Assignment-1

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1 Question 1

[ ]:
```
from scipy.stats import beta
import numpy as np
import matplotlib.pyplot as plt
```

1.1 Case 1: (a,b) = (1,1)

[ ]:
```
a0, b0 = 1, 1
N = [0, 1, 2, 3, 8, 15, 50, 500]
X = [0, 1, 2, 2, 4, 6, 24, 263]
a = a0*np.ones(8) + X
b = b0*np.ones(8) + N - X

print(a,'\n',b)
```

```
[  1.   2.   3.   3.   5.   7.  25. 264.]
 [  1.   1.   1.   2.   5.  10.  27. 238.]
```

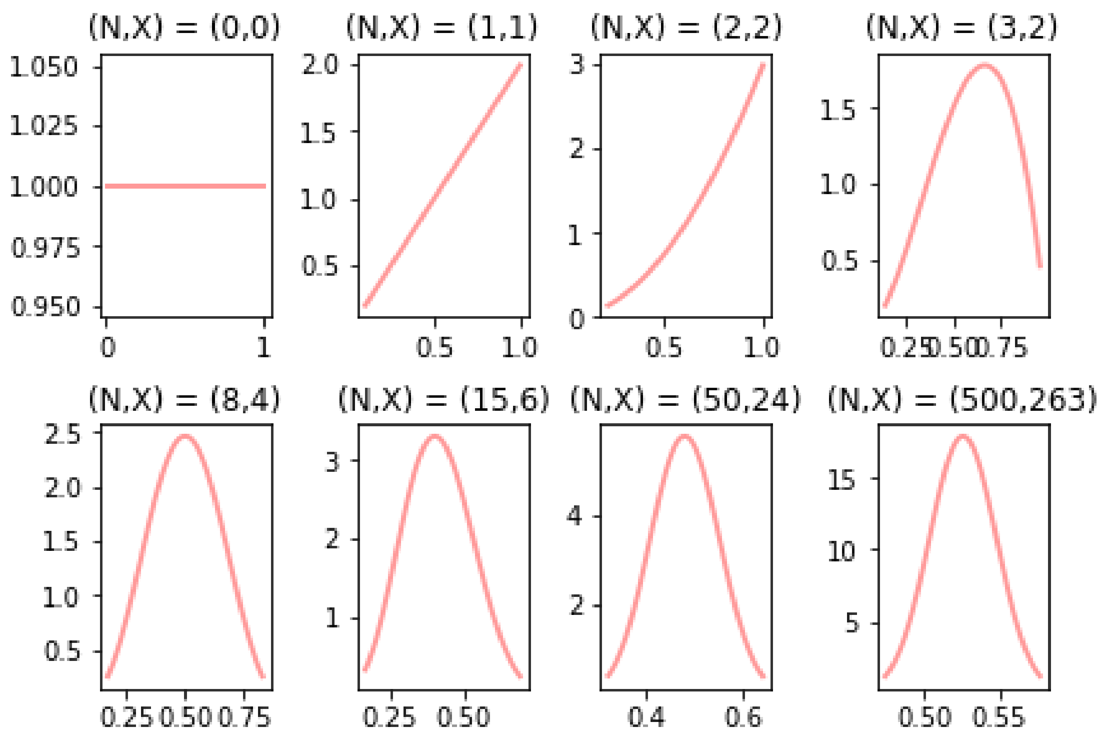
[ ]:
```
figure, axes = plt.subplots(2,4, constrained_layout=True)

for i in np.arange(8):

    x = np.linspace(beta.ppf(0.01, a[i], b[i]),
                    beta.ppf(0.99, a[i], b[i]), 100)

    axes[i//4,i%4].plot(x, beta.pdf(x, a[i], b[i]),
        'r-', lw=2, alpha=0.4, label='beta pdf')
    axes[i//4,i%4].set_title("(N,X) = (%s,%s)"%(N[i],X[i]))

plt.show()
```



1.2 Case 2 : (a,b)=(10,5)

[ ]:
```
a0, b0 = 10, 5
a = a0*np.ones(8) + X
b = b0*np.ones(8) + N - X

print(a,'\n',b)
```

```
[ 10.  11.  12.  12.  14.  16.  34. 273.]
 [  5.   5.   5.   6.   9.  14.  31. 242.]
```

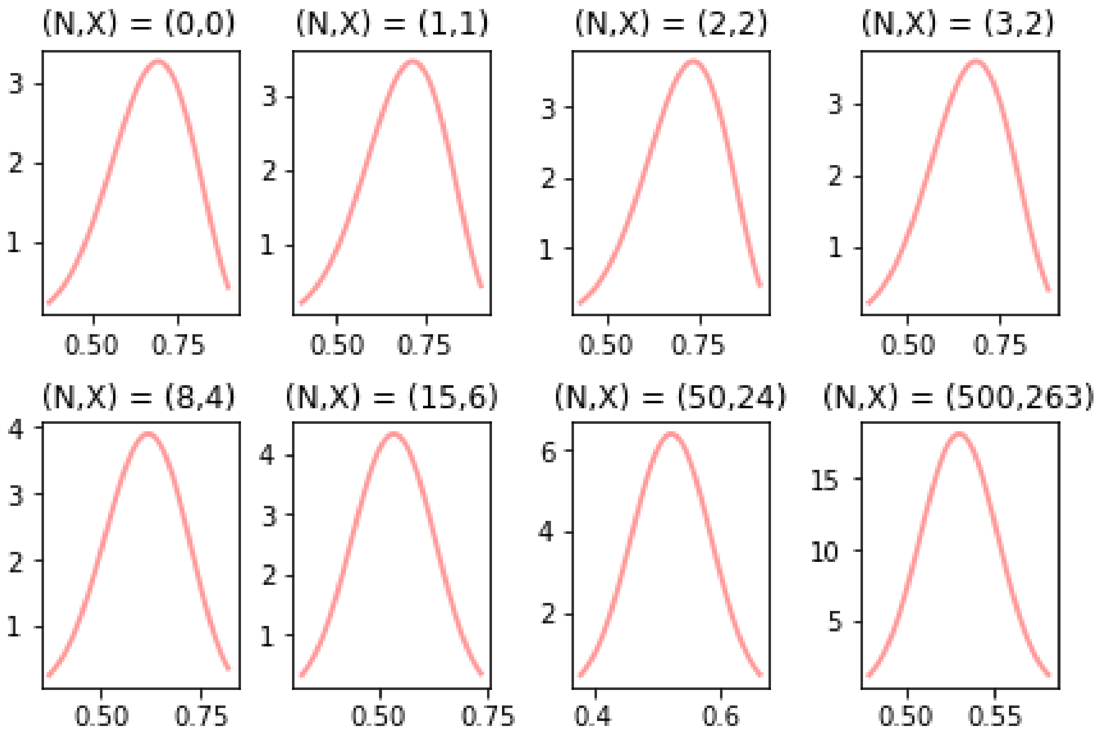
[ ]:
```
figure, axes = plt.subplots(2,4, constrained_layout=True)

for i in np.arange(8):

    x = np.linspace(beta.ppf(0.01, a[i], b[i]),
                    beta.ppf(0.99, a[i], b[i]), 100)

    axes[i//4,i%4].plot(x, beta.pdf(x, a[i], b[i]),
        'r-', lw=2, alpha=0.4, label='beta pdf')
    axes[i//4,i%4].set_title("(N,X) = (%s,%s)"%(N[i],X[i]))

plt.show()
```


(N,X) = (0,0)
(N,X) = (1,1)
(N,X) = (2,2)
(N,X) = (3,2)
(N,X) = (8,4)
(N,X) = (15,6)
(N,X) = (50,24)
(N,X) = (500,263)
0.50
0.75
0.4
0.6
0.55
1
2
3
4
6
5
10
15

二. 证明. $P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$

1). 多项分布的共轭先验是狄里克雷分布.

① Multinomial:

$$P(X_i = n_i, i=1,\dots,k|\underline{\theta}) = \begin{cases} n!\prod_{i=1}^{k}\frac{\theta_i^{n_i}}{n_i!}, & \sum_{i=1}^{k} n_i = n. \\ 0. & \text{otherwise}. \end{cases}$$

②. Dirichlet:

$$P(\theta|\underline{\alpha}) = \frac{1}{B(\underline{\alpha})}\prod_{i=1}^{K}\theta_i^{\alpha_i-1}, \quad \sum_i \theta_i = 1.$$

③. $P(x) = \int_0^1 P(x|\theta)P(\theta)d\theta$

$$= \int_0^1 \left(n!\prod_{i=1}^{k}\frac{\theta_i^{n_i}}{n_i!}\right)\left(\frac{1}{B(\underline{\alpha})}\prod_{i=1}^{K}\theta_i^{\alpha_i-1}\right)d\theta$$

$$= \int_0^1 \frac{n!}{B(\underline{\alpha})}\prod_{i=1}^{k}\frac{1}{n_i!}\prod_{i=1}^{k}\theta_i^{\alpha_i+n_i-1}d\theta$$

$$= \frac{n!}{B(\underline{\alpha})}\left(\prod_{i=1}^{k}\frac{1}{n_i!}\right)B(\underline{\alpha}+\underline{n})\underbrace{\int_0^1\frac{1}{B(\underline{\alpha}+\underline{n})}\prod_{i=1}^{k}\theta_i^{\alpha_i+n_i-1}d\theta}_{=1}$$

$$= \frac{B(\underline{\alpha}+\underline{n})}{B(\underline{\alpha})}n!\left(\prod_{i=1}^{k}\frac{1}{n_i}\right)$$

④. $P(\theta|X) = \frac{P(X|\theta)P(\theta)}{P(X)}$

$$= \frac{\left(\cancel{n!}\prod_{i=1}^{k}\frac{\theta_i^{n_i}}{\cancel{n_i!}}\right)\left(\cancel{\frac{1}{B(\underline{\alpha})}}\prod_{i=1}^{k}\theta_i^{\alpha_i-1}\right)}{\cancel{\frac{B(\underline{\alpha}+\underline{n})}{B(\underline{\alpha})}}\cancel{n!}\left(\prod_{i=1}^{k}\cancel{\frac{1}{n_i}}\right)}$$

$$= \frac{1}{B(\underline{\alpha}+\underline{n})}\prod_{i=1}^{k}\theta_i^{\alpha_i+n_i-1}.$$

$$= Dir(\underline{\alpha}+\underline{n}).$$

So the conjugate prior distribution for the Multinomial likelihood is Dirichlet distribution, and the posterior hyperparameters are: $\underline{\alpha}+\underline{n}$, where $\underline{\alpha}=(\alpha_1,\dots,\alpha_k)$, $\underline{n}=(n_1,\dots,n_k)$.

2）泊松分布的共轭分布是伽马分布

① Poisson.

$$f(x_i;\lambda) = Pr(X=x_i) = \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

i.i.d. $$f(\underline{x};\lambda) = \prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!}$$

② Gamma.

$$f(\lambda;\alpha,\beta) = \frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)}$$

for $\lambda > 0$; $\alpha, \beta > 0$.

③

$$P(\underline{x}) = \int_0^{+\infty} P(\underline{x}|\lambda) P(\lambda;\alpha,\beta) \, d\lambda$$

$$= \int_0^{+\infty} \left(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{x_i!} \right) \left(\frac{\beta^{\alpha} \lambda^{\alpha-1} e^{-\beta\lambda}}{\Gamma(\alpha)} \right) d\lambda$$

$$= \int_0^{+\infty} \frac{\beta^{\alpha}}{\prod_{i=1}^{n} x_i! \, \Gamma(\alpha)} \lambda^{\alpha + \sum_{i=1}^{n} x_i - 1} e^{-(\beta+n)\lambda} \, d\lambda$$

$$= \frac{\beta^{\alpha}}{\prod_i x_i! \, \Gamma(\alpha)} \frac{\Gamma(\alpha + \sum x_i)}{(\beta+n)^{\alpha+\sum x_i}} \underbrace{\int_0^{+\infty} \frac{(\beta+n)^{\alpha+\sum x_i} \lambda^{\alpha+\sum x_i - 1}}{\Gamma(\alpha+\sum x_i)} e^{-(\beta+n)\lambda} \, d\lambda}_{=1}$$

$$= \frac{\beta^{\alpha}}{(\beta+n)^{\alpha+\sum x_i}} \cdot \frac{\Gamma(\alpha+\sum x_i)}{\Gamma(\alpha)} \cdot \frac{1}{\prod_i x_i!} .$$

(2) .

$$P(\lambda | \underline{x}) = \frac{P(\underline{x} | \lambda) \, P(\lambda)}{P(x)}$$

$$
= \frac{\left(\prod_{i=1}^{n} \frac{\lambda^{x_i} e^{-\lambda}}{\cancel{x_i!}}\right)\left(\frac{\cancel{\beta^{\alpha}}\lambda^{\alpha-1} e^{-\beta\lambda}}{\cancel{\Gamma(\alpha)}}\right)}{\frac{\cancel{\beta^{\alpha}}}{(\beta+n)^{\alpha+\Sigma\alpha_i}} \frac{\Gamma(\alpha+\Sigma x_i)}{\cancel{\Gamma(\alpha)}} \frac{1}{\cancel{\prod_i x_i!}}}
$$

$$
= \frac{(\beta+n)^{\alpha+\Sigma x_i}}{\Gamma(\alpha+\Sigma x_i)} \lambda^{\alpha+\Sigma x_i - 1} e^{-(\beta+n)\lambda}
$$

$$
\sim \text{Gamma}(\alpha+\Sigma x_i, \beta+n)
$$

So the conjugate prior for the poisson distribution is γ dis.

3) 指数分布的共轭先验是 γ 分布.

①. exponential.

$$f(x_i;\lambda)=\begin{cases}\lambda e^{-\lambda x_i} & x_i\geq 0\\ 0 & x_i<0\end{cases}$$

$$f(\underline{x};\lambda) \overset{i.i.d.}{=\!=} \prod_{i=1}^{n} f(x_i;\lambda)$$

②. γ.

$$f(\lambda;\alpha,\beta)=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}$$

③. $$p(x)=\int_0^{+\infty} f(\underline{x};\lambda)f(\lambda;\alpha,\beta)\,d\lambda$$

$$=\int_0^{+\infty}\left(\prod_{i=1}^{n}\lambda e^{-\lambda x_i}\right)\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha-1}e^{-\beta\lambda}\,d\lambda$$

$$=\int_0^{+\infty}\frac{\beta^{\alpha}}{\Gamma(\alpha)}\lambda^{\alpha+n-1}e^{-(\beta+\Sigma x_i)\lambda}\,d\lambda$$

$$=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{(\beta+\Sigma x_i)^{\alpha+n}}\underbrace{\int_0^{+\infty}\frac{(\beta+\Sigma x_i)^{\alpha+n}}{\Gamma(\alpha+n)}\lambda^{\alpha+n-1}e^{-(\beta+\Sigma x_i)\lambda}\,d\lambda}_{=1}$$

$$=\frac{\beta^{\alpha}}{\Gamma(\alpha)}\frac{\Gamma(\alpha+n)}{(\beta+\Sigma x_i)^{\alpha+n}}$$

④. $$P(\lambda|x)=\frac{P(x|\lambda)P(\lambda)}{p(x)}$$

$$=\frac{\left(\prod_{i=1}^{n}\lambda e^{-\lambda x_i}\right)\cancel{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}\lambda^{\alpha-1}e^{-\beta\lambda}}{\cancel{\frac{\beta^{\alpha}}{\Gamma(\alpha)}}\frac{\Gamma(\alpha+n)}{(\beta+\Sigma x_i)^{\alpha+n}}}$$

$$= \frac{(\beta + \Sigma x_i)^{\alpha+n}}{\Gamma(\alpha+n)} \lambda^{\alpha+n-1} e^{-(\beta+\Sigma x_i)\lambda}$$

$$\sim \gamma(\alpha+n, \beta+\Sigma x_i).$$

So the conjugate prior distribution for the exponential distribution is γ distribution.

4). 方差已知的正态分布的共轭先验是正态分布.

①. normal, given σ^2

$$P(x_i|\mu) = \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i-\mu)^2/2\sigma^2}.$$

$$P(X|\mu) \xlongequal{i.i.d} \prod_{i=1}^{n} P(x_i|\mu)$$

②. prior $\sim N(\mu_0, \sigma_0^2)$.

③. $P(X) = \int_{-\infty}^{+\infty} P(X|\mu) P(\mu)\, d\mu$

$$= \int_{-\infty}^{+\infty} \left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i-\mu)^2/2\sigma^2} \right) \frac{1}{\sqrt{2\pi}\sigma_0} e^{-(\mu-\mu_0)^2/2\sigma_0^2} d\mu$$

$$= \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{\frac{n+1}{2}} \sigma^n \sigma_0} e^{-\left[\frac{(\mu-\mu_0)^2}{2\sigma_0^2} + \Sigma \frac{(x_i-\mu)^2}{2\sigma^2}\right]} d\mu.$$

$$= \int_{-\infty}^{+\infty} \frac{1}{(2\pi)^{\frac{n+1}{2}} \sigma^n \sigma_0} e^{-\frac{1}{2s^2}(\mu - m)^2} d\mu$$

$$= \frac{1}{(2\pi)^{\frac{n+1}{2}} \sigma^n \sigma_0} \sqrt{2\pi} \cdot s \cdot C \underbrace{\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi} s} e^{-\frac{1}{2s^2}(\mu - m)^2} d\mu}_{=1}$$

$$= \frac{\sqrt{2\pi}\, s}{(2\pi)^{\frac{n+1}{2}} \sigma^n \sigma_0} \cdot C$$

where
$$\boxed{\begin{cases} s^2 = \left(\dfrac{n\sigma_0^2 + \sigma^2}{\sigma_0^2 \sigma^2}\right)^{-1} \\ m = \left(\sigma_0^2 \sum x_i + \sigma^2 \mu_0\right) / \left(n\sigma_0^2 + \sigma^2\right) \end{cases}}$$

C is a known constant.

④ $$P(\mu|x) = \frac{P(x|\mu) P(\mu|\mu_0, \sigma_0^2)}{P(x)}$$

$$= \frac{\left(\prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-(x_i - \mu)^2 / 2\sigma^2}\right) \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{(\mu - \mu_0)^2}{2\sigma_0^2}}}{\frac{\sqrt{2\pi}\, s}{(2\pi)^{\frac{n+1}{2}} \sigma^n \sigma_0} C}$$

$$= \frac{\frac{1}{(2\pi)^{\frac{n+1}{2}}\sigma^n\sigma_0} \cdot C \cdot e^{-\frac{1}{2s^2}(\mu-m)^2}}{\frac{\sqrt{2\pi}\,s}{(2\pi)^{\frac{n+1}{2}}\sigma^n\sigma_0}\,C}$$

$$= \frac{1}{\sqrt{2\pi}\,s}\,e^{-\frac{1}{2s^2}(\mu-m)^2}$$

$$\sim N(s^2, m).$$

$$\boxed{\begin{cases} s^2 = \left(\dfrac{n}{\sigma^2} + \dfrac{1}{\sigma_0^2}\right)^{-1} \\[2ex] m = \dfrac{\sigma_0^2 \sum x_i + \sigma^2 \mu_0}{n\sigma_0^2 + \sigma^2} \end{cases}}$$

So the conjugate prior distribution for Nomal with known σ^2 is a Normal distribution.

5). 均值已知的正态分布的共轭先验是逆γ分布.

① Normal with known μ.

$$P(x_i|\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x_i-\mu)^2}.$$

② inverse Gamma

$$f(\sigma^2|\alpha,\beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\beta/\sigma^2}.$$

③ $$P(x) = \int_0^{+\infty} \prod_{i=1}^{n} P(x_i|\sigma^2) P(\sigma^2|\alpha,\beta)\, d\sigma^2$$

$$= \int_0^{+\infty} \frac{1}{(2\pi)^{\frac{n}{2}}\sigma^n} e^{-\frac{1}{2\sigma^2}\sum_i^n (x_i-\mu)^2} \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\sigma^2}\right)^{\alpha+1} e^{-\beta/\sigma^2}\, d\sigma^2$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{+\infty} \left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{n}{2}+1} e^{-\frac{1}{\sigma^2}\left(\beta+\frac{\sum_i (x_i-\mu)^2}{2}\right)}\, d\sigma^2$$

$$= \frac{1}{(2\pi)^{\frac{n}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+\frac{n}{2})}{\left(\beta+\frac{\sum (x_i-\mu)^2}{2}\right)^{\alpha+\frac{n}{2}}}$$

④ $$P(\sigma^2|x) = \frac{P(x|\sigma^2) P(\sigma^2|\alpha,\beta)}{P(x)}$$

$$= \frac{\cancel{\frac{1}{(2\pi)^{\frac{n}{2}}} \frac{\beta^\alpha}{\Gamma(\alpha)}} \left(\frac{1}{\sigma^2}\right)^{\alpha+\frac{n}{2}+1} e^{-\frac{1}{\sigma^2}\left(\beta+\sum(x_i-\mu)^2/2\right)}}{\cancel{\frac{1}{(2\pi)^{\frac{n}{2}}}} \frac{\beta^\alpha}{\quad} \Gamma(\alpha+\frac{n}{2})}$$

$(2\pi)^{n/2}$ $\cancel{\Gamma(\alpha)}$ $\left(\beta + \frac{\sum(x_i-\mu)^2}{2}\right)^{\alpha+\frac{n}{2}}$

$$\sim \text{Inverse Gamma}\left(\alpha+\frac{n}{2},\ \beta+\frac{\sum_i (x_i-\mu)^2}{2}\right).$$

So the conjugate prior for the Normal with known μ is an Inverse Gamma distribution.