Analytic Number Theory Fall 2025 - Homework

Homework 4 is due Wednesday, January 7th **Name:**

Directions

- Your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

Homework 4

(1) Prove that, for $Re(s) > 1$, we have

$$\log \zeta(s) = s \int_2^\infty \frac{\pi(x)}{x(x^s - 1)} dx.$$

where $\pi(x) = \sum_{p \leq x} 1$ is the prime counting function.

(2) In this problem, we prove the functional equation of Riemann zeta function via contour integral, which interprets another way for the analytic continuation. Please refer to this paper.

(a) Consider the contour integral

$$g(s) = \frac{1}{2\pi i} \int_C \frac{z^{s-1}}{e^{-z} - 1} dz.$$

Show that $\zeta(s) = \Gamma(1-s) g(s)$ for $Re(s) > 1$.

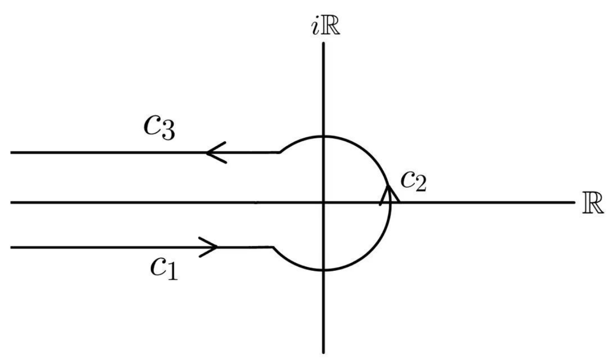

FIGURE 1. Contour C: C_2 is a circle with radius ϵ, C_1,C_3 are two rays in different directions on the real axis.

(b) Next we consider a series of contour integrals $g_k(s)$ to approximate $g(s)$.

$$g_k(s) = \frac{1}{2\pi i} \int_{C_k} \frac{z^{s-1}}{e^{-z} - 1} dz$$

where the contour C_k is displayed in Figure 2. Use the residue theorem to show that

$$g_k(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2}) \sum_{n=1}^{k} \frac{1}{n^{1-s}}.$$

(Pay attention to the direction of C_k.)

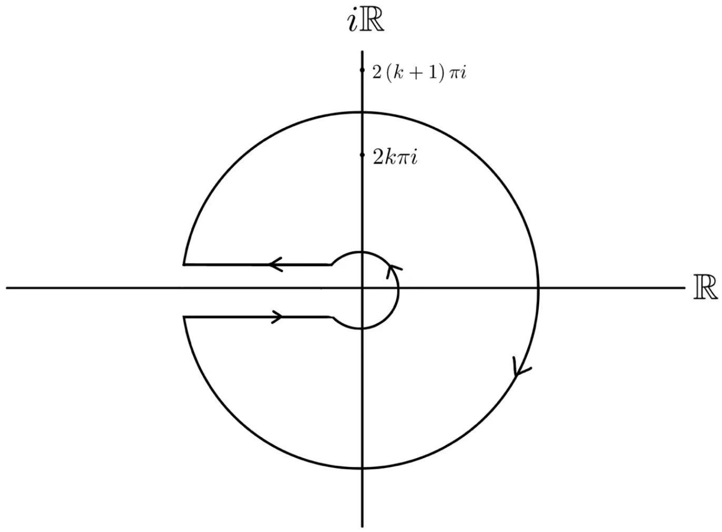

FIGURE 2. Contour C_k: The radius of the inside circle is ϵ, choose the radius of the outside circle to include $2k\pi i$ but exclude $2(k+1)\pi i$ inside the contour, the rest parts are similar to contour C.

(c) Prove that $\lim_{k\to\infty} g_k(s) = g(s)$ for $Re(s) < 0$, then deduce the functional equation of $\zeta(s)$

$$\zeta(s) = 2^s \pi^{s-1} sin(\frac{\pi s}{2})\Gamma(1-s)\zeta(1-s).$$

(d) What is the value of $\zeta(-1)$? Can you say something interesting about it?

(3) We've already known that $s = 1$ is a single pole of $\zeta(s)$. In this problem, we focus on the Laurent expansion of $\zeta(s)$ at $s = 1$.

(a) Prove the Laurent expansion of $\zeta(s)$ at $s = 1$ is given by

$$\zeta(s) = \frac{1}{s-1} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!}(s-1)^n.$$

where

$$\gamma_n = \sum_{k=1}^{\infty} (\frac{\log^n k}{k} - \frac{\log^{n+1}(k+1) - \log^{n+1}(k)}{n+1})$$

are defined to be Stieltjes constants.

(b) Show that γ_0 is exactly the Euler-Mascheroni constant γ, that's why the Stieltjes constants are also called generalized Euler-Mascheroni constants. Then use this result to calculate the Cauchy principal value of $\zeta(s)$ at $s = 1$, i.e. you need to calculate $PV[\zeta(1)] := \lim_{\epsilon\to 0} \frac{\zeta(1+\epsilon)+\zeta(1-\epsilon)}{2}$.

(c) Prove the integral formula of γ_n in terms of $\zeta(s)$

$$\gamma_n = \frac{(-1)^n n!}{2\pi} \int_0^{2\pi} \zeta(e^{ix}+1)e^{-inx}dx.$$

(4) Read the paper "Selberg, Atle. 'An elementary proof of the prime-number theorem.' Annals of Mathematics (1949)" and briefly outline the steps of the elementary proof of the prime number theorem.