Root Locus

Friday, July 8, 2022 10:36 AM

Input → System parameter K → Output

System = $\frac{s^2+s+1}{s^3+4s^2+Ks+1}$

Unknown parameter affects poles

poles of system are values of s when

$s^3+4s^2+Ks+1=0$

if K moves, poles moves

So now we have question

Design

What value of K should I choose to meet my system performance requirements?

$s^3+4s^2+0s+1=0$

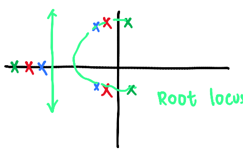
$s^3+4s^2+1s+1=0$

$s^3+4s^2+2s+1=0$

: as $K\to\infty$

Effects of variation

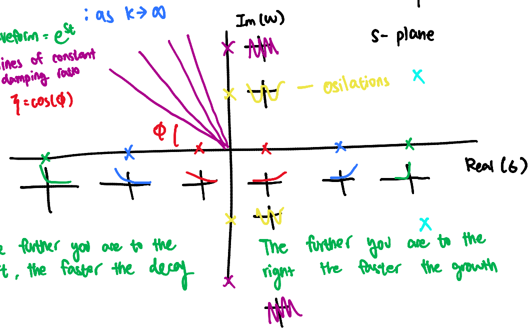
What is the effect of a variation of K on my system



Waveform = e^{st}

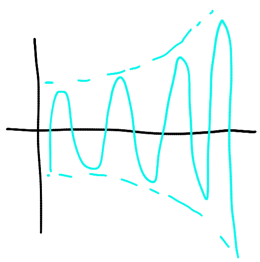
lines of constant damping ratio

$\zeta=\cos(\phi)$



The further you are to the left, the faster the decay

The further you are to the right the faster the growth



System = $\frac{\text{zeros}}{\text{poles}}$

↳ unforced response

↳ tied to stability

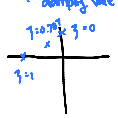
Side Note: damping ratio ≠ damping coeff

ζ : unitless damping vale

b : value of the damping term

[N·s/m]

$b\cdot\dot{x}$ = Force

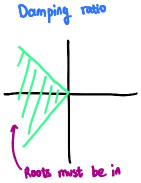


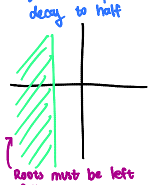
Why do we care about this? Requirements can be

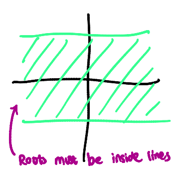
Damping ratio

time for exponential decay to half

Natural frequency



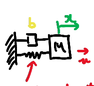




or any combination of

Example: Design time

$\frac{x(s)}{u(s)}=\frac{1}{s^2+bs+1}$



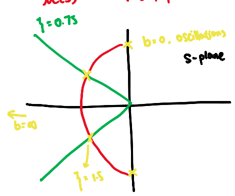
Requirement

System shall have

$\zeta\geq0.75$

Spring department: spring constant = 1

Mass department: Mass = 1

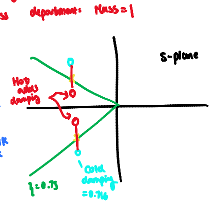


If Hot: K= 0.9 Knom

Cold: K= 1.1 Knom

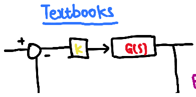
We can use the root locus to see how sensitive the system is

$TF=\frac{1}{s^2+1.5s+K}$ for 0.9K to 1.1K

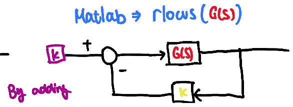


Sketching a Root locus by hand

Textbooks



Matlab ⇒ rlocus(G(s))



10 Rules of root locus

From previous

$Q(s)=s$

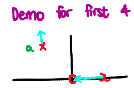
$P(s)=s^3+4s^2+1$

Start from this form: $1+KG(s)=0$ ⇒ $1+K\frac{Q(s)}{P(s)}$

Rule 1) There are n lines (loci) where n is degree of Q or P, whichever is greater

Rule 2) As K increases from 0 to ∞, the roots move from the poles of G(s) to the zeros of G(s)

Demo for first 4 rules

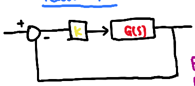


1) 3 poles, 3 line

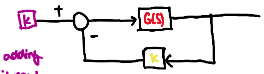
2) 2 lines will → ∞

3) ... more as points

Textbooks

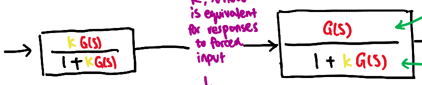


Matlab ⇒ rlocus (G(s))



By adding k, it now is equivalent for responses to forced input

↳ same stability

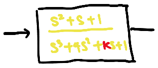


Different numerator

same denominator

$$\underbrace{1 + kG(s) = 0}$$

The basic form for drawing root locus



1. Group all k terms
2. Divide by non-k terms

Not in the correct form

$s^3+4s^2+ks+1=0$

How do roots move as k goes $0 \to \infty$?

We need to put it in correct form

$$\underbrace{s^3+4s^2+1} + k(s) = 0$$

$$\frac{s^3+4s^2+1}{s^3+4s^2+1} + \frac{k(s)}{s^3+4s^2+1} = 0$$

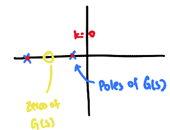
$$1 + k\underbrace{\frac{s}{s^3+4s^2+1}}_{G(s)} = 0$$

Start from this form: P(s)

Rule 1) There are n lines (loci) where n is degree of Q or P, whichever is greater

Rule 2) As k increases from 0 to ∞, the roots move from the poles of G(s) to the zeros of G(s)

$P(s)\ 1 + k\frac{Q(s)}{\cancel{P(s)}} = 0 \Rightarrow P(s) + kQ(s) = 0$



poles of G(s) are when P(s) = 0

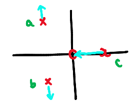
zeros of G(s) are when Q(s) = 0

when k=0, P(s) = 0

as $k \to \infty$, $P(s) + \underbrace{\infty Q(s)}_{\text{Dominates}} = 0$

★ Closed loop poles travel from poles of G(s) to zeros of G(s) as $k \to \infty$

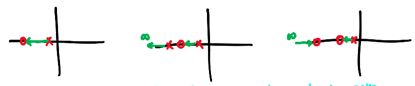
Demo for first 4 rules



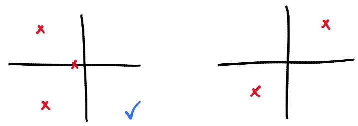
1) 3 poles, 3 line
2) 2 lines will → ∞
3) a & b move as pairs
4) each roots will not cross

what if # of poles ≠ # of zeros?

If P(s) = Q(s) If P(s) > Q(s) If P(s) < Q(s)

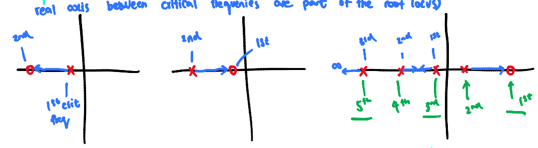


Rule 3) When roots are complex they occur in conjugate pairs



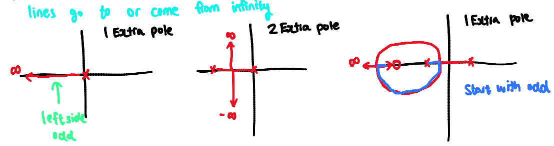
Rule 4) At no time will the roots cross its path (for signal path)

Rule 5) The portion of the real axis to the left of an odd number of open loop poles and zeros are part of the loci (every other space on the real axis between critical frequencies are part of the root locus)



Rule 6) Line leave (breakout) and enter (break in) the real axis at 90°

Rule 7) If there are not enough poles or zeros to make a pair then the extra lines go to or come from infinity



Rule 8) Line go to infinity along asymptotes

The angles of the asymptotes, $\phi_A = \frac{(2q+1)}{n-m} \cdot 180°$

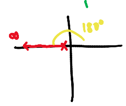
$q: 0, 1, 2 \ldots (n-m-1)$

The centroids of the asymptotes, $= \frac{\sum \text{Finite Poles} - \sum \text{Finite Zeros}}{n-m}$

Remember that n−m equals # poles − # zeros, number of lines that goes to infinity

If 1 line → ∞

$\phi_A = \frac{2 \cdot 0 + 1}{1} \cdot 180 = 180°$

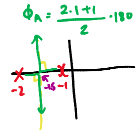


If 2 line → ∞

$\phi_A = \frac{2 \cdot 0 + 1}{2} \cdot 180 = 90°$

$\phi_A = \frac{2 \cdot 1 + 1}{2} \cdot 180 = 270° = -90°$

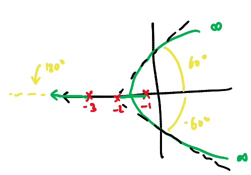
centroid $= \frac{(-2-1)}{2} = -1.5$



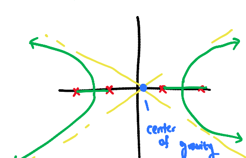
If 3 lines → ∞

Centroid $= \frac{-6}{3} = -2$

$\phi_A = 60°, 180°, -60°$

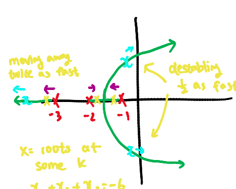


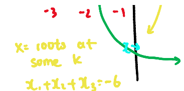
If 4 lines → ∞



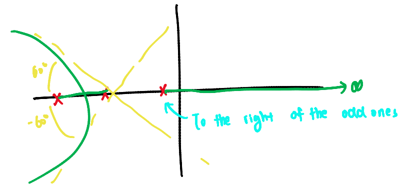
Rule 9) If there is atleast two lines to infinity, then the sum of all roots is constant (Good for understanding)

Sum of roots = −1 −2 −3 = −6





Rule 10) K going from 0 to negative infinity can be drawn by reversing Rule 5 and adding 180° to asymptote angles



What about all the other rules?