Solution to Homework 5

Xingfeng Lin

2025 Fall

1. See the solution to Exercise 4 in Homework 3.

2. Since $\Lambda * \mathbf{1} = log$, we have

$$\begin{aligned}\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{u^s}{s^2}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds &= \frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{u^s}{s^2}\left(\sum_{n=1}^{\infty}\frac{\Lambda(n)}{n^s}\right)ds \\ &= \frac{1}{2\pi i}\sum_{n=1}^{\infty}\Lambda(n)\int_{2-i\infty}^{2+i\infty}s^{-2}\left(\frac{u}{n}^s\right)ds.\end{aligned}$$

By residue theorem, the integral inside the summation is

$$\int_{2-i\infty}^{2+i\infty}s^{-2}\left(\frac{u}{n}^s\right)ds = \begin{cases} 2\pi i log\frac{u}{n}, & \frac{u}{n} > 1 \\ 0, & 0 < \frac{u}{n} \leq 1. \end{cases}$$

Therefore we have

$$\frac{1}{2\pi i}\int_{2-i\infty}^{2+i\infty}\frac{u^s}{s^2}\left(-\frac{\zeta'(s)}{\zeta(s)}\right)ds = logu\sum_{n<u}\Lambda(n) - \sum_{n<u}\Lambda(n)logn.$$

3. Note that $log\, lcm[1,\ldots,n] = \psi(n)$, where $\psi(n) = \sum_{m\leq n}\Lambda(m)$. By Prime Number Theorem, we have $\psi(n) \sim n$.

4. When $x \geq 2^{16}$, we have the following estimate from arXiv:1002.0442

$$\frac{x}{logx - 1} \leq \pi(x) \leq \frac{x}{logx - \frac{11}{10}}.$$

This proves there are at least $\frac{2^{n-1}}{2n}$ primes in $[2^{n-1}, 2^n)$ for large n. Finitely many cases where n are small can be verified by hand.

5. We prove that

$$\sum_{p\leq x}\frac{1}{p} = loglogx + O(1).$$

We will prove this by estimating $\sum_{p\leq x}\frac{logp}{p}$ and apply Abel's summation. Note that

$$\begin{aligned}\sum_{p\leq x}\frac{logp}{p} &= \sum_{n\leq x}\frac{\Lambda(n)}{n} + O(1) \\ &= \frac{1}{x}\sum_{n\leq x}\Lambda(n)\frac{x}{n} + O(1) \\ &= \frac{1}{x}T(x) + O(1),\end{aligned}$$

where $T(x) := \sum_{n\le x} logn$. The last equality follows from the fact that $\Lambda * \mathbf{1} = log$. By Euler's summation, we obtain an estimate of T:

$$T(x) = xlogx - x + O(logx).$$

This implies that

$$\sum_{p\le x} \frac{logp}{p} = logx + O(1). \tag{1}$$

Now let

$$a_n = \begin{cases} \frac{logp}{p}, & n = p\, for\, some\, prime\, p \\ 0, & otherwise, \end{cases}$$

$f(x) := \frac{1}{logx}$. By Abel's summation, we have

$$\begin{aligned} \sum_{p\le x} \frac{1}{p} &= \sum_{n\le x} a_n f(n) \\ &= \frac{1}{logx} A(x) + \int_2^x \frac{1}{tlog^2 t} A(t)dt + O(1), \end{aligned}$$

where $A(t) := \sum_{n\le t} a_n = \sum_{p\le t} \frac{logp}{p}$. By (1), we have

$$\sum_{p\le x} \frac{1}{p} = loglogx + O(1).$$

6. For $x, z > 0$, put $\pi(x,z) := \#\{n \le x : (n, P_z) = 1\}$, where $P_z := \prod_{p\le z} p$. We have

$$\pi(x) \le \pi(x,z) + z.$$

We estimate $\pi(x,z)$ as follows.

$$\begin{aligned} \pi(x,z) &= \sum_{n\le x} \sum_{d|n, d|P_z} \mu(d) \\ &= \sum_{d|P_z} \mu(d)[\frac{x}{d}] \\ &= x \prod_{p\le z} (1 - \frac{1}{p}) + O(2^z). \end{aligned}$$

Note that

$$\prod_{p\le z} (1 - \frac{1}{p})^{-1} \ge \sum_{n\le z} \frac{1}{n} = logz + \gamma + O(\frac{1}{z}).$$

We obtain

$$\pi(x,z) = O(\frac{x}{logz}) + O(2^z).$$

Taking $z = logx$ results in

$$\pi(x) = O(\frac{x}{loglogx}).$$