Some correspondences in Algebraic Geometry

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July 13, 2024

Contents

Notations and Definitions

- Let k denote algebraically closed fields, $\mathbb{A}^n = \mathbb{A}^n(k) = k^n$, and $\mathbb{P}^n = \mathbb{P}^n(k) = \{[k_1 : \ldots, k_{n+1}]\}$ where $k_1, \ldots, k_{n+1} \in k$.
- For an ideal $I \subset k[X_1, \ldots, X_n]$, $V_a(I) := \{p \in \mathbb{A}^n \mid \forall f \in I, f(p) = 0\}$, shorthanded as $V(I)$.

Notations and Definitions

- A subset $X \subset \mathbb{A}^n$ is algebraic if $X = V(S)$ for some set of polynomials $S \subset k[X_1, \ldots, X_n]$. $V(S)$ means $V(I)$ where I is the ideal generated by S. An algebraic set X is called a variety if it is irreducible (could not be break down into a union of two algebraic subsets of X).
- For any subset X of an affine variety, $I(X)$ is defined to be the ideal of polynomials in n variables such that for any polynomial $F \in I(X)$, F vanishes at all points of X.

Hilbert's Nullstellensatz

Weak Nullstellensatz

If I is a proper ideal of $k[X_1, \dots, X_n]$, then $V(I) \neq \varnothing$.

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Key idea: The maximal ideals of $k[X_1, \ldots, X_n]$ corresponds to points in $\mathbb{A}^n$, i.e. all maximal ideals $\mathfrak{m} \subset k[X_1, \ldots, X_n]$ takes the form $\mathfrak{m} = (X_1 - a_1, \ldots, X_n - a_n)$ for $a_1, \ldots, a_n \in k$.

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First correspondence:

Maximal ideals of polynomial rings $\leftrightharpoons$ Points in the affine plane

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Corollary 2

There is a one-to-one correspondence between prime ideals and irreducible algebraic sets, which we later call "affine varieties" (note that prime ideals are radical).

Coordinate Rings

Definition

Let $V \subset \mathbb{A}^n$ be a non-empty affine variety, then $I(V)$ is prime in $k[X_1, \ldots, X_n]$. Define $\Gamma(V) := k[X_1, \ldots, X_n]/I(V)$, so $\Gamma(V)$ is an integral domain. We call $\Gamma(V)$ to be the *coordinate ring* of V.
A *polynomial function* is a function $f : V \to k$ such that there exists a polynomial F and $F(p) = f(p)$ for all $p \in V$.

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Polynomial maps

A map $\varphi : V \to W$ between two affine varieties $V \subset \mathbb{A}^n$ and $W \subset \mathbb{A}^m$ is called a polynomial map if there are polynomials $F_1, \ldots, F_m \in k[X_1, \ldots, X_n]$ such that $\varphi : p \mapsto (F_1(p), \ldots, F_m(p))$

Coordinate Rings

Correspondence

There is a natural one-to-one correspondence between polynomial maps $\varphi : V \to W$ and ring homomorphisms $\tilde{\varphi} : \Gamma(W) \to \Gamma(V)$. One direction is given by $\varphi \mapsto f \circ \varphi$ for $f \in \Gamma(W)$. For the other direction, we define $\varphi : p = (x_1, \ldots, x_n) \mapsto (f_1(p), \ldots, f_m(p))$, where $\overline{f_i} = \tilde{\varphi}(\overline{X_i})$.

General Varieties

Definition

Let $X = \mathbb{P}^{n_1} \times \cdots \times \mathbb{P}^{n_r} \times \mathbb{A}^m$. A set $U \subset X$ is open if $X \setminus U$ is an algebraic subset of X.

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Definition

Let V be a non-empty irreducible algebraic subset of X. Any open subset W of V will be called a variety. It is given by the induced topology from V.

General Varieties

Definition

We define $k(X) = k(V)$ to be the field of rational functions on X, and if $P \in X$, we define $\mathcal{O}_P(X) = \mathcal{O}_P(V)$, the local ring of X at P, to be the ring of rational functions that are defined at P.

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Definition

Let X be a variety, U a non-empty open subset of X. We define $\Gamma(U)$ to be the set of rational functions that are defined at each $P \in U$, i.e. $\Gamma(U) = \bigcap_{P \in U} \mathcal{O}_P(X)$. If $U = X$ is an affine variety, then $\Gamma(X)$ is the coordinate ring of X, so the notation is consistent.

An example of using correspondences

Problem 6.17 [Fulton, Algebraic Curves]

Show that $X = \mathbb{A}^2 \setminus \{(0,0)\}$ is not an affine variety.

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Hint

How about looking at the coordinate ring of X?

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Proof.

By definition, $\Gamma(\mathbb{A}^2) \subset \Gamma(X)$, so it suffices to prove $\Gamma(X) \subset \Gamma(\mathbb{A}^2)$.
Let $f/g \in \Gamma(X)$, where f, g polynomials. Since $k[X, Y]$ is a UFD, we can assume f and g share no components.
Want to show: g is a constant (since for any proper ideal I, $V(I) \neq \varnothing$).

An example of using correspondences

Proof.

Suppose g is not constant, then $g(x,y)$ has infinitely many solutions in $\mathbb{A}^2(k)$. By Bezout's theorem, since f,g shares no components they can only have a finite number of common solutions. Then we can pick $(x_0,y_0) \neq (0,0)$ such that $f(x_0,y_0) \neq 0$ but is a zero for g. Then $f/g \notin \Gamma(X)$. Then g is a constant, so $\Gamma(X) \subset \Gamma(\mathbb{A}^2)$.

An example of using correspondences

Proof.

Assume X is an affine variety. Consider the inclusion map $i : X \hookrightarrow \mathbb{A}^2$. It corresponds to $\tilde{i} : \Gamma(\mathbb{A}^2) \to \Gamma(X)$, defined by $f \mapsto f|_X$ (the restriction homomorphism).
Surjectivity: Since X is affine, we can extend $f \in \Gamma(X)$ to $f' \in \Gamma(\mathbb{A}^2)$ by sending $f \mapsto f$ (the same polynomial).
Injectivity: X is dense in $\mathbb{A}^2$, so if $f|_X = g|_X$, then $f = g$.
Thus $\tilde{i}$ is an isomorphism, which implies $\tilde{i}^{-1}$ is an isomorphism as well.

An example of using correspondences

Proof.

By the correspondence theorem, the corresponding polynomial map for $\tilde{i}$ should be i. From correspondence theorem we know $i : X \to \mathbb{A}^2$ should be defined by

$$i : (x_1, x_2) \mapsto (f_1(x_1, x_2), f_2(x_1, x_2)), \text{ where } f_i \in \Gamma(X), \overline{f_i} = \tilde{i}(\overline{Y_i}).$$

By this definition we can see that (1) i is a polynomial map; (2) i^{-1} is a polynomial map; (3) $i \circ i^{-1} = \mathrm{id}$.

An example of using correspondences

Proof.

Therefore i is an isomorphism of affine varieties X and $\mathbb{A}^2$. However i is defined to be the inclusion map so it is not surjective. Therefore by contradiction, X is not an affine variety. $\square$