Analytic Number Theory Fall 2025 - Homework
Homework 2 is due Wednesday, November 19th **Name:**

Directions

- Your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

Homework 2

(1) Complete the following questions about Euler's summation formula, where $x \geqslant 2$
   (a) Describe Euler's summation formula without the need for proof.
   (b) Prove the asymptotic formula for harmonic series
   $$\sum_{n \leqslant x} \frac{1}{n} = \log x + \gamma + O(\frac{1}{x}),$$
   where γ is Euler–Mascheroni constant.
   (c) Prove that
   $$\sum_{n \leqslant x} \frac{\log n}{n} = \frac{1}{2} \log^2 x + A + O(\frac{\log x}{x}),$$
   where A is a constant.
   (d) Reviewing Theorem 3.3 in Apostol's book, using similar methods and the conclusions of questions (b) and (c), prove the following formula
   $$\sum_{n \leqslant x} \frac{\tau(n)}{n} = \frac{1}{2} \log^2 x + 2\gamma \log x + O(1),$$
   where, $\tau(n)$ denotes the number of positive divisors of n.

(2) For each real x the symbol $\lfloor x \rfloor$ denotes the greatest integer $\leq x$, answer the following questions
   (a) Prove that $\lfloor x \rfloor + \lfloor x + \frac{1}{2} \rfloor = \lfloor 2x \rfloor$, and more generally,
   $$\sum_{k=0}^{n-1} \lfloor x + \frac{k}{n} \rfloor = \lfloor nx \rfloor.$$
   (b) Let n be a positive integer, show that $\lfloor \sqrt{n} + \sqrt{n+1} \rfloor = \lfloor \sqrt{4n+2} \rfloor$.

(3) Let $f(x) = a_0 + a_1 x + a_2 x^2 + ... + a_n x^n$ be a polynomial with integer coefficients, where $a_n > 0$ and $n \geq 1$. Prove that $f(x)$ is composite for infinitely many integers x.

(4) Let $S = \{1,\ 5,\ 9,\ 13,\ ...\}$ denote the set of all positive integers of the form $4n + 1$. An element p of S is called an $S - prime$ if $p > 1$ and if the only divisors of p, among the elements of S, are 1 and p. (For example, 49 is an $S - prime$.) An element $n > 1$ in S which is not an $S - prime$ is called an $S - composite$.

(a) Prove that every $S - composite$ is a product of $S - primes$.

(b) Find the smallest $S - composite$ that can be expressed in more than one way as a product of $S - primes$.

This example shows that unique factorization does not hold in S.

(5) I recommend visiting Professor Paul Garrett's personal page https://www-users.cse.umn.edu/~garrett/, where you can find notes on courses such as number theory, abstract algebra, real analysis, and complex analysis for reference.

Choose **one** of the following two tasks to complete:

(a) Find a note on the **number theory** section called **Riemann's explicit formula** on Paul Garrett's homepage.

(b) Find any paper about Dirichlet L-function or Riemann Zeta function. (On arxiv, MathSciNet, or other websites)

No matter which task you choose, you need to submit **two** things to me:

1. PDF files of papers or notes;

2. Describe the content and main steps covered in the paper or notes, and write them **briefly**.

(The purpose of setting this question is to encourage everyone to search for papers or materials and read them more, without having to write too much to submit.)