Analytic Number Theory Homework 3

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1. (a) *Proof.* Note that $2^4, 2^6$ are not equal to 1 modulo 13. Therefore 2 has order 12 in $(\mathbb{Z}/13)^\times$. $\square$

   (b) *Proof.* Let $P(n)$ denote the number of primitive roots modulo n. If $P(n) > 0$, then we have $(\mathbb{Z}/n)^\times \simeq \mathbb{Z}/\varphi(n)$. This implies that
$$\begin{aligned} P(n) &= \#\{g \in \mathbb{Z}/\varphi(n) : ord(g) = \varphi(n)\} \\ &= \varphi(\varphi(n)). \end{aligned}$$
$\square$

2. (a) Suppose $k, a \in \mathbb{Z}_{>0}, (k,a) = 1$. Then there exist infinitely many primes p satisfying $p \equiv a \, mod \, k$.

   (b) *Proof.* Pick $k = 10^{n+1}, a = 1$. Then by Dirichlet's theorem, there are infinitely many primes with the last $n+1$ digits $(00\cdots01)$. $\square$

   (c) *Proof.* Let $S_1 = \{ak+1 : k = 0,1,2,\dots\}$. For any distinct primes $p_1, \dots, p_{m-1} \in S_1, p_m \in S$, $p_1 \cdots p_{m-1}p_m \in S$ is a product of m distinct prims. By Dirichlet's theorem, S_1, S are infinite sets, and thus there are infinitely many choices of $\{p_1, \dots, p_{m-1}\}$ and p_m. $\square$

3. (a) Let $\pi(x)$ denote the number of prime numbers not greater than x, for $x > 1$. Then we have
$$\lim_{x \longrightarrow +\infty} \frac{\pi(x) logx}{x} = 1.$$

   (b) *Proof.* As a corollary of the Prime Number Theorem, we have $\pi(x) = o(x)$. Noting that $\pi(p_n) = n$, we have
$$\frac{nlogn}{p_n} = \frac{\pi(p_n) logp_n}{p_n} + \frac{\pi(p_n)}{p_n} log \frac{\pi(p_n)}{p_n}.$$
By Prime Number Theorem and its corollary, we have $\frac{\pi(p_n) logp_n}{p_n} \to 1$, and $\frac{\pi(p_n)}{p_n} log \frac{\pi(p_n)}{p_n} \to 0$, as $n \to \infty$, completing the proof. $\square$

   (c) Since $\mathbb{Q}_{>0}$ is dense in $\mathbb{R}_{>0}$, it suffices to show that every $\frac{n}{m} \in \mathbb{Q}, n, m \in \mathbb{Z}_{>0}$, is a limit of S. Let $a_k = nklog(nk), b_k = mklog(mk), k = 1, 2, \dots$. Then we have
$$\lim_{k \longrightarrow \infty} \frac{a_k}{b_k} = \frac{n}{m}.$$
By (b), we have $\frac{a_k}{p_{nk}} \to 1$, $\frac{b_k}{p_{mk}} \to 1$, as $k \to \infty$. Therefore we have
$$\lim_{k \longrightarrow \infty} \frac{p_{nk}}{p_{mk}} = \frac{n}{m}.$$

4. *Proof.* For $a > 1$, consider the rectangle γ with vertices $c-iT, c+iT, l-iT, l+iT$, where $l < 0$. Since $\frac{a^z}{z}$ is holomorphic inside γ except for a pole at $z=0$ with residue 1, integrating along γ gives

$$\int_{c-iT}^{c+iT} \frac{a^z}{z} dz - 2\pi i = \int_{c-iT}^{l-iT} + \int_{l-iT}^{l+iT} + \int_{l+iT}^{c+iT} \frac{a^z}{z} dz.$$

Estimation of the three terms on the right-hand side gives

$$\left| \int_{c-iT}^{c+iT} \frac{a^z}{z} dz - 2\pi i \right| \leq \frac{2a^c}{Tloga} + \frac{2Ta^l}{|l|}.$$

Let $l \to -\infty$. We then have

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{a^z}{z} dz - 1 \right| \leq \frac{a^c}{\pi Tloga}.$$

For $a = 1$, we can calculate the integral directly,

$$\begin{aligned} \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{z} dz &= \frac{1}{\pi} \int_0^T \frac{c}{c^2+x^2} dx \\ &= \frac{1}{2} - \frac{1}{\pi} arctan(\frac{c}{T}). \end{aligned}$$

Therefore we have

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{1}{z} - \frac{1}{2} \right| \leq \frac{c}{\pi T}.$$

For $0 < a < 1$, consider the rectangle γ with vertices $c-iT, c+iT, l-iT, l+iT$, where $l > c$. Since $\frac{a^z}{z}$ is holomorphic inside γ, integrating along γ gives

$$\int_{c-iT}^{c+iT} \frac{a^z}{z} dz = \int_{c-iT}^{l-iT} + \int_{l-iT}^{l+iT} + \int_{l+iT}^{c+iT} \frac{a^z}{z} dz.$$

Estimation of the three terms on the right-hand side gives

$$\begin{aligned} \left| \int_{c-iT}^{c+iT} \frac{a^z}{z} dz \right| &= \frac{2}{T} \int_c^\infty a^t dt + \frac{2Ta^l}{T} \\ &= \frac{2a^c}{Tlog(1/a)} + \frac{2Ta^l}{T}. \end{aligned}$$

Let $l \to +\infty$. We then have

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{a^z}{z} dz \right| \leq \frac{a^c}{\pi Tlog(1/a)}.$$

Letting $T \to +\infty$ in those three cases, we obatin

$$\frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{a^z}{z} dz = \begin{cases} 1, & a > 1 \\ \frac{1}{2}, & a = 1 \\ 0, & 0 < a < 1. \end{cases}$$

The proof is now complete. ☐