Problem Set Linearization

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Exercise 1. For the Brock and Mirman model in use Uhlig's notation to analytically find the values of the following matrices: F, G, H, L, M, N as functions of the parameters. Given these find the values of P & Q, also as functions of the parameters. Imposing our calibrated parameter values, plot the three-dimensional surface plot for the policy function $K' = \Phi(K, z)$. Compare this with the closed form solution and the solution you found using the grid search method in exercise 8 from the DSGE chapter.

Solution. The Euler equation from the Brock and Mirman model is

$$\frac{1}{e^{z_t}K_t^{\alpha} - K_{t+1}} = \beta E_t \left\{ \frac{\alpha e^{z_{t+1}} K_{t+1}^{\alpha-1}}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\},$$

which can be rewritten as

$$E_t \left\{ \frac{\beta \alpha e^{z_{t+1}} K_{t+1}^{\alpha-1} \left(e^{z_t} K_t^{\alpha} - K_{t+1}\right)}{e^{z_{t+1}} K_{t+1}^{\alpha} - K_{t+2}} \right\} = 1.$$

Denote $\mathcal{I} = \{K_{t+2}, K_{t+1}, K_t, z_{t+1}, z_t\}$. In the steady state with $\overline{z} = 0$, we have

$$\overline{K} = (\alpha\beta)^{\frac{1}{1-\alpha}}.$$

Apply a first-order Taylor expansion with respect to $K_{t+2}, K_{t+1}, K_t, z_{t+1}, z_t$ by evaluating at the steady state, then we can recover the Uhlig matrices:

$$F = \frac{\beta\alpha\overline{K}^{\alpha-1}}{\overline{K}^{\alpha} - \overline{K}}$$

$$G = -\frac{\beta\alpha\overline{K}^{\alpha-1}\left(\overline{K}^{\alpha-1} + \alpha\right)}{\overline{K}^{\alpha} - \overline{K}}$$

$$H = \frac{\beta\alpha^2\overline{K}^{2\alpha-2}}{\overline{K}^{\alpha} - \overline{K}}$$

$$L = -\frac{\beta\alpha\overline{K}^{\alpha}}{\overline{K}^{\alpha} - \overline{K}}$$

$$M = \frac{\beta\alpha\overline{K}^{2\alpha-1}}{\overline{K}^{\alpha} - \overline{K}}$$

and $N = \rho$. The matrices P, Q are such that

$$FP^2 + GP + H = 0$$

$$FQN + (FP + G)Q + (LN + M) = 0.$$

Thus we can solve for

$$P = \frac{-G \pm \sqrt{G^2 - 4FH}}{2F}$$

$$Q = -\frac{LN + M}{FN + FP + G}$$

Note that under the calibrated parameters, $P = 2.55, 0.40$ and we pick the one with $P < 1$. The policy function can be linearly approximated by

$$\tilde{K}_{t+1} = P\tilde{K}_t + Q\tilde{z}_t,$$

i.e.,

$$K_{t+1} = P\left(K_t - \overline{K}\right) + Qz_t + \overline{K}.$$

From Problem Set 1 we know the closed form solution is

$$K_{t+1} = \alpha\beta e^{z_t} K_t^{\alpha}.$$

Both three-dimensional surface plot are shown in PS_Linear.ipynb.

Exercise 2. Repeat the above exercise using $k \equiv \ln K$ in place of K as the endogenous state variable.

Solution. Note that $k \equiv \ln K$ and hence $K = e^k$. Now the Euler equation can be written as

$$E_t\left\{\frac{\beta\alpha e^{z_{t+1}} e^{k_{t+1}(\alpha-1)}\left(e^{z_t}e^{k_t\alpha} - e^{k_{t+1}}\right)}{e^{z_{t+1}}e^{k_{t+1}\alpha} - e^{k_{t+2}}}\right\} = 1.$$

Differentiation with respect to $k_{t+2}, k_{t+1}, k_t, z_{t+1}, z_t$ by evaluating at the steady state gives rise to the Uhlig matrices:

$$F = \frac{\beta\alpha e^{\bar{k}\alpha}}{\left(e^{\bar{k}\alpha} - e^{\bar{k}}\right)} = \frac{\beta\alpha\overline{K}^{\alpha}}{\overline{K}^{\alpha} - \overline{K}}$$

$$G = -\frac{\beta\alpha e^{\bar{k}\alpha}\left(e^{\bar{k}(\alpha-1)} + \alpha\right)}{e^{\bar{k}\alpha} - e^{\bar{k}}} = -\frac{\beta\alpha\overline{K}^{\alpha}\left(\overline{K}^{\alpha-1} + \alpha\right)}{\overline{K}^{\alpha} - \overline{K}}$$

$$H = \frac{\beta\alpha^2 e^{\bar{k}(2\alpha-1)}}{e^{\bar{k}\alpha} - e^{\bar{k}}} = \frac{\beta\alpha^2\overline{K}^{2\alpha-1}}{\overline{K}^{\alpha} - \overline{K}}$$

$$L = -\frac{\beta\alpha e^{\bar{k}\alpha}}{e^{\bar{k}\alpha} - e^{\bar{k}}} = -\frac{\beta\alpha\overline{K}^{\alpha}}{\overline{K}^{\alpha} - \overline{K}}$$

$$M = \frac{\beta\alpha e^{\bar{k}(2\alpha-1)}}{e^{\bar{k}\alpha} - e^{\bar{k}}} = \frac{\beta\alpha\overline{K}^{2\alpha-1}}{\overline{K}^{\alpha} - \overline{K}}$$

By the same procedure we can obtain P and Q. Now we have the approximation

$$\tilde{k}_{t+1} = P\tilde{k}_t + Q\tilde{z}_t,$$

i.e.,

$$\ln K_{t+1} - \ln\overline{K} = P\left(\ln K_t - \ln\overline{K}\right) + Qz_t$$

Equivalently,

$$K_{t+1} = \exp\left\{P\left(\ln K_t - \ln\overline{K}\right) + Qz_t + \ln\overline{K}\right\}$$

The three-dimensional plot is drawn in PS_Linear.ipynb.

Exercise 3. Do the necessary tedious matrix algebra necessary to transform equation (5) into (8).

Solution. From equation (5) we have

$$E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} = 0.$$

Plug in $\tilde{Z}_{t+1} = N\tilde{Z}_t + \varepsilon_{t+1}$, $\tilde{X}_t = P\tilde{X}_{t-1} + Q\tilde{Z}_t$ and $\tilde{X}_{t+1} = P\tilde{X}_t + Q\tilde{Z}_{t+1} = P\left(P\tilde{X}_{t-1} + Q\tilde{Z}_t\right) + Q\left(N\tilde{Z}_t + \varepsilon_{t+1}\right)$, then

$$\begin{aligned}
0 &= E_t\left\{F\tilde{X}_{t+1} + G\tilde{X}_t + H\tilde{X}_{t-1} + L\tilde{Z}_{t+1} + M\tilde{Z}_t\right\} \\
&= E_t\left\{F\left(P^2\tilde{X}_{t-1} + PQ\tilde{Z}_t + QN\tilde{Z}_t + Q\varepsilon_{t+1}\right) + G\left(P\tilde{X}_{t-1} + Q\tilde{Z}_t\right) + H\tilde{X}_{t-1} + L\left(N\tilde{Z}_t + \varepsilon_{t+1}\right) + M\tilde{Z}_t\right\} \\
&= \left(FP^2 + GP + H\right)\tilde{X}_{t-1} + \left(FQN + (FP + G)Q + (LN + M)\right)\tilde{Z}_t
\end{aligned}$$

Therefore,

$$FP^2 + GP + H = 0$$

$$FQN + (FP + G)\,Q + (LN + M) = 0.$$

Exercise 4. For the baseline tax model, find the steady state values of k, c, r, w, ℓ, T, y and i, numerically. Assume $u(c_t, \ell_t) = \frac{c_t^{1-\gamma}-1}{1-\gamma} + a\frac{(1-\ell_t)^{1-\xi}-1}{1-\xi}$ and $F(K_t, L_t, z_t) = K_t^{\alpha}\left(L_t e^{z_t}\right)^{1-\alpha}$. Use the following parameter values: $\gamma = 2.5, \xi = 1.5, \beta = .98, \alpha = .40, a = .5, \delta = .10, \overline{z} = 0, \rho_z = .9$ and $\tau = .05$.

Solution. The characterizing Euler equations are

$$c_t^{-\gamma} = \beta E_t\left\{c_{t+1}^{-\gamma}\left[(r_{t+1} - \delta)(1-\tau) + 1\right]\right\}$$

$$a\,(1 - l_t)^{-\xi} = c_t^{-\gamma} w_t (1 - \tau)$$

with definition equations giving rise to r_t, w_t, T_t, c_t once we have k_t, l_t solved:

$$r_t = \alpha k_t^{\alpha-1} l_t^{1-\alpha} e^{z_t(1-\alpha)}$$

$$w_t = (1-\alpha)\, k_t^{\alpha} l_t^{-\alpha} e^{z_t(1-\alpha)}$$

$$T_t = \tau\left[w_t l_t + (r_t - \delta)\, k_t\right]$$

$$c_t = (1-\tau)\left[w_t l_t + (r_t - \delta)\, k_t\right] + k_t + T_t - k_{t+1}.$$

We can solve for the steady state using LinApp_FindSS.py. The numerical steady states found are

$$\overline{k} = 4.23, \overline{c} = 0.86, \overline{r} = 0.12, \overline{w} = 1.33, \overline{\ell} = 0.58, \overline{T} = 0.04, \overline{y} = 1.28, \overline{i} = 0.42$$

Exercise 5. For the same model as above, find $\frac{\partial y}{\partial x}$ for $y \in \{\overline{k}, \overline{c}, \overline{r}, \overline{w}, \overline{\ell}, \overline{T}, \overline{y}, \overline{i}\}$ and $x \in \{\delta, \tau, \overline{z}, \alpha, \gamma, \xi, \beta, a\}$ using numerical techniques.

Solution. I use the centered differentiation with $h = 10^{-6}$. The results are presented in PS_Linear.ipynb.

Exercise 6. For the same model as above, let $X_t = \{k_{t-1}, \ell_{t-1}\}$. Find the values of F, G, H, L, M, N, P and Q.

Solution. We need to modify X to include l in addition to k, as required. The values of the Uhlig matrices are reported in PS_Linear.ipynb.

For Exercise 7, 8, and 9, see PS_Linear.ipynb.