Analytic Number Theory Fall 2025 - Homework

Homework 3 is due Wednesday, December 17th **Name:**

Directions

- Your work is graded on the quality of your writing and explanation as well as the validity of the mathematics.

Homework 3

(1) Review the definition of primitive root and address the following issues
  (a) Show that 2 is a primitive root modulo 13.
  (b) If the primitive root modulo n exists, prove that the number of such primitive roots is exactly $\varphi(\varphi(n))$.

(2) Review Dirichlet's theorem on arithmetic progressions and use it to complete the following problems
  (a) State the content of Dirichlet's theorem.
  (b) Prove that for any integer n, there exists a prime p such that at least n digits in p are zero.
  (c) If a and b are relatively prime integers, S is the arithmetic progression

$$S = \{ak + b : k = 0, 1, 2 \cdots\}.$$

  Prove that for any integer m, S contains an infinite subset which can be written as the product of m distinct primes.

(3) Review the Prime Number Theorem and complete the following problems.
  (a) State the content of Prime Number Theorem.
  (b) Let p_n be the n^{th} prime number, prove that

$$\lim_{n\to\infty} \frac{p_n}{n \log n} = 1.$$

  (c) Using the conclusion from (b), prove that $S := \left\{ \dfrac{p}{q} : p, q \textit{ are primes} \right\}$ is dense in $\mathbb{R}^+$.

(4) Read Chapter 11 of Apostol's book. If $c > 0$, Define $\int_{c-\infty i}^{c+\infty i}$ to mean $\lim\limits_{T\to+\infty} \int_{c-iT}^{c+iT}$. Let x be a positive real number. Show that

$$\frac{1}{2\pi i} \int_{c-\infty i}^{c+\infty i} \frac{x^z}{z} dz = \begin{cases} 1, \; if \; a > 1 \\ \frac{1}{2}, \; if \; a = 1 \\ 0, \; if \; 0 < a < 1 \end{cases}$$

Moreover, prove that

$$\left| \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \frac{x^z}{z} dz - 1 \right| \le \frac{x^c}{\pi T \log(a)}, \; if \; a > 1,$$

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^z}{z}dz-\frac{1}{2}\right|\leq\frac{c}{\pi T},\ if\ a=1,$$

and

$$\left|\frac{1}{2\pi i}\int_{c-iT}^{c+iT}\frac{x^z}{z}dz\right|\leq\frac{x^c}{\pi T\log\left(\frac{1}{a}\right)},\ if\ 0<a<1.$$