Aalto University
School of Electrical
Engineering

ELEC-E7120 Wireless Systems

Homework for Unit 3

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Problem 3.1 (1.5 points). *AWGN channel capacity in different working regimes*

The (Shannon) capacity of an AWGN channel is given by:

$$C_{AWGN} = W \log_2\left(1 + \frac{P}{N_0 W}\right) \quad [bit/s]$$

where P $[W]$ is the received signal power, W $[Hz]$ is the communication bandwidth, and N_0 $[W/Hz]$ is the Power Spectral Density (PSD) of the additive background noise at the receiver side.

a) Find the **spectral efficiency of the AWGN channel** for a fixed received signal power and PSD of additive noise, when the bandwidth grows large (i.e., when $W \to \infty$). What kind of behavior are you able to observe in the so-called **power-limited region**?

b) Consider now the case when the bandwidth remains fixed, and the power grows large. Determine the **spectral efficiency of the AWGN channel** when the received signal power grows large (i.e., when $\boldsymbol{P} \to \infty$). What kind of behavior is now observed in the so-called **bandwidth-limited region**?

c) Let us assume that the bandwidth of the AWGN channel is $\boldsymbol{W} = \mathbf{40}\ \boldsymbol{MHz}$, the total received signal power $\boldsymbol{P} = \mathbf{5}\ \boldsymbol{mW}$, and noise PSD $N_0 = 2 \times 10^{-9}\ W/Hz$. How much does capacity increase by doubling the received power? What happens to the channel capacity when the channel bandwidth is doubled? What looks more convenient from a Wireless System designer perspective? Is this system working in the bandwidth-limited or power-limited region? Please, justify your answer briefly.

d) Let us now consider that bandwidth of the AWGN channel is reduced to $\boldsymbol{W} = \mathbf{1}\ \boldsymbol{MHz}$, total received signal power is increased to $\boldsymbol{P} = \mathbf{100}\ \boldsymbol{mW}$, and noise PSD maintained at $N_0 = 2 \times 10^{-9}\ W/Hz$. How much does capacity increase by doubling the received power? How much does capacity increase by doubling the channel bandwidth? What looks more convenient from a Wireless System designer perspective? Is this system working in the bandwidth-limited or power-limited region? Please, give a brief justification of your observation.

a. The (Shannon) capacity of an AWGN channel is given by

$$C_{AWGN} = W \log_2\left(1+\frac{P}{N_0 W}\right)$$

The spectral efficiency of the AWGN channel is

$$SE = \frac{C}{W} = \log_2\left(1+\frac{P}{N_0 W}\right)$$

In the so-called bandwidth-limited region, when the bandwidth grows large (i.e., when $W \rightarrow \infty$), the capacity of AWGN channel increases, but the SNR decreases because the received signal power is fixed. Therefore, the spectral efficiency SE = C/W decreases in the power limited region as the SNR decreases. In addition, when $W \rightarrow \infty$, the spectral efficiency

$$SE = \log_2\left(1+\frac{P}{N_0 W}\right) \rightarrow 0$$

b. In the so-called bandwidth-limited region, when the bandwidth remains fixed, and the power grows large, the capacity of AWGN channel is also increased because the capacity is logarithmically related to the received signal power. This means that more information can be transmitted per unit bandwidth. The spectral efficiency (SE) is the ratio of capacity to bandwidth. Since the bandwidth remains constant and the capacity increases, the spectral efficiency also increases. When $W \rightarrow \infty$

$$SE = \log_2\left(1+\frac{P}{N_0 W}\right) \rightarrow \infty$$

c. When the bandwidth of the AWGN channel $W = 40\ MHz$, the total received signal power $P = 5\ mW$, and noise PSD $N_0 = 2 \times 10^{-9}\ W/Hz$

$$C_{AWGN} = 40\times 10^6 \times \log_2\left(1+\frac{5\times 10^{-3}}{2\times 10^{-9}\times 40\times 10^6}\right) = 3.50\times 10^6 \text{ bit}/s$$

Doubling the received power:

$$C_1 = 40\times 10^6 \times \log_2\left(1+\frac{2\times 5\times 10^{-3}}{2\times 10^{-9}\times 40\times 10^6}\right) = 6.80\times 10^6 \text{ bit}/s$$

The capacity increase 3.3×10^6 bit/s.

Doubling the bandwidth

$$C_2 = 2\times40\times10^6\times\log_2\left(1+\frac{5\times10^{-3}}{2\times10^{-9}\times2\times40\times10^6}\right)=3.55\times10^6 \text{ bit}/s$$

The capacity increase 0.05×10^6 bit/s.

According to the analysis, increasing the received power has a better effect on increasing the capacity of AWGN channel. So, the system is working in the power-limited region and increasing the received power can significantly increase the channel capacity. Increasing the bandwidth also decreases the SNR, so the channel capacity gain is small.

d. When the bandwidth of the AWGN channel $W = 1\ MHz$, the total received signal power $P = 100\ mW$, and noise PSD $N_0 = 2\times10^{-9}\ W/Hz$

$$C_{AWGN} = 1\times10^6\times\log_2\left(1+\frac{100\times10^{-3}}{2\times10^{-9}\times1\times10^6}\right)=5.67\times10^6 \text{ bit}/s$$

Doubling the received power:

$$C_1 = 1\times10^6\times\log_2\left(1+\frac{2\times100\times10^{-3}}{2\times10^{-9}\times1\times10^6}\right)=6.66\times10^6 \text{ bit}/s$$

The capacity increase 1×10^6 bit/s.

Doubling the bandwidth

$$C_2 = 2\times1\times10^6\times\log_2\left(1+\frac{5\times10^{-3}}{2\times10^{-9}\times2\times1\times10^6}\right)=9.40\times10^6 \text{ bit}/s$$

The capacity increase 3.73×10^6 bit/s.

Increasing the received power is more convenient. The system is working in the bandwidth-limited region. In a bandwidth-limited channel, the bandwidth is limited while the received power can be large enough, so increasing the bandwidth performance better.

Problem 3.2 (1 point). *Bit error probability analysis in AWGN & Rayleigh fading channels*

a) For a **Rayleigh fading channel** (i.e., when received SNR follows an exponential distribution), and in case of binary signalling, it is possible to demonstrate that the probability of error (P_e) is given by the following expression:

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\text{SNR}}{1+\text{SNR}}}\right) \tag{1}$$

It is possible to demonstrate as well that the exact expression above can be asymptotically approximated by

$$P_e \approx \frac{1}{4\text{SNR}} \tag{2}$$

at high SNR values (i.e., when SNR $\to \infty$). Show how to obtain approximation in (2) from expression (1). What kind of asymptotic behaviour does it show? Hint: Use Taylor series expansions.

b) Use the inequality

$$\left(1 - \frac{1}{z^2}\right)\frac{1}{z\sqrt{2\pi}} \le e^{\frac{z^2}{2}} Q(z) \le \frac{1}{z\sqrt{2\pi}} \tag{3}$$

to find a suitable approximation for the probability of error in an **AWGN channel** when SNR grows large in case of binary signalling (BPSK). What kind of asymptotic behaviour does it show?

c) Find the SNR values required to obtain a bit error probability of 10^{-6} in case of:

- AWGN channel, and
- Rayleigh fading channel.

For this, assume BPSK signalling and express the result in both dB and linear. Use the expressions derived in (**a**) and (**b**). Analyse the obtained results and take out your own conclusions.

a. For

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\mathrm{SNR}}{1+\mathrm{SNR}}}\right)$$

Let

$$x = \frac{1}{\mathrm{SNR}}$$

Hence

$$P_e = \frac{1}{2}\left(1 - \sqrt{\frac{\frac{1}{x}}{1+\frac{1}{x}}}\right) = \frac{1}{2}\left(1 - \sqrt{\frac{1}{x+1}}\right) = \frac{1}{2} - \frac{1}{2}(x+1)^{-\frac{1}{2}}$$

Use Taylor series expansions to $(x+1)^{\frac{1}{2}}$

$$(x+1)^{-\frac{1}{2}} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots$$

$$P_e = \frac{1}{2} - \frac{1}{2}(x+1)^{-\frac{1}{2}} = \frac{1}{2} - \frac{1}{2}\left(1 - \frac{1}{2}x + \frac{3}{8}x^2 + \cdots\right) = \frac{1}{4}x - \frac{3}{16}x^2 + \cdots$$

At high SNR values (i.e., when SNR $\to \infty$), we can ignore terms other than x, hence

$$P_e \approx \frac{1}{4}x = \frac{1}{4\mathrm{SNR}}$$

b. In BPSK

$$P_e = Q\left(\sqrt{2\mathrm{SNR}}\right)$$

In inequality

$$\left(1 - \frac{1}{z^2}\right)\frac{1}{z\sqrt{2\pi}} \le e^{\frac{z^2}{2}} Q(z) \le \frac{1}{z\sqrt{2\pi}}$$

Let $z = \sqrt{2\mathrm{SNR}}$, we have

$$\left(1 - \frac{1}{2\mathrm{SNR}}\right)\frac{1}{2\sqrt{\pi \mathrm{SNR}}} \le e^{\mathrm{SNR}} Q\left(\sqrt{2\mathrm{SNR}}\right) \le \frac{1}{2\sqrt{\pi \mathrm{SNR}}}$$

Since $e^{\mathrm{SNR}} > 0$, divide both sides of the inequality by e^{SNR}

$$\left(1-\frac{1}{2\mathrm{SNR}}\right)\frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}} \le Q\left(\sqrt{2\mathrm{SNR}}\right) \le \frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}}$$

When SNR grows large

$$\frac{1}{2\mathrm{SNR}} \to 0$$
$$1-\frac{1}{2\mathrm{SNR}} \to 1$$

Hence

$$\frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}} \le Q\left(\sqrt{2\mathrm{SNR}}\right) \le \frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}}$$

The suitable approximation for the probability of error can be written as

$$P_e = Q\left(\sqrt{2\mathrm{SNR}}\right) = \frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}}$$

When both sides of a function's inequality converge to the same function, then the function also converges to this function.

c. In AWGN channel

$$P_e = \frac{1}{2\sqrt{\pi\mathrm{SNR}}}e^{-\mathrm{SNR}}$$

Let $P_e = 10^{-6}$

$$\mathrm{SNR} = 11.34$$

Convert to dB

$$\mathrm{SNR} = 10.54\ \mathrm{dB}$$

In Rayleigh channel

$$P_e = \frac{1}{4\mathrm{SNR}}$$

Let $P_e = 10^{-6}$

$$\mathrm{SNR} = \frac{1}{4P_e} = \frac{1}{4\times 10^{-6}} = 250000$$

Convert to dB

$$\mathrm{SNR} = 53.98\ \mathrm{dB}$$

From the results, it can be seen that AWGN channels require much lower SNR than Rayleigh channels if they want to maintain the same bit error probability.

Problem 3.3 (1.5 points). *Cross link interference in TDD system*

Consider the parameters below for a scenario where there is a transmission and reception between the base stations (BS) and mobile stations (MS) as shown in Figure 1.

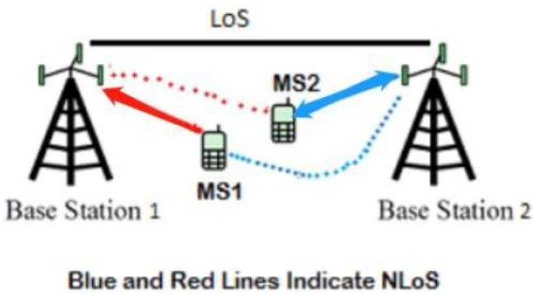

Figure 1: Link between BS 1 and BS 2 is Line-of-Sight (black line). All the other links between BS-MS and MS-MS are Non-Line-of-Sight (blue and red lines).

The bandwidth is 10 MHz, and one user is served per cell. Mobile station 1 (MS1) is served by BS1, while MS2 is served by BS2.

Transmit power per MS = 23 dBm. Transmit power per BS = 30 dBm. Noise power = -174 dBm. Distance between BSs = 100 m. Distance from BS to MS= 35 m.

MS1 transmits in UL all the time. MS2 uses three different configurations: 1) 100% in UL; 2) 50% in UL and 50% in DL; 3) 100% in DL.

Answer appropriately the following questions:

a) What is the **maximum data rate** that MS1 can achieve in each configuration of the different TDD frame configurations used in cell 2? (Use Shannon's formula, including the effect of both noise and interference)
b) Estimate the **upper bound for the mean data rate** that MS1 can achieve in uplink
c) Explain, with your own words, the effect of the frame configuration of cell 2 on the data rate of cell 1? Would this effect be observed if, instead of TDD, the duplexing both uplink and downlink transmissions would take place in the frequency domain

Use the following path loss models:

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- LoS Path Loss: PL(d) = 16.9 $\log_{10}$(d) + 38.8 [dB], 'd' in meters
- NLoS Path Loss: PL(d) = 43.3 $\log_{10}$(d) + 17.5 [dB], 'd' in meters

a. maximum data rate

1) MS2 100% in UL

Received signal power

$$\begin{aligned} P_s &= MS - PL_{NLoS1} \\ &= 23 - \left[43.3 \times \log_{10}(35) + 17.5\right] \\ &= -61.36 \text{ dBm} \\ &= 7.31 \times 10^{-7} \text{ mW} \end{aligned}$$

BS1 is subjected to interference power from the uplink transmission of MS2

$$\begin{aligned} P_{i1} &= MS - PL_{NLoS2} \\ &= 23 - \left[43.3 \times \log_{10}(35) + 17.5\right] \\ &= -61.36 \text{ dBm} \\ &= 7.31 \times 10^{-7} \text{ mW} \end{aligned}$$

Noise power

$$N_0 = -174 \text{ dBm} = 3.98 \times 10^{-18} \text{ mW}$$

Shannon's formula

$$\begin{aligned} C_1 &= W \log_2\left(1 + \frac{P_s}{P_{i1} + N_0 W}\right) \\ &= 10 \times 10^6 \times \log_2\left(1 + \frac{7.31 \times 10^{-7}}{7.31 \times 10^{-7} + 3.98 \times 10^{-18}}\right) \\ &= 1 \times 10^7 \text{ bit/s} \end{aligned}$$

2) MS2 50% in UL and 50% in DL

Received signal power

$$\begin{aligned} P_s &= MS - PL_{NLoS1} \\ &= 23 - \left[43.3 \times \log_{10}(35) + 17.5\right] \\ &= -61.36 \text{ dBm} \\ &= 7.31 \times 10^{-7} \text{ mW} \end{aligned}$$

BS1 is subjected to interference power from the uplink transmission of MS2 and the downlink transmission from BS2.

$$\begin{aligned} P_{iB} &= BS - PL_{LoS} \\ &= 30 - \left[16.9 \times \log_{10}(100) + 38.8\right] \\ &= -42.60 \text{ dBm} \\ &= 5.50 \times 10^{-5} \text{ mW} \end{aligned}$$

Noise power

$$N_0 = -174 \text{ dBm} = 3.98 \times 10^{-18} \text{ mW}$$

When MS2 in UL

$$\begin{aligned} C_{2,1} &= W \log_2\left(1 + \frac{P_s}{P_{i1} + N_0 W}\right) \\ &= 10 \times 10^6 \times \log_2\left(1 + \frac{7.31 \times 10^{-7}}{7.31 \times 10^{-7} + 3.98 \times 10^{-18}}\right) \\ &= 1 \times 10^7 \text{ bit/s} \end{aligned}$$

When MS2 in DL

$$\begin{aligned} C_{2,2} &= W \log_2\left(1 + \frac{P_s}{P_i + N_0 W}\right) \\ &= 10 \times 10^6 \times \log_2\left(1 + \frac{7.31 \times 10^{-7}}{5.50 \times 10^{-5} + 3.98 \times 10^{-18}}\right) \\ &= 1.90 \times 10^5 \text{ bit/s} \end{aligned}$$

3) MS2 100% in DL

Received signal power

$$\begin{aligned} P_s &= MS - PL_{NLoS1} \\ &= 23 - \left[43.3 \times \log_{10}(35) + 17.5\right] \\ &= -61.36 \text{ dBm} \\ &= 7.31 \times 10^{-7} \text{ mW} \end{aligned}$$

BS1 is subjected to interference power from the downlink transmission of BS2

$$\begin{aligned} P_{iB} &= BS - PL_{LoS} \\ &= 30 - \left[16.9 \times \log_{10}(100) + 38.8\right] \\ &= -42.60 \text{ dBm} \\ &= 5.50 \times 10^{-5} \text{ mW} \end{aligned}$$

Noise power

$$N_0 = -174 \text{ dBm} = 3.98\times10^{-18} \text{ mW}$$

Shannon's formula

$$\begin{aligned} C_3 &= W\log_2\left(1+\frac{P_s}{P_{iB}+N_0W}\right) \\ &= 10\times10^6\times\log_2\left(1+\frac{7.31\times10^{-7}}{5.50\times10^{-5}+3.98\times10^{-18}}\right) \\ &= 1.90\times10^5 \text{ bit/s} \end{aligned}$$

b. Upper bound for the mean data rate

Assuming that each configuration of MS2 lasts for t

$$R_{mean} = \frac{C_1 t + C_{2,1}\dfrac{t}{2} + C_{2,2}\dfrac{t}{2} + C_3 t}{3t} = \frac{C_1 + \dfrac{C_{2,1}}{2} + \dfrac{C_{2,2}}{2} + C_3}{3} = 5.095\times10^6 \text{ bit/s}$$

c. The frame configuration of cell 2 has an impact on cell 1, reducing the data rate of MS1 in cell 1 when MS2 is in both uplink and downlink. Since the transmission power of the base station is higher, the reduction of the MS1 uplink data rate is greater when MS2 is in the downlink.

If duplexing is performed in the frequency domain, the uplink and downlink can operate independently of each other and the interference in both cells is reduced because the uplink and downlink operate in different frequency bands. But this reduces the bandwidth of the system and will also reduce the data rate.