of (5.2.5) is usually obtained by the Crank-Nicolson method.

From the point of view of linear algebra, the finite difference algebraic equations resulting from (5.2.5) are well conditioned (with a proper choice of β), as compared to the original equations (5.2.1) and (5.2.2). This is due to the well-conditioned eigenvalues given by (5.2.6). All other solution schemes for incompressible flows without using the artificial compressibility must employ special approaches as discussed below.

5.3 PRESSURE CORRECTION METHODS

5.3.1 SEMI-IMPLICIT METHOD FOR PRESSURE-LINKED EQUATIONS (SIMPLE)

It is well known that, if the finite difference equation is written in control volume grids (Section 1.4) for continuity $\mathrm{v}_{i,i} = 0$, this will lead to nonphysical, checkerboard-type oscillations of velocity in each one-dimensional direction (same values repeated at every other node, assuming that the velocity distribution between the adjacent nodes is linear). As a consequence, the mass is not conserved, thus causing the pressure to undergo similar oscillations. This is particularly true when pressure becomes constant ($p_{,i} = 0$) for the same reason as $\mathrm{v}_{i,i} = 0$. These difficulties can be shown to be remedied by using staggered grids [velocity nodes staggered with respect to pressure nodes (Figure 5.3.1)]

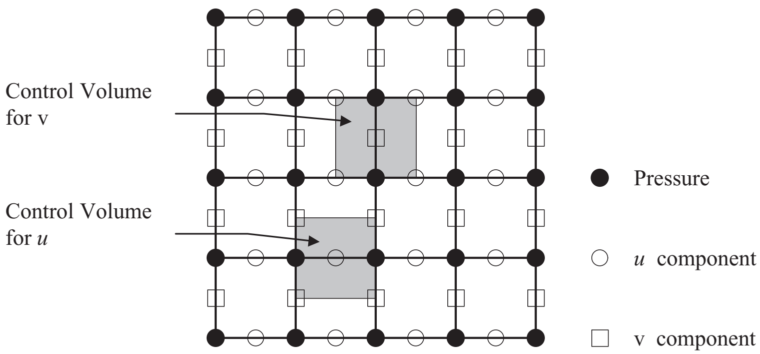

Figure 5.3.1 Computational domain for staggered grid.

in the algorithm known as SIMPLE [Patankar and Spalding, 1972]. In this method, the predictor-corrector procedure with successive pressure correction steps is used:

$$p = \overline{p} + p' \tag{5.3.1}$$

where p is the actual pressure, $\overline{p}$ is the estimated pressure, and p' is the pressure correction. Likewise, the actual velocity components in two-dimensions are

$$u = \overline{u} + u' \tag{5.3.2a}$$

$$\mathrm{v} = \overline{\mathrm{v}} + \mathrm{v}' \tag{5.3.2b}$$

The pressure corrections are related to the velocity corrections by approximate momentum equations,

$$\rho \frac{\partial u'}{\partial t} = -\frac{\partial p'}{\partial x} \tag{5.3.3a}$$

$$\rho \frac{\partial \mathrm{v}'}{\partial t} = -\frac{\partial p'}{\partial y} \tag{5.3.3b}$$

or

$$u' = -\frac{\Delta t}{\rho}\frac{\partial p'}{\partial x} \tag{5.3.4a}$$

$$\mathrm{v}' = -\frac{\Delta t}{\rho}\frac{\partial p'}{\partial y} \tag{5.3.4b}$$

Combining (5.3.2) and (5.3.4) and substituting the result into the continuity equation, we obtain the so-called pressure-correction Poisson equation of an elliptic form,

$$p'_{,ii} = -\frac{\rho}{\Delta t}\left(\frac{\partial\, \mathrm{v}_i}{\partial x_i} - \frac{\partial\, \overline{\mathrm{v}}_i}{\partial x_i}\right) = \frac{\rho}{\Delta t}\frac{\partial\, \overline{\mathrm{v}}_i}{\partial x_i}, \qquad (i = 1,\ 2) \tag{5.3.5}$$

where we set $\frac{\rho}{\Delta t}\frac{\partial \mathrm{v}_i}{\partial x_i} = 0$ to enforce the mass conservation at the current iteration step. An iterative procedure is used to obtain a solution as follows [Raithby and Schneider, 1979].

(a) Guess the pressure $\overline{p}$ at each grid point.
(b) Solve the momentum equation to find $\overline{\mathrm{v}}_i$ at the staggered grid $(i+1/2, i-1/2, j+1/2, j-1/2)$, discretized in control volumes and control surfaces (Section 1.4) as shown in Figure 5.3.1.
(c) Solve the pressure correction equation (5.3.5) to find p' at $(i, j), (i, j-1), (i, j+1), (i-1, j), (i+1, j)$. Since the corner grid points are avoided, the scheme is "semi-implicit," not fully implicit, as shown in Figure 5.3.1.
(d) Correct the pressure and velocity using (2.2.9b), (5.3.2), and (5.3.4).

$$\begin{aligned} p &= \overline{p} + p' \\ u &= \overline{u} - \frac{\Delta t}{2\rho\,\Delta x}(p'_{i+1,j} - p'_{i-1,j}) - \frac{\Delta t}{\rho}\left(A^{(1)}_{i+\frac{1}{2},j} - A^{(1)}_{i-\frac{1}{2},j}\right) \\ \mathrm{v} &= \overline{\mathrm{v}} - \frac{\Delta t}{2\rho\,\Delta y}(p'_{i,j+1} - p'_{i,j-1}) - \frac{\Delta t}{\rho}\left(A^{(2)}_{i,j+\frac{1}{2}} - A^{(2)}_{i,j-\frac{1}{2}}\right) \end{aligned} \tag{5.3.6}$$

where

$$A^{(1)} = (\rho\,\mathrm{v}'_k \mathrm{v}'_1)_{,k} - \mu\left(\mathrm{v}'_{1,kk} + \frac{1}{3}\mathrm{v}'_{k,k1}\right) \quad (k = 1, 2)$$

$$A^{(2)} = (\rho\,\mathrm{v}'_k \mathrm{v}'_2)_{,k} - \mu\left(\mathrm{v}'_{2,kk} + \frac{1}{3}\mathrm{v}'_{k,k2}\right) \quad (k = 1, 2)$$

with μ being the dynamic viscosity.
(e) Replace the previous intermediate values of pressure and velocity $(\overline{p}, \overline{\mathrm{v}}_i)$ with the new corrector values (p, v_i) and return to (b).
(f) Repeat Steps (b) through (e) until convergence.

Often the convergence of the above process is not satisfactory because of the tendency for overestimation of p'. A remedy to this difficulty may be found by the use of under-relaxation parameter α,

$$p = \overline{p} + \alpha p' \tag{5.3.7}$$

However, in many cases a proper choice of α is not easy ($\alpha \cong 0.8$ is often used). Thus, a further corrective measure is to use SIMPLER (SIMPLE revised) in which a complete Poisson equation is used for pressure corrections.

$$\nabla^2 p = -\rho(\mathrm{v}_{i,j}\mathrm{v}_j)_{,i} \tag{5.3.8a}$$

or

$$\nabla^2 p = 2\rho\left(\frac{\partial u}{\partial x}\frac{\partial \mathrm{v}}{\partial y} - \frac{\partial \mathrm{v}}{\partial x}\frac{\partial u}{\partial y}\right) \tag{5.3.8b}$$

Here u and v will be replaced by (5.3.2) and subsequently (5.3.5) replaced by (5.3.8).

Instead of using the time-dependent formulation described above, it is convenient to use a steady state approach with finite volume discretizations as shown in Figure 5.3.2.

$$a_p\phi_p = \alpha\left(\sum a_{nb}\phi_{nb} + b\right) + (1-\alpha)a_p\phi^0_p \tag{5.3.9}$$

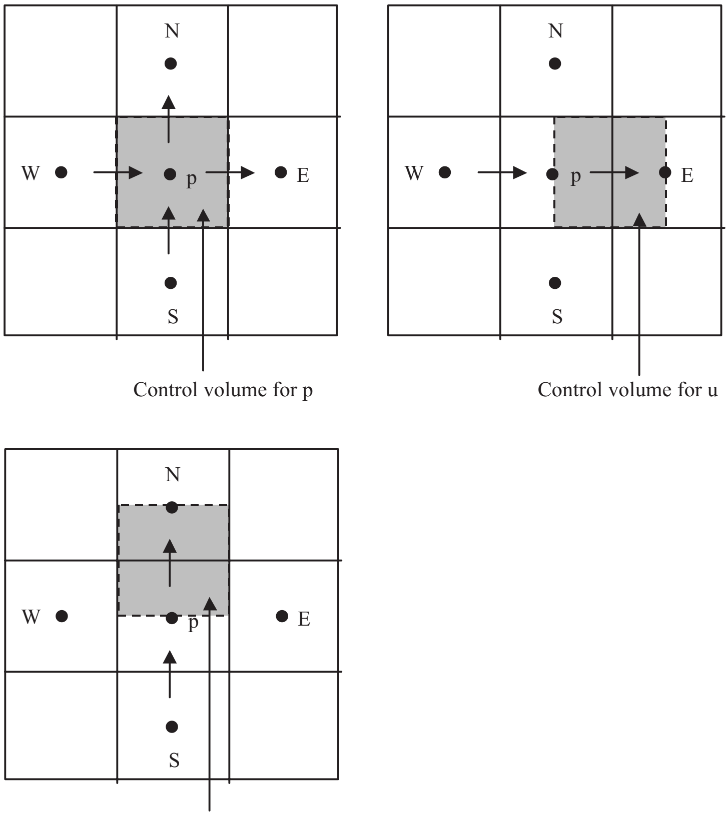

Figure 5.3.2 Computational domain for steady-state problems.

where ϕ is any conservation variable and α is the under-relaxation parameter with the subscripts p and nb denoting the node under consideration and neighbor contributions, respectively.

Convergence of (5.3.9) may be improved using SIMPLEC proposed by Van Doormaal and Raithby [1984] in which more "consistent" approximation of (5.3.9) is implemented:

$$\left(a_e - \sum a_{nb}\right) u'_e = A_e(p'_p - p'_E) \tag{5.3.10}$$

with

$$u_e = u'_e + d_e(p'_p - p'_E) \tag{5.3.11}$$

$$u'_e = \frac{\sum a_{nb} u_{nb} + b}{a_e}, \qquad d_e = \frac{A_e}{a_e - \sum a_{nb}} \tag{5.3.12}$$

Examples of computations reported by Van Doormaal and Raithby [1984] show that SIMPLEC is most effective, followed by SIMPLER and SIMPLE.

5.3.2 PRESSURE IMPLICIT WITH SPLITTING OF OPERATORS

We note that the SIMPLE method requires an iterative procedure. To obtain solutions without iterations, and with large time steps and less computing effort, Issa [1985] proposed the PISO (Pressure Implicit with Splitting of Operators) scheme. In this scheme, the conservation of mass is designed to be satisfied within the predictor-corrector steps.

The governing equations consist of the momentum equation and pressure correction equation written as follows:

Momentum

$$\frac{\rho}{\Delta t}\left(\mathrm{v}_j^{n+1} - \mathrm{v}_j^n\right) = -s_{ij,i}^{n+1} - p_{,j}^{n+1} \tag{5.3.13}$$

Pressure Corrector

$$p_{,jj}^{n+1} = -\frac{\rho}{\Delta t}\left(\mathrm{v}_{j,j}^{n+1} - \mathrm{v}_{j,j}^n\right) - s_{ij,ij}^{n+1} \tag{5.3.14}$$

where $s_{ij,ij}$ refers to the derivatives of the sum of convection and viscous diffusion terms, $s_{ij,i}$.

$$s_{ij,i} = (\rho \mathrm{v}_i \mathrm{v}_j)_{,i} - \tau_{ij,i} \tag{5.3.15a}$$

$$\tau_{ij} = \mu(\mathrm{v}_{i,j} + \mathrm{v}_{j,i}) - \frac{2\mu}{3}\mathrm{v}_{k,k}\delta_{ij} \tag{5.3.15b}$$

(a) Predictor

$$\frac{\rho}{\Delta t}\left(\mathrm{v}_j^* - \mathrm{v}_j^n\right) = -s_{ij,i}^* - p_{,j}^n \tag{5.3.16}$$

(b) Corrector I

$$p_{,jj}^* = -\frac{\rho}{\Delta t}\left(\mathrm{v}_{j,j}^* - \mathrm{v}_{j,j}^n\right) - s_{ij,ij}^* = \frac{\rho}{\Delta t}\mathrm{v}_{j,j}^n - s_{ij,ij}^* \tag{5.3.17}$$

$$\frac{\rho}{\Delta t}\left(\mathrm{v}_j^{**} - \mathrm{v}_j^n\right) = -s_{ij,i}^* - p_{,j}^* \tag{5.3.18}$$

with $\mathrm{v}_{j,j}^*$ set equal to zero in (5.3.17) in order to enforce the conservation of mass.

(c) Corrector II

$$p_{,jj}^{**} = \frac{\rho}{\Delta t}\mathrm{v}_{j,j}^n - s_{ij,ij}^{**} \tag{5.3.19}$$

$$\frac{\rho}{\Delta t}\left(\mathrm{v}_j^{***} - \mathrm{v}_j^n\right) = -s_{ij,i}^{**} - p_{,j}^{**} \tag{5.3.20}$$

with $\mathrm{v}_{j,j}^{**} = 0$ being once again enforced in (5.3.19). Thus, in the above process, there are no iterative steps involved.

In order to increase stability and accuracy, we may split $s_{ij,i}$ into diagonal and non-diagonal terms.

$$s_{ij,i} = s_{ij,i}^{(D)} + s_{ij,i}^{(N)} = A_{ji}^{(D)}\mathrm{v}_i + s_{ij,i}^{(N)} \tag{5.3.21}$$

To illustrate this splitting of diagonal term, consider a one-dimensional case

$$s_{ij,i} \Rightarrow \frac{\partial}{\partial x}\left(\rho u \phi - k\frac{\partial \phi}{\partial x}\right) \tag{5.3.22}$$

or

$$s_{ij,i} \Rightarrow \frac{(\rho u \phi)_{i+1} - (\rho u \phi)_{i-1}}{2\Delta x} - \frac{(k\phi)_{i+1} - 2(k\phi)_i + (k\phi)_{i-1}}{\Delta x^2}$$

$$\Rightarrow \frac{1}{\Delta x_i}\left[(\rho u \phi)_{i+\frac{1}{2}} - (\rho u \phi)_{i-\frac{1}{2}} - \frac{k_{i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}(\phi_{i+1} - \phi_i) + \frac{k_{i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}}(\phi_i - \phi_{i-1})\right]$$

Construct an upwind scheme to get

$$(\rho u \phi)_{i+\frac{1}{2}} - (\rho u \phi)_{i-\frac{1}{2}} \Rightarrow \begin{cases} (\rho u)_{i+\frac{1}{2}}\phi_i - (\rho u)_{i-\frac{1}{2}}\phi_{i-1} & \text{for } (+u) \\ (\rho u)_{i+\frac{1}{2}}\phi_{i+1} - (\rho u)_{i-\frac{1}{2}}\phi_i & \text{for } (-u) \end{cases}$$

Then we arrive at

$$\left.\begin{aligned} (\rho u \phi)_{i+\frac{1}{2}} &= (\rho u)^+_{i+\frac{1}{2}}\phi_i + (\rho u)^-_{i+\frac{1}{2}}\phi_{i+1} \\ (\rho u \phi)_{i-\frac{1}{2}} &= (\rho u)^+_{i-\frac{1}{2}}\phi_{i-1} + (\rho u)^-_{i-\frac{1}{2}}\phi_i \end{aligned}\right\} \qquad \text{with } (\rho u)^{\pm} = \frac{1}{2}(\rho u \pm |\rho u|)$$

Thus $s_{ij,i}$ can be written as

$$s_{ij,i} \Rightarrow \frac{1}{\Delta x_i}(\alpha\phi_{i+1} + \beta\phi_i + \gamma\phi_{i-1}) \tag{5.3.23}$$

where

$$\alpha = (\rho u)^-_{i+\frac{1}{2}} - \frac{k_{i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}}, \qquad \beta = (\rho u)^+_{i+\frac{1}{2}} - (\rho u)^-_{i-\frac{1}{2}} + \frac{k_{i+\frac{1}{2}}}{\Delta x_{i+\frac{1}{2}}} + \frac{k_{i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}},$$

$$\gamma = -(\rho u)^+_{i-\frac{1}{2}} - \frac{k_{i-\frac{1}{2}}}{\Delta x_{i-\frac{1}{2}}}$$

Rewriting (5.3.23), we have

$$\begin{bmatrix} \beta & \alpha & & \\ \gamma & \beta & \alpha & \\ & \gamma & \beta & \alpha \\ & & \gamma & \beta \end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix} = \begin{bmatrix} \beta & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & \beta & 0 \\ 0 & 0 & 0 & \beta \end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix} + \begin{bmatrix} 0 & \alpha & 0 & 0 \\ \gamma & 0 & \alpha & 0 \\ 0 & \gamma & 0 & \alpha \\ 0 & 0 & \gamma & 0 \end{bmatrix}\begin{bmatrix}\phi_1\\ \phi_2\\ \phi_3\\ \phi_4\end{bmatrix} \tag{5.3.24}$$

or for multidimensions, we write (5.3.24) as

$$s_{ij,i} = A^{(D)}_{ji}\mathrm{v}^*_i + s^{(N)}_{ij,i} \tag{5.3.25}$$

Note that $s_{ij,i}$ is diagonally dominant for low Mach number flows,

$$(\rho u)_{i+\frac{1}{2}} > (\rho u)_{i-\frac{1}{2}}$$

or

$$\beta > |\alpha| + |\gamma|$$

If Mach number increases (high speed or compressible flow), then $(\rho u)_{i-\frac{1}{2}} > (\rho u)_{i+\frac{1}{2}} > 0$, or

$$\beta < |\alpha| + |\gamma|.$$

This implies that the diagonal dominance diminishes at high speed or compressible flows. We discuss a remedy for this problem in Section 6.3.3 on the PISO scheme for compressible flows.

With the splitting of $s_{ij,i}$ into the diagonal and nondiagonal parts, we proceed as follows:

(a) Predictor

$$\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)\mathrm{v}_i^* = -s_{ij,i}^{*(N)} - p_{,j}^n + \frac{\rho}{\Delta t}\mathrm{v}_j^n \tag{5.3.26}$$

(b) Corrector I

$$\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)(\mathrm{v}_i^{**} - \mathrm{v}_i^*) = -\left(p_{,j}^* - p_{,j}^n\right) \tag{5.3.27}$$

$$\left[\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)^{-1}(p^* - p^n)_{,j}\right]_{,i} = \mathrm{v}_{i,i}^* \tag{5.3.28}$$

Solve $(p^* - p^n)$ and insert the result into (5.3.26) to obtain new v_i^{**}.

(c) Corrector II

$$\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)\mathrm{v}_i^{**} - \frac{\rho}{\Delta t}\mathrm{v}_i^n = -s_{ij,i}^{*(N)} - p_{,j}^* \tag{5.3.29}$$

$$\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)\mathrm{v}_i^{***} - \frac{\rho}{\Delta t}\mathrm{v}_j^n = -s_{ij,i}^{**(N)} - p_{,j}^{**} \tag{5.3.30}$$

Subtracting (5.3.29) from (5.3.30), we obtain

$$\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)(\mathrm{v}_i^{***} - \mathrm{v}_i^{**}) = -\left(s_{ij,i}^{**(N)} - s_{ij,i}^{*(N)}\right) - (p^{**} - p^*)_{,j} \tag{5.3.31}$$

For $\mathrm{v}_{i,i}^{***} = 0$, we must have

$$\left[\left(\frac{\rho}{\Delta t}\delta_{ij} + A_{ji}^{(D)}\right)^{-1}(p^{**} - p^*)_{,i}\right]_{,j} = -\left(\frac{\rho}{\Delta t}\delta_{ik} + A_{ki}^{(D)}\right)^{-1}\left(s_{ij,k}^{**(N)} - s_{ij,k}^{*(N)}\right)_{,j} + \mathrm{v}_{i,i}^{**} \tag{5.3.32}$$

Solution of (5.3.32) leads to

$$\mathrm{v}_i^{***} = \mathrm{v}_i^{n+1} \tag{5.3.33}$$

$$p^{**} = p^{n+1} \tag{5.3.34}$$

This completes the splitting process in which the v_i^{***} and p^{**} fields imply the exact solution v_i^{n+1} and p^{n+1}. For additional information on this procedure, see Issa, Gosman, and Watkins [1986].

5.3.3 MARKER-AND-CELL (MAC) METHOD

This is one of the earliest methods developed for the solution of incompressible flows, although its use in the original form is no longer pursued, but it has been altered to other more efficient schemes. The basic idea of MAC as originally introduced by Harlow and Welch [1965] is one of the pressure correction schemes developed on a staggered mesh, seeking to trace the paths of fictitious massless marker particles introduced on the free surface. The solution is advanced in time by solving the momentum equations for velocity components using the current estimates of the pressure distributions. The pressure is improved by numerically solving the Poisson equation,

$$p_{,ii} = f \tag{5.3.35}$$

with

$$f = S - \frac{\partial D}{\partial t} \tag{5.3.36}$$

$$S = \lfloor -(\rho \mathrm{v}_i \mathrm{v}_j)_{,i} + \mu \mathrm{v}_{j,ii} \rfloor_{,j} \tag{5.3.37}$$

$$D = \mathrm{v}_{i,i} \tag{5.3.38}$$

Here, the correction in pressure is required to compensate for the nonzero dilatation D (5.3.38) at the current iteration level. The Poisson equation is then solved for the revised pressure field. The improved pressure may then be used in the momentum equations for a better solution at the present time step. If D does not vanish, cyclic process of solving the momentum equations and the Poisson equation is repeated until the velocity field is divergent free.

The original MAC method was based on an explicit time-marching scheme. Subsequently, implicit schemes have been implemented by various authors [Briley, 1974; Ghia, Hankey, and Hodge, 1979].

5.4 VORTEX METHODS

Two-Dimensional Vorticity Transport Equation

In the previous sections, we dealt with primitive variables, v_i and p. An alternative approach is to use the vortex methods in which we utilize the vorticity and stream functions as variables.

$$\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{v} \tag{5.4.1}$$

$$\mathbf{v} = \varepsilon_{ij} \psi_{,j} \mathbf{i}_i \tag{5.4.2}$$

where $\boldsymbol{\omega}$ is the vorticity vector, ε_{ij} is the second order tensor of the permutation symbol for 2-D,

$$\varepsilon_{ij} = \begin{cases} 1 & \text{for } \varepsilon_{12} \\ -1 & \text{for } \varepsilon_{21} \\ 0 & \text{otherwise} \end{cases}$$

and ψ is the stream function.

For incompressible two-dimensional flows, the scalar vorticity transport equation is written as

$$\frac{\partial \omega}{\partial t} + \omega_{,i} \mathrm{v}_i = \nu \omega_{,ii} \tag{5.4.3}$$

where $\omega = \omega_3$ is the component of the vorticity vector ω in the direction normal to the x-y plane. Auxiliary equations required are

$$\nabla^2 \psi = -\omega \tag{5.4.4}$$

$$\nabla^2 p = 2\rho \left(\frac{\partial u}{\partial x} \frac{\partial \mathrm{v}}{\partial y} - \frac{\partial u}{\partial y} \frac{\partial \mathrm{v}}{\partial x} \right) \tag{5.4.5}$$

or

$$\nabla^2 p = 2\rho \left[\frac{\partial^2 \psi}{\partial x^2} \frac{\partial^2 \psi}{\partial y^2} - \left(\frac{\partial^2 \psi}{\partial x \partial y} \right)^2 \right] \tag{5.4.6}$$

It is seen that the variables v_i, p, ψ, and ω may be computed using equations (5.4.1) through (5.4.6).

For simplicity, let us consider a wall located at $y = 0$. Referring to Figure 5.4.1, we have

$$\left(\frac{\partial p}{\partial x} \right)_{\text{wall}} = -\mu \left(\frac{\partial \omega}{\partial y} \right)_{\text{wall}} \tag{5.4.7}$$

or

$$\frac{p_{i+1,1} - p_{i-1,1}}{2\Delta x} = -\mu \frac{-3\omega_{i,1} + 4\omega_{i,2} - \omega_{i,3}}{2\Delta y} \tag{5.4.8}$$

Here, the pressure must be specified on the wall surface. The pressure at the adjacent point can be determined with a first order, one-sided difference expression for $\partial p/\partial x$ in (5.4.7). Thereafter, (5.4.8) can be used to determine the pressure at all other wall points.

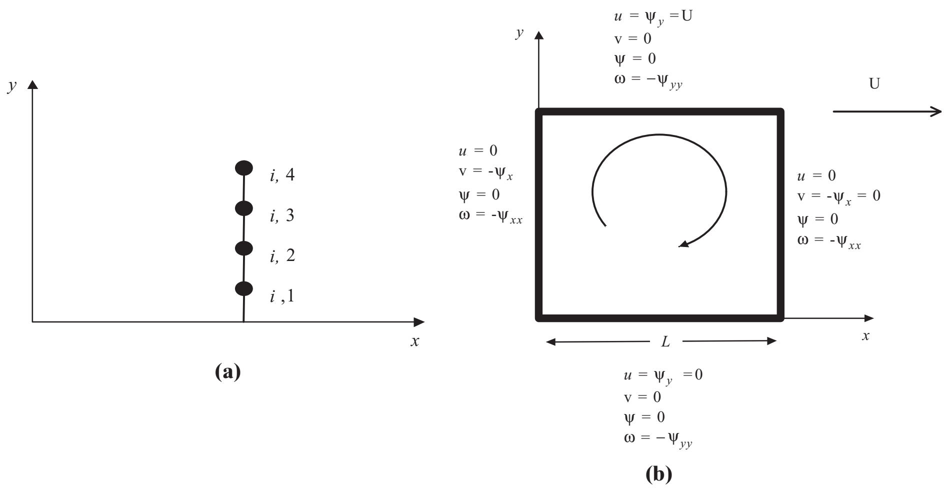

Figure 5.4.1 Illustration of vortex methods. **(a)** Grid point normal to a flat plate. **(b)** Driven cavity problem.

Notice that for the simultaneous solutions of (5.4.2), (5.4.3), (5.4.4), and (5.4.6), we may use the finite difference schemes presented in Chapter 3. For example, the nonlinear terms on the right-hand side of (5.4.6) may be represented as

$$\nabla^2 p = 2\rho_{i,j}\left[\left(\frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{(\Delta x)^2}\right)\left(\frac{\psi_{i,j+1} - 2\psi_{i,j} - \psi_{i,j-1}}{(\Delta y)^2}\right) - \left(\frac{\psi_{i+1,j+1} - \psi_{i+1,j-1} - \psi_{i-1,j+1} + \psi_{i-1,j-1}}{4\Delta x \Delta y}\right)^2\right] \tag{5.4.9}$$

where the alternative mixed derivative may be chosen as shown in Section 3.5.

For a steady state problem, the Poisson equation for pressure is solved once, that is, after the steady-state values of ω and ψ have been computed.

For time dependent problems, the solution of the vorticity transport equation and the Poisson equation requires that boundary conditions for ψ and ω be specified. At the wall, ψ is a constant and may be set equal to a reference value, that is, $\psi = 0$. To find ω at the wall surface, we write ψ in terms of Taylor series about the wall point $(i, 1)$,

$$\psi_{i,2} = \psi_{i,1} + \left.\frac{\partial\psi}{\partial y}\right|_{i,1} \Delta y + \frac{1}{2}\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1} (\Delta y)^2 + \cdots \tag{5.4.10}$$

where

$$\left.\frac{\partial\psi}{\partial y}\right|_{i,1} = u_{i,1} = 0 \tag{5.4.11a}$$

$$\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1} = \left.\frac{\partial u}{\partial y}\right|_{i,1} \tag{5.4.11b}$$

$$\omega_{i,1} = \left.\frac{\partial v}{\partial x}\right|_{i,1} - \left.\frac{\partial u}{\partial y}\right|_{i,1} = -\left.\frac{\partial^2\psi}{\partial y^2}\right|_{i,1} \tag{5.4.11c}$$

thus, rewriting (5.4.10) as

$$\psi_{i,2} = \psi_{i,1} - \frac{1}{2}\omega_{i,1}\Delta y^2 + O(\Delta y^3) \tag{5.4.12}$$

$$\omega_{i,1} = \frac{2(\psi_{i,1} - \psi_{i,2})}{\Delta y^2} + O(\Delta y)$$

$$u_{i,1} = \left.\frac{\partial\psi}{\partial y}\right|_{i,2} = \frac{-3\psi_{i,1} + 4\psi_{,2} + \psi_{i,3}}{4\Delta y} \tag{5.4.13}$$

Three-Dimensional Vorticity Transport Equations

For three-dimensional problems, the vorticity transport equations are of the form [Chung, 1996]:

$$\frac{\partial\boldsymbol{\omega}}{\partial t} + (\mathbf{v}\cdot\boldsymbol{\nabla})\boldsymbol{\omega} - (\boldsymbol{\omega}\cdot\boldsymbol{\nabla})\mathbf{v} = \nu\nabla^2\boldsymbol{\omega} \tag{5.4.14}$$

$$\mathbf{v} = \boldsymbol{\nabla}\psi \times \hat{\mathbf{n}} = \boldsymbol{\nabla} \times \boldsymbol{\Psi} \tag{5.4.15}$$

with

$$\mathrm{v}_i = \varepsilon_{ijk}\Psi_{k,j} \tag{5.4.16}$$

$$\begin{aligned} i &= 1 \quad \mathrm{v}_1 = \Psi_{3,2} - \Psi_{2,3} \\ i &= 2 \quad \mathrm{v}_2 = \Psi_{1,3} - \Psi_{3,1} \\ i &= 3 \quad \mathrm{v}_3 = \Psi_{2,1} - \Psi_{1,2} \\ \Psi_\mathrm{k} &= \hat{n}_k\psi \end{aligned} \tag{5.4.17}$$

and

$$\boldsymbol{\omega} = -\nabla^2\boldsymbol{\Psi} \tag{5.4.18}$$

Note that $\boldsymbol{\nabla}\psi$ is perpendicular to the velocity vector $\mathbf{v}$ and $\hat{\mathbf{n}}$ is perpendicular to the plane $\boldsymbol{\nabla}\psi$ and $\mathbf{v}$, whereas $\boldsymbol{\Psi}$ is known as the three-dimensional stream function vector. The geometric properties of the stream function vector are presented in Section 12.2.

Another approach is to use the fourth order stream function vector equation of the form

$$\frac{\partial}{\partial t}\nabla^2\boldsymbol{\Psi} + (\boldsymbol{\nabla} \times \boldsymbol{\Psi} \cdot \boldsymbol{\nabla})\nabla^2\boldsymbol{\Psi} - (\nabla^2\boldsymbol{\Psi} \cdot \boldsymbol{\nabla})(\boldsymbol{\nabla} \times \boldsymbol{\Psi}) = \nu\nabla^4\boldsymbol{\Psi} \tag{5.4.19}$$

with the boundary conditions extended to three-dimensional geometries.

Solutions may be obtained from either (5.4.14) or (5.4.19) using the definitions given by (5.4.15) and (5.4.18). These and other subjects on applications in three-dimensional stream function vector components are further detailed in Section 12.2.

The Curl of Vorticity Transport Equations

We have noted that the advantage of the vorticity transport equation(s) is the numerical stability accrued from removing pressure gradient terms from the solution process. However, the velocity must be calculated from solving simultaneously (5.4.14) through (5.4.18) or from (5.4.19). These steps can be eliminated if we take a curl of the vorticity transport equation (5.4.14), in which the velocity is the only variable. This subject will be discussed in Section 12.2.1.

5.5 SUMMARY

The incompressible flow analysis based on the artificial compressibility method and the pressure-based formulation using SIMPLE, SIMPLER, SIMPLEC, and PISO have been presented. It was shown that these methods are devised in order to ensure the conservation of mass so that pressure oscillations can be prevented. Vortex methods in which pressure terms are absent are preferred in dealing with rotational incompressible flows as they are computationally efficient. Accurate physics of fluids can be obtained without difficulties which may arise from inaccurate pressure calculations in other methods.

The current trend appears to be in favor of preconditioning of the time-dependent term of the density-based formulation so that both compressible and incompressible flows can be treated. This is because, in many practical situations, high- and low-speed regions are coupled particularly in high-speed boundary layer flows and the analysis capable of handling both compressible and incompressible flows is frequently in demand. Details of the preconditioning process for the combined density- and pressure-based formulations for the incompressible flow analysis are presented in Section 6.4.

Since the solution of incompressible flows can be obtained as a part of the compressible flow formulation, it appears that more attention is given to the compressible flow analysis. This leads to a motivation toward attempting to develop a general purpose program, anticipating that the results of incompressible flows arise automatically when the flow velocity decreases at low Mach number. This topic is addressed in Section 6.5.

The theoretical basis for three-dimensional vorticity transport equations is examined. Numerical examples for the three-dimensional vortex methods based on the three-dimensional stream function vector components will be discussed in Section 12.2.

Although not presented in this chapter, other methods have been used in the past. One of the significant developments in the late 1950s was the particle-in-cell (PIC) method [Evans and Harlow, 1957, 1959], particularly efficient in the flows with large distortions (see Section 16.4.3). Recent developments dealing with multiphase incompressible flows will be presented in Chapter 25.

REFERENCES

Briley, W. R. [1974]. Numerical method for predicting three-dimensional steady viscous flow in ducts. *J. Comp. Phys.*, 14, 8–28.

Chorin, A. J. [1967]: A numerical method for solving incompressible viscous flow problems. *J. Comp. Phys.*, 2, 12–26.

Chung, T. J. [1996]. *Applied Continuum Mechanics*. New York: Cambridge University Press.

Evans, M. W. and Harlow, F. H. [1957]. The particle-in-cell method for hydrodynamic calculations. Los Alamos Scientific Laboratory Report No. LA-2139.

———. [1959]. Calculation of unsteady supersonic flow past a circular cylinder. *ARS Journal,* 29, 46–51.

Ghia, K. N., Hankey Jr., W. L., and Hodge, J. K. [1979]. Use of primitive variables in the solution of incompressible Navier-Stokes equations. *AIAA J.,* 17, 298–301.

Harlow, F. H. and Welch, J. E. [1965]. Numerical calculation of time-dependent viscous incompressible flow of fluid with free surface. *Phys. Fluids*, 8, 2182–89.

Issa, R. [1985]. Solution of the implicitly discretized fluid flow equations by operator splitting. *J. Comp. Phys.,* 62, 40–65.

Issa, R. I., Gosman, A. D., and Watkins, A. P. [1986]. The computation of compressible and incompressible recirculating flows by a non-iterative implicit scheme. *J. Comp. Phys.*, 62, 66–82.

Kwak, D. C., Chang, J. L. C., Shanks, S. P., and Chakravarthy, S. K. [1986]. A three-dimensional incompressible Navier-Stokes solver using primitive variables. *AIAA Journal.*, 24, 390–96.

Merkle, C. L., Sullivan, J. Y., Buelow, P. E. O., and Ventateswaran, S. [1998]. Computation of flows with arbitrary equations of state. *AIAA J.*, 36, 4, 515–21.

Patankar, S. V. and Spalding, D. B. [1972]: A calculation procedure for heat, mass and momentum transfer in three-dimensional parabolic flows. *Int. J. Heat Mass Transfer*, 15, 1787–1806.

Raithby, G. D. and Schneider, G. E. [1979]. Numerical solution of problems in incompressible fluid flow: Treatment of the velocity-pressure coupling. *Num. Heat Transfer*, 2, 417–40.

Van Doormaal, J. P. and Raithby, G. D. [1984]. Enhancements of the SIMPLE methods for predicting incompressible fluid flows. *Num. Heat Transfer.*, 7, 147–63.