A3 Characteristic function

The characteristic function $\Phi(t)$ of a probability density function $f(x)$:

$$\Phi(t) \stackrel{\text{def}}{=} E[e^{itx}] = \int_{-\infty}^{\infty} e^{itx} f(x)\, dx \tag{A3.1}$$

has some interesting properties. In fact, $\Phi(t)$ is the Fourier transform of $f(x)$. This implies that the characteristic function of the *convolution* $f_1 * f_2$ of two density functions f_1 and f_2 is the *product* of the two corresponding characteristic functions Φ_1 and Φ_2. The convolution, defined by

$$f_1 * f_2(x) = \int_{-\infty}^{\infty} f_1(x-\xi) f_2(\xi)\, d\xi, \tag{A3.2}$$

is the density distribution of the *sum* of two random variables $x_1 + x_2$ when the density functions of x_1 and x_2 are resp. f_1 and f_2. The *convolution theorem* of Fourier analysis states that the Fourier transform of a convolution equals the product of the Fourier transforms of the contributing terms. This product rule also applies to convolutions of n functions.

Another interesting property of the characteristic function is that its series expansion in powers of t generates the *moments* of the distribution. The characteristic function is therefore often called the *moment-generating function*. Since

$$e^{itx} = \sum_{n=0}^{\infty} \frac{(itx)^n}{n!}, \tag{A3.3}$$

it follows that

$$\Phi(x) = E[e^{itx}] = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} E[x^n] = \sum_{n=0}^{\infty} \frac{(it)^n}{n!} \mu_n. \tag{A3.4}$$

The moments are also given by the *derivatives* of the characteristic function at $t = 0$:

$$\Phi^{(n)}(0) = \frac{d^n \Phi}{dt^n}\Big|_{t=0} = i^n \mu_n. \tag{A3.5}$$

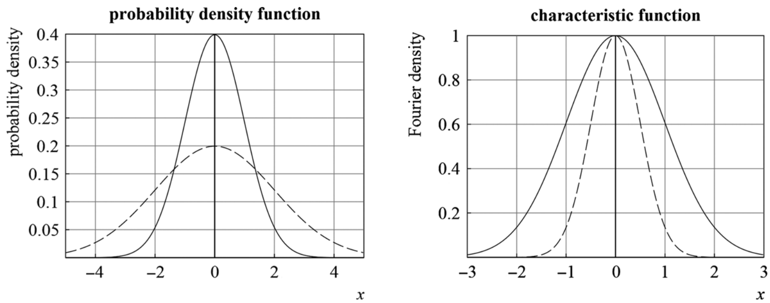

Figure A3.1 Left: a probability density function (in this case a normal distribution); right: its characteristic function. The dashed curve has a standard deviation twice that of the drawn curve.

The μ_n are the moments, not the central moments. But you can always choose the origin of x at the position of the mean.

A special case is the variance σ^2:

$$\sigma^2 = -\frac{d^2\Phi}{dt^2}(0). \tag{A3.6}$$

Figure A3.1 shows the relation between a density function and its characteristic function. The density function is normalized by its integral; the characteristic function is always equal to 1 for $t = 0$. The broader the density function, the narrower the characteristic function.