A1 Combining uncertainties

Why do squared uncertainties add up in sums?

We wish to determine the sum $f = x + y$ of two quantities, each of which are drawn from a probability distribution, with

$$E[x] = \mu_x; \quad E[(x - \mu_x)^2] = \sigma_x^2, \tag{A1.1}$$

$$E[y] = \mu_y; \quad E[(y - \mu_y)^2] = \sigma_y^2. \tag{A1.2}$$

The quantity $f = x + y$ has the expectation

$$\mu = \mu_x + \mu_y \tag{A1.3}$$

and a variance

$$\begin{aligned} \sigma_f^2 &= E[(f - \mu)^2] = E[(x - \mu_x + y - \mu_y)^2] \\ &= E[(x - \mu_x)^2 + (y - \mu_y)^2 + 2(x - \mu_x)(y - \mu_y)] \\ &= \sigma_x^2 + \sigma_y^2 + 2E[(x - \mu_x)(y - \mu_y)]. \end{aligned} \tag{A1.4}$$

If x and y are independent of each other (i.e., the deviations from the means of x and y are statistically independent samples), then the last term vanishes.[1] In that case the squared uncertainties (the variances) indeed add up to yield the squared uncertainty of the sum.

From the derivation we see immediately that squared uncertainties no longer simply add up when the deviations of the two contributing quantities are correlated. The quantity $E[(x - \mu_x)(y - \mu_y)]$ is the *covariance* of x and y. The covariance is often expressed relative to the variances themselves as the *correlation coefficient* ρ_{xy}:

$$\mathrm{cov}\,(x, y) = E[(x - \mu_x)(y - \mu_y)] \tag{A1.5}$$

$$\rho_{xy} = \frac{\mathrm{cov}\,(x, y)}{\sigma_x \sigma_y}. \tag{A1.6}$$

[1] Strictly, the requirement is that the two quantities are uncorrelated, i.e., their covariance is zero. This is a less severe requirement than being independent.

The complete equation for a sum thus is

$$\operatorname{var}(x+y) = \operatorname{var}(x) + \operatorname{var}(y) + 2\operatorname{cov}(x,y). \tag{A1.7}$$

For a *difference* $f = x - y$ the equation is

$$\operatorname{var}(x-y) = \operatorname{var}(x) + \operatorname{var}(y) - 2\operatorname{cov}(x,y). \tag{A1.8}$$

For a product, resp. a quotient, the same equations are valid for the *relative* deviations:

$$\frac{\operatorname{var} f}{f^2} = \frac{\operatorname{var} x}{x^2} + \frac{\operatorname{var} y}{y^2} \pm 2\frac{\operatorname{cov}(x,y)}{xy}, \tag{A1.9}$$

where the plus sign is valid for $f = xy$ and the minus sign for $f = x/y$.

The general equation for the variance of a function $f(x_1, x_2, \ldots)$ is

$$\operatorname{var}(f) = \sum_{i,j} \frac{\partial f}{\partial x_i}\frac{\partial f}{\partial x_j} \operatorname{cov}(x_i, x_j). \tag{A1.10}$$

Here $\operatorname{cov}(x_i, x_i) = \operatorname{var}(x_i)$. This equation follows directly by taking the square of

$$df = \sum_i \frac{\partial f}{\partial x_i}\, dx_i.$$

The assumption is made that deviations are small, so that only the first order in a Taylor expansion need be considered.

Here is an example of the use of covariances. Suppose we have performed (with a computer program) a least-squares analysis of $f(x) = ax + b$ on a number of data points with the result:[2]

$$a = 2.30526; \quad b = 5.21632;$$

$$\sigma_a = 0.00312; \quad \sigma_b = 0.0357; \quad \rho_{ab} = 0.7326.$$

These results will be used for an inter- or extrapolation: *What value and standard deviation is expected for $f(10)$?*

For this purpose we first determine the values, variances and covariances for the two quantities ax and b we want to add. In this case x is a multiplying

[2] Note that the numbers are given in too many digits. This is good practice for intermediate results of a statistical analysis, since unnecessary rounding errors are thus avoided.

factor that appears quadratically in $\mathrm{var}\,(ax)$ and linearly in $\mathrm{cov}\,(ax, b)$:

$$ax = 23.0526; \quad b = 5.21632; \quad f = 28.26892;$$

$$\mathrm{var}\,(ax) = 0.00312^2 \times 10^2; \quad \mathrm{var}\,(b) = 0.0357^2;$$

$$\mathrm{cov}\,(ax, b) = 10 \times 0.7326 \times 0.00312 \times 0.0357.$$

Insertion in Eq. (A1.7) then gives $\mathrm{var}\,(f) = 0.00388$. Had we disregarded the covariance, $\mathrm{var}\,(f)$ would have appeared to be equal to 0.00225. The s.d. of f now is 0.0623 and we give the result as $f = 28.27 \pm 0.06$.