FROM CLASSICAL TO MODERN: BRIDGING VARIETIES AND SCHEMES

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Abstract. For a long time, varieties were the central objects of study in classical algebraic geometry. In EGA, Grothendieck revolutionized the field by defining schemes, which had since become foundation of modern algebraic geometry. In this talk, I will begin with a brief introduction to varieties, followed by definitions and examples of schemes. Finally, I will present a fully faithful functor t from varieties over a field k to schemes over a field k. This functor gives us a rigorous way to view schemes as natural extensions of varieties.

Contents

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References and Cautions

Many, if not all, theorems and examples mentioned in this talk can be found in Hartshorne's *Algebraic Geometry*. Since this handout is mainly for the convenience of audience to navigate along this dense talk, proofs are omitted.

For the sake of time, discussions of projective varieties and projective schemes are ignored. However, one should keep in mind that when the theorem works for varieties, it doesn't restrict to only affine/quasi-affine varieties. Proj is equally important as Spec.

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Conventions and Notation

Throughout this talk, we adopt the following conventions unless stated otherwise:

- The symbol $\subset$ denotes inclusion (not necessarily proper), while $\subsetneq$ denotes proper inclusion.
- All rings are assumed to be commutative with identity $1 \neq 0$. A morphism of rings preserve the multiplicative identity 1.
- All fields are algebraically closed and of characteristic zero. For your convenience, you may assume such a field $k = \mathbb{C}$.
- All maps between topological spaces are continuous.
- Closures of a set S is denoted as $\overline{S}$.

1. Varieties

1.1. **Affine Varieties.** We define the *affine n-space* to be $\mathbb{A}^n$ over a field k to be simply k^n. Let $A = k[x_1, \ldots, x_n]$, the polynomial ring of n variables. One can view A as a collection of functions $\mathbb{A}^n \to k$ by evaluation.

For $f \in A$ a polynomial, we define $Z(f)$ to be the zero set of f. For a subset $T \subset A$, we define $Z(T)$ to be the common zeroes of all elements of T. Explicitly,

$$Z(T) = \{P \in \mathbb{A}^n \mid f(P) = 0 \text{ for all } f \in T\}.$$

In particular, let $I \subset A$ be the ideal generated by elements of T, then $Z(T) = Z(I)$. Sets of the form $Z(T)$ are called *algebraic sets*. We define a topology on $\mathbb{A}^n$ by declaring the closed sets to be all algebraic sets $Z(T)$. This defines a topology, as one may check $Z(T_1) \cup Z(T_2) = Z(T_1 T_2)$ and $\bigcap Z(T_\alpha) = Z(\bigcup T_\alpha)$, $\varnothing = Z(1)$, and $\mathbb{A}^n = Z(0)$. This topology is called the *Zariski's topology* .

Example 1.1. Consider $\mathbb{A}^1$, where $A = k[x]$. A is a PID, and any element $f \in A$ could be factored as

$$f(x) = c(x - a_1)(x - a_2) \cdots (x - a_n).$$

Therefore, $Z(f) = \{a_1, \ldots, a_n\}$. Any polynomial has finitely many zeroes, so the Zariski's topology coincides with the cofinite topology. In particular, this topology is not Hausdorff, and the space is irreducible (in the topological sense).

Definition 1.2 (Affine Variety)**.** An *affine variety* Y is an irredicuble subset of $\mathbb{A}^k$, considered in the Zariski's topology. A *quasi-affine variety* is an open subset of an affine variety.

Theorem 1.3. *There is a one-to-one correspondence between algebraic sets in $\mathbb{A}^n$ and radical ideals in A. This correspondence is established as follows. For an algebraic set Y, consider $I(Y)$, the elements in A that vanish at all points in Y. For a radical ideal $\mathfrak{a}$ of A, consider $Z(\mathfrak{a})$. Furthermore, an algebraic set is irreducible if and only if it corresponds to a prime ideal.*

The preceding theorem is a consequence of Hilbert's Nullstellensatz, which tells us for a radical $\mathfrak{a} \subset A$, $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}} = \mathfrak{a}$.

Theorem 1.4 (Hilbert's Nullstellensatz)**.** *Let $\mathfrak{a}$ be an ideal in A. Let $f \in A$ be a polynomial that vanishes at all points of $Z(\mathfrak{a})$. Then $f^r \in \mathfrak{a}$ for some integer $r > 0$.*

Example 1.5. Consider $A = k[x_1, \ldots, x_n]$. Maximal ideals of A corresponds to minimal irreducible algebraic sets, which then are described by individual points in $\mathbb{A}^n$.

1.2. **Regular Functions.**

Definition 1.6 (Regular Functions)**.** Let Y be an affine variety. A function $f \colon Y \to k$ is *regular at P* if there exists an open set $U \subset Y$, and $g, h \in A$ which h doesn't vanish on U, such that $f = g/h$ on U. A function f is *regular* if it is regular at all $P \in Y$.

Theorem 1.7 (Continuity of Regular Functions)**.** *Regular functions are continuous, given the Zariski's topology and identifying k as $\mathbb{A}^1$.*

1.3. Varieties over k.

Definition 1.8 (Varieties over k)**.** A *variety* over k is any affine/quasi-affine, projective/quasi-projective variety over the field k.

Definition 1.9 (Morphisms of Varieties)**.** Let X, Y be varieties over k. $\varphi\colon X \to Y$ a *morphism* of varieties is a continuous map such that for all open $V \subset Y$, $f\colon V \to k$ a regular function pulls back to a regular function on $\varphi^{-1}(V) \subset X$.

We then define $\mathcal{O}(Y)$ to be the *ring of regular functions* on Y, $\mathcal{O}_{P,Y}$ (or simply $\mathcal{O}_P$) to be the *ring of germs of regular functions* on Y near P. *Germs* are equivalence classes of functions where two functions represent the same germ if they agree on the intersection. Explicitly, elements in the germs have the form $\langle U, f\rangle$, and $\langle U, f\rangle = \langle V, g\rangle$ if and only if $f|_{U\cap V} = g|_{U\cap V}$. Note that $\mathcal{O}_P$ is a local ring – its unique maximal ideal is given by $\mathfrak{m}_P$ the germs that vanish at P, since if $f(P) \neq 0$ then $1/f$ is regular near P. The residue field $\mathcal{O}_P/\mathfrak{m}_P$ is therefore isomorphic to k.
Let $K(Y)$ be the function field: it is the equivalence class of regular functions on Y with the same equivalence relation as before. $K(Y)$ is not the same as $\mathcal{O}_P$, as in $K(Y)$ we allow all open sets, meanwhile in $\mathcal{O}_P$ we only consider open sets that contain P.

Theorem 1.10. *Let $Y \subset \mathbb{A}^n$ an affine variety, with coordinate ring $A(Y)$.*

1. $\mathcal{O}(Y) \cong A(Y)$.
2. *Let $\mathfrak{m}_P \subset A(Y)$ be the ideal of functions vanishing at P. Then $P \to \mathfrak{m}_P$ gives a correspondence*
$$\{\textit{points of } Y\} \longleftrightarrow \{\textit{maximal ideals of } A(Y)\}\,.$$
3. $\mathcal{O}_P \cong A(Y)_{\mathfrak{m}_P}$.
4. $K(Y) \cong \mathrm{Frac}(A(Y))$.

2. Schemes

To build our way to schemes, we need a series of definitions.

Definition 2.1 (Presheaf)**.** Let X be a topological space. A *presheaf* $\mathcal{F}$ of abelian groups (or your favorite algebraic structure) on X consists of the data

- for every open subset $U \subset X$, an abelian group (or your favorite algebraic structure) $\mathcal{F}(U)$,
- for every inclusion $V \subset U$ of open sets of X, a morphism of abelian groups (or your favorite algebraic structure) $\rho_{UV} : \mathcal{F}(U) \to \mathcal{F}(V)$,

subject to the following conditions

- $\mathcal{F}(\varnothing) = 0$,
- ρ_{UU} is the identity map, and
- if $W \subset V \subset U$ open sets, then $\rho_{UW} = \rho_{VW} \circ \rho_{UV}$. This condition is also called as the *cocycle condition.*

In short, a presheaf of abelian groups is a *contravariant* functor from **Top** to **Ab**. We also use the notation $\Gamma(U, \mathcal{F})$ and $\mathcal{F}(U)$ interchangably. For $s \in \mathcal{F}(U)$, $s|_V$ means $\rho_{UV}(s)$. The maps ρ_{UV} are called *restriction maps.*

Definition 2.2 (Sheaf)**.** A *sheaf* $\mathcal{F}$ on X is a presheaf that satisfy the following additional properties:

- if U is an open set, and $\{V_i\}$ an open covering of U, and if $s \in \mathcal{F}(U)$ is an element such that $s|_{V_i} = 0$ for all i, then $s = 0$;
- if $s_i \in \mathcal{F}(V_i)$ for each i, such that for all i and j, $s_i|_{V_i\cap V_j} = s_j|_{V_j\cap V_i}$, then there exists $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. The above constraint tells us that this s is unique.

Example 2.3. Consider $\mathbb{R}^n$ with the standard topology. Continuous functions on $\mathbb{R}^n$ form a sheaf. Explicitly, take $\mathcal{F}(U)$ to be the continuous functions defined on the open $U \subset \mathbb{R}^n$.

Definition 2.4 (Stalk)**.** Let $\mathcal{F}$ be a presheaf on X. The *stalk* $\mathcal{F}_P$ at $P \in X$ is the direct limit of the groups $\mathcal{F}(U)$ for all open sets U containing P, via the restriction maps ρ. Explicitly, the elements in the stalk $\mathcal{F}_P$ are of the form $[(U, s)]$ where $s \in \mathcal{F}(U)$, and $[(U, s)] = [(V, t)]$ if and only if there exists an open $W \subset U \cap V$ such that $s|_W = t|_W$.

Definition 2.5 (Morphisms between presheaves)**.** Let $\mathcal{F}$ and $\mathcal{G}$ be presheaves on X. A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is a collection of morphisms of your favorite algebraic structures such that whenever $U \supset V$, the following diagram commutes.

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \big\downarrow{\scriptstyle \rho^{\mathcal{F}}_{UV}} & & \big\downarrow{\scriptstyle \rho^{\mathcal{G}}_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

In other words, a morphism between presheaves is a *natural transformation* from $\mathcal{F}$ to $\mathcal{G}$.

Theorem 2.6. *A morphism $\varphi\colon \mathcal{F} \to \mathcal{G}$ is injective (resp. surjective) if and only if the induced map $\varphi_P\colon \mathcal{F}_P \to \mathcal{G}_P$ on stalks are injective (resp. surjective) for all $P \in X$.*

Definition 2.7 (Direct image and inverse image sheaf)**.** Let $f\colon X \to Y$ be a map of topological spaces. For any sheaf $\mathcal{F}$, define the *direct image sheaf* $f_*\mathcal{F}$ on Y by $(f_*\mathcal{F})(V) = \mathcal{F}(f^{-1}(V))$ for any open set $V \subset Y$. For any sheaf $\mathcal{G}$ on Y, we define the *inverse image sheaf* $f^{-1}\mathcal{G}$ to be the sheafification of $U \mapsto \varinjlim_{V \supset f(U)} \mathcal{G}(V)$, where U is open in X and limit taken over all open subsets V containing $f(U)$.

2.1. **Affine Schemes.** Consider a ring A, and $\operatorname{Spec} A$ the set of all prime ideals of A. Let $\mathfrak{a}$ be an ideal of A and define $V(\mathfrak{a}) \subset \operatorname{Spec} A$ to be the set of all prime ideals which contain $\mathfrak{a}$. Define a topology on $\operatorname{Spec} A$ by declaring the closed sets to be $V(\mathfrak{a})$'s, for any ideal $\mathfrak{a} \subset A$. One observe that

- If $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals, then $V(\mathfrak{a}\mathfrak{b}) = V(\mathfrak{a}) \cup V(\mathfrak{b})$.
- If $\{\mathfrak{a}_i\}$ is any set of ideals of A, then $V(\sum \mathfrak{a}_i) = \bigcap V(\mathfrak{a}_i)$.
- If $\mathfrak{a}, \mathfrak{b} \subset A$ two ideals, $V(\mathfrak{a}) \subset V(\mathfrak{b})$ if and only if $\sqrt{\mathfrak{a}} \supset \sqrt{\mathfrak{b}}$.

Also, $V(A) = \varnothing$, and $V((0)) = \operatorname{Spec} A$. This confirms that it indeed gives us a topology. Now, we define a sheaf of rings $\mathcal{O}$ on $\operatorname{Spec} A$. For an open subset $U \subset \operatorname{Spec} A$, Define $\mathcal{O}(U)$ to be the set of functions $s\colon U \to \bigsqcup_{\mathfrak{p} \in U} A_{\mathfrak{p}}$ such that $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for each $\mathfrak{p}$, such that s is locally a quotient of elements of A. Explicitly, for each $\mathfrak{p} \in U$, there exists a neighbourhood V of $\mathfrak{p}$, contained in U, and $a, f \in A$, such that for each $\mathfrak{q} \in V$, $f \notin \mathfrak{q}$, and $s(\mathfrak{q}) = a/f$ in $A_{\mathfrak{q}}$.

Definition 2.8 (Spectrum of Rings)**.** Let A be a ring. The *spectrum* of A is the pair consisting of the topological space $\operatorname{Spec} A$ together with the sheaf of rings $\mathcal{O}$ defined above. We write $(\operatorname{Spec} A, \mathcal{O})$.

Let $D(f)$ to be the complement of $V((f))$. Then such open sets form a base for the topology on $\operatorname{Spec} A$.

Theorem 2.9. *Let A be a ring, and* $(\operatorname{Spec} A, \mathcal{O})$ *its spectrum.*

1. *For any $\mathfrak{p} \in \operatorname{Spec} A$, the stalk $\mathcal{O}_{\mathfrak{p}}$ of the sheaf $\mathcal{O}$ is isomorphic to the local ring $A_{\mathfrak{p}}$.*
2. *For any element $f \in A$, the ring $\mathcal{O}(D(f))$ is isomorphic to the localized ring A_f.*
3. *In particular, $\Gamma(\operatorname{Spec} A, \mathcal{O}) \cong A$.*

One may spot some similarities with Theorem 1.10. In Theorem 1.5, we saw the correspondence between points and maximal ideals of $k[x]$ and points in $\mathbb{A}^1$. Points, in the usual sense, are described by $\operatorname{MaxSpec}(k[x])$, the collection of all *maximal* ideals of $k[x]$. Here we look at all *prime* ideals, not only maximal ones. This gives us an idea that the space $\operatorname{Spec} A$ should be larger than our usual space of points. In fact, $k[x]$ is a PID, so nonzero prime ideals are maximal.

Definition 2.10 (Ringed Spaces)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ consisting of a topological space X and a sheaf of rings $\mathcal{O}_X$ on X. A morphism of ringed spaces from $(X, \mathcal{O}_X)$ to $(Y, \mathcal{O}_Y)$ is a pair $(f, f^\sharp)$ of a continuous map $f\colon X \to Y$ and a map $f^\sharp\colon \mathcal{O}_Y \to f_*\mathcal{O}_X$ of sheaves of rings on Y.

Definition 2.11 (Locally Ringed Spaces)**.** A ringed space $(X, \mathcal{O}_X)$ is a *locally ringed space* if for each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is a local ring. A morphism of locally ringed spaces is a morphism $(f, f^\sharp)$ of rined spaces, such that for each point $P \in X$, the induced map of local rings $f^\sharp_P\colon \mathcal{O}_{Y,f(P)} \to O_{X,P}$ is a *local homomorphism* of local rings, i.e., inverse image of the maximal ideal of $\mathcal{O}_{X,P}$ is exactly the maximal ideal of $\mathcal{O}_{Y,f(P)}$.

Example 2.12. Consider the spaces $X, Y = \mathbb{R}^n$ with the standard topology. Let $\mathcal{O}_X, \mathcal{O}_Y$ be the sheaf of smooth functions on X, Y respectively. Let $(f, f^\sharp)$ be a morphism of locally ringed spaces between $(X, \mathcal{O}_X)$ and $(Y, \mathcal{O}_Y)$. For each point $P \in X$, the stalk $\mathcal{O}_{X,P}$ is the collection of germs of smooth functions around P; similarly, the stalk $\mathcal{O}_{Y,f(P)}$ is the collection of germs of smooth functions around $f(P)$. They are both local rings, with the unique maximal ideal to be elements which vanish at P, or $f(P)$, respectively. Then the local homomorphism condition simply says that the inverse image of the germs that vanish at P (under $f^\sharp_P$) is exactly the set of germs that vanish at $f(P)$. This suggests that $f^\sharp_P$ is closely related to the pullback of functions $Y \to \mathbb{R}$ to $X \to Y \to \mathbb{R}$, justifying our choice of notation $(f, f^\sharp)$.
In fact, in the above case, we can conclude something stronger: the map $f^\sharp$ is actually the pullback via f! We fix $U \subset Y$ an open subset. Then we have a morphism of rings

$$f^\sharp(U)\colon \mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U) = \mathcal{O}_x(f^{-1}(U)).$$

Let $\phi \in \mathcal{O}_Y(U)$, i.e., ϕ a smooth function defined on the open $U \subset Y$. We want to calculate $f^\sharp(U)(\phi)$. Fix $P \in f^{-1}(U)$, and let $c = \phi(f(P))$. Treat c as a constant function in $\mathcal{O}_Y(U)$, we see $(\phi - c) \in \mathfrak{m}_{f(P)}$, i.e., $\phi - c$ vanishes at $f(P)$. Now, the local morphism condition says that for

$$f^\sharp_P\colon \mathcal{O}_{Y,f(P)} \to \mathcal{O}_{X,P}$$

we have $(f^\sharp_P)^{-1}(\mathfrak{m}_P) = \mathfrak{m}_{f(P)}$. Since $\phi - c$ vanishes at $f(P)$, it represents a germ in $\mathfrak{m}_{f(P)}$, so $f^\sharp_P(\phi - c) = f^\sharp_P(\phi) - f^\sharp_P(c)$ is represented by a germ in $\mathfrak{m}_P$. Notice that $\mathcal{O}_{Y,f(P)}$ and $\mathcal{O}_{X,P}$ both contain a natural copy of $\mathbb{R}$, and in our convention a ring homomorphism preserves 1. Therefore our map $f^\sharp_P$ is a map of $\mathbb{R}$-algebras, i.e., $f^\sharp_P(c) = cf^\sharp_P(1) = c = \phi(f(P))$. This requires

$$f^\sharp_P(\phi)(P) = f^\sharp_P(c)(P) = \phi(f(P)),$$

so $f^\sharp(U)$ indeed maps ϕ to the pullback $\phi \circ f$!

Example 2.13. Extending the preceding example, we may view a smooth manifold M as a locally ringed space $(M, \mathcal{C}^\infty_M)$. For any open set U, we define the ring of sections $\mathcal{C}^\infty_M(U)$ to be the ring of smooth functions $f\colon U \to \mathbb{R}$. Since smoothness is a local property defined over charts, this indeed forms a sheaf, and for every point $P \in M$, there is a neighbourhood isomorphic (as locally ringed spaces) to $(\mathbb{R}^n, \mathcal{C}^\infty_{\mathbb{R}^n})$. It is called the local model of a manifold M. The map of sheaves $f^\sharp$ here is therefore the pullback.

Theorem 2.14. *Let A, B be rings.*

1. $(\operatorname{Spec} A, \mathcal{O})$ *is a locally ringed space.*
2. $\varphi\colon A \to B$ *a ring homomorphism, then φ induces a natural morphism of locally ringed spaces*
$$(f, f^\sharp)\colon (\operatorname{Spec} B, \mathcal{O}_{\operatorname{Spec} B}) \to (\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A}).$$
3. *Any morphism of locally ringed spaces from* $\operatorname{Spec} B$ *to* $\operatorname{Spec} A$ *is induced by a ring homomorphism* $\varphi\colon A \to B$.

Definition 2.15 (Schemes)**.** An *affine scheme* is a locally ringed space $(X, \mathcal{O}_X)$ which is isomorphic (as a locally ringed space) to the spectrum of some ring. A scheme is a locally ringed space $(X, \mathcal{O}_X)$ in which every point has an open neighbourhood U such that the topological space U, together with the restricted sheaf $\mathcal{O}_X|_U$, is an affine scheme. We call X the underlying topological space of the scheme $(X, \mathcal{O}_X)$, and $\mathcal{O}_X$ its structure sheaf. A morphism of schemes is a morphism of locally ringed spaces.

By abuse of notation, we would write X to denote $(X, \mathcal{O}_X)$. The underlying topological space is denoted by $\operatorname{sp}(X)$. In short, a scheme is a locally ringed space with local model $(\operatorname{Spec} A, \mathcal{O}_{\operatorname{Spec} A})$ for some ring A.

Definition 2.16 (Scheme over S)**.** A scheme X over another scheme S is itself together with a morphism of schemes $X \to S$. A morphism $f\colon X \to Y$ between schemes over S is a morphism of schemes compatible with given morphisms to S. In other words, the below diagram commutes.

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "p_X"'] & & Y \arrow[dl, "p_Y"] \\ & S & \end{tikzcd}$$

2.2. **Scheme over k.** A *scheme over k* is a scheme X with a morphism $(f, f^\sharp)\colon (X, \mathcal{O}_X) \to (\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$. Notice that $\operatorname{Spec} k$ only has one point, (0) the zero ideal, so the topological map is the trivial map. The space $\operatorname{Spec} k$ then only has one nonempty open set, which is itself $\operatorname{Spec} k = \{(0)\}$. Consider the map of sheaves $f^\sharp\colon \mathcal{O}_{\operatorname{Spec} k} \to f_*\mathcal{O}_X$. Then

$$f^\sharp(\{(0)\})\colon \mathcal{O}_{\operatorname{Spec} k}(\{(0)\}) \to \mathcal{O}_X(f^{-1}(\{(0)\})) = \mathcal{O}_X(X).$$

The domain is isomorphic to k, and the codomain is the ring of global sections $\mathcal{O}_X(X)$; so it says $\mathcal{O}_X(X)$ is a k-algebra. The local homomorphism condition says that for any $x \in X$,

$$f_x^\sharp\colon k_{(0)} \cong k \to \mathcal{O}_{X,x}$$

is a map of local rings. Then the inverse image of the maximal ideal of the stalk $\mathcal{O}_{X,x}$ is the maximal ideal of k, which is the zero ideal (0).

3. Construction of the functor t

Theorem 3.1. *Let k be an algebraically closed field. There is a natural fully faithful functor*

$$t\colon \mathbf{Var}(k) \to \mathbf{Sch}(k)$$

from the category of varieties over k to schemes over k. For any variety V, its topological space is homeomorphic to the set of closed points of $\operatorname{sp}(t(V))$*, and its sheaf of regular functions is obtained by restricting the structure sheaf of* $t(V)$ *via this homeomorphism.*

Before constructing t, I want to discuss some intuitions and significance of it. First of all, why *closed* points? We defined a topology on $\operatorname{Spec} A$ for a ring A to have closed sets of the form $V(\mathfrak{a})$ where $\mathfrak{a}$ is an ideal in A. For a maximal ideal $\mathfrak{m}$, $V(\mathfrak{m}) = \{\mathfrak{m}\}$. In general, for any $\mathfrak{p} \in \operatorname{Spec} A$, the closure $\overline{\{\mathfrak{p}\}} = V(\mathfrak{p})$. Therefore the closed points are exactly maximal ideals in $\operatorname{Spec} A$. In fact, one may compare Theorem 1.10 and Theorem 2.9 to expect the statement on relationship of sheaves. Furthermore, of t gives a one-to-one correspondence between the open sets in V and $t(V)$, even though their topologies are not the same.
For a detailed construction, one can refer to Hartshorne's Proposition 2.6, in Chapter 2. Below I give the brief ideas of the construction.

3.1. **Sketch of Construction.** Let X be a topological spacce. Define $t(X)$ to be the set of nonempty irreducible closed subsets of X. We observe

- if $Y \subset X$ is closed, then $t(Y) \subset t(X)$, by taking the intersection with Y;
- $t(Y_1 \cup Y_2) = t(Y_1) \cup t(Y_2)$, and $t(\bigcap Y_i) = \bigcap t(Y_i)$.

Therefore, we may define a topology on $t(X)$ by taking as closed sets the subsets of the form $t(Y)$, where $Y \subset X$ a closed subset. Define $\alpha\colon X \to t(X)$ by $P \mapsto \overline{\{P\}}$. The map α induces a correspondence between open sets in X and open sets in $t(X)$. Then there is a natural structure sheaf $\mathcal{O}_{t(X)}$ on $t(X)$ using this correspondence.
Now, an affine variety $Z(I) \subset \mathbb{A}^n$ has a corresponding affine scheme $\operatorname{Spec} A(Z(I)) = \operatorname{Spec} k[x_1, \ldots, x_n]/I$. For $X = Z(I)$ and $Y = Z(J)$ affine varieties, we have

$$\operatorname{Hom}_{\mathbf{Var}(k)}(X, Y) \cong \operatorname{Hom}_{k-\mathrm{alg}}(\mathcal{O}_Y(Y), \mathcal{O}_X(X)),$$

which then by the equivalence of **AffSch** with $\mathbf{CRing}^{\mathrm{op}}$ described in Theorem 2.14, we have

$$\mathrm{Hom}_{\mathbf{Var}(k)}(X,Y) \cong \mathrm{Hom}_{\mathbf{Sch}(k)}(\operatorname{Spec}\mathcal{O}_X(X), \operatorname{Spec}\mathcal{O}_Y(Y)).$$