Problem Set DSGE

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Exercise 1. For the Brock and Mirman model, find the value of A in the policy function.

Solution. In the Brock and Mirman model, households solve the following dynamic program

$$V\left(K_t, z_t\right) = \max_{K_{t+1}} \ln\left(e^{z_t} K_t^{\alpha} - K_{t+1}\right) + \beta E_t \left\{V\left(K_{t+1}, z_{t+1}\right)\right\}$$

which yields the Euler equation

$$\frac{1}{e^{z_t} K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\}.$$

We guess that the policy function takes the following form

$$K_{t+1} = A e^{z_t} K_t^{\alpha}.$$

Plugging it into the Euler equation we obtain

$$\begin{aligned}\frac{1}{e^{z_t} K_t^{\alpha} - A e^{z_t} K_t^{\alpha}} &= \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - A e^{z_{t+1}} K_{t+1}^{\alpha}} \right\} \\ &= \frac{\alpha\beta}{1-A} E_t \left\{ \frac{1}{A e^{z_t} K_t^{\alpha}} \right\} = \frac{\alpha\beta}{1-A} \frac{1}{A e^{z_t} K_t^{\alpha}}\end{aligned}$$

which gives rise to

$$A = \alpha\beta.$$

Exercise 2. For the model in section 3 of the notes consider the following functional forms:

$$u\left(c_t, \ell_t\right) = \ln c_t + a \ln\left(1 - \ell_t\right)$$

$$F\left(K_t, L_t, z_t\right) = e^{z_t} K_t^{\alpha} L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Can you use the same tricks as in Exercise 1 to solve for the policy function in this case? Why or why not?

Solution. The characterizing equations are:

$$c_t = (1-\tau)\left[w_t\ell_t + (r_t-\delta)\,k_t\right] + k_t + T_t - k_{t+1}$$

$$\frac{1}{c_t} = \beta E_t\left\{\frac{1}{c_{t+1}}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\}$$

$$\frac{a}{1-l_t} = \frac{1}{c_t}w_t(1-\tau)$$

$$r_t = \alpha e^{z_t}k_t^{\alpha-1}l_t^{1-\alpha}$$

$$w_t = (1-\alpha)\,e^{z_t}k_t^{\alpha}l_t^{-\alpha}$$

$$\tau\left[w_t\ell_t + (r_t-\delta)\,k_t\right] = T_t$$

$$z_t = (1-\rho_z)\,\overline{z} + \rho_z z_{t-1} + \epsilon_t^z; \quad \epsilon_t^z \sim \text{ i.i.d. } \left(0,\sigma_z^2\right)$$

Now we cannot solve the problem via a social planner's problem, because the competitive equilibrium under a distortionary tax may not be socially optimal.

Exercise 3. For the model in section 3 consider the following functional forms:

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\ln\left(1-\ell_t\right)$$

$$F\left(K_t,L_t,z_t\right) = e^{z_t}K_t^{\alpha}L_t^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms.

Solution. The intertemporal and intratemporal Euler equations for household now become

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\}$$

$$\frac{a}{1-l_t} = c_t^{-\gamma}w_t(1-\tau)$$

and other characterizing equations remain unchanged.

Exercise 4. For the model in section 3 consider the following functional forms:

$$u\left(c_t,\ell_t\right) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi}$$

$$F(K_t, L_t, z_t) = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}}$$

Write out all the characterizing equations for the model using these functional forms.

Solution. Now the household Euler equations become:

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\}$$

$$a(1-l_t)^{-\xi} = c_t^{-\gamma} w_t (1-\tau)$$

and firm's first order conditions become:

$$r_t = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}-1} \alpha K_t^{\eta-1}$$

$$w_t = e^{z_t}\left[\alpha K_t^{\eta} + (1-\alpha)L_t^{\eta}\right]^{\frac{1}{\eta}-1} (1-\alpha) L_t^{\eta-1}$$

while the household budget constraint, government budget balance, and law of motion for technology shocks remain unchanged.

Exercise 5. For the model in section 3 abstract from the labor/leisure decision and consider the following functional forms:

$$u(c_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma}$$

$$F(K_t, L_t, z_t) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha}$$

Write out all the characterizing equations for the model using these functional forms. Assume $l_t = 1$. Write out the steady state versions of these equations. Solve algebraically for the steady state value of k as a function of the steady state value of z and the parameters of the model. Numerically solve for the steady state values of all variables using the following parameter values: $\gamma = 2.5, \beta = .98, \alpha = .40, \delta = .10, \bar{z} = 0$ and $\tau = .05$. Also solve for the steady state values of output and investment. Compare these values with the ones implied by the algebraic solution.

Solution. Now the characterizing equations are:

$$c_t = (1-\tau)\left[w_t + (r_t - \delta)k_t\right] + k_t + T_t - k_{t+1}$$

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1}-\delta)(1-\tau)+1\right]\right\}$$

$$r_t = \alpha k_t^{\alpha-1} e^{z_t(1-\alpha)}$$

$$w_t = (1-\alpha) k_t^{\alpha} e^{z_t(1-\alpha)}$$

$$\tau\left[w_t+\left(r_t-\delta\right)k_t\right]=T_t$$

$$z_t=\left(1-\rho_z\right)\overline{z}+\rho_z z_{t-1}+\epsilon_t^z;\quad \epsilon_t^z\sim\ \text{i.i.d.}\ \left(0,\sigma_z^2\right).$$

In the steady state, the characterizing equations become:

$$\overline{c}=\left(1-\tau\right)\left[\overline{w}+\left(\overline{r}-\delta\right)\overline{k}\right]+\overline{T}$$

$$1=\beta\left[\left(\overline{r}-\delta\right)\left(1-\tau\right)+1\right]$$

$$\overline{r}=\alpha\overline{k}^{\alpha-1}e^{\overline{z}(1-\alpha)}$$

$$\overline{w}=\left(1-\alpha\right)\overline{k}^{\alpha}e^{\overline{z}(1-\alpha)}$$

$$\tau\left[\overline{w}+\left(\overline{r}-\delta\right)\overline{k}\right]=\overline{T}.$$

This is a system of five equations in five unknowns $(\overline{c},\overline{k},\overline{w},\overline{r},\overline{T})$. The analytical solution is

$$\overline{r}=\frac{1}{1-\tau}\left(\frac{1}{\beta}-1\right)+\delta$$

$$\overline{k}=\left[\frac{\overline{r}}{\alpha e^{\overline{z}(1-\alpha)}}\right]^{\frac{1}{\alpha-1}}=\left[\frac{\frac{1}{1-\tau}\left(\frac{1}{\beta}-1\right)+\delta}{\alpha e^{\overline{z}(1-\alpha)}}\right]^{\frac{1}{\alpha-1}}.$$

For the numerical solution, see the jupyter notebook PS_DSGE.ipynb. The numerical solution indeed coincides with the analytical solution.

Exercise 6. For the model in section 3 consider the following functional forms:

$$u\left(c_t,\ell_t\right)=\frac{c_t^{1-\gamma}-1}{1-\gamma}+a\frac{\left(1-\ell_t\right)^{1-\xi}-1}{1-\xi}$$

$$F\left(K_t,L_t,z_t\right)=K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha}$$

Repeat the above exercise.

Solution. The steady state characterizing equations are:

$$\overline{c}=\left(1-\tau\right)\left[\overline{w}\overline{l}+\left(\overline{r}-\delta\right)\overline{k}\right]+\overline{T}$$

$$1=\beta\left[\left(\overline{r}-\delta\right)\left(1-\tau\right)+1\right]$$

$$a\left(1-\overline{l}\right)^{-\xi}=\overline{c}^{-\gamma}\overline{w}(1-\tau)$$

$$\overline{r}=\alpha\overline{k}^{\alpha-1}\overline{l}^{1-\alpha}e^{\overline{z}(1-\alpha)}$$

$$\overline{w} = (1-\alpha)\,\overline{k}^{\alpha}\overline{l}^{-\alpha}e^{\overline{z}(1-\alpha)}$$

$$\tau\left[\overline{w}\overline{l} + (\overline{r}-\delta)\,\overline{k}\right] = \overline{T}.$$

Now this is a system of six equations in six unknowns $(\overline{c}, \overline{l}, \overline{k}, \overline{w}, \overline{r}, \overline{T})$. The analytical solution for steady state capital is

$$\overline{k} = \left[\frac{\overline{r}}{\alpha\left(\overline{l}e^{\overline{z}}\right)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}} = \left[\frac{\frac{1}{1-\tau}\left(\frac{1}{\beta}-1\right)+\delta}{\alpha\left(\overline{l}e^{\overline{z}}\right)^{1-\alpha}}\right]^{\frac{1}{\alpha-1}}.$$

The numerical solution in PS_DSGE.ipynb indeed coincides with the analytical solution.

Exercise 7. For the steady state of the baseline tax model in section 3.7.3 use numerical differentiation to solve for the full set of comparative statics and sign them where possible. Find $\frac{\partial y}{\partial x}$ for $y \in \{\overline{k}, \overline{\ell}, \overline{y}, \overline{w}, \overline{r}, \overline{T}, \overline{i}, \overline{c}\}$ and $x \in \{\alpha, \beta, \gamma, \delta, \xi, \tau, a, \overline{z}\}$. Using the same parameter values as in Exercise 6, solve for the numerical values of the comparative statics.

Solution. See PS_DSGE.ipynb.