Analytic Number Theory Homework 4

Xingfeng Lin

2025 Fall

1. *Proof.* Let p_n denote the n-th prime number. Then, for $Re(s) > 1$,

$$
\begin{aligned}
s\int_2^{\infty} \frac{\pi(x)}{x(x^s-1)}dx &= \sum_{n=1}^{\infty} n\int_{p_n}^{p_{n+1}} \frac{s}{x(x^s-1)}dx \\
&= \sum_{n=1}^{\infty} nlog(1-p_{n+1}^{-s}) - \sum_{n+1}^{\infty} nlog(1-p_n^{-s}) \\
&= -\sum_{n=1}^{\infty} log(1-p_n^{-s}) \\
&= \log\zeta(s).
\end{aligned}
$$

□

2. In this exercise, we extend the ζ-function to $\mathbb{C}$ and establish the Riemann's functional equation.

   (a) *Proof.* By Cauchy's theorem, we know that g is independent of ε. By definition, we have

$$2\pi i g(s) = \int_{C_{\varepsilon,1}} + \int_{C_{\varepsilon,2}} + \int_{C_{\varepsilon,3}} \frac{z^{s-1}}{e^{-z}-1}dz.$$

Let $z = Re^{i\theta}$, then

$$
\begin{aligned}
g(s) &= -\frac{e^{-i\pi s}}{2\pi i}\int_{\varepsilon}^{\infty} \frac{R^{s-1}}{e^R-1}dR + \int_{C_{\varepsilon,2}} \frac{z^{s-1}}{e^{-z}-1}dz + \frac{e^{i\pi s}}{2\pi i}\int_{\varepsilon}^{\infty} \frac{R^{s-1}}{e^R-1}dR \\
&= \frac{sin\pi s}{\pi}\int_{\varepsilon}^{\infty} \frac{R^{s-1}}{e^R-1}dR + \int_{C_{\varepsilon,2}} \frac{z^{s-1}}{e^{-z}-1}dz.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\left|\int_{C_{\varepsilon,2}} \frac{z^{s-1}}{e^{-z}-1}dz\right| &\le \int_{C_{\varepsilon,2}} \frac{|z^{s-1}|}{\frac{1}{2}|z|}dz \\
&\le 4\pi\varepsilon^{Re(s)-1}.
\end{aligned}
$$

For $Re(s) > 1$, letting $\varepsilon \to 0^+$, we have

$$\lim_{\varepsilon\to 0^+}\left|\int_{C_{\varepsilon,2}} \frac{z^{s-1}}{e^{-z}-1}dz\right| = 0$$

Therefore,

$$
\begin{aligned}
g(s) &= \frac{sin\pi s}{\pi}\int_0^\infty \frac{R^{s-1}}{e^R-1}dR \\
&= \frac{sin\pi s}{\pi}\int_0^\infty R^{s-1}\sum_{n=1}^\infty e^{-nR}dR \\
&= \frac{sin\pi s}{\pi}\sum_{n=1}^\infty\int_0^\infty R^{s-1}e^{-nR}dR \\
&= \frac{sin\pi s}{\pi}\Gamma(s)\zeta(s).
\end{aligned}
$$

By Euler's reflection formula

$$
\Gamma(s)\Gamma(1-s) = \frac{\pi}{sin\pi s},
$$

we have $\zeta(s) = \Gamma(1-s)g(s)$, $Re(s) > 1$. □

(b) *Proof.* Since $f_s(z) := \frac{z^{s-1}}{e^{-z}-1}$ is meromorphic inside $C_{\varepsilon,k}$, with simple poles $i2n\pi$, $n \neq 0$, by residue theorem,

$$
\begin{aligned}
g_k(s) &= \sum_{n=1}^k (Res(f_s, 2in\pi) + Res(f_s, -2in\pi)) \\
&= \sum_{n=1}^k -(2n\pi)^{s-1}(e^{i\frac{s-1}{2}\pi} + e^{-i\frac{s-1}{2}\pi}) \\
&= 2^s\pi^{s-1}sin(\pi s/2)\sum_{n=1}^k \frac{1}{n^{1-s}}.
\end{aligned}
$$

□

(c) *Proof.* We claim that g is an entire function. To verify this fact, it suffices to show $\int_{C_{\varepsilon,1}} \frac{z^{s-1}}{e^{-z}-1}dz$ converges locally uniformly in $s \in \mathbb{C}$, i.e. $\int_\varepsilon^\infty \frac{R^{s-1}}{e^R-1}dR$ converges locally uniformly in $s \in \mathbb{C}$. This follows from $|R^{s-1}| \lesssim_{Re(s)} R^{-2}(e^R - 1)|$, as $R \to +\infty$. Therefore g is holomorphic on $\mathbb{C}$. Recall that

$$
g(s) = \frac{sin\pi s}{\pi}\int_0^\infty \frac{R^{s-1}}{e^R-1}dR.
$$

We know g has zeros $2, 3, \ldots$ of order 1. By (a) we can define the meromorphic extension of ζ on $\mathbb{C}$ by $\zeta(s) := \Gamma(1-s)g(s)$, which is holomorphic in $\mathbb{C} - \{1\}$. To establish the functional equation of ζ, we now prove that $\lim_k g_k(s) = g(s)$, $Re(s) < 0$.

Recall that $g_{k(s)} = \frac{1}{2\pi i}\int_{C_{\varepsilon,k}} \frac{z^{s-1}}{e^{-z}-1}dz$. Let the outer circle of $C_{\varepsilon,k}$ have radius $r_k = (2k+1)\pi$, and denote the outer circle by C_k^0. For $z \in C_k^0$, we have $|e^z - 1| \geq C$, where $C > 0$ is independent of $z \in C_k^0$, since $cos(r_k) = -1$. Therefore,

$$
\left|\int_{C_k^0} \frac{z^{s-1}}{e^{-z}-1}dz\right| \leq \int_{-\pi}^{\pi} \frac{e^{\pi|Im(s)|}r_k^{Re(s)}}{C}d\theta \to 0,
$$

as $k \to \infty$, since $Re(s) > 0$, and thus we have $\lim_k g_k(s) = g(s)$, $Re(s) < 0$.

From (b) we deduce that

$$g(s) = lim_k g_k(s) = 2^s \pi^{s-1} sin(\pi s/2)\zeta(1-s),$$

for $Re(s) < 0$, and therefore,

$$\zeta(s) = 2^s \pi^{s-1} sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Both sides of the equality are meromorphic functions on $\mathbb{C}$ and agree on $Re(s) < 0$, hence the equality holds for $s \in \mathbb{C}$. $\square$

(d) By the functional equation we have $\zeta(-1) = -\frac{1}{12}$. One can also derive $\sum_n n = -1/12$ by nonsense calculation ignoring the divergence of the series. It is mysterious that the nonsense calculation reaches the right answer, and that this is verified in experimental physics:
the vacuum energy in a one-dimensional cavity formally contains $\frac{1}{2}\sum_n n$. If one puts the value $-1/24$ using ζ, this will lead to the Casimir force, which has been experimentally verified.

3. In this exercise we study the Laurent expansion of ζ.

(a) *Proof.* Put $g(s) := \zeta(s) - \frac{1}{s-1}$. Then we have $\gamma_n = (-1)^n g^{(n)}(1)$. Note that

$$g(s) = 1 - s\int_1^\infty \frac{\{t\}}{t^{s+1}}dt,$$

Immediately we have $\gamma_0 = g(1) = \sum_{k=1}^\infty(\frac{1}{k} - log(1+\frac{1}{k}))$. For $n \geq 1$, we have

$$g^{(n)}(1) = (-1)^n \int_1^\infty \frac{\{t\}(nlog^{n-1}t - log^n t)}{t^2}dt,$$

for $Re(s) > 0$. On the other hand, We have to show that

$$\lim_{x\longrightarrow+\infty}\sum_{1\leq k\leq x}(\frac{log^n k}{k} - \frac{log^{n+1}(k+1) - log^{n+1}k}{n+1}) = \int_1^\infty \frac{\{t\}(nlog^{n-1}t - log^n t)}{t^2}dt,$$

which would complete the proof. Simplify the sum

$$\sum_{1\leq k\leq x}(\frac{log^n k}{k} - \frac{log^{n+1}(k+1) - log^{n+1}k}{n+1}) = \sum_{1<k\leq x}\frac{log^n k}{k} - \frac{1}{n+1}log^{n+1}([x]+1),$$

and apply Euler's summation formula, we obtain

$$\sum_{1\leq k\leq x}(\frac{log^n k}{k} - \frac{log^{n+1}(k+1) - log^{n+1}k}{n+1}) = \int_1^x \frac{\{t\}(nlog^{n-1}t - log^n t)}{t^2}dt + \frac{log^{n+1}(\frac{x}{[x]+1})}{n+1} + \frac{([x]-x)log^n x}{x}.$$

Letting $x \to +\infty$, we finally obtain

$$\gamma_n = \sum_{k=1}^\infty(\frac{log^n k}{k} - \frac{log^{n+1}(k+1) - log^{n+1}k}{n+1}).$$

$\square$

(b) *Proof.* From (a), we know that $\gamma_0 = \sum_{k=1}^{\infty}(\frac{1}{k} - log(1+\frac{1}{k})) = \gamma$. Therefore the Stieltjes constants are also called generalized Euler-Mascheroni constants. From the Laurent expansion, we have

$$\zeta(1+\varepsilon) = \frac{1}{\varepsilon} + \sum_{n=0}^{\infty} \frac{(-1)^n \gamma_n}{n!} \varepsilon^n$$
$$\zeta(1-\varepsilon) = \frac{-1}{\varepsilon} + \sum_{n=0}^{\infty} \frac{\gamma_n}{n!} \varepsilon^n,$$

which gives

$$\frac{1}{2}(\zeta(1+\varepsilon) + \zeta(1-\varepsilon)) = \gamma_0 + \sum_{n=1}^{\infty} \frac{\gamma_{2n}}{(2n)!} \varepsilon^{2n}.$$

Letting $\varepsilon \to 0$, we obtain

$$PV[\zeta(1)] = \gamma_0.$$

□

(c) *Proof.* By the residue theorem, we have

$$\begin{aligned}\frac{(-1)^n \gamma_n}{n!} &= \frac{1}{2\pi i} \int_C \frac{\zeta(s)}{(s-1)^{n+1}} ds \\ &= \frac{1}{2\pi} \int_0^{2\pi} \zeta(1+e^{ix}) e^{-inx} dx,\end{aligned}$$

where C is a circle of radius 1 centered at 1. □

(d) Here we sketch the elementary proof of the Prime Number Theorem by Selberg, 1949.

Let

$$\theta(x) = \sum_{p \le x} logp$$
$$R(x) = \theta(x) - x.$$

Then we have the following Selberg's identity:

Theorem 1.

$$\theta(x)logx + \sum_{p \le x} logp\, \theta(\frac{x}{p}) = 2xlogx + O(x).$$

To prove the prime number theorem, it is equivalent to prove $R(x) = o(x)$. So we have the following estimates of R:

Theorem 2.

$$R(x)logx = \sum_{pq \le x} \frac{logplogq}{logpq} R(\frac{x}{pq}) + O(xloglogx) \tag{1}$$

$$|R(x)| \le \frac{1}{logx} \sum_{n \le x} |R(\frac{x}{n})| + O(x\frac{loglogx}{logx}) \tag{2}$$

$$\sum_{n \le x} \frac{R(n)}{n^2} = O(1). \tag{3}$$

Since $\theta(x) = O(x)$, There exist $\alpha_0 > 0$ such that $|R(x)| < \alpha x$, $\forall x > 1$. Theorems above will deduce that

$$|R(x)| < \alpha_0(1 - \frac{\alpha_0^2}{K})x,$$

for sufficiently large x, where K is a positive constant independent of x, α_0. Iterate this process, we obtain a sequence $\{\alpha_n\}_{n=0}^{\infty}$ of positive real numbers satisfying

$$\alpha_{n+1} = \alpha_n(1 - \frac{\alpha_n^2}{K}),$$

and for each n, there exist some $X_n > 0$ such that $|R(x)| < \alpha_{n+1}x$, $\forall x > X_n$. Since $\alpha_n \to 0, n \to \infty$, We obtain

$$R(x) = o(x).$$

This completes (the sketch of) the proof of the prime number theorem.