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# AD\_excitation

Xingyu Zhang  
2022.11.4

# background

- Ground state

$$|\Psi(A)\rangle = \dots - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \dots$$

$$\langle \Psi(\bar{A}) | h | \Psi(A) \rangle = \begin{array}{c} \begin{array}{ccc} & \boxed{A} & \boxed{A} \\ & | & | \\ \circ & \boxed{h} & \circ \\ & | & | \\ & \boxed{\bar{A}} & \boxed{\bar{A}} \\ & | & | \\ \circ & & \circ \end{array} \end{array}$$

(Left circle is labeled  $l$ , right circle is labeled  $r$ )

- Quasiparticle ansatz(single-mode approximation)

$$|\Phi(B)_k\rangle = \sum_n e^{ikn} \dots - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{B} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \begin{array}{c} \boxed{A} \\ | \end{array} - \dots$$

$\dots \quad s_{n-1} \quad s_n \quad s_{n+1} \quad \dots$

# Steps summary and difficulties

graph summation



$$\frac{\partial}{\partial B^\dagger} [\langle \Phi(B)_k | \mathcal{H} | \Phi(B)_k \rangle - \omega_k (\langle \Phi(B)_k | \Phi(B)_k \rangle - 1)] = 0$$



Only depend on  $B$

$$H_{eff}(k)B = \omega N_{eff}B$$



Orthogonal Parameterization of  $\Phi(B)$

$$H_{beff}(k)X = \omega N_{beff}X$$



ordinary eigenvalue problem

$$H_{Nbeff}(k)X = \omega X$$

# Correct graph summation

- Correct series summation

$s_1$  is series summation of

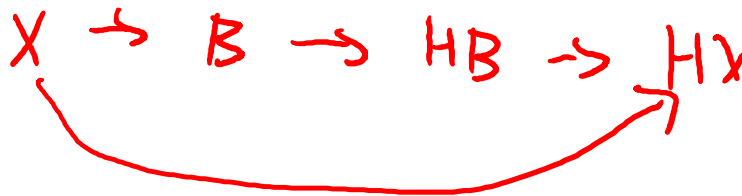
$$\begin{array}{|c|} \hline - \\ \hline \end{array} + \begin{array}{|c|} \hline I \\ \hline \end{array} + \begin{array}{|c|} \hline II \\ \hline \end{array} + \begin{array}{|c|} \hline III \\ \hline \end{array} + \dots + \begin{array}{|c|} \hline III \dots I \\ \hline \end{array} + \dots$$

$s_2$  is series summation of

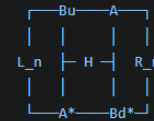
$$\begin{array}{|c|} \hline e^{i0k} \\ \hline \end{array} + e^{i1k} \begin{array}{|c|} \hline I \\ \hline \end{array} + e^{i2k} \begin{array}{|c|} \hline II \\ \hline \end{array} + e^{i3k} \begin{array}{|c|} \hline III \\ \hline \end{array} + \dots + e^{ink} \begin{array}{|c|} \hline III \dots I \\ \hline \end{array} + \dots$$

They are divergent

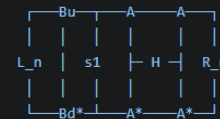
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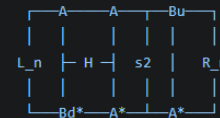
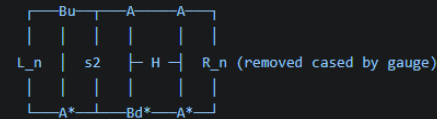
1. Bu, Bd\* and H on the same site:



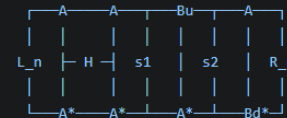
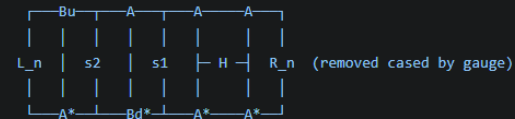
2. Bu and Bd\* are on the same site but away from the site of H



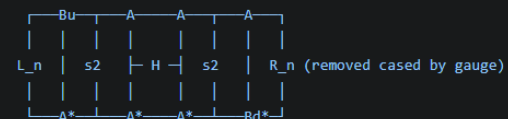
3. one of Bu and Bd\* on the same site of H



4. Bu and Bd\* are on the different sites and away from the site of H on the same side



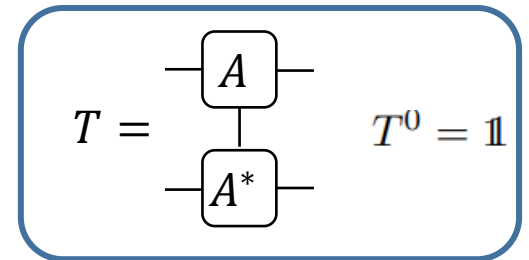
5. Bu and Bd\* are on the different sites and away from the site of H on the different side



# Geometric Sums of Transfer Matrices(1/3)

- Geometric Sums

$$\langle y| = \langle x| \sum_{n=0}^{\infty} T^n \quad |y\rangle = \sum_{n=0}^{\infty} T^n |x\rangle$$



- decomposition

$$T = \sum_{j=0}^{D^2-1} \lambda_j |j\rangle \langle j| \quad T^n = |0\rangle \langle 0| + \sum_{j=1}^{D^2-1} \lambda_j^n |j\rangle \langle j|$$

$$\langle j|k\rangle = \delta_{jk}$$

$$\lambda_0 = 1 \quad |\lambda_{j>0}| < 1$$

$$\begin{aligned} \sum_{n=0}^{\infty} T^n &= \sum_{n=0}^{\infty} |0\rangle \langle 0| + \sum_{j=1}^{D^2-1} \sum_{n=0}^{\infty} \lambda_j^n |j\rangle \langle j| \\ &= |\mathbb{N}| |0\rangle \langle 0| + \sum_{j=1}^{D^2-1} (1 - \lambda_j)^{-1} |j\rangle \langle j| \end{aligned}$$

divergent

# Geometric Sums of Transfer Matrices(2/3)

- projectors

$$P = |0\rangle\langle 0|$$

$$Q = \mathbb{1} - |0\rangle\langle 0|$$

$$\mathcal{T} = \sum_{j=1}^{D^2-1} \lambda_j |j\rangle\langle j| = QT = TQ = T - P.$$

$$\sum_{n=0}^{\infty} T^n = |\mathbb{N}| |0\rangle\langle 0| + Q(\mathbb{1} - \mathcal{T})^{-1}Q$$

$$\langle y| = |\mathbb{N}| \langle x|0\rangle \langle 0| + \langle x|Q(\mathbb{1} - \mathcal{T})^{-1}$$

$$|y\rangle = |\mathbb{N}| |0\rangle \langle 0|x\rangle + (\mathbb{1} - \mathcal{T})^{-1}Q|x\rangle.$$

The diverging contributions can typically be safely discarded, as they correspond to a constant (albeit infinite) offset of some extensive observable (e.g. the Hamiltonian).

# Geometric Sums of Transfer Matrices(3/3)

- Linear solve

$$\begin{aligned} (y | (\mathbb{1} - \mathcal{T}) &= (x | Q \\ (\mathbb{1} - \mathcal{T}) | y) &= Q | x \end{aligned}$$

$$\underline{\mathbb{I}} + \underline{\mathbb{I}} + \underline{\mathbb{I}} + \underline{\mathbb{I}} \dots \dots \underline{\mathbb{I}} \quad \times$$

$$\begin{aligned} & \begin{matrix} P & & T-P \\ \downarrow & & \downarrow \end{matrix} \\ (\underline{\mathbb{I}} - \lambda)(\underline{\mathbb{I}}) + (\underline{\mathbb{I}} - \lambda)(\underline{\mathbb{I}}) + (\underline{\mathbb{I}} - \lambda)(\underline{\mathbb{I}})^2 \end{aligned}$$

$$\begin{aligned} \underline{\mathbb{I}} &= \underline{\mathbb{I}} \\ (\underline{\mathbb{I}} - \lambda) &= (\underline{\mathbb{I}} - \lambda) \end{aligned}$$

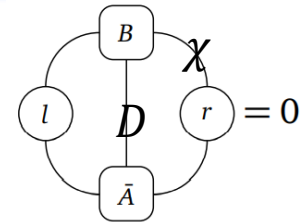
$$\begin{aligned} (\underline{\mathbb{I}} - \lambda)(\underline{\mathbb{I}} - \lambda) &= \underline{\mathbb{I}} - \lambda \underline{\mathbb{I}} - \lambda \underline{\mathbb{I}} + \lambda^2 \underline{\mathbb{I}} \\ (\underline{\mathbb{I}} - \lambda)^n &= \underline{\mathbb{I}}^n - \lambda \underline{\mathbb{I}} \end{aligned}$$

$$\left( \begin{array}{c|c|c} \text{Bu} & & \text{A} \text{ A} \\ \hline \text{I} & \text{S}_1 & \text{O} \\ \hline \text{Bd} & & \text{A}^+ \text{ A}^+ \end{array} \right) = \left( \begin{array}{c|c|c} \text{Bu} & & \text{I} \\ \hline \text{I} & \text{I} & \text{O} \\ \hline \text{Bd} & & \text{A} \text{ A} \end{array} \right) - \left( \begin{array}{c|c} \text{Bu} & \\ \hline \text{I} & \text{O} \\ \hline \text{Bd} & \text{A} \text{ A} \end{array} \right) \quad e$$



# Orthogonal Parameterization of $\Phi(B)$ (1/2)

- $\Phi(B)$  is Orthogonal to  $\psi(A)$
- Gauge Invariant of Tangent vector



$$|\Phi(B; A)\rangle = B^i \frac{\partial}{\partial A_i} |\Psi(A)\rangle = \sum_n \dots - \underset{\dots}{\boxed{A}} - \underset{s_{n-1}}{\boxed{A}} - \underset{s_n}{\boxed{B}} - \underset{s_{n+1}}{\boxed{A}} - \underset{\dots}{\boxed{A}} - \dots$$

$$-\underset{\quad}{\boxed{B}}- \rightarrow -\underset{\quad}{\boxed{B}}- + \underset{\quad}{\boxed{\chi}}- \underset{\quad}{\boxed{A}}- - \underset{\quad}{\boxed{A}}- \underset{\quad}{\boxed{\chi}}-$$

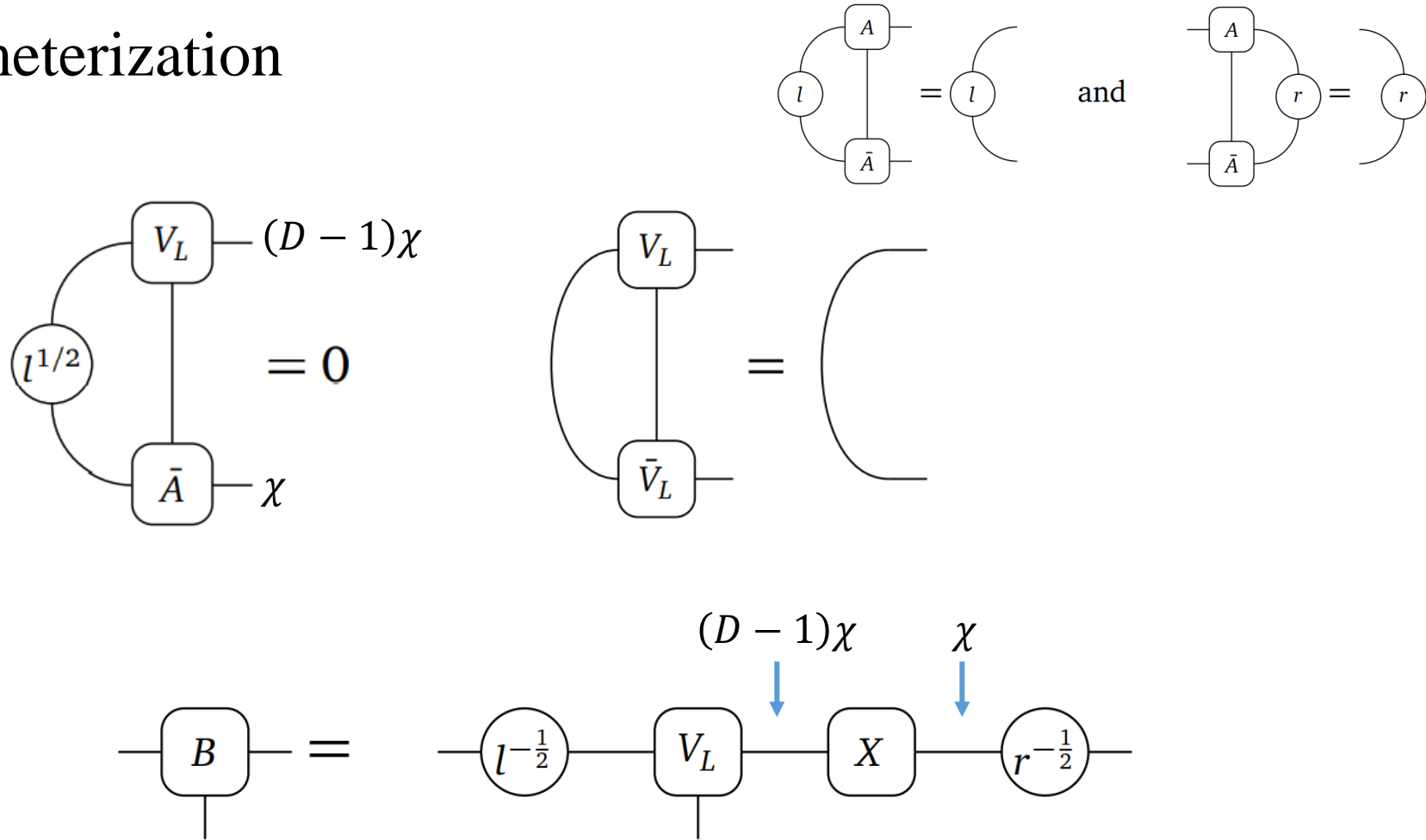
Parameters  $\underline{D\chi^2} \rightarrow \underline{D\chi^2 - 1} \rightarrow \underline{D\chi^2 - \chi^2}$

- Left gauge-fixing condition

$$\begin{array}{c} \boxed{B} \\ \curvearrowright \\ \text{\textit{l}} \\ \curvearrowleft \\ \boxed{\bar{A}} \end{array} = \begin{array}{c} \boxed{A} \\ \curvearrowright \\ \text{\textit{l}} \\ \curvearrowleft \\ \boxed{\bar{B}} \end{array} = 0$$

# Orthogonal Parameterization of $\Phi(B)$ (2/2)

- parameterization

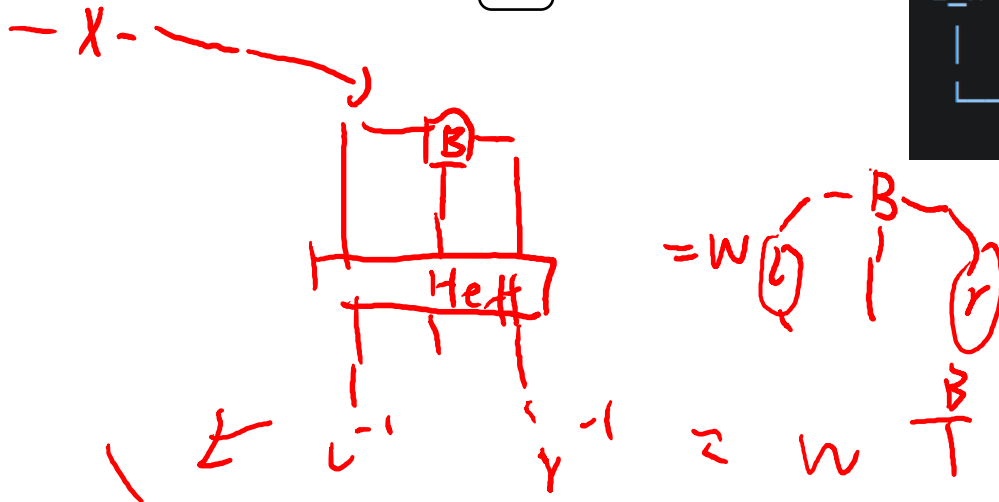
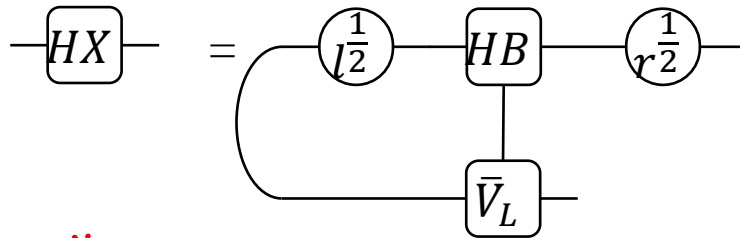
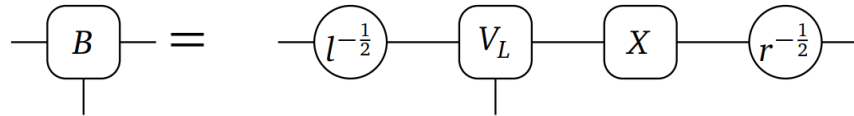


# test

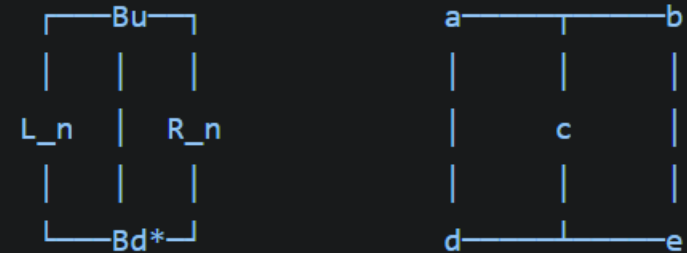
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# ordinary eigenvalue problem

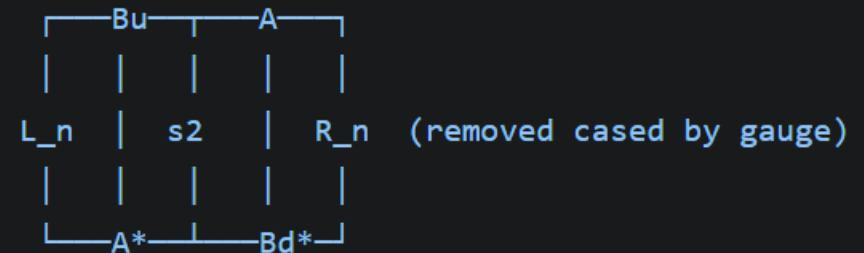
- $N_{eff} \rightarrow N_{beff}$



1. Bu, Bd\* on the same site



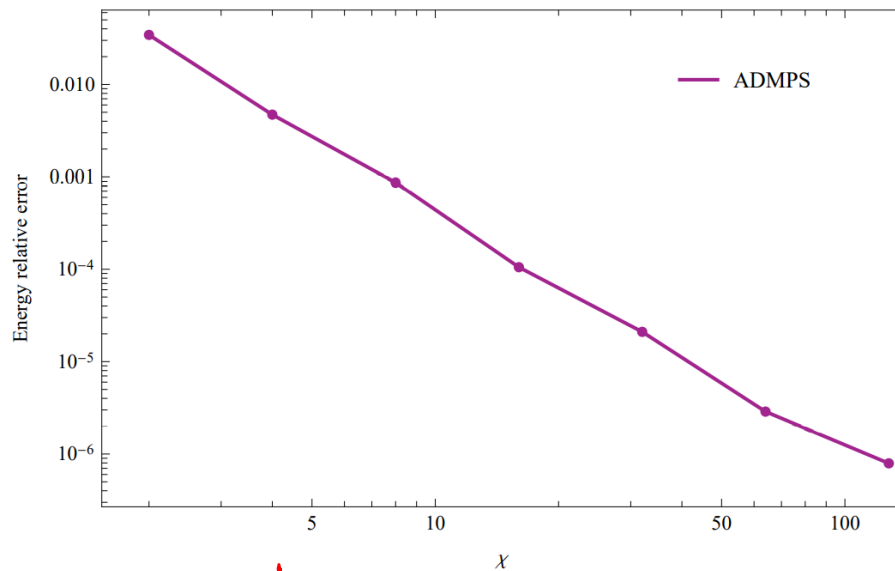
2. Bu, Bd\* on the different sites



# Ground state by AD

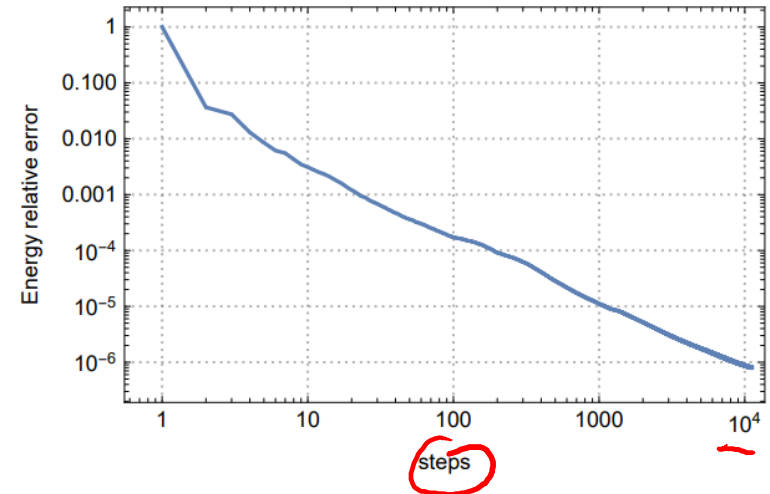
- Heisenberg  $S = 1/2$

error exponentially-dependent on  $\chi$

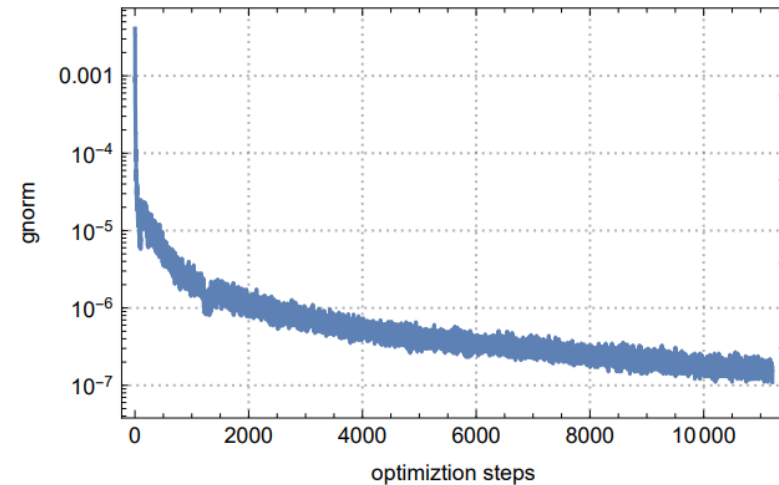


Handwritten red notes: a vertical line, a horizontal line, and a curve that levels off, possibly indicating a plateau or convergence.

E-steps  $\chi=128$

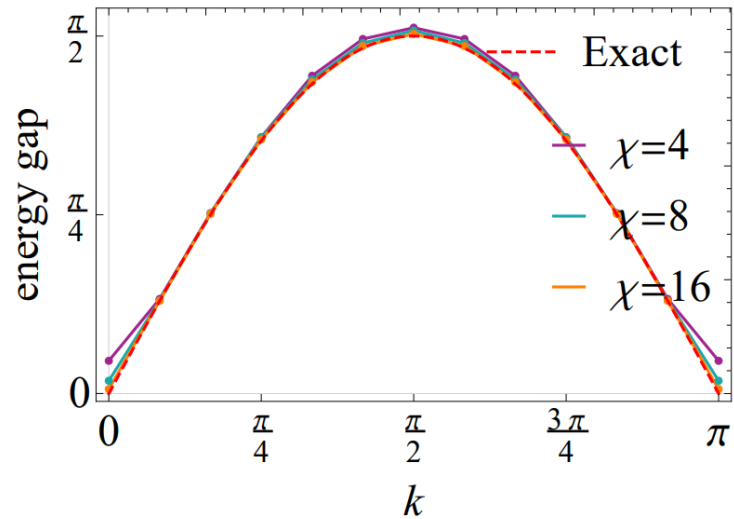


gnorm-steps  $\chi=128$

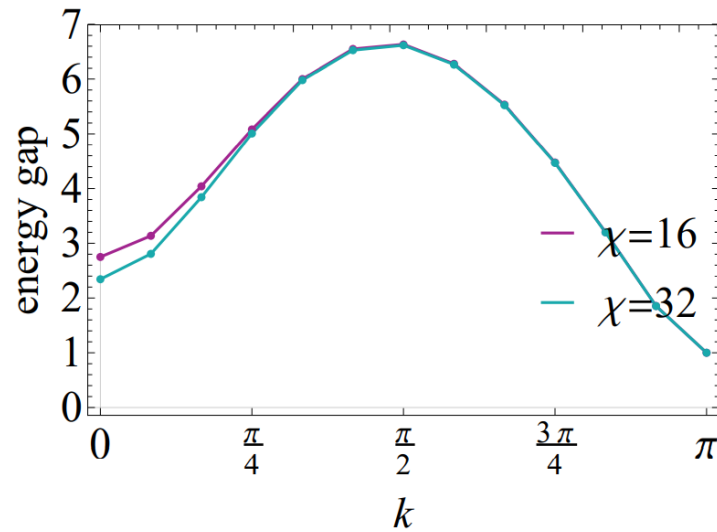


# gap

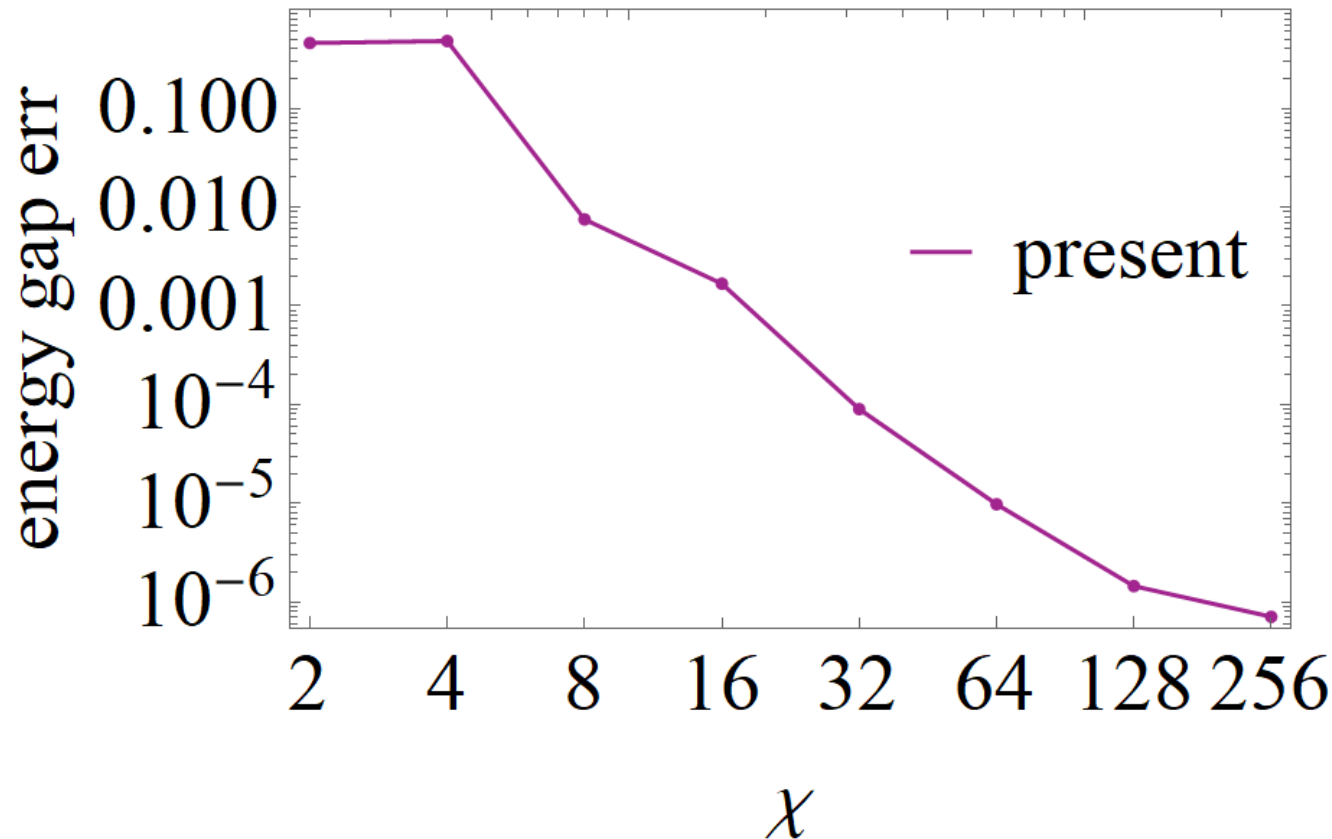
- Heisenberg  $S = 1/2$



- Heisenberg  $S = 1$



# $k = \pi$ Haldane gap error



$10^{-8}$

$(I = C)$

208

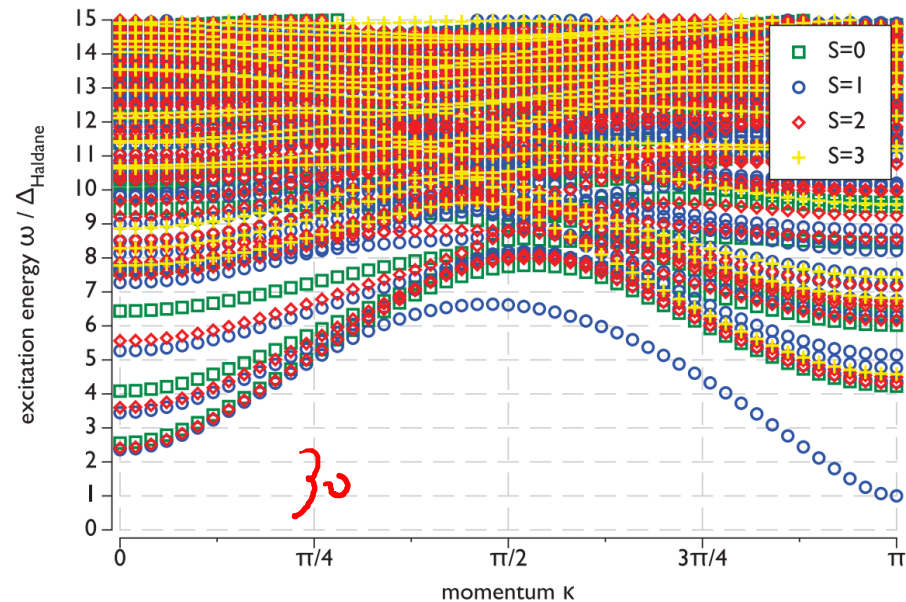
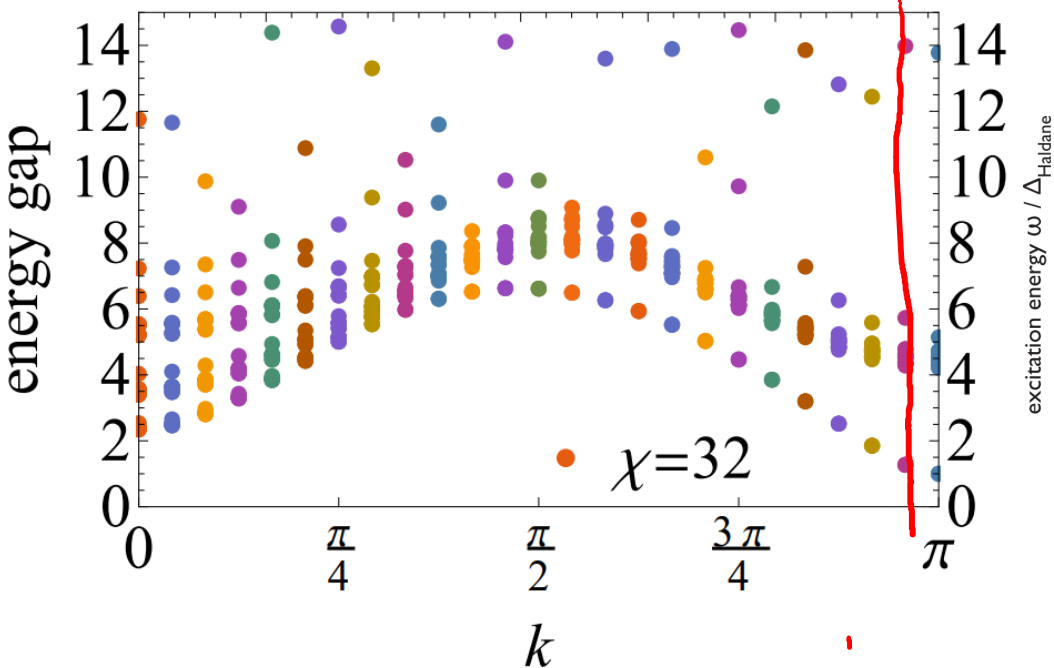


For  $\chi = 256$

AD for ground state  $\sim 8h$  10000 steps

Eigsolve for excitation state  $\sim 500s$

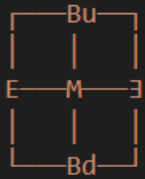
# Heisenberg $S = 1$ excitation spectrum



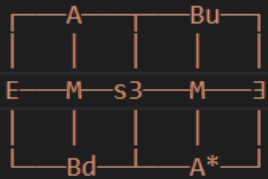
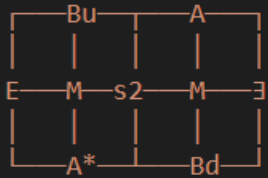


# MPO graph summation

1. Bu and Bd on the same site of M



2. B and dB on different sites of M



s2

$$e^{i_0 k} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + e^{i_1 k} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + e^{i_2 k} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + e^{i_3 k} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + \dots + e^{i_n k} \begin{array}{|c|} \hline \text{---} \\ \hline \end{array} + \dots$$

# MPO

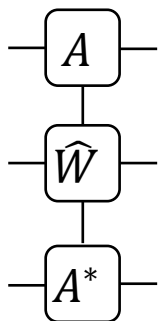
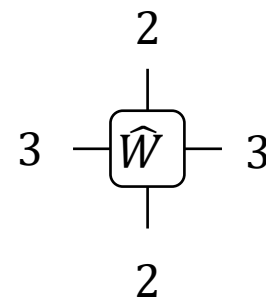
- TFI sing

$$H_{\text{TFI}} = -J \sum_j \sum_{n>0} \lambda^{n-1} X_j X_{j+n} - h \sum_j Z_j$$

$$\hat{W} = \begin{bmatrix} \mathbb{1} & 0 & 0 \\ -JX & \lambda \mathbb{1} & 0 \\ -hZ & X & \mathbb{1} \end{bmatrix} \quad \lambda < 1$$

$$\hat{w}_L = [-hZ \quad X \quad \mathbb{1}]$$

$$\hat{w}_R = [\mathbb{1} \quad -JX \quad -hZ]^T$$



The  $\Xi$  MPO transfer matrix contains Jordan blocks and that the dominant eigenvalue is one and of **twofold** algebraic degeneracy.

$$\text{Overlap}(E, \Xi) = 0 \quad E \Xi \Xi = 0$$

MPO transfer matrices technically do not have well defined fixed points.  $\rightarrow$  **quasi** fixed points

# Fixed point equations

- Left and right environment  $\mathbb{E}\mathbb{X}$

$$\begin{aligned} \mathbb{C}_a &= \mathbb{C}_a \mathbb{I}^{aa} + \mathbb{E}_a & \mathbb{E}_a &= \sum_{b>a} \mathbb{C}_b \mathbb{I}^{ba} \\ \mathbb{D}_a &= \mathbb{I}^{aa} \mathbb{D}_a + \mathbb{D}_a & \mathbb{D}_a &= \sum_{b<a} \mathbb{I}^{ab} \mathbb{D}_b \end{aligned}$$

- $\mathbb{I}^{aa} = 0$

$$\begin{aligned} \mathbb{C}_a &= \mathbb{E}_a \\ \mathbb{D}_a &= \mathbb{D}_a \end{aligned}$$

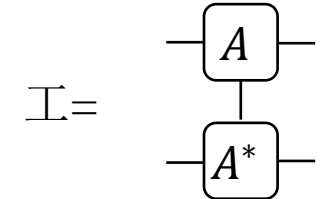
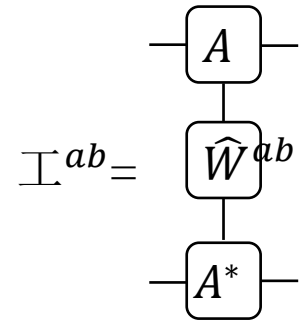
- $\mathbb{I}^{aa} = \lambda_a \mathbb{I}$

$$\begin{aligned} \mathbb{C}_a &= \lambda_a \mathbb{C}_a \mathbb{I} + \mathbb{E}_a & \mathbb{C}_a (1 - \lambda_a \mathbb{I}) &= \mathbb{E}_a \\ \mathbb{D}_a &= \lambda_a \mathbb{I} \mathbb{D}_a + \mathbb{D}_a & (1 - \lambda_a \mathbb{I}) \mathbb{D}_a &= \mathbb{D}_a \end{aligned}$$

- $\mathbb{I}^{aa} = \mathbb{I}$

$$\begin{aligned} \mathbb{C}_a (1 - \mathbb{I}) &= \mathbb{E}_a \\ (1 - \mathbb{I}) \mathbb{D}_a &= \mathbb{D}_a \end{aligned}$$

$$\begin{aligned} \mathbb{C}_a (1 - \mathbb{I} - \mathbb{D}) &= \mathbb{E}_a - \mathbb{E}_a \mathbb{D} \\ (1 - \mathbb{I} - \mathbb{D}) \mathbb{D}_a &= \mathbb{D}_a - \mathbb{D} \mathbb{D}_a \\ \text{energy} &= \mathbb{C}_1 \mathbb{D} = \mathbb{E}_1 \mathbb{D} \end{aligned}$$



$$(y| = (x| \sum_{n=0}^{\infty} T^n \quad |y) = \sum_{n=0}^{\infty} T^n |x)$$

$$\begin{aligned} (y| &= |\mathbb{N}| (x|0) (0| + (x|Q(\mathbb{1} - \mathcal{T})^{-1} \\ |y) &= |\mathbb{N}| |0) (0|x) + (\mathbb{1} - \mathcal{T})^{-1} Q|x). \end{aligned}$$

$$\frac{(\overline{I})}{\mid e - e'}$$

$$(\overline{II})$$

$e$

$$\frac{(\overline{I} \neq \overline{I})}{\overline{I} \neq \overline{I}}$$

$w =$

$$k \quad (\overline{I})$$

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# 2D excitation spectrum on Helix

Xingyu Zhang  
2023.1.5

# Review and contents

$$|\Phi(B)_k\rangle = \sum_n e^{ikn} \dots \text{---} \underset{\dots}{\boxed{A}} \text{---} \underset{s_{n-1}}{\boxed{A}} \text{---} \underset{s_n}{\boxed{B}} \text{---} \underset{s_{n+1}}{\boxed{A}} \text{---} \underset{\dots}{\boxed{A}} \text{---}$$

single-mode approximation  $\longrightarrow$  1D excitation spectrum

Graph summation  $\xrightarrow{\text{Orthogonal Parameterization}}$  Eigen equation

Two site Hamiltonian  
 MPO (**much simpler!** quasi fixed points)  $\longrightarrow$  2D excitation spectrum

1D to 2D

cylinders  $\longrightarrow$  **Large** unit cell;  $k_y$  is **not** a good quantum number  
 helix  $\longrightarrow$  **Single** unit cell;  $k_y$  is a good quantum number

# Direct 2D single-mode approximation

## Spin excitation spectra of the spin-1/2 triangular Heisenberg antiferromagnets from tensor networks

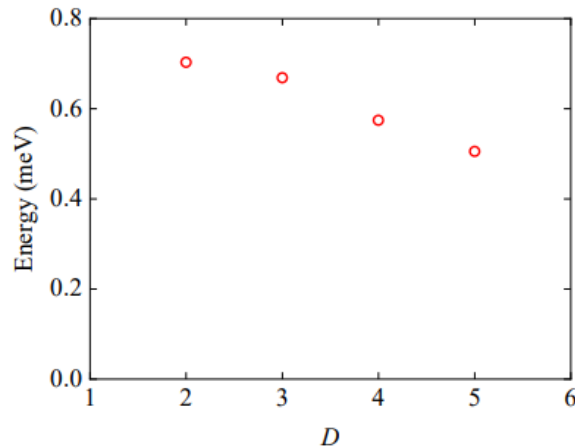
Run-Ze Chi,<sup>1,2,\*</sup> Yang Liu,<sup>1,2,\*</sup> Yuan Wan,<sup>1,3</sup> Hai-Jun Liao,<sup>1,3,†</sup> and T. Xiang<sup>1,2,4,‡</sup>

<sup>1</sup>*Beijing National Laboratory for Condensed Matter Physics and Institute of Physics,  
Chinese Academy of Sciences, Beijing 100190, China.*

<sup>2</sup>*School of Physical Sciences, University of Chinese Academy of Sciences, Beijing 100049, China.*

<sup>3</sup>*Songshan Lake Materials Laboratory, Dongguan, Guangdong 523808, China.*

<sup>4</sup>*Beijing Academy of Quantum Information Sciences, Beijing, China.*



$$|\Phi_{\kappa_x \kappa_y}(B)\rangle = \sum_{m,n} e^{i(\kappa_x m + \kappa_y n)}$$

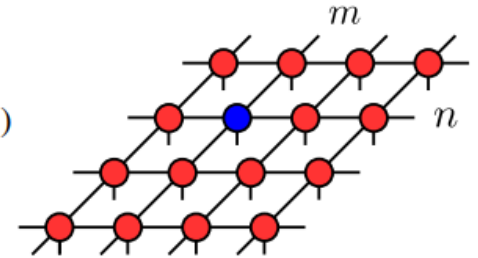
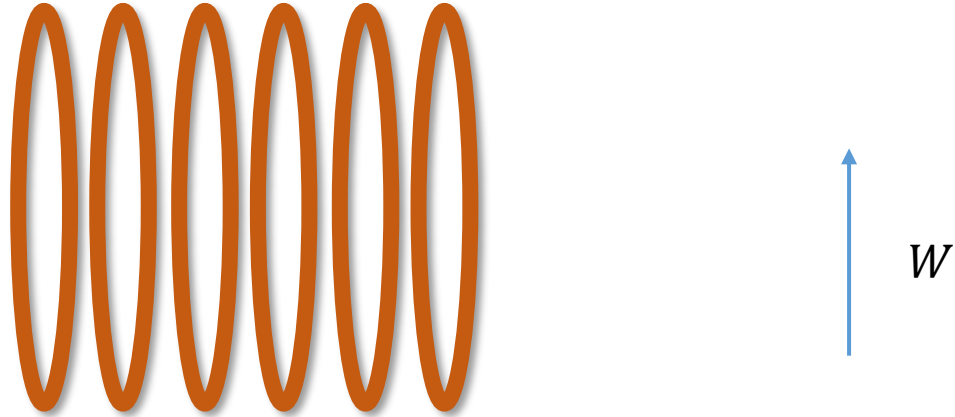


FIG. S2. **Minimal spectral gap as a function of bond dimension  $D$ .** The lowest spectral gap of the XXZ model occurs at the K point (see Fig. S1) in the Brillouin zone. The gap values are obtained by contracting the effective Hamiltonian tensor network states of the excited states using the corner transfer matrix renormalization group method with a bond dimension  $\chi = 50$  for  $D = 2, 3, 4$  and  $\chi = 60$  for  $D = 5$ .

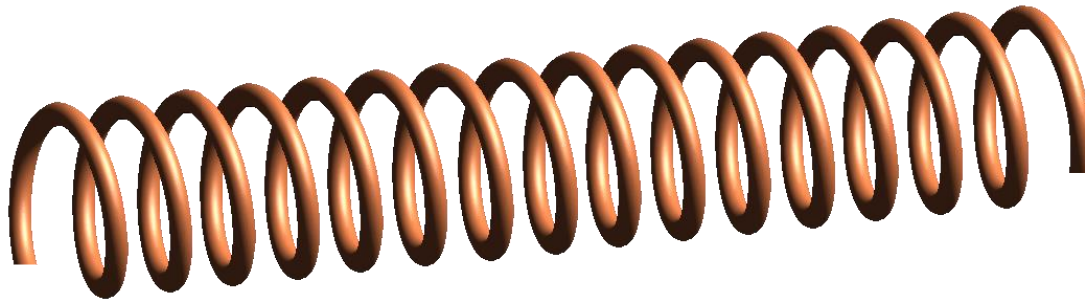
- High computational cost
- No innovation point

# 1D $\rightarrow$ 2D map

- Cylinder



- Helix

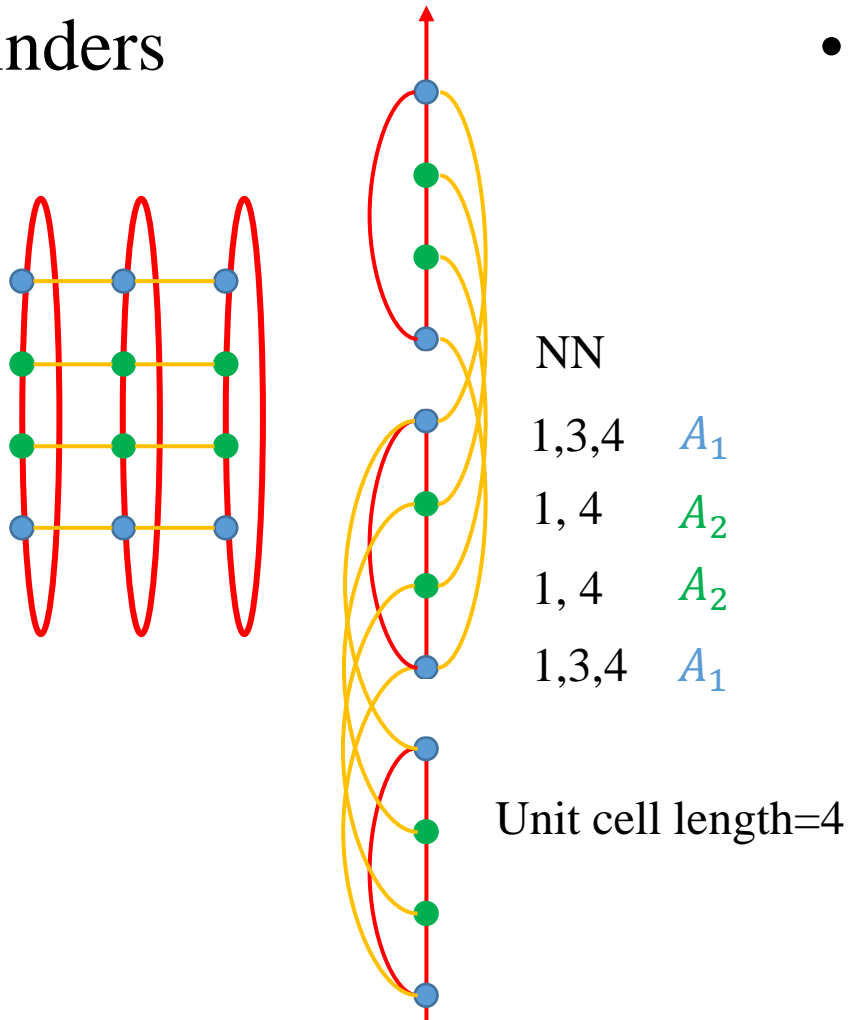




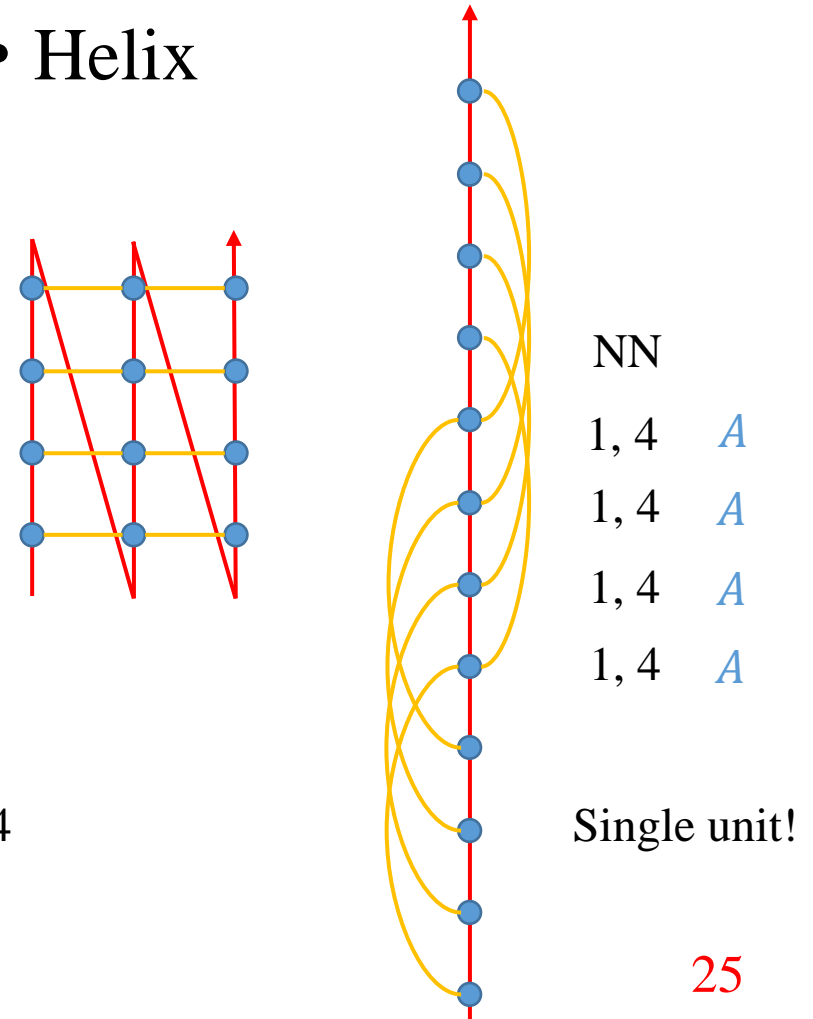
# 1D→2D map

$W = 4$  for example

## • Cylinders



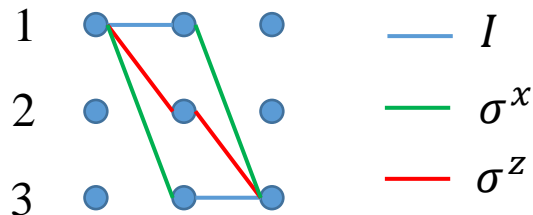
## • Helix



# MPO from matrix product (MP) diagram

- 1D TFIsing

$$\begin{aligned}
 H &= \sum_i -\sigma_i^z \sigma_{i+1}^z - \lambda \sigma_i^x \\
 &= -\sigma_1^z \sigma_2^z I_3 I_4 \dots - \sigma_1^x I_2 I_3 I_4 \\
 &\quad - I_1 \sigma_2^z \sigma_3^z I_4 \dots - I_1 \sigma_2^x I_3 I_4 \\
 &\quad - I_1 I_2 \sigma_3^z \sigma_4^z \dots - I_1 I_2 \sigma_3^x I_4
 \end{aligned}$$

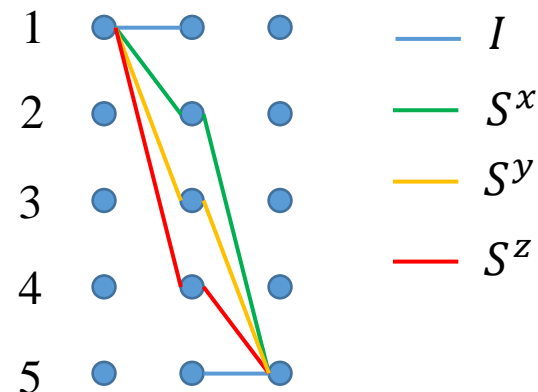


$$M = \begin{bmatrix} I & -\sigma^z & -\sigma^x \\ 0 & 0 & \sigma^z \\ 0 & 0 & I \end{bmatrix}$$

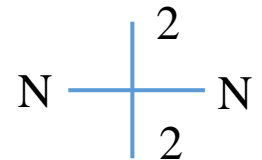
$N = 3$

- 1D Heisenberg

$$H = \sum_i S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z$$



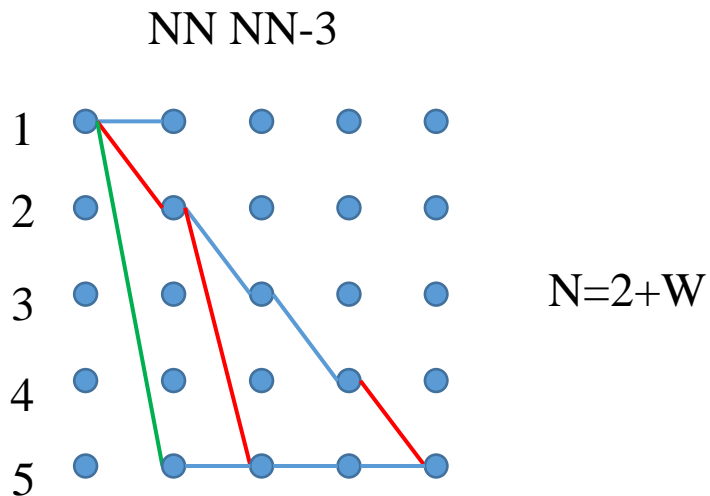
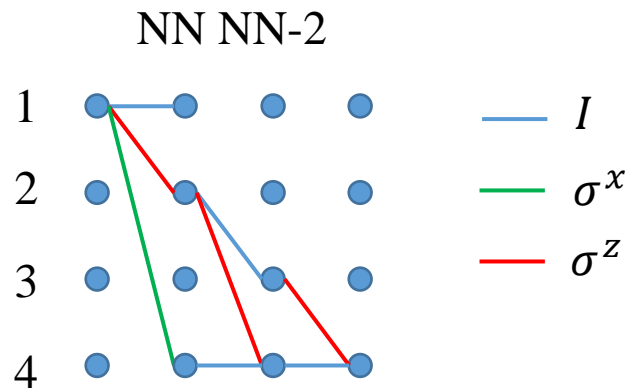
$$M = \begin{bmatrix} I & S^x & S^y & S^z & 0 \\ 0 & 0 & 0 & 0 & S^x \\ 0 & 0 & 0 & 0 & S^y \\ 0 & 0 & 0 & 0 & S^z \\ 0 & 0 & 0 & 0 & I \end{bmatrix}$$



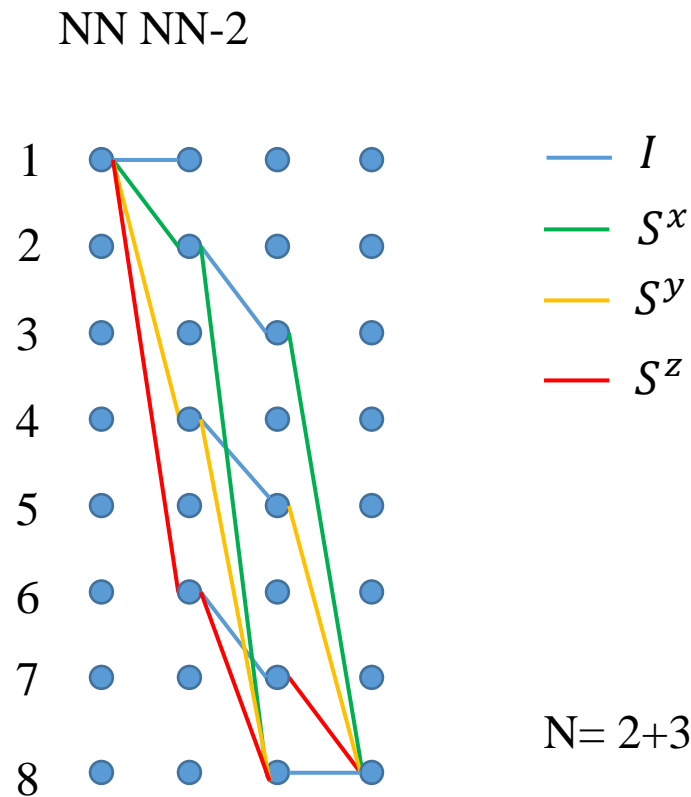
$N = 5$

# MPO from matrix product (MP) diagram

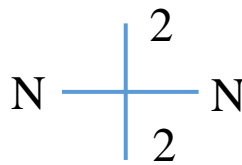
- TFIsing



- Heisenberg



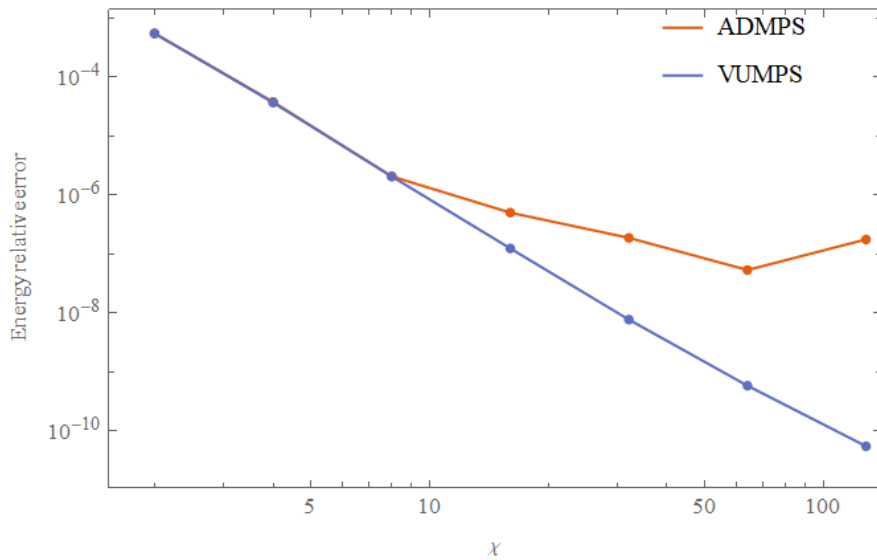
$$N=2+3W$$



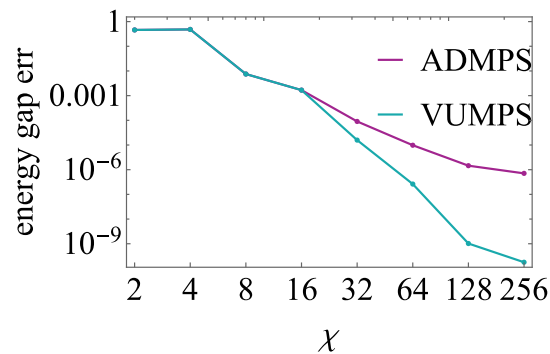
# Ground state vumps

- 1D TFI sing NN

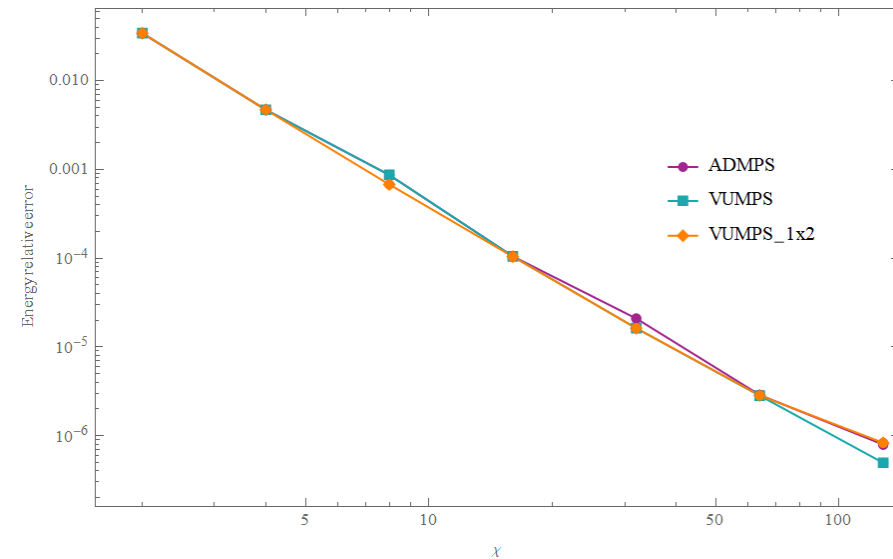
$$\lambda_c = 1$$



$S = 1$  Haldane gap



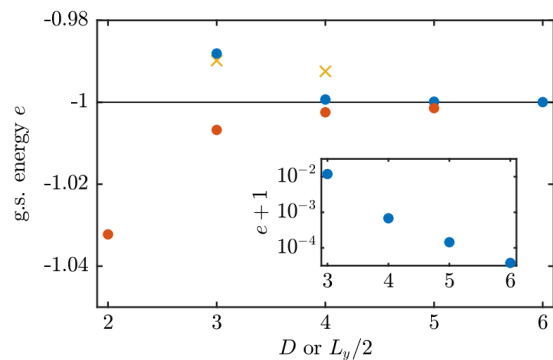
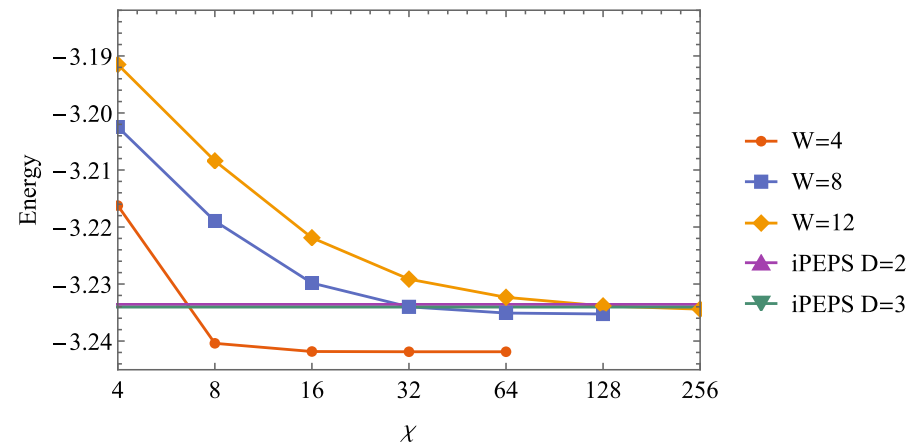
- 1D Heisenberg NN



# Ground state vumps

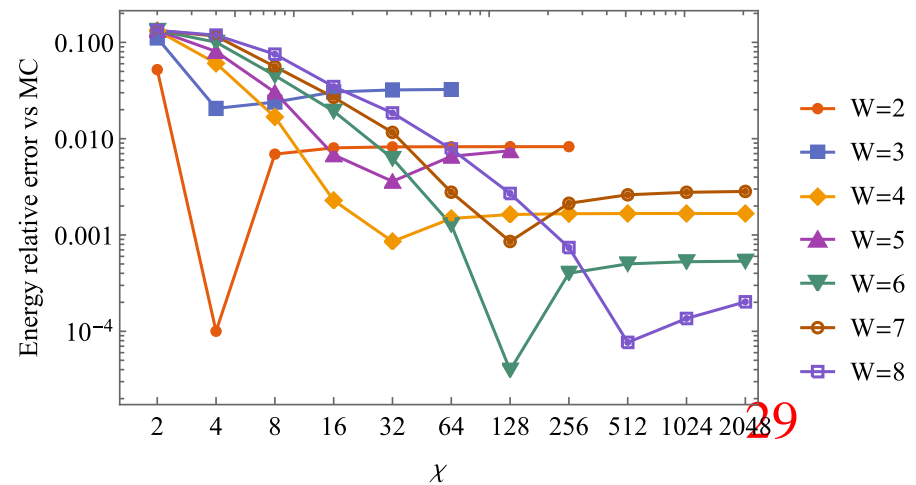
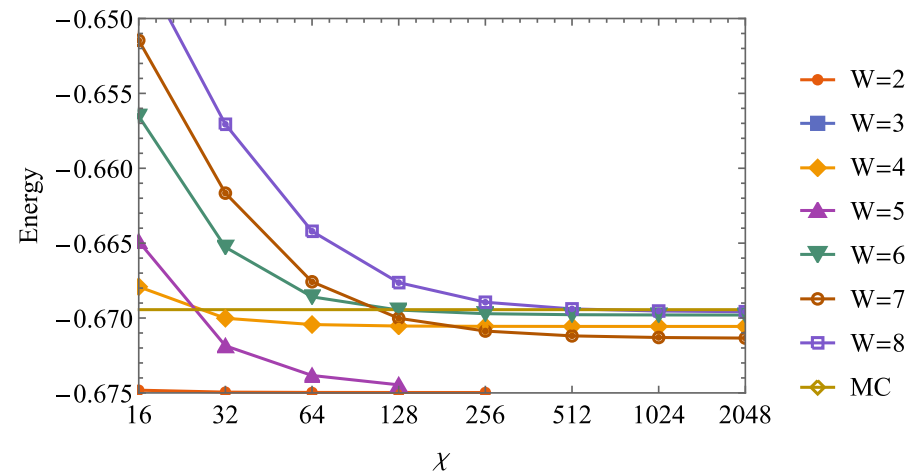
- 2D TFI sing

$$\lambda_c = 3.04438$$



- Different approach with  $D$  v.s.  $W$
- Nonmonotonicity with  $W$  and  $\chi$

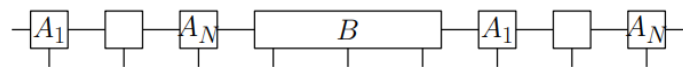
- 2D Heisenberg NN



# excitation spectrum

## • Cylinder

$$|\Phi_{p_x}(B)\rangle = \sum_n e^{ip_x n} T_x^n$$



with

$$\begin{aligned} \text{---} \boxed{B} \text{---} &= \text{---} \boxed{B_1} \text{---} \cdots \text{---} \boxed{A_N} \text{---} + \cdots \\ &\quad \cdots + \text{---} \boxed{A_1} \text{---} \cdots \text{---} \boxed{B_N} \text{---} . \end{aligned}$$

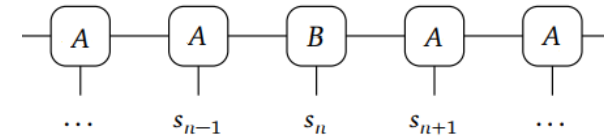
No  $k_y$  because  $k_y$  is **not** a good quantum number

Recover by directly calculating

$$\frac{1}{2\pi\delta(p_x - p'_x)} \frac{\langle \Phi_{p'_x}(B) | T_y | \Phi_{p_x}(B) \rangle}{\langle \Psi(A) | T_y | \Psi(A) \rangle}$$

## • Helix

$$|\Phi_{k_x, k_y}(B)\rangle = \sum_n e^{ik_x \lfloor n/W \rfloor + ik_y (n \bmod W)} T_x T_y^n$$



$k_y$  is a good quantum number

# translation operator

$$|\Phi(B)_k\rangle = \sum_n e^{ikn} \dots \text{---} \underset{\dots}{\boxed{A}} \text{---} \underset{s_{n-1}}{\boxed{A}} \text{---} \underset{s_n}{\boxed{B}} \text{---} \underset{s_{n+1}}{\boxed{A}} \text{---} \underset{\dots}{\boxed{A}} \text{---}$$

$$\begin{aligned}
 |\Phi(B)_k\rangle &= e^{ik \cdot 0} \begin{array}{c} B \\ \bullet \\ | \quad | \quad | \quad | \quad | \end{array} \\
 &+ e^{ik \cdot 1} \begin{array}{c} | \quad \bullet \quad | \quad | \quad | \end{array} \\
 &+ e^{ik \cdot 2} \begin{array}{c} | \quad | \quad \bullet \quad | \quad | \end{array} \\
 &+ e^{ik \cdot 3} \begin{array}{c} | \quad | \quad | \quad \bullet \quad | \end{array}
 \end{aligned}
 \quad \xrightarrow{\quad} \quad
 \begin{aligned}
 T|\Phi(B)_k\rangle &= e^{ik \cdot 0} \begin{array}{c} B \\ \bullet \\ | \quad | \quad | \quad | \quad | \end{array} \\
 &+ e^{ik \cdot 1} \begin{array}{c} | \quad | \quad \bullet \quad | \quad | \end{array} \\
 &+ e^{ik \cdot 2} \begin{array}{c} | \quad | \quad | \quad \bullet \quad | \end{array} \\
 &+ e^{ik \cdot 3} \begin{array}{c} | \quad | \quad | \quad | \quad \bullet \end{array}
 \end{aligned}$$

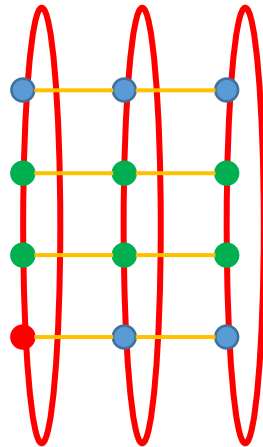
$$e^{-ik}|\Phi(B)_k\rangle = T|\Phi(B)_k\rangle$$

# translation operator $T_y$

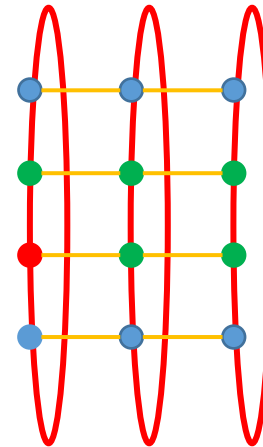
$W = 4$  for example

- Cylinder

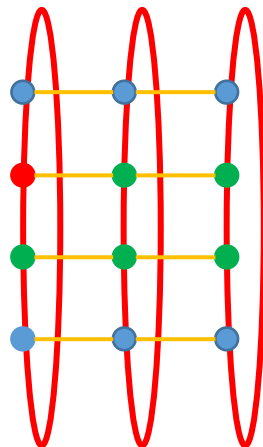
$$T_y \left| \Phi(B)_{k_x, k_y} \right\rangle = e^{ik_x \cdot 0 + ik_y \cdot 0}$$



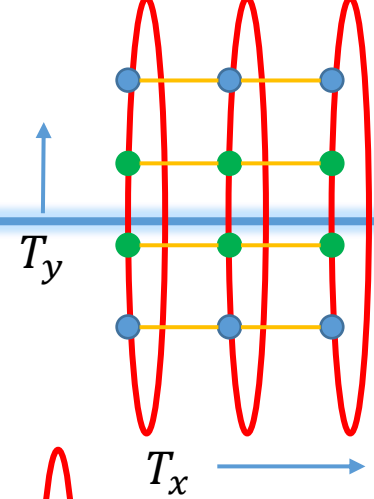
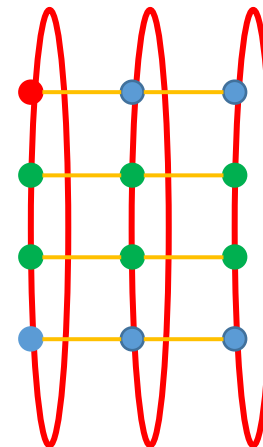
$$+ e^{ik_x \cdot 0 + ik_y \cdot 1}$$



$$+ e^{ik_x \cdot 0 + ik_y \cdot 2}$$



$$+ e^{ik_x \cdot 0 + ik_y \cdot 3}$$



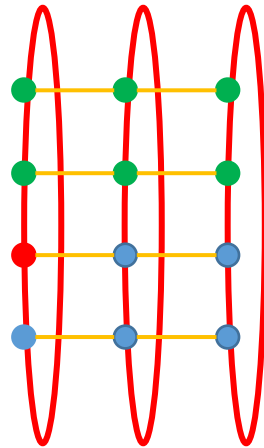


# translation operator $T_y$

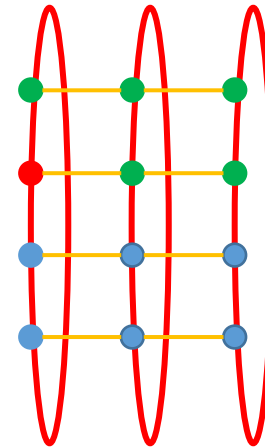
$W = 4$  for example

## • Cylinder

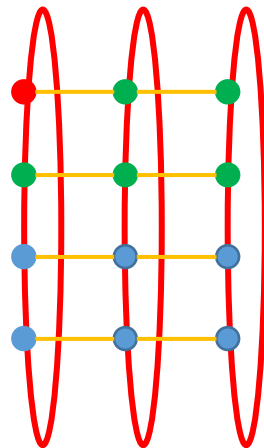
$$T_y \left| \Phi(B)_{k_x, k_y} \right\rangle = e^{ik_x \cdot 0 + ik_y \cdot 0}$$



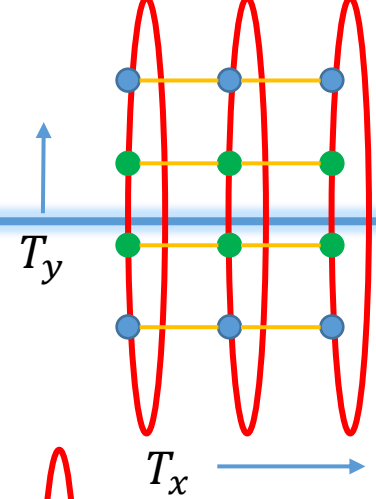
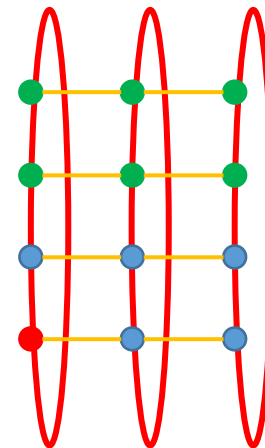
$$+ e^{ik_x \cdot 0 + ik_y \cdot 1}$$



$$+ e^{ik_x \cdot 0 + ik_y \cdot 2}$$



$$+ e^{ik_x \cdot 0 + ik_y \cdot 3}$$



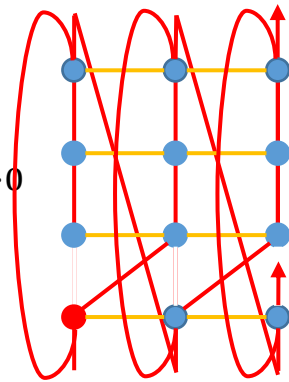
$k_y$  is **not** a good quantum number!

# translation operator $T_y$

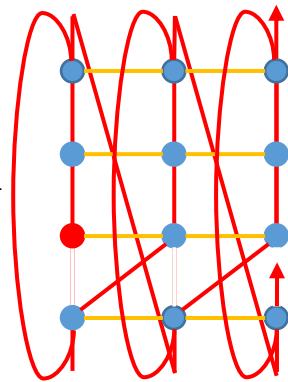
$W = 4$  for example

- Helix

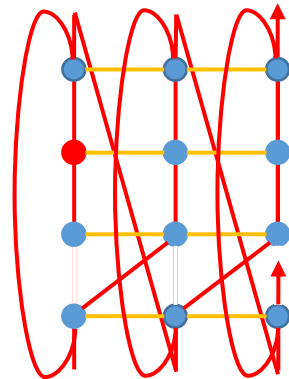
$$T_y \left| \Phi(B)_{k_x, k_y} \right\rangle = e^{ik \cdot 0 + ik_y \cdot 0}$$



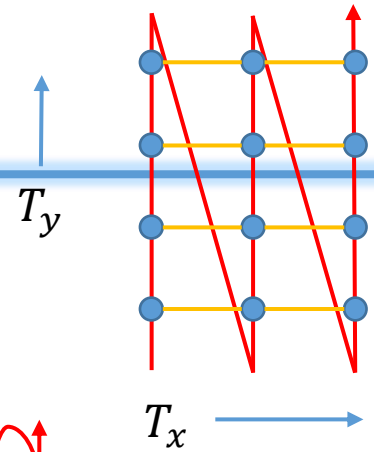
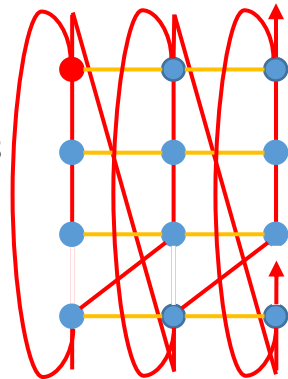
$$+ e^{ik \cdot 0 + ik_y \cdot 1}$$



$$+ e^{ik \cdot 0 + ik_y \cdot 2}$$



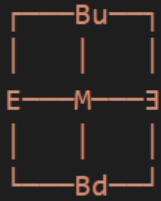
$$+ e^{ik \cdot 0 + ik_y \cdot 3}$$



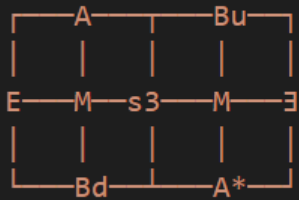
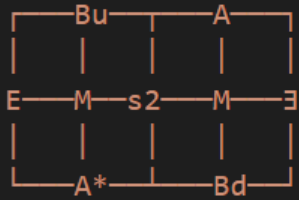
$k_y$  is a good quantum number!

# Graph summation

1. Bu and Bd on the same site of M



2. B and dB on different sites of M

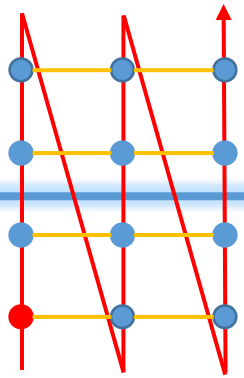


$s_2 = \text{sum of } e^{ik} \text{ series:}$

$$e^{i0k} \begin{array}{|c|} \hline \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \hline \end{array} + e^{i1k} \begin{array}{|c|} \hline \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \hline \end{array} + e^{i2k} \begin{array}{|c|} \hline \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \hline \end{array} + e^{i3k} \begin{array}{|c|} \hline \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \hline \end{array} + \dots + e^{ink} \begin{array}{|c|} \hline \text{---} \\ | \\ \text{---} \\ | \\ \text{---} \\ \hline \end{array} + \dots$$

# Graph summation

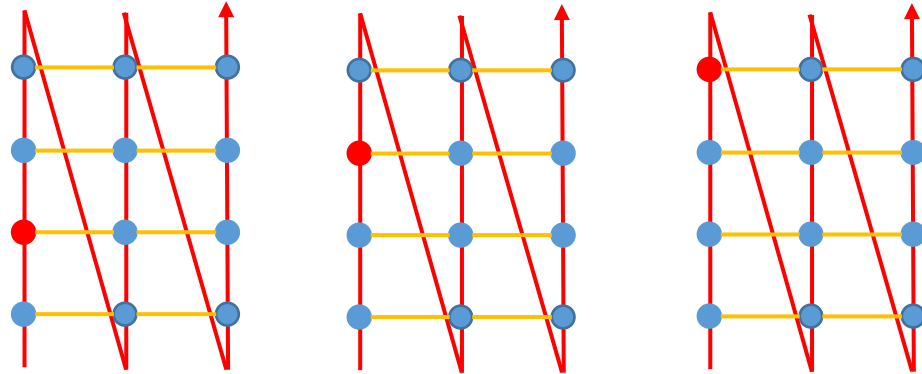
$$| \rangle = e^{ik_x \cdot 0 + ik_y \cdot 0}$$



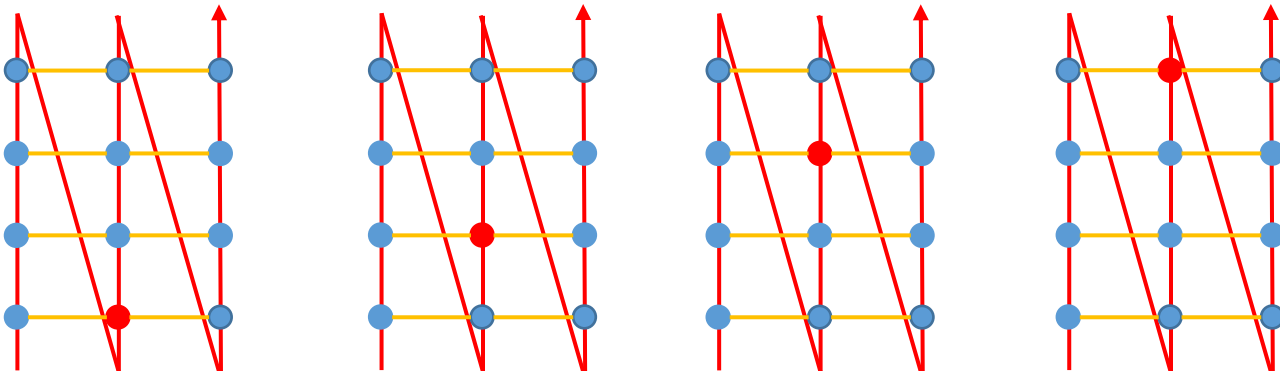
$W = 4$  for example

$$S^{+1} = e^{ik_x \cdot 0 + ik_y \cdot 1} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 2} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 3} \Xi^2$$

$\langle | =$



$$+ e^{ik_x \cdot 1 + ik_y \cdot 0} \Xi^3 + e^{ik_x \cdot 1 + ik_y \cdot 1} \Xi^4 + e^{ik_x \cdot 1 + ik_y \cdot 2} \Xi^5 + e^{ik_x \cdot 1 + ik_y \cdot 3} \Xi^6$$



# Graph summation

$$\begin{aligned}
 S^{+1} &= \left( e^{ik_x \cdot 0 + ik_y \cdot 1} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 2} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 3} \Xi^2 + e^{ik_x \cdot 1 + ik_y \cdot 0} \Xi^3 \right) \cdot \\
 &\quad \left( \Xi + e^{ik_x \cdot 1} \Xi^4 + e^{ik_x \cdot 2} \Xi^8 + \dots \right) \\
 &\equiv \left( e^{ik_x \cdot 0 + ik_y \cdot 1} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 2} \Xi + e^{ik_x \cdot 0 + ik_y \cdot 3} \Xi^2 + e^{ik_x \cdot 1 + ik_y \cdot 0} \Xi^3 \right) S_4^+
 \end{aligned}$$

$$\begin{aligned}
 S^+ &= S^{+1} + S^{+2} + S^{+3} + S^{+4} \\
 &= \left( (3e^{ik_x \cdot 0 + ik_y \cdot 1} + e^{ik_x \cdot 1 - ik_y \cdot 3}) \Xi + (2e^{ik_x \cdot 0 + ik_y \cdot 2} + 2e^{ik_x \cdot 1 - ik_y \cdot 2}) \Xi \right. \\
 &\quad \left. + (e^{ik_x \cdot 0 + ik_y \cdot 3} + 3e^{ik_x \cdot 1 - ik_y \cdot 1}) \Xi^2 + 4e^{ik_x \cdot 1 + ik_y \cdot 0} \Xi^3 \right) S_4^+
 \end{aligned}$$

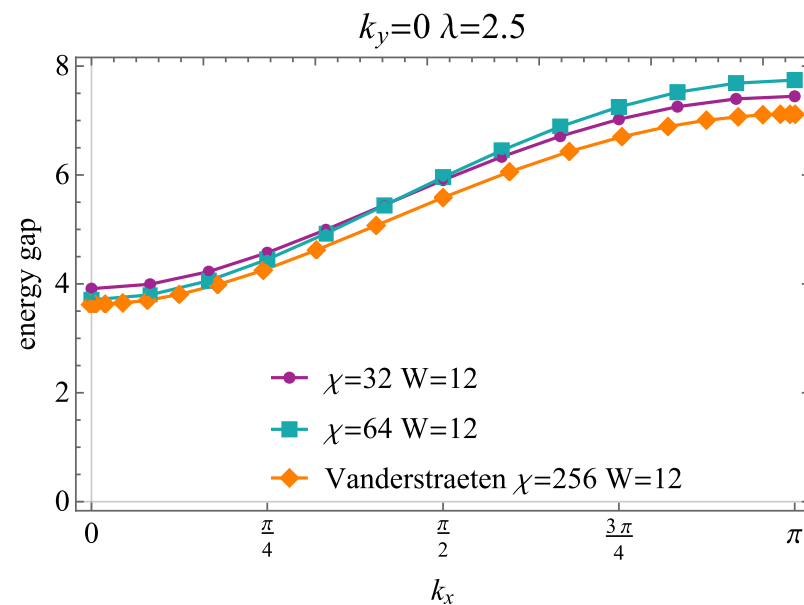
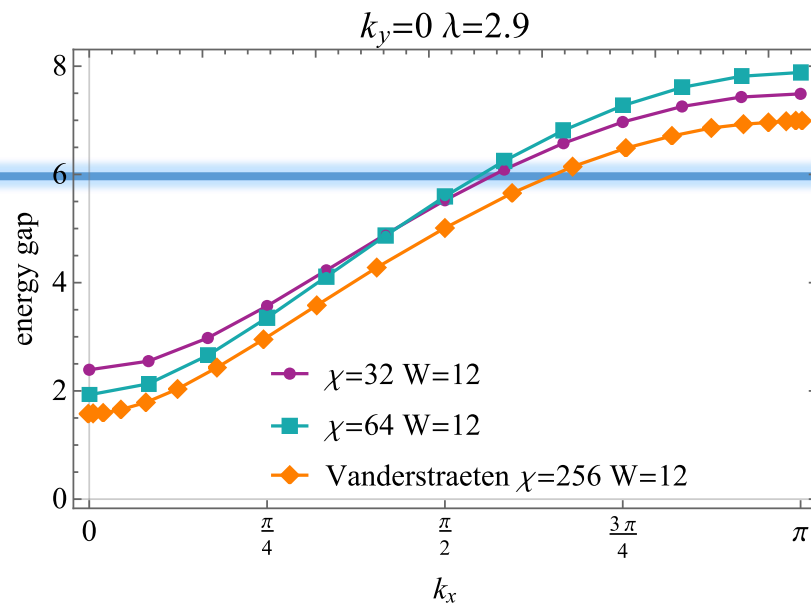
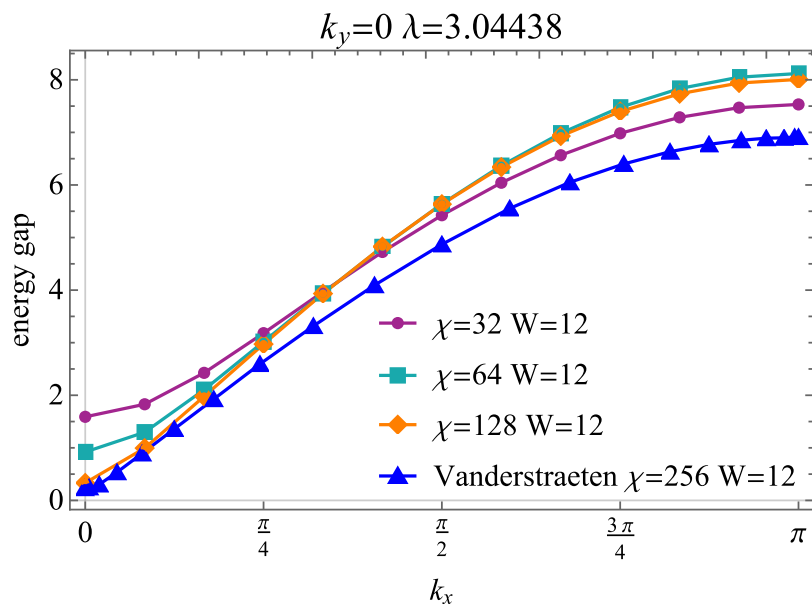
Same way:

$$\begin{aligned}
 S^- &= S^{-1} + S^{-2} + S^{-3} + S^{-4} \\
 &= \left( (3e^{ik_x \cdot 0 - ik_y \cdot 1} + e^{-ik_x \cdot 1 + ik_y \cdot 3}) \Xi + (2e^{ik_x \cdot 0 - ik_y \cdot 2} + 2e^{-ik_x \cdot 1 + ik_y \cdot 2}) \Xi \right. \\
 &\quad \left. + (e^{ik_x \cdot 0 - ik_y \cdot 3} + 3e^{-ik_x \cdot 1 + ik_y \cdot 1}) \Xi^2 + 4e^{-ik_x \cdot 1 + ik_y \cdot 0} \Xi^3 \right) S_4^-
 \end{aligned}$$

$$S_4^- = \left( \Xi + e^{-ik_x \cdot 1} \Xi^4 + e^{-ik_x \cdot 2} \Xi^8 + \dots \right)$$

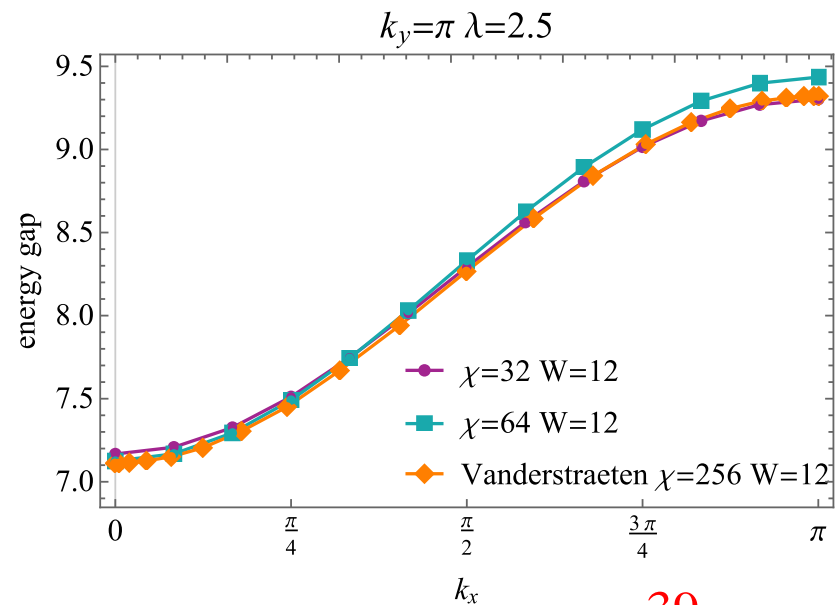
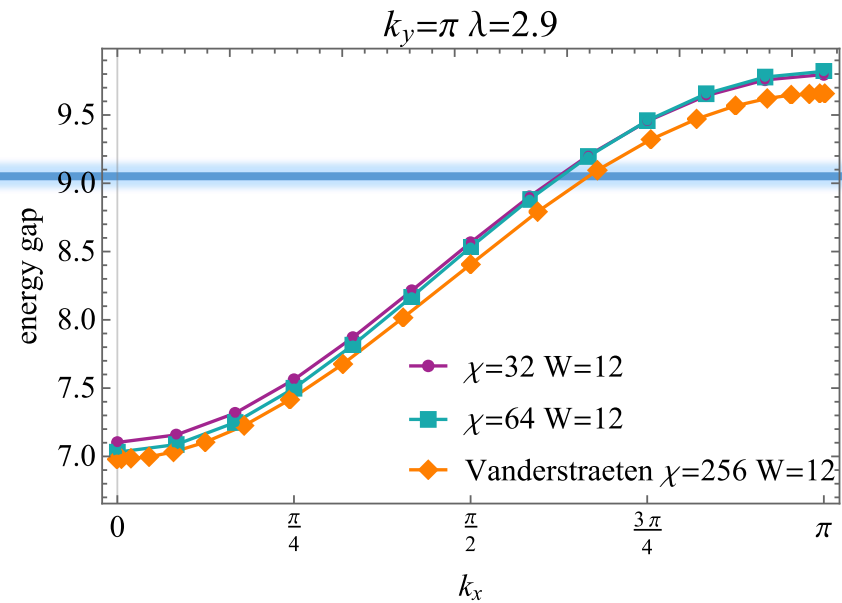
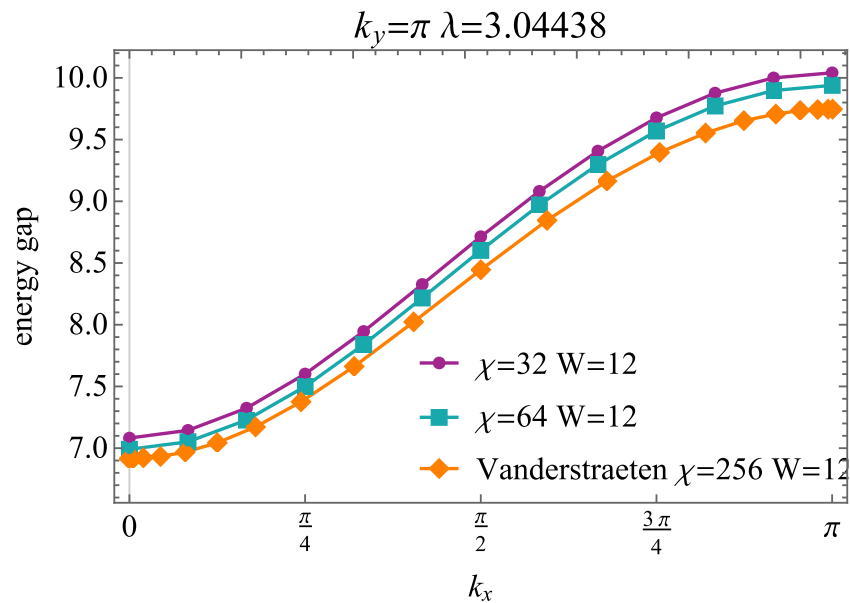
# result

- 2D TFIsing  $k_y = 0$

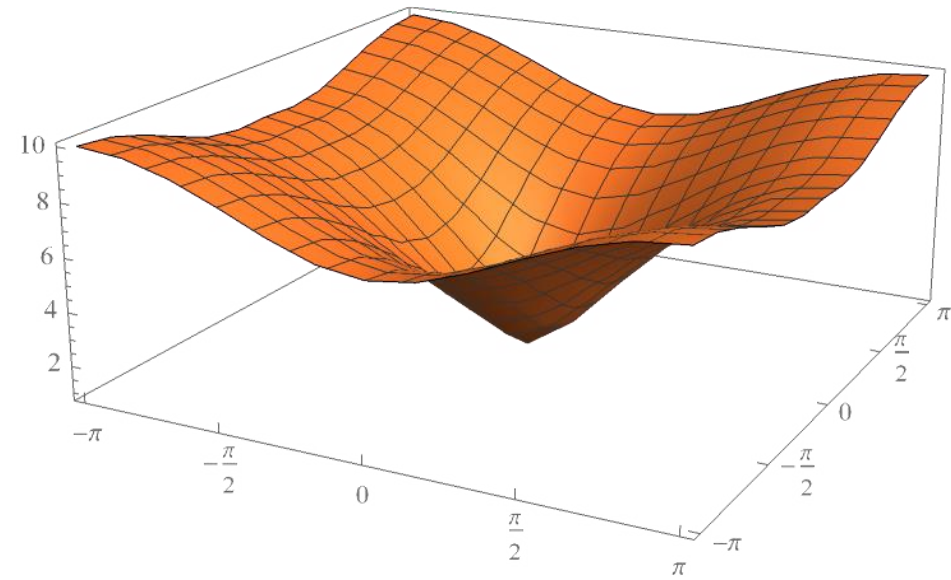


# result

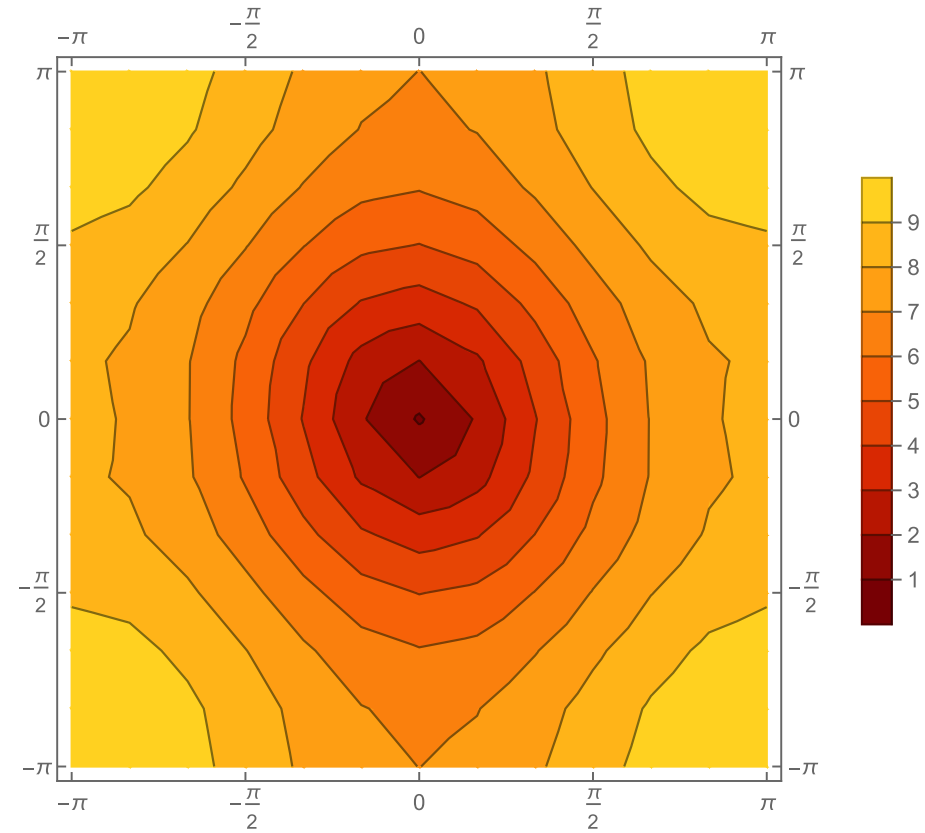
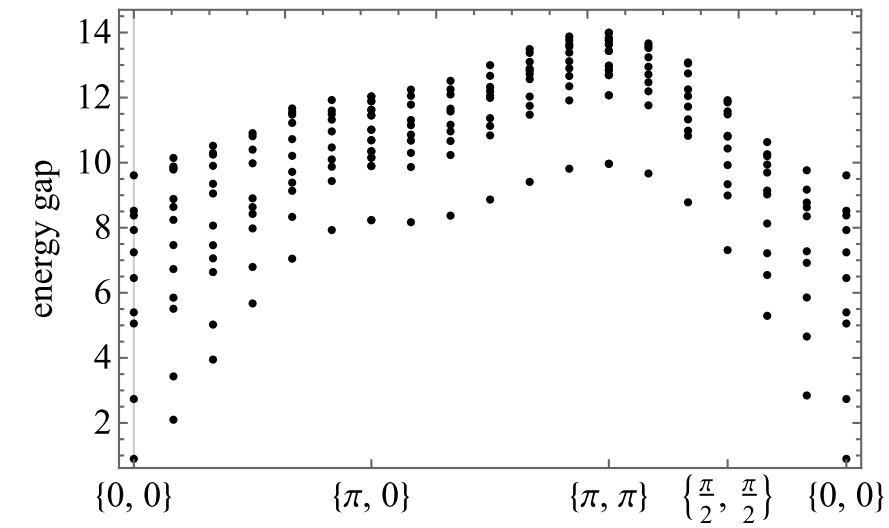
- 2D TFIsing  $k_y = \pi$



# result



$\lambda = 3.04438$





# outlook

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- 1D anti-Heisenberg, Large unit cell
- 2D anti-Heisenberg
- Kitaev

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# Thank you for listening!

Q&A?