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Entanglement-enhanced information transmission over a quantum channel with correlated noise

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We show that entanglement is a useful resource to enhance the mutual information of the depolarizing channel when the noise on consecutive uses of the channel has some partial correlations. We obtain a threshold in the degree of memory above which a higher amount of classical information is transmitted with entangled signals.

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The classical capacity of quantum channels, i.e., the amount of classical information that can be reliably transmitted by quantum states in the presence of a noisy environment has received renewed interest in recent years [1]. One of the main focuses of such interest is the study of entanglement as a useful resource to enhance the classical channel capacity. Indeed, the theory does not rule out the possibility that by entangling multiple uses of the channel, a larger amount of classical information per use can be reliably transmitted. This property is known as superadditivity (a more precise definition will be provided later in the text). Attention so far has been paid to memoryless channels, i.e., channels in which independent noise acts on each use. The absence of superadditivity has been first proved analytically for the case of two entangled uses of the depolarizing channel [2], and then extended to a broader class of memoryless channels [3]. A different related problem, which we will not consider here, is the entanglement-assisted classical capacity. In [4] it has been shown that prior entanglement between sender and receiver can increase the classical capacity of some noisy memoryless quantum channels.

In this paper we will turn our attention to a different class of channels, in which correlated noise acts on consecutive uses, i.e., to channels with partial memory. For such channels our results show that a higher mutual information can indeed be achieved above a certain memory threshold by entangling two consecutive uses of the channel. In the following, each use of the channel will be a qubit, i.e., it will be a quantum state belonging to a two-dimensional Hilbert space. The action of transmission channels is described by Kraus operators [5] A_i, satisfying $\Sigma_i A_i^\dagger A_i = \mathbb{1}$, such that if we send through the channel a qubit in a state described by the density operator π, the corresponding output state is given by the map

$$\pi \rightarrow \Phi(\pi) = \sum_i A_i \pi A_i^\dagger . \tag{1}$$

An interesting class of Kraus operators acting on individual qubits can be expressed in terms of the Pauli operators $\sigma_{x,y,z}$

$$A_i = \sqrt{p_i}\sigma_i , \tag{2}$$

with $\Sigma_i p_i = 1$, $i = 0,x,y,z$ and $\sigma_0 = \mathbb{1}$. A noise model for these Kraus operators is, for instance, a random rotation of an angle π around axes $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ with probability p_x, p_y, p_z on the qubit state, or the identity with probability p_0.

In the simplest scenario the transmitter can send one qubit at a time along the channel. In this case the codewords will be restricted to be the tensor products of the states of the individual qubits. Quantum mechanics, however, allows also the possibility to entangle multiple uses of the channel. For this more general strategy it has been shown that the amount of reliable information that can be transmitted per use of the channel is given by [1]

$$C_n = \frac{1}{n} \sup_{\mathcal{E}} I_n(\mathcal{E}), \tag{3}$$

where $\mathcal{E} = \{P_i, \pi_i\}$ with $P_i \geqslant 0$, $\Sigma P_i = 1$ is the input ensemble of states π_i, transmitted with *a priori* probabilities P_i, of n—generally entangled—qubits, and $I_n(\mathcal{E})$ is the mutual information

$$I_n(\mathcal{E}) = S(\rho) - \sum_i P_i S(\rho_i), \tag{4}$$

where the index n stands for the number of uses of the channel. In the above equation

$$S(\chi) = -\mathrm{Tr}(\chi \log_2 \chi) \tag{5}$$

is the von Neumann entropy, $\rho_i = \Phi(\pi_i)$ are the density operators describing the output states, and $\rho = \Sigma_i P_i \rho_i$. The advantage of the expression (4) is that it includes an optimization over all possible POVMs (positive operator value measures) at the output, including collective ones. Therefore no explicit maximization procedure for the decoding at the output of the channel is needed.

The interest for the possibility of using entangled states as channel inputs is motivated by the fact that it cannot generally be excluded that $I_n(\mathcal{E})$ is superadditive in the presence of entanglement, i.e., we might have $I_{n+m} > I_n + I_m$ and, therefore, $C_n > C_1$. In this scenario, the classical capacity C of the channel is defined as

$$C = \lim_{n\rightarrow\infty} C_n . \tag{6}$$

So far the main objects of investigation have been memoryless channels. By definition, a channel is memoryless when its action on arbitrary signals π_s, consisting of n qubits (including entangled ones), is given by

$$\Phi(\pi_s)=\sum_{i_1\cdots i_n}(A_{i_n}\otimes\cdots\otimes A_{i_1})\pi_s(A_{i_1}^\dagger\otimes\cdots\otimes A_{i_n}^\dagger). \quad (7)$$

In the case of Pauli channels a more general situation is described by Kraus operators of the following form:

$$A_{k_1\ldots k_n}=\sqrt{p_{k_1\ldots k_n}}\sigma_{k_1}\ldots\sigma_{k_n}, \quad (8)$$

with $\Sigma_{k_1\ldots k_n}p_{k_1\ldots k_n}=1$. The quantity $p_{k_1\ldots k_n}$ can be interpreted as the probability that a given random sequence of rotations of an angle π along axes $k_1\ldots k_n$ is applied to the sequence of n qubits sent through the channel. For a memoryless channel, $p_{k_1\ldots k_n}=p_{k_1}p_{k_2}\cdots p_{k_n}$. An interesting generalization is described by Markov chains defined as

$$p_{k_1\ldots k_n}=p_{k_1}p_{k_2|k_1}\cdots p_{k_n|k_{n-1}}, \quad (9)$$

where $p_{k_n|k_{n-1}}$ can be interpreted as the conditional probability that a π rotation around the k_n axis is applied to the nth qubit given that a π rotation around the k_{n-1} axis was applied on the $(n-1)$th qubit. Here we will consider the case of two consecutive uses of a channel with partial memory, i.e., we will assume $p_{k_n|k_{n-1}}=(1-\mu)p_{k_n}+\mu\delta_{k_n,k_{n-1}}$. This means that with probability μ the same rotation is applied to both qubits while with probability $1-\mu$ the two rotations are uncorrelated.

This noise model can describe situations in which time correlations are present in the system. For instance, μ could depend on the time lapse between the two channel uses. If the two qubits are sent at a very short time interval, the properties of the channel, which determine the direction of the random rotations, will be unchanged, and it is, therefore, reasonable to assume that the action on both qubits will take the form

$$A_k^c=\sqrt{p_k}\sigma_k\sigma_k. \quad (10)$$

If on the other hand, the time interval between the channel uses is such that the channel properties have changed, then the actions will be

$$A_{k_1,k_2}^u=\sqrt{p_{k_1}}\sqrt{p_{k_2}}\sigma_{k_1}\sigma_{k_2}. \quad (11)$$

An intermediate case, as mentioned above, is described by actions of the form

$$A_{k_1,k_2}^i=\sqrt{p_{k_1}[(1-\mu)p_{k_2}+\mu\delta_{k_2|k_1}]}\sigma_{k_2}\sigma_{k_2}. \quad (12)$$

It is straightforward to verify that the Bell states, defined in the basis $|0\rangle,|1\rangle$ of the eigenstates of the σ_z operators as

$$|\Phi_\pm\rangle=\frac{1}{\sqrt{2}}\{|00\rangle\pm|11\rangle\}, \quad (13)$$

$$|\Psi_\pm\rangle=\frac{1}{\sqrt{2}}\{|01\rangle\pm|10\rangle\},$$

are eigenstates of the operators A_k^c and, therefore, will pass undisturbed through the channel. If used as equiprobable signal states they maximize I_2, as we will have $I_2=2$. Furthermore, it is immediate to verify that the value $I_2=2$ cannot be achieved by any ensemble of tensor product input states. This situation is reminiscent of the so-called noiseless codes, where collective states are used to encode and protect quantum information against collective noise [6].

In the following we will concentrate our attention to the depolarizing channel, for which $p_0=1-p$ and $p_i=p/3$, $i=x,y,z$. We will consider an ensemble of orthogonal input states parametrized as follows

$$|\pi_1\rangle=\cos\vartheta|00\rangle+\sin\vartheta|11\rangle,$$

$$|\pi_2\rangle=\sin\vartheta|00\rangle-\cos\vartheta|11\rangle,$$

$$|\pi_3\rangle=\cos\vartheta|01\rangle+\sin\vartheta|10\rangle,$$

$$|\pi_4\rangle=\sin\vartheta|01\rangle-\cos\vartheta|10\rangle. \quad (14)$$

Although it is not *a priori* certain that this is the optimal choice for all values of μ, we know that it maximizes C_2 with $\vartheta=0$ for $\mu=0$ (uncorrelated noise), and with $\vartheta=\pi/4$ for $\mu=1$ (fully correlated noise). We will, therefore, optimize the ansatz (14) by looking for the value $\vartheta(\mu)$, which maximizes I_2 as a function of μ.

We will now show that there is a threshold value μ_t for which $I_2(\vartheta=\pi/4,\mu_t)=I_2(\vartheta=0,\mu_t)$. Below the threshold value, $I_2(\vartheta=0,\ \mu<\mu_t)>I_2(\vartheta=\pi/4,\ \mu<\mu_t)$, while above it $I_2(\vartheta=\pi/4,\ \mu>\mu_t)>I_2(\vartheta=0,\ \mu>\mu_t)$. To this goal, it is useful to use the Bloch representation [7] for the input states

$$\pi=\frac{1}{4}\left\{\mathbb{1}\otimes\mathbb{1}+\mathbb{1}\otimes\sum_k \beta_k^{(2)}\sigma_k+\sum_k \beta_k^{(1)}\sigma_k\otimes\mathbb{1}+\sum_{kl}\chi_{kl}\sigma_k\otimes\sigma_l\right\}, \quad (15)$$

where the Bloch vectors and tensor are defined, respectively, as $\beta_i=\mathrm{Tr}(\pi\sigma_i)$, $\chi_{ij}=\mathrm{Tr}(\pi\sigma_i\sigma_j)$. We will express the action of the channel in terms of the so-called shrinking factor [8] $\eta=1-4p/3$.

It is straightforward to verify that for $\mu=0$,

$$\sum_{k_1,k_2}A_{k_1,k_2}\mathbb{1}\otimes\sigma_jA_{k_1,k_2}^\dagger=\eta\mathbb{1}\otimes\sigma_j,$$

$$\sum_{k_1,k_2}A_{k_1,k_2}\sigma_j\otimes\mathbb{1}A_{k_1,k_2}^\dagger=\eta\sigma_j\otimes\mathbb{1},$$

$$\sum_{k_1,k_2}A_{k_1,k_2}\sigma_k\otimes\sigma_jA_{k_1,k_2}^\dagger=\eta^2\sigma_k\otimes\sigma_j, \quad (16)$$

while for $\mu=1$

$$\sum_{k_1,k_2} A_{k_1,k_2} \mathbb{1}\otimes\sigma_j A^\dagger_{k_1,k_2} = \eta \mathbb{1}\otimes\sigma_j ,$$

$$\sum_{k_1,k_2} A_{k_1,k_2} \sigma_j \otimes \mathbb{1} A^\dagger_{k_1,k_2} = \eta \sigma_j\otimes\mathbb{1},$$

$$\sum_{k_1,k_2} A_{k_1,k_2}\sigma_k\otimes\sigma_j A^\dagger_{k_1,k_2} = \delta_{kj}\sigma_k\otimes\sigma_j + (1-\delta_{kj})\,\eta\sigma_k\otimes\sigma_j . \tag{17}$$

It is interesting to note that both for $\mu=0$ and for $\mu=1$, the components of the Bloch vectors $\beta_k^{(i)}$ of the input states are shrunk isotropically by the shrinking factor η. The difference between the two cases is the action on the Bloch tensor χ. The input state π_1 is transformed by the action of the depolarizing channel with partial memory defined in Eq. (12) into the output density operator ρ_1,

$$\rho_1 = \frac{1}{4}\{\mathbb{1}\otimes\mathbb{1} + \eta\cos 2\vartheta(\mathbb{1}\otimes\sigma_z + \sigma_z\otimes\mathbb{1}) + [\mu+(1-\mu)\,\eta^2] \times[\sigma_z\otimes\sigma_z + \sin 2\vartheta(\sigma_x\otimes\sigma_x - \sigma_y\otimes\sigma_y)]\}, \tag{18}$$

whose eigenvalues are

$$\lambda_{1,2} = \frac{1}{4}(1-\mu)(1-\eta^2), \tag{19}$$

$$\lambda_{3,4} = \frac{1}{4}\{1+\mu+\eta^2(1-\mu) \pm 2\sqrt{\eta^2\cos^2 2\vartheta + [\eta^2(1-\mu)+\mu]^2\sin^2 2\vartheta}\}. \tag{20}$$

Notice that the first two eigenvalues are degenerate and do not depend on ϑ. The same eigenvalues are obtained for the output states ρ_2,ρ_3,ρ_4. The von Neumann entropy $S(\rho_i)$ is minimized as a function of ϑ when the term under the square root in the expression for $\lambda_{3,4}$ is maximum. The mutual information is then maximized for equiprobable states π_i corresponding to the minimum von Neumann entropy. Therefore for $\eta^2 > [\eta^2(1-\mu)+\mu]^2$ the mutual information is maximal for uncorrelated states $\vartheta=0$, while for $\eta^2 < [\eta^2(1-\mu)+\mu]^2$ it is maximal for the Bell states. The threshold value μ_t is a function of the shrinking factor and for $0<\eta<1$ takes the form

$$\mu_t = \frac{\eta}{1+\eta}. \tag{21}$$

Therefore, for channels with $\mu<\mu_t$ the most convenient choice within the ansatz (14) corresponds to uncorrelated states, while for $\mu>\mu_t$, to maximally entangled states. At the threshold value, any set of states of the form (14) leads to the same value for the mutual information. As an example, the behavior of the mutual information is plotted in Fig. 1. It is interesting to notice that, within the ansatz (14), for any value of μ, the mutual information is optimized by either maximally entangled or completely unentangled states. We

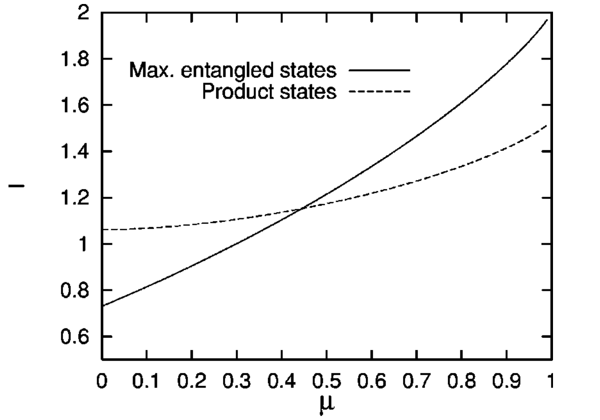

FIG. 1. Mutual information for product states and for maximally entangled states as a function of the degree of memory of the channel, for $\eta=0.8$.

have used so far the z axis as the axis of quantization for the system; however, due to the symmetry of the channel, we would have obtained the same results using x or y as the axis of quantization.

Notice that so far we have restricted our attention to input states of the form (14). We will now show that the product states that are less deteriorated when transmitted through the channel are the eigenstates of $\sigma_{z1}\sigma_{z2}$ or $\sigma_{y1}\sigma_{y2}$ or $\sigma_{x1}\sigma_{x2}$. This suggests that no different choice of product signal states can achieve a higher I_2 than our ansatz (14). From Eqs. (16) and (17) it follows that the output density operator corresponding to an arbitrary input product state takes the form

$$\Phi(\pi) = \frac{1}{4}\Bigg[\mathbb{1}\otimes\mathbb{1} + \eta\Bigg(\mathbb{1}\otimes\sum_i \beta_{2i}\sigma_{2i} + \sum_i \beta_{1i}\sigma_{1i}\otimes\mathbb{1}\Bigg) + (\mu+(1-\mu)\,\eta^2)\sum_i \beta_{1i}\beta_{2i}\sigma_{1i}\otimes\sigma_{2i} + (\mu\eta+(1-\mu)\,\eta^2)\sum_{i\neq j}\beta_{1i}\beta_{2j}\sigma_{1i}\otimes\sigma_{2j}\Bigg]. \tag{22}$$

A measure of the degree of purity of the state at the output of the channel is given by $\mathrm{Tr}(\rho^2)$. It is straightforward to show that for the above state we have

$$\mathrm{Tr}[\Phi(\pi)^2] = \frac{1}{4}\Bigg[1+2\eta^2+[\mu+(1-\mu)\,\eta^2]^2\sum_i \beta_{1i}^2\beta_{2i}^2 + [\mu\eta+(1-\mu)\,\eta^2]^2\sum_{i\neq j}\beta_{1i}^2\beta_{2j}^2\Bigg]. \tag{23}$$

The above expression is maximized when both Bloch vectors point in the same x, y, or z direction. It is straightforward to verify that these states maximize also the fidelity, defined as $\mathrm{Tr}[\pi\Phi(\pi)]$. Moreover, we have numerical evidence that for any value of μ and η, the input product states that maximize the mutual information are still of this form. Therefore, no better choice of product states leads to a higher mutual information than that achieved by the ansatz (14). Finally we would like to point out that for input product states, the mutual information I_2 is larger for $\mu=1$ than for $\mu=0$,

$$I_2(\mu=1,\ \vartheta=0)=1+\frac{1}{2}\{(1+\eta)\log_2(1+\eta)+(1-\eta)\log_2(1-\eta)\},$$
$$I_2(\mu=0,\ \vartheta=0)=\{(1+\eta)\log_2(1+\eta)+(1-\eta)\log_2(1-\eta)\}. \quad (24)$$

This is due to the fact that the Bloch tensor χ is multiplied by a larger shrinking factor when the noise is collective. In other words, in the presence of perfect memory with two uses of the channel, it is possible to achieve a higher mutual information than in the case of memoryless channels even if we restrict to product states.

In conclusion, in this paper we have analyzed for the first time, to the best of our knowledge, the problem of the classical capacity of quantum channels with time correlated noise. This problem is of great interest not only from the theoretical viewpoint but also from the experimental one as time correlated noise is not rare in real physical quantum transmission channels. For the specific case of a quantum depolarizing channel with collective noise, we have shown that the transmission of classical information can be enhanced by employing maximally entangled states as carriers of information rather than product states. This result broadens the class of situations in which the use of entanglement enhances the efficiency in communications and information processing.

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