THE CANONICAL LINE BUNDLE AND THE ADJUNCTION FORMULA

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ABSTRACT. In this end-of-semester project, we study the theory of holomorphic vector bundles over complex manifolds, establishing the correspondence between divisors and line bundles. As a main computation, we derive the adjunction formula for hypersurfaces. We then apply it to complex projective spaces $\mathbb{CP}^m$ to compute the genus of various algebraic curves.

CONTENTS

1. HOLOMORPHIC VECTOR BUNDLES

In parallel to the theory of (smooth) vector bundles on a smooth manifold, we define holomorphic vector bundles on a complex manifold as follows.

Definition 1.1. Let X be a complex manifold. A *holomorphic vector bundle* of rank r on X is a complex manifold E together with a holomorphic map $\pi \colon E \to X$ and the structure of an r-dimensional $\mathbb{C}$-vector space on any fibre $\pi^{-1}(x)$ such that there exists an open cover $\{U_\alpha\}$ and biholomorphic maps $\varphi_\alpha \colon \pi^{-1}(U_\alpha) \cong U_\alpha \times \mathbb{C}^r$ (called *local trivializations*) such that the diagram

$$\begin{array}{ccc} \pi^{-1}(U_\alpha) & \xrightarrow{\quad \varphi_\alpha \quad} & U_\alpha \times \mathbb{C}^r \\ & {\scriptstyle \pi} \searrow \quad \swarrow {\scriptstyle \mathrm{pr}_1} & \\ & U_\alpha & \end{array} \tag{1.1}$$

commutes, and that the induced map $\pi^{-1}(x) \cong \mathbb{C}^r$ is $\mathbb{C}$-linear.

In particular, a *holomorphic line bundle* is a holomorphic vector bundle of rank 1. As for smooth manifolds, a holomorphic vector bundle over a complex manifold X could also be given in terms of an open cover $\{U_\alpha\}$ of X and a collection of holomorphic transition functions

$$\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_r(\mathbb{C}) \tag{1.2}$$

for any α, β, satisfying the *cocycle condition*: for $p \in U_\alpha \cap U_\beta \cap U_\gamma$,

$$\tau_{\alpha\beta} \circ \tau_{\beta\gamma} \circ \tau_{\gamma\alpha}(p) = \mathrm{id}. \tag{1.3}$$

As described by [Lee12, Exer. 10-8], for each canonical construction of vector spaces (e.g., direct sums, self tensor products, duals), there is a corresponding construction for vector bundles in which the fibre is canonically isomorphic to the canonical construction of the fibres as vector spaces. We define morphisms in the category of holomorphic vector bundles over X as follows.

Definition 1.2. Let X be a complex manifold, and $\pi_E \colon E \to X$ and $\pi_F \colon F \to X$ be two holomorphic vector bundles over X. A *morphism* between holomorphic vector bundles E and F is a holomorphic map $\psi \colon E \to F$ with $\pi_E = \pi_F \circ \psi$ such that the induced map $\psi(x) \colon \pi_E^{-1}(x) \to \pi_F^{-1}(x)$ is a map of $\mathbb{C}$-vector spaces.

In particular, an isomorphism $\varphi \colon E \to F$ is a morphism that admits an inverse morphism $\psi \colon F \to E$. As in the case of vector spaces, this is equivalent to φ being bijective (Theorem B.1). Throughout the writing, I would repeatedly use the following fact to identify isomorphic holomorphic line bundles by comparing transition functions.

Fact 1.3 (cf. Theorem B.2). Two holomorphic vector bundles E and F with $\{U_i\}$ as a common refinement for the respective open covers satisfying the local trivialization property, and g_{ij}^E, g_{ij}^F the transition functions for respective bundles. Then E and F are isomorphic if and only if there exist holomorphic maps $h_i \colon U_i \to \mathrm{GL}_r(\mathbb{C})$ such that on every overlap U_{ij},

$$g_{ij}^F = h_i \circ g_{ij}^E \circ h_j^{-1}. \tag{1.4}$$

1.1. **Picard group.** Fix a complex manifold X. Let $\mathrm{Pic}(X)$ be the set of *isomorphism classes* of holomorphic line bundles on X.

Proposition 1.4. *The set* $\mathrm{Pic}(X)$ *has a group structure with tensor product.*

Proof. It suffices to define the group operation on the level of line bundles. Let $\mathcal{L}_1, \mathcal{L}_2$ be respective representatives from two elements in $\mathrm{Pic}(X)$, where $\mathcal{L}_1$ is described by holomorphic cocycles $(\{V_\alpha\}, \tau_{\alpha\beta})$ and $\mathcal{L}_2$ is described by holomorphic cocycles $(\{W_\alpha\}, \tau'_{\alpha\beta})$. By taking a common refinement of the open covers for $\mathcal{L}_1$ and $\mathcal{L}_2$, we may assume both bundles are trivialized over the same cover $\{U_\alpha\}$. Since $\mathcal{L}_1, \mathcal{L}_2$ are line bundles, the maps $\tau_{\alpha\beta}$ and $\tau'_{\alpha\beta}$ are

$$U_\alpha \cap U_\beta \to \mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^*. \tag{1.5}$$

Fix a point $x \in X$. Since we constructed $\mathcal{L}_1 \otimes \mathcal{L}_2$ such that the fibre is canonically isomorphic to the tensor products of the fibres, the transition functions of $\mathcal{L}_1 \otimes \mathcal{L}_2$ is given by

$$\tau''_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{GL}_1(\mathbb{C} \otimes_\mathbb{C} \mathbb{C}) \cong \mathbb{C}^*. \tag{1.6}$$

This is because the original transition functions act by scalar multiplication $\tau_{\alpha\beta}(x)(v) = \lambda v$ and $\tau'_{\alpha\beta}(x)(w) = \mu v$ where $\lambda, \mu \in \mathbb{C}^*$, so

$$\tau_{\alpha\beta}(x) \otimes \tau'_{\alpha\beta}(x)(v \otimes w) = (\lambda v) \otimes (\mu w) = (\lambda\mu)(v \otimes w). \tag{1.7}$$

The identity element is the isomorphism class represented by $\mathcal{L}_0$ the trivial line bundle, described by the transition functions

$$\tau_{\alpha\beta} \colon U_\alpha \cap U_\beta \to 1 \in \mathrm{GL}_1(\mathbb{C}), \tag{1.8}$$

and the inverse of a line bundle $\mathcal{L}$ is given by its dual $\mathcal{L}^*$. To see why, recall that in linear algebra, a vector space automorphism $F\colon V \to V$ has a corresponding dual automorphism $F^*\colon V^* \to V^*$, where the first V^* is the dual of the second V, vice versa. This map, in matrix form, is exactly $F^\top$. In order to get the arrow in correct direction, we simply invert the map F^*. Then the matrix representation becomes $(F^\top)^{-1}$. In the case of line bundles, this translates to multiplication by the inverse of the original scalar. Therefore $\mathcal{L} \otimes \mathcal{L}^*$ is trivial. $\qquad\square$

1.2. **The holomorphic line bundle $\mathcal{O}_M(H)$.** Let H be a complex submanifold of a complex manifold M of codimension 1. By definition, there exists a holomorphic atlas $\{(U_\alpha, \varphi_\alpha)\}$ of M such that

$$\varphi_\alpha\colon U_\alpha \cap H \xrightarrow{\sim} \varphi_\alpha(U_\alpha) \cap \mathbb{C}^{n-1}.$$

Proposition 1.5. *There is a natural holomorphic line bundle $\mathcal{O}_M(H) \to M$ together with holomorphic section $s\colon M \to \mathcal{O}_M(H)$ such that $s^{-1}(0) = H$.*

Proof. On any U_α, we define $f_\alpha\colon U_\alpha \to \mathbb{C}$ to be φ_α composed with the n-th coordinate function z_n in $\mathbb{C}^n$. Then f_α is holomorphic. To define a line bundle, we pick the open sets $\{U_\alpha\}$ and define holomorphic transition functions

$$\tau_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^*, \quad \tau_{\alpha\beta}(p) = \frac{f_\alpha(p)}{f_\beta(p)}.$$

It is easily seen that they satisfy the cocycle condition. Therefore the data defines a holomorphic line bundle, which we call $\mathcal{O}_M(H)$. We define a global section $s\colon M \to \mathcal{O}_M(H)$ by defining $s_\alpha = f_\alpha$ over each chart U_α of M. For $U_\alpha \cap U_\beta \neq \varnothing$, we verify

$$s_\alpha = f_\alpha = \left(\frac{f_\alpha}{f_\beta}\right) \cdot f_\beta = \tau_{\alpha\beta} \cdot s_\beta, \tag{1.9}$$

so the local sections glues into the global section s. By construction, $s(p) = 0$ if and only if locally the n-th coordinate of $p \in M$ is zero, i.e., $p \in H$. $\qquad\square$

Theorem 1.6. *Any line bundle $\mathcal{L}$ which admits a holomorphic section vanishing (to order 1) along H is isomorphic to $\mathcal{O}_M(H)$.*

Proof. It suffices to show for any line bundle $\mathcal{L}$ with holomorphic transition functions $\{g_{\alpha\beta}\colon U_\alpha \cap U_\beta \to \mathbb{C}^*\}$, there exists u_α holomorphic and nonvanishing on U_α (looking at the common refinement of the respective charts) such that

$$g_{\alpha\beta} = \frac{u_\alpha}{u_\beta} \cdot \tau_{\alpha\beta}. \tag{1.10}$$

Let $t\colon M \to \mathcal{L}$ be a holomorphic section vanishing to order 1 along H, and t_α the local representation of t in chart U_α. In this fixed chart U_α, since t_α and s_α both vanish exactly at the same set $H \cap U_\alpha$, their ratio is a nonzero holomorphic unit u_α, i.e., $t_\alpha = u_\alpha \cdot s_\alpha$. Since

$$t_\alpha = g_{\alpha\beta} \cdot t_\beta, \tag{1.11}$$

we have

$$u_\alpha \cdot s_\alpha = g_{\alpha\beta} \cdot (u_\beta \cdot s_\beta) \iff g_\alpha = \left(\frac{u_\alpha}{u_\beta}\right) \cdot \left(\frac{s_\alpha}{s_\beta}\right) = \frac{u_\alpha}{u_\beta} \cdot \tau_{\alpha\beta}. \tag{1.12}$$

$\qquad\square$

1.3. **Relationship with the conormal line bundle.**

Definition 1.7. Let $H \subset M$ be a complex submanifold. The *normal bundle* of H in M is the holomorphic vector bundle $\mathcal{N}_{H/M}$ on H which is the cokernel of the natural injection $TH \subset TM|_H$. The *conormal bundle* of H, denoted $\mathcal{N}_{H/M}^\vee$, is the *dual* of $\mathcal{N}_{H/M}$.

The above definition for $\mathcal{N}_{H/M}^{\vee}$ is the same as the description given in the problem set (the kernel of the natural restriction map $T^*M|_H \to T^*H$). In fact, by this description, we see that the conormal bundle at $p \in H$ is exactly the annihilator of T_pH. Let's use the same atlas for M as before that gives H the complex submanifold structure. For any point $p \in H \cap U_\alpha$, the tangent space T_pH is exactly the kernel of the differential $df_\alpha(p)$. To see why, imagine a curve $\gamma(t)$ lying entirely in the submanifold H, passing through p at $t = 0$. By definition, $v \in T_pH$ is $\gamma'(0)$ for some curve γ. Since H is the zero set of f_α, by the chain rule

$$\frac{d}{dt} f_\alpha(\gamma(t))\Big|_{t=0} = df_\alpha(p) \cdot \gamma'(0) = 0 \cdot v = 0, \tag{1.13}$$

so $T_pH \subset \ker df_\alpha(p)$. Since the kernel has dimension $n - 1$ and T_pH has the same dimension, we conclude they are equal. This says $\{df_\alpha\}$ forms a basis of local sections for the conormal bundle $\mathcal{N}_{H/M}^{\vee}$ over each open set $U_\alpha \cap H$. One can also prove a slightly more general statement [Huy05, Exer. 2.2.12].

Proposition 1.8. *The two holomorphic line bundles $\mathcal{O}_M(-H)|_H$ and $\mathcal{N}_{H/M}^{\vee}$ are isomorphic.*

Proof. The transition functions $\tau_{\alpha\beta}|_H$ for $\mathcal{O}_M(H)|_H$ are restrictions of $\tau_{\alpha\beta}$ to H. In $\mathcal{O}_M(H)$, we have

$$f_\alpha = \tau_{\alpha\beta} \cdot f_\beta. \tag{1.14}$$

Applying differential operator d to both sides, we get

$$df_\alpha = d\tau_{\alpha\beta} \cdot f_\beta + \tau_{\alpha\beta} \cdot df_\beta. \tag{1.15}$$

For $p \in H \cap U_\alpha \cap U_\beta$, f_β vanish, so it simplifies to

$$df_\alpha(p) = \tau_{\alpha\beta}(p) \cdot df_\beta(p). \tag{1.16}$$

Then the bundle transition functions of $\mathcal{N}_{H/M}^{\vee}$ is defined by the *inverse* collection $\left\{\tau_{\alpha\beta}^{-1}\right\}$, since the above describes transitions between local sections df_α and df_β. This tells us that the transition functions for $\mathcal{N}_{H/M}^{\vee}$ and $\mathcal{O}_M(-H)|_H$ are the same. Thus they are isomorphic. One could also say, as in [GH78], that the above formula tells us there exists a nonzero global section of $\mathcal{N}_{H/M}^{\vee} \otimes \mathcal{O}_M(H)$. Then $\mathcal{N}_{H/M}^{\vee} \otimes \mathcal{O}_M(H)$ is trivial, concluding the same result. $\qquad\square$

2. Divisor and the canonical line bundle

2.1. **$\mathcal{O}_M$ for a divisor.** Let $D = \sum_{i=1}^N n_j H_j$ be a divisor. We may define

$$\mathcal{O}_M(D) = \bigotimes_j \mathcal{O}_M(H_j)^{\otimes n_j}.$$

Theorem 2.1. *If a line bundle $\mathcal{L}$ on M admits a meromorphic section whose zeroes and poles (with multiplicities) define the divisor D, then $\mathcal{L} \cong \mathcal{O}_M(D)$.*

Proof. As in the proof of Proposition 1.8, we calculate the transition functions $\tau_{\alpha\beta}$ of $\mathcal{O}_M(D)$ and compare it with those of $\mathcal{L}$. Fix an open cover $\{U_\alpha\}$. For each codimension 1 submanifold H_j, let $f_\alpha^{(j)} : U_\alpha \to \mathbb{C}$ be the local holomorphic function defining H_j (vanishing to order 1). By the discussion in Proposition 1.4, the transition functions of $\mathcal{O}_M(D)$ are given by

$$\tau_{\alpha\beta} = \prod_{j=1}^N \left(\frac{f_\alpha^{(j)}}{f_\beta^{(j)}}\right)^{n_j}. \tag{2.1}$$

Let $g_{\alpha\beta}$ be the transition functions of $\mathcal{L}$. Let $s_\alpha : U_\alpha \to \mathbb{C}$ be the local section of s on the chart U_α, where s is a global meromorphic section whose divisor is exactly D. We may factor

$$s_\alpha(z) = u_\alpha(z) \prod_{j=1}^N \left(f_\alpha^{(j)}(z)\right)^{n_j}, \tag{2.2}$$

where $u_\alpha \colon U_\alpha \to \mathbb{C}^*$ is a nowhere vanishing holomorphic function. By this formula, one can immediately see the transition functions of two bundles are related:

$$g_{\alpha\beta} = \frac{u_\alpha}{u_\beta} \cdot \tau_{\alpha\beta}. \tag{2.3}$$

Therefore, the two line bundles are isomorphic. □

Corollary 2.2. *If D_1, D_2 are linearly equivalent divisors on X, then $\mathcal{O}_M(D_1) \cong \mathcal{O}_M(D_2)$.*

Proof. To say two divisors are linearly equivalent means there exists a meromorphic function f on X such that

$$D_1 + \operatorname{div} f = D_2. \tag{2.4}$$

Then the map $s \mapsto f \cdot s$ defines an isomorphism $\mathcal{O}_M(D_1) \to \mathcal{O}_M(D_2)$. □

2.2. **The canonical line bundle $\mathcal{K}_M$.**

Definition 2.3. The *canonical line bundle* $\mathcal{K}_M$ over M is the top exterior power of the cotangent vector bundle T^*M, i.e., $\mathcal{K}_M = \bigwedge_n T^*M$.

The term "canonical" reflects the functoriality of the construction. Independence of the chosen basis is apparent from an observation in linear algebra, and that the determinant are invariant under changes of basis, i.e., $\det(AB) = \det(A) \det(B)$.

Lemma 2.4 (cf. Lemma A.1). *Let $A \colon V \to W$ be a map of n-dimensional k-vector spaces. Then the natural map*

$$\bigwedge_n A \colon \bigwedge_n V \to \bigwedge_n W$$

is the determinant $\det(A)$.

3. $\mathcal{O}(n)$ AND THE CANONICAL LINE BUNDLE

3.1. **Construction of $\mathcal{O}(n)$.** Here I follow the steps outlined by [Huy05].

Proposition 3.1. *Let $\mathcal{O}(-1) \subset \mathbb{CP}^m \times \mathbb{C}^{m+1}$ be the subset of all pairs (ℓ, z) where $z = \lambda \ell$ for some $\lambda \in \mathbb{C}$. The set $\mathcal{O}(-1)$ has a natural holomorphic line bundle structure over $\mathbb{CP}^m$.*

Proof. The map $\pi \colon \mathcal{O}(-1) \to \mathbb{CP}^m$ is the projection onto the first factor. Let $\{(U_i, \varphi_i)\}_{i=0}^m$ be the holomorphic atlas defined by

$$U_i := \{[z_0 : z_1 : \cdots : z_m] \mid z_i \neq 0\}, \tag{3.1}$$

and $\varphi_i \colon U_i \to \mathbb{C}^m$ given by

$$\varphi_i \colon [z_0 : \cdots : z_m] \mapsto \left(\frac{z_0}{z_i}, \cdots, \frac{z_{i-1}}{z_i}, \frac{z_{i+1}}{z_i}, \cdots, \frac{z_m}{z_i} \right). \tag{3.2}$$

A canonical trivialization of $\mathcal{O}(-1)$ over U_i is given by $\psi_i \colon \pi^{-1}(U_i) \cong U_i \times \mathbb{C}$, where $\psi_i(\ell, z) = (\ell, z_i)$. The transition maps are then

$$\psi_{ij}(\ell) \colon \quad \mathbb{C} \to \mathbb{C} \tag{3.3}$$

$$w \mapsto \frac{z_i}{z_j} \cdot w \tag{3.4}$$

where $\ell = [z_0 : \cdots : z_m]$. The collection $\{(U_i, \psi_i)\}_{i=0}^m$ gives a holomorphic atlas for $\mathcal{O}(-1)$. □

Definition 3.2. The line bundle $\mathcal{O}(1)$ is the dual of $\mathcal{O}(-1)$. For $n \in \mathbb{Z}$ positive, define $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$, and for $n \in \mathbb{Z}$ negative, define $\mathcal{O}(-1)^{\otimes n}$. Define $\mathcal{O}(0)$ to be the trivial line bundle $\mathbb{CP}^m \times \mathbb{C} \to \mathbb{CP}^m$.

The line bundle $\mathcal{O}(-1)$ is called the tautological line bundle, for it generates $\operatorname{Pic}(\mathbb{CP}^m)$.

3.2. **Concrete description of $\mathcal{O}(-1)$.** We want to understand $\mathcal{O}(-1)$ more concretely, e.g., in terms of its defining divisor. This will help us to shape a geometric intuition for $\mathcal{O}(n)$'s in general.

Observe first that in $\mathcal{O}(-1)$, the fibre $\pi^{-1}(\ell)$ for $\ell = [w_0 : \cdots : w_m]$ is naturally homeomorphic to the line $L = \lambda(w_0, \ldots, w_m)$ in $\mathbb{C}^{m+1}$. Then by construction, the fibre for ℓ of the line bundle $\mathcal{O}(-1)$ is L. By Theorem 2.1, to identify the divisor D such that $\mathcal{O}_{\mathbb{CP}^m}(D) \cong \mathcal{O}(-1)$, it suffices to construct a global meromorphic section s of $\mathcal{O}(-1)$ whose divisor is D.

Let $\{U_i\}_{i=0}^m$ be the cover described in Proposition 3.1. We define the meromorphic section $s \colon \mathbb{CP}^m \to \mathcal{O}(-1)$ locally as follows: for every line $\ell \in \mathbb{CP}^m$, we pick the unique $v \in \mathrm{pr}_2(\pi^{-1}(\ell)) = L \subset \mathbb{C}^{m+1}$ such that $z_0(v) = 1$, where z_i stands for the i-th coordinate function of $\mathbb{C}^{m+1}$, and then define $s_i(\ell) = s|_{U_i}(\ell) = z_0(v)$. In U_0, we pick

$$v = \left(1, \frac{w_1}{w_0}, \cdots, \frac{w_m}{w_0}\right). \tag{3.5}$$

Therefore

$$s_0(\ell) = s|_{U_0}(\ell) = 1. \tag{3.6}$$

Similarly, we have

$$s_i(\ell) = s|_{U_i}(\ell) = \frac{w_i}{w_0}. \tag{3.7}$$

The transition functions described in Proposition 3.1 tells us s indeed defines a global meromorphic section. Locally, the divisor of s_i is exactly $-(H_0 \cap U_i)$, where H_0 is the hyperplane $\{w_0 = 0\}$. Thus the divisor of s is $-H_0$. We then conclude that

$$\mathcal{O}(-1) \cong \mathcal{O}_{\mathbb{CP}^m}(-H_0). \tag{3.8}$$

Since divisors H_i, H_j are linearly equivalent by meromorphic z_i/z_j, we have

$$\mathcal{O}_{\mathbb{CP}^m}(-H_i) \cong \mathcal{O}_{\mathbb{CP}^m}(-H_j). \tag{3.9}$$

This fact would become useful in the next section.

3.3. **Description of $\mathcal{K}_{\mathbb{CP}^m}$.** In fact, on $M = \mathbb{CP}^m$, the canonical bundle $\mathcal{K}_M$ coincides with the line bundle $\mathcal{O}(-(m+1))$. We relate them together by finding the defining divisor for $\mathcal{K}_M$ and then apply Theorem 2.1.

Theorem 3.3. $\mathcal{K}_{\mathbb{CP}^m} \cong \mathcal{O}(-(m+1))$.

Proof. We examine a meromorphic m-form on U_0

$$\omega = \frac{dz_1}{z_1} \wedge \frac{dz_2}{z_2} \wedge \cdots \wedge \frac{dz_m}{z_m}. \tag{3.10}$$

Observe that ω is meromorphic, with local divisor

$$\mathrm{div}\,\omega = -((H_1 + \cdots + H_m) \cap U_0). \tag{3.11}$$

Switching to the local coordinates w_i of U_j, we have

$$z_j = 1/w_0, \qquad z_i = w_i/w_0, \quad i \neq j. \tag{3.12}$$

Then

$$\frac{dz_j}{z_j} = \frac{d\left(\frac{1}{w_0}\right)}{\frac{1}{w_0}} = -\frac{dw_0}{w_0}, \tag{3.13}$$

and for $i \neq j$,

$$\frac{dz_i}{z_i} = \frac{d\left(\frac{w_i}{w_0}\right)}{\frac{w_i}{w_0}} = \frac{dw_i}{w_i} - \frac{dw_0}{w_0}. \tag{3.14}$$

Then in U_j,

$$\omega = (-1)^j \cdot \left(\frac{dw_0}{w_0}\right) \wedge \left(\frac{dw_i}{w_i} - \frac{dw_0}{w_0}\right) \wedge \cdots \tag{3.15}$$

$$= (-1)^j \cdot \left(\frac{dw_0}{w_0}\right) \wedge \cdots \wedge \left(\widehat{\frac{dw_j}{w_j}}\right) \wedge \left(\frac{dw_m}{w_m}\right). \tag{3.16}$$

This tells us that under an affine change of coordinates, the local expressions of ω transform by the Jacobian determinant, which is the transition function of the canonical bundle $\mathcal{K}_M$. Hence the local meromorphic forms glue to a global meromorphic section ω of $\mathcal{K}_M$. Then the divisor of ω is

$$\operatorname{div}\omega = -(H_0 + H_1 + \cdots + H_m), \tag{3.17}$$

so by Theorem 2.1 and eq. (3.8),

$$\mathcal{K}_M = \mathcal{O}_M(\operatorname{div}\omega) = \mathcal{O}(-(m+1)). \tag{3.18}$$

$\square$

4. The adjunction formula and additional consequences

4.1. **The adjunction formula.** Recall in section 1.3, for a codimension 1 submanifold H of M, we have a short exact sequence

$$0 \to \mathcal{N}_{H/M}^{\vee} \to T^*M|_H \to T^*H \to 0 \tag{4.1}$$

by definition. In the theory of vector spaces, we have the following fact.

Lemma 4.1. *Let*

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0 \tag{4.2}$$

be a short exact sequence of vector spaces. Then

$$\det(B) \cong \det(A) \otimes \det(C). \tag{4.3}$$

Proof. Let $a_1 \wedge \cdots \wedge a_m \in \det(A)$, and $c_1 \wedge \cdots \wedge c_k \in \det(C)$. Then

$$(a_1 \wedge \cdots \wedge a_m), (c_1 \wedge \cdots \wedge c_k) \mapsto f(a_1) \wedge \cdots \wedge f(a_m) \wedge g^{-1}(c_1) \wedge \cdots \wedge g^{-1}(c_k) \tag{4.4}$$

induces the isomorphism. $\square$

Then by passing the short exact sequence of line bundles through the determinant map, one obtain

$$\det(T^*M|_H) \cong \det(\mathcal{N}_{H/M}^{\vee}) \otimes \det(T^*H), \tag{4.5}$$

which is equivalent to

$$\det(T^*H) \cong (\det(\mathcal{N}_{H/M}^{\vee}))^{\vee} \otimes \det(T^*M|_H). \tag{4.6}$$

By definition, $\det(T^*M|_H) = \mathcal{K}_M|_H$, $\det(T^*H) = \mathcal{K}_H$, and by Proposition 1.8 we have $\det(\mathcal{N}_{H/M}^{\vee}) = \mathcal{N}_{H/M}^{\vee} \cong \mathcal{O}_M(-H)|_H$. Substituting these, one obtain

Theorem 4.2 (The Adjunction Formula)**.**

$$\mathcal{K}_H \cong (\mathcal{K}_M \otimes \mathcal{O}_M(H))|_H. \tag{4.7}$$

4.2. Euler characteristic of an algebraic curve. Let $C \subset \mathbb{CP}^2$ be an algebraic curve of degree d. As [Huy05, Prop. 2.4.1] describes, such an algebraic curve yields a global holomorphic section of $\mathcal{O}(d)$ over $\mathbb{CP}^2$. Thus $\mathcal{O}_{\mathbb{CP}^2}(C) \cong \mathcal{O}(d)$. By Theorem 3.3, we also have $\mathcal{K}_{\mathbb{CP}^2} \cong \mathcal{O}(-2-1) = \mathcal{O}(-3)$. Then by Theorem 4.2, we have

$$\mathcal{K}_C \cong (\mathcal{O}(-3) \otimes \mathcal{O}(d))|_C \cong \mathcal{O}(d-3)|_C. \tag{4.8}$$

Recall $\mathcal{O}(1)$ corresponds to a hyperplane H in the sense that there exists a global section s that vanish at H of order 1. Since a hyperplane is described by a homogeneous monomial of degree 1, Bézout's theorem says that $\#(C \cap H) = d$. This says exactly

$$\deg(\mathcal{O}(1)|_C) = d, \tag{4.9}$$

which implies

$$\deg \mathcal{K}_C = \deg(\mathcal{O}(d-3)|_C) = d(d-3), \tag{4.10}$$

since one would need to count each intersection $(d-3)$ times. A nonzero meromorphic section s of $\mathcal{K}_C$ is simply a meromorphic 1-form. Then by definition, $\operatorname{div} s = \operatorname{div} C$ is a canonical divisor; therefore

$$\deg \mathcal{K}_C = \deg \operatorname{div} C = -\chi(C) = d(3-d). \tag{4.11}$$

4.3. Genus of an algebraic curve defined by two transverse algebraic curves. Let C be an algebraic curve in $\mathbb{CP}^3$ cut out *transversely* by two homogeneous polynomials C_1 and C_2 of degrees d_1 and d_2. We know $\mathcal{K}_{\mathbb{CP}^3} \cong \mathcal{O}(-3-1) \cong \mathcal{O}(-4)$, and $\mathcal{O}_{\mathbb{CP}^3}(C_1) = \mathcal{O}(d_1)$. As in the previous section, applying Theorem 4.2 gives

$$\mathcal{K}_{C_1} \cong (\mathcal{O}(-4) \otimes \mathcal{O}(d_1))|_{C_1} \cong \mathcal{O}(d_1 - 4)|_{C_1}. \tag{4.12}$$

Since $\mathcal{O}_{\mathbb{CP}^3 \cap C_1}(C_2) \cong \mathcal{O}_{\mathbb{CP}^3 \cap C_1}(d_2)$, applying Theorem 4.2 again gives

$$\mathcal{K}_{C_1 \cap C_2} \cong (\mathcal{O}(d_1 - 4)|_{C_1} \otimes \mathcal{O}(d_2))|_{C_1 \cap C_2} \cong \mathcal{O}(d_1 + d_2 - 4)|_{C_1 \cap C_2}. \tag{4.13}$$

As in the previous section (or by Bézout's theorem), for each restriction we need to multiply the order by the degree of the algebraic curve. Therefore

$$\deg \mathcal{K}_{C_1 \cap C_2} = d_1 \cdot d_2 \cdot (d_1 + d_2 - 4). \tag{4.14}$$

On the other hand, we know the degree of a canonical divisor is always

$$\deg \mathcal{K}_{C_1 \cap C_2} = 2g - 2. \tag{4.15}$$

Combining the two equations, we find

$$g = \frac{d_1 \cdot d_2 \cdot (d_1 + d_2 - 4) + 2}{2}. \tag{4.16}$$

4.4. Genus of an algebraic curve of bidegree (d_1, d_2). Let $C \subset \mathbb{CP}^1 \times \mathbb{CP}^1$ cut out smoothly by a polynomial $F(x_0, x_1; y_0, y_1)$ of homogeneous bidegree (d_1, d_2). Let pr_1, pr_2 be projection maps to respective components.

$$\begin{array}{c} \mathbb{CP}^1 \times \mathbb{CP}^1 \\ {}^{\mathrm{pr}_1}\swarrow \qquad \searrow {}^{\mathrm{pr}_2} \\ \mathbb{CP}^1 \qquad\qquad\qquad \mathbb{CP}^1 \end{array} \tag{4.17}$$

Denote $M = \mathbb{CP}^1 \times \mathbb{CP}^1$. The tangent bundle is then the direct sum of pullback bundles

$$TM \cong \mathrm{pr}_1^* T\mathbb{CP}^1 \oplus \mathrm{pr}_1^* T\mathbb{CP}^1. \tag{4.18}$$

Taking the duals and then the determinant of both side of the equation, we obtain

$$\mathcal{K}_M \cong \det(\mathrm{pr}_1^* T^*\mathbb{CP}^1) \otimes \det(\mathrm{pr}_2^* T^*\mathbb{CP}^1) \cong \mathrm{pr}_1^* \mathcal{K}_{\mathbb{CP}^1} \otimes \mathrm{pr}_2^* \mathcal{K}_{\mathbb{CP}^1}. \tag{4.19}$$

By Theorem 3.3, we have $\mathcal{K}_{\mathbb{CP}^1} \cong \mathcal{O}(-1-1) \cong \mathcal{O}(-2)$. Then

$$\mathcal{K}_M \cong \mathrm{pr}_1^* \mathcal{O}(-2) \otimes \mathrm{pr}_2^* \mathcal{O}(-2) =: \mathcal{O}(-2, -2). \tag{4.20}$$

A homogeneous polynomial of bidegree (d_1, d_2) gives a global section of the line bundle

$$\mathcal{O}_M(C) = \mathrm{pr}_1^* \, \mathcal{O}(d_1) \otimes \mathrm{pr}_2^* \, \mathcal{O}(d_2). \tag{4.21}$$

Then by Theorem 4.2, we have

$$\mathcal{K}_C \cong (\mathcal{O}(-2, -2) \otimes \mathcal{O}_M(C))|_C. \tag{4.22}$$

Since tensor product of vector spaces commutes with pullback,

$$\mathcal{K}_C \cong \mathrm{pr}_1^* \, \mathcal{O}(d_1 - 2) \otimes \mathrm{pr}_2^* \, \mathcal{O}(d_2 - 2) =: \mathcal{O}(d_1 - 2, d_2 - 2). \tag{4.23}$$

To compute $\deg \mathcal{K}_C$, first note that

$$\deg \mathcal{K}_C = \deg \mathcal{O}(d_1 - 2, d_2 - 2) = \deg(\mathrm{pr}_1^* \, \mathcal{O}(d_1 - 2)) + \deg(\mathrm{pr}_1^* \, \mathcal{O}(d_1 - 2)). \tag{4.24}$$

We use the strategy in the last two sections – intersect with hyperplanes. Consider

$$H_1 = \{x_0 = 0\} \times \mathbb{CP}^1, \qquad H_2 = \mathbb{CP}^1 \times \{y_0 = 0\}.$$

The hyperplane H_1 defines $\mathcal{O}(1, 0)$, and the hyperplane H_2 defines $\mathcal{O}(0, 1)$. Then

$$\mathcal{O}(d_1 - 2, d_2 - 2) \cong \mathcal{O}(1, 0)^{\otimes (d_1 - 2)} \otimes \mathcal{O}(0, 1)^{\otimes (d_2 - 2)}. \tag{4.25}$$

Since C has homogeneous degree d_2 in (y_0, y_1), analogous to the previous two sections, we have

$$\deg \mathcal{O}(1, 0)|_C = \#(C \cap H_1) = d_2, \tag{4.26}$$

as we set $x_0 = 0$. Similarly, since C has homogeneous degree d_1 in (x_0, x_1),

$$\deg \mathcal{O}(0, 1)|_C = \#(C \cap H_2) = d_1. \tag{4.27}$$

Assembling the formulae above,

$$\deg \mathcal{K}_C = d_1 \cdot (d_2 - 2) + d_2 \cdot (d_1 - 2) = 2d_1 d_2 - 2d_1 - 2d_2. \tag{4.28}$$

Since C defines a canonical divisor,

$$\deg \mathcal{K}_C = 2g - 2. \tag{4.29}$$

Combining the above two equalities, one obtain

$$g = (d_1 - 1)(d_2 - 1). \tag{4.30}$$

Appendix A. Linear algebra

Lemma A.1. *Let $A\colon V \to W$ be a map of n-dimensional k-vector spaces. Then the natural map*

$$\bigwedge_{i=1}^{n} A\colon \bigwedge_{i=1}^{n} V \to \bigwedge_{i=1}^{n} W$$

is the determinant $\det(A)$.

Proof. Write the matrix A in terms of bases $\{v_i\}_{i=1}^{n}$ of V and $\{w_i\}_{i=1}^{n}$ of W. Note that

$$Av_i = (\text{the } i\text{-th column of } A) = \sum_{j=1}^{n} a_{ji} w_j.$$

Then

$$(Av_1) \wedge \cdots \wedge (Av_n) = \left(\sum_{j=1}^{n} a_{j1} w_j \right) \wedge \cdots \wedge \left(\sum_{j=1}^{n} a_{jn} w_j \right) \tag{A.1}$$

$$= \left(\sum_{\sigma \in S_n} \operatorname{sgn}(\sigma)\, a_{\sigma(1),1} \cdots a_{\sigma(n),n} \right) w_1 \wedge \cdots \wedge w_n \tag{A.2}$$

$$= \det(A)\, w_1 \wedge \cdots \wedge w_n, \tag{A.3}$$

by uniqueness of determinant and wedge product. $\qquad\square$

Appendix B. Vector bundle

Theorem B.1. *In the category of holomorphic vector bundles over X, a morphism $f\colon E \to F$ is an isomorphism if and only if f is bijective.*
Proof.

($\Longrightarrow$): Let $f\colon E \to F$ be an isomorphism of holomorphic vector bundles. Then there exists $f^{-1}\colon F \to E$ an inverse morphism. Then the map f has to be a bijection as sets.

($\Longleftarrow$): Let $f\colon E \to F$ be a bijective morphism of holomorphic vector bundles. Since f is bijective, we may take $\tilde{f}$ to be the inverse as sets. We verify that $\tilde{f}$ is indeed a morphism of holomorphic vector bundles. Since $\pi_F \circ f = \pi_E$, by construction $\pi_E \circ \tilde{f} = \pi_F$, and by linear algebra a bijective linear map between vector spaces has its inverse in **Set** linear. Then it remains to verify that $\tilde{f}$ is holomorphic $F \to E$. Over U_i in the common refinement of the respective open covers realizing local trivializations, take local trivializations Φ_i of E and Ψ_i of F:

$$\Phi_i\colon \pi_E^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^r, \tag{B.1}$$

$$\Psi_i\colon \pi_F^{-1}(U_i) \xrightarrow{\cong} U_i \times \mathbb{C}^r. \tag{B.2}$$

In local coordinates, f looks like

$$f_i\colon U_i \times \mathbb{C}^r \to U_i \times \mathbb{C}^r$$
$$(p, v) \mapsto (p, A_i(p)(v)),$$

where $A_i(p)$ is an $r \times r$ matrix of holomorphic functions depending on p. Invertibility of f tells invertibility over fibres, so $\det(A_i(p)) \neq 0$ for all p. In local coordinates, $\tilde{f}$ looks exactly like

$$\tilde{f}_i\colon U_i \times \mathbb{C}^r \to U_i \times \mathbb{C}^r$$
$$(p, w) \mapsto (p, A_i^{-1}(p)(w)),$$

and $A_i^{-1}(p)$ is holomorphic.

$\qquad\square$

Theorem B.2. *Let E and F be two holomorphic vector bundles with transition functions g_{ij}^E and g_{ij}^F respectively. Let $\{U_i\}$ be a common refinement of the respective open covers realizing the local trivializations. Then E and F are isomorphic if and only if there exists holomorphic maps $h_i \colon U_i \to \mathrm{GL}_r(\mathbb{C})$ such that on every overlap U_{ij},*

$$g_{ij}^F = h_i \circ g_{ij}^E \circ h_j^{-1}. \tag{B.3}$$

Proof. Fix U_i in the open cover. Holomorphic vector bundles E and F are isomorphic if and only if there exists a bijective morphism $\varphi \colon E \to F$. When restricted to U_i, E and F are trivialized. Then in the local coordinates, φ realized locally as φ_i defines a holomorphic map $h_i \colon U_i \to \mathrm{GL}_r(\mathbb{C})$ such that for $p \in U_i$ and $v \in \mathbb{C}^r$,

$$\varphi_i(p, v) = (x, h_i(p)(v)). \tag{B.4}$$

Since φ is bijective, h_i must have image contained in $\mathrm{GL}_r(\mathbb{C})$. Now consider a point $p \in U_i \cap U_j$. Let v_j be an element in the projection onto second factor $(\mathbb{C}^r)$ in the local trivialization defined over U_j. Since

$$v_i = g_{ij}^E(p)v_j, \tag{B.5}$$

one have

$$w_i = h_i(p)v_i = h_i(p) \circ g_{ij}^E(p)v_j. \tag{B.6}$$

Alternatively, we may first apply h_j in the U_j chart to v_j, then use transition function to move back to U_i:

$$w_i = g_{ij}^F(p) \circ h_j(p)v_j, \tag{B.7}$$

we should get the same result since φ is defined globally. Then

$$h_i(p) \circ g_{ij}^E(p) = g_{ij}^F(p) \circ h_j(p) \tag{B.8}$$

for all $p \in U_i \cap U_j$. Rearranging the terms, one obtain

$$g_{ij}^F = h_i \circ g_{ij}^E \circ h_j^{-1}. \tag{B.9}$$

One could easily go from the bottom to the top: define $\varphi \colon E \to F$ locally, then use the condition to check they glue together a global morphism. $\qquad\square$

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