From Classical to Modern: Bridging Varieties and Schemes

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Affine Space

The *affine* n*-space* over a field k: $\mathbb{A}^n = k^n$ as a set.
Let $A = k[x_1, \ldots, x_n]$, the polynomial ring of n variables.

Zariski Topology

For $f \in A$ a polynomial, let $Z(f)$ to be the zero set of f.
For a subset $T \subset A$, we define $Z(T)$ to be the common zeroes of all elements of T:

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

$Z(T) = Z(I)$ if I is generated by T.
Sets of the form $Z(T)$ are called *algebraic sets*.

Zariski Topology

Define a topology on $\mathbb{A}^n$: **closed** sets are algebraic sets $Z(T)$.

- $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$
- $\bigcap Z(T_\alpha) = Z(\bigcup T_\alpha)$
- $\varnothing = Z(1)$, and $\mathbb{A}^n = Z(0)$.

Affine Space

Example

Consider $\mathbb{A}^1$, where $A = k[x]$.
A is a PID, and any element $f \in A$ could be factored as

$$f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n).$$

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NOT Hausdorff, irreducible

Affine Variety

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There is a one-to-one **order-reversing** correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A.

$$\textit{irreducible}\text{ algebraic set} \iff \textit{prime}\text{ ideal.}$$

Affine Variety

Example

Consider $A = k[x_1, \ldots, x_n]$. Maximal ideals of A corresponds to minimal irreducible algebraic sets: points in $\mathbb{A}^n$.

Regular Functions

Definition (Regular Functions)

Let Y be an affine variety. A function $f\colon Y \to k$ is *regular at* P if there exists an open set $U \subset Y$ and $g, h \in A$ where h doesn't vanish on U, such that $f = g/h$ on U. A function f is *regular* if it is regular at all $P \in Y$.

Theorem (Continuity of Regular Functions)

Regular functions are continuous, given the Zariski's topology and identifying k as $\mathbb{A}^1$.

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Definition (Morphisms of Varieties)

Let X, Y be varieties over k. $\varphi \colon X \to Y$ a *morphism* of varieties is a continuous map such that for all open $V \subset Y$, $f \colon V \to k$ a regular function pulls back to a regular function on $\varphi^{-1}(V) \subset X$.

Schemes

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subject to the following conditions

- $\mathcal{F}(\varnothing) = 0$,
- ρ_{UU} is the identity map, and
- if $W \subset V \subset U$ open sets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$.

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- if U is an open set, and $\{V_i\}$ an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;
- if $s_i \in \mathcal{F}(V_i)$ for each i, such that for all i and j, $s_i|_{V_i \cap V_j} = s_j|_{V_j \cap V_i}$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. The first property tells us that this s is unique.

Schemes

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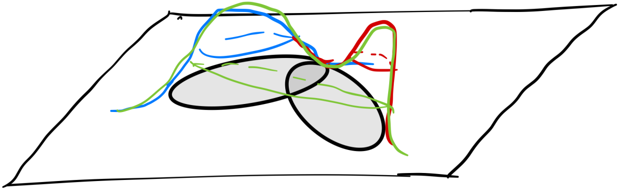
Consider $\mathbb{R}^n$ with the standard topology. Smooth functions on $\mathbb{R}^n$ form a sheaf. Explicitly, take $\mathcal{F}(U)$ to be the continuous functions defined on the open $U \subset \mathbb{R}^n$. Denote this sheaf of smooth functions as $\mathcal{C}^{\infty}_{\mathbb{R}^n}$.



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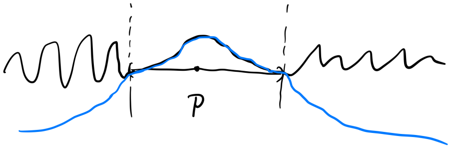

Morphism between presheaves

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Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A morphism $\varphi \colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms $\varphi \colon \mathcal{F}(U) \to \mathcal{F}(V)$ making the following diagram commute.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow \rho^{\mathcal{F}}_{UV} & & \downarrow \rho^{\mathcal{G}}_{UV} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

Direct image & inverse image sheaf

Let $f\colon X \to Y$ be a map of topological spaces.

- For any sheaf $\mathcal{F}$ on X, the *direct image sheaf* $f_*\mathcal{F}$ on Y is defined by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subset Y$.

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- For any sheaf $\mathcal{G}$ on Y, the *inverse image sheaf* $f^{-1}\mathcal{G}$ is the sheafification of
$$U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V),$$
where U is open in X and limit taken over all open subsets V containing $f(U)$.

Locally Ringed Spaces

Definition (Ringed Spaces)

A *ringed space* is a pair $(X, \mathcal{O}_X)$ of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f\colon X \to Y$ and a map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

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Definition (Locally Ringed Spaces)

A *locally ringed space* is a ringed space where the stalk of the structure sheaf $\mathcal{O}_X$ at any point is a local ring. A morphism between locally ringed spaces is a morphism of ringed spaces inducing local morphisms between stalks.

Example (Manifold as locally ringed spaces)

Consider the spaces $X, Y = \mathbb{R}^n$ with the standard topology, equipped with the sheaf of smooth functions. Let $(f, f^\sharp)$ be a morphism of locally ringed spaces between $(X, \mathcal{C}_X^\infty)$ and $(Y, \mathcal{C}_Y^\infty)$.

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From the local homomorphism condition, one can derive that the map $f^\sharp$ is indeed the pullback map of smooth functions.

Affine Schemes

Consider a ring A, and $\operatorname{Spec} A$ the set of all prime ideals of A.

Zariski Topology

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Let $\mathfrak{a}$ be an ideal of A and define $V(\mathfrak{a}) \subset \operatorname{Spec} A$ to be the set of all prime ideals which contain $\mathfrak{a}$.
Define a topology on $\operatorname{Spec} A$ by declaring the closed sets to be $V(\mathfrak{a})$'s, for any ideal $\mathfrak{a} \subset A$. One observe that

- If $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
- If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
- If $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals, $V(\mathfrak{a}) \subset V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$.

Also, $V(A) = \varnothing$, and $V((0)) = \operatorname{Spec} A$. This confirms that it indeed gives us a topology.

Structure Sheaf

A sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$: for an open subset $U \subset \operatorname{Spec} A$, define $\mathcal{O}(U)$ to be the set of functions

$$s\colon U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$$

such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and s is locally a quotient of elements of A.

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such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, and s is locally a quotient of elements of A.
Explicitly, for each $\mathfrak{p} \in U$, there exists a neighbourhood V of $\mathfrak{p}$, contained in U, and $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{q}}$.

Spectrum

Definition (Spectrum of Rings)

Let A be a ring. The *spectrum* of A is the pair consisting of the topological space $\operatorname{Spec} A$ together with the sheaf of rings $\mathcal{O}$ defined above. We write $(\operatorname{Spec} A, \mathcal{O})$.

Rings and Affine Schemes

Theorem

We have an equivalence of categories

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where $\mathbf{CRing}^{\mathrm{op}} \to \mathbf{AffSch}$ *is given by* $A \mapsto \operatorname{Spec} A$, *and* $\mathbf{AffSch} \to \mathbf{CRing}^{\mathrm{op}}$ *is given by* $X \mapsto \Gamma(X, \mathcal{O}_X)$.

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A scheme is a locally ringed space $(X, \mathcal{O}_X)$ locally isomorphic to an affine scheme. We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf. A morphism of schemes is a morphism of locally ringed spaces.

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Notation:

- X is $(X, \mathcal{O}_X)$;
- the underlying topological space is $\mathrm{sp}(X)$.

Schemes over k

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A scheme X over k is itself together with a morphism of schemes $X \to \operatorname{Spec} k$.

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A scheme X over k is itself together with a morphism of schemes $X \to \operatorname{Spec} k$. A morphism $f\colon X \to Y$ between schemes over k is a morphism of schemes compatible with given morphisms to $\operatorname{Spec} k$. In other words, the below diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & \operatorname{Spec} k & \end{tikzcd}$$

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Sheaf map $\implies$ $\Gamma(X, \mathcal{O}_X)$ is a k-algebra

Main theorem

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from the category of varieties over k to schemes over k.

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Let k be an algebraically closed field. There is a natural fully faithful functor

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from the category of varieties over k to schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of $\mathrm{sp}(t(V))$, and its sheaf of regular functions is obtained by restricting the structure sheaf of $t(V)$ via this homeomorphism.

Construction of t

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Define a topology on $t(X)$ by taking as closed sets the subsets of the form $t(Y)$, where $Y \subset X$ a closed subset.
The map $\alpha \colon X \to t(X)$ by $P \mapsto \overline{\{P\}}$ induces a correspondence between open sets in X and open sets in $t(X)$. Then there is a natural structure sheaf $\mathcal{O}_{t(X)}$ on $t(X)$.

Construction of t

Now, let k be an algebraically closed field. An affine variety $Z(I) \subset \mathbb{A}^n$ has a corresponding affine scheme $\operatorname{Spec} k[x_1, \ldots, x_n]/I$.

Fully-faithfullness of t

For $X = V(I)$ and $Y = V(J)$ affine varieties, we have

$$\begin{aligned}\mathrm{Hom}_{\mathbf{Var}(k)}(X, Y) &= \mathrm{Hom}_{k-\mathsf{alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)) \\ &= \mathrm{Hom}_{\mathbf{Sch}}(\mathrm{Spec}\, \mathcal{O}_X(X), \mathrm{Spec}\, \mathcal{O}_Y(Y)).\end{aligned}$$