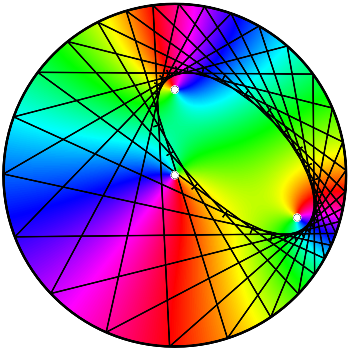

# Finding Ellipses: Blaschke Products, Poncelet's Theorem, and the Numerical Range

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# Background

# Finite Blaschke Products

A Blaschke factor is a complex-valued function $\phi_a(z) = \frac{z-a}{1-\bar{a}z}$, where $a \in \mathbb{D}$, the *open* unit disk in the complex plane. By complex analysis, $\phi_a$'s are exactly the automorphisms of the unit disk (up to rotation, i.e. multiplying by $e^{i\theta}$). Finite Blaschke products are finite products of Blaschke factors.

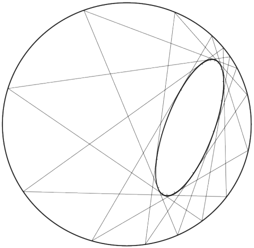
As a result, composing Blaschke products gives Blaschke products.

In this project, we are specifically interested in Blaschke products which have a zero at the origin, i.e. of the form

Figure: Phase plot of a degree 5 Blaschke product.

$$B(z) = z\frac{z-a_1}{1-\bar{a}_1 z}\frac{z-a_2}{1-\bar{a}_2 z}\cdots\frac{z-a_n}{1-\bar{a}_n z}.$$

# Poncelet's Porism



- We focused on ellipses inscribed in the unit circle. The ellipses for which we get closed polygons can be made are called Poncelet ellipses. These give rise to Blaschke products

- Poncelet's Theorem states that if two conics are such that one is lying in the other, and if there exists a closed polygon formed by drawing successive tangents from a point on the outer conic to the inner one, returning to the starting point after a finite number of steps, then this property holds for every point on the outer conic. More remarkably, the number of sides of the polygon (i.e., the number of tangents) is the same regardless of the starting point.

# The Numerical Range

- Toeplitz-Hausdorff's work on numerical range
- The numerical range of a complex $n \times n$ matrix $A$ is
$$\left\{\langle Ax, x\rangle : x \in \mathbb{C}^n, ||x|| = 1\right\}$$
- For each Blaschke product, there is an associated matrix $n \times n$ matrix $A = [\lambda_{ij}]$, where we let $\lambda_1, \ldots, \lambda_n \in \mathbb{D}$ and $n \geq 2$, given by,
$$\lambda_{ij} = \begin{cases} \lambda_j & \text{if } i = j, \\ \left(\prod_{k=i+1}^{j-1}(-\lambda_k)\right)\sqrt{1-|\lambda_i|^2}\sqrt{1-|\lambda_j|^2} & \text{if } i < j, \\ 0 & \text{if } i > j. \end{cases}$$
- $A$ has a corresponding $n+1 \times n+1$ unitary 1-dilation
- The numerical range of $A$ is equal to the intersection of the numerical range of the unitary 1-dilations of $A$

# The Connection

**Well established theorem**

*For any Blaschke product $B$, the is a corresponding matrix $A$ has the following property.*
*If $|z| = |w| = 1$ and $B(z) = B(w)$ then the line joining $z$ and $w$ is tangent to the boundary of numerical range of $A$. This curve is called the Blaschke curve of $B$.*

**Theorem (Daepp, Gorkin, Shaffer, Voss)**

*The Blaschke curve of a degree 3 Blaschke product is always an ellipse.*

**Theorem (Interpolation)**

*Every ellipse inscribed in a convex n-gon inscribed in the unit circle is a Blaschke curve.*

# One more theorem

**Theorem (Fujimura, Gorkin, Wagner)**

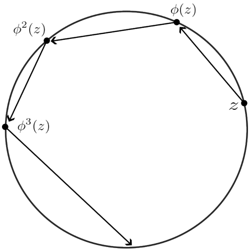
*The Blaschke curve of a degree 4 Blaschke product $B$ is an ellipse if and only if there is a decomposition $B = B_1 \circ B_2$ with $B_1$ and $B_2$ degree 2 Blaschke products.*

# Results

# Blaschke products conjugate to $z^n$

### Theorem (Frantz 2018)

*Choose a Blaschke factor $\phi$ and a point $z \in \mathbb{T}$. The polygon obtained by joining $\phi^i(z) \leftrightarrow \phi^{i+1}(z)$ is tangent to a unique ellipse determined by $\phi$.*
*Ellipses that arise in this way are called ellipses of Möbius type.*




### Algebraic proof of Poncelet's theorem

There is an elegant proof of Poncelet's theorem due to Griffiths & Harris 1977, using the group structure of elliptic curves.
This proof fails in certain degenerate cases.

# Blaschke products conjugate to $z^n$

Both of these statements can be connected to Blaschke products.
We have found new characterisations of Blaschke products of the form $\phi_a \circ [z \mapsto z^n] \circ \phi_b$, where $\phi_a$ and $\phi_b$ are Blaschke factors.
A Blaschke product is conjugate to $z^n$ if and only if it's Blaschke curve $C$ satisfies one of the equivalent conditions:

- $C$ is an ellipse of Möbius type.
- $C$ is doubly tangent to the unit circle, as a curve in $\mathbb{PC}^2$.
- $C$ is of the degenerate form where Griffiths and Harris' proof of Poncelet's theorem fails.

**Lemma**

*Let $B$ be a Blaschke curve and $\phi$ a Blaschke factor. Then the Blaschke curves of $B$ and $B \circ \phi$ are projectively equivalent.*

# Darboux ellipses

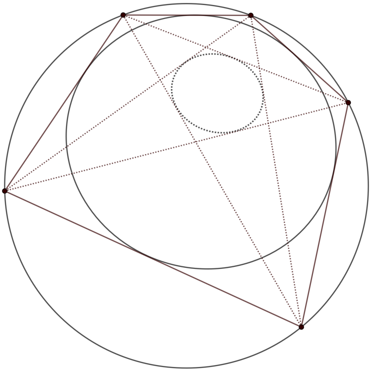


Figure: A Poncelet ellipse and its Darboux ellipse

# Roots of $B$ and Darboux ellipses

Given a polygon circumscribing a Poncelet ellipse, we can obtain a new polygon by joining every $n$th vertex. It is known that for every $n$, these polygons also circumscribe a unique ellipse, called a Darboux ellipse.

### Known case

Suppose $B$ is conjugate to $z^n$ and $B(0) = 0$. In this case:

- The foci of the ellipse are roots of the Blaschke product.
- The other roots of the Blaschke product are foci of the corresponding Darboux ellipses.

### Conjecture

Suppose $B$ is a Blaschke product with $B(0) = 0$ whose Blaschke curve is an ellipse. Then the roots of the Blaschke product are the foci of the Darboux ellipses.

The conjecture holds in degree 3, and appears to hold (numerically) in degree 4.

# Perimeter of Triangles formed by Blaschke products

- We focus on degree 3 Blaschke products on the unit circle. These form 3-to-1 maps, and if we draw lines connecting the points on the unit circle that are identified by the Blaschke product, we obtain triangles corresponding to every point on the Blaschke product.
- In the course of this program, we were able to prove that the perimeter of the triangle deforms continuously as we traverse the unit circle, that is, small perturbations of points on the unit circle, give rise to small changes in the perimeter of triangles formed.

# Bounds for the perimeter of triangles formed by Blaschke products

- From computations, we conjecture that the perimeter of the triangles formed increase continuously as we move from points identified with 1 and travel up along the unit circle to -1 and decreases as we continue downwards towards 1.
- We also conjecture that reasonable upper and lower bounds for the perimeters of the triangles are $3\sqrt{3}$ and 4 respectively.

# Curvatures and Eccentricities of Blaschke-Ellipses

A Blaschke Product whose Poncelet curve is an Ellipse can be represented as the loci of points:

$$|w-a|+|w-b| \;=\gamma, \text{ where } \gamma \text{ is the length of the major axis}$$

Then for any Blaschke-Ellipse with foci $a$ and $b$, length of *semi*-minor axis $\alpha$, and distance between foci $|b-a|=d$:

- A Degree-3 Blaschke Product with $\gamma=|1-\bar{a}b|$,

$$Eccentricity, e(a,b)=\frac{|b-a|}{|1-\bar{a}b|}, \quad Curvature, \kappa(t)=\frac{4\alpha\gamma}{(\gamma^2-d^2 sin^2 t)^{\frac{3}{2}}}$$

- A **Decomposable** Degree-4 Blaschke Product with $\gamma=|1-\bar{a}b|\sqrt{\dfrac{2-|a|^2-|b|^2}{1-|a|^2|b|^2}}$, then

$$Eccentricity, e(a,b)=\frac{d}{\gamma}, \quad Curvature, \kappa(t)=\frac{4\alpha\gamma}{(\gamma^2-d^2 sin^2 t)^{\frac{3}{2}}}$$

# Curvatures and Eccentricities of Blaschke-Ellipses

For a Degree-3 Blaschke product, the eccentricity is the *pseudohyperbolic metric* with nice properties:

- $e(a,b) = \rho(a,b) = \rho(U(a), U(b))\ \forall a, b \in \mathbb{D}$, *and all Unitary U*
- $e(a,b) = \rho(a,b) = \rho(\phi_\alpha(a), \phi_\alpha(b))\ \forall a, b, \alpha \in \mathbb{D}$, *and all disk automorphisms* $\phi_\alpha$
- $e(a,b) = \rho(a,b) \leq \dfrac{\rho(a,\alpha) + \rho(\alpha,b)}{1 + \rho(a,\alpha)\rho(\alpha,b)}$

Item (3) establishes that changing the foci by adding an $\epsilon > 0$ continuously increases the eccentricity to a maximum of 1.
For all Blaschke ellipses the maximum and minimum curvature are

$$\kappa_{max} = \frac{4\alpha\gamma}{(\gamma^2 - d^2)^{\frac{3}{2}}}, \kappa_{min} = \frac{4\alpha}{\gamma^2}$$

This investigation can be extended to any higher-degree Blaschke-Ellipses.

# Chapple-Euler and Fuss's Theorem

When is a circle inscribed in an $n$-gon that is itself inscribed in a circle?

**Theorem**

*If a circle of radius $r$ is inscribed in a quadrilateral that is itself inscribed in a circle of radius $R$, and if $d$ denotes the distance between the centers of the circles, then*

$$\frac{1}{(R-d)^2} + \frac{1}{(R+d)^2} = \frac{1}{r^2}.$$

- Degree 3 case is trivial
- Degree 4 found by Fujimara, Gorkin, Wagner

Can we generalize Fuss's Theorem in terms of the zeros of a Blaschke product for $n > 3$?

- First, determine when the curve associated with a degree $n > 3$ Blaschke product is an ellipse in terms of it's zeros

# Chapple-Euler and Fuss's Theorem

- Let $B$ be a degree $n-1$ Blaschke product with zeros $\lambda_i$, where $n > 3$.
- If the zeros of $B$ $\lambda_3, \ldots, \lambda_{n-1}$ satisfy,

$$s^2 \prod_{i=3}^{n-1} \lambda_i = 4 \left[ \prod_{i=1}^{n-1} \lambda_i - \det(H) - \det(K) \right]$$

- Then the $n$-Poncelet curve associated with $B_1$, where $B_1(z) = zB(z)$, is an $n$-Poncelet ellipse with foci at $\lambda_1$ and $\lambda_2$
- This ellipse may be written as $|z - \lambda_1| + |z - \lambda_2| = \sqrt{d + |\lambda_1 - \lambda_2|^2}$, where $d = \sum_{j<k} |\lambda_{jk}|^2$ and $\lambda_{jk}$ are the elements of $A$

So what of the bicentric polygon conditions for $n > 3$?

- A circle is inscribed in a $n$-gon that is itself inscribed in $\mathbb{T}$...
- If the circumscribed circle is described by the numerical range of the compression of the shift matrix $A$ associated with the Blaschke product $B_1(z) = zB(z)$, $B$ has zeros which satisfy the conditions above, and $\lambda_1 = \lambda_2$.

# Degree 6 Blaschke Products

Let $B_n$ denote a Blaschke product of degree $n$. There are many examples that the Blaschke curve of $B_2 \circ B_3$ generally are not ellipses.

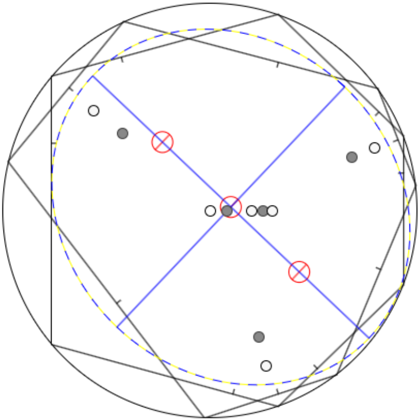


Figure: $B_2$ with zeroes at 0 and $i/2$, and $B_3$ with zeroes at 0, $1/5$, and $3/10$.

# Degree 6 Blaschke Products

However, it still remains interesting to investigate compositions in the other order, i.e. $B_3 \circ B_2$.

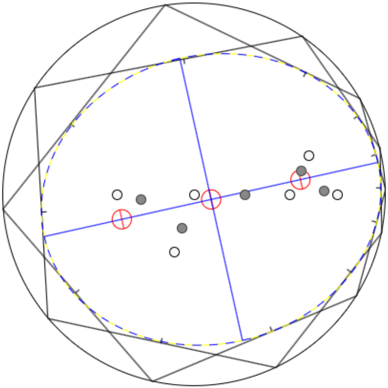


Figure: $B_2$ with zeroes at 0 and $1/2$, and $B_3$ with zeroes at 0, $i/5$, and $3/10$.

# Decomposing Blaschke Products

One may be thinking: would the Blaschke curves of $B_{p_1} \circ \cdots \circ B_{p_n}$ be ellipses, where $\{p_i\}_{i=1}^{n}$ is a list of nonincreasing prime numbers?
In fact, in degree 8 this fails.

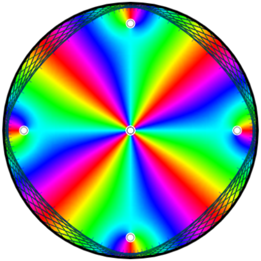


Figure: $B_8(z) = z^4\left(\frac{z^4 - 0.84^4}{1 - 0.84^4 z^4}\right)$. Credit: *Decomposable Blaschke Products of Degree* $2^n$, by Asuman Güven Aksoy, Francesca Arici, M. Eugenia Celorrio, and Pamela Gorkin

# Afterword

The images in this presentation were generated using A.E. Shaffer's applet at `https://pubapps.bucknell.edu/static/aeshaffer/v1`.