两期模型(Two Periods Model)

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$$\max\left\{u(C_1)+\beta\cdot u(C_2)\right\}$$

s.t.

$$\begin{cases}\text{Period1: } C_1+B_1-B_0=r_0\cdot B_0+Q_1 \text{ ①}\\ \text{Period2: } C_2+B_2-B_1=r_1\cdot B_1+Q_2 \text{ ②}\end{cases}$$

此处C_t是消费，B_t是持有的债券，Q_t是禀赋收入，r_t是利率，B_0为初始资产，可假设为0.

Transversality Condition(TVC)：B_2=0.

$$\begin{cases}\text{TVC:} \quad B_T\geq 0 & \text{(不允许负资产)}\\ \text{NPG:} \quad \frac{B_T}{(1+r)^T}\geq 0 & \text{(负资产的增速不能太快)}\end{cases}$$

1 求解模型

由②$\Rightarrow B_1=\frac{C_2-Q_2}{(1+r_1)}$.代入①$\Rightarrow C_1+\frac{C_2}{(1+r_1)}=(1+r_0)\cdot B_0+Q_1+\frac{Q_2}{1+r_1}$.

$$\max\left\{u(C_1)+\beta\cdot u(C_2)\right\}$$

s.t.

$$C_1+\frac{C_2}{(1+r_1)}=(1+r_0)\cdot B_0+Q_1+\frac{Q_2}{(1+r_1)}$$

1.1 Lagrange求解

定义

$$L=u(C_1)+\beta u(C_2)+\lambda\cdot[(1+r_0)\cdot B_0+Q_1+\frac{Q_2}{(1+r_1)}-C_1-\frac{C_2}{(1+r_1)}]$$

其中λ为Lagrange乘子.

1.2 求 First-order-condition(FOC)

$$\begin{cases} C_1 : u^{'}(C_1) - \lambda = 0 ③ \\ C_2 : \beta \cdot u^{'}(C_2) - \frac{\lambda}{1+r_1} = 0 ④ \\ \lambda : C_1 + \frac{C_2}{(1+r_1)} = (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)} ⑤ \end{cases}$$

定义

$$\bar{Y} \equiv (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)}$$

将③带入④得:

$$\begin{cases} \beta(1+r_1)u^{'}(C_2) = u'(C_1) \\ C_1 + \frac{C_2}{(1+r_1)} = \bar{Y} \end{cases}$$

(*)假设：$u(C_t) = \ln C_t$,$B_0 = 0, r_0 = r_1 = r$，则

$$\Rightarrow \begin{cases} \beta \cdot (1+r) \cdot \frac{1}{C_2} = \frac{1}{C_1} ⑥ \\ C_1 + \frac{C_2}{1+r} = Q_1 + \frac{Q_2}{1+r} ⑦ \end{cases}$$

由⑥可得$C_2 = \beta(1+r)C_1$，再代入⑦，可以得到:

$$C_1 + \beta \cdot C_1 = Q_1 + \frac{Q_2}{(1+r)}$$

进一步地，我们可得:

$$\begin{cases} C_1 = \frac{1}{(1+\beta)}[Q_1 + \frac{Q_2}{(1+r)}] \\ C_2 = \frac{\beta \cdot (1+r)}{(1+\beta)} \cdot [Q_1 + \frac{Q_2}{(1+r)}] \end{cases}$$

2 分析不同的收入冲击类型对消费、储蓄的影响

问题导向：⟶分析不同类型收入冲击的影响

收入冲击的类型:

$$\begin{cases} ①\text{暂时性收入冲击:} & \Delta Q_1 = 1, \Delta Q_2 = 0 \\ ②\text{预期未来收入冲击:} & \Delta Q_1 = 0, \Delta Q_2 = 1 \\ ③\text{永久性收入冲击:} & \Delta Q_1 = 1, \Delta Q_1 = 0 \end{cases}$$

为了便于分析，我们假设$\beta = 1$，则

$$\begin{cases} C_1 = \frac{1}{2}[Q_1 + \frac{Q_2}{(1+r)}] \\ C_2 = (1+r)C_1 = \frac{(1+r)}{2}[Q_1 + \frac{Q_2}{(1+r)}] \end{cases}$$

进一步，我们得到

$$\begin{cases} \Delta C_1 = \frac{1}{2}[\Delta Q_1 + \frac{\Delta Q_2}{(1+r)}] \\ \Delta C_2 = \frac{(1+r)}{2}[\Delta Q_1 + \frac{\Delta Q_2}{(1+r)}] \end{cases}$$

由储蓄定义$S_t = B_t - B_{t-1}$

$$\Rightarrow \begin{cases} S_1 = B_1 - B_0 = -C_1 + Q_1 - 0 = -\frac{Q_2}{2(1+r)} + \frac{Q_1}{2} \\ S_2 = B_2 - B_1 = 0 + (C_1 - Q_1) = -\frac{Q_1}{2} + \frac{Q_2}{2(1+r)} \end{cases}$$

$$\Rightarrow \begin{cases} \Delta S_1 = -\frac{\Delta Q_2}{2(1+r)} + \frac{\Delta Q_1}{2} \\ \Delta S_2 = -\frac{\Delta Q_1}{2} + \frac{\Delta Q_2}{2(1+r)} \end{cases}$$

①$\Delta Q_1 = 1, \Delta Q_2 = 0$

$$\begin{cases} \Delta C_1 = \frac{1}{2} \\ \Delta C_2 = \frac{(1+r)}{2} \approx \frac{1}{2} \\ \Delta S_1 = \frac{1}{2} \\ \Delta S_2 = -\frac{1}{2} \end{cases}$$

②$\Delta Q_1 = 0, \Delta Q_2 = 1$

$$\begin{cases} \Delta C_1 = \frac{1}{2(1+r)} \approx \frac{1}{2} \\ \Delta C_2 = \frac{1}{2} \\ \Delta S_1 = -\frac{1}{2(1+r)} \\ \Delta S_2 = \frac{1}{2(1+r)} \end{cases}$$

③$\Delta Q_1 = 1, \Delta Q_1 = 0$

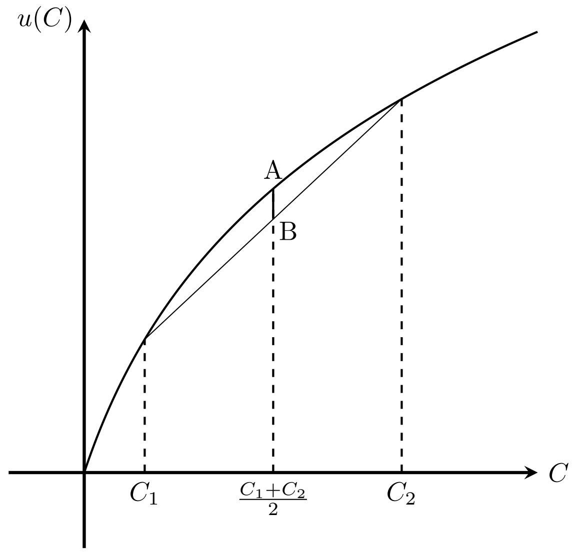
$$\begin{cases} \Delta C_1 = \frac{2+r}{2(1+r)} \approx 1 \\ \Delta C_2 = \frac{2+r}{2} \approx 1 \\ \Delta S_1 = \frac{r}{2(1+r)} \approx 0 \\ \Delta S_2 = -\frac{r}{2(1+r)} \approx 0 \end{cases}$$

Rmk:

① 为什么要消费平滑⇐边际效用递减⇐ $u(C_t)$是凹函数.

$$u\mid_A = u(\frac{C_1 + C_2}{2})$$

$$u\mid_B = \frac{1}{2}(u(C_1) + u(C_2))$$



$$
\begin{aligned}
& u\mid_A \geq u\mid_B \\
\Rightarrow \quad & u(\frac{C_1+C_2}{2}) \geq \frac{1}{2}(u(C_1)+u(C_2)) \\
\Rightarrow \quad & u(\frac{C_1+C_2}{2})+u(\frac{C_1+C_2}{2}) \geq u(C_1)+u(C_2)
\end{aligned}
$$

② 如何实现消费平滑？$\Leftarrow$跨期借贷.

3 考虑耐用品(Durable goods)

(假设$u(C_t)=\ln(C_t), r_0=r_1=r, B_0=0$)

$$\max\{\ln(C_1)+\beta\cdot\ln[(1-\delta)C_1+C_2]\}$$

其中δ为折旧率，$\delta\in[0,1]$

s.t.

$$C_1+\frac{C_2}{(1+r)}=Q_1+\frac{Q_2}{(1+r)}, \quad B_2=0(\text{TVC})$$

$\circledast$求解

3.1 Lagrange求解

定义

$$L=\ln(C_1)+\beta\cdot\ln[(1-\delta)C_1+C_2]+\lambda\cdot[Q_1+\frac{Q_2}{(1+r)}-C_1-\frac{C_2}{(1+r)}]$$

其中λ为预算约束对应的Lagrange乘子

3.2 求出FOC

$$\begin{cases} C_1: & \frac{1}{C_1}+\beta(1-\delta)\frac{1}{(1-\delta)C_1+C_2}-\lambda=0 \text{⑥} \\ C_2: & \beta\cdot\frac{1}{(1-\delta)C_1+C_2}-\lambda\cdot\frac{1}{(1+r)}=0 \text{⑦} \\ \lambda: & C_1+\frac{C_2}{(1+r)}=Q_1+\frac{Q_2}{(1+r)} \text{⑧} \end{cases}$$

$(*)$假设$\beta=1$，则

$$\begin{cases} \frac{1}{C_1}+(1-\delta)\cdot\frac{1}{(1-\delta)C_1+C_2}=\lambda & \Leftarrow \text{⑥} \\ \frac{1}{(1-\delta)C_1+C_2}=\lambda\cdot\frac{1}{(1+r)} & \Leftarrow \text{⑦} \end{cases}$$

$$\Rightarrow \quad \frac{1}{C_1}+(1-\delta)\cdot\frac{1}{(1-\delta)C_1+C_2}=\frac{(1+r)}{(1-\delta)C_1+C_2}$$

$$\Rightarrow \quad (1-\delta)C_1+C_2+(1-\delta)C_1=(1+r)C_1$$

$$\Rightarrow C_2=(2\delta-1)\cdot C_1+r\cdot C_1=(r+2\delta-1)C_1$$

代入⑧，得：

$$C_1+\frac{(r+2\delta-1)}{(1+r)}\cdot C_1=Q_1+\frac{Q_2}{(1+r)}$$

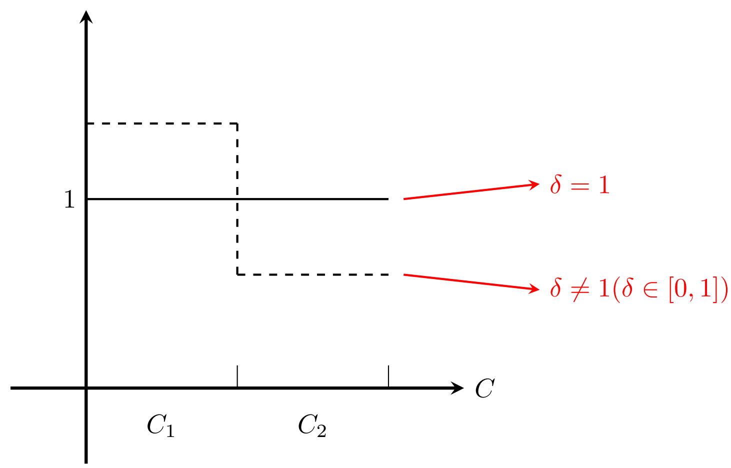
$$\Rightarrow \begin{cases} C_1=\frac{(1+r)Q_1}{2(r+\delta)}+\frac{Q_2}{2(r+\delta)} \\ C_2=(r+2\delta-1)\cdot[\frac{(1+r)Q_1}{2(r+\delta)}+\frac{Q_2}{2(r+\delta)}] \end{cases}$$

(当$\delta=1$时，与上述两期模型一致)

$$\begin{cases} \Delta C_1=\frac{1}{2(r+\delta)}\cdot[(1+r)\Delta Q_1+\Delta Q_2] \\ \Delta C_2=(r+2\delta-1)\cdot[\frac{1+r}{2(r+\delta)}\Delta Q_1+\frac{1}{2(r+\delta)}\Delta Q_2] \end{cases}$$

$$\begin{cases} \text{①}\ \Delta Q_1=1,\Delta Q_2=0 \\ \text{②}\ \Delta Q_1=0,\Delta Q_2=1 \\ \text{③}\ \Delta Q_1=1,\Delta Q_2=1 \end{cases}$$

⊛分析δ对消费、储蓄得影响。取$r=0,\ Q_1=Q_2=1$.
$C_1=\frac{Q_1}{2\delta}+\frac{Q_2}{2\delta}=\frac{1}{2\delta}+\frac{1}{2\delta}=\frac{1}{\delta},C_2=\frac{2\delta-1}{2\delta}\cdot Q_1+\frac{2\delta-1}{2\delta}\cdot Q_2=\frac{2\delta-1}{2\delta}+\frac{2\delta-1}{2\delta}=\frac{2\delta-1}{\delta}$
当$\delta=1$时,则$C_1=1,C2=1$
当$\delta\neq$时,则$C_1=\frac{1}{\delta},C_2=\frac{2\delta-1}{\delta}=2-\frac{1}{\delta}$



$\delta\downarrow,C_1\uparrow,C_2\downarrow$

Rmk:
折旧率越低，越提前消费.

4 考虑消费惯性(Habit formation).

$$\max\left\{u(C_1)+\beta\cdot u(C_2-\alpha\cdot C_1)\right\}$$

此处α为消费惯性系数，$\alpha\in(0,1)$.

s.t.

$$\begin{cases}C_1+B_1-B_0=r\cdot B_0+Q_1\\ C_2+B_2-B_1=r\cdot B_1+Q_2\end{cases}$$

假设$B_0=0,B_2=0$(TVC)，$r=0,\beta=1,Q_1=Q_2=Q,u(C_t)=\ln(C_t)$.

$$\Rightarrow\begin{cases}C_1+B_1=Q\\ C_2=B_1+Q\end{cases}$$

$$\Rightarrow C_1+C_2=2Q$$

⊛求解:

4.1 Lagrange求解

定义

$$L = [\ln(C_1) + \ln(C_2 - \alpha \cdot C_1)] + \lambda \cdot [2Q - C_1 - C_2]$$

此处λ为预算约束对应的Lagrange乘子.

4.2 求FOC

$$\begin{cases} C_1: & \frac{1}{C_1} - \alpha \cdot \frac{1}{C_2 - \alpha \cdot C_1} - \lambda = 0 \\ C_2: & \frac{1}{C_2 - \alpha \cdot C_1} - \lambda = 0 \end{cases}$$

$$\Rightarrow \quad \frac{1}{C_1} - \frac{\alpha}{C_2 - \alpha \cdot C_1} = \frac{1}{C_2 - \alpha \cdot C_1}$$

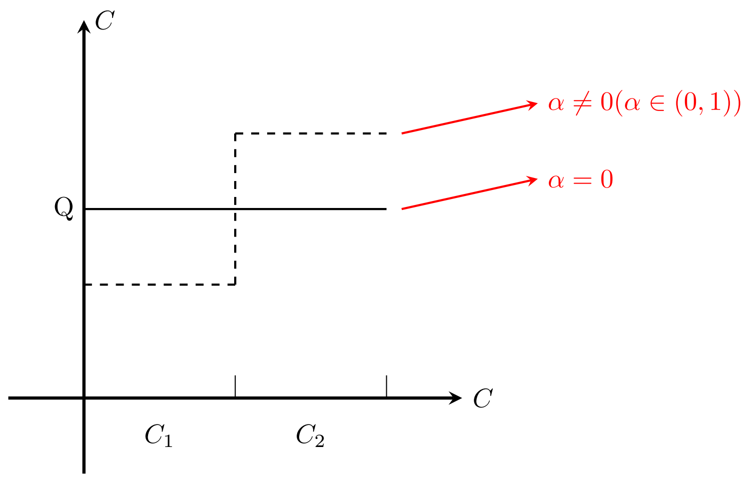
$$\Rightarrow \quad (C_2 - \alpha \cdot C_1) - \alpha C_1 = C_1$$

$$\Rightarrow \quad C_2 = (2\alpha + 1) \cdot C_1$$

由$C_1 + C_2 = 2Q$

$$\begin{cases} C_1 = \frac{1}{1+\alpha} Q \\ C_2 = \frac{2\alpha+1}{1+\alpha} Q \end{cases}$$

当$\alpha = 0$时，$C_1 = C_2 = Q$



Rmk:

消费惯性越大，越倾向未来消费.

5 封闭⇒开放经济

$$\max\{\ln(C_1)+\beta\cdot\ln(C_2)\}$$

s.t.

$$\begin{cases} C_1+B_1^*-B_0^*=r_0\cdot B_0^*+Q_1 \\ C_2+B_2^*-B_1^*=r_1\cdot B_1^*+Q_2 \end{cases}$$

当$B_0^*=0, B_2^*=0(\text{TVC}), r_0=r_1=r^*(\text{小国开放}), \beta=1$时，则

$$\Rightarrow \max\{\ln(C_1)+\ln(C_2)\}$$

s.t.

$$C_1+\frac{C_2}{(1+r^*)}=Q_1+\frac{Q_2}{(1+r^*)}$$

⊛求解:

5.1 Lagrange求解

定义

$$L=[\ln(C_1)+\ln(C_2)]+\lambda\cdot[Q_1+\frac{Q_2}{(1+r^*)}-C_1-\frac{C_2}{(1+r^*)}]$$

其中λ为Lagrange乘子

5.2 求出FOC

$$\begin{cases} C_1: \quad \frac{1}{C_1}-\lambda=0 \\ C_2: \quad \frac{1}{C_2}-\lambda\cdot\frac{1}{(1+r^*)}=0 \\ \lambda: \quad C_1+\frac{C_2}{(1+r^*)}=Q_1+\frac{Q_2}{(1+r^*)} \end{cases}$$

$\Rightarrow C_1 = \frac{1}{2}[Q_1 + \frac{Q_2}{(1+r^*)}], \quad C_2 = \frac{(1+r^*)}{2}[Q_1 + \frac{Q_2}{(1+r^*)}]$

$\nearrow$ Trade Balance

$$TB_1 = Q_1 - C_1 = \frac{1}{2}Q_1 - \frac{Q_2}{2(1+r^*)}$$

$$CA_1 = TB_1 + r^* \cdot B_0^* = TB_1$$

$$TB_2 = Q_2 - C_2 = \frac{1}{2}Q_2 - \frac{(1+r^*)}{2}Q_1$$

$$CA_2 = TB_2 + r^* \cdot B_1^* = Q_2 - C_2 + r^* \cdot (-\frac{Q_2 - C_2}{(1+r^*)}) = \frac{Q_2}{2(1+r^*)} - \frac{1}{2}Q_1$$

5.3 不同类型冲击下的影响

不同类型的收入冲击：

①$\Delta Q_1 = 1, \Delta Q_2 = 0.$

$$\Delta C_1 = \frac{1}{2}[\Delta Q_1 + \frac{\Delta Q_2}{2(1+r^*)}] = \frac{1}{2}$$

$$\Delta C_2 = \frac{(1+r^*)}{2}\Delta Q_1 + \Delta Q_2 = \frac{(1+r^*)}{2} \approx \frac{1}{2}$$

$$\Delta TB_1 = \frac{1}{2}\Delta Q_1 - \frac{\Delta Q_2}{2(1+r^*)} = \frac{1}{2}$$

$$\Delta CA_1 = \frac{1}{2}$$

$$\Delta TB_2 = \frac{1}{2}\Delta Q_2 - \frac{(1+r^*)}{2}\Delta Q_1 = -\frac{(1+r^*)}{2} \approx -\frac{1}{2}$$

$$\Delta CA_2 = \frac{\Delta Q_2}{2(1+r^*)} - \frac{1}{2}\Delta Q_1 = -\frac{1}{2}$$

②$\Delta Q_1 = 0, \Delta Q_2 = 1.$

$$\Delta C_1 = \frac{1}{2(1+r^*)} \approx \frac{1}{2}$$

$$\Delta C_2 = \frac{1}{2}$$

$$\Delta TB_1 = -\frac{1}{2(1+r^*)} \approx -\frac{1}{2}$$

$$\Delta CA_1 = -\frac{1}{2(1+r^*)} \approx -\frac{1}{2}$$

$$\Delta TB_2 = \frac{1}{2}$$

$$\Delta CA_2 = \frac{1}{2(1+r^*)} \approx \frac{1}{2}$$

③$\Delta Q_1 = 1, \Delta Q_2 = 1.$

$$\begin{aligned}
\Delta C_1 &= \frac{2+r^*}{2(1+r^*)} \approx 1 \\
\Delta C_2 &= \frac{2+r^*}{2} \approx 1 \\
\Delta TB_1 &= \frac{1}{2} - \frac{1}{2(1+r^*)} = \frac{r^*}{2(1+r^*)} \approx 0 \\
\Delta CA_1 &= \frac{r^*}{2(1+r^*)} \approx 0 \\
\Delta TB_2 &= \frac{1}{2} - \frac{(1+r^*)}{2} = -\frac{r^*}{2} \approx 0 \\
\Delta CA_2 &= \frac{1}{2(1+r^*)} - \frac{1}{2} = -\frac{r^*}{2(1+r^*)} \approx 0
\end{aligned}$$

Rmk:

①考虑耐用品、消费习惯的影响

$$u(C_2) = \ln(C_2) \begin{array}{l} \nearrow \ln(C_2 + (1-\delta)C_1) \\ \\ \searrow \ln(C_2 - \alpha C_1) \end{array}$$

②如何实现消费平滑？$\Leftarrow$跨期借贷.