A: Basic vector and matrix operations

A.1 Vector operations

Vector addition: The addition of two N-dimensional vectors

$$\mathbf{a}_{N\times 1} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_N \end{bmatrix} \quad \text{and} \quad \mathbf{b}_{N\times 1} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix}, \tag{A.1}$$

is defined as the entry-wise sum of **a** and **b**, resulting in a vector of the same dimension denoted by

$$\mathbf{a} + \mathbf{b} = \begin{bmatrix} a_1 + b_1 \\ a_2 + b_2 \\ \vdots \\ a_N + b_N \end{bmatrix}. \tag{A.2}$$

Subtraction of two vectors is defined in a similar fashion.

Vector multiplication by a scalar: Multiplying a vector **a** by a scalar γ returns a vector of the same dimension whose every element is multiplied by γ

$$\gamma\mathbf{a} = \begin{bmatrix} \gamma a_1 \\ \gamma a_2 \\ \vdots \\ \gamma a_N \end{bmatrix}. \tag{A.3}$$

Vector transpose: The transpose of a *column vector* **a** (with vertically stored elements) is a *row vector* with the same elements which are now stored horizontally, denoted by

$$\mathbf{a}^T = \begin{bmatrix} a_1 & a_2 & \cdots & a_N \end{bmatrix}. \tag{A.4}$$

Similarly the transpose of a row vector is a column vector, and we have in general that $\left(\mathbf{a}^T\right)^T = \mathbf{a}$.

Inner product of two vectors: The inner product (or dot product) of two vectors $\mathbf{a}$ and $\mathbf{b}$ (of the same dimensions) is simply the sum of their component-wise product, written as

$$\mathbf{a}^T\mathbf{b} = \sum_{n=1}^{N} a_n b_n. \tag{A.5}$$

The inner product of $\mathbf{a}$ and $\mathbf{b}$ is also often written as $\langle \mathbf{a}, \mathbf{b} \rangle$.

Inner product rule and correlation: The inner product rule between two vectors provides a measurement of

$$\mathbf{a}^T\mathbf{b} = \|\mathbf{a}\|_2 \, \|\mathbf{b}\|_2 \cos(\theta), \tag{A.6}$$

where θ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$. Dividing both vectors by their length gives the *correlation* between the two vectors

$$\frac{\mathbf{a}^T\mathbf{b}}{\|\mathbf{a}\|_2 \, \|\mathbf{b}\|_2} = \cos(\theta), \tag{A.7}$$

which ranges between -1 and 1 (when the vectors point in completely opposite or parallel directions respectively).

Outer product of two vectors: The outer product of two vectors $\mathbf{a}_{N\times 1}$ and $\mathbf{b}_{M\times 1}$ is an $N \times M$ matrix $\mathbf{C}$ defined as

$$\mathbf{C} = \begin{bmatrix} a_1b_1 & a_1b_2 & \cdots & a_1b_M \\ a_2b_1 & a_2b_2 & \cdots & a_2b_M \\ \vdots & \vdots & \ddots & \vdots \\ a_Nb_1 & a_Nb_2 & \cdots & a_Nb_M \end{bmatrix}. \tag{A.8}$$

The outer product of $\mathbf{a}$ and $\mathbf{b}$ is also written as $\mathbf{a}\mathbf{b}^T$. Unlike the inner product, the outer product does not hold the commutative property meaning that the outer product of $\mathbf{a}$ and $\mathbf{b}$ does *not* necessarily equal the outer product of $\mathbf{b}$ and $\mathbf{a}$.

A.2 Matrix operations

Matrix addition: The addition of two $N \times M$ matrices

$$\mathbf{A}_{N\times M} = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,M} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} & a_{N,2} & \cdots & a_{N,M} \end{bmatrix} \text{ and } \mathbf{B}_{N\times M} = \begin{bmatrix} b_{1,1} & b_{1,2} & \cdots & b_{1,M} \\ b_{2,1} & b_{2,2} & \cdots & b_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ b_{N,1} & b_{N,2} & \cdots & b_{N,M} \end{bmatrix}, \tag{A.9}$$

is defined again as the entry-wise sum of $\mathbf{A}$ and $\mathbf{B}$, given as

$$\mathbf{A} + \mathbf{B} = \begin{bmatrix} a_{1,1} + b_{1,1} & a_{1,2} + b_{1,2} & \cdots & a_{1,M} + b_{1,M} \\ a_{2,1} + b_{2,1} & a_{2,2} + b_{2,2} & \cdots & a_{2,M} + b_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1} + b_{N,1} & a_{N,2} + b_{N,2} & \cdots & a_{N,M} + b_{N,M} \end{bmatrix}. \tag{A.10}$$

Subtraction of two matrices is defined in a similar fashion.

Matrix multiplication by a scalar: Multiplying a matrix $\mathbf{A}$ by a scalar γ returns a matrix of the same dimension whose every element is multiplied by γ

$$\gamma\mathbf{A} = \begin{bmatrix} \gamma a_{1,1} & \gamma a_{1,2} & \cdots & \gamma a_{1,M} \\ \gamma a_{2,1} & \gamma a_{2,2} & \cdots & \gamma a_{2.M} \\ \vdots & \vdots & \ddots & \vdots \\ \gamma a_{N,1} & \gamma a_{N,2} & \cdots & \gamma a_{N,M} \end{bmatrix}. \tag{A.11}$$

Matrix transpose: The transpose of an $N \times M$ matrix $\mathbf{A}$ is formed by putting the transpose of each column of $\mathbf{A}$ into the corresponding row of $\mathbf{A}^T$, giving the $M \times N$ transpose matrix as

$$\mathbf{A}^T = \begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{N,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{N,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,M} & a_{2,M} & \cdots & a_{M,N} \end{bmatrix}. \tag{A.12}$$

Again, we have $\left(\mathbf{A}^T\right)^T = \mathbf{A}$.

Matrix multiplication: The product of two matrices $\mathbf{A}_{N\times M}$ and $\mathbf{B}_{M\times P}$ is an $N \times P$ matrix defined via the sum of M outer product matrices, as

$$\mathbf{C} = \mathbf{AB} = \sum_{m=1}^{M} \mathbf{a}_m \mathbf{b}^m, \tag{A.13}$$

where $\mathbf{a}_m$ and $\mathbf{b}^m$ respectively denote the mth column of $\mathbf{A}$ and the mth row of $\mathbf{B}$.

The pth column of $\mathbf{C}$ can be found via multiplying $\mathbf{A}$ by the pth column of $\mathbf{B}$,

$$\mathbf{c}_p = \mathbf{A}\mathbf{b}_p = \sum_{m=1}^{M} \mathbf{a}_m b_{m,p}. \tag{A.14}$$

The nth row of $\mathbf{C}$ can be found via multiplying the nth row of $\mathbf{A}$ by $\mathbf{B}$,

$$\mathbf{c}^n = \mathbf{a}^n\mathbf{B} = \sum_{m=1}^{M} a_{n,m} \mathbf{b}^m. \tag{A.15}$$

The (n, p) th entry of $\mathbf{C}$ is found by multiplying the nth row of $\mathbf{A}$ by the pth column of $\mathbf{B}$,

$$c_{n,p} = \mathbf{a}^n \mathbf{b}_p. \tag{A.16}$$

Note that vector inner and outer products are special cases of matrix multiplication.

Entry-wise product: The entry-wise product (or Hadamard product) of two matrices $\mathbf{A}_{N\times M}$ and $\mathbf{B}_{N\times M}$ is defined as

$$\mathbf{A} \circ \mathbf{B} = \begin{bmatrix} a_{1,1}b_{1,1} & a_{1,2}b_{1,2} & \cdots & a_{1,M}b_{1,M} \\ a_{2,1}b_{2,1} & a_{2,2}b_{2,2} & \cdots & a_{2.M}b_{2,M} \\ \vdots & \vdots & \ddots & \vdots \\ a_{N,1}b_{N,1} & a_{N,2}b_{N,2} & \cdots & a_{N,M}b_{N,M} \end{bmatrix}. \tag{A.17}$$

In other words, the (n, m) th entry of $\mathbf{A} \circ \mathbf{B}$ is simply the product of the (n, m) th entry of $\mathbf{A}$ and the (n, m) th entry of $\mathbf{B}$.