Fermionic tensor network methods

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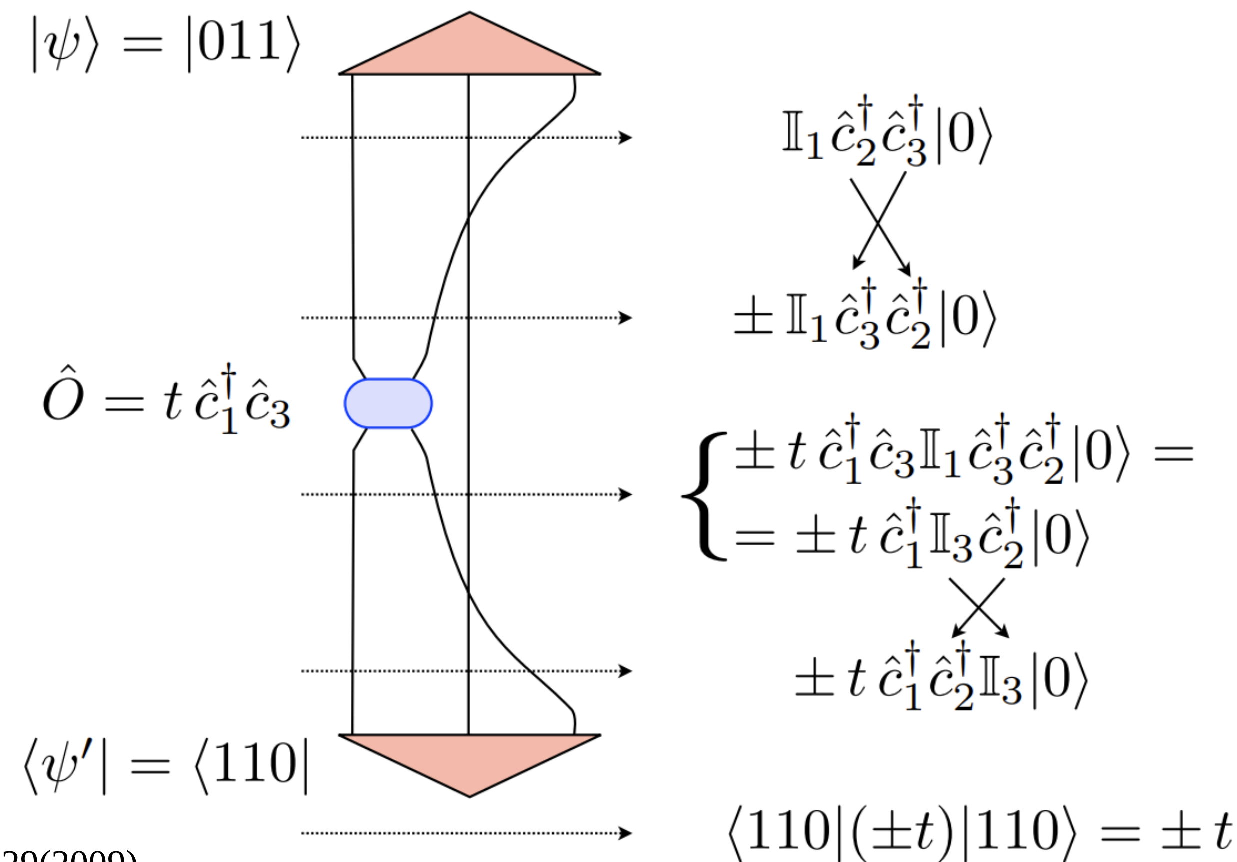
Content

- Swap gate method
- Z2-graded Hilbert spaces method
- Appendix: Category theory

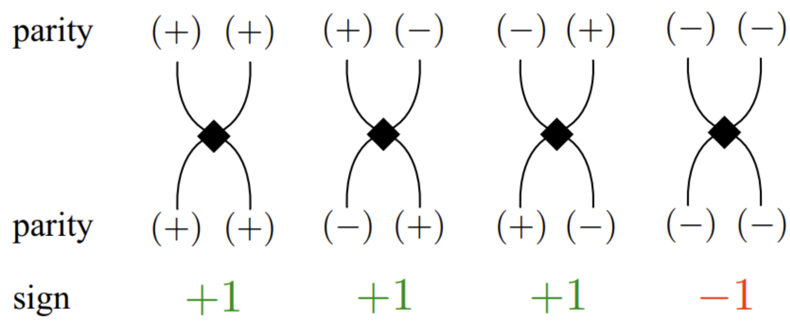
Fermionic system

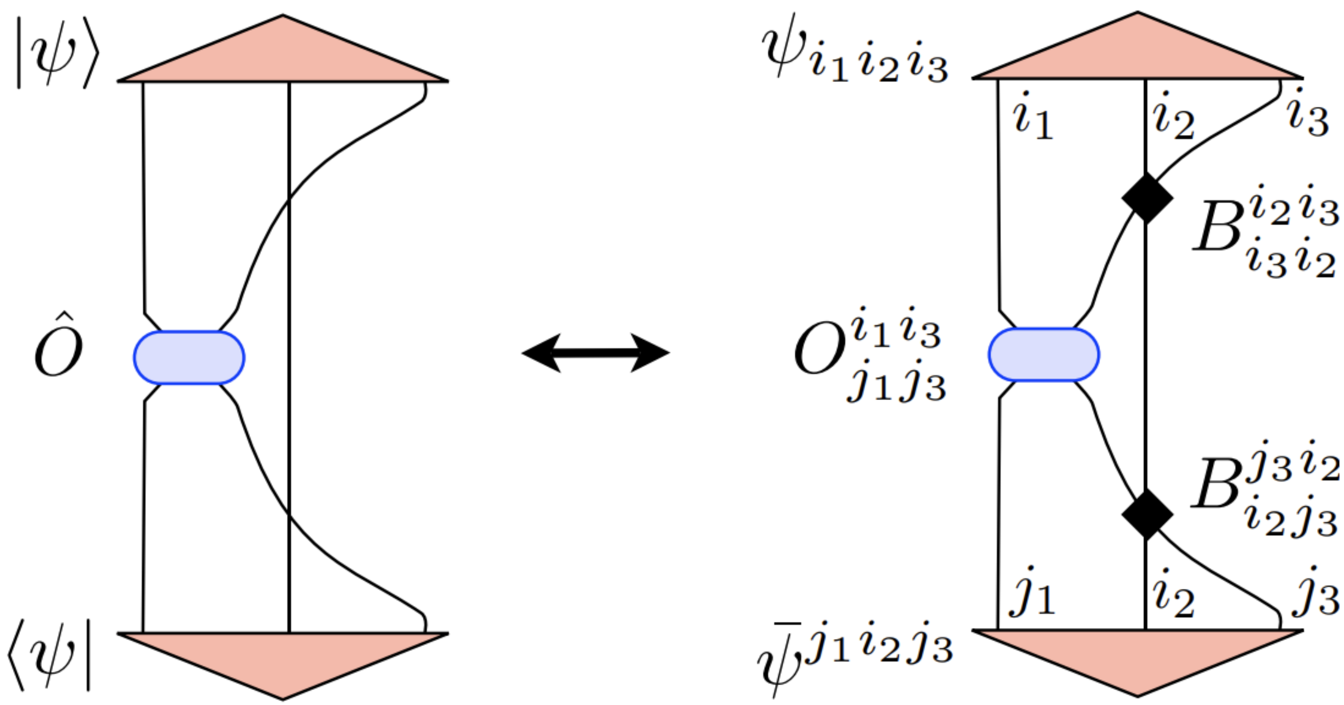
- State $|i_1 i_2 i_3\rangle \equiv \hat{c}_1^{\dagger i_1} \hat{c}_2^{\dagger i_2} \hat{c}_3^{\dagger i_3} |0\rangle$
- overlap $\langle \psi' | \hat{O} | \psi \rangle = \langle 110 | t \hat{c}_1^\dagger \hat{c}_3 | 011 \rangle = t \langle 0 | \hat{c}_2 \hat{c}_1 \hat{c}_1^\dagger \hat{c}_3 \hat{c}_2^\dagger \hat{c}_3^\dagger | 0 \rangle$



Corboz, PRB,80, 165129(2009)

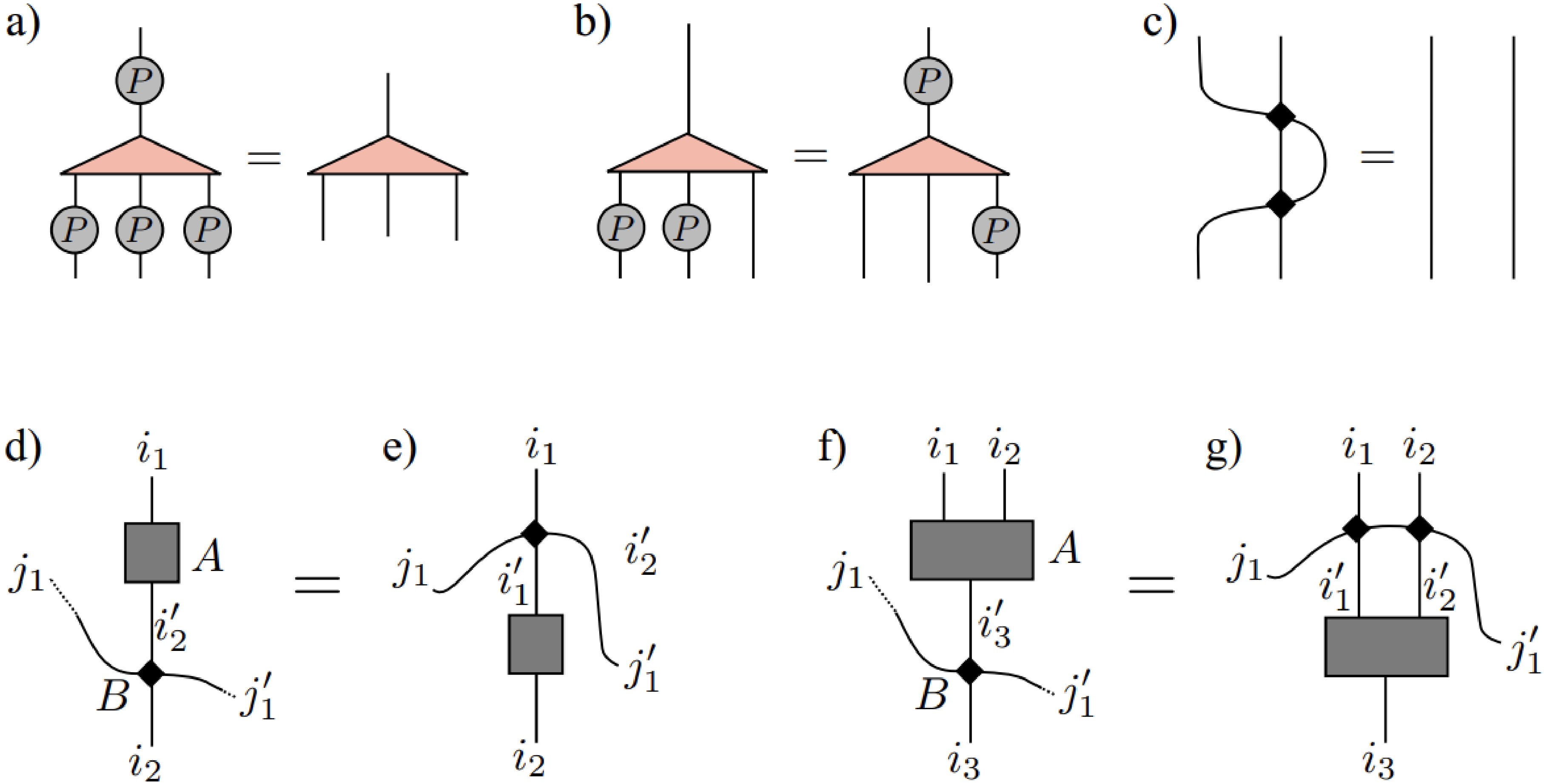
Swap gate



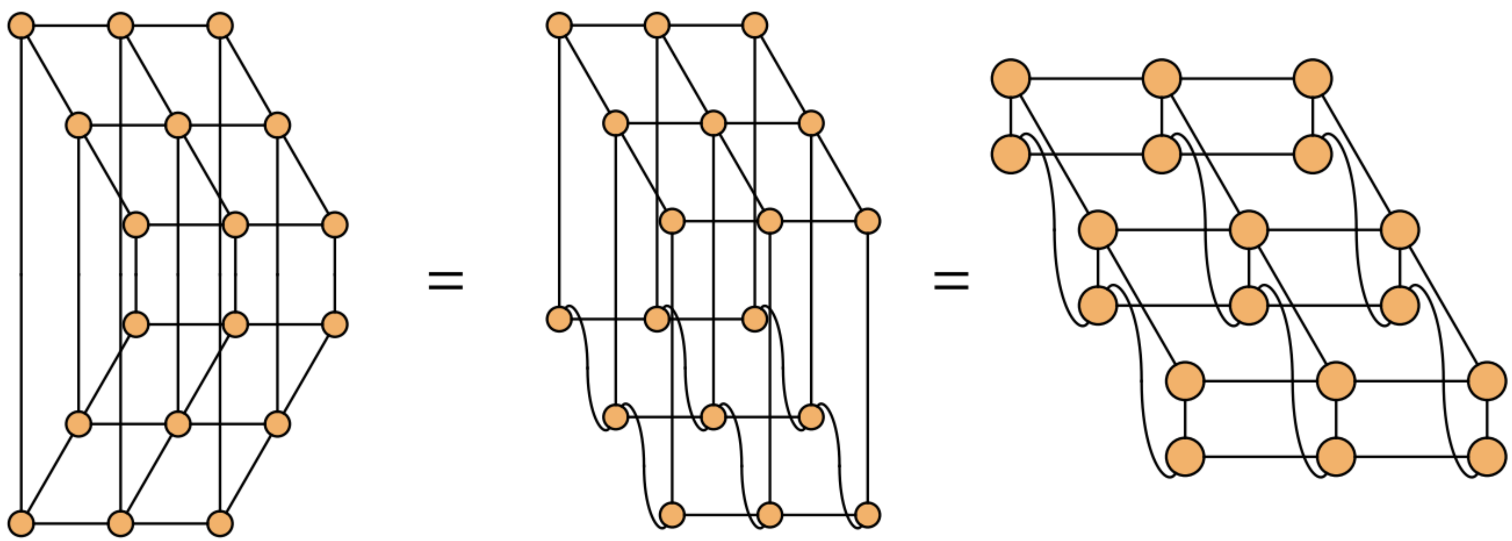


Corboz, PRB,80, 165129(2009)

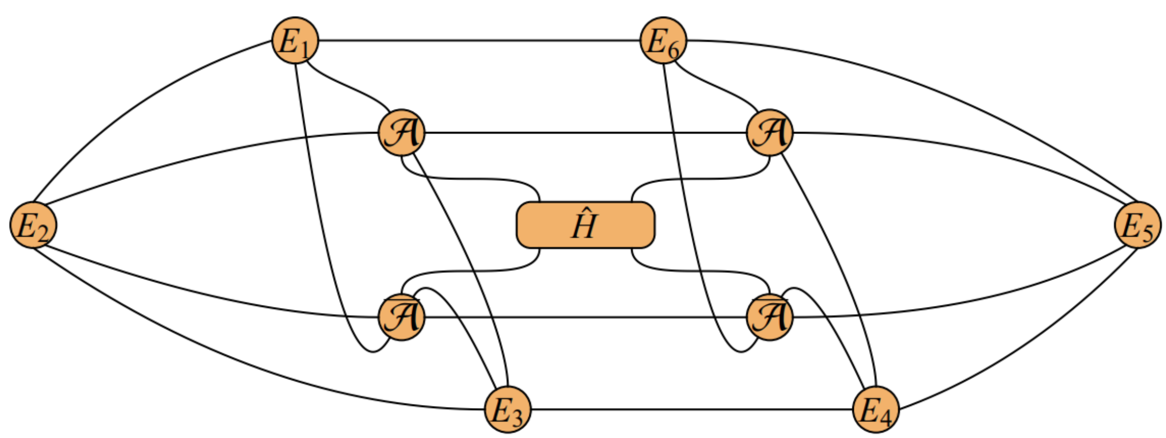
Jump move


a)
P
P
P
P
=
b)
P
P
=
P
P
c)
=
d)
i_1
A
j_1
i'_2
B
j'_1
i_2
=
e)
i_1
j_1
i'_1
i'_2
j'_1
i_2
f)
i_1 i_2
A
j_1
i'_3
B
j'_1
i_3
=
g)
i_1 i_2
j_1
i'_1
i'_2
j'_1
i_3

2D tensor network

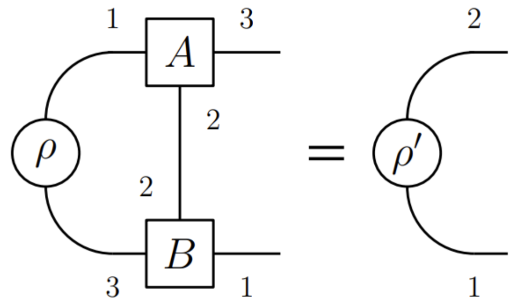


$\langle \Psi | \hat{H} | \Psi \rangle =$

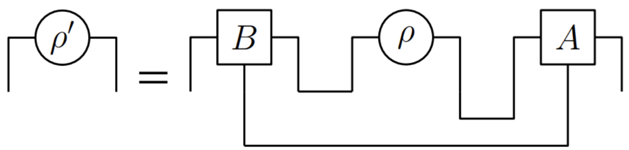


Fermionic tensor contraction

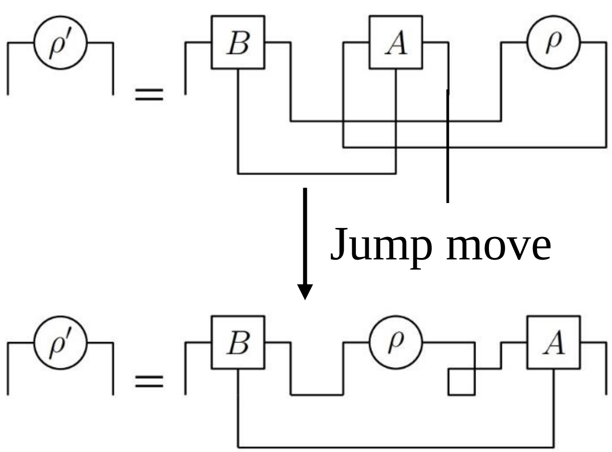
- A simple example



$B \to \rho \to A$



$B \to A \to \rho$



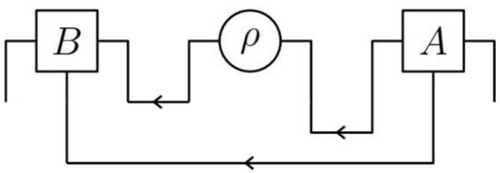
Self-crossing → a extra swap gate,
different order → different result

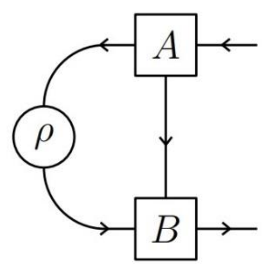
Mortier, arXiv:2404.14611(2024)

Two ways out of this impasse

- crossing-free
  - 1D: planar contraction √
  - 2D: planar contraction ×
    - Swap gate → planar connection
- an orientation (arrow) to all tensor legs
  - It's natural in symmetric tensor network

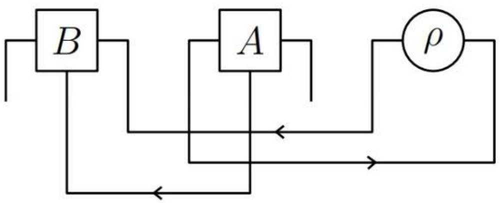
$B \to \rho \to A$





,

$B \to A \to \rho$

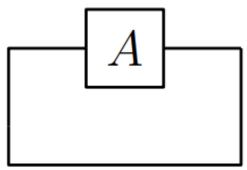


Extra fermionic minus signs for →

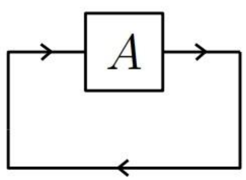
Mortier, arXiv:2404.14611(2024)

Supertrace

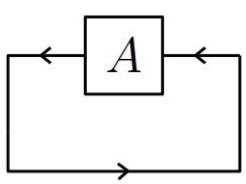
- trace



$\mathrm{tr}(A) = A_{ii}$



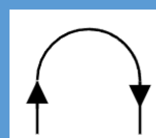
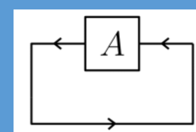
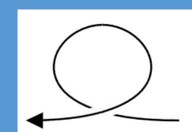
$\mathrm{tr}(A) = A_{ii}$ Regular trace



$\mathrm{tr}(A) = A_{ii}(-1)^{|i|}$ Supertrace

What is different?

- Whether place fermionic minus sign to $\longrightarrow$

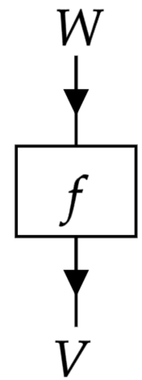
			
swap gate	+	+	-
arrow	-	-	+

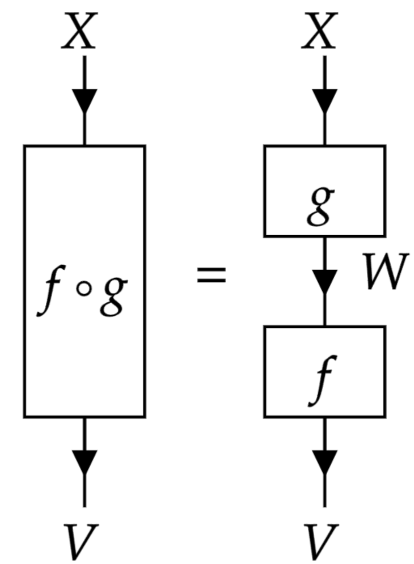
right evaluation **supertrace** **twist**

Penrose graphical calculus

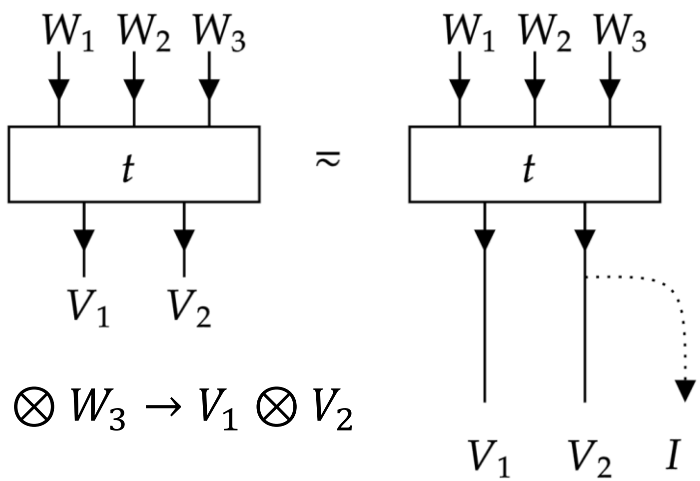
- **Morphism** $f: {\color{red}W} \rightarrow {\color{green}V}$

  domain(source) $\rightarrow$ **codomain(target)**





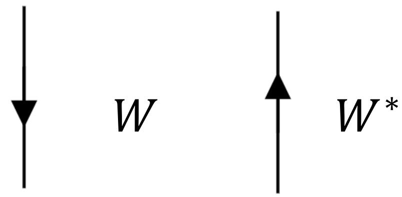
- **Identity**



$W_1 \otimes W_2 \otimes W_3 \rightarrow V_1 \otimes V_2$

$W_1 \otimes W_2 \otimes W_3 \rightarrow V_1 \otimes V_2 \otimes I$

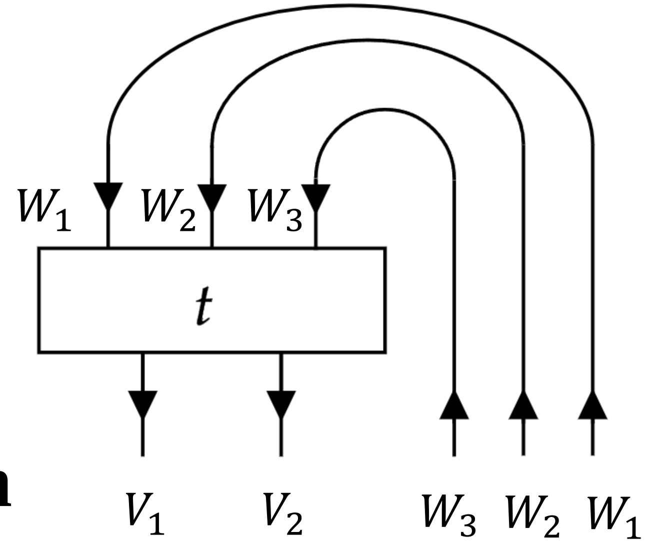
- **Dual**



$W_1 \otimes W_2 \rightarrow V_1 \otimes V_2 \otimes W_3^*$

Duality

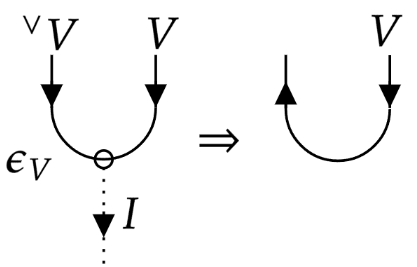
- $$\begin{array}{c} W_1 \otimes W_2 \otimes W_3 \to V_1 \otimes V_2 \\ \Big\downarrow \eta \\ I \to V_1 \otimes V_2 \otimes W_3^* \otimes W_2^* \otimes W_1^* \end{array}$$

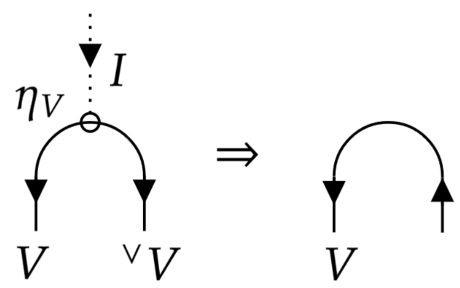


- **Left evaluation and coevaluation**

$\epsilon_V : {}^\vee V \otimes V \to I$

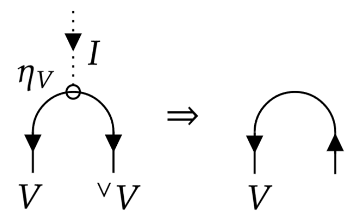
$\eta_V : I \to V \otimes {}^\vee V$



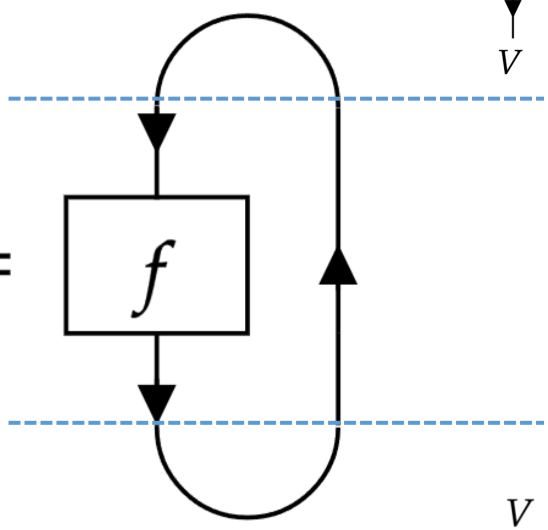


trace

$$\mathrm{tr_r}(f) = \tilde{\epsilon}_V \circ (f \otimes \mathrm{id}_{V^*}) \circ \eta_V$$

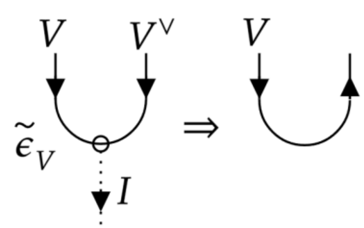


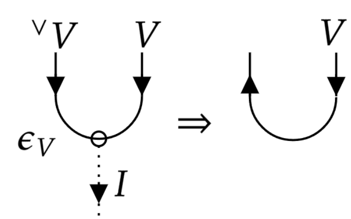
$$\mathrm{tr_r}(f) \quad =$$



$$\epsilon_V : V^* \otimes V \to \mathbb{C} : \langle m| \otimes_{\mathrm{g}} |n\rangle \to \langle m|n\rangle = \delta_{m,n}$$

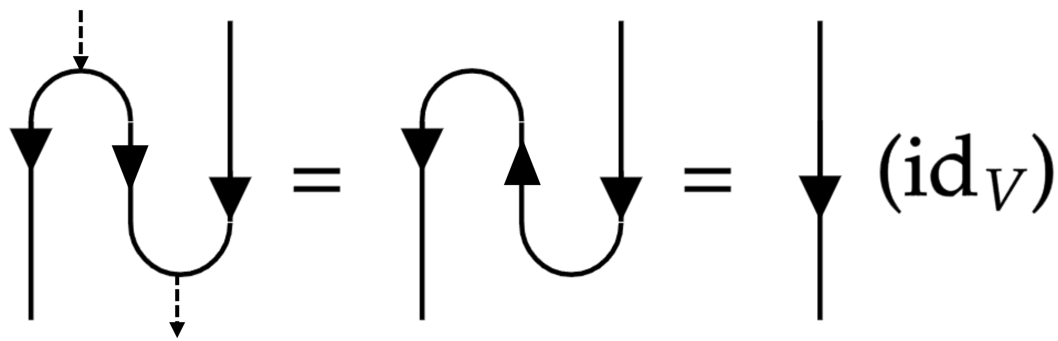
$$\tilde{\epsilon}_V : V \otimes V^* \to \mathbb{C} : |n\rangle \otimes_{\mathrm{g}} \langle m| \to (\pm 1)^{|n|}\delta_{m,n}$$



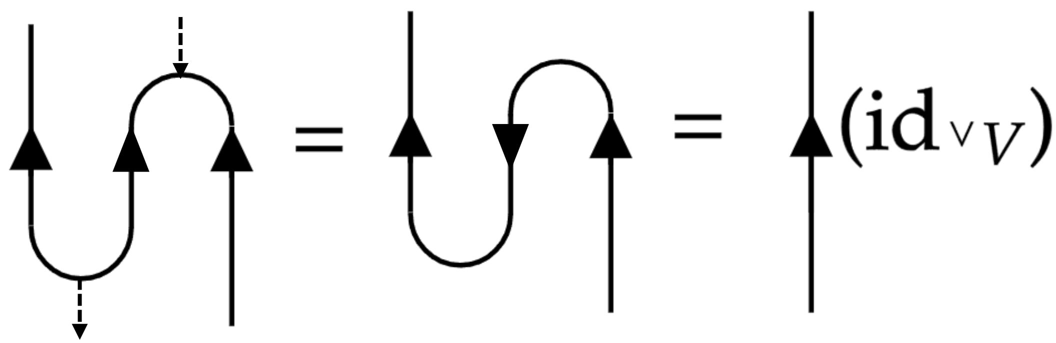


snake rules

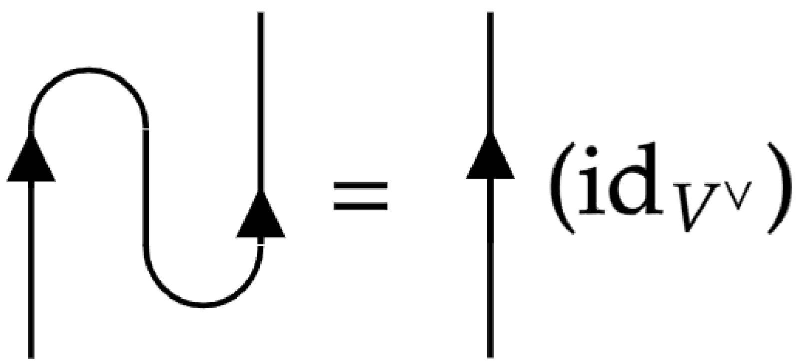
- $\rho_V \circ (\mathrm{id}_V \otimes \epsilon_V) \circ (\eta_V \otimes \mathrm{id}_V) \circ \lambda_V^{-1} = \mathrm{id}_V$

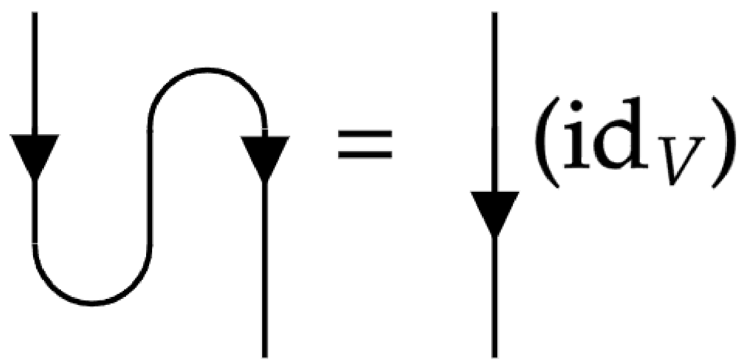


- $\lambda_{{}^\vee V}^{-1} \circ (\epsilon_V \otimes \mathrm{id}_{{}^\vee V}) \circ (\mathrm{id}_{{}^\vee V} \otimes \eta_V) \circ \rho_{{}^\vee V}^{-1} = \mathrm{id}_{{}^\vee V}$



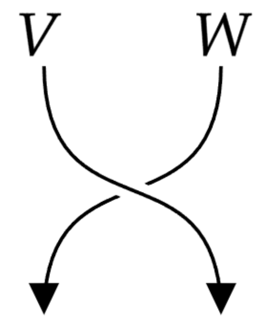
Right snake rules





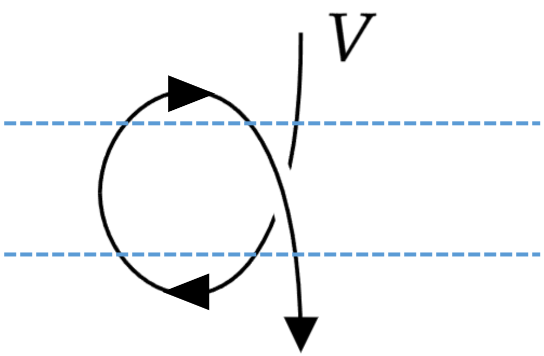
twist

- **Braiding** $\tau_{V,W}: V \otimes W \to W \otimes V$

$\tau_{V,W} =$ 

$$\tau_{V,W}: |m\rangle \otimes_{\mathrm{g}} |n\rangle \mapsto (-1)^{|m||n|} |n\rangle \otimes_{\mathrm{g}} |m\rangle$$

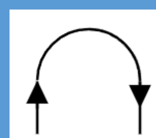
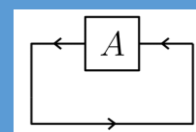
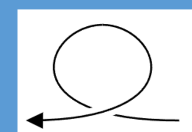
- **twist** $\theta^{\mathrm{l}}_V = (\epsilon_V \otimes \mathrm{id}_V)(\mathrm{id}_{V^*} \otimes \tau_{V,V})(\tilde{\eta}_V \otimes \mathrm{id}_V)$



$$\theta_V: |n\rangle \mapsto (\mp 1)^{|n|} |n\rangle$$

What is different?

- Whether place fermionic minus sign to $\longrightarrow$

			
swap gate	+	+	-
arrow	-	-	+

right evaluation **supertrace** **twist**

canonical contraction map

- $\mathcal{C} : V^* \otimes V \to \mathbb{C} : \langle i| \, |j\rangle \mapsto \mathcal{C}\left(\langle i| \, |j\rangle\right) = \langle i|j\rangle = \delta_{ij}$
- $\mathcal{F} : V_1 \otimes V_2 \to V_2 \otimes V_1 : |i\rangle_1 \otimes |j\rangle_2 \mapsto (-1)^{|i||j|} \, |j\rangle_2 \otimes |i\rangle_1$

- $\tilde{\mathcal{C}} = \mathcal{C} \circ \mathcal{F} : V \otimes V^* \to \mathbb{C}$

  $\tilde{\mathcal{C}}\left(|i\rangle \langle j|\right) = (-1)^{|i||j|} \mathcal{C}\left(\langle j| \, |i\rangle\right) = (-1)^{|i||j|} \delta_{ij}$

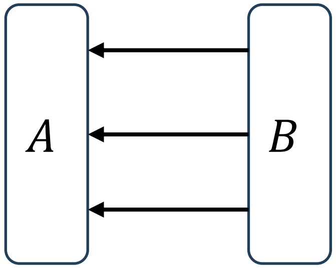
Inner product

- $A, B \in V_1 \otimes V_2 \otimes V_3 \quad A = A_{\alpha\beta\gamma} |\alpha\rangle |\beta\rangle |\gamma\rangle$ and $B = B_{\delta\varepsilon\zeta} |\delta\rangle |\varepsilon\rangle |\zeta\rangle$

$$\bar{A} = A_{\alpha\beta\gamma}{}^{*} \langle\gamma| \langle\beta| \langle\alpha| \in V_3^* \otimes V_2^* \otimes V_1^*$$

$$\langle A, B\rangle = A_{\alpha\beta\gamma}{}^{*} B_{\delta\varepsilon\zeta} \langle\alpha|\delta\rangle \langle\beta|\varepsilon\rangle \langle\gamma|\zeta\rangle = A_{\alpha\beta\gamma}{}^{*} B_{\alpha\beta\gamma}$$

$$\langle A, B\rangle = \mathcal{C}(\bar{A} \otimes B)$$

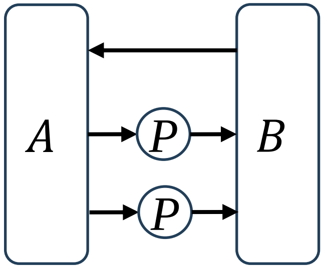


- $A = A_{\alpha\beta\gamma} |\alpha\rangle \langle\beta| \langle\gamma|$ and $B = B_{\delta\varepsilon\zeta} |\delta\rangle \langle\varepsilon| \langle\zeta|$

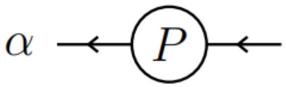
$$\begin{aligned}\langle A, B\rangle &= A_{\alpha\beta\gamma}{}^{*} B_{\delta\varepsilon\zeta} \langle\alpha|\delta\rangle \langle\varepsilon|\beta\rangle \langle\zeta|\gamma\rangle \\ &= A_{\alpha\beta\gamma}{}^{*} |\gamma\rangle |\beta\rangle \langle\alpha| (-1)^{|\varepsilon|}(-1)^{|\zeta|} B_{\delta\varepsilon\zeta} |\delta\rangle \langle\varepsilon| \langle\zeta|\end{aligned}$$

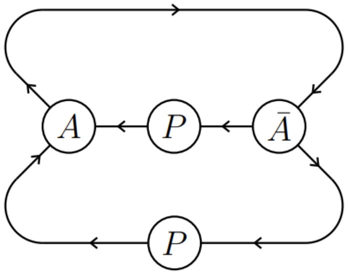
$$\mathcal{P}(B) = (-1)^{|\varepsilon|}(-1)^{|\zeta|} B_{\delta\varepsilon\zeta} |\delta\rangle \langle\varepsilon| \langle\zeta|$$

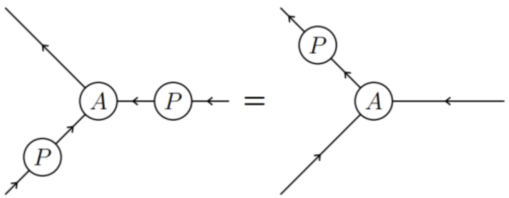
$$\langle A, B\rangle = \mathcal{C}(\bar{A}\mathcal{P}(B))$$



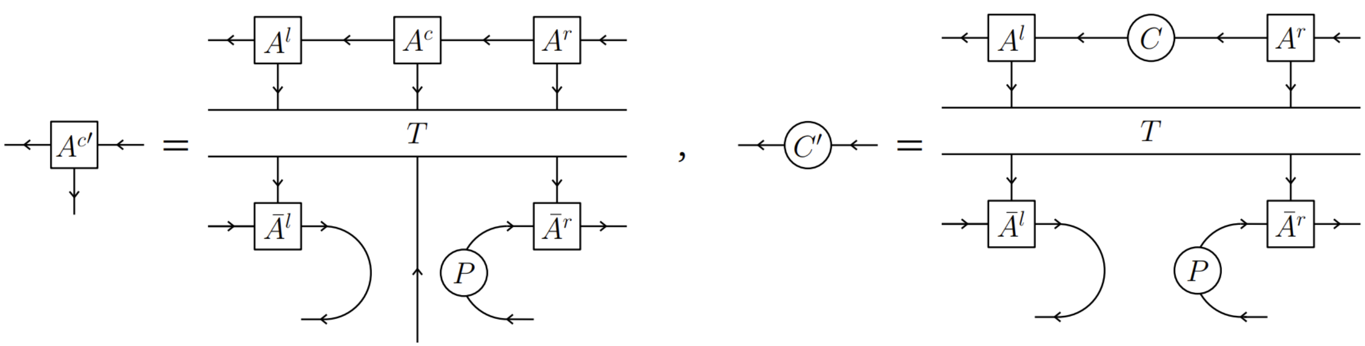
Implement

- P (parity) tensor  $= (-1)^{|\alpha|}\delta_\alpha^\beta \left|\alpha\right\rangle \left\langle\beta\right|$
- Contraction

$|\langle A, A\rangle|^2 = \mathcal{C}(\bar{A}\mathcal{P}(A)) =$ 

$\mathcal{P}(A) =$ 

- VUMPS

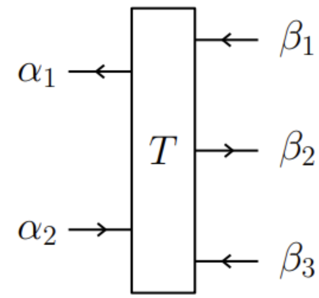


Hermitian conjugation

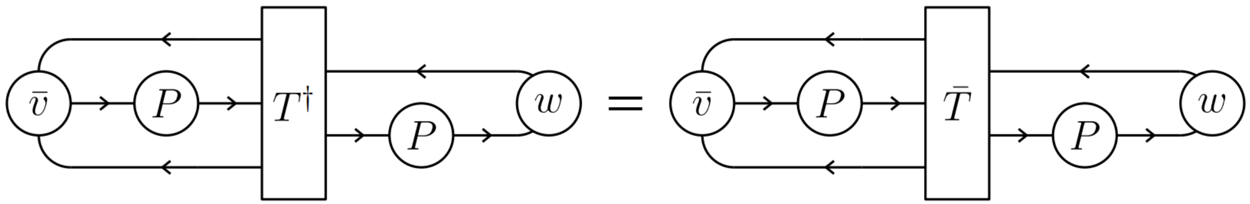
- A example

$$T : V_1 \otimes V_2^* \otimes V_3 \to W_1 \otimes W_2^* \qquad\qquad T^\dagger : W_1 \otimes W_2^* \to V_1 \otimes V_2^* \otimes V_3$$

$$T = T^{\beta_1\beta_2\beta_3}_{\alpha_1\alpha_2} \left|\alpha_1\right\rangle \left\langle\alpha_2\right| \left\langle\beta_3\right| \left|\beta_2\right\rangle \left\langle\beta_1\right| =$$



$$\langle v, T^\dagger w\rangle = \langle T v, w\rangle \longrightarrow \mathcal{C}(\bar{v}\,\mathcal{P}(T^\dagger(w))) = \mathcal{C}(\overline{T(v)}\,\mathcal{P}(w))$$

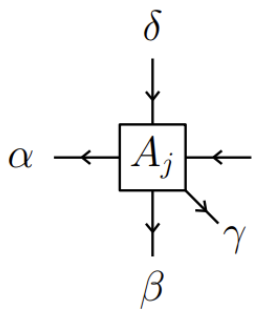


- Hermitian

$$T^{\beta_1\beta_2\ldots\beta_n}_{\alpha_1\alpha_2\ldots\alpha_n} = T^{\alpha_1\alpha_2\ldots\alpha_n\,*}_{\beta_1\beta_2\ldots\beta_n}$$

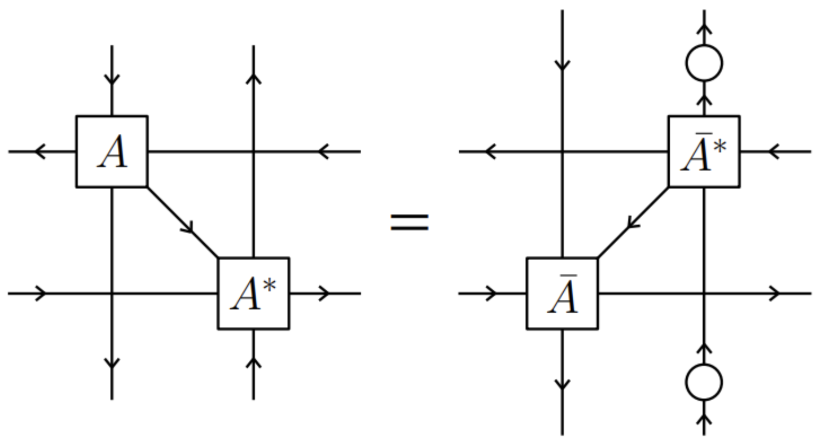
Fermionic PEPS

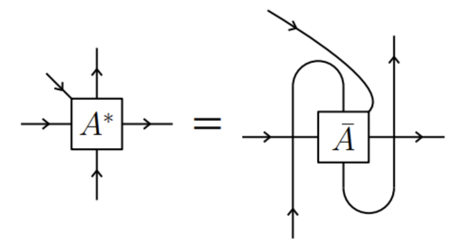
- iPEPS

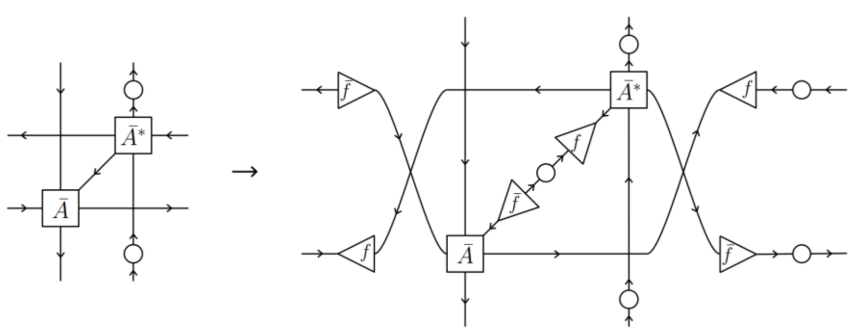


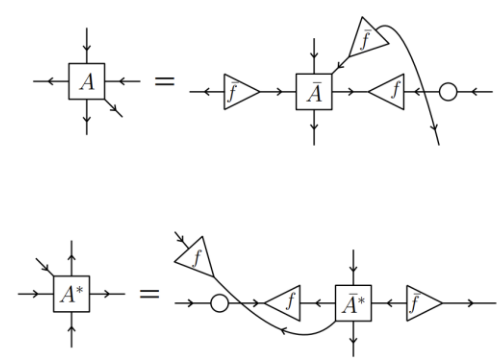
$$= \left(A_j\right)_{\alpha\beta\gamma\delta\varepsilon} |\alpha\rangle |\beta\rangle |\gamma\rangle \langle\varepsilon| \langle\delta|$$

- Hermitian









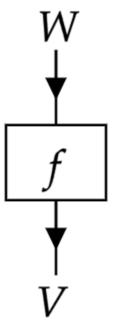
Thank you for listening!

Q&A?

Appendix: Category

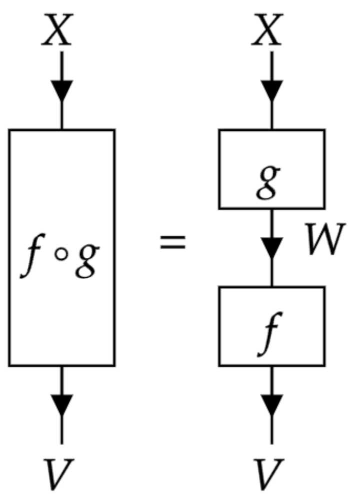
Category

- **Object** $\mathrm{Ob}(C)\ V, W, \ldots$
- **Morphism** $f: \color{red}{W} \rightarrow \color{green}{V}$ and $f \in \mathrm{Hom}_C(W, V)$



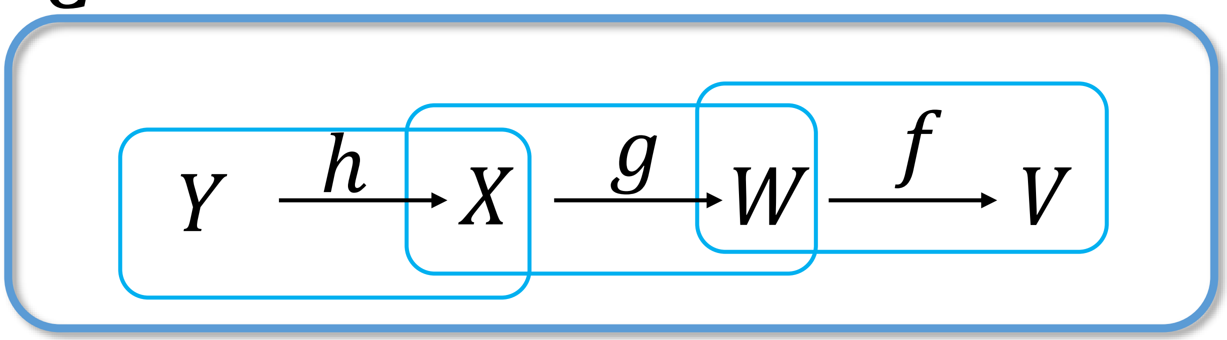
domain(source) → **codomain(target)**

Penrose graphical calculus

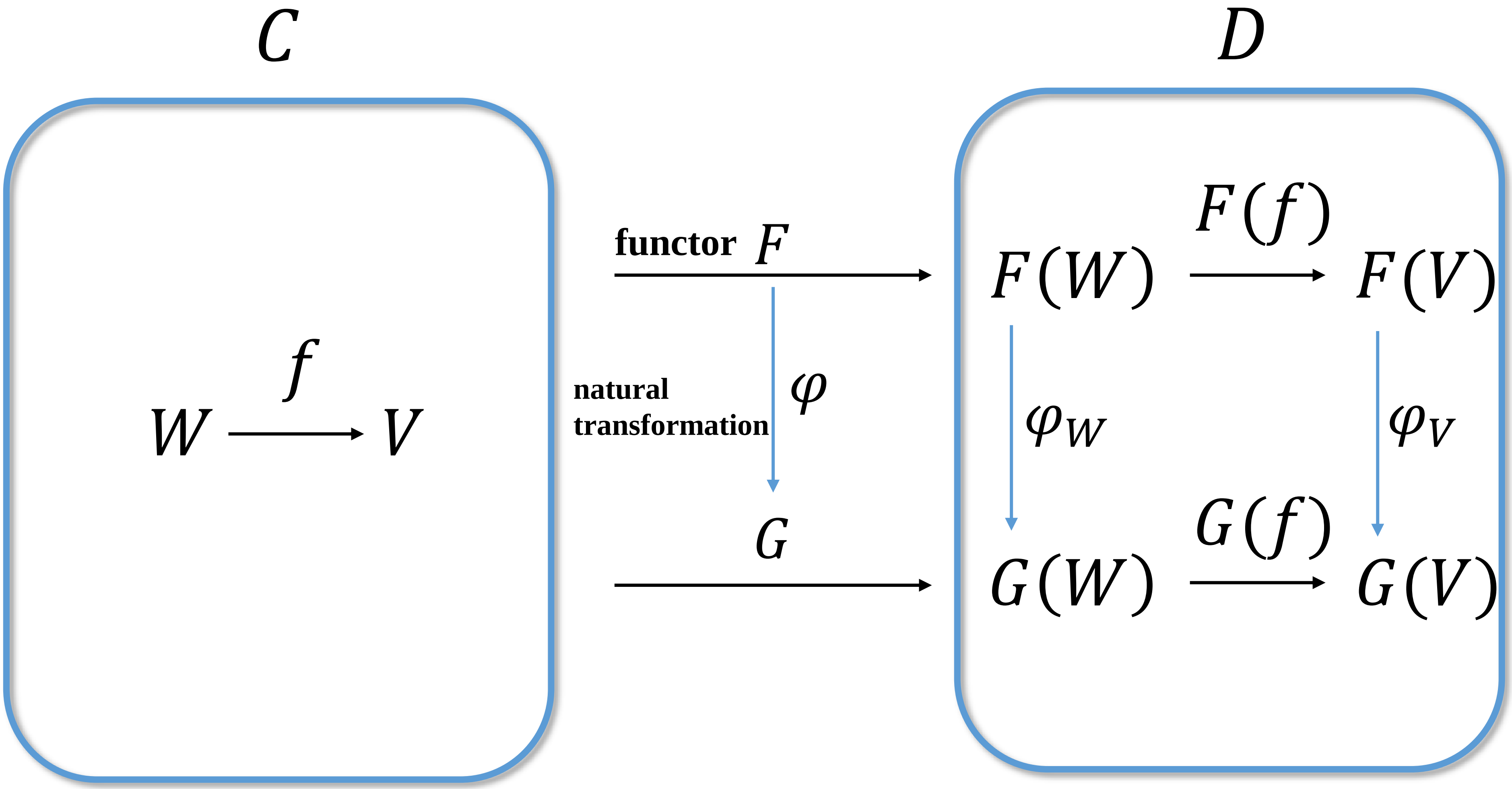


- **Composition** $f \circ (g \circ h) = (f \circ g) \circ h$
- **Identity** $\mathrm{id}_V: V \rightarrow V$ s.t. $f \circ \mathrm{id}_W = f = \mathrm{id}_V \circ f$

C



Functor and natural transformation



$$\varphi_V \circ F(f) = G(f) \circ \varphi_W$$

Other concept

- **Endomorphism** $f \in \mathrm{End}_C(V) = \mathrm{Hom}_C(V, V)$
- **Isomorphism** for $f: W \to V, \exists\, f^{-1}: V \to W$ s. t.
  - $f^{-1} \circ f = \mathrm{id}_W$
  - $f \circ f^{-1} = \mathrm{id}_V$
- $\forall \varphi_V$ is isomorphism
  - φ is **natural isomorphism**
  - F and G are **isomorphic**

Basic category theory

- object, morphism, composition
    - identity morphism, endomorphism, isomorphism
    - Penrose graphical calculus
- functor, natural transformation
    - natural isomorphism, morphic

Category product $C \times C'$

- Object $\mathrm{Ob}(C \times C') = \mathrm{Ob}(C) \times \mathrm{Ob}(C')$
- Morphism

$$\mathrm{Hom}_{C\times C'}((W,W'),(V,V')) = \mathrm{Hom}_C(W,V) \times \mathrm{Hom}_{C'}(W',V')$$

  - Composition $(f,f') \circ (g,g') = (f \circ f', g \circ g')$
  - Identity $\mathrm{id}_{V,V'} = (\mathrm{id}_V, \mathrm{id}_{V'})$
- Functor $F \times F' : (C \times C') \to (D \times D')$

Tensor Category

- Functor $\otimes$
  - On objects $\mathrm{Ob}(C) \times \mathrm{Ob}(C) \to \mathrm{Ob}(C)$
  - On morphisms $\mathrm{Hom}_C(W_1, V_1) \times \mathrm{Hom}_C(W_2, V_2) \to \mathrm{Hom}_C(W_1 \otimes W_2, V_1 \otimes V_2)$
- Identity(unit object) I
- Natural isomorphisms
  - **Left unitor** $\lambda_V : I \otimes V \to V$
  - **Right unitor** $\rho_V : V \otimes I \to V$
  - **Associator** $\alpha_{V_1,V_2,V_3} : (V_1 \otimes V_2) \otimes V_3 \to V_1 \otimes (V_2 \otimes V_3)$
    - Pentagon equation $(((V_1 \otimes V_2) \otimes V_3) \otimes V_4)$ to $(V_1 \otimes (V_2 \otimes (V_3 \otimes V_4)))$
    - Triangle equation $((V_1 \otimes I) \otimes V_2)$ to $(V_1 \otimes (I \otimes V_2))$

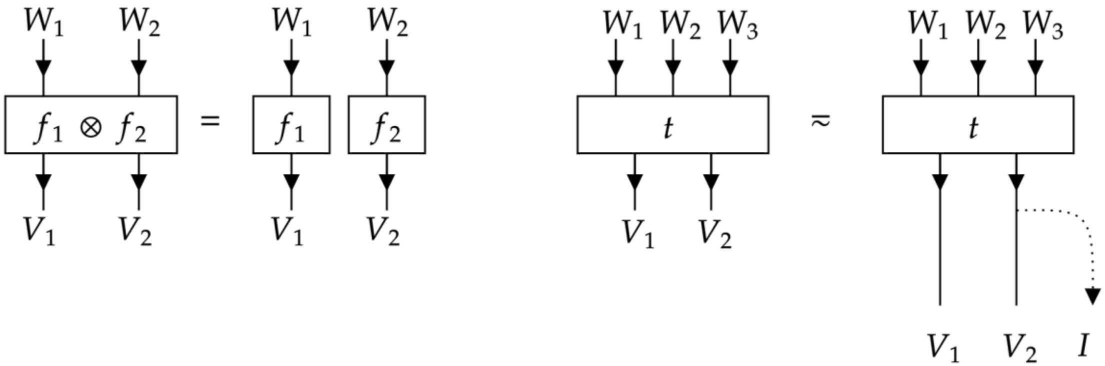
Functor and natural isomorphisms

$$\begin{tikzcd}[column sep=huge] C & C \\ V \arrow[r, shift left=3, "\otimes"] \arrow[r, shift right=3, "1_C"'] \arrow[r, phantom, "\Downarrow \varphi"] & \begin{tikzcd} I \otimes V \arrow[d, "\lambda_V"] & V \otimes I \arrow[d, "\rho_V"] \\ V & V \end{tikzcd} \end{tikzcd}$$

$$\begin{tikzcd}[column sep=huge] V_1, V_2, V_3 \arrow[r, shift left=3, "\otimes \times 1_C"] \arrow[r, shift right=3, "1_C \times \otimes"'] \arrow[r, phantom, "\Downarrow \alpha"] & \begin{tikzcd} (V_1 \otimes V_2) \otimes V_3 \arrow[d, "\alpha_{V_1, V_2, V_3}"] \\ V_1 \otimes (V_2 \otimes V_3) \end{tikzcd} \end{tikzcd}$$

Other concept

- **Strict**
  - $I \otimes V = V = V \otimes I$
  - $(V_1 \otimes V_2) \otimes V_3 = V_1 \otimes (V_2 \otimes V_3)$
  - $\lambda_V, \rho_V, \alpha_{V_1,V_2,V_3}$ are identity morphisms
  - Example
    - **Vect** vector spaces over a field $\Bbbk$
    - **F-move** Symmetric tensors
- Mac Lane's coherence theorem: $\lambda_V, \rho_V, \alpha_{V_1,V_2,V_3}$ all commute



Example: $\mathbf{SVect}_{\Bbbk}$

- vector spaces with a Z_2 grading $V = V_0 \oplus V_1$
- For $f \in \mathrm{Hom}_{\mathbf{SVect}}(W, V)$ has $f(W_0) \subset V_0$ and $f(W_1) \subset V_1$
- Graded tensor product
  - $$(V \otimes_{\mathrm{g}} W) = \underbrace{(V \otimes_{\mathrm{g}} W)_0}_{(V_0 \otimes W_0) \oplus (V_1 \otimes W_1)} \oplus \underbrace{(V \otimes_{\mathrm{g}} W)_1}_{(V_0 \otimes W_1) \oplus (V_1 \otimes W_0)}$$

  $$(V \otimes_{\mathrm{g}} W)_0 = (V_0 \otimes W_0) \oplus (V_1 \otimes W_1) \qquad (V \otimes_{\mathrm{g}} W)_1 = (V_0 \otimes W_1) \oplus (V_1 \otimes W_0)$$

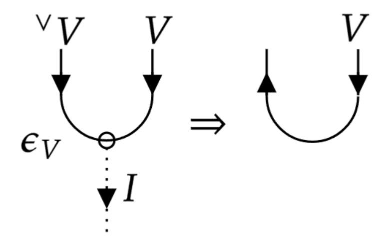
- Unit object $I_0 = \Bbbk$ and $I_1 = 0 \rightarrow V_1 = 0$

Monoidal(Tensor) Category

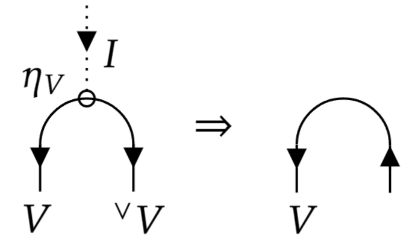
- Product
- Left/right unitor, associator
  - Pentagon equation
  - Triangle equation
  - Mac Lane’s coherence theorem
- Strict

Duality

- **Dual space** V^*
  - Evaluating the action of dual vector on a vector can, because of linearity, be interpreted as a morphism from $V^* \otimes V$ to I
- **Left evaluation** $\epsilon_V : {}^{\vee}V \otimes V \to I$

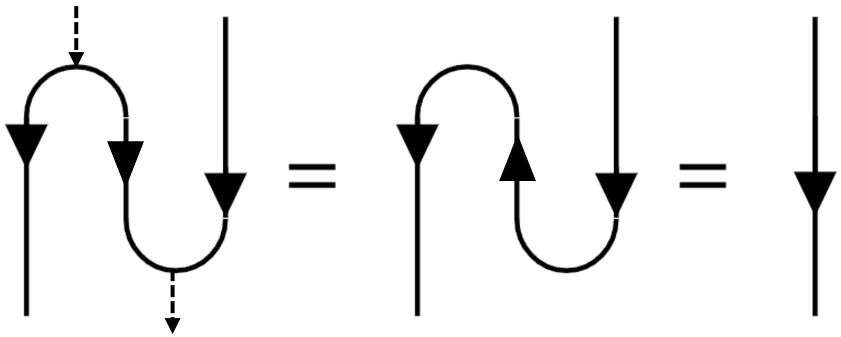


- **Left coevaluation** $\eta_V : I \to V \otimes {}^{\vee}V$



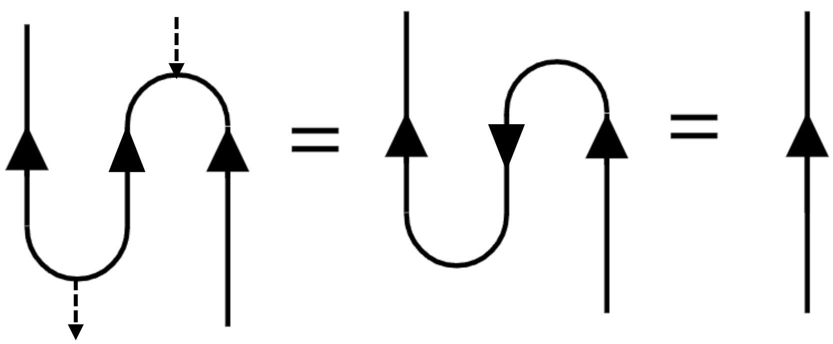
snake rules

- $\rho_V \circ (\mathrm{id}_V \otimes \epsilon_V) \circ (\eta_V \otimes \mathrm{id}_V) \circ \lambda_V^{-1} = \mathrm{id}_V$



(id_V)

- $\lambda_{^\vee V}^{-1} \circ (\epsilon_V \otimes \mathrm{id}_{^\vee V}) \circ (\mathrm{id}_{^\vee V} \otimes \eta_V) \circ \rho_{^\vee V}^{-1} = \mathrm{id}_{^\vee V}$



$(\mathrm{id}_{^\vee V})$

Example

- vector spaces V, dual V^*
  - $|n\rangle$ for V $\langle m|$ for V^*
- Evaluation $\epsilon_V : {}^\vee V \otimes V \to \mathbb{C} : \langle m| \otimes |n\rangle \mapsto \delta_{m,n}$
- Coevaluation $\eta_V : \mathbb{C} \to V \otimes {}^\vee V : \alpha \mapsto \alpha \sum_n |n\rangle \otimes \langle n|$

- General tensor map ${}^\vee(V \otimes W) = {}^\vee W \otimes {}^\vee V$

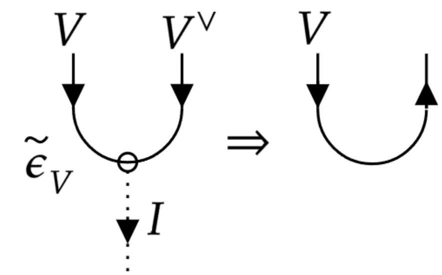
$$t : W_1 \otimes W_2 \otimes \ldots \otimes W_{N_2} \to V_1 \otimes V_2 \otimes \ldots \otimes V_{N_1}$$

$$\downarrow$$

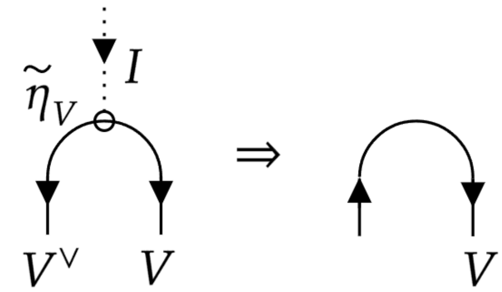
$$V_1 \otimes V_2 \otimes \ldots \otimes V_{N_1} \otimes W^*_{N_2} \otimes \ldots \otimes W^*_1.$$

Right one

- **Right evaluation** $\tilde{\epsilon}_V : V \otimes V^\vee \to I$



- **Right coevaluation** $\tilde{\eta}_V : I \to V^\vee \otimes V$

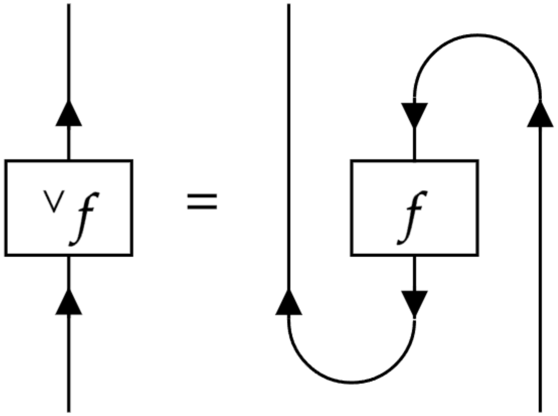


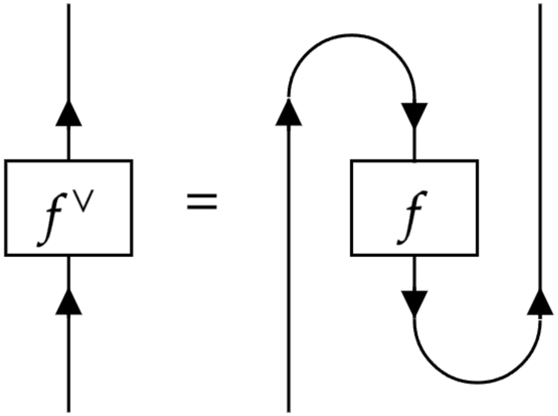
- **Rigid category**
  - **Pivotal category** $X^* = {}^\vee X = X^\vee$

    $f^* = {}^\vee f = f^\vee \in \mathrm{Hom}(V^*, W^*)$
  - **Pivotal structure** $\delta_X : X \to X^{**}$

Transpose

- $f \in \mathrm{Hom}(W, V)$
  - $^{\vee}f \in \mathrm{Hom}(^{\vee}V, {}^{\vee}W)$ $\qquad$ $f^{\vee} \in \mathrm{Hom}(V^{\vee}, W^{\vee})$

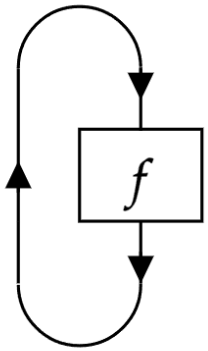




Trace

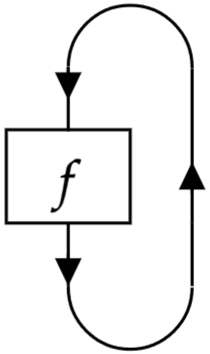
- **Left trace**

$$\mathrm{tr_l}(f) = \epsilon_V \circ (\mathrm{id}_{V^*} \otimes f) \circ \tilde{\eta}_V =$$



- **Right trace**

$$\mathrm{tr_r}(f) = \tilde{\epsilon}_V \circ (f \otimes \mathrm{id}_{V^*}) \circ \eta_V =$$



- **Spherical category** $\mathrm{tr}_l(f) = \mathrm{tr}_r(f) = \mathrm{tr}(f)$
- **(quantum) dimension** $\dim(V) = \mathrm{tr}(\mathrm{id}_V)$

Example **SVect**

- Left
  - evaluation $\epsilon_V : V^* \otimes V \to \mathbb{C} : \langle m| \otimes_{\mathrm{g}} |n\rangle \to \langle m|n\rangle = \delta_{m,n}$
  - coevaluation $\eta_V : \mathbb{C} \to V \otimes V^* : \alpha \to \alpha \sum_n |n\rangle \otimes_{\mathrm{g}} \langle n|$
- Right
  - evaluation $\tilde{\epsilon}_V : V \otimes V^* \to \mathbb{C} : |n\rangle \otimes_{\mathrm{g}} \langle m| \to (\pm 1)^{|n|} \delta_{m,n}$
  - Coevaluation $\tilde{\eta}_V : \mathbb{C} \to V^* \otimes V : \alpha \to \sum_n (\pm 1)^{|n|} \langle n| \otimes_{\mathrm{g}} |n\rangle$
- Trace $\mathrm{tr}(f) = \sum_n (\pm 1)^{|n|} \langle n|f|n\rangle$ + regular trace – supertrace
- spherical category $\dim(V) = \dim(V_0) \pm \dim(V_1)$

Duality

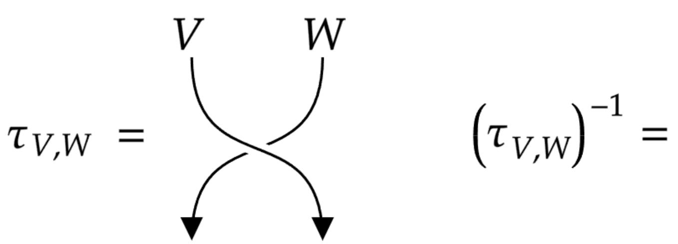
- Dual
- Left(Right) (co)evaluation
  - Snake rules
  - Transpose
- Rigid category, pivotal category, pivotal structure
- Trace, spherical category, quantum dimension
  - Supertrace

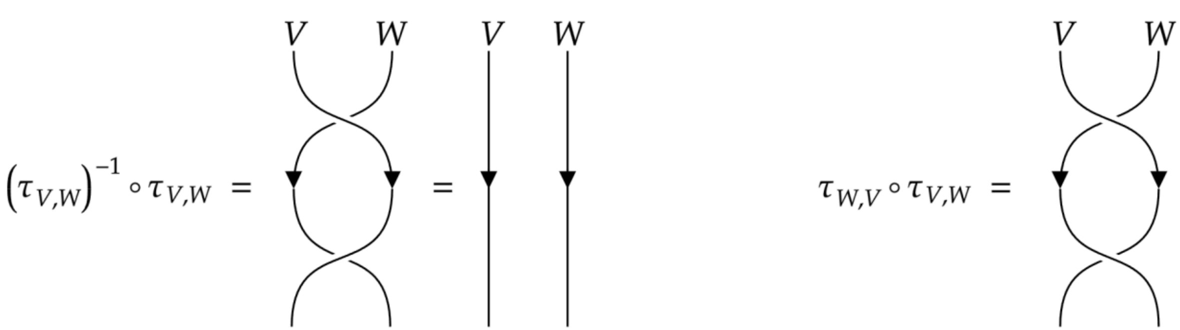
Braided category

- **Braiding** $\tau_{V,W} : V \otimes W \to W \otimes V_{V,W \in \mathrm{Ob}(C)}$
  - **hexagon equation**

$$\tau_{U,V\otimes W} = (\mathrm{id}_V \otimes \tau_{U,W})(\tau_{U,V} \otimes \mathrm{id}_W)$$

- **symmetric tensor category**





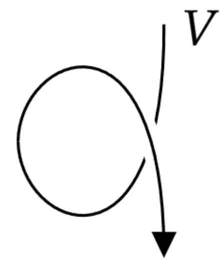
$$\tau_{W,V} \circ \tau_{V,W} = \mathrm{id}_{V\otimes W}$$

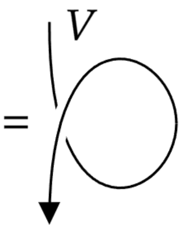
Balanced categories

- **Twist** θ a natural transformation from the identity functor 1_C to itself

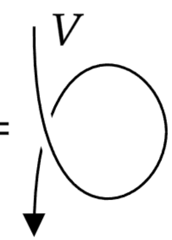
$$\theta_V \circ f = f \circ \theta_W$$

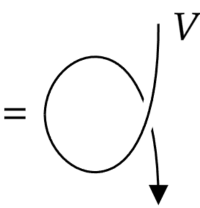
- Left

$$\theta^{\mathrm{l}}_V = (\epsilon_V \otimes \mathrm{id}_V)(\mathrm{id}_{V*} \otimes \tau_{V,V})(\tilde{\eta}_V \otimes \mathrm{id}_V) =$$ 

$$\left(\theta^{\mathrm{l}}_V\right)^{-1} =$$ 

- Right

$$\theta^{\mathrm{r}}_V = (\mathrm{id}_V \otimes \tilde{\epsilon}_V)(\tau_{V,V} \otimes \mathrm{id}_{V*})(\mathrm{id}_V \otimes \epsilon_V) =$$ 

$$\left(\theta^{r}_V\right)^{-1} =$$ 

- Ribbon Category $\theta^l = \theta^r$
- Compact closed Category: braiding is symmetric

Example

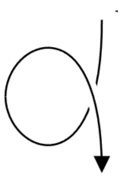
- **SVect** $|m\rangle \in V$ and $|n\rangle \in W$
  - Koszul sign rule

$$\tau_{V,W} : |m\rangle \otimes_{\mathrm{g}} |n\rangle \mapsto (-1)^{|m||n|} |n\rangle \otimes_{\mathrm{g}} |m\rangle$$

  - Symmetric $\tau_{W,V} \circ \tau_{V,W} = \mathrm{id}_{V \otimes W}$

- $\tilde{\epsilon}_V : V \otimes V^* \to \mathbb{C} : |n\rangle \otimes_{\mathrm{g}} \langle m| \mapsto (\pm 1)^{|n|} \delta_{m,n}$

$$\theta_V : |n\rangle \mapsto (\mp 1)^{|n|} |n\rangle$$



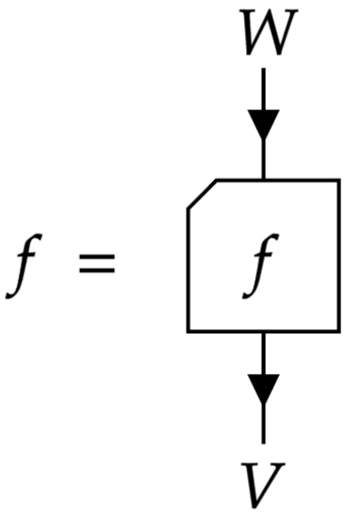
Braided category

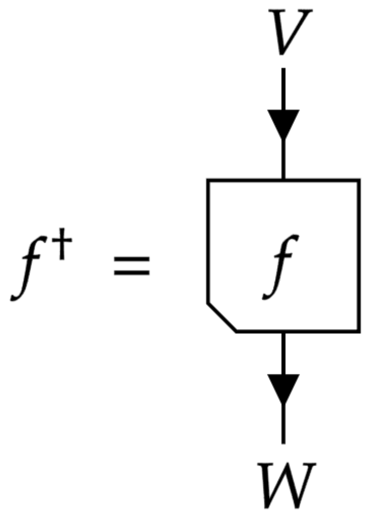
- Braiding, hexagon equation
- Symmetric tensor category
- Balanced categories, twist, tortile(ribbon) category, compact closed category

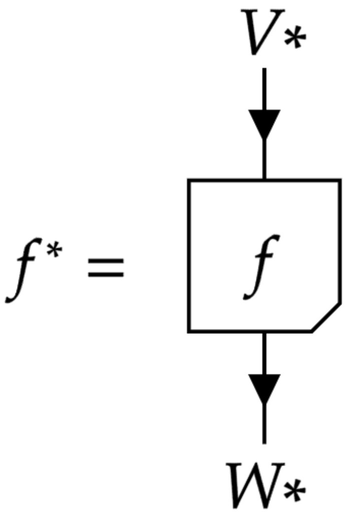
dagger categories

- Dagger $f : W \to V \qquad f^\dagger : V \to W$
  - $\mathrm{id}_V^\dagger = \mathrm{id}_V$
  - $(f \circ g)^\dagger = f^\dagger \circ^{\mathrm{op}} g^\dagger = g^\dagger \circ f^\dagger$
  - $(f^\dagger)^\dagger = f$
- Unitary $f^{-1} = f^\dagger$
- Hermitian $f^\dagger = f$
- Isometry $f^\dagger \circ f = \mathrm{id}_W$

Compare with transpose







Horizontal mirror

Rotate 180°

Dual

- Relationship between left and right

$$\epsilon_V : V^* \otimes V \to I \qquad \tilde{\epsilon}_V = (\eta_V)^\dagger$$

$$\eta_V : I \to V \otimes V^* \qquad \tilde{\eta}_V = (\epsilon_V)^\dagger$$

- **dagger compact category**
  - Same sign in **SVect,** regular trace

dagger categories

- Dagger, unitary, Hermitian, isometry
- Dagger compact category