PHYSICAL REVIEW LETTERS

VOLUME 77 | 19 AUGUST 1996 | NUMBER 8

Separability Criterion for Density Matrices

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(Received 8 April 1996)

A quantum system consisting of two subsystems is *separable* if its density matrix can be written as $\rho = \sum_A w_A \rho'_A \otimes \rho''_A$, where ρ'_A and ρ''_A are density matrices for the two subsystems, and the positive weights w_A satisfy $\sum w_A = 1$. In this Letter, it is proved that a necessary condition for separability is that a matrix, obtained by partial transposition of ρ, has only non-negative eigenvalues. Some examples show that this criterion is more sensitive than Bell's inequality for detecting quantum inseparability. [S0031-9007(96)00911-8]

PACS numbers: 03.65.Bz, 03.65.Ca

A striking quantum phenomenon is the inseparability of composite quantum systems. Its most famous example is the violation of Bell's inequality, which may be detected if two distant observers, who independently *measure* subsystems of a composite quantum system, *report* their results to a common site where that information is analyzed [1]. However, even if Bell's inequality is satisfied by a given composite quantum system, there is no guarantee that its state can be *prepared* by two distant observers who receive *instructions* from a common source. For this to be possible, the density matrix ρ has to be separable into a sum of direct products,

$$\rho = \sum_A w_A \rho'_A \otimes \rho''_A, \tag{1}$$

where the positive weights w_A satisfy $\sum w_A = 1$, and where ρ'_A and ρ''_A are density matrices for the two subsystems. A separable system always satisfies Bell's inequality, but the converse is not necessarily true [2–5]. In this Letter, I shall derive a simple algebraic test, which is a *necessary* condition for the existence of the decomposition (1). I shall then give some examples showing that this criterion is more restrictive than Bell's inequality, or than the α-entropy inequality [6].

The derivation of this separability condition is best done by writing the density matrix elements explicitly, with all their indices [1]. For example, Eq. (1) becomes

$$\rho_{m\mu,n\nu} = \sum_A w_A (\rho'_A)_{mn} (\rho''_A)_{\mu\nu} . \tag{2}$$

Latin indices refer to the first subsystem, Greek indices to the second one (the subsystems may have different dimensions). Note that this equation can always be satisfied if we replace the quantum density matrices by classical Liouville functions (and the discrete indices are replaced by canonical variables **p** and **q**). The reason is that the only constraint that a Liouville function has to satisfy is being non-negative. On the other hand, we want quantum density matrices to have non-negative *eigenvalues*, rather than non-negative elements, and the latter condition is more difficult to satisfy.

Let us now define a new matrix,

$$\sigma_{m\mu,n\nu} \equiv \rho_{n\mu,m\nu} . \tag{3}$$

The Latin indices of ρ have been transposed, but not the Greek ones. This is not a unitary transformation but, nevertheless, the σ matrix is Hermitian. When Eq. (1) is valid, we have

$$\sigma = \sum_A w_A (\rho'_A)^T \otimes \rho''_A. \tag{4}$$

Since the transposed matrices $(\rho'_A)^T \equiv (\rho'_A)^*$ are non-negative matrices with unit trace, they can also be legitimate density matrices. It follows that *none of* *the eigenvalues of σ is negative.* This is a necessary condition for Eq. (1) to hold.

Note that the eigenvalues of σ are invariant under separate unitary transformations, U' and U'', of the bases used by the two observers. In such a case, ρ transforms as

$$\rho \longrightarrow (U' \otimes U'')\rho(U' \otimes U'')^\dagger, \tag{5}$$

and we then have

$$\sigma \longrightarrow (U'^T \otimes U'')\sigma(U'^T \otimes U'')^\dagger, \tag{6}$$

which also is unitary transformation, leaving the eigenvalues of σ invariant.

As an example, consider a pair of spin-$\frac{1}{2}$ particles in a Werner state (an impure singlet), consisting of a single fraction x and a random fraction $(1 - x)$ [7]. Note that the "random fraction" $(1 - x)$ also includes singlets, mixed in equal proportions with the three triplet components. We have

$$\rho_{m\mu,n\nu} = xS_{m\mu,n\nu} + (1 - x)\delta_{mn}\delta_{\mu\nu}/4\,, \tag{7}$$

where the density matrix for a pure singlet is given by

$$S_{01,01} = S_{10,10} = -S_{01,10} = -S_{10,01} = \tfrac{1}{2}\,, \tag{8}$$

and all the other components of S vanish. (The indices 0 and 1 refer to any two orthogonal states, such as "up" and "down.") A straightforward calculation shows that σ has three eigenvalues equal to $(1 + x)/4$, and the fourth eigenvalue is $(1 - 3x)/4$. This lowest eigenvalue is positive if $x < \frac{1}{3}$, and the separability criterion is then fulfilled. This result may be compared with other criteria: Bell's inequality holds for $x < 1/\sqrt{2}$, and the α-entropic inequality [6] for $x < 1/\sqrt{3}$. These are, therefore, much weaker tests for detecting inseparability than the condition that was derived here.

In this particular case, it happens that this necessary condition is also a sufficient one. It is indeed known that if $x < \frac{1}{3}$ it is possible to write ρ as a mixture of unentangled product states [8]. This result suggests that the necessary condition derived above (σ has no negative eigenvalue) might also be sufficient for any ρ. Some time after this Letter was submitted for publication, a proof of this conjecture was indeed obtained [9] for composite systems having dimensions 2×2 and 2×3. However, for higher dimensions, the present necessary condition was shown *not* to be a sufficient one.

As a second example, consider a mixed state introduced by Gisin [5]. With the present notations, it consists of a fraction x of the pure state $a|01\rangle + b|10\rangle$ (with $|a|^2 + |b|^2 = 1$), and fractions $(1 - x)/2$ of the pure states $|00\rangle$ and $|11\rangle$. The nonvanishing elements of ρ thus are

$$\rho_{00,00} = \rho_{11,11} = (1 - x)/2\,, \tag{9}$$

$$\rho_{01,01} = x|a|^2, \tag{10}$$

$$\rho_{10,10} = x|b|^2, \tag{11}$$

$$\rho_{01,10} = \rho^*_{10,01} = xab^*. \tag{12}$$

It is easily seen that the σ matrix has a negative determinant, and therefore a negative eigenvalue, when

$$x > (1 + 2|ab|)^{-1}. \tag{13}$$

This is a lower limit than the one for a violation of Bell's inequality, which requires [5]

$$x > [1 + 2|ab|(\sqrt{2} - 1)]^{-1}. \tag{14}$$

An even more striking example is the mixture of a singlet and a maximally polarized pair:

$$\rho_{m\mu,n\nu} = xS_{m\mu,n\nu} + (1 - x)\delta_{m0}\delta_{n0}\delta_{\mu0}\delta_{\nu0}. \tag{15}$$

For any positive x, however small, this state is inseparable, because σ has a negative eigenvalue $(-x/2)$. On the other hand, the Horodecki criterion [10] gives a very generous domain to the validity of Bell's inequality: $x \leq 0.8$.

The weakness of Bell's inequality as a test for inseparability is attributable to the fact that the only use made of the density matrix ρ is for computing the probabilities of the various outcomes of tests that may be performed on the subsystems of a *single* composite system. On the other hand, an experimental verification of that inequality necessitates the use of *many* composite systems, all prepared in the same way. However, if many such systems are actually available, we may also test them collectively, for example, two by two, or three by three, etc., rather than one by one. If we do that, we must use, instead of ρ (the density matrix of a single system), a *new* density matrix, which is $\rho \otimes \rho$, or $\rho \otimes \rho \otimes \rho$, in a higher-dimensional space. It then turns out that there are some density matrices ρ that satisfy Bell's inequality, but for which $\rho \otimes \rho$, or $\rho \otimes \rho \otimes \rho$, etc., violate that inequality [11].

This result raises a new question: Can we get stronger inseparability criteria by considering $\rho \otimes \rho$, or higher tensor products? It is easily seen that no further progress can be achieved in this way. If ρ is separable as in Eq. (1), so is $\rho \otimes \rho$. Moreover, the partly transposed matrix corresponding to $\rho \otimes \rho$ simply is $\sigma \otimes \sigma$, so that if no eigenvalue of σ is negative, then $\sigma \otimes \sigma$ too has no negative eigenvalue.

I am grateful to R. Horodecki and R. Jozsa for pointing out that Eq. (4) could be used instead of a longer derivation that appeared in an earlier version of this

Letter. This work was supported by the Gerard Swope Fund and the Fund for Encouragement of Research.

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[1] A. Peres, *Quantum Theory: Concepts and Methods* (Kluwer, Dordrecht, 1993), Chaps. 5 and 6.
[2] R. F. Werner, Phys. Rev. A **40**, 4277 (1989).
[3] S. Popescu, Phys. Rev. Lett. **72**, 797 (1994); **74**, 2619 (1995).
[4] N. D. Mermin, in *Quantum Mechanics without Observer*, edited by R. K. Clifton (Kluwer, Dordrecht, 1996), pp. 57–71.
[5] N. Gisin, Phys. Lett. A **210**, 151 (1996).
[6] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **210**, 377 (1996).
[7] Werner considered only the case $x = \frac{1}{2}$. These more general states were introduced by J. Blank and P. Exner, Acta Univ. Carolinae, Math. Phys. **18**, 3 (1977).
[8] C. H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. Smolin, and W. K. Wootters, Phys. Rev. Lett. **76**, 722 (1996).
[9] M. Horodecki, P. Horodecki, and R. Horodecki, "Separability of Mixed Quantum States: Necessary and Sufficient Conditions" Report No. quant-ph/9605038.
[10] R. Horodecki, P. Horodecki, and M. Horodecki, Phys. Lett. A **200**, 340 (1995).
[11] A. Peres, "Collective Tests for Quantum Nonlocality" Phys. Rev. A (to be published).