4

Description of radiation

We turn now to the subject of radiation transport. As much as possible, the present goal is to demystify this subject. Photons are just particles like the others that make up our systems; they just happen to go faster and farther, and are therefore often of special importance in carrying energy and momentum from one place to another. In kinetic theory we introduce the phase-space distribution function for the atoms, develop the theory of the Boltzmann transport equation, and come up with some satisfactory approximate methods for solving it. Radiation transport is exactly the same; the transport equation is about the same, and the approximate methods are about the same as well. The difference is that the subject of radiation transport was elaborated by different people than was kinetic theory, using an entirely different notation, and we have that difference with us today. In the last two or three decades yet another community has joined the discussion of radiation transport, and these are the nuclear engineers, who have evolved a collection of methods for describing neutron transport, methods that are useful for photons as well as neutrons. The present discussion will not attempt to show, Rashomon-like, the same physical concepts from the varied points of view of several disciplines. We will stick with one, mainly the astrophysical notation found, for example, in Mihalas and Mihalas (1984). The elementary definitions of the radiation field quantities are found in many astrophysics books. One good treatment is Mihalas's *Stellar Atmospheres* (1978), and this is also found in Mihalas and Mihalas (1984). The coordinate-free equations are presented by Pomraning (1973) and by Cox and Giuli in the first volume of *Principles of Stellar Structure* (1968).

4.1 Intensity; flux; energy density; stress tensor

Astrophysical radiation transport begins with the intensity I_ν as the fundamental concept instead of the phase space distribution function; the content is equivalent, as we will see. Historically, this might be due to the fact that you can discuss the intensity entirely in the wave picture, and never make reference to those pesky quanta. With sufficient effort it is possible to derive the transport equation directly from Maxwell's equations of electromagnetism and avoid quantum theory entirely. In the present day this seems like a silly thing to do, and certainly more trouble than it is worth. As soon as one has to deal with matter–radiation interactions the quantum picture quickly becomes indispensable. So our present discussion will begin with the classical intensity, I_ν, but we will notice right away that it has an interpretation in terms of a distribution function for photons.

So let us define the intensity. It is a function of three spatial coordinates, two angle coordinates, the radiation frequency, and time, seven coordinates in all. (Polarization of the radiation is described with an additional coordinate, but for the moment we will lump the polarizations together.) The angle coordinates specify in which direction the radiation is going, the spatial coordinates specify the location of the intensity measurement. The frequency coordinate says that we measure the intensity in a small spectral band around one frequency in the spectrum. The intensity is in power units, energy per unit time, but is also expressed per unit area, per unit solid angle, and per unit frequency bandwidth.

The per-unit-area and per-unit-solid-angle attributes work in this way: We make an ideal apparatus, a pinhole camera, to measure intensity by setting up a screen at the place where we want to know the intensity that is opaque except for an aperture that has an area A_1. This is illustrated in Figure 4.1. This screen is oriented perpendicular to the direction we want to measure. The aperture has a shutter, and we will open the shutter at time t and leave it open for a duration Δt. On the downstream side of the screen we arrange for the matter to be cleared away, so the radiation streams freely through. At a suitable distance away from the screen we set up a second screen, which is used as a detector. The second screen has to be at a distance from the first screen that is much larger than the size of the aperture. At a location on the second screen exactly in line with the ray through the aperture that is going in the desired direction, there is a sensitive detector area of size A_2. This is a clever detector that responds only to the frequency bandwidth $\Delta\nu$ around frequency ν. The size of A_2 as viewed from A_1 has to be large compared with the diffraction angle for the aperture radius. The definition of intensity I_ν is that the energy collected by this detector is

$$\Delta E = I_\nu \frac{A_1 A_2}{r^2} \Delta t \Delta\nu. \tag{4.1}$$

The combination A_2/r^2 is the solid angle $d\Omega$ subtended by A_2 at the aperture, so we say the intensity is the energy crossing a unit area at a given point per unit time per unit frequency and per unit solid angle in the direction of interest. In this construction it is obvious that we can increase the distance r by a factor f,

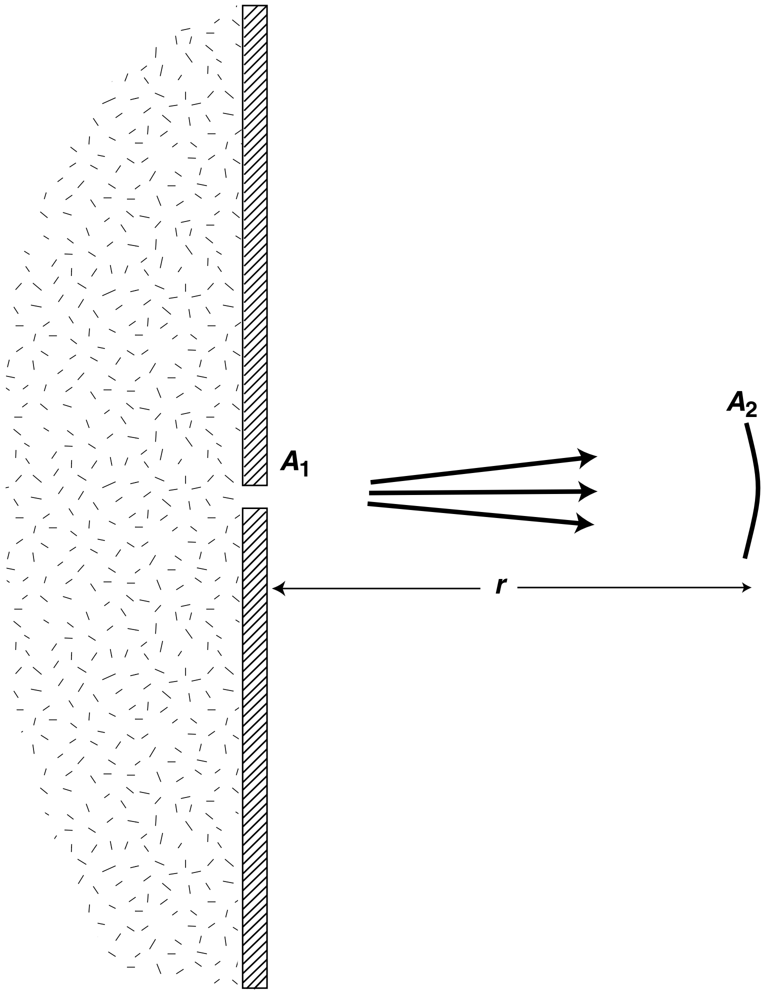

Fig. 4.1 Illustration of the ideal apparatus that serves to define the specific intensity.

and enlarge both of the lateral dimensions of the detector by the same factor thus increasing A_2 by a factor f^2, without changing $d\Omega$, and therefore the intensity is really a property of the radiation field at the place where we put the aperture.

The photons that comprise our bundle of radiant energy that will be registered by the detector travel a distance $c\Delta t$ during the time the shutter is open. Thus at any given time they occupy a cylindrical volume $c\Delta t$ long with a cross section A_1, and therefore a total volume $cA_1\Delta t$. Dividing ΔE by the volume tells us that there is a contribution to the radiation energy per unit volume of $(I_\nu/c)(A_2/r^2)\Delta\nu$ from the solid angle A_2/r^2 and the frequency band $\Delta\nu$. Thus I_ν/c is the radiation energy density per unit solid angle per unit frequency. If this quantity is integrated over all directions for the radiation, weighted by $d\Omega$, we have the spectral energy density, the total radiation energy density per bandwidth in the spectrum. This quantity is sometimes denoted by u_ν and sometimes by E_ν; we will adopt the latter notation, so we have

$$E_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\Omega. \tag{4.2}$$

If our ideal detector responds equally to energy of all frequencies, then we measure the frequency-integrated intensity and frequency-integrated energy density. One notation for this is to simply drop the suffix ν. Thus the total radiation energy density of whatever kind is

$$E = \frac{1}{c}\int d\nu \int_{4\pi} I_\nu \, d\Omega. \tag{4.3}$$

Of course, we may have dropped the suffix for notational simplicity when we still are discussing the spectral density; the context must make it clear.

If we want to emphasize the particle picture of the radiation, we can introduce the quanta. Each one of the little guys has an energy $h\nu$. When we count an energy ΔE then this consists of $\Delta E/h\nu$ photons. Therefore $I_\nu/h\nu$ is the number of photons crossing a unit area per unit frequency per unit solid angle in the specified direction per unit time, and $I_\nu/h\nu c$ is the number density of photons per unit solid angle per unit frequency. We can make the connection with kinetic theory closer still by observing that the photon momentum is $h\nu/c$, and that the momentum space volume element, in spherical momentum-space coordinates, is

$$d^3\mathbf{p} = \left(\frac{h}{c}\right)^3 \nu^2 d\nu d\Omega. \tag{4.4}$$

That means that if we divide $I_\nu/h\nu c$ by $h^3\nu^2/c^3$ we get the phase space density:

$$f = \frac{dN}{dV d^3\mathbf{p}} = \frac{I_\nu}{h^4\nu^3/c^2}, \tag{4.5}$$

which is the usual unknown in the Boltzmann transport equation.

Along with the intensity, we will often work with its angle moments. The first of these has been introduced already, the energy density:

$$E_\nu = \frac{1}{c}\int_{4\pi} I_\nu d\Omega. \tag{4.6}$$

The next moment is the vector flux

$$\mathbf{F}_\nu = \int_{4\pi} \mathbf{n} I_\nu d\Omega. \tag{4.7}$$

If we consider an oriented area element represented by the vector $\mathbf{A}$, and form $\mathbf{F} \cdot \mathbf{A}$, then we see that it is the sum of contributions like $\mathbf{n} \cdot \mathbf{A} I_\nu d\Omega$, which gives the signed flow of energy from one side of the element to the other, positive if going toward $\mathbf{A}$, and negative for going toward $-\mathbf{A}$. In other words, $\mathbf{F}_\nu$ is a proper energy flux for radiation with the frequency ν per unit bandwidth. Notice that the flux moment does not contain a factor c as does the energy density. The ratio of $\mathbf{F}_\nu$ to E_ν is the "fluid velocity" of the radiation considered by itself as a fluid. From the integrals it is evident that its magnitude must be no larger than c. As a concept this radiation fluid velocity has not been found to be especially useful, although the parameter $\mathbf{F}/(cE)$ does enter some approximation schemes.

The third moment is defined by

$$\ _\nu = \frac{1}{c} \int_{4\pi} \mathbf{nn} I_\nu d\Omega. \tag{4.8}$$

By appealing to our knowledge of kinetic theory we can recognize this as the pressure tensor, i.e., the pressure per unit bandwidth due to the radiation of frequency ν, to be exact. We have to be careful to remember that lower case p is the material pressure, and capital   refers to radiation. Notice that the factor $1/c$ that was present in the definition of energy density and absent in the definition of flux is present again in the definition of the radiation pressure.

In much of the astrophysical radiative transfer literature the moments are defined as averages over solid angle, i.e., are divided by 4π, and the factors of c are omitted. Thus the quantities J_ν, $\mathbf{H}_\nu$ and K_ν are defined by

$$J_\nu = \frac{1}{4\pi} \int_{4\pi} I_\nu d\Omega, \tag{4.9}$$

$$\mathbf{H}_\nu = \frac{1}{4\pi} \int_{4\pi} \mathbf{n} I_\nu d\Omega, \tag{4.10}$$

and

$$\mathsf{K}_\nu = \frac{1}{4\pi} \int_{4\pi} \mathbf{nn} I_\nu d\Omega. \tag{4.11}$$

These definitions remove some of the 4πs and cs from radiative transfer theory, at the expense of introducing those factors into the radiation-hydrodynamic equations. We will stick with E, $\mathbf{F}$, and   .

The pressure tensor $\ _\nu$ is represented by a 3×3 matrix. The trace of that matrix is defined as the sum of its diagonal elements. If we perform that operation on (4.8), the effect is to replace the factor $\mathbf{nn}$ with $\mathbf{n} \cdot \mathbf{n}$ inside the integral. But $\mathbf{n} \cdot \mathbf{n}$ is 1 since $\mathbf{n}$ is a unit vector, and therefore

$$\mathrm{Tr}(\ _\nu) = E_\nu. \tag{4.12}$$

Deep inside an opaque material all directions look the same, since the material far enough away from any given observation point to have different temperature, density, and so on is hidden from view by the opacity of the intervening material. Thus the radiation field tends to be isotropic, being the same in every direction. This makes the pressure tensor a scalar tensor, i.e., one which is a factor times the unit matrix. The reasoning goes like this: the diagonal elements should all be the same since we can turn x into y and vice-versa by making a suitable rotation of 90 degrees about the z axis, and this rotation does nothing to the isotropic intensity. Likewise for x and z and y and z. The off-diagonal elements should vanish, since otherwise a reflection like $x \to -x$ would flip the signs of P_{xy} and P_{xz}; but this reflection leaves the intensity unchanged and therefore the tensor elements should stay the same. When the pressure tensor is a scalar tensor, say

$$ {}_{\nu} = P_{\nu} \ , \tag{4.13}$$

then the trace is

$$ \mathrm{Tr}(\ {}_{\nu}) = 3P_{\nu}. \tag{4.14}$$

Since the trace is E_ν according to (4.12), the diagonal elements are all equal to $E_\nu/3$:

$$ {}_{\nu} = \frac{1}{3}E_{\nu} \ . \tag{4.15}$$

(The unit matrix is denoted by not I to avoid confusing it with the intensity.) This relation for pressure and energy density is exactly that for a $\gamma = 4/3$ ideal gas. Radiation *is* a $\gamma = 4/3$ ideal gas if the mean free paths are short enough that transport effects are small. This works very well inside stars.

For a final remark on the intensity concept, let us return to the topic of wave-particle duality and the coarse-graining that was slipped in earlier when we said, “take a distance large enough to avoid the diffraction effects.” There will not be such a distance unless the system is quite large compared with the wavelength. And if we have chosen a truly tiny bandwidth $\Delta\nu$, then the wave train will be at least $1/\Delta\nu$ in duration, since otherwise the light pulse has sidebands outside the chosen bandwidth. It can easily happen that $1/\Delta\nu$ is longer than the shutter time Δt we selected. In other words, the Fourier relations between localization in coordinate space and spreading in Fourier space, and between location in time and spreading in frequency, will make the classical definition of the intensity given here nonsense unless the system is very large compared with the wavelength and the times of interest are much longer than the wave period. Radiative transfer is a geometrical optics concept that makes no allowance for wave optics.

4.2 The transport equation; absorptivity, emissivity

The idea of the transport equation is very simple: the intensity does not change as a bundle of radiation travels along. If we think again of that cylindrical bundle of radiation that contains a total energy $I_\nu A_1 \Delta\nu \Delta t d\Omega$, we see that at a later time $t + \tau$ the same bundle of radiation is located at a different place, but occupies the same bandwidth $\Delta\nu$ and fills the same solid angle $d\Omega$; furthermore it will take the same time Δt to pass by the new location. In other words, the intensity at the displaced location at the later time is also I_ν, unless, that is, radiation has been gained or lost by the bundle through interaction with the matter it had to pass through.

We need a notation for the direction of the radiation propagation. We will use the unit vector $\mathbf{n}$ for that. The vector $\mathbf{n}$ can be defined by two angles, such as colatitude and longitude. We might say

$$\mathbf{n} = \sin\theta \cos\phi \mathbf{e}_x + \sin\theta \sin\phi \mathbf{e}_y + \cos\theta \mathbf{e}_z, \tag{4.16}$$

with θ for the colatitude and ϕ for the longitude. In 3-D space the vectors $\mathbf{n}$ trace out a unit sphere; the element of area on that sphere is $d\Omega$. In terms of the angle coordinates, it is

$$d\Omega = \sin\theta \, d\theta d\phi. \tag{4.17}$$

The angle θ goes from 0 to π, and ϕ goes from 0 to 2π. It is easily checked that the total solid angle, the surface area of the unit sphere, is 4π. There is a conceptual pitfall in using these coordinates θ and ϕ for angle space. We might confuse them for the angles that are part of the spherical coordinate set in real space! In fact, they are completely independent. One set is for momentum space and the other is for configuration space or real space. Some authors use Θ and Φ for the momentum coordinates and θ and ϕ for the real space coordinates, hoping to avoid confusion that way.

The other new notation we need is the name of the position vector, i.e., the vector (x, y, z) in real space. We will use $\mathbf{r}$ for that. The boldface matters here; vector $\mathbf{r}$ is (x, y, z) while scalar r is the distance between two points. Returning to the bundle of radiation, we now have the notation to say that I_ν does not change as the bundle moves over the time τ:

$$I_\nu(\mathbf{r} + \mathbf{n}c\tau, \mathbf{n}, t + \tau) = I_\nu(\mathbf{r}, \mathbf{n}, t). \tag{4.18}$$

Next we Taylor-expand the left-hand side around the point $\mathbf{r}$ and time t, and discard the terms of order τ^2 or higher. Subtract the term $I_\nu(\mathbf{r}, \mathbf{n}, t)$, divide by $c\tau$, and we have it:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = 0. \tag{4.19}$$

(The operator ∇ is the spatial gradient operator, and it will keep that meaning for us hereafter.) This is the radiation transport equation, minus the source and sink terms, which come next. If this resembles the Boltzmann transport equation, that is not an accident. The latter equation might have extra terms involving the momentum derivatives of the phase space density, which are proportional to the forces acting on the particles. There are no forces acting on our photons, so those terms are missing here.

Next we consider first the absorption and then the emission. The empirical law is that radiation impinging on a thin slab of matter is attenuated by a small fraction. This is a definite fraction that does not depend on the intensity, so if the intensity doubles, so does the amount of energy removed from the beam. The fraction is also proportional to the thickness of the slab, if this is not too large. So the intensity change is

$$\Delta I_\nu = -k_\nu c\tau I_\nu \tag{4.20}$$

if none of the absorbed energy is replaced. The proportionality coefficient k_ν is the absorptivity, or absorption coefficient.[1] If this model of pure attenuation is applied to a thicker slab (thickness L), the result is

$$I_\nu = I_\nu^0 \exp(-k_\nu L), \tag{4.21}$$

which is called Beer's law after an application in atmospheric physics. In the atomic model of radiation–matter action, we would say that there is a probability per unit length k_ν that a photon will interact with the matter. For a thin slab $c\tau$ thick, there is a probability $1 - k_\nu c\tau$ of no interaction, so the photon makes it through to the other side, and a probability $k_\nu c\tau$ of having an interaction, in which case it is gone from this beam. The mean loss to I_ν is then $k_\nu c\tau I_\nu$, as before.

Optical depth, a well-loved concept in astrophysics, is defined to be the exponent in Beer's law: $k_\nu L$, or $\int k_\nu dl$ in general. The usual notation for optical depth is τ, perhaps with a subscript ν, and we will adopt this in later sections when we are no longer using τ for the lag time in discussing the transport equation.

The emissivity determines the amount of energy added by a thin slab to a beam of radiation that is passing through it. Normally this does not depend on what the

[1] There is a more precise terminology that is used when the process of scattering, in which the radiation is modified and redirected by its interaction with the matter, is distinguished from true absorption, in which the radiation is actually removed. Then the coefficient of absorption gives the probability of the second process alone, while the combined probability of both processes is called the *extinction* coefficient.

intensity of the beam is, but it does depend on the thickness of the slab:

$$\Delta I_\nu = +j_\nu c\tau. \tag{4.22}$$

The emissivity or emission coefficient j_ν and the absorptivity k_ν are not in the same units. Absorptivity has the dimensions of inverse length, while emissivity has the dimensions of I_ν divided by length, i.e., energy per unit volume per unit bandwidth per unit solid angle per unit time.

The consideration of what the absorptivity and emissivity actually are will be taken up in Chapter 8; this is a topic in atomic physics. We should mention here that there are processes that are nonlinear in the intensity, for example, multiphoton absorption, for which the energy loss is proportional to two or more factors of the intensity for different frequencies and directions. This problem is so specialized that it can be treated in its own context should the need arise. One important process that modifies this discussion is stimulated emission. This will be treated at more length later, but the end result is that j_ν contains a part that is proportional to the intensity in the beam to which j_ν is contributing, i.e., to I_ν. This is exactly accounted for by making a subtraction from k_ν ("negative absorption"). In thermodynamic equilibrium the subtracted piece is $\exp(-h\nu/kT)$ times the original value, and therefore the subtraction can be performed at the outset. We will talk more about scattering in Chapter 8 as well. It is normally a linear process, i.e., scattering contributes something to k_ν, and the photons thus removed are returned at other angles and frequencies (perhaps), and therefore appear as a term in j_ν involving an integration of I_ν over angle and frequency. Stimulated emission raises its head here as well, making the out-scattering and in-scattering terms quadratic in the intensity. For the present, we will suppose that k_ν has been corrected for stimulated emission, and that j_ν has lumped into it any of the more complex processes. We proceed to put the loss and gain terms into the intensity budget, Taylor expand and cancel as before, and end up with the radiation transport equation in the most general form we need right now:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = j_\nu - k_\nu I_\nu. \tag{4.23}$$

If we prefer to call out the scattering processes explicitly, then the equation takes this form (see Section 12.3)

$$\begin{aligned}\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu &= j_\nu - k_\nu I_\nu \\ &\quad + \sigma_\nu \int_0^\infty d\nu' \int_{4\pi} d\Omega' \left\{ -R(\nu'\mathbf{n}', \nu\mathbf{n})\frac{\nu}{\nu'} I_\nu \left[1 + \frac{c^2 I_{\nu'}}{2h\nu'^3}\right] \right. \\ &\quad \left. + R(\nu\mathbf{n}, \nu'\mathbf{n}') I_{\nu'} \left[1 + \frac{c^2 I_\nu}{2h\nu^3}\right]\right\},\end{aligned} \tag{4.24}$$

in which j_ν and k_ν now pertain only to absorptive processes, σ_ν is the scattering coefficient and the R function is the (normalized) scattering redistribution function in frequency and angle. As we will discuss in Sections 8.3 and 12.3, the frequency shift in scattering is often negligible and the cross product terms due to stimulated scattering cancel out; furthermore, isotropic scattering can be a good approximation. In this case the transport equation with scattering is simpler:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n}\cdot\nabla I_\nu = j_\nu + \sigma_\nu J_\nu - (k_\nu + \sigma_\nu) I_\nu. \tag{4.25}$$

4.3 Radiation moment equations

Earlier we discussed the first three angle moments of the intensity, namely the energy density, the flux, and the pressure tensor. If we chose to, we could add tensors of higher rank to this set, the moments of rank 3, 4, ... We can also take moments of the transport equation by integrating over angles after multiplying by 1, $\mathbf{n}$, $\mathbf{nn}$, ... Sad to say, each moment of the transport equation introduces the next higher moment of the intensity, so the set of moment equations up through a given order is always one equation short of having as many equations as there are unknowns. The system of equations must be closed by using an *ad hoc* relation that gives the highest moment, say, as an expression involving lower moments. The kinetic theory of gases and plasma physics are richer in closure theories than is radiation transport; in the latter case no closure theory beyond the pressure tensor has had any currency. Our discussion here will therefore be limited to the first two moments of the transport equation. In the kinetic theory of gases these are precisely the moments that lead to Euler's equations after the pressure tensor is approximated.

We will start taking integrals of the transport equation (4.23), but two points must be mentioned first. The transport equation is written above in a coordinate-free notation. Its simplicity in that case is somewhat deceptive. If a curvilinear system of spatial coordinates is used then it is not correct to think of $\mathbf{n}$ as a constant vector during spatial transport. Instead we have to imagine that the vector $\mathbf{n}$ is moved along remaining parallel to itself in a physical sense, which means that its components along the three coordinate directions are changing as it goes. This would take us deeply into Riemannian geometry, which cannot be undertaken in this book. This is avoided if we always refer to a Cartesian system of fixed-space coordinates, for then the components of $\mathbf{n}$ truly are constant.

The second point is that isotropic k_ν and j_ν is by no means the only possibility. The absorptivity can depend on direction if the absorbers are preferentially oriented a certain way, as is the case with ice crystals in cirrus clouds, for example, since the crystals tend to float with their flat sides horizontal. Another example is given by a highly magnetized plasma, for which the normal processes of atomic absorption are highly modified by the Zeeman effect and depend on the photon direction with respect to the magnetic field. Anisotropy of the emissivity is more common, since scattering is in most cases different for different angles between the incoming and outgoing photons; since the intensity itself may be quite anisotropic, the scattered intensity is anisotropic too. An additional reason for anisotropy will be discussed below, which is that absorption and emission that occur isotropically in the rest frame of the material fluid are not isotropic in the fixed frame owing to the Doppler and aberration effects. Having said this, we will proceed to ignore these anisotropies for now, but you were warned.

As a first step to taking angle moments we take advantage of the constancy of **n** and take it inside the spatial derivative in the transport equation:

$$\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \nabla \cdot (\mathbf{n} I_\nu) = j_\nu - k_\nu I_\nu. \tag{4.26}$$

Now we multiply successively by 1 and **n**, and integrate over angles. The factor **n** passes through the divergence by the same argument we just gave. We immediately obtain

$$\frac{\partial E_\nu}{\partial t} + \nabla \cdot \mathbf{F}_\nu = 4\pi j_\nu - k_\nu c E_\nu, \tag{4.27}$$

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t} + c\nabla \cdot \quad_\nu = -k_\nu \mathbf{F}_\nu. \tag{4.28}$$

Since E_ν, $\mathbf{F}_\nu$, and $\quad_\nu$ are a proper scalar, vector, and tensor, respectively (not in space-time as we discuss later, but just in space), we are free to use curvilinear coordinates for these moment equations, which are written in coordinate-free form.

Rather than commenting on the meaning of these equations we pass on to their frequency-integrated form:

$$\frac{\partial E}{\partial t} + \nabla \cdot \mathbf{F} = \int d\nu \, (4\pi j_\nu - k_\nu c E_\nu), \tag{4.29}$$

$$\frac{1}{c}\frac{\partial \mathbf{F}}{\partial t} + c\nabla \cdot \quad = -\int d\nu \, k_\nu \mathbf{F}_\nu. \tag{4.30}$$

For a nonrelativistic gas the first two moment equations express mass conservation and momentum conservation. The second of the radiation moment equations is indeed an expression of conservation of radiation momentum. But photons have no rest mass, and the first equation here is not about the conservation of that. Rather, it is about the conservation of relative mass, which is to say, of energy. The terms on the left-hand side of the first equation have the normal conservation law form for an energy density and an energy flux. The integral on the right-hand side represents the rates of gain and loss of radiation energy per unit volume. Emissivity is an energy gain for the radiation; it would be a loss for material energy. Likewise the absorptivity term is a loss of radiation energy; it would be a gain for the material.

Turning to the second equation, the first thing we need to do is divide by c:

$$\frac{1}{c^2}\frac{\partial \mathbf{F}}{\partial t} + \nabla \cdot \quad = -\int d\nu\, k_\nu c \frac{\mathbf{F}_\nu}{c^2}. \tag{4.31}$$

Now we read off the terms. The radiation momentum density is $\mathbf{F}/c^2$ and the radiation momentum flux (pressure) is . On the right-hand side we can regard $k_\nu c$ as the absorption probability per unit time and $\mathbf{F}_\nu/c^2$ as the spectral momentum density, so the integral is the momentum lost per unit time by the radiation and transferred to the matter. There is an important caveat with (4.31): the neglected anisotropy of absorption and emission introduces an important correction to this term if the material is not at rest; this is treated in Section 6.3.

4.4 Diffusion approximation

The diffusion approximation is by far the most important approximate treatment of radiation transport; it pertains to the limit in which radiation is treated as an ideal fluid with small corrections. The approximation becomes accurate when the photon mean free paths are small compared with other length scales, i.e., when $k_\nu L \gg 1$, where L is a typical length. The diffusion approximation is found to be so much simpler than solving the full transport equation that every effort is made to adapt it to problems where $k_\nu L < 1$, for which it is not expected to be accurate. In this case the goal is somewhat different: it is understood that the results will not be precise, but they may well be qualitatively correct, and the error, perhaps 20–30%, may be tolerable.

The following development of the diffusion approximation is motivated by the discussion in Cox and Giuli (1968), Section 6.3, and a similar discussion in Schwarzschild's book *Structure and Evolution of the Stars* (1958).

The diffusion approximation is an expansion in a small parameter – the mean free path or $1/k_\nu$ – truncated after the first two terms. We begin by rearranging the transport equation in this way:

$$I_\nu = \frac{j_\nu}{k_\nu} - \frac{1}{k_\nu}\left(\frac{1}{c}\frac{\partial I_\nu}{\partial t} + \mathbf{n} \cdot \nabla I_\nu\right). \tag{4.32}$$

Since we suppose that k_ν is large the second expression on the right-hand side should be a small correction to the first term. So we use this equation as a basis for obtaining I_ν by successive approximations. The first approximation is

$$I_\nu^0 = \frac{j_\nu}{k_\nu}. \tag{4.33}$$

This is a local balance approximation which says that the radiation is in equilibrium with its sources. If the source and sink terms are those for matter at a temperature T, then I_ν^0 should be the thermodynamic equilibrium radiation field, which is the Planck function $B_\nu(T)$. The next step of the successive approximations is to put I_ν^0 in for I_ν on the right-hand side of (4.32), which leads to

$$I_\nu^1 = \frac{j_\nu}{k_\nu} - \frac{1}{k_\nu}\left(\frac{1}{c}\frac{\partial j_\nu/k_\nu}{\partial t} + \mathbf{n}\cdot\nabla\frac{j_\nu}{k_\nu}\right). \tag{4.34}$$

For the standard diffusion approximation we stop here. We could include more terms, but there is no good reason to do so. If k_ν is indeed large, then the next term would be too small to care about. If k_ν is not large, then we anticipate that including more terms will make the answer worse rather than better. The infinite series is really an asymptotic expansion, and including one term after another will make the answer better for a while, then it will start to diverge. Thus we may as well stop after the second term. You will notice that this series contains no reference whatsoever to the possible existence of a nearby boundary that may have a large effect on the intensity; perhaps there is vacuum just a short distance away, and therefore some of the radiation that would have come from that direction if the medium had gone on indefinitely is actually missing. A helpful way of looking at this is that the presence of a boundary modifies the solution for a certain distance into the interior (see Figure 4.2); this region of boundary influence is the boundary layer. The diffusion expansion will never give the right answer in the boundary layer; if we are lucky the error is tolerable. Farther into the interior of the problem is a region where the boundary influence is not felt. It is here that the diffusion approximation is valid.

At first sight we would say that the thickness of a boundary layer should be one or two mean free paths, since that is the distance a photon is likely to penetrate before being absorbed. Indeed, in many cases this is a good estimate. But we must mention here that scattering complicates this discussion considerably. Scattering is described with an albedo, which is the probability that the photon survives one interaction with the matter. If the albedo is very close to unity, then a photon will survive a large number of scatterings before finally being destroyed. In this case the thickness of the boundary layer becomes the distance the photon can move in a random walk with that many steps; this can be many mean free paths, depending on the albedo. Thus with scattering the boundary layers are thicker and encompass a large part of what we might have thought was the interior of the problem. In

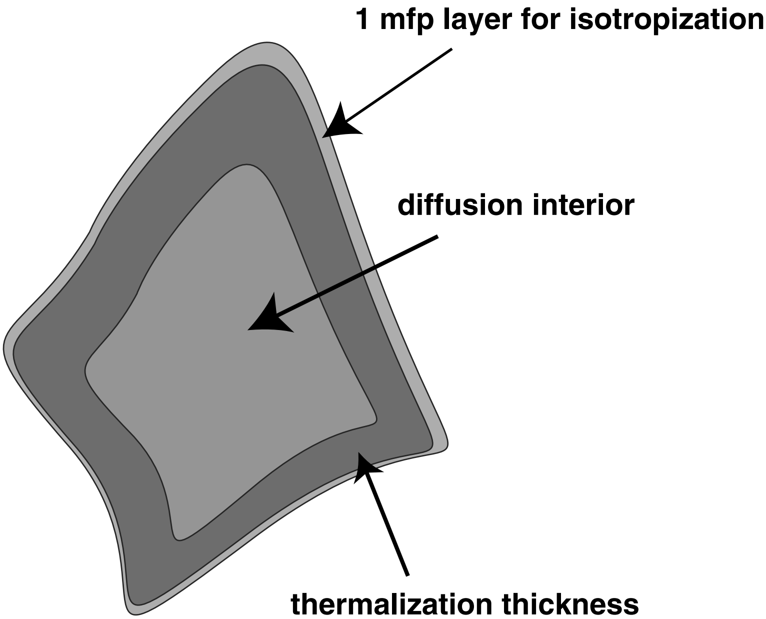

Fig. 4.2 Illustration of a generic radiative transfer problem showing the three qualitatively different regions: the isotropization layer within about one mean free path (mfp) of the outside boundary; the thermalization layer, perhaps many mean free paths thick, interior to which the radiation field is well approximated as Planckian; and the diffusion interior, this region of near equilibrium. Only in the innermost region is diffusion a good approximation.

the limit of unit albedo the problem is all boundary layer. There is no interior at all in this case, just as there is none for problems in potential theory – Laplace's equation does not allow an interior region free from the influence of the boundary. Fortunately for us, the scattering albedo in the large majority of practical problems is either small, or in any case not too close to unity, so the boundary layers are only a small number of mean free paths thick.

We turn again to the diffusion approximation (4.34) and consider what the radiation moments become in this case. For the energy density:

$$E_\nu = \frac{4\pi}{c}\frac{j_\nu}{k_\nu} - \frac{4\pi}{k_\nu c^2}\frac{\partial}{\partial t}\left(\frac{j_\nu}{k_\nu}\right). \tag{4.35}$$

The gradient term cancels in E_ν since $\mathbf{n}$ is odd and its angle average vanishes. The integrals for $\mathbf{F}_\nu$ and $\ _\nu$ are facilitated by noticing that the angle average of the tensor $\mathbf{nn}$ is $\ /3$. The $\mathbf{n}\cdot\nabla()$ term can be changed to $\nabla\cdot(\mathbf{n}())$, and then when another factor of $\mathbf{n}$ is put in, for obtaining the flux, the term becomes $\nabla\cdot(\mathbf{nn}())$. As noted, averaging over angles changes $\mathbf{nn}$ into $\ /3$, and we can use the fact that the divergence of a scalar tensor is the gradient of one of the diagonal elements. In the flux integral this particular term is the only one that survives:

$$\mathbf{F}_\nu = -\frac{4\pi}{3k_\nu}\nabla\frac{j_\nu}{k_\nu}. \tag{4.36}$$

For the pressure moment the odd term goes away and the even terms survive. All the even terms lead to the angle average of **nn**, and so

$${}_\nu = \frac{1}{3}\left[\frac{4\pi}{c}\frac{j_\nu}{k_\nu} - \frac{4\pi}{k_\nu c^2}\frac{\partial}{\partial t}\left(\frac{j_\nu}{k_\nu}\right)\right]. \tag{4.37}$$

This result is crucial: the first two terms in the diffusion approximation lead to an isotropic pressure tensor. The intensity itself is not isotropic because of the first order gradient term, but to this order there is no anisotropic correction to I that is even in **n**. This would appear in the next term in the asymptotic series, of order $1/k_\nu^2$. Thus the diffusion approximation leads to Eddington's approximation

$${}_\nu = \frac{1}{3}E_\nu\ . \tag{4.38}$$

It is important to distinguish Eddington's approximation from the diffusion approximation. The diffusion approximation is stronger, i.e., a more severe approximation. Eddington's approximation follows from the diffusion approximation, as we have seen, but the reverse is not true. In the example discussed above, where it was noted that scattering with an albedo very close to unity results in thick boundary layers in which the radiation field is strongly influenced by the boundary, it would be found that Eddington's approximation would be valid in most of the boundary layer except the one or two means free paths next to the boundary. Scattering quickly produces isotropy of the intensity, even though the mean intensity might be far from the diffusion value.

We will leave the diffusion approximation aside for now and look at the implications of Eddington's approximation for the radiation moment equations. We substitute (4.38) into (4.28) and find

$$\frac{1}{c}\frac{\partial \mathbf{F}_\nu}{\partial t} + \frac{c}{3}\nabla E_\nu = -k_\nu \mathbf{F}_\nu. \tag{4.39}$$

Equations (4.27) and (4.39) form a closed set for the moments E_ν and $\mathbf{F}_\nu$, so these can be solved, possibly in conjunction with the hydrodynamic equations. The radiation equations themselves form a hyperbolic system. If we drop the j_ν and k_ν terms temporarily (not a good idea in general, since Eddington's approximation is based on $k_\nu L \gg 1$!) we can get a single equation for E_ν by combining the time derivative of (4.27) with the divergence of (4.39),

$$\frac{\partial^2 E_\nu}{\partial t^2} - \frac{c^2}{3}\nabla^2 E_\nu = 0. \tag{4.40}$$

This is the wave equation for a wave speed of $c/\sqrt{3}$. It is the wrong vacuum solution, of course. In reality, taking the j_ν and k_ν terms into account, the $c/\sqrt{3}$ waves can never be observed unless the equations are applied to a case for which Eddington's approximation is not valid. Before the wave can have propagated one wavelength it will have been absorbed, since $k_\nu \times$ wavelength should be large. The $\partial\mathbf{F}/\partial t$ term serves to keep the information propagation bounded; no signal will propagate faster than $c/\sqrt{3}$. But if this limit is being exercised, the solution is probably wrong.

As an alternative to the hyperbolic system we consider modifying the second moment equation (4.39) by discarding the $\partial\mathbf{F}_\nu/\partial t$ term to get

$$\frac{c}{3}\nabla E_\nu = -k_\nu \mathbf{F}_\nu. \tag{4.41}$$

By doing this we have lost the finite propagation speed, and we have also lost the radiation momentum density; when radiation imparts some momentum to the matter using this picture, there is no compensation in radiation momentum, so there is an error in the total momentum budget. The radiation equations are now substantially simpler. The flux need not be kept as a separate variable, but it can be eliminated between (4.27) and (4.41), which take the combined form

$$\frac{\partial E_\nu}{\partial t} - \nabla \cdot \left(\frac{c}{3k_\nu}\nabla E_\nu\right) = 4\pi j_\nu - k_\nu c E_\nu. \tag{4.42}$$

This is an equation of parabolic type, like the equation for the diffusion of heat. If the absorption and emission terms are dropped temporarily, then the terms that are left correspond to a radiation wave that spreads according to $x \sim \sqrt{ct/(3k_\nu)}$, which is plausible. Even so, the propagation of radiation from a pulse at $t = 0$ can be faster than the speed of light at early time when $k_\nu ct < 1$, and the transport in reality is not diffusive. This is related to the fact that there is no limit to the flux given by (4.41). *Ad hoc* methods for preventing numerical calculations from giving unphysical results on this account when they adopt an equation like (4.42) are referred to as flux limiting. These are discussed in Section 11.5. For additional reading on the merits of the Eddington closure of the radiation moment equations see Mihalas and Mihalas (1984) and also the literature that has developed on flux limiting (see Section 11.5), especially Pomraning (1982).

4.5 Coupling terms in Euler's equations

The topic of the coupling of radiation and matter concerns the source-sink terms in the conservation equations. In this discussion we generally follow Mihalas and Mihalas (1984), Section 94, with some parts from Castor (1972). We have already discussed how the right-hand side of the transport equation, i.e., $j_\nu - k_\nu I_\nu$, is the energy gained by the radiation field at the expense of the matter per unit volume per unit time per unit bandwidth and per unit solid angle. This is quite general, even when there are anisotropies, Doppler shifts, and so forth. Therefore the quantities g^0 and $\mathbf{g}$ defined by

$$g^0 = \int d\nu \int_{4\pi} d\Omega \, (j_\nu - k_\nu I_\nu) \tag{4.43}$$

and

$$\mathbf{g} = \frac{1}{c} \int d\nu \int_{4\pi} d\Omega \, \mathbf{n}(j_\nu - k_\nu I_\nu) \tag{4.44}$$

are the correct energy and momentum exchange rates. We should put the negatives of these on the right-hand side of the material total energy and momentum equations:

$$\frac{\partial}{\partial t}\left(\rho e + \frac{1}{2}\rho u^2\right) + \nabla \cdot \left(\rho \mathbf{u} h + \frac{1}{2}\rho \mathbf{u} u^2\right) = -g^0, \tag{4.45}$$

$$\frac{\partial \rho \mathbf{u}}{\partial t} + \nabla \cdot (\rho \mathbf{uu}) + \nabla p = -\mathbf{g}. \tag{4.46}$$

We have left out the nonradiative body force and heat deposition terms for simplicity. Summing the frequency-integrated radiation moment equations and the material equations gives the overall conservation laws

$$\frac{\partial}{\partial t}\left(\rho e + E + \frac{1}{2}\rho u^2\right) + \nabla \cdot \left(\rho \mathbf{u} h + \frac{1}{2}\rho \mathbf{u} u^2 + \mathbf{F}\right) = 0, \tag{4.47}$$

$$\frac{\partial}{\partial t}\left(\rho \mathbf{u} + \frac{\mathbf{F}}{c^2}\right) + \nabla \cdot (\rho \mathbf{uu} + \quad) + \nabla p = 0. \tag{4.48}$$

Here again we see that we have to choose between keeping the awkward radiation momentum density term or not having exact momentum conservation.

One remark should be made about the radiation–matter energy exchange. We often see treatments that use $-\nabla \cdot \mathbf{F}$ for the matter heating rate. That is incorrect, since it fails to account for the rate of change of the radiation energy density. In fact, $-\nabla \cdot \mathbf{F}$ is a contribution to the rate of increase of the density of matter energy plus radiation energy, as shown by (4.47). The correct matter heating rate due to the radiation is $-g^0$. In a steady state $\nabla \cdot \mathbf{F}$ and g^0 are equal, as shown by (4.29) and (4.43).