Local Unitary Equivalence of Multipartite Pure States

B. Kraus
Institute for Theoretical Physics, University of Innsbruck, A-6020 Innsbruck, Austria
(Received 20 October 2009; published 14 January 2010)

Necessary and sufficient conditions for the equivalence of arbitrary n-qubit pure quantum states under local unitary (LU) operations are derived. First, an easily computable standard form for multipartite states is introduced. Two generic states are shown to be LU equivalent iff their standard forms coincide. The LU-equivalence problem for nongeneric states is solved by presenting a systematic method to determine the LU operators (if they exist) which interconvert the two states.

DOI: 10.1103/PhysRevLett.104.020504 PACS numbers: 03.67.−a, 03.65.Ud

Multipartite states occur in many applications of quantum information, like one-way quantum computing, quantum error correction, and quantum secret sharing [1,2]. Furthermore, the theory of many-body states plays also an important role in other fields of physics [3]. The existence of those practical and abstract applications is due to the subtle properties of multipartite entangled states. Thus, one of the main goals in quantum-information theory is to gain a better understanding of the nonlocal properties of quantum states. Whereas the bipartite case is well understood, the multipartite case is much more complex. Even though a big theoretical effort has been undertaken where several entanglement measures for multipartite states have been introduced [4], different classes of entangled states have been identified [5], and a normal form of multipartite states has been presented [6], we are far from completely understanding the nonlocal properties of multipartite states [7].

One way to gain insight into the entanglement properties of quantum states is to consider their interconvertibility. That is, given two states $|\Psi\rangle$, $|\Phi\rangle$ the question is whether or not $|\Psi\rangle$ can be transformed into $|\Phi\rangle$ by local operations [7]. One particularly interesting case, which is also investigated in this Letter, is the local unitary (LU) equivalence of multipartite states. We say that a n-partite state $|\Psi\rangle$ is LU equivalent to $|\Phi\rangle$ ($|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$) if there exist local unitary operators $U_1 \cdots U_n$ such that $|\Psi\rangle = U_1 \otimes \cdots \otimes U_n|\Phi\rangle$. Note that two states which are LU equivalent are equally useful for any kind of application and they possess precisely the same amount of entanglement. This is why understanding the interconvertibility of quantum states by LU operations is part of the solution to the more general problem of characterizing the different types of entangled quantum states.

Several approaches have been used to tackle this long-standing problem. For instance in [8] a standard form for 3-qubit states has been presented. For n-qubit states the so-called local polynomial invariants have been introduced [9]. However, even though it is known that it is sufficient to consider only a finite set of them, this complete finite set is known only for very few simple cases.

Here, we derive necessary and sufficient conditions for the existence of LU operations which transform two n-qubit states into each other. For generic states, states where none of the single qubit reduced states is completely mixed, the conditions can be easily computed. For arbitrary n-qubit states a systematic method to determine the unitaries (in case they exist) which interconvert the states is presented.

The sequel of the Letter is organized as follows. First, we introduce a standard form of multipartite states, which we use in order to derive easily computable necessary and sufficient conditions for the LU equivalence of generic multipartite states. As in the bipartite case, it is shown that two generic states are LU equivalent iff their standard forms coincide. For nongeneric states it is shown that whenever one of the single qubit reduced states is not completely mixed, the problem of LU equivalence of n-qubit states can be reduced to the one of $(n-1)$-qubit states. Then, a systematic method to determine the local unitaries (if they exist) which interconvert two arbitrary states is presented. It is shown that the states are LU equivalent iff there exists a solution to a finite set of equations. The number of variables involved in those equations depends on the entanglement properties of the states. The case with the largest number of variables occurs for the sometimes-called maximally entangled states of n qubits, where any bipartition of $\lceil n/2 \rceil$ qubits is maximally entangled with the rest. It is known, however, that only for certain values of n such states exist [10]. The power of this method is illustrated by considering several examples.

Throughout this Letter the following notation is used. By X, Y, Z we denote the Pauli operators. The subscript of an operator will always denote the system it is acting on, or the system it is describing. The reduced states of system $i_1 \cdots i_k$ of $|\Psi\rangle$ ($|\Phi\rangle$) will always be denoted by $\rho_{i_1\cdots i_k}$ ($\sigma_{i_1\cdots i_k}$) respectively, i.e., $\rho_{i_1\cdots i_k} = \mathrm{tr}_{\neg i_1 \cdots \neg i_k}(|\Psi\rangle\langle\Psi|)$. We denote by $\mathbf{i}$ the classical bit string $(i_1 \cdots i_n)$ with $i_k \in \{0,1\}\ \forall\, k \in \{1 \cdots n\}$ and $|\mathbf{i}\rangle \equiv |i_1 \cdots i_n\rangle$ denotes the computational basis. Normalization factors as well as the tensor product symbol will be omitted whenever it does not cause any confusion.

Let us start by introducing a unique standard form of multipartite states (see also [11]). Let $|\Psi\rangle$ be a n-qubit state. As a first step we apply local unitaries U_i^1 such that all the single qubit reduced states of the state $|\Psi_t\rangle = U_1^1 \otimes \cdots U_n^1|\Psi\rangle$ are diagonal in the computational basis, i.e., $\mathrm{tr}_{\neg i}(|\Psi_t\rangle\langle\Psi_t|) = D_i = \mathrm{diag}(\lambda_i^1, \lambda_i^2)$. We call any such decomposition trace decomposition of the state $|\Psi\rangle$. A sorted trace decomposition is then defined as a trace decomposition with $\lambda_i^1 \geq \lambda_i^2$. Note that transforming a state into its sorted trace decomposition, which we will denote by $|\Psi_{\mathrm{st}}\rangle$ in the following, can be easily done by computing the spectral decomposition of all the single qubit reduced states. The sorted trace decomposition of a generic state $|\Psi\rangle$ with $\rho_i \neq \mathbb{1}\ \forall\, i$ is unique up to local phase gates. That is $U_1 \cdots U_n|\Psi_{\mathrm{st}}\rangle$ is a sorted trace decomposition of $|\Psi\rangle$ iff (up to a global phase α_0) $U_i = Z_i(\alpha_i) \equiv \mathrm{diag}(1, e^{i\alpha_i})$. In order to make the sorted trace decomposition of generic states unique we impose the following condition on the phases α_i, $i \in \{0 \cdots n\}$. We write $|\Psi_{\mathrm{st}}\rangle = \sum_{\mathbf{i}} \lambda_{\mathbf{i}}|\mathbf{i}\rangle$, and define the set $S = \{\mathbf{i}\colon \lambda_{\mathbf{i}} \neq 0\}$ and $\bar{S}$ denotes the set of the first (in lexicographic order) linearly independent vectors in S [12]. The global phase α_0 is chosen to make $\lambda_{\mathbf{i_0}}$ real and positive where $\mathbf{i_0} = \mathbf{0}$ in case $\lambda_{\mathbf{0}} \neq 0$ else $\mathbf{i_0}$ denotes the first (in lexicographic order) linearly dependent vector in S. After that, the n phases are chosen to make the coefficients $e^{i\alpha_0}\lambda_{\mathbf{i}}$ for $\mathbf{i} \in \bar{S}$ real and positive [13]. The so-defined standard form, which will be denoted by $|\Psi_s\rangle$ in the following, is unique. If $\rho_i = \mathbb{1}$, for some system i, the standard form can be similarly defined [11], however it will not be unique then. Because of the definition any state is LU equivalent to its standard form [14].

We employ now the standard form to derive a criterion for the LU equivalence of generic multipartite states. First of all note that $|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$ iff $|\Psi_s\rangle \simeq_{\mathrm{LU}} |\Phi_s\rangle$. Using then that the standard form is unique we obtain the following theorem.

Theorem 1.—Let $|\Psi\rangle$ be an n qubit state with $\rho_i \neq \mathbb{1}\ \forall\, i$. Then $|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$ iff the standard form of $|\Psi\rangle$ is equivalent to the standard form of $|\Phi\rangle$, i.e., $|\Psi_s\rangle = |\Phi_s\rangle$.

Thus, similarly to the bipartite case, two generic states are LU equivalent iff their standard forms coincide, which can be easily checked. Furthermore, if the states are LU equivalent then $|\Psi\rangle = \bigotimes_i U_i|\Phi\rangle$ with $U_i = (U_s^i)^\dagger V_s^i$, where U_s^i, (V_s^i) denote the local unitaries which transform $|\Psi\rangle$ ($|\Phi\rangle$) into its standard form respectively.

In order to study now the nongeneric cases, we will rewrite the necessary and sufficient condition derived above. For a generic state, $|\Psi\rangle$ it is easy to verify that $|\Psi_s\rangle = |\Phi_s\rangle$ iff there exists a bit string $\mathbf{k} = (k_1 \cdots k_n)$, local phase gates $Z_i(\alpha_i)$, and a global phase α_0 such that

$$e^{i\alpha_0}\bigotimes_i Z_i(\alpha_i)X_i^{k_i}\bar{W}_i|\Psi\rangle = \bigotimes_i \bar{V}_i|\Phi\rangle, \tag{1}$$

where $\bar{W}_i$ ($\bar{V}_i$) are local unitaries which diagonalize ρ_i (σ_i). That is $\bigotimes_i \bar{W}_i|\Psi\rangle$ and $\bigotimes_i \bar{V}_i|\Phi\rangle$ are trace decompositions of $|\Psi\rangle$ and $|\Phi\rangle$ respectively. For generic states k_i is chosen such that the order of the eigenvalues of the single qubit reduced states of $\bigotimes_i X_i^{k_i}\bar{W}_i|\Psi\rangle$ and $\bigotimes_i \bar{V}_i|\Phi\rangle$ coincides. In order to check then whether or not there exist phases α_i such that Eq. (1) is satisfied, we make use of the following lemma, whose proof is presented in [16,17]. There, we will consider four n-qubit states. The systems, each composed out of n qubits will be denoted by A, B, C, D, respectively. The ith qubit of system A will be denoted by A_i, etc. Furthermore, we will use the notation $|\chi_i\rangle = (|0110\rangle - |1001\rangle)_{A_i,B_i,C_i,D_i}$ and $P_{AC}^i = \sum_{\mathbf{k}}|\mathbf{k}\rangle\langle\mathbf{k}\,\mathbf{k}|_{A_1,C_1\cdots A_{i-1},C_{i-1},A_{i+1},C_{i+1}\cdots A_n,C_n}$ and similarly we define P_{BD}^i for systems B, D. For a state $|\Psi\rangle$ we define $K_\Psi \equiv \{\mathbf{k}\ \text{such that}\ \langle\mathbf{k}|\Psi\rangle = 0\}$ and $|\Psi_{\{\bar{\alpha}_i\}}\rangle = |\Psi\rangle + 2e^{-i\bar{\alpha}_0}\sum_{\mathbf{k}\in K_\Psi} e^{-i\sum_{i=1}^n \bar{\alpha}_i k_i}|\mathbf{k}\rangle$ for some phases $\bar{\alpha}_i$ and $|\Psi_{\mathbf{0}}\rangle = |\Psi\rangle + 2\sum_{\mathbf{k}\in K_\Psi}|\mathbf{k}\rangle$.

Lemma 2.—Let $|\Psi\rangle$, $|\Phi\rangle$ be n-qubit states. Then, there exist local phase gates $Z_i(\alpha_i)$ and a phase α_0 such that $|\Psi\rangle = e^{i\alpha_0}\bigotimes_{i=1}^n Z_i(\alpha_i)|\Phi\rangle$ iff there exist phases $\{\bar{\alpha}_i\}_{i=0}^n$ such that (i) $|\langle\mathbf{i}|\Psi_{\mathbf{0}}\rangle| = |\langle\mathbf{i}|\Phi_{\{\bar{\alpha}_i\}}\rangle|\,\forall\,\mathbf{i}$ and (ii) $\langle\chi|_i P_{AC}^i P_{BD}^i|\Psi_{\mathbf{0}}\rangle_A|\Psi_{\mathbf{0}}\rangle_B|\Phi_{\{\bar{\alpha}_i\}}\rangle_C|\Phi_{\{\bar{\alpha}_i\}}\rangle_D = 0\ \forall\, i \in \{1 \cdots n\}$.

Let us now consider the nongeneric case. Obviously, two arbitrary states $|\Psi\rangle$, $|\Phi\rangle$ are LU equivalent iff there exist local unitaries $\bar{V}_k$, $\bar{W}_k$, a bit string $\mathbf{k}$, and phases α_i such that Eq. (1) is fulfilled. We will show now how $\bar{V}_k$, $\bar{W}_k$ can be determined by imposing necessary conditions of LU equivalence.

First of all, we note that for any state $|\Psi\rangle$ with $\rho_i \neq \mathbb{1}$ for some system i, k_i as well as $\bar{V}_i$ and $\bar{W}_i$ can be easily determined as follows. If $|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$ then all the reduced states must be LU equivalent, in particular $D_i = \mathrm{diag}(\lambda_1^i, \lambda_2^i) = \bar{W}_i\rho_i\bar{W}_i^\dagger = \bar{V}_i\sigma_i\bar{V}_i^\dagger$, for some unitaries $\bar{W}_i$, $\bar{V}_i$. Analogously to the generic case, this equation determines $\bar{W}_i$ and $\bar{V}_i$ (and $k_i = 0$) uniquely up to a phase gate. Thus, for this case we have that $|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$ iff there exist two phases α_i and α_0 and local unitaries U_j such that

$${}_i\langle l|\bar{W}_i\Psi_s\rangle = e^{i(\alpha_0+\alpha_i l)}\bigotimes_{j\neq i} U_j\,{}_i\langle l|\bar{V}_i\Phi_s\rangle \quad \text{for } l \in \{0, 1\}, \tag{2}$$

where $\bar{W}_i$, and $\bar{V}_i$ are chosen such that $D_i = \mathrm{diag}(\lambda_1^i, \lambda_2^i) = \bar{W}_i\rho_i\bar{W}_i^\dagger = \bar{V}_i\sigma_i\bar{V}_i^\dagger$. Hence, if there is one system where the reduced state is not proportional to the identity then we can reduce the problem of LU equivalence of n-qubit states to the one of $(n-1)$-qubit states. This statement can be easily generalized to the case where more than one single qubit reduced state is not completely mixed.

Let us now consider the more complicated case, where some $\rho_i = \mathbb{1}$. There, it is obviously no longer possible to determine $\bar{V}_i$, $\bar{W}_i$ by imposing the necessary condition of LU equivalence, $\rho_i = U_i\sigma_i U_i^\dagger$. However, we will show next which necessary condition can be used in order to determine them. Before we do so, we explain the problem which might occur if $\rho_i = \mathbb{1}$ by considering a simple example. Let $|\Psi\rangle$ and $|\Phi\rangle$ denote two states with $\rho_{12} = \sigma_{12} = \mathbb{1} - \lambda|\Psi^-\rangle\langle\Psi^-|$, for some $\lambda \neq 0$. Then we find that $\rho_{12} = U_1 U_2 \sigma_{12} U_1^\dagger U_2^\dagger$ iff $U_1 = U_2$, which implies that $|\Psi\rangle \simeq_{LU} |\Phi\rangle$ iff there exist local unitaries U_1, $U_3 \cdots U_n$ such that $|\Psi\rangle = U_1 U_1 \cdots U_n |\Phi\rangle$. Thus, the unitary U_2 depends on U_1. Or, stated differently, $\bar{W}_2$ (and α_2) depends on $U_1 = [Z_1(\alpha_1) X_1^{k_1} \bar{W}_1]^\dagger$ in Eq. (1), where we set $V_1 = V_2 = \mathbb{1}$. In general we might neither be able to determine the phase α_2, nor $\bar{W}_2$ as a function of U_1 alone. However, the next lemma shows that any $\bar{W}_k$ can be determined as a function of a few unitaries and $\bar{V}_k$ can always be determined directly form the state $|\Phi\rangle$. We will see that the number of unitaries which are required to define $\bar{W}_k$ depends on the entanglement properties of the state.

Lemma 3.—If $|\Psi\rangle = U_1 \cdots U_n |\Phi\rangle$ and if there exist systems $n_1 \cdots n_l$ and k such that $\rho_{n_1 \cdots n_l, k} \neq \rho_{n_1 \cdots n_l} \otimes \mathbb{1}_k$ then $\bar{V}_k$ in Eq. (1) can be determined from the state $|\Phi\rangle$ and $\bar{W}_k$ can be determined as a function of the unitaries $U_{n_1} \cdots U_{n_l}$.

Proof.—Without loss of generality we assume $n_1 = 1 \cdots n_l = l$ and write $|\Psi\rangle = \sum |\mathbf{i}\rangle_{1\ldots l} |\Psi_\mathbf{i}\rangle_{l+1 \cdots n}$ and $|\Phi\rangle = \sum |\mathbf{i}\rangle_{1\ldots l} |\Phi_\mathbf{i}\rangle_{l+1 \cdots n}$, where $\mathbf{i} = (i_1 \cdots i_l)$. Since $\sigma_{1 \cdots l, k} = \sum |\mathbf{i}\rangle\langle \mathbf{j}| \mathrm{tr}_{\neg k}(|\Phi_\mathbf{i}\rangle\langle\Phi_\mathbf{j}|) \neq \sigma_{1 \ldots l} \otimes \mathbb{1}$, there exist at least two tuples $\mathbf{i}$ and $\mathbf{j} = (j_1 \cdots j_l)$ such that the 2×2 matrix $X_\mathbf{i}^\mathbf{j} \equiv \mathrm{tr}_{\neg k}(|\Phi_\mathbf{i}\rangle\langle\Phi_\mathbf{j}|) \not\propto \mathbb{1}$. Thus, at least one of the two Hermitian operators $Y_\mathbf{i}^\mathbf{j} = X_\mathbf{i}^\mathbf{j} + (X_\mathbf{i}^\mathbf{j})^\dagger$ and $Z_\mathbf{i}^\mathbf{j} = iX_\mathbf{i}^\mathbf{j} - i(X_\mathbf{i}^\mathbf{j})^\dagger$ is not proportional to the identity. Without loss of generality we assume that $\mathbb{1} \not\propto Y_\mathbf{i}^\mathbf{j} = \mathrm{tr}_{\neg k}[(|\mathbf{i}\rangle\langle\mathbf{j}| + \mathrm{H.c})|\Phi\rangle\langle\Phi|]$. Using that $|\Psi\rangle = U_1 \cdots U_n |\Phi\rangle$ we have

$$U_k Y_\mathbf{i}^\mathbf{j} U_k^\dagger = \mathrm{tr}_{\neg k}[(|\mathbf{i}\rangle\langle\mathbf{j}| + \mathrm{H.c}) U_1^\dagger \cdots U_l^\dagger |\Psi\rangle\langle\Psi| U_1 \cdots U_l]. \tag{3}$$

Since $Y_\mathbf{i}^\mathbf{j}$ is Hermitian we can diagonalize it as well as the right-hand side of Eq. (3) and obtain $U_k \bar{V}_k^\dagger D \bar{V}_k U_k^\dagger = \bar{W}_k^\dagger (U_1 \cdots U_l) D (U_1 \cdots U_l) \bar{W}_k (U_1 \cdots U_l)$, which is true iff $D = X^{i_k} D (U_1 \cdots U_l) X^{i_k}$, with $i_k \in \{0, 1\}$ and $U_k = e^{i\alpha_0'} \bar{W}_k^\dagger (U_1 \cdots U_l) Z(\alpha_k') X^{i_k} \bar{V}_k$, for some phases α_0', α_k'. Thus, we have $|\Psi\rangle = U_1 \cdots U_n |\Phi\rangle$ iff there exists $i_k \in \{0, 1\}$, and α_0 and α_k such that $e^{i\alpha_0} X^{i_k} Z(\alpha_k) \times \bar{W}_k (U_1 \cdots U_l) |\Psi\rangle = U_1 \cdots \bar{V}_k \cdots U_n |\Phi\rangle$, where $\bar{V}_k$ is the unitary which diagonalizes $Y_\mathbf{i}^\mathbf{j}$ and can therefore be determined directly from the state $|\Phi\rangle$ and $\bar{W}_k(U_1 \cdots U_l)$ diagonalizes the right-hand side of Eq. (3). ■

Note that the proof of Lemma 3 is constructive. The idea was to impose the necessary condition for LU equivalence given in Eq. (3) for any l tuples $\mathbf{i}$, $\mathbf{j}$. Since the 2×2 matrices occurring in this equation are Hermitian, one can, similarly to the previous cases, determine the unitaries $\bar{V}_k$, $\bar{W}_k$ by diagonalizing these matrices. In contrast to before we will find here, that $\bar{W}_k$ might depend on $U_1 \cdots U_l$.

We use now Lemma 3 to present a constructive method to compute all local unitaries as functions of a few variables. If some unitary U_i cannot be determined in this way, we write $U_i = e^{-i\gamma_i Z_i} e^{-i\beta_i X_i} e^{-i\alpha_i Z_i}$ (up to a phase) and choose without loss of generality $\bar{V}_i = \mathbb{1}$, $k_i = 0$, and $\bar{W}_i = e^{i\beta_i X_i} e^{i\gamma_i Z_i}$ in Eq. (1). We will say then that we consider U_i as a variable.

The constructive method to compute now $\bar{V}_k$ and $\bar{W}_k$ in Eq. (1) is as follows: (1) If there exists a system i such that $\rho_i \neq \mathbb{1}$ compute $\bar{V}_i$, $\bar{W}_i$ using that $\bar{W}_i \rho_i \bar{W}_i^\dagger = \bar{V}_i \sigma_i \bar{V}_i^\dagger = D_i$ ($k_i = 0$). Furthermore, compute $\bar{V}_k$ and $\bar{W}_k(U_i)$ for any system k with $\rho_{ik} \neq \rho_i \otimes \mathbb{1}$ using Lemma 3. (2) For all systems i for which $\rho_i \neq \mathbb{1}$ apply the unitaries $\bar{W}_i$ ($\bar{V}_i$) to $|\Psi\rangle$ ($|\Phi\rangle$) respectively and measure system i in the computational basis thereby reducing the number of systems [see Eq. (2)]. After this step we have $\rho_i = \mathbb{1}\ \forall\, i$. Then we continue as follows: (3) Consider the two-qubit reduced states: (3a) There exist systems i, j such that $\rho_{ij} \neq \mathbb{1}$. Without loss of generality we choose $i = 1$, consider U_1 as variable, and set $\bar{V}_1 = \mathbb{1}$, $k_1 = 0$ and $\bar{W}_1 = e^{i\beta_1 X_1} e^{i\gamma_1 Z_1}$. Then, compute $\bar{V}_j$ and $\bar{W}_j(U_1)$ using Lemma 3 for any system j with $\rho_{1j} \neq \mathbb{1}$. Let us denote by J_2 the set of systems j, for which $\rho_{1j} \neq \mathbb{1}$. (3b) If there exists no system i, j such that $\rho_{ij} \neq \mathbb{1}$ consider U_1 and U_2 as variables and set $\bar{V}_i = \mathbb{1}$, $k_i = 0$ and $\bar{W}_i = e^{i\beta_i X_i} e^{i\gamma_i Z_i}$, for $i = 1, 2$. Furthermore, set $J_2 = \{2\}$. (4) Consider the three-qubit reduced states: (4a) There exists a system k such that $\rho_{1jk} \neq \rho_{1j} \otimes \mathbb{1}$ for some $j \in J_2$. Compute $\bar{V}_k$ and $\bar{W}_k(U_1, U_j)$ using Lemma 3 for any system k with $\rho_{1jk} \neq \rho_{1j} \otimes \mathbb{1}$. (4b) If there exists no system k such that $\rho_{1jk} \neq \rho_{1j} \otimes \mathbb{1}$ include U_3 as variable. (5) Continue in this way until all unitaries are either determined as functions of a few unitaries, or are free parameters. If at some point it is not possible to choose $\bar{V}_k$ or $\bar{W}_k$ unitary, e.g., if the eigenvalues of the operators occurring in Eq. (3) do not coincide, the states are not LU equivalent.

Once all unitaries $\bar{V}_i$ are determined and all unitaries $\bar{W}_i$ are determined as functions of a few variables, we have that $|\Psi\rangle \simeq_{LU} |\Phi\rangle$ iff there exists a bit string $\mathbf{k}$ and phases $\{\alpha_i\}_{i=0}^n$, such that Eq. (1) is fulfilled. In order to check the existence of the local phase gates in Eq. (1) (for some bit string $\mathbf{k}$), we use Lemma 2. It is important to note here that the state on the right-hand side of Eq. (1) is completely determined, thus, the set K_Ψ in Lemma 2 can be determined and therefore this lemma can be applied. The states are LU equivalent iff the conditions in Lemma 2 are fulfilled for some bit string $\mathbf{k}$. The unitaries which interconvert the states are, up to the symmetry of the states, uniquely determined and are given by $U_i = \bar{W}_i^\dagger Z(\alpha_i) X^{k_i} \bar{V}_i$ (up to a global phase) [18].

Note that this method introduces a natural classification of multipartite states. We have to add a unitary U_k as a variable only if $\rho_{i_1 \cdots i_l, k} = \rho_{i_1 \cdots i_l} \otimes \mathbb{1}_k$ for some properly chosen $i_1 \cdots i_l$. The worst case, i.e., the case which involves the largest number of variables, occurs for maximally entangled states (all bipartite splittings are maximally entangled). In this case, which exists only for certain instances of n, we have $\lceil n/2 \rceil$ unitaries as variables. For a pure state the equation above holds iff for any outcome of any von Neumann measurement on systems

$i_1 \cdots i_l$, system k is maximally entangled with the remaining systems. Thus, two states $|\Psi\rangle$, $|\Phi\rangle$ with $\rho_{i_1\cdots i_l,k} = \rho_{i_1\cdots i_l} \otimes \mathbb{1}$ and $\sigma_{i_1\cdots i_l,k} \neq \sigma_{i_1\cdots i_l} \otimes \mathbb{1}$ can neither be LU equivalent nor possess the same entanglement. Hence, the method presented above suggests that in order to characterize the nonlocal properties of multipartite states, one should first identify the class (as described above) to which the state belongs to and then determine within this class the entanglement of the state. It might well be that the different classes lead to different applications. For instance, the states used for error correction and one-way quantum computing have the property that all single qubit reduced states are completely mixed.

In order to illustrate the power of this method we consider first the simplest examples of two and three-qubit states. The standard form of a two-qubit state is $|\Psi\rangle = \lambda_1|00\rangle + \lambda_2|11\rangle$. Thus, the method above tells us that if $\lambda_1 \neq \lambda_2$, i.e., $\rho_i \neq \mathbb{1}$, then $|\Psi\rangle \simeq_{\mathrm{LU}} |\Phi\rangle$ iff the Schmidt coefficients λ_i are the same. For $\lambda_1 = \lambda_2$ it is straightforward to show that the unitaries U_i, which are obtained using the method above for the states $|\Phi^+\rangle \equiv |00\rangle + |11\rangle$ and some LU-equivalent state $V_1V_2|\Phi^+\rangle$ are $U_1 = V_1W$ and $U_2 = V_2W^*$ for any unitary W. The reason why the unitaries U_i are not completely determined by V_i is due to the symmetry of $|\Phi^+\rangle$. For three qubits the method is almost equally simple. First, we transform both states into their trace decomposition. If none of the reduced states is completely mixed, we simply compare their standard forms (Theorem 1). If there exists some i such that $\rho_i \neq \mathbb{1}$, we know that $U_i = Z(\alpha_i)$. We measure system i in the computational basis and are left with two two-qubit states [see Eq. (2)]. For the remaining case, where $\rho_i = \mathbb{1}\ \forall\, i$ it can be easily shown that $|\Psi\rangle$ is LU equivalent to the GHZ state, $|\Psi_0\rangle = |000\rangle + |111\rangle$ [17]. Even without using this fact also in this case the right unitaries can be directly computed using the presented method (for, details see [17]).

The LU-equivalence classes of up to 5-qubit states are investigated in [17]. We will show there, for instance, that for 4-qubit states with $\rho_{ij} = \mathbb{1}$ for some i, j (which is the hardest class of states using the method presented above), the LU-equivalence class is determined by only three parameters. Thus, the entanglement of those states is completely determined by the fact that system ij is maximally entangled to the other two qubits and those three parameters, to which also an operational meaning will be given [17]. This example shows already that the method presented here does not only give necessary and sufficient conditions for the LU equivalence of arbitrary multipartite states, but also leads to a new insight into their entanglement properties. Moreover, the results presented here lead also to a criterion of LU equivalence for certain mixed and also d-level states [17] and to conditions for the existence of more general operations transforming one state into the other, namely local operations and classical communication (LOCC). This is due to the fact that two multipartite states, having the same marginal one-party entropies, are either LU equivalent, or LOCC incomparable [19].

The author would like to thank Hans Briegel for continuous support and interest in this work and acknowledges support of the FWF (Elise Richter Program).

Note added.—Recently, the author became aware of a similar standard form presented in Ref. [20].

[1] D. Gottesman, Ph.D. thesis, Caltech [arXiv:quant-ph/9705052].

[2] R. Raussendorf and H. J. Briegel, Phys. Rev. Lett. **86**, 5188 (2001).

[3] For a recent reviews, see L. Amico *et al.*, Rev. Mod. Phys. **80**, 517 (2008); F. Verstraete *et al.*, Adv. Phys. **57**, 143 (2008), and references therein.

[4] See, for instance, V. Coffman, J. Kundu, and W. K. Wootters, Phys. Rev. A **61**, 052306 (2000); F. Verstraete, M. Popp, and J. I. Cirac, Phys. Rev. Lett. **92**, 027901 (2004); J. Eisert and H. J. Briegel, Phys. Rev. A **64**, 022306 (2001).

[5] W. Dür, G. Vidal, and J. I. Cirac, Phys. Rev. A **62**, 062314 (2000); F. Verstraete *et al.*, Phys. Rev. A **65**, 052112 (2002).

[6] F. Verstraete, J. Dehaene, and B. DeMoor, Phys. Rev. A **68**, 012103 (2003).

[7] For a recent review, see R. Horodecki *et al.*, Rev. Mod. Phys. **81**, 865 (2009), and references therein.

[8] A. Acin *et al.*, Phys. Rev. Lett. **85**, 1560 (2000).

[9] M. Grassl, M. Rotteler, and T. Beth, Phys. Rev. A **58**, 1833 (1998).

[10] See, for instance, S. Bravyi, Phys. Rev. A **67**, 012313 (2003); J. Preskill, *Quantum Computation*, Lecture Notes for Physics Vol. 219 (California Institute of Technology, Pasadena CA, 1998), http://www.theory.caltech.edu/people/preskill/ph219/.

[11] C. Kruszynska and B. Kraus, Phys. Rev. A **79**, 052304 (2009).

[12] We sort the vectors in S in lexicographic order and include the first vector ($\neq 0$) in the set $\bar{S}$. A subsequent vector is included in $\bar{S}$ if it is linearly independent of the vectors in $\bar{S}$.

[13] If there are less than n linearly independent vectors in S, say k, then k phases can be defined like that, the other phases leave the state invariant and can therefore be chosen arbitrarily.

[14] Note that this standard form coincides for the simplest case of two qubits with the Schmidt decomposition [15] and can be generalized to d-level systems.

[15] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, U.K., 2000).

[16] B. Kraus, arXiv: quant-ph/09095152v1.

[17] B. Kraus (to be published).

[18] Note that the phases α_i can be easily computed.

[19] C. H. Bennett *et al.*, Phys. Rev. A **63**, 012307 (2000).

[20] See H. A. Carteret *et al.*, J. Math. Phys. (N.Y.) **41**, 7932 (2000), and references therein.