Analytic Number Theory Homework 2

Xingfeng Lin

2025 Fall

1. (a) If f has a continuous derivative f' on the interval $[y,x]$, where $0 < y < x$, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

(b) *Proof.* By Euler's summation formula, we have

$$\begin{aligned}\sum_{n\leq x}\frac{1}{n} &= 1+\sum_{1<n\leq x}\frac{1}{n}\\ &= 1+logx-\int_1^x\frac{t-[t]}{t^2}dt+O(\frac{1}{x})\\ &= logx+1-\int_1^\infty\frac{t-[t]}{t^2}dt+O(\frac{1}{x}).\end{aligned}$$

It suffices to show that $1-\int_1^\infty\frac{t-[t]}{t^2}dt=\gamma$, where $\gamma:=\lim_n(\sum_{k=1}^n\frac{1}{k}-logn)$. In fact,

$$\begin{aligned}\int_1^\infty\frac{t-[t]}{t^2}dt &= \sum_{n=1}^\infty\int_n^{n+1}\frac{t-n}{t^2}dt\\ &= \sum_{n=1}^\infty(log(n+1)-logn-\frac{1}{n+1})\\ &= -\lim_n(\sum_{k=1}^n\frac{1}{k}-logn)+1\\ &= 1-\gamma.\end{aligned}$$

Therefore, we have

$$\sum_{n\leq x}\frac{1}{n} = logx+\gamma+O(\frac{1}{x}).$$

□

(c) *Proof.* Let $f(t)=\frac{logt}{t}$. By Euler's summation formula,

$$\begin{aligned}\sum_{n\leq x}\frac{logn}{n} &= 1+\int_1^x\frac{logt}{t}dt+\frac{logx}{x}([x]-x)+\int_1^x(t-[t])f'(t)dt\\ &= \frac{1}{2}log^2x+1+\int_1^\infty\frac{(t-[t])(1-logt)}{t^2}dt-\int_x^\infty(t-[t])f'(t)dt+O(\frac{logx}{x})\end{aligned}$$

Note that

$$\left|\int_x^\infty (t-[t])f'(t)dt\right| \le \int_x^\infty |f'(t)|dt$$
$$= -\int_x^\infty f'(t)dt$$
$$= \frac{logx}{x}.$$

Let $A = 1 + \int_1^\infty \frac{(t-[t])(1-logt)}{t^2}dt$, where the integral converges to a constant. Then we have

$$\sum_{n\le x}\frac{logn}{n} = \frac{1}{2}log^2x + A + O(\frac{logx}{x}).$$

□

(d) *Proof.* Rewrite the summation as

$$\sum_{n\le x}\frac{\tau(n)}{n} = \sum_{n\le x}\sum_{d|n}\frac{1}{n}$$
$$= \sum_{q,d:qd\le x}\frac{1}{qd}$$
$$= 2\sum_{q\le\sqrt{x}}\frac{1}{q}\sum_{d\le\frac{x}{q}}\frac{1}{d} - (\sum_{n\le\sqrt{x}}\frac{1}{n})^2$$

By (b),

$$\sum_{n\le x}\frac{\tau(n)}{n} = 2\sum_{q\le\sqrt{x}}\frac{1}{q}(\log(\frac{x}{q}) + \gamma + O(\frac{q}{x})) - (\log\sqrt{x} + \gamma + O(\frac{1}{\sqrt{x}}))^2$$
$$= -2\sum_{q\le\sqrt{x}}\frac{logq}{q} + 2(\gamma + logx)\sum_{q\le\sqrt{x}}\frac{1}{q} - \frac{1}{4}log^2x - \gamma logx + O(1).$$

By (b)(c), the first two terms are

$$= \frac{3}{4}log^2x + 3\gamma logx + O(1).$$

Therefore,

$$\sum_{n\le x}\frac{\tau(n)}{n} = \frac{1}{2}log^2x + 2\gamma logx + O(1).$$

□

2. (a) Let $[x]$ denote the largest integer not greater than x. We have

$$[nx] = \sum_{t\le nx} 1$$
$$= \sum_{k=0}^{n-1}\sum_{m:nm-k\le nx} 1$$
$$= \sum_{k=0}^{n-1}\sum_{m\le x+\frac{k}{n}} 1$$
$$= \sum_{k=0}^{n-1}[x+\frac{k}{n}].$$

This proves the equality and, in particular, for $n = 2$.

(b) *Proof.* Note that $\sqrt{4n+1} \le \sqrt{n}+\sqrt{n+1} \le \sqrt{4n+2}$, we have $[\sqrt{4n+1}] \le [\sqrt{n}+\sqrt{n+1}] \le [\sqrt{4n+2}]$. If $k \in \mathbb{Z}_{>0}$, and $k \le \sqrt{4n+2}$, then there must be $k^2 \le 4n+1$ since $k^2 \equiv 0,1 \, mod \, 4$. This implies that $[\sqrt{4n+1}] \ge [\sqrt{4n+2}]$. Therefore we have $[\sqrt{n}+\sqrt{n+1}] = [\sqrt{4n+2}]$. □

3. *Proof.* If otherwise, there exists $N \in \mathbb{Z}_{>0}$ such that $f(x)$ is a prime for every $x \ge N$. In particular, $f(N+kf(N))$ are primes for all $k \in \mathbb{Z}_{>0}$, but

$$f(N+kf(N)) \equiv a_0 + a_1 N + \cdots + a_n N^n \equiv 0 \, mod \, f(N),$$

leading to a contradiction. □

4. (a) *Proof.* By induction we know that it suffices to show: if $n \in S$ is an S-composite, then n is divided by some S-prime.

In fact, pick p to be the smallest integer among $\{m \in S : m|n\}$. If p is not an S-prime, then there exists $p' \in S$ with $1 < p' < p$ and $p'|p$, but this leads to $p'|n$, contradicting the minimality of p. Hence p is an S-prime. □

(b) If n is an S-composite, decompose n into prime numbers, $n = p_1 \cdots p_k p_{k+1} \cdots p_{k+l}$, where $p_i \equiv 3 \, mod \, 4, i = 1, \ldots, k$, $p_j \equiv 1 \, mod \, 4$, $j = k+1, ..., k+l$. Then k must be even and, if n can be expressed in more than one way as a product of distinct S-primes, $k \ge 4$, and there are at least two distinct primes turning up in $\{p_i\}_{i=1}^k$. Therefore, $n \ge 3^2 7^2 = 441$. Indeed, $441 = 21 \cdot 21 = 9 \cdot 49$, where $9, 21, 49$ are S-primes. Therefore the smallest S-composite is 441.

5. Here I describe the content and main steps covered in the notes Riemann's Explicit Formula by Paul Garrett.

The notes illustrate the relation between prime numbers and the Riemann ζ-function, together with discussions on analytic continuation and integral presentation of the ζ-function, using ξ-function and θ-function. I will split these content into three sections.

Riemann's Explicit Formula. Let $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$. We state the Riemann-Hadamard product as follows:

Theorem 1. *For all $s \in \mathbb{C}$, we have*

$$(s-1)\zeta(s) = e^{a+bs} \prod_{\rho} (1 - \frac{s}{\rho}) e^{s/\rho} \prod_{n=1}^{\infty} (1 + \frac{s}{2n}) e^{\frac{-s}{2n}}, \tag{1}$$

where the product over ρ ranges over all nontrivial zeros of ζ, a, b.

Combined with Euler product, by taking logrithmic derivatives of both sides of (1), we obtain

$$\sum_p logp \sum_{m=1}^{\infty} \frac{1}{p^{ms}} = \frac{1}{s-1} - b - \sum_{\rho} (\frac{1}{s-\rho} + \frac{1}{\rho}) - \sum_{n=1}^{\infty} (\frac{1}{s+2n} - \frac{1}{2n}). \tag{2}$$

The left-hand side needs $Re(s) > 1$ for convergence, while the right-hand side converges for all $s \in \mathbb{C}$ apart from the visible poles at 1, the nontrivial zeros ρ, and the trivial zeros $-2n, n \ge 1$ (this is a consequence of the Riemann-von Mangoldt formula stated as Theorem 2 below).

The left-hand side is appropriate for direct estimation, and operations like integration, while the right-hand side has its advantage of exhibiting poles concretely.

Theorem 2. *For $T > 0$, let $N(T)$ denote the number of nontrivial zeros of ζ in $|Im\rho| \le T$. Then,*

$$N(T) = \frac{T}{2\pi} log \frac{T}{2\pi} - \frac{T}{2\pi} + O(logT).$$

Now apply to (2) the Perron identity

$$\frac{1}{2\pi i} \int_{\sigma - i\infty}^{\sigma + i\infty} \frac{Y^t}{t} dt = \begin{cases} 1, & Y > 1 \\ \frac{1}{2}, & Y = 1 \\ 0, & 0 \le Y < 1, \end{cases}$$

where $\sigma > 0$. We can obtain Riemann's explicit formula:

Theorem 3. *Let $X > 1$, then*

$$\sum_{p^m < X} logp = X - (b+1) - \lim_{T \to \infty} \sum_{\rho : |Im\rho| \le T} \frac{X^\rho}{\rho} + \sum_{n=1}^{\infty} \frac{X^{-2n}}{2n}.$$

□

Functional Equation of ζ. To extend ζ to $Re(s) > 0$, one approach is to do an elementary estimate

$$\begin{aligned} \zeta(s) - \frac{1}{s-1} &= \zeta(s) - \int_1^\infty t^{-s} dt \\ &= \sum_{n=1}^{\infty} \left(\frac{1}{n^s} - \frac{1}{s-1} (\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}}) \right) \\ &= O_s(\sum_{n=1}^{\infty} \frac{1}{n^{s+1}}). \end{aligned}$$

However, this approach is under-powered since it is done by direct estimation, giving nothing more than the convergence of $\zeta(s) - \frac{1}{s-1}$. We introduce another approach using the functioncal equation (and ξ-function). We have the following result:

Theorem 4. *The completed zeta function (ξ-function)*

$$\xi(s) = \pi^{-\frac{s}{2}} \Gamma(\frac{s}{2}) \zeta(s)$$

has an analytic continuation to $s \in \mathbb{C}$, except for $s = 0, 1$, and satisfies the functional equation

$$\xi(1-s) = \xi(s).$$

Moreover, $s(s-1)\xi(s)$ is entire and bounded in vertical strips of finite width.

θ-Function and Integral Presentation of ζ. Let $\theta(z) = \sum_{n \in \mathbb{Z}} e^{\pi i n^2 z} = 1 + 2\sum_{n=1}^{\infty} e^{\pi i n^2 z}, z \in \mathbb{H}$, where $\mathbb{H} := \{z \in \mathbb{C} : Im(z) > 0\}$ is the complex upper-half plane. Simple calculation gives

$$\frac{\theta(iy) - 1}{2} = \sum_{n=1}^{\infty} e^{-\pi n^2 y},$$

for $y > 0$. This deduces the following result, presenting the ξ-function (or ζ-function) as an integral.

Theorem 5. *For $Re(s) > 1$,*

$$\xi(s) = \int_0^\infty \frac{\theta(iy) - 1}{2} y^{s/2} \frac{dy}{y}.$$

$\square$