

# 两期模型(Two Periods Model)

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$$\max \{u(C_1) + \beta \cdot u(C_2)\}$$

s.t.

$$\begin{cases} \text{Period1: } C_1 + B_1 - B_0 = r_0 \cdot B_0 + Q_1 \text{ ①} \\ \text{Period2: } C_2 + B_2 - B_1 = r_1 \cdot B_1 + Q_2 \text{ ②} \end{cases}$$

此处 $C_t$ 是消费,  $B_t$ 是持有的债券,  $Q_t$ 是禀赋收入,  $r_t$ 是利率,  $B_0$ 为初始资产, 可假设为0.

Transversality Condition(TVC):  $B_2=0$ .

$$\begin{cases} \text{TVC: } B_T \geq 0 & (\text{不允许负资产}) \\ \text{NPG: } \frac{B_T}{(1+r)^T} \geq 0 & (\text{负资产的增速不能太快}) \end{cases}$$

## 1 求解模型

$$\text{由②} \Rightarrow B_1 = \frac{C_2 - Q_2}{(1+r_1)}. \text{代入①} \Rightarrow C_1 + \frac{C_2}{(1+r_1)} = (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{1+r_1}.$$

$$\max \{u(C_1) + \beta \cdot u(C_2)\}$$

s.t.

$$C_1 + \frac{C_2}{(1+r_1)} = (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)}$$

### 1.1 Lagrange求解

定义

$$L = u(C_1) + \beta u(C_2) + \lambda \cdot [(1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)} - C_1 - \frac{C_2}{(1+r_1)}]$$

其中 $\lambda$ 为Lagrange乘子.

## 1.2 求 First-order-condition(FOC)

$$\begin{cases} C_1 : u'(C_1) - \lambda = 0 \textcircled{3} \\ C_2 : \beta \cdot u'(C_2) - \frac{\lambda}{1+r_1} = 0 \textcircled{4} \\ \lambda : C_1 + \frac{C_2}{(1+r_1)} = (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)} \textcircled{5} \end{cases}$$

定义

$$\bar{Y} \equiv (1+r_0) \cdot B_0 + Q_1 + \frac{Q_2}{(1+r_1)}$$

将③带入④得:

$$\begin{cases} \beta(1+r_1)u'(C_2) = u'(C_1) \\ C_1 + \frac{C_2}{(1+r_1)} = \bar{Y} \end{cases}$$

(\*)假设:  $u(C_t) = \ln C_t$ ,  $B_0 = 0$ ,  $r_0 = r_1 = r$ , 则

$$\Rightarrow \begin{cases} \beta \cdot (1+r) \cdot \frac{1}{C_2} = \frac{1}{C_1} \textcircled{6} \\ C_1 + \frac{C_2}{1+r} = Q_1 + \frac{Q_2}{1+r} \textcircled{7} \end{cases}$$

由⑥可得  $C_2 = \beta(1+r)C_1$ , 再代入⑦, 可以得到:

$$C_1 + \beta \cdot C_1 = Q_1 + \frac{Q_2}{(1+r)}$$

进一步地, 我们可得:

$$\begin{cases} C_1 = \frac{1}{(1+\beta)} \left[ Q_1 + \frac{Q_2}{(1+r)} \right] \\ C_2 = \frac{\beta(1+r)}{(1+\beta)} \left[ Q_1 + \frac{Q_2}{(1+r)} \right] \end{cases}$$

## 2 分析不同的收入冲击类型对消费、储蓄的影响

问题导向:  $\rightarrow$  分析不同类型收入冲击的影响

收入冲击的类型:

$$\begin{cases} \textcircled{1} \text{暂时性收入冲击:} & \Delta Q_1 = 1, \Delta Q_2 = 0 \\ \textcircled{2} \text{预期未来收入冲击:} & \Delta Q_1 = 0, \Delta Q_2 = 1 \\ \textcircled{3} \text{永久性收入冲击:} & \Delta Q_1 = 1, \Delta Q_2 = 0 \end{cases}$$

为了便于分析, 我们假设  $\beta = 1$ , 则

$$\begin{cases} C_1 = \frac{1}{2} \left[ Q_1 + \frac{Q_2}{(1+r)} \right] \\ C_2 = (1+r)C_1 = \frac{(1+r)}{2} \left[ Q_1 + \frac{Q_2}{(1+r)} \right] \end{cases}$$

进一步，我们得到

$$\begin{cases} \Delta C_1 = \frac{1}{2}[\Delta Q_1 + \frac{\Delta Q_2}{(1+r)}] \\ \Delta C_2 = \frac{(1+r)}{2}[\Delta Q_1 + \frac{\Delta Q_2}{(1+r)}] \end{cases}$$

由储蓄定义  $S_t = B_t - B_{t-1}$

$$\Rightarrow \begin{cases} S_1 = B_1 - B_0 = -C_1 + Q_1 - 0 = -\frac{Q_2}{2(1+r)} + \frac{Q_1}{2} \\ S_2 = B_2 - B_1 = 0 + (C_1 - Q_1) = -\frac{Q_1}{2} + \frac{Q_2}{2(1+r)} \end{cases}$$

$$\Rightarrow \begin{cases} \Delta S_1 = -\frac{\Delta Q_2}{2(1+r)} + \frac{\Delta Q_1}{2} \\ \Delta S_2 = -\frac{\Delta Q_1}{2} + \frac{\Delta Q_2}{2(1+r)} \end{cases}$$

$$\textcircled{1} \Delta Q_1 = 1, \Delta Q_2 = 0$$

$$\begin{cases} \Delta C_1 = \frac{1}{2} \\ \Delta C_2 = \frac{(1+r)}{2} \approx \frac{1}{2} \\ \Delta S_1 = \frac{1}{2} \\ \Delta S_2 = -\frac{1}{2} \end{cases}$$

$$\textcircled{2} \Delta Q_1 = 0, \Delta Q_2 = 1$$

$$\begin{cases} \Delta C_1 = \frac{1}{2(1+r)} \approx \frac{1}{2} \\ \Delta C_2 = \frac{1}{2} \\ \Delta S_1 = -\frac{1}{2(1+r)} \\ \Delta S_2 = \frac{1}{2(1+r)} \end{cases}$$

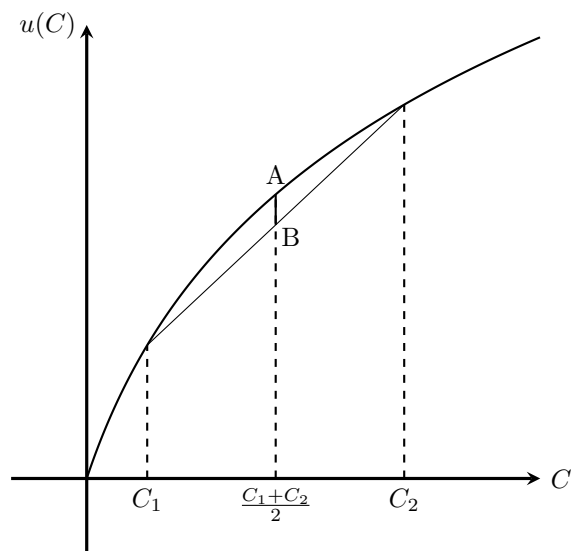
$$\textcircled{3} \Delta Q_1 = 1, \Delta Q_2 = 1$$

$$\begin{cases} \Delta C_1 = \frac{2+r}{2(1+r)} \approx 1 \\ \Delta C_2 = \frac{2+r}{2} \approx 1 \\ \Delta S_1 = \frac{r}{2(1+r)} \approx 0 \\ \Delta S_2 = -\frac{r}{2(1+r)} \approx 0 \end{cases}$$

Rmk:

① 为什么要消费平滑  $\Leftarrow$  边际效用递减  $\Leftarrow u(C_t)$  是凹函数.

$$u|_A = u\left(\frac{C_1 + C_2}{2}\right)$$



$$\begin{aligned}
 u|_B &= \frac{1}{2} (u(C_1) + u(C_2)) \\
 u|_A &\geq u|_B \\
 \Rightarrow u\left(\frac{C_1 + C_2}{2}\right) &\geq \frac{1}{2} (u(C_1) + u(C_2)) \\
 \Rightarrow u\left(\frac{C_1 + C_2}{2}\right) + u\left(\frac{C_1 + C_2}{2}\right) &\geq u(C_1) + u(C_2)
 \end{aligned}$$

② 如何实现消费平滑?  $\leftarrow$  跨期借贷.

### 3 考虑耐用品(Durable goods)

(假设  $u(C_t) = \ln(C_t)$ ,  $r_0 = r_1 = r$ ,  $B_0 = 0$ )

$$\max \{ \ln(C_1) + \beta \cdot \ln[(1 - \delta)C_1 + C_2] \}$$

其中  $\delta$  为折旧率,  $\delta \in [0, 1]$

s.t.

$$C_1 + \frac{C_2}{(1+r)} = Q_1 + \frac{Q_2}{(1+r)}, \quad B_2 = 0(\text{TVC})$$

⊗ 求解

### 3.1 Lagrange求解

定义

$$L = \ln(C_1) + \beta \cdot \ln[(1 - \delta)C_1 + C_2] + \lambda \cdot \left[ Q_1 + \frac{Q_2}{(1 + r)} - C_1 - \frac{C_2}{(1 + r)} \right]$$

其中 $\lambda$ 为预算约束对应的Lagrange乘子

### 3.2 求出FOC

$$\begin{cases} C_1 : \quad \frac{1}{C_1} + \beta(1 - \delta) \frac{1}{(1 - \delta)C_1 + C_2} - \lambda = 0 \textcircled{6} \\ C_2 : \quad \beta \cdot \frac{1}{(1 - \delta)C_1 + C_2} - \lambda \cdot \frac{1}{(1 + r)} = 0 \textcircled{7} \\ \lambda : \quad C_1 + \frac{C_2}{(1 + r)} = Q_1 + \frac{Q_2}{(1 + r)} \textcircled{8} \end{cases}$$

(\*)假设 $\beta = 1$ ，则

$$\begin{aligned} & \begin{cases} \frac{1}{C_1} + (1 - \delta) \cdot \frac{1}{(1 - \delta)C_1 + C_2} = \lambda \quad \Leftarrow \textcircled{6} \\ \frac{1}{(1 - \delta)C_1 + C_2} = \lambda \cdot \frac{1}{(1 + r)} \quad \Leftarrow \textcircled{7} \end{cases} \\ \Rightarrow & \frac{1}{C_1} + (1 - \delta) \cdot \frac{1}{(1 - \delta)C_1 + C_2} = \frac{(1 + r)}{(1 - \delta)C_1 + C_2} \\ \Rightarrow & (1 - \delta)C_1 + C_2 + (1 - \delta)C_1 = (1 + r)C_1 \\ \Rightarrow & C_2 = (2\delta - 1) \cdot C_1 + r \cdot C_1 = (r + 2\delta - 1)C_1 \end{aligned}$$

代入⑧，得：

$$\begin{aligned} & C_1 + \frac{(r + 2\delta - 1)}{(1 + r)} \cdot C_1 = Q_1 + \frac{Q_2}{(1 + r)} \\ \Rightarrow & \begin{cases} C_1 = \frac{(1 + r)Q_1}{2(r + \delta)} + \frac{Q_2}{2(r + \delta)} \\ C_2 = (r + 2\delta - 1) \cdot \left[ \frac{(1 + r)Q_1}{2(r + \delta)} + \frac{Q_2}{2(r + \delta)} \right] \end{cases} \end{aligned}$$

(当 $\delta = 1$ 时，与上述两期模型一致)

$$\begin{cases} \Delta C_1 = \frac{1}{2(r + \delta)} \cdot [(1 + r)\Delta Q_1 + \Delta Q_2] \\ \Delta C_2 = (r + 2\delta - 1) \cdot \left[ \frac{1 + r}{2(r + \delta)} \Delta Q_1 + \frac{1}{2(r + \delta)} \Delta Q_2 \right] \end{cases}$$

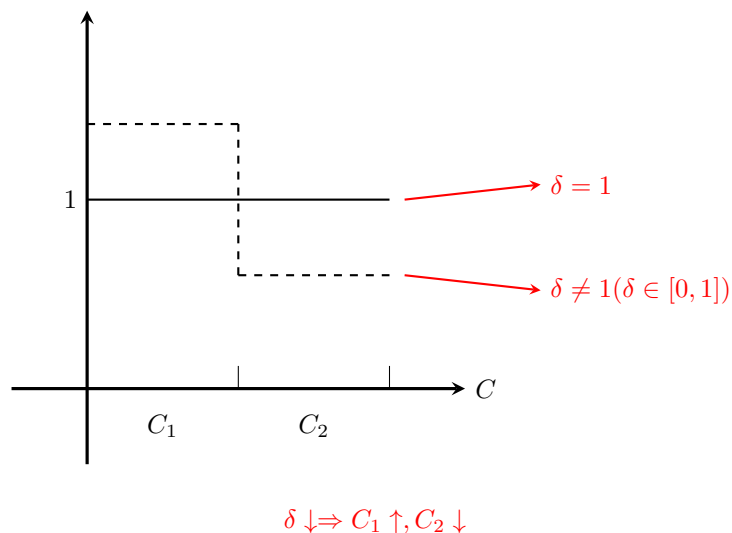
$$\begin{cases} \textcircled{1} \Delta Q_1 = 1, \Delta Q_2 = 0 \\ \textcircled{2} \Delta Q_1 = 0, \Delta Q_2 = 1 \\ \textcircled{3} \Delta Q_1 = 1, \Delta Q_2 = 1 \end{cases}$$

⊗分析 $\delta$ 对消费、储蓄的影响。取 $r = 0, Q_1 = Q_2 = 1$ .

$$C_1 = \frac{Q_1}{2\delta} + \frac{Q_2}{2\delta} = \frac{1}{2\delta} + \frac{1}{2\delta} = \frac{1}{\delta}, \quad C_2 = \frac{2\delta-1}{2\delta} \cdot Q_1 + \frac{2\delta-1}{2\delta} \cdot Q_2 = \frac{2\delta-1}{2\delta} + \frac{2\delta-1}{2\delta} = \frac{2\delta-1}{\delta}$$

当 $\delta = 1$ 时, 则 $C_1 = 1, C_2 = 1$

当 $\delta \neq 1$ 时, 则 $C_1 = \frac{1}{\delta}, C_2 = \frac{2\delta-1}{\delta} = 2 - \frac{1}{\delta}$



Rmk:

折旧率越低, 越提前消费。

#### 4 考虑消费惯性(Habit formation).

$$\max \{u(C_1) + \beta \cdot u(C_2 - \alpha \cdot C_1)\}$$

其中 $\alpha$ 为消费惯性系数,  $\alpha \in (0, 1)$ .

s.t.

$$\begin{cases} C_1 + B_1 - B_0 = r \cdot B_0 + Q_1 \\ C_2 + B_2 - B_1 = r \cdot B_1 + Q_2 \end{cases}$$

假设 $B_0 = 0, B_2 = 0$ (TVC),  $r = 0, \beta = 1, Q_1 = Q_2 = Q, u(C_t) = \ln(C_t)$ .

$$\Rightarrow \begin{cases} C_1 + B_1 = Q \\ C_2 = B_1 + Q \end{cases}$$

$$\Rightarrow C_1 + C_2 = 2Q$$

⊗求解:

## 4.1 Lagrange求解

定义

$$L = [\ln(C_1) + \ln(C_2 - \alpha \cdot C_1)] + \lambda \cdot [2Q - C_1 - C_2]$$

此处 $\lambda$ 为预算约束对应的Lagrange乘子.

## 4.2 求FOC

$$\begin{cases} C_1 : \frac{1}{C_1} - \alpha \cdot \frac{1}{C_2 - \alpha \cdot C_1} - \lambda = 0 \\ C_2 : \frac{1}{C_2 - \alpha \cdot C_1} - \lambda = 0 \end{cases}$$

$$\Rightarrow \frac{1}{C_1} - \frac{\alpha}{C_2 - \alpha \cdot C_1} = \frac{1}{C_2 - \alpha \cdot C_1}$$

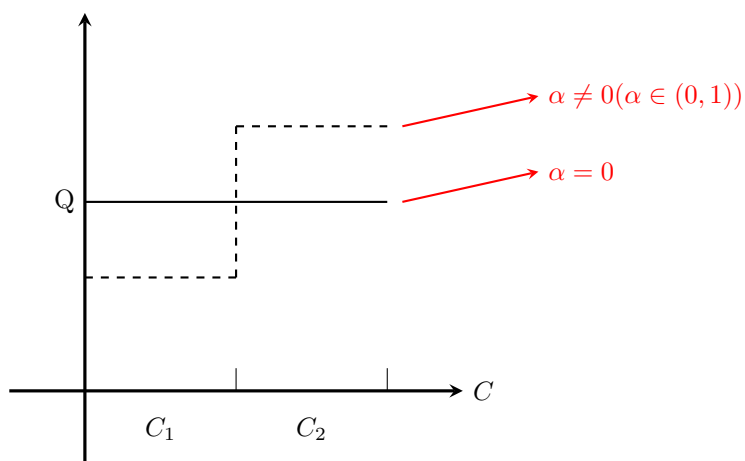
$$\Rightarrow (C_2 - \alpha \cdot C_1) - \alpha C_1 = C_1$$

$$\Rightarrow C_2 = (2\alpha + 1) \cdot C_1$$

由 $C_1 + C_2 = 2Q$

$$\begin{cases} C_1 = \frac{1}{1+\alpha}Q \\ C_2 = \frac{2\alpha+1}{1+\alpha}Q \end{cases}$$

当 $\alpha = 0$ 时,  $C_1 = C_2 = Q$



Rmk:

消费惯性越大，越倾向未来消费。

## 5 封闭 $\Rightarrow$ 开放经济

$$\max\{\ln(C_1) + \beta \cdot \ln(C_2)\}$$

s.t.

$$\begin{cases} C_1 + B_1^* - B_0^* = r_0 \cdot B_0^* + Q_1 \\ C_2 + B_2^* - B_1^* = r_1 \cdot B_1^* + Q_2 \end{cases}$$

当 $B_0^* = 0, B_2^* = 0$ (TVC),  $r_0 = r_1 = r^*$ (小国开放),  $\beta = 1$ 时, 则

$$\Rightarrow \max\{\ln(C_1) + \ln(C_2)\}$$

s.t.

$$C_1 + \frac{C_2}{(1+r^*)} = Q_1 + \frac{Q_2}{(1+r^*)}$$

⊗求解:

### 5.1 Lagrange求解

定义

$$L = [\ln(C_1) + \ln(C_2)] + \lambda \cdot [Q_1 + \frac{Q_2}{(1+r^*)} - C_1 - \frac{C_2}{(1+r^*)}]$$

其中 $\lambda$ 为Lagrange乘子

### 5.2 求出FOC

$$\begin{cases} C_1: & \frac{1}{C_1} - \lambda = 0 \\ C_2: & \frac{1}{C_2} - \lambda \cdot \frac{1}{(1+r^*)} = 0 \\ \lambda: & C_1 + \frac{C_2}{(1+r^*)} = Q_1 + \frac{Q_2}{(1+r^*)} \end{cases}$$



$$\Rightarrow C_1 = \frac{1}{2}[Q_1 + \frac{Q_2}{(1+r^*)}], \quad C_2 = \frac{(1+r^*)}{2}[Q_1 + \frac{Q_2}{(1+r^*)}]$$

↗ Trade Balance

$$TB_1 = Q_1 - C_1 = \frac{1}{2}Q_1 - \frac{Q_2}{2(1+r^*)}$$

$$CA_1 = TB_1 + r^* \cdot B_0^* = TB_1$$

$$TB_2 = Q_2 - C_2 = \frac{1}{2}Q_2 - \frac{(1+r^*)}{2}Q_1$$

$$CA_2 = TB_2 + r^* \cdot B_1^* = Q_2 - C_2 + r^* \cdot (-\frac{Q_2 - C_2}{(1+r^*)}) = \frac{Q_2}{2(1+r^*)} - \frac{1}{2}Q_1$$

### 5.3 不同类型冲击下的影响

不同类型的收入冲击下：

$$\Delta C_1 = \frac{1}{2}\Delta Q_1 + \frac{1}{2(1+r^*)}\Delta Q_2$$

$$\Delta C_2 = \frac{(1+r^*)}{2}\Delta Q_1 + \frac{1}{2}\Delta Q_2$$

$$\Delta TB_1 = \frac{1}{2}\Delta Q_1 - \frac{1}{2(1+r^*)}\Delta Q_2$$

$$\Delta CA_1 = \Delta TB_1$$

$$\Delta TB_2 = \frac{1}{2}\Delta Q_2 - \frac{(1+r^*)}{2}\Delta Q_1$$

$$\Delta CA_2 = \frac{1}{2(1+r^*)}\Delta Q_2 - \frac{1}{2}\Delta Q_1$$

$$\textcircled{1} \Delta Q_1 = 1, \Delta Q_2 = 0.$$

$$\Delta C_1 = \frac{1}{2}$$

$$\Delta C_2 = \frac{(1+r^*)}{2} \approx \frac{1}{2}$$

$$\Delta TB_1 = \frac{1}{2}$$

$$\Delta CA_1 = \Delta TB_1 = \frac{1}{2}$$

$$\Delta TB_2 = -\frac{(1+r^*)}{2} \approx -\frac{1}{2}$$

$$\Delta CA_2 = -\frac{1}{2}$$

$$\textcircled{2} \Delta Q_1 = 0, \Delta Q_2 = 1.$$

$$\Delta C_1 = \frac{1}{2(1+r^*)} \approx \frac{1}{2}$$

$$\Delta C_2 = \frac{1}{2}$$

$$\Delta TB_1 = -\frac{1}{2(1+r^*)} \approx -\frac{1}{2}$$

$$\Delta CA_1 = \Delta TB_1 = -\frac{1}{2(1+r^*)} \approx -\frac{1}{2}$$

$$\Delta TB_2 = \frac{1}{2}$$

$$\Delta CA_2 = \frac{1}{2(1+r^*)} \approx \frac{1}{2}$$

$$\textcircled{3} \Delta Q_1 = 1, \Delta Q_2 = 1.$$

$$\Delta C_1 = \frac{2+r^*}{2(1+r^*)} \approx 1$$

$$\Delta C_2 = \frac{2+r^*}{2} \approx 1$$

$$\Delta TB_1 = \frac{1}{2} - \frac{1}{2(1+r^*)} = \frac{r^*}{2(1+r^*)} \approx 0$$

$$\Delta CA_1 = \Delta TB_1 = \frac{r^*}{2(1+r^*)} \approx 0$$

$$\Delta TB_2 = \frac{1}{2} - \frac{(1+r^*)}{2} = -\frac{r^*}{2} \approx 0$$

$$\Delta CA_2 = \frac{1}{2(1+r^*)} - \frac{1}{2} = -\frac{r^*}{2(1+r^*)} \approx 0$$

Rmk:

①考虑耐用品、消费习惯的影响

$$\nearrow \ln(C_2 + (1-\delta)C_1)$$

$$u(C_2) = \ln(C_2)$$

$$\searrow \ln(C_2 - \alpha C_1)$$

②如何实现消费平滑?  $\Leftarrow$  跨期借贷.