

Directions

- Your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

Homework 1

(1) Let A be the set of all arithmetical functions f with $f(1) \neq 0$, Show that

- A is closed under Dirichlet convolution.
- A satisfies the commutative and associative laws.
- A contains an identity element, namely the Dirac function $\delta(n)$.
- For a given function $f \in A$, there exists a unique $f^{-1} \in A$, such that

$$f * f^{-1} = f^{-1} * f = \delta(n).$$

(Please construct the function f^{-1} by induction)

- A forms an abelian group under Dirichlet convolution.

(2) (a) Let f and g be multiplicative. Prove that $f * g$ is also multiplicative.

- If f and g are completely multiplicative, does the conclusion still hold? Provide a proof or counterexample.

(3) Let $f(n) = \lfloor \sqrt{n} - \sqrt{n-1} \rfloor$. Prove that f is multiplicative, but not completely multiplicative.

(4) Find all integers n such that

- $\varphi(n) = \frac{n}{2}$,
- $\varphi(n) = \varphi(2n)$,

(5) Prove that

$$\sum_{d^2|n} \mu(d) = \mu^2(n)$$

(6) Let $\tau(n)$ denotes the number of positive divisors of n . Prove that

$$\sum_{d|n} \tau(d)^3 = \left(\sum_{d|n} \tau(d) \right)^2.$$

(7) Choose one of the following two tasks to complete:

- Read the attached paper and write the conclusion and main steps;
- Find a paper on arithmetical functions $\mu(n)$, $\varphi(n)$, $\sigma(n)$ or $\tau(n)$, provide links and main conclusions.