Analytic Number Theory-Homework-1

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(1) (a) For all $\xi, q \in A$, we have $(\xi * q)(i) = \xi(i)q(i) \neq 0$. Thus $\xi * q \in A$.

(b) For all $\xi, q, h \in A$, we have

$$(\xi * q)(n) = \sum_{d|n} \xi(d) q\left(\frac{n}{d}\right) = \sum_{d|n} q(d)\xi\left(\frac{n}{d}\right) = (q * \xi)(n).$$

Thus $\xi * q = q * \xi$.
We also have

$$\xi * (q * h)(n) = \sum_{d|n} \xi(d)(q * h)\left(\frac{n}{d}\right) = \sum_{d_1 d_2 d_3 = n} \xi(d_1)q(d_2)h(d_3)$$

$$= \sum_{d|n} (\xi * q)(d) h\left(\frac{n}{d}\right) = (\xi * q) * h(n).$$

Thus $\xi * (q * h) = (\xi * q) * h$.

(c) By definition,

$$\delta(n) = \begin{cases} 1, & n = 1, \\ 0, & \text{otherwise}, \end{cases}$$

so $\delta(n) \in A$. Moreover, for any $f \in A$,

$$(\xi * \delta)(n) = \sum_{d|n} \xi(d)\delta\left(\frac{n}{d}\right) = \xi(n).$$

Thus, by (b), $\delta * \xi = \xi * \delta = \xi$, so δ is the identity element under $*$.

(d) We construct ξ^{-1} inductively. Define $\xi^{-1}(1) := \xi(1)^{-1}$. Suppose $\xi^{-1}(k)$ has been defined for $k = 1, \ldots, n-1$. Define

$$\xi^{-1}(n) = -\frac{1}{\xi(n)} \sum_{\substack{d|n \\ d<n}} \xi^{-1}(d)\xi\left(\frac{n}{d}\right), \quad n \neq 1.$$

Then we have $\xi * \xi^{-1} = \xi^{-1} * \xi = \delta$ by construction.

(e) This is clear from parts (a)–(d).

(2) (a) For $m, n \in \mathbb{Z}_{>0}$, $(m, n) = 1$, if $d \mid mn$, then d can be uniquely written as $d = d_1 d_2$, where $d_1 \mid m$, $d_2 \mid n$. Thus

$$(\xi * q)(mn) = \sum_{d|mn} \xi(d)q\left(\frac{mn}{d}\right) = \sum_{d_1|m,\, d_2|n} \xi(d_1)q\left(\frac{n}{d_1}\right)\xi(d_2)q\left(\frac{m}{d_2}\right)$$

$$= \left(\sum_{d_1|n} \xi(d_1)q\left(\frac{n}{d_1}\right)\right)\left(\sum_{d_2|m} \xi(d_2)q\left(\frac{m}{d_2}\right)\right) = (\xi * q)(m)(\xi * q)(n).$$

Therefore, $\xi * q$ is multiplicative.

(b) No. Consider $I(n) = n$ for all $n \in \mathbb{Z}_{>0}$, which is completely multiplicative, but $I * I$ is not completely multiplicative, since

$$(I * I)(4) = 12, \quad \text{while} \quad ((I * I)(2))^2 = 16.$$

(3) For $n \in \mathbb{Z}_{>0}$, there exists $d \in \mathbb{Z}_{>0}$ such that $d^2 < n \leq (d+1)^2$ (i.e., $d^2 \leq n-1 < (d+1)^2$).

If $n = (d+1)^2$, then $f(n) = \lfloor \sqrt{n} \rfloor - \lfloor \sqrt{n-1} \rfloor = d+1-d = 1$; if $n < (d+1)^2$, then $f(n) = d-d = 0$.

Thus we have

$$f(n) = \begin{cases} 1, & n \text{ is a square,} \\ 0, & \text{otherwise.} \end{cases}$$

For $m, n \in \mathbb{Z}_{>0}$ with $(m,n) = 1$, mn is a square if and only if m and n are squares. Thus f is multiplicative. However, $f(4) = 1 \neq f(2)f(2) = 0$, so f is not completely multiplicative.

(4) (a) Suppose $n = 2^m \cdot p_1^{k_1} \cdots p_r^{k_r}$, where $p_1, \ldots, p_r$ are odd primes and $m, k_1, \ldots, k_r \in \mathbb{Z}_{>0}$. Then

$$\varphi(n) = n \cdot \prod_{p \mid n} \left(1 - \frac{1}{p}\right) \Rightarrow \prod_{p \mid n} \left(1 - \frac{1}{p}\right) = \frac{1}{2}.$$

Thus $n = 2^m$, $m \in \mathbb{Z}_{>0}$.

(b) If n is odd, then $\varphi(2n) = \varphi(2) \cdot \varphi(n) = \varphi(n)$. If n is even, write $n = 2^m \cdot p_1^{k_1} \cdots p_r^{k_r}$ with $m > 0$, then

$$\varphi(2n) = 2n \cdot \frac{1}{2} \cdot \prod_{i} \left(1 - \frac{1}{p_i}\right) = 2 \cdot \varphi(n).$$

Therefore, $\varphi(n) = \varphi(2n)$ if and only if n is odd.

(5) Suppose $n = p_1 \cdots p_r \cdot q_1^{a_1} \cdots q_s^{a_s}$, where $a_1, \ldots, a_s \geq 2$. Then

$$\sum_{d \mid n} \mu(d) = \sum_{d \mid q_1 \cdots q_s} \mu(d) = \delta(q_1 \cdots q_s) = \begin{cases} 1, & s = 0, \text{ i.e., } n \text{ is square-free,} \\ 0, & \text{otherwise.} \end{cases}$$

Thus

$$\sum_{d \mid n} \mu(d) = \mu^2(n).$$

(6) Suppose $n = \prod_{i=1}^{r} p_i^{a_i}$, $a_i \geq 1$. Then we have

$$\sum_{d \mid n} \tau(d)^3 = \prod_{i=1}^{r} \left(1 + 2^3 + \cdots + (a_i+1)^3\right),$$

$$\left(\sum_{d \mid n} \tau(d)\right)^2 = \prod_{i=1}^{r} \left(1 + 2 + \cdots + (a_i+1)\right)^2.$$

Since

$$\sum_{k=1}^{m} k^3 = \left(\sum_{k=1}^{m} k\right)^2, \quad \forall m > 0,$$

it follows that $\sum_{d \mid n} \tau(d)^3 = \left(\sum_{d \mid n} \tau(d)\right)^2$.

(7) (a) **Theorem 1:** For every $c > 0$,

$$\#\{n \leq x \mid \varphi(\sigma(n)) \geq cn\} \leq \frac{\pi^2}{6c} \cdot \frac{x}{\log_4 x} + O\left(\frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right).$$

Theorem 2: Suppose $f : \mathbb{R}^+ \to \mathbb{R}^+$ is a non-decreasing function satisfying $f(x) = o(\log_4 x)$. Let

$$P_f(x) := \{n \leq x \mid \varphi(\sigma(n)) \geq \frac{n}{f(n)}\}.$$

Then

$$|P_f(x)| = O\left(\frac{x f(x)}{\log_4 x} + \frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right) = o(x).$$

Main Steps:

We have:

$$\varphi(\sigma(n)) = \sigma(n) \prod_{p|\sigma(n)} \left(1 - \frac{1}{p}\right).$$

Step 1: Estimate $\prod_{p|\sigma(n)} \left(1 - \frac{1}{p}\right)$:
For every $y \in \mathbb{N}^*$, denote

$$P(y) := \prod_{p \le y} p.$$

If $P(y) \mid \sigma(n)$, then for all $p \le y$, $p \mid \sigma(n)$. Thus,

$$\prod_{p|\sigma(n)} \left(1 - \frac{1}{p}\right) \le \prod_{p \le y} \left(1 - \frac{1}{p}\right) < \frac{1}{\log y} \quad \text{(by Mertens' theorem)}.$$

Step 2: Estimate $\#\{n \le x \mid \sigma(n) < \delta n\}$:
We have:

$$\#\{n \le x \mid \sigma(n) \ge \delta n\} = \sum_{\substack{n \le x \\ \sigma(n) \ge \delta n}} 1 \le \frac{1}{\delta} \sum_{n \le x} \frac{\sigma(n)}{n} = \frac{\pi^2 x}{6\delta} + O\left(\frac{\log^2 x}{\delta}\right).$$

So,

$$\#\{n \le x \mid \sigma(n) < \delta n\} = [x] - \#\{n \le x \mid \sigma(n) \ge \delta n\} = x\left(1 - \frac{\pi^2}{6\delta}\right) + O\left(\frac{\log^2 x}{\delta}\right).$$

Step 3: Estimate $\#\{n \le x \mid P(y) \mid \sigma(n)\}$:
Define

$$S_p(x) := \#\{n \le x \mid p \nmid \sigma(n)\}.$$

Then,

$$\#\{n \le x \mid P(y) \nmid \sigma(n)\} \le \sum_{p \le y} S_p(x).$$

Lemma 1: For any prime p and $x \ge e^p$,

$$S_p(x) = O\left(x \left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{p-1}}\right),$$

where the implied constant is absolute.
By Lemma 1,

$$\sum_{p \le y} S_p(x) = O\left(\sum_{p \le y} x \left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{p-1}}\right) = O\left(x \left(\frac{\log_2 x}{\log x}\right)^{\frac{1}{y}} \cdot \frac{y}{\log y}\right).$$

Step 4: Estimate $\#\{n \le x \mid \varphi(\sigma(n)) \ge cn\}$:
Choose $y = \log_3 x$, $\delta = c \log_4 x$. Then,

$$\#\{n \le x \mid \varphi(\sigma(n)) < cn\} \ge \#\{n \le x \mid \sigma(n) \le \delta n\} + \#\{n \le x \mid P(y) \mid \sigma(n)\} - x.$$

$$\ge x - \frac{\pi^2}{6c} \cdot \frac{x}{\log_4 x} + O\left(\frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right),$$

which implies:

$$\#\{n \le x \mid \varphi(\sigma(n)) \ge cn\} \le \frac{\pi^2}{6c} \cdot \frac{x}{\log_4 x} + O\left(\frac{x \log_3 x}{(\log x)^{\frac{1}{\log_3 x}} \log_4 x}\right).$$

Similarly, Theorem 2 follows by replacing $\delta = c \log_4 x$ with $\delta = \frac{\log_4 x}{f(x)}$.

(b) Paper: The Summatory Function of the Möbius Function Involving the Greatest Common Divisor by Isao Kiuchi and Sumaia Saad Eddin (2024).

Link:https://arxiv.org/pdf/2403.02792

This paper studies a new summatory function based on the Möbius function $\mu(n)$ and the greatest common divisor. For $x > 5$, the authors define

$$S_\mu(x) := \sum_{mn \le x} \mu(\gcd(m, n)).$$

They derive asymptotic formulas for $S_\mu(x)$, obtain precise estimates for its error term, and provide conditional refinements under the Riemann Hypothesis (RH).

Main Results:

1. Asymptotic formula (Theorem 1):

The authors prove that

$$S_\mu(x) = \frac{x}{\zeta^2(2)}\left(\log x + 2\gamma - 1 - 4\frac{\zeta'(2)}{\zeta(2)}\right) + O\Big(x^{1/2}\log^2 x\Big),$$

where γ is the Euler–Mascheroni constant and $\zeta(s)$ denotes the Riemann zeta-function.

2. Improved error term (Theorem 2):

Without assuming RH, the error term satisfies

$$E_\mu(x) = O\Big(x^{1/2}\exp\big(-A(\log x \log\log x)^{1/3}\big)\Big),$$

for some constant $A > 0$. This strengthens the bound $O(x^{1/2}\log^2 x)$ by including an exponential decay factor.

3. Conditional estimate under RH (Theorem 3 and Corollary 1):

Assuming the Riemann Hypothesis, the error term can be improved further:

$$E_\mu(x) \ll x^{439/1032+\varepsilon} \approx x^{0.4254+\varepsilon},$$

which represents a modest improvement over the $x^{1/2}$ order of magnitude.