D: Convex geometry

D.1 Definitions of convexity

In this section we describe two further definitions of convexity (beyond the second order definition described in Section 2.1.3) that can be used for verifying convexity of a given scalar valued function $g : \mathbb{R}^N \to \mathbb{R}$. Additional care must be taken when the domain of g is not the entire $\mathbb{R}^N$ but a subset $\mathcal{D}$ of it. Specifically, the domain of a convex function g must be a convex set itself. A set $\mathcal{D}$ is convex if for any $\mathbf{w}_1$ and $\mathbf{w}_2$ in it, $\mathcal{D}$ also contains the line segment connecting $\mathbf{w}_1$ and $\mathbf{w}_2$. This line segment can be expressed via

$$\lambda \mathbf{w}_1 + (1 - \lambda)\,\mathbf{w}_2, \tag{D.1}$$

where each value for λ in the unit interval $[0,\ 1]$ uniquely corresponds to one point on the line segment. Examples of a convex and a non-convex set are illustrated in Fig. D.1.

D.1.1 Zeroth order definition of a convex function

A function g is convex if and only if any line segment connecting two points on the graph of g lies *above* its graph. Figure D.2 illustrates this definition of a convex function.

Stating this geometric fact algebraically, g is convex if and only if for all $\mathbf{w}_1$ and $\mathbf{w}_2$ in the domain of g and all $\lambda \in [0,\ 1]$, we have

$$g\left(\lambda \mathbf{w}_1 + (1 - \lambda)\,\mathbf{w}_2\right) \leq \lambda g\left(\mathbf{w}_1\right) + (1 - \lambda)\, g\left(\mathbf{w}_2\right). \tag{D.2}$$

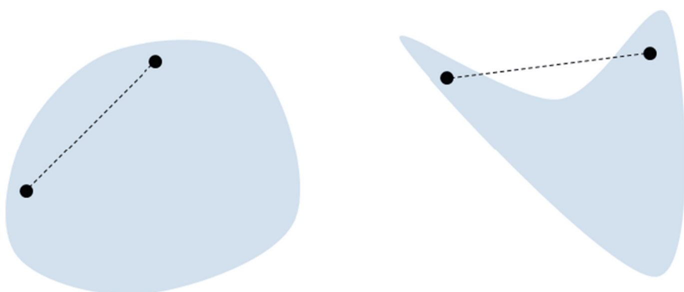

Fig. D.1 (left) A convex set contains the line segment connecting any two points inside it. (right) A non-convex set does not satisfy this property.

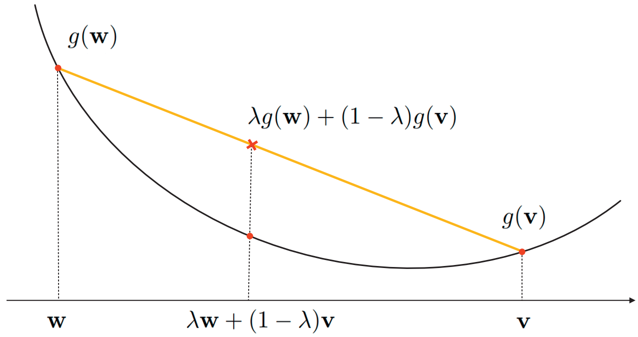

Fig. D.2 The line segment (shown in orange) connecting two points on the graph of a convex function.

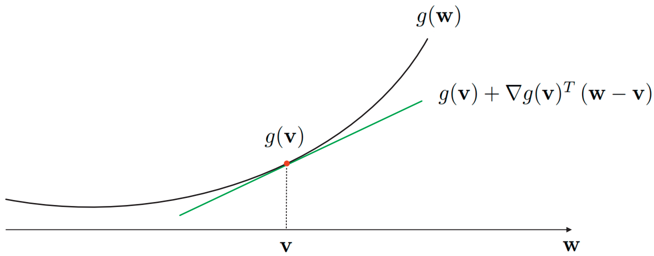

Fig. D.3 A differentiable convex function is one whose tangent plane at any point $\mathbf{v}$ lies below its graph.

D.1.2 First order definition of a convex function

A *differentiable* function g is convex if and only if at each point $\mathbf{v}$ in its domain the tangent plane (generated by its first order Taylor approximation) lies *below* the graph of g. This definition is shown in Fig. D.3.

Algebraically, it says that a differentiable g is convex if and only if for all $\mathbf{w}$ and $\mathbf{v}$ in its domain, we have

$$g(\mathbf{w}) \geq g(\mathbf{v}) + \nabla g(\mathbf{v})^T (\mathbf{w} - \mathbf{v}). \tag{D.3}$$

Note that this definition of convexity only applies for differentiable functions.