Homework 1

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1. What are all the points in the representational space $\mathbb{R}^3$ that are the homogeneous coordinates of the origin in the physical space $\mathbb{R}^2$?

All the points on $l = \begin{pmatrix} 0 \\ 0 \\ n \end{pmatrix}$, ($n \in \mathbb{R}$ and $n \neq 0$)

2. Are all points at infinity in the physical plane $\mathbb{R}^2$ the same? Justify your answer.

No, the points at infinity in the physical plane has homogeneous coordinates representation $\mathbf{X} = \begin{pmatrix} b \\ -a \\ 0 \end{pmatrix}$, the points at infinity have different directions controlled by (a, b), only when the points at infinity have the same a/b they are the same.

3. Argue that the matrix rank of a degenerate conic can never exceed 2

A degenerate conic $\mathbf{C}$ is represented by it's two intersected lines $\mathbf{l}$ and $\mathbf{m}$: $\mathbf{C} = \mathbf{l}\mathbf{m}^T + \mathbf{m}\mathbf{l}^T$. Since $rank(\mathbf{l}\mathbf{m}^T) = 1$ and $rank(\mathbf{m}\mathbf{l}^T) = 1$, $rank(\mathbf{C}) \leq rank(\mathbf{l}\mathbf{m}^T) + rank(\mathbf{m}\mathbf{l}^T) = 2$. Thus, the matrix rank of such C can never exceed 2.

4. Derive in just 3 steps the intersection of two lines $\mathbf{l}_1$ and $\mathbf{l}_2$ with $\mathbf{l}_1$ passing through the points $(0, 0)$ and $(3, 5)$, and with $\mathbf{l}_2$ passing through the points $(-3, 4)$ and $(-7, 5)$. How many steps would take you if the second line passed through $(-7, -5)$ and $(7, 5)$?

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix}$$

$$\mathbf{l}_2 = \begin{pmatrix} -3 \\ 4 \\ 1 \end{pmatrix} \times \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix}$$

the intersection of those two lines in representation space is $\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} -5 \\ 3 \\ 0 \end{pmatrix} \times \begin{pmatrix} -1 \\ -4 \\ 13 \end{pmatrix} = \begin{pmatrix} 39 \\ 65 \\ 23 \end{pmatrix}$,

in $\mathbb{R}^2$ physical space, the intersection is $\mathbf{x} = (\frac{39}{23}, \frac{65}{23})$

If the second line passed through $(-7,-5)$ and $(7,5)$, we have $\mathbf{l}_2 = \begin{pmatrix} -7 \\ 5 \\ 1 \end{pmatrix} \times \begin{pmatrix} 7 \\ 5 \\ 1 \end{pmatrix} = \begin{pmatrix} -10 \\ 14 \\ 0 \end{pmatrix}$, since parameter vectors $\mathbf{l}_1$ and $\mathbf{l}_2$ are ideal points, the intersection of lines $\mathbf{l}_1$ and $\mathbf{l}_2$ is $\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$, we only need two steps in this case.

5. Let l_1 be the line passing through points $(0,0)$ and $(5,-3)$ and l_2 be the line passing through points $(-5,0)$ and $(0,-3)$. Find the intersection between these two lines. Comment on your answer.

$$\mathbf{l}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 5 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix}$$

$$\mathbf{l}_2 = \begin{pmatrix} -5 \\ 0 \\ 1 \end{pmatrix} \times \begin{pmatrix} 0 \\ -3 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix}$$

the intersection of those two lines in representation space is $\mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} 3 \\ 5 \\ 0 \end{pmatrix} \times \begin{pmatrix} 3 \\ 5 \\ 15 \end{pmatrix} = \begin{pmatrix} 75 \\ -45 \\ 0 \end{pmatrix}$, it's an ideal point, so these two lines are parallel.

6. As you know, when a point p is on a conic C, the tangent to the conic at that point is given by $\mathrm{l} = \mathrm{Cp}$. That raises the question as to what Cp would correspond to when p was outside the conic. As you'll see later in class, when p is outside the conic, Cp is the line that joins the two points of contact if you draw tangents to C from the point p. This line is referred to as the polar line. Now let our conic C be an ellipse that is centered at the coordinates (3, 2), with $a = 1$ and $b = 1/2$, where a and b, respectively, are the lengths of semi-major and semi-minor axes. For simplicity, assume that the major axis is parallel to x-axis and the minor axis is parallel to y-axis. Let p be the origin of the $\mathbb{R}^2$ physical plane. Find the intersections points of the polar line with x- and y-axes.

All points $(x,y) \in \mathbb{R}^2$ on the ellipse satisfy $\frac{(x-3)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$, which is also $x^2 + 4y^2 - 6x - 16y + 24 = 0$. The conic $\mathbf{C}$ is $\mathbf{C} = \begin{pmatrix} 1 & 0 & -3 \\ 0 & 4 & -8 \\ -3 & -8 & 24 \end{pmatrix}$.

$\mathbf{p}$ in representation space is $(0,0,1)^T$, thus the polar line is $\mathbf{l} = \mathbf{Cp} = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix}$. The x-axis is $y = 0$, which is $\mathbf{l}_x = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ and y-axis is $x = 0$, which is $\mathbf{l}_y = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$, the intersection of the polar line with

x-axis in representation space is:
$\mathbf{x} = \mathbf{l} \times \mathbf{l}_x = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -24 \\ 0 \\ -3 \end{pmatrix}$, it's $(8, 0)$ in physical space.

The intersection point of the polar line with y-axis in representation space is:
$\mathbf{x} = \mathbf{l} \times \mathbf{l}_x = \begin{pmatrix} -3 \\ -8 \\ 24 \end{pmatrix} \times \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 24 \\ 8 \end{pmatrix}$, it's $(0, 3)$ in physical space.

7. Find the intersection of two lines whose equations are given by $x = 1/2$ and $y = -1/3$.

The two lines can be represented by parameter vectors $\mathbf{l}_1 = \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix}$ and $\mathbf{l}_1 = \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix}$, the intersection in representation space is $\mathbf{x} = \mathbf{l}_1 \times \mathbf{l}_2 = \begin{pmatrix} 1 \\ 0 \\ -1/2 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 1/3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ -1/3 \\ 1 \end{pmatrix}$, so the intersection is $(\frac{1}{2}, -\frac{1}{3})$ in physical space.