How to Solve Deterministic Models

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2022 年 4 月 17 日

$$\max\sum_{t=0}^{\infty}\beta^t\cdot u(c_t)$$

s.t.

$$c_t+k_{t+1}=(1-\delta)k_t+z_t\cdot k_t^{\alpha}$$

此处z_t为生产技术冲击

$$\lim_{t\to\infty}k_t\geq 0(\mathrm{TVC}),k_0\ \text{ given.}$$

1 Social Planner下的Ramsey模型

1.1 求解模型

1.1.1 Lagrange求解

定义

$$L=\max\sum_{t=0}^{\infty}\beta^t\cdot u(c_t)+\sum_{t=0}^{\infty}\lambda_t\cdot\beta^t[(1-\delta)k_t+z_t\cdot k_t^{\alpha}-k_{t+1}-c_t]$$

此处λ_t为t期预算约束对应的Lagrange乘子.

1.1.2 求FOC:

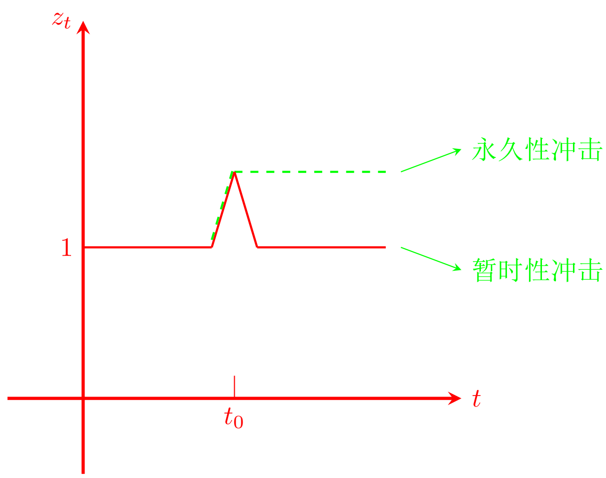
$$\begin{cases}c_t: \quad \beta^t\cdot u^{'}(c_t)-\beta^t\cdot\lambda_t=0 \text{ ①}\\ k_{t+1}: \quad -\beta^t\cdot\lambda_t+\beta^{t+1}\cdot\lambda_{t+1}\cdot[(1-\delta)+\alpha\cdot z_{t+1}\cdot k_{t+1}^{\alpha-1}]=0 \text{ ②}\\ \lambda_t: \quad (1-\delta)k_t+z_t\cdot k_t^{\alpha}=k_{t+1}+c_t \text{ ③}\end{cases}$$

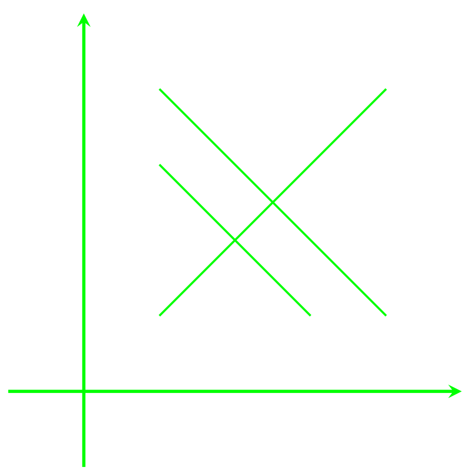
将①代入②消去λ_t,则

$$-\beta^t \cdot u^{'}(c_t)+\beta^{t+1}u^{'}(c_{t+1})[(1-\delta)+\alpha\cdot z_{t+1}\cdot k_{t+1}^{\alpha-1}]=0$$
$$\Rightarrow u^{'}(c_t)=\beta\cdot u^{'}(c_{t+1})[(1-\delta)+\alpha\cdot z_{t+1}\cdot k_{t+1}^{\alpha-1}] ④$$

$$\begin{cases}\text{方程:③,④}\\ \text{变量:}c_t,k_{t+1}\end{cases}$$

问:不同类型的z_t冲击下,c_t与k_{t+1}的变化路径?





⇒数值解!

(*)假设$u(c_t)=\frac{c_t^{1-\sigma}-1}{1-\sigma}$.

(*)Steady State:

$$c_t=c_{t+1}=c, k_t=k_{t+1}=k.$$

$$\begin{cases} c_t^{-\sigma} = \beta \cdot c_{t+1}^{-\sigma}[(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1}] \\ c_t + k_{t+1} = (1-\delta)k_t + z_t \cdot k_t^{\alpha} \end{cases}$$

$$\Rightarrow \begin{cases} c^{-\sigma} = \beta \cdot c^{-\sigma} \cdot [(1-\delta) + \alpha \cdot z \cdot k^{\alpha-1}] ⑤ \\ c + k = (1-\delta)k + z \cdot k^{\alpha} ⑥ \end{cases}$$

(*)假设$z = 1, \sigma = 2, \alpha = 0.5, \beta = 0.98, \delta = 0.1.$

⑤$\Rightarrow 1 = \beta \cdot [(1-\delta) + \alpha \cdot k^{\alpha-1}] \Rightarrow k = [\frac{1}{\alpha}(\frac{1}{\beta} - (1-\delta))]^{\frac{1}{\alpha-1}}$

⑥$\Rightarrow c = z \cdot k^{\alpha} - \delta \cdot k = k^{\alpha} - \delta \cdot k.$

(*)假设求解T期(T足够大使得模型回到均衡或新的均衡).

2 推广：带有劳动的Ramsey模型

$$\max \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, n_t)$$

s.t.

$$c_t + k_{t+1} = (1-\delta)k_t + z_t \cdot k_t^{\alpha} \cdot n_t^{1-\alpha}$$

$$\lim_{t \to \infty} k_t \geq 0(\text{TVC}), k_0 \, given$$

⊛EX:Dynare plus Matlab!→test accurary

⊛求解

2.1 Lagrange求解

定义

$$L = \sum_{t=0}^{\infty} \beta^t \cdot u(c_t, n_t) + \sum_{t=0}^{\infty} \beta^t \cdot \lambda_t \cdot [(1-\delta)k_t + z_t \cdot k_t^{\alpha} \cdot n_t^{1-\alpha} - c_t - k_{t+1}]$$

此处λ_t为t期预算约束对应的Lagrange乘子

2.2 求FOC:

$$\begin{cases} c_t: \quad \beta^t \cdot u_c(c_t, n_t) - \beta^t \cdot \lambda_t = 0 ① \\ n_t: \quad \beta^t \cdot u_n(c_t, n_t) + \beta^t \cdot \lambda_t \cdot (1-\alpha) \cdot z_t \cdot k_t^{\alpha} \cdot n_t^{-\alpha} = 0 ② \\ k_{t+1}: \quad \beta^{t+1} \cdot \lambda_{t+1} \cdot [(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1} \cdot n_{t+1}^{1-\alpha}] - \beta^t \cdot \lambda_t = 0 ③ \end{cases}$$

将①代入②消去λ_t,得

$$u_n(c_t, n_t) + u_c(c_t, n_t) \cdot (1-\alpha) \cdot z_t \cdot k_t^{\alpha} \cdot n_t^{-\alpha} = 0 \text{ ④}$$

将①代入③

$$\Rightarrow \begin{array}{l} \beta \cdot u_c(c_{t+1}, n_{t+1}) \cdot [(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1} \cdot n_{t+1}^{1-\alpha}] = u_c(c_t, n_t) \text{ ⑤} \\ c_t + k_{t+1} = (1-\delta)k_t + z_t \cdot k_t^{\alpha} \cdot n_t^{1-\alpha} \text{ ⑥} \end{array}$$

$$\begin{cases} \text{方程:④⑤⑥} \\ \text{变量:} c_t, n_t, k_{t+1} \end{cases}$$

问:不同类型的z_t冲击下,c_t, n_{t+1}与k_{t+1}的变化路径?
⇒数值解!
(*)假设$u(c_t, n_t) = \frac{c_t^{1-\sigma}-1}{1-\sigma} - \psi \cdot \frac{n_t^{1+\eta}}{1+\eta}$

$$\Rightarrow u_c(c_t, n_t) = c_t^{-\sigma}, u_n(c_t, n_t) = -\psi \cdot n_t^{\eta};$$

代入方程得:

$$\begin{aligned} \text{代入④} &\Rightarrow -\psi \cdot n_t^{\eta} + c_t^{-\sigma} \cdot (1-\alpha) \cdot z_t \cdot k_t^{\alpha} \cdot n_t^{-\alpha} = 0 \\ &\Rightarrow \psi \cdot n_t^{\eta} \cdot c_t^{\sigma} = (1-\alpha) \cdot z_t \cdot k_t^{\alpha} \cdot n_t^{-\alpha} \text{ ⑦} \\ \text{代入⑤} &\Rightarrow c_t^{-\sigma} = \beta \cdot c_{t+1}^{-\sigma} \cdot [(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1} \cdot n_{t+1}^{1-\alpha}] \text{ ⑧} \end{aligned}$$

$$\begin{cases} \text{方程:⑥⑦⑧} \\ \text{变量:} c_t, n_t, k_{t+1} \end{cases}$$

(*)Steady State:

$$\begin{aligned} c_t = c_{t+1} = c, n_t = n_{t+1} = n, k_t = k_{t+1} = k, z = 1 \\ \Rightarrow \psi \cdot n^{\eta} \cdot c^{\sigma} = (1-\alpha) \cdot z \cdot k^{\alpha} \cdot n^{-\alpha} \text{ ⑨} \\ \cancel{c^{-\sigma}} = \beta \cdot \cancel{c^{-\sigma}} \cdot [(1-\delta) + \alpha \cdot z \cdot k^{\alpha-1} \cdot n^{1-\alpha}] \text{ ⑩} \\ c + k = (1-\delta) \cdot k + z \cdot k^{\alpha} \cdot n^{1-\alpha} \text{ ⑪} \end{aligned}$$

$$\begin{cases} \text{参数：} & \psi, \eta, \sigma, \alpha, \delta \\ \text{稳态：} & c, n, k \end{cases}$$

$$\text{由⑩} \Rightarrow \quad \frac{k}{n} = [\frac{1}{\alpha z}(\frac{1}{\beta} - (1-\delta))]^{\frac{1}{\alpha-1}}$$

$$\text{由⑪} \Rightarrow \quad \frac{c}{n} = z \cdot (\frac{k}{n})\alpha - \delta \cdot \frac{k}{n}$$

$$\text{由⑨} \Rightarrow \quad \psi \cdot n^{\eta+\sigma} \cdot (\frac{c}{n})^{\sigma} = (1-\alpha) \cdot z \cdot (\frac{k}{n})^{\alpha}$$

$$\Rightarrow n = [\frac{(1-\alpha)}{\psi} \cdot z \cdot (\frac{k}{n})^{\alpha} \cdot (\frac{c}{n})^{-\sigma}]^{\frac{1}{\eta+\sigma}}$$

(以ψ较准$n = \frac{1}{3}$)

$$\Rightarrow \begin{cases} \text{已知:} & n = \frac{1}{3}, \eta = 1, \sigma = 2, \alpha = 0.5, \delta = 0.1, z = 1 \\ \text{未知:} & \psi, c, k \end{cases}$$

$$\text{由⑩} \Rightarrow \ k = \dots$$

$$\text{再由⑪} \Rightarrow \ c = \dots$$

$$\text{最后由⑨} \Rightarrow \ \psi = \dots$$

3 带有借贷约束的禀赋经济模型(Endowment Economy)

$$\max \sum_{t=0}^{\infty} \beta^t \cdot [\frac{C_t^{1-\sigma} - 1}{1-\sigma}]$$

s.t.

$$\begin{cases} C_t + (1+r) \cdot \underset{\nearrow \text{债务}}{B_{t-1}} = Q_t + B_t \rightarrow \lambda_t \cdot \beta^t \\ B_t \leq m \cdot Q_t \quad \rightarrow \mu_t \cdot \beta^t \\ \lim_{t \to \infty} B_t = 0 (\text{TVC}), B_0 \text{ given.} \end{cases}$$

⊛求解

3.1 Lagrange求解

定义

$$L=\sum_{t=0}^{\infty}\beta^t\cdot(\frac{C_t^{1-\sigma}-1}{1-\sigma})+\sum_{t=0}^{\infty}\beta^t\cdot\lambda_t\cdot[Q_t+B_t-(1+r)\cdot B_{t-1}-C_t]+\sum_{t=0}^{\infty}\beta^t\cdot\mu_t\cdot[m\cdot Q_t-B_t]$$

此处λ_t,μ_t为Lagrange乘子

3.2 求F0C:

$$\begin{cases} C_t:\beta^t\cdot C_t^{-\sigma}-\beta^t\cdot\lambda_t=0 \\ B_t:\beta^t\cdot\lambda_t-\beta^{t+1}\lambda_{t+1}\cdot(1+r)-\beta^t\cdot\mu_t=0 \\ \lambda_t:C_t+(1+r)\cdot B_{t-1}=Q_t+B_t,\lambda_t\geq 0 \end{cases}$$

(Kuhn-Tucker)松弛条件：$B_t\leq m\cdot Q_t,\mu_t\geq 0,\mu_t\cdot[m\cdot Q_t-B_t]=0$

$$\Rightarrow C_t^{-\sigma}=\beta\cdot(1+r)\cdot C_{t+1}^{-\sigma}+\mu_t$$

3.3 均衡系统:$\{c_t,\mu_t,B_t\}$3个

$$\begin{cases} C_t^{-\sigma}=\beta\cdot(1+r)\cdot C_{t+1}^{-\sigma}+\mu_t \text{①} \\ C_t+(1+r)\cdot B_{t-1}=Q_t+B_t \text{②} \\ B_t=m\cdot Q_t \quad \text{or} \quad \mu_t=0 \text{③} \end{cases}$$

$$\Rightarrow \begin{cases} m\cdot Q_t-B_t=\max(0,-\xi_t^3) \text{④} \\ \mu_t=\max(0,\xi_t^3) \text{⑤} \end{cases}$$

$$\Rightarrow \quad \begin{array}{c} [\text{①},\text{②},\text{④},\text{⑤}]\rightarrow 4\text{个} \\ \Updownarrow \\ [c_t,B_t,\mu_t,\xi_t]\rightarrow 4\text{个} \end{array}$$

⊛Steady State

$$C^{-\delta}=\beta\cdot(1+r)\cdot C^{-\delta}+\mu$$

$$C+(1+r)\cdot B=Q+B$$

$$m\cdot Q-B=\max(0,-\xi^3)$$

$$\mu=\max(0,\xi^3)$$

⊛Set $\sigma = 2, r = 5\%, \beta = 0.945 (\beta \cdot (1+r) < 1 \Rightarrow \mu > 0), Q = 1, m = 1$

$$
\begin{aligned}
&\mu = [1 - \beta \cdot (1+r)] \cdot C^{-\sigma} \\
&\xi = \mu^{\frac{1}{3}} \\
&B = m \cdot Q \\
&C = Q - r \cdot B
\end{aligned}
$$

Note: numerical solutions by *borrowing_constraint.mod*

4 扩展一：Iheversible Investment(Ramsey Model)

$$\max \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma}$$

s.t.

$$
\begin{cases}
c_t + k_{t+1} = (1-\delta) \cdot k_t + z_t \cdot k_t^{\alpha} (\beta^t \cdot \lambda_t) \\
k_{t+1} \geq (1-\delta) \cdot k_t \quad \rightarrow (\beta^t \cdot \mu_t) \qquad \Rightarrow i_t \geq 0 (\text{irreversible investment}). \\
\lim\limits_{t \to \infty} k_t \geq 0 \quad (\text{TVC}), \quad k_0 \text{ given.}
\end{cases}
$$

4.1 Lagrange求解:

定义

$$L = \sum_{t=0}^{\infty} \beta^t \cdot \frac{c_t^{1-\sigma} - 1}{1-\sigma} + \sum_{t=0}^{\infty} \beta^t \lambda_t \cdot [(1-\delta) \cdot k_t + z_t \cdot k_t^{\alpha} - c_t - k_{t+1}] + \sum_{t=0}^{\infty} \beta^t \cdot \mu_t \cdot [k_{t+1} - (1-\delta) \cdot k_t]$$

4.2 求FOC:

$$
\begin{cases}
c_t : \cancel{\beta^t} c_t^{-\sigma} - \cancel{\beta^t} \cdot \lambda_t = 0 \\
k_{t+1} : \beta^{t+1} \cdot \lambda_{t+1} \cdot [(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1}] - \beta^t \cdot \lambda_t + \beta^t \cdot \mu_t - \beta^{t+1} \cdot \mu_{t+1} \cdot (1-\delta) = 0
\end{cases}
$$

$$\Rightarrow \quad \beta \cdot c_{t+1}^{-\sigma} \cdot [(1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1}] - c_t^{-\sigma} + \mu_t - \beta \cdot \mu_{t+1} \cdot (1-\delta) = 0$$

$$\Rightarrow \quad c_t^{-\sigma} - \mu_t = \beta \cdot (c_{t+1}^{-\sigma} \cdot ((1-\delta) + \alpha \cdot z_{t+1} \cdot k_{t+1}^{\alpha-1}) - (1-\delta)\mu_{t+1}) \text{①}$$

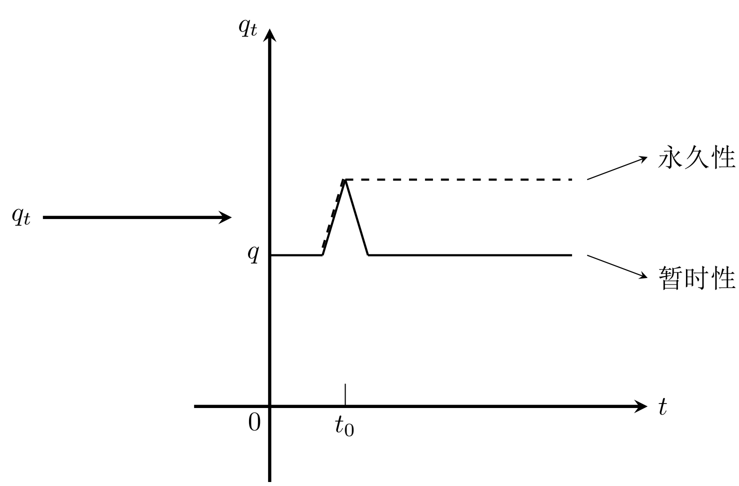
$$c_t + k_{t+1} = (1-\delta)\cdot k_t + z_t \cdot k_t^{\alpha} \text{ ②}$$
$$k_{t+1} - (1-\delta)\cdot k_t = \max(0, -\xi_t^3) \text{ ③}$$
$$\mu_t = \max(0, \xi_t^3) \text{ ④}$$
$$[\text{①}, \text{②}, \text{③}, \text{④}] \to 4\text{个}$$
$$\Updownarrow$$
$$[c_t, k_{t+1}, \mu_t, \xi_t] \to 4\text{个}$$

5 扩展二：Collateral Constraints

$$\max \sum_{t=0}^{\infty} \beta^t \cdot [\ln(c_t) + j\cdot \ln(h_t)] \to j \Rightarrow j_t \quad \text{housing demand shocks}$$

s.t.

$$\begin{cases} c_t + q_t \cdot h_t = y + b_t - (1+r)\cdot b_{t-1} + q_t \cdot (1-\delta_h)\cdot h_{t-1} \\ b_t \le m \cdot q_t \cdot h_t \end{cases}$$



⊛具体模型

$$\max \sum_{t=0}^{\infty} \beta^t \cdot [\ln(c_t) + j\cdot \ln(h_t)]$$

s.t.

$$\begin{cases} c_t + q_t \cdot h_t + (1+r)\cdot b_{t-1} = y + b_t + q_t \cdot (1-\delta_h)\cdot h_{t-1} \qquad \text{⑥} \qquad \to \beta^t \cdot \lambda_t \\ b_t \le m \cdot q_t \cdot h_t \qquad \to \beta^t \cdot \mu_t \end{cases}$$

⊛求解

5.1 Lagrange求解

定义

$$L=\sum_{t=0}^{\infty}\beta^t\cdot[\ln(c_t)+j\cdot\ln(h_t)]+\sum_{t=0}^{\infty}\beta^t\cdot\lambda_t[y+b_t+q_t\cdot(1-\delta_h)\cdot h_{t-1}-(1+r)\cdot b_{t-1}-q_t\cdot h_t-c_t]$$
$$+\sum_{t=0}^{\infty}\beta^t\cdot\mu_t\cdot[m\cdot q_t\cdot h_t-b_t]$$

此处λ_t,μ_t为Lagrange乘子

5.2 求FOC:

$$\begin{cases} c_t: & \beta^t\cdot\frac{1}{c_t}-\beta^t\cdot\lambda_t=0 \text{ ①} \\ b_t: & \beta^t\cdot\lambda_t-\beta^{t+1}\cdot\lambda_{t+1}\cdot(1+r)-\beta^t\cdot\mu_t=0 \text{ ②} \\ h_t: & \beta^t\cdot\frac{j}{h_t}+\beta^{t+1}\cdot\lambda_{t+1}\cdot q_{t+1}\cdot(1-\delta_h)-\beta^t\cdot\lambda_t\cdot q_t+\beta^t\cdot\mu_t\cdot m\cdot q_t=0 \text{ ③} \end{cases}$$

$$\text{将①代入②}\Rightarrow\frac{1}{c_t}=\beta\cdot\frac{(1+r)}{c_{t+1}}+\mu_t \text{ ④}$$
$$\text{将①代入③}\Rightarrow\frac{j}{h_t}+\beta\cdot\frac{1}{c_{t+1}}\cdot q_{t+1}\cdot(1-\delta_h)-\frac{1}{c_t}\cdot q_t+m\cdot q_t\cdot\mu_t=0$$
$$\Rightarrow\quad q_t=j\cdot\frac{c_t}{h_t}+\beta\cdot(1-\delta_h)\cdot\frac{c_t}{c_{t+1}}\cdot q_{t+1}+m\cdot q_t\cdot\mu_t \text{ ⑤}$$
$$m\cdot q_t\cdot h_t-b_t=\max(0,-\xi_t^3) \text{ ⑦}$$
$$\mu_t=\max(0,\xi_t^3) \text{ ⑧}$$

$$[\text{④},\text{⑤},\text{⑥},\text{⑦},\text{⑧}]\rightarrow 5\text{个}$$
$$\Updownarrow$$
$$[c_t,h_t,b_t,\mu_t,\xi_t]\rightarrow 5\text{个}$$

⊛Steady State

$q = 1, y = 1, \beta \cdot (1+r) < 1$ $(h = 1 \Rightarrow j)$

$$\frac{1}{c} = \beta \cdot \frac{(1+r)}{c} + \mu$$
$$\frac{j}{h} + \beta \cdot (1-\delta_h) \cdot \frac{q}{c} - \frac{q}{c} + m \cdot q \cdot \mu = 0$$
$$C + q \cdot h + (1+r) \cdot b = y + b + q \cdot (1-\delta_h) \cdot h$$
$$m \cdot q \cdot h - b = 0$$
$$\mu = \xi^3$$