

Bayesian Methods for Machine Learning

Lecture 7 – Deep generative models

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CentraleSupélec, 2020-2021

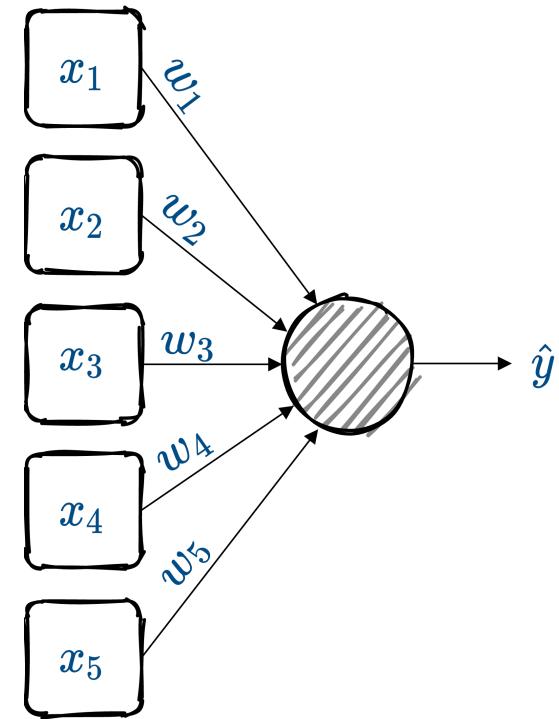
Introduction

Artificial neuron

$$\hat{y} = f(\mathbf{x}; \theta) = \sigma(\mathbf{w}^T \mathbf{x} + b) = \sigma\left(\sum_i w_i x_i + b\right),$$

where

- \mathbf{x} is the input vector;
- \hat{y} the scalar output;
- \mathbf{w} is the weight vector;
- b is the scalar bias;
- σ is a non-linear activation function;
- $\theta = \{\mathbf{w}, b\}$ are the neuron's parameters.



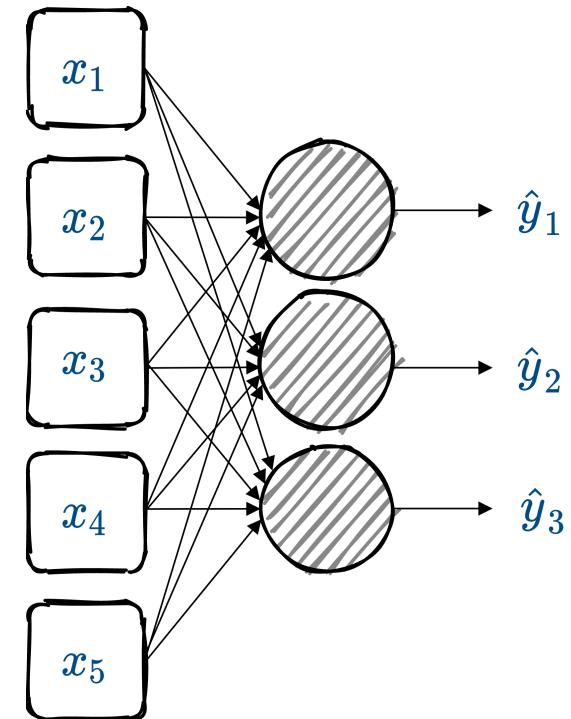
Layer

Neurons can be composed in parallel to form a **layer** with multiple outputs:

$$\hat{\mathbf{y}} = f(\mathbf{x}; \theta) = \sigma (\mathbf{W}^T \mathbf{x} + \mathbf{b}),$$

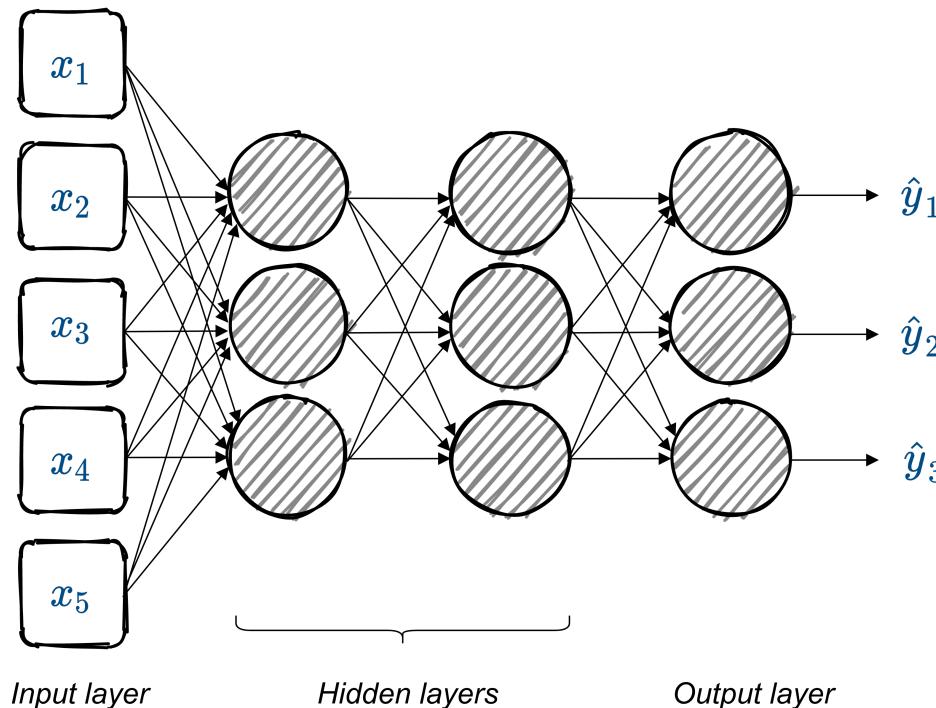
where we have now

- an **element-wise** activation function,
- an **output vector** $\hat{\mathbf{y}}$,
- a **weight matrix** \mathbf{W} ,
- a **bias vector** \mathbf{b} ,
- such that $\theta = \{\mathbf{W}, \mathbf{b}\}$.



Multi-layer Perceptron

Similarly, layers can be composed in **series**, to form a **multi-layer Perceptron**, or **feed-forward fully-connected** neural network.



The model parameters are the weight matrices and bias vectors of all layers.

Supervised learning

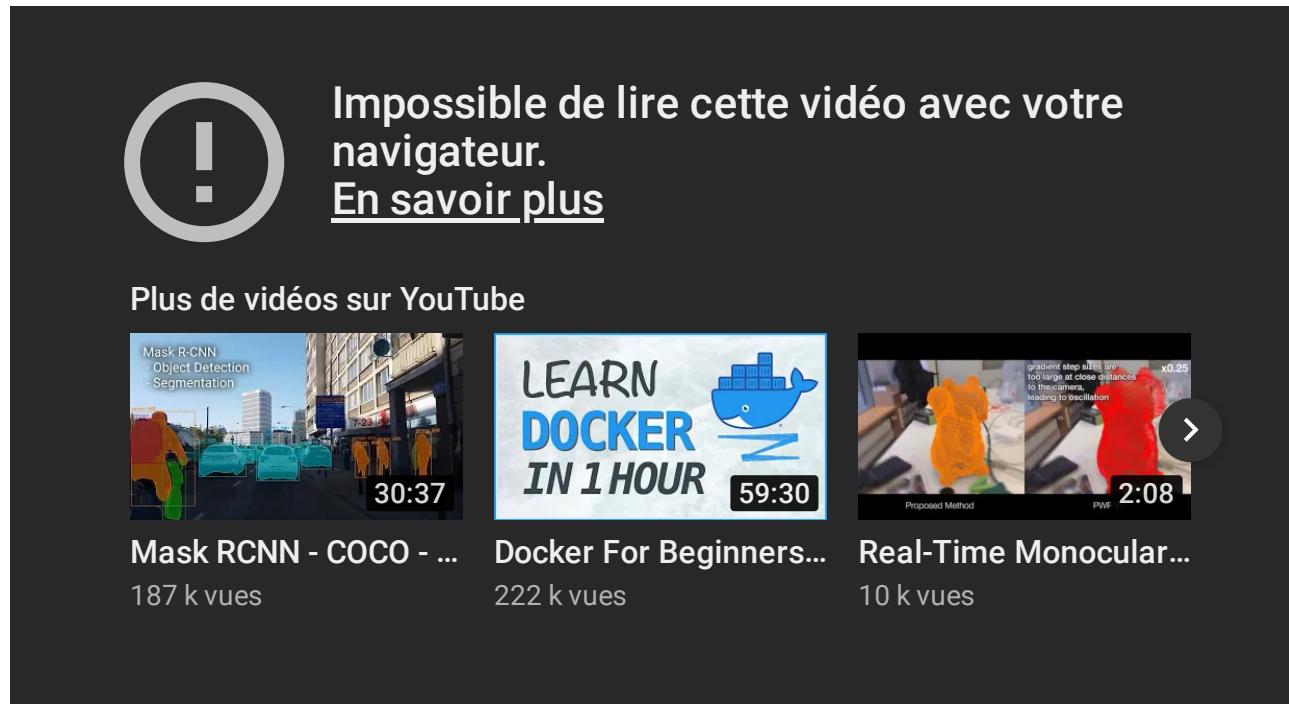
- **Training dataset** (step 0): $\mathcal{D} = \{\mathbf{x}_i, \mathbf{y}_i\}_{i=1}^N$ where the \mathbf{x}_i 's are the inputs, and \mathbf{y}_i 's the labels.
- **Model** (step 1): A neural network $f(\cdot; \theta)$.
- **Loss function** (step 2):

$$\mathcal{L}(\theta) = \frac{1}{N} \sum_{(\mathbf{x}_i, \mathbf{y}_i) \in \mathcal{D}} \ell(\mathbf{y}_i, f(\mathbf{x}_i; \theta)),$$

where $\ell(\cdot, \cdot)$ is task-dependent.

- **Training** (step 3): Minimize $\mathcal{L}(\theta)$ with (variants of) gradient descent and backpropagation.
- **Evaluation** (step 4): Test the performance on examples that were not seen during training.

A great number of successful practical applications of machine and deep learning rely on **supervised learning** methods.



The Cityscapes dataset for semantic segmentation of urban images contains **5k images** with high quality pixel-level annotations.

"Annotation and quality control required more than **1.5 h on average for a single image**" (Cordts et al., 2016).

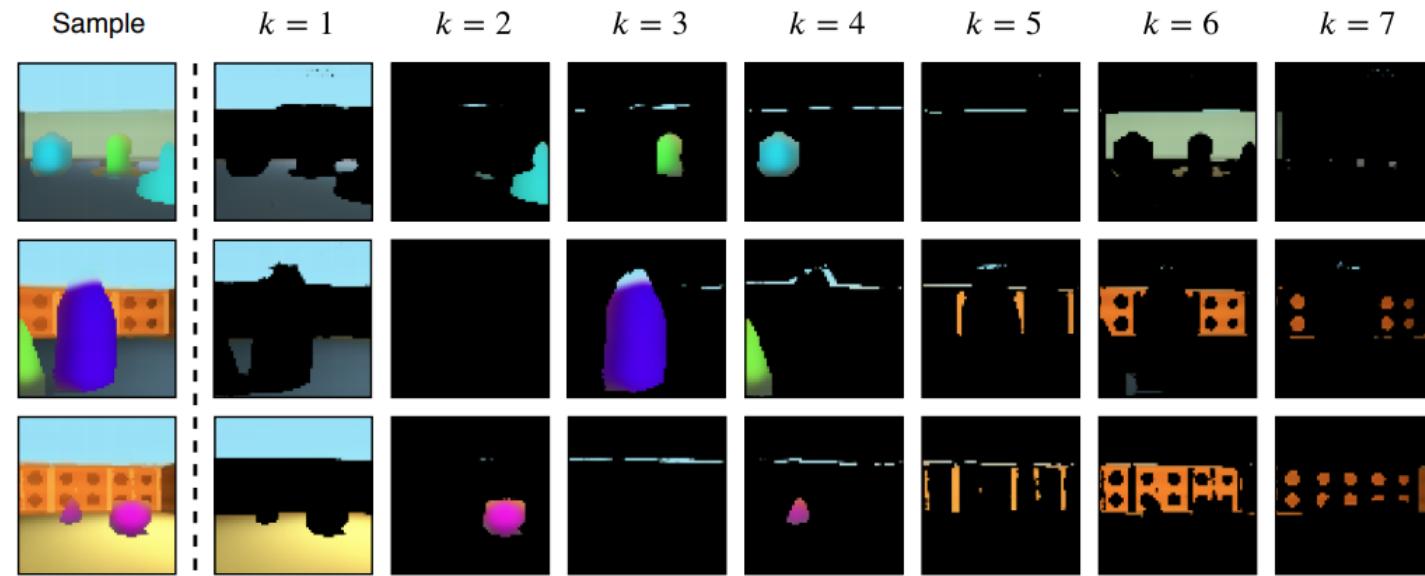
More than 300 days of annotation!

Zhao et al., "ICNet for Real-Time Semantic Segmentation on High-Resolution Images", ECCV 2018

Unsupervised learning

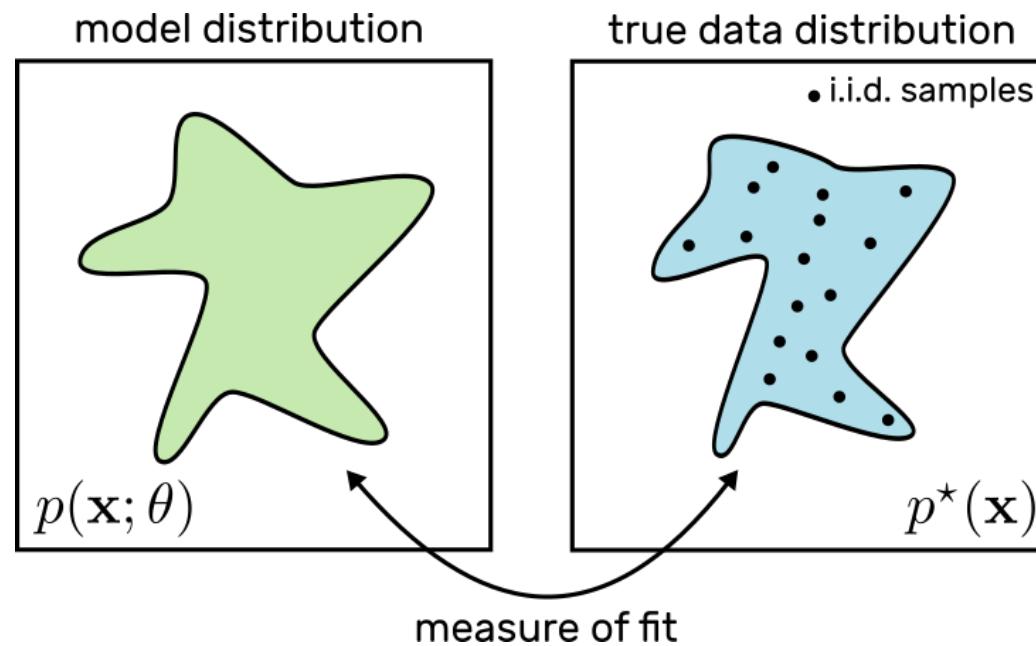
We need **unsupervised** methods that can learn to unveil the **underlying (hidden) structure** of the data without requiring ground-truth labels.

Semi-supervised methods are also of great interest.



Component-by-component scene generation with GENESIS, a generative model of 3D scenes capable of both decomposing and generating scenes by capturing relationships between scene components

Generative modeling



The goal is to tune the parameters θ of the model distribution $p(\mathbf{x}; \theta)$ so that it is as close as possible to the true data distribution $p^*(\mathbf{x})$, according to some measure of fit.

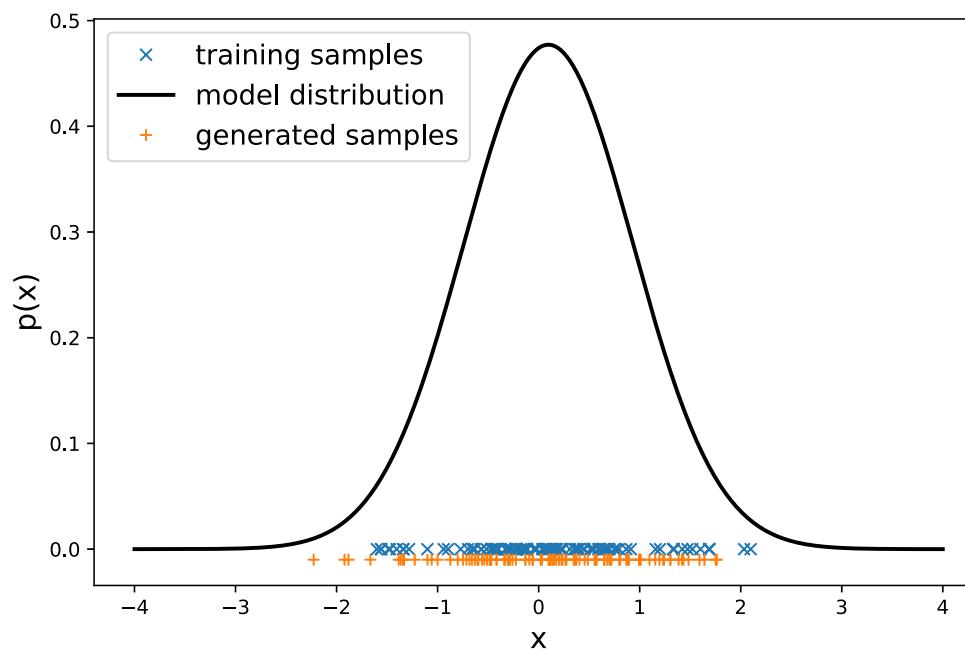
For instance, minimizing the Kullback-Leibler (KL) divergence $D_{\text{KL}}(p^*(\mathbf{x}) \parallel p(\mathbf{x}; \theta))$ is equivalent to maximum likelihood estimation.

1-dimensional example

We assume a simple Gaussian generative model of data $x \in \mathbb{R}$:

$$p(x; \theta) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(\frac{(x - \mu)^2}{2\sigma^2}\right),$$

where the parameters $\theta = \{\mu, \sigma^2\}$ are estimated in the maximum likelihood sense.



10^6 -dimensional example



Deep generative models (DGM)

- The model distribution $p(\mathbf{x}; \theta)$ is somehow defined by means of a neural network.
- Two seminal models: Variational autoencoders (VAEs) (Kingma and Welling, 2014; Rezende et al., 2014) and generative adversarial networks (GANs) (Goodfellow et al., 2014).
- DGM contain **hundreds of thousands of parameters**, trained in a scalable way using **large datasets of unlabeled high-dimensional data**.



I. Goodfellow et al., "Generative Adversarial Networks", NeurIPS 2014.

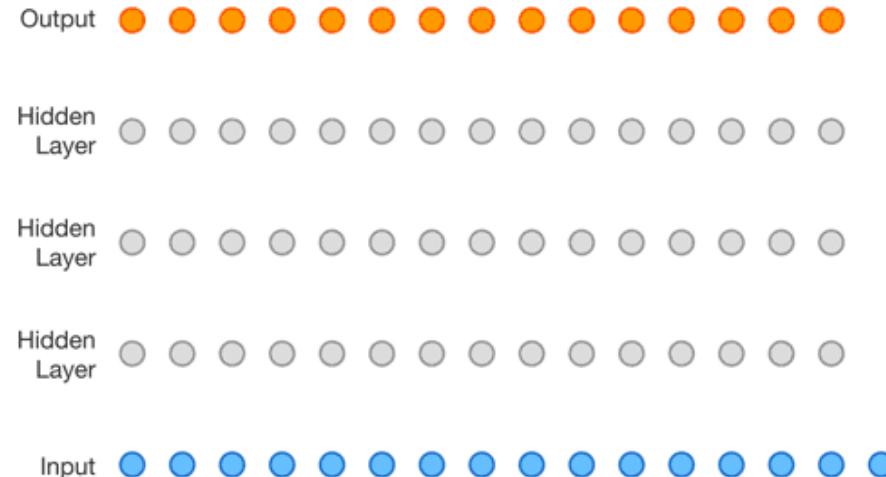
D.P. Kingma and M. Welling, "Auto-Encoding Variational Bayes", ICLR 2014.

D.J. Rezende et. al, " Stochastic backpropagation and approximate inference in deep generative models", ICML 2014.

Several flavors of DGMs

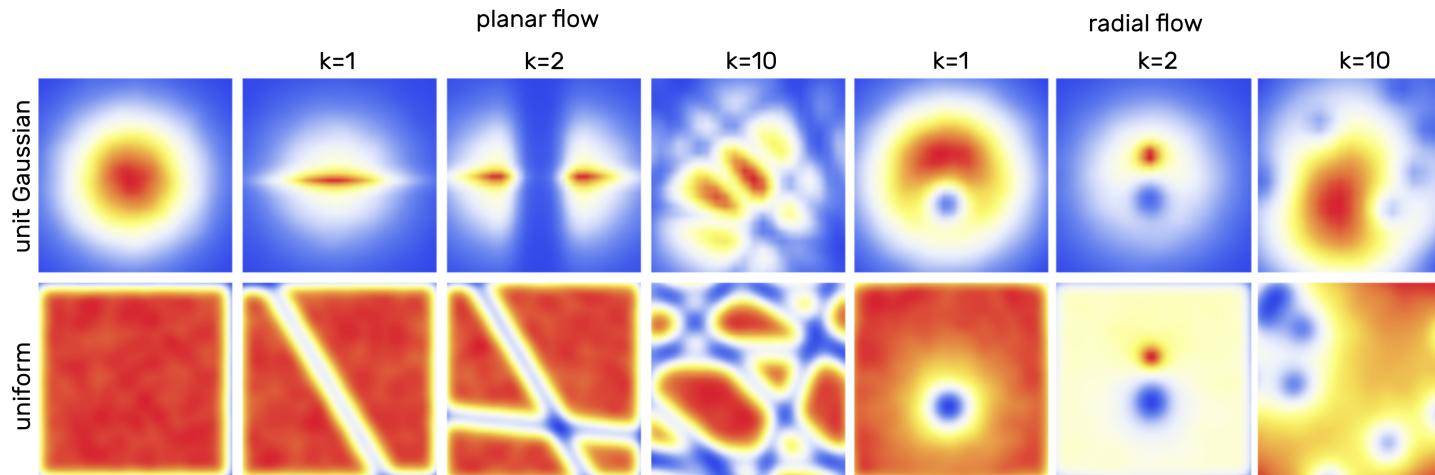
Autoregressive DGMs define the model distribution recursively, using the chain rule:

$$p(\mathbf{x}; \theta) = p(x_1) \prod_{i=2}^D p(x_i | x_{1:i-1}; \theta).$$



Flow-based DGMs transform a simple distribution into a complex one by applying a sequence of invertible mappings:

$$\mathbf{x} = f_K \circ \dots \circ f_1(\mathbf{z}_0), \quad \mathbf{z}_0 \sim p_0(\mathbf{z}_0).$$

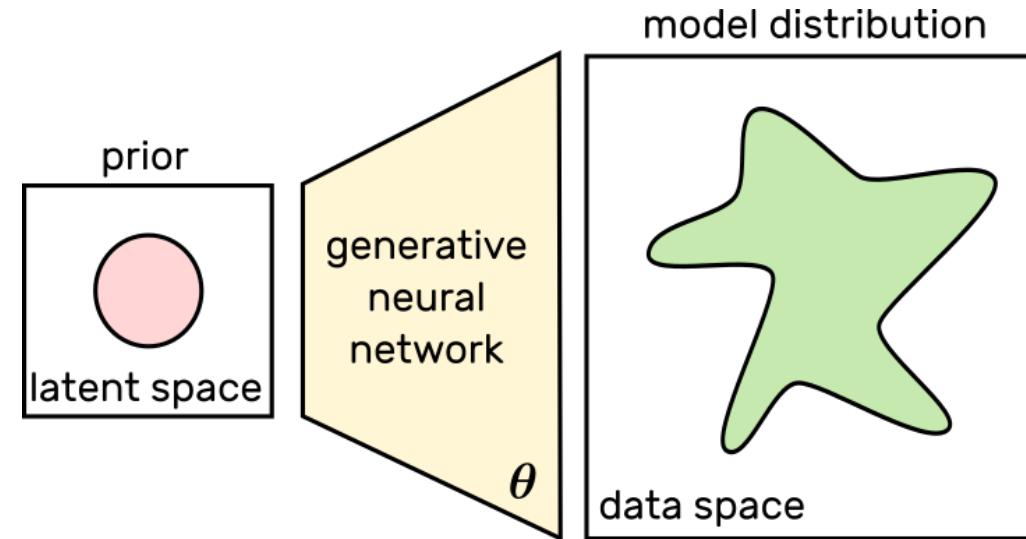


As mappings are invertible, we can express $p(\mathbf{x}; \theta)$ analytically from the Jacobian of the inverse transforms:

$$p_k(\mathbf{z}_k) = p_{k-1}(f_k^{-1}(\mathbf{z}_k)) \left| \det \frac{df_k^{-1}}{d\mathbf{z}_k} \right|.$$

Latent-variable-based DGMs define the model distribution as a marginal distribution, by introducing a low-dimensional latent random vector:

$$p(\mathbf{x}; \theta) = \int p(\mathbf{x}, \mathbf{z}; \theta) d\mathbf{z} = \int p(\mathbf{x}|\mathbf{z}; \theta)p(\mathbf{z})d\mathbf{z}.$$



GANs and VAEs are two examples of latent-variable-based DGM, but $p(\mathbf{x}|\mathbf{z}; \theta)$ is only defined explicitly (i.e. analytically) for VAEs.

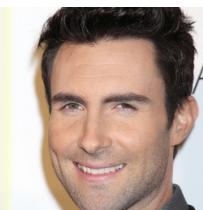
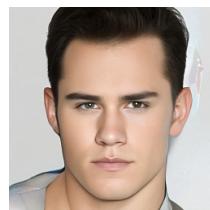
Today we will focus on VAEs.

VAE for face generation

Generated samples

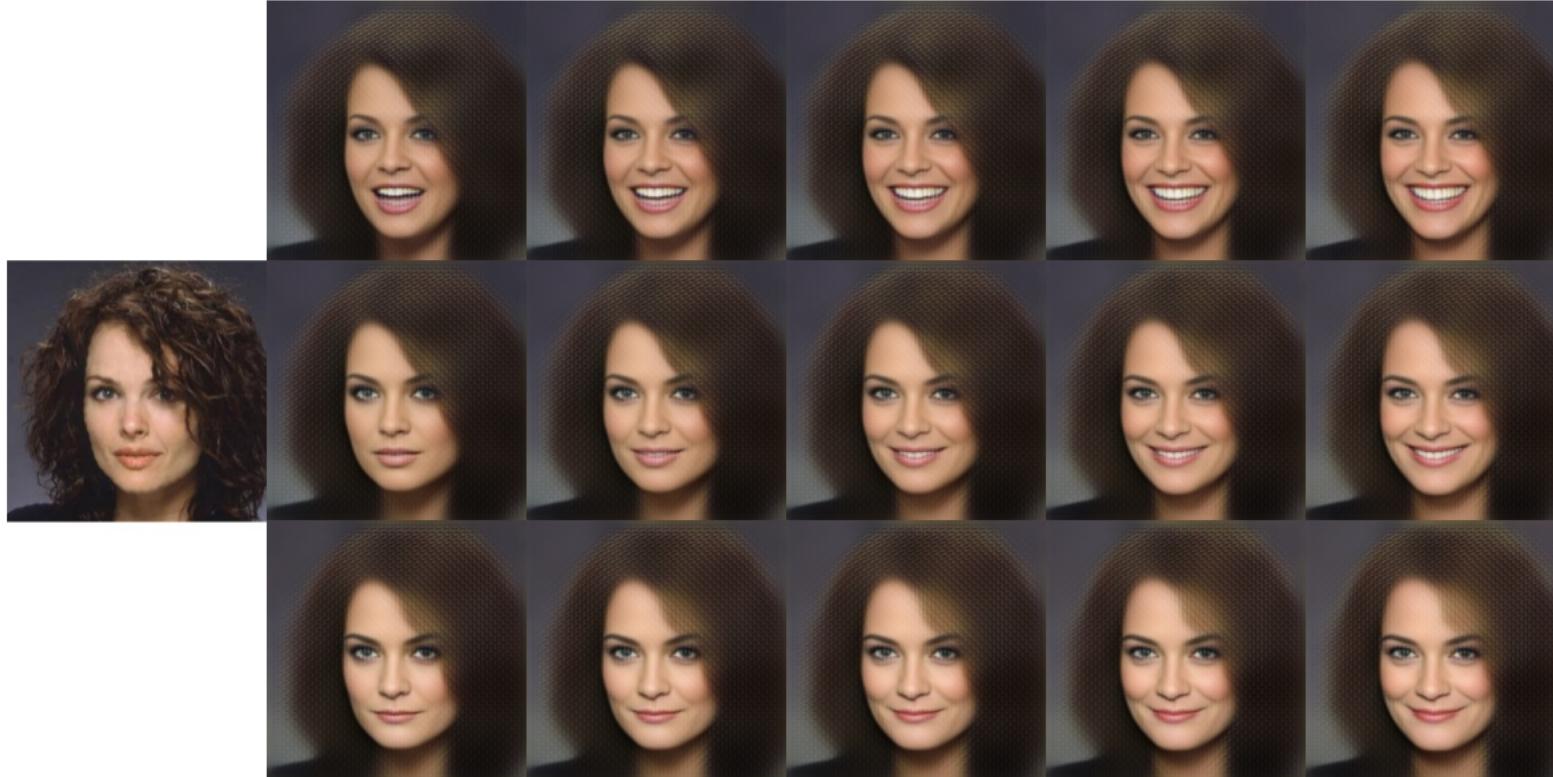


Retrieved images from the training set



Top retrieved images from the training set are visualized for samples generated by NVAE in each row. The generated instances do not exist in the training set.

VAE for modifying faces by tuning latent factors



Modifying two latent factors: smiling and mouth open

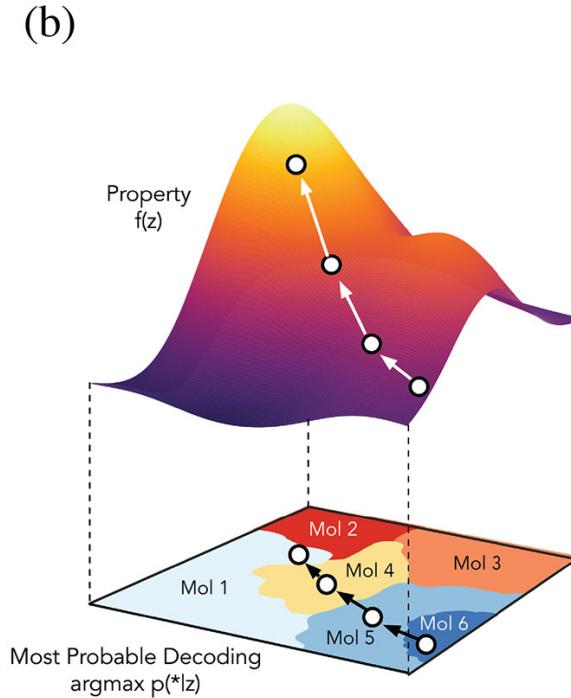
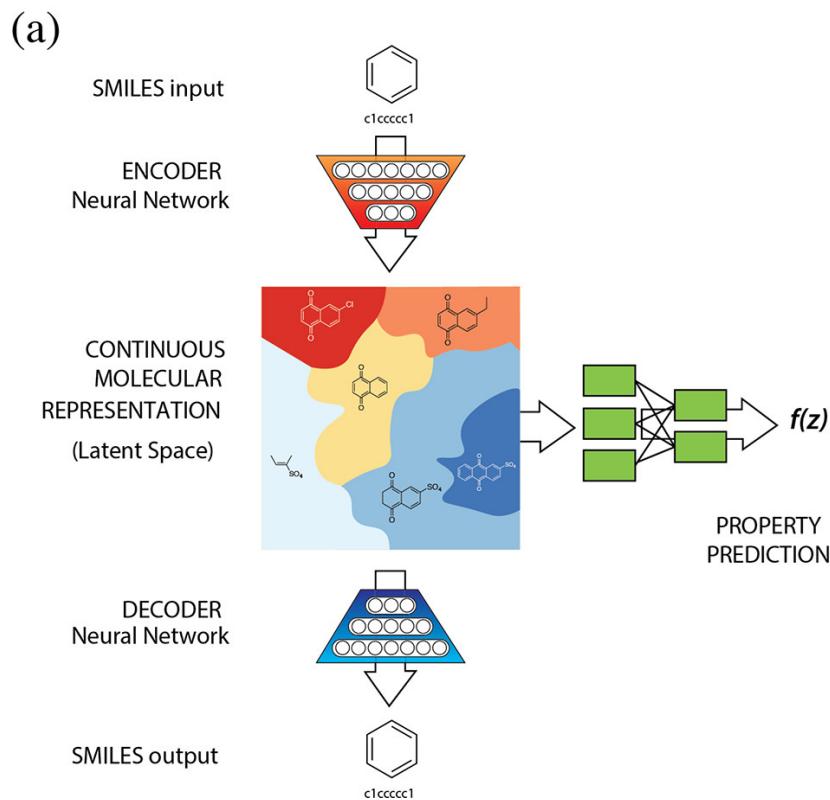
VAE for sentence generation from a continuous space

“ **i want to talk to you .** ”
“ *i want to be with you .* ”
“ *i do n’t want to be with you .* ”
i do n’t want to be with you .
she did n’t want to be with him .

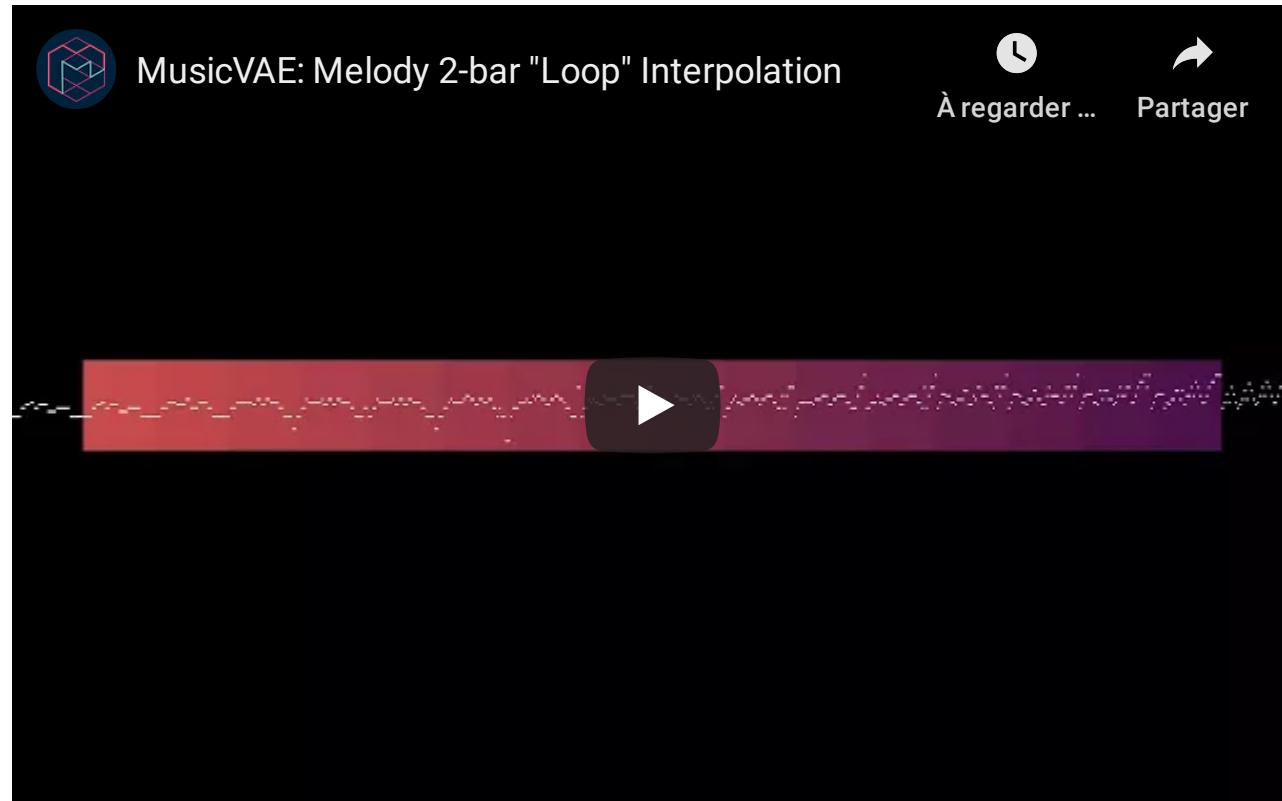
he was silent for a long moment .
he was silent for a moment .
it was quiet for a moment .
it was dark and cold .
there was a pause .
it was my turn .

Interpolation in a recurrent VAE latent space

VAE for automatic design of new molecules



VAE for symbolic music generation



Gradual blending of 2 different melodies.

Variational autoencoders

D.P. Kingma and M. Welling, [Auto-Encoding Variational Bayes](#), ICLR 2014.

D.J. Rezende et. al, [Stochastic backpropagation and approximate inference in deep generative models](#), ICML 2014.

Generative model

Let $\mathbf{x} \in \mathbb{R}^D$ and $\mathbf{z} \in \mathbb{R}^K$ be two random vectors (typically $K \ll D$). The generative model is defined by:

$$p(\mathbf{x}; \theta) = \int p(\mathbf{x}|\mathbf{z}; \theta)p(\mathbf{z})d\mathbf{z}.$$

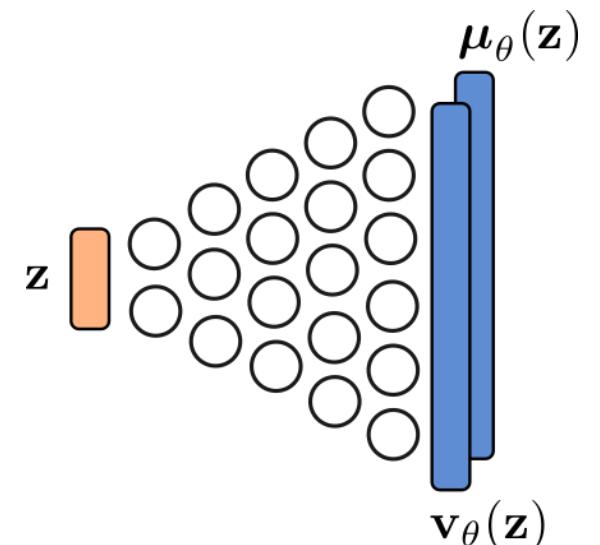
- The **prior** is a standard Gaussian distribution:

$$p(\mathbf{z}) = \mathcal{N}(\mathbf{z}; \mathbf{0}, \mathbf{I}).$$

- The **likelihood** is parametrized with a **generative/decoder neural network**, e.g.

$$p(\mathbf{x}|\mathbf{z}; \theta) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_\theta(\mathbf{z}), \text{diag}\{\mathbf{v}_\theta(\mathbf{z})\}),$$

where θ denotes the parameters of the decoder network.



Pause on the notebook

How to estimate the VAE generative model parameters?

KL divergence minimization

- We have defined our model distribution $p(\mathbf{x}; \theta) = \int p(\mathbf{x}, \mathbf{z}; \theta) d\mathbf{z} = \int p(\mathbf{x}|\mathbf{z}; \theta)p(\mathbf{z}) d\mathbf{z}$.

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- We take the Kullback-Leibler (KL) divergence as a measure of fit:

$$D_{\text{KL}}(p \parallel q) = \mathbb{E}_p[\ln(p) - \ln(q)] \geq 0.$$

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Equivalent to **maximum marginal likelihood** estimation.

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- **Monte Carlo estimate:** The expectation is approximated by an empirical average, using i.i.d samples drawn from the true intractable distribution,

$$\mathbb{E}_{p^*(\mathbf{x})}[f(\mathbf{x}; \theta)] \approx \frac{1}{N} \sum_{i=1}^N [f(\mathbf{x}_n; \theta)].$$

The approximation error decreases as N increases (law of large numbers).

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- This is a general principle called **empirical risk minimization** in statistical learning theory.

Maximum likelihood estimation

- To sum up, we want to minimize the KL divergence between $p^*(\mathbf{x})$ and $p(\mathbf{x}; \theta)$ w.r.t θ (the weights and biases of the generative network), which is equivalent to solving:

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- We cannot compute this integral analytically, because the integrand is highly non-linear in \mathbf{z} .
- We will resort to **variational inference** techniques, that we have already encountered in this course.

Variational inference

- For any distribution $q(\mathbf{z}|\mathbf{x}; \phi)$, we have (Neal and Hinton, 1999; Jordan et al. 1999)

$$\ln p(\mathbf{x}; \theta) = \mathcal{L}(\mathbf{x}; \phi, \theta) + D_{\text{KL}}(q(\mathbf{z}|\mathbf{x}; \phi) \parallel p(\mathbf{z}|\mathbf{x}; \theta)),$$

where $\mathcal{L}(\mathbf{x}; \phi, \theta)$ is the **evidence lower bound** (ELBO), defined by

$$\mathcal{L}(\mathbf{x}; \phi, \theta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x}; \phi)} [\ln p(\mathbf{x}, \mathbf{z}; \theta) - \ln q(\mathbf{z}|\mathbf{x}; \phi)].$$

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Problem #1

$$\max_{\theta} \mathcal{L}(\mathbf{x}; \phi, \theta),$$

where $\mathcal{L}(\mathbf{x}; \phi, \theta) \leq \ln p(\mathbf{x}; \theta)$

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Problem #1

$$\max_{\theta} \mathcal{L}(\mathbf{x}; \phi, \theta),$$

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Problem #2

$$\max_{\phi} \mathcal{L}(\mathbf{x}; \phi, \theta)$$

$$\Leftrightarrow \min_{\phi} D_{\text{KL}}(q(\mathbf{z}|\mathbf{x}; \phi) \parallel p(\mathbf{z}|\mathbf{x}; \theta))$$

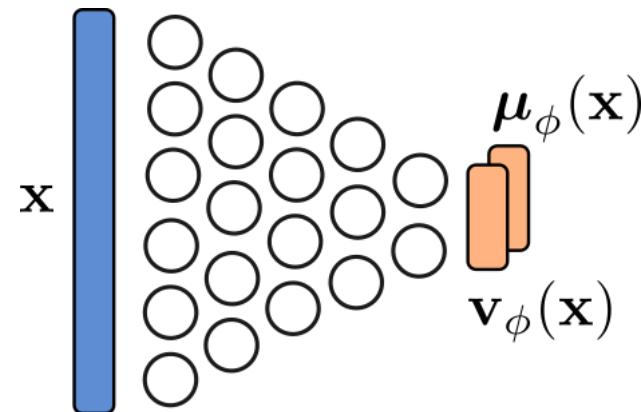
To fully define the objective function, we need to specify the inference model $q(\mathbf{z}|\mathbf{x}; \phi)$.

Inference model

The **inference model** (approximate posterior) is typically defined by:

$$q(\mathbf{z}|\mathbf{x}; \phi) = \mathcal{N} (\mathbf{z}; \boldsymbol{\mu}_\phi(\mathbf{x}), \text{diag} \{ \mathbf{v}_\phi(\mathbf{x}) \}) ,$$

where the mean and variance vectors are provided by the **encoder** neural network.



ELBO

The ELBO is now fully defined:

$$\begin{aligned}\mathcal{L}(\mathbf{x}; \phi, \theta) &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln p(\mathbf{x}, \mathbf{z}; \theta) - \ln q(\mathbf{z}|\mathbf{x}; \phi)] \\ &= \underbrace{\mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln p(\mathbf{x}|\mathbf{z}; \theta)]}_{\text{reconstruction accuracy}} - \underbrace{D_{\text{KL}}(q(\mathbf{z}|\mathbf{x}; \phi) \parallel p(\mathbf{z}))}_{\text{regularization}}.\end{aligned}$$

- prior: $p(\mathbf{z}) = \mathcal{N}(\mathbf{z}; \mathbf{0}, \mathbf{I})$
- likelihood model: $p(\mathbf{x}|\mathbf{z}; \theta) = \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_\theta(\mathbf{z}), \text{diag}\{\mathbf{v}_\theta(\mathbf{z})\})$
- inference model: $q(\mathbf{z}|\mathbf{x}; \phi) = \mathcal{N}(\mathbf{z}; \boldsymbol{\mu}_\phi(\mathbf{x}), \text{diag}\{\mathbf{v}_\phi(\mathbf{x})\})$

Pause on the notebook

Back to the computation of the ELBO

$$\mathcal{L}(\mathbf{x}; \phi, \theta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln p(\mathbf{x}|\mathbf{z};\theta)] - D_{\text{KL}}(q(\mathbf{z}|\mathbf{x};\phi) \parallel p(\mathbf{z})).$$

Let's first focus on the second term of the ELBO which is called the **regularization term** because it constrains the inference model to be not too far from the prior.

$$\begin{aligned} D_{\text{KL}}(q(\mathbf{z}|\mathbf{x};\phi) \parallel p(\mathbf{z})) &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln q(\mathbf{z}|\mathbf{x};\phi) - \ln p(\mathbf{z})] \\ &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln \mathcal{N}(\mathbf{z}; \boldsymbol{\mu}_\phi(\mathbf{x}), \text{diag}\{\mathbf{v}_\phi(\mathbf{x})\}) - \ln \mathcal{N}(\mathbf{z}; \mathbf{0}, \mathbf{I})] \\ &= \sum_{k=1}^K \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} \left[\ln \mathcal{N}(z_k; \mu_{k,\phi}(\mathbf{x}), v_{k,\phi}(\mathbf{x})) - \ln \mathcal{N}(z_k; 0, 1) \right] \\ &= \dots \end{aligned}$$

$$\mathcal{L}(\mathbf{x}; \phi, \theta) = \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln p(\mathbf{x}|\mathbf{z};\theta)] - D_{\text{KL}}(q(\mathbf{z}|\mathbf{x};\phi) \parallel p(\mathbf{z})).$$

Let's now focus on the second term, which is called the **reconstruction accuracy term**.

$$\begin{aligned}\mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} [\ln p(\mathbf{x}|\mathbf{z};\theta)] &= \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} \left[\ln \mathcal{N}(\mathbf{x}; \boldsymbol{\mu}_\theta(\mathbf{z}), \text{diag}\{\mathbf{v}_\theta(\mathbf{z})\}) \right] \\ &= \sum_{d=1}^D \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} \left[\ln \mathcal{N}(x_d; \mu_{d,\theta}(\mathbf{z}), v_{d,\theta}(\mathbf{z})) \right] \\ &= -\frac{1}{2} \sum_{d=1}^D \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} \left[\ln (2\pi v_{d,\theta}(\mathbf{z})) + \frac{(x_d - \mu_{d,\theta}(\mathbf{z}))^2}{v_{d,\theta}(\mathbf{z})} \right]\end{aligned}$$

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A common approach consist in choosing $v_{d,\theta}(\mathbf{z}) = 1$ for all $d \in \{1, \dots, D\}$, such that the reconstruction accuracy term involves the **mean squared error** between the data \mathbf{x} and the mean vector $\boldsymbol{\mu}_\theta(\mathbf{z})$ provided by the decoder network.

Reconstruction accuracy term

$$\mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)}[\ln p(\mathbf{x}|\mathbf{z};\theta)] = -\frac{1}{2} \sum_{d=1}^D \mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)} \left[\ln (2\pi v_{d,\theta}(\mathbf{z})) + \frac{(x_d - \mu_{d,\theta}(\mathbf{z}))^2}{v_{d,\theta}(\mathbf{z})} \right].$$

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- **Problem 1** - The expectation cannot be computed analytically.

Reconstruction accuracy term

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- **Problem 1** - The expectation cannot be computed analytically.
- **Solution 1** - Approximate it with a Monte Carlo estimate:

$$\mathbb{E}_{q(\mathbf{z}|\mathbf{x};\phi)}[\ln p(\mathbf{x}|\mathbf{z};\theta)] \approx -\frac{1}{2R} \sum_{d=1}^D \sum_{r=1}^R \left[\ln (2\pi v_{d,\theta}(\mathbf{z}_r)) + \frac{(x_d - \mu_{d,\theta}(\mathbf{z}_r))^2}{v_{d,\theta}(\mathbf{z}_r)} \right],$$

where $\{\mathbf{z}_r\}_{r=1}^R$ are i.i.d samples drawn from $q(\mathbf{z}|\mathbf{x}; \phi)$.

Note that in practice, we choose $R = 1$.

Reconstruction accuracy term

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- **Problem 2** - This sampling operation is not differentiable w.r.t. ϕ .

Reconstruction accuracy term

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Note that in practice, we choose $R = 1$.

- **Problem 2** - This sampling operation is not differentiable w.r.t. ϕ .
- **Solution 2** - Use the so-called **reparametrization trick**.

Reparametrization trick

- The reparameterization trick consists in rewriting the sampling operation of \mathbf{z}_r from $q(\mathbf{z}|\mathbf{x}; \phi)$ as a invertible transformation of another random sample $\boldsymbol{\epsilon}_r$, drawn from a distribution $p(\boldsymbol{\epsilon})$ which is independent of \mathbf{x} and ϕ .

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- In our case, we have $q(\mathbf{z}|\mathbf{x}; \phi) = \mathcal{N}(\boldsymbol{\mu}_\phi(\mathbf{x}), \text{diag}\{\mathbf{v}_\phi(\mathbf{x})\})$.
- A simple reparametrization is given by:

$$\mathbf{z}_r = \boldsymbol{\mu}_\phi(\mathbf{x}) + \text{diag}\{\mathbf{v}_\phi(\mathbf{x})\}^{1/2} \boldsymbol{\epsilon}_r,$$

where $\boldsymbol{\epsilon}_r$ is drawn from $\mathcal{N}(\mathbf{0}, \mathbf{I})$.

Finally, the full ELBO

- Putting all together, we end-up with the following expression of the variational free energy:

$$\begin{aligned}\mathcal{L}(\mathbf{x}; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}})) + \frac{(x_d - \mu_{d,\theta}(\tilde{\mathbf{z}}))^2}{v_{d,\theta}(\tilde{\mathbf{z}})} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}) - \mu_{k,\phi}^2(\mathbf{x}) - v_{k,\phi}(\mathbf{x}) \right] + cst(\phi, \theta),\end{aligned}$$

where $\tilde{\mathbf{z}}$ is drawn from $q(\mathbf{z}|\mathbf{x}; \phi)$ using the **reparametrization trick**.

Finally, the full ELBO

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- This objective function is differentiable with respect to ϕ and θ . It can be optimized with gradient-ascent-based techniques, where gradients are computed by backpropagation.

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- This objective function is differentiable with respect to ϕ and θ . It can be optimized with gradient-ascent-based techniques, where gradients are computed by backpropagation.
- Do not forget that we actually maximize a sample average over our training dataset:

$$\max_{\phi, \theta} \left\{ \mathbb{E}_{p^*(\mathbf{x})} [\mathcal{L}(\mathbf{x}; \phi, \theta)] \approx \frac{1}{N} \sum_{n=1}^N \mathcal{L}(\mathbf{x}_n; \phi, \theta) \right\}.$$

Training procedure

Step 0: Do the math

$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$

Training procedure

Step 1: Pick an example in the dataset

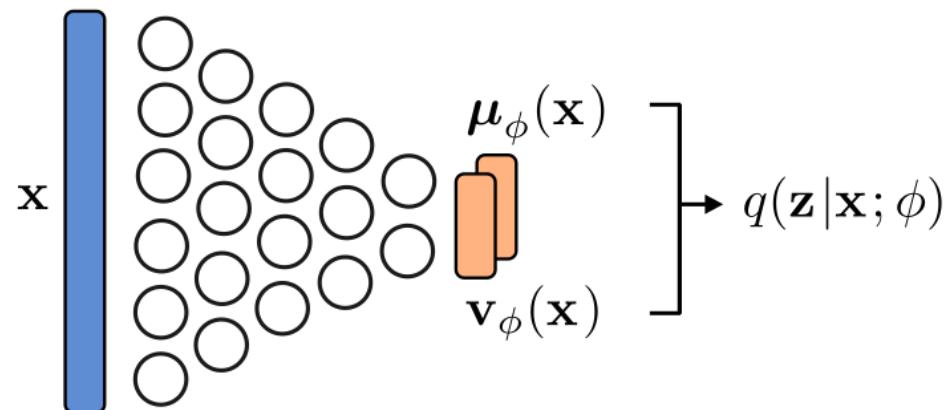
$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$



Training procedure

Step 2: Map through the encoder

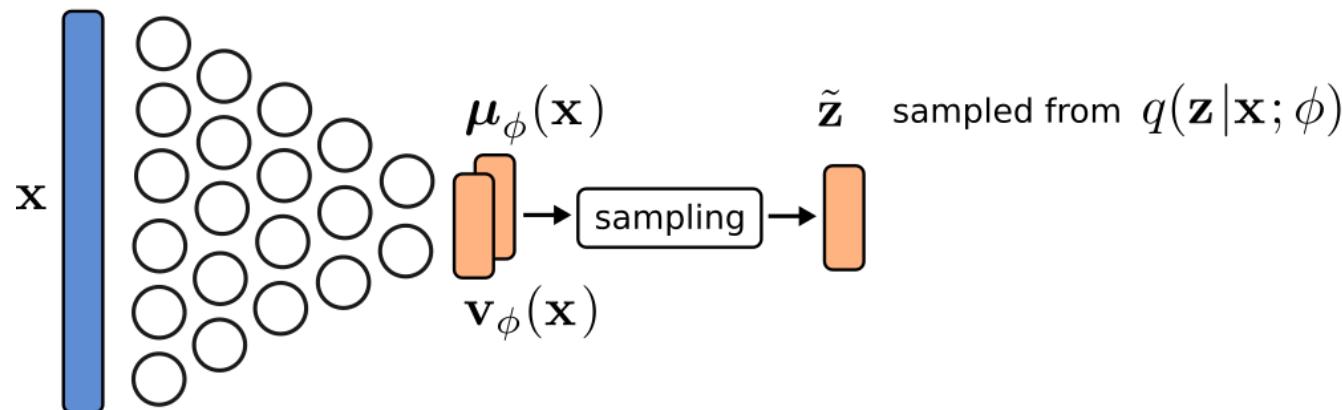
$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$



Training procedure

Step 3: Sample

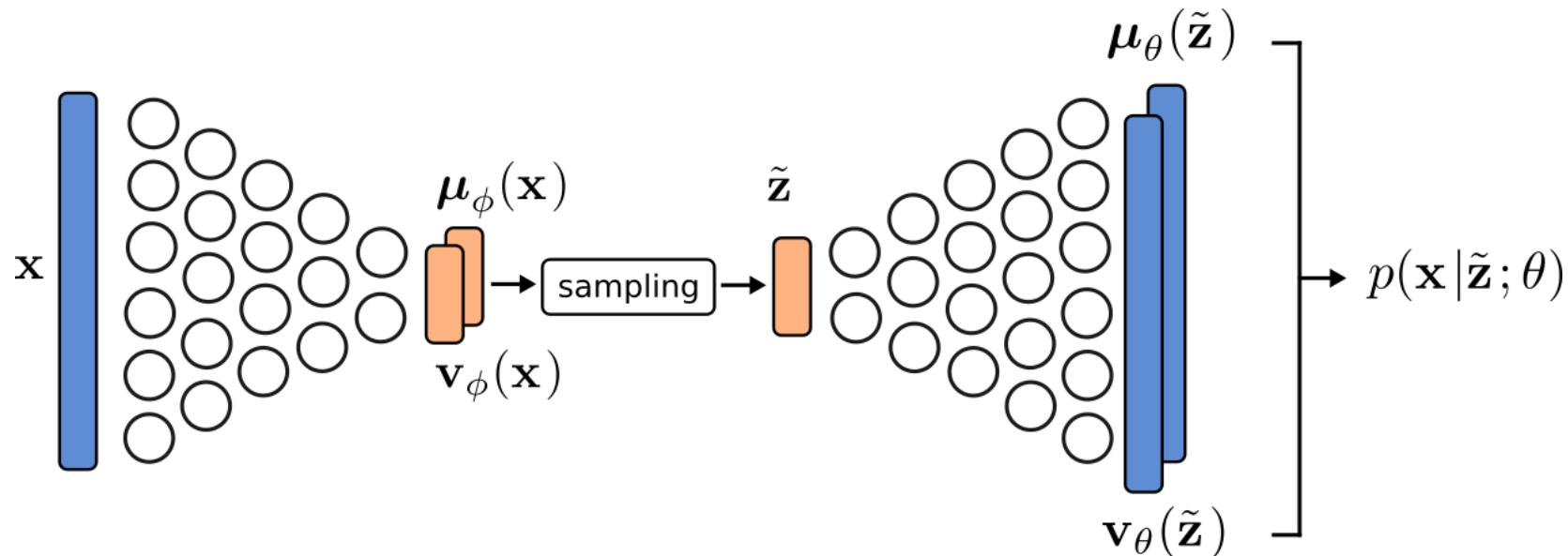
$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$



Training procedure

Step 4: Map through the decoder

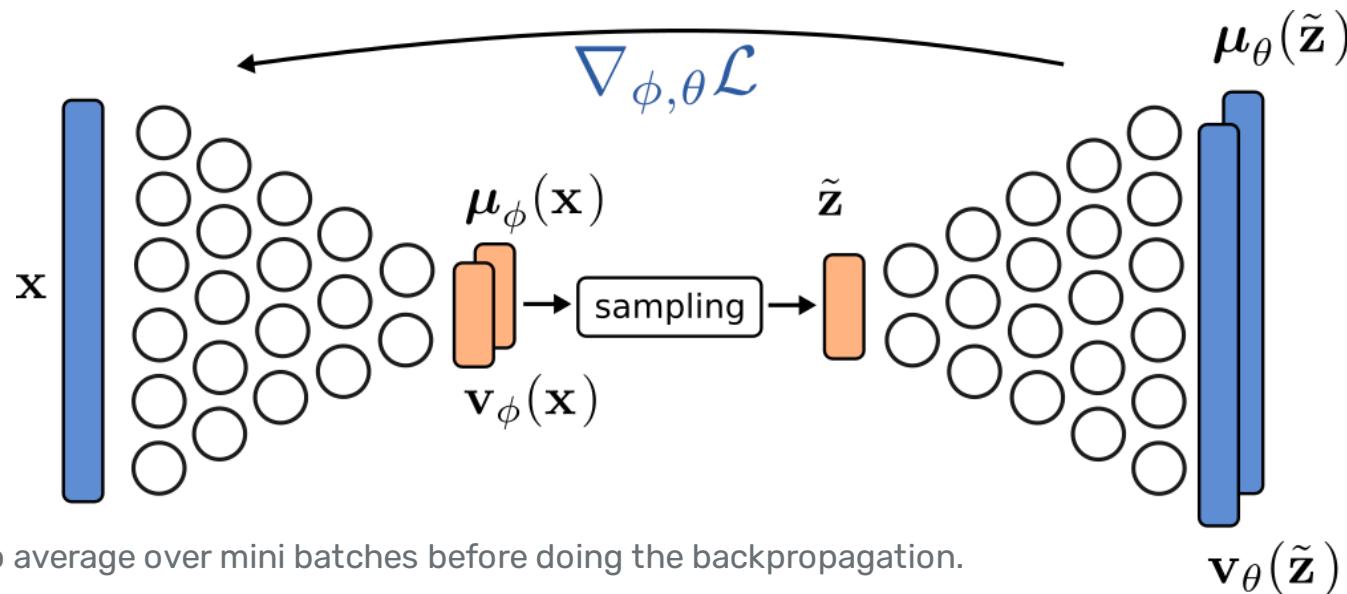
$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$



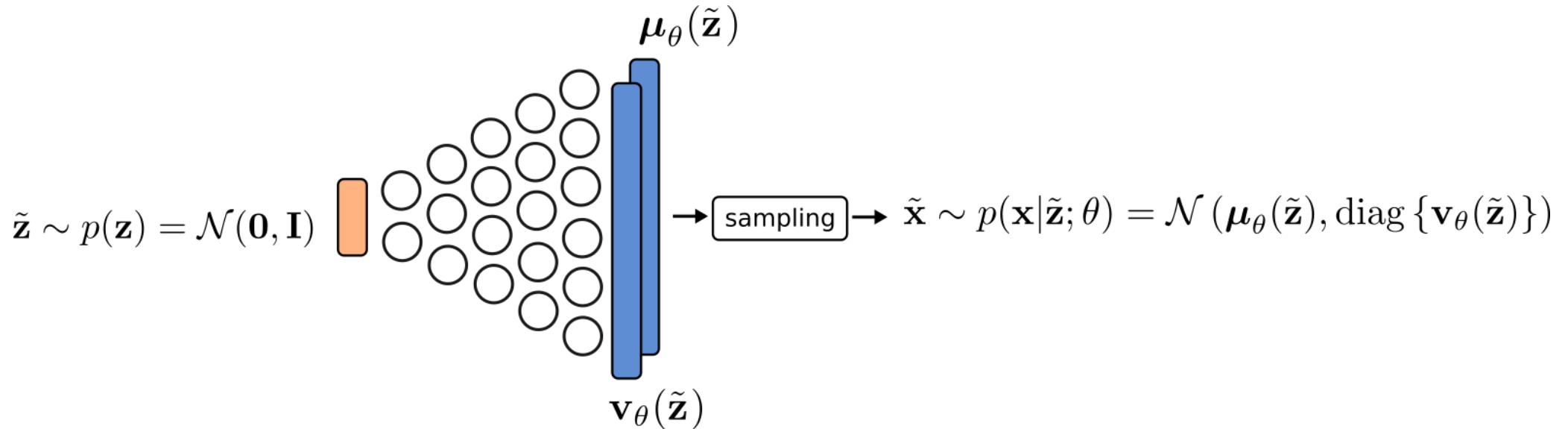
Training procedure

Step 5: Backpropagate and gradient step

$$\begin{aligned}\mathcal{L}(\mathbf{x}_n; \phi, \theta) = & -\frac{1}{2} \sum_{d=1}^D \left[\ln(v_{d,\theta}(\tilde{\mathbf{z}}_n)) + \frac{(x_{n,d} - \mu_{d,\theta}(\tilde{\mathbf{z}}_n))^2}{v_{d,\theta}(\tilde{\mathbf{z}}_n)} \right] \\ & + \frac{1}{2} \sum_{k=1}^K \left[\ln v_{k,\phi}(\mathbf{x}_n) - \mu_{k,\phi}^2(\mathbf{x}_n) - v_{k,\phi}(\mathbf{x}_n) \right] + cst(\phi, \theta),\end{aligned}$$



At test time (after training)

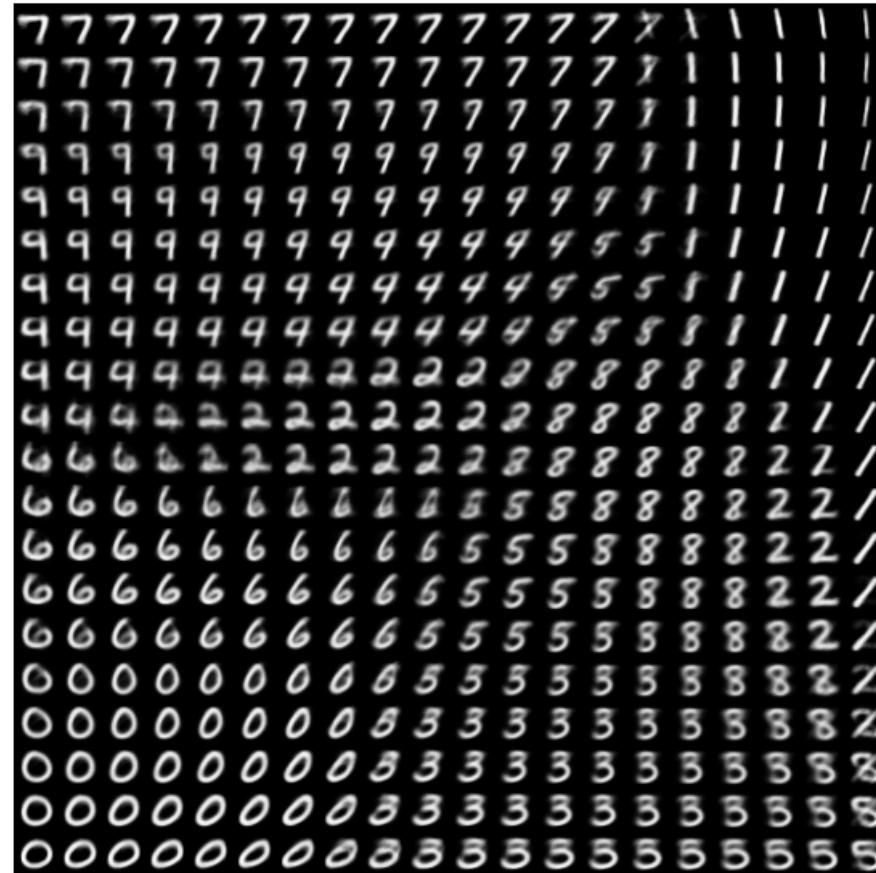


- Note that the encoder was only introduced in order to estimate the parameters of the decoder.
- We do not need the encoder for generating new samples.
- But it is useful if we need to do inference.

Generated MNIST samples



VAE with $K = 16$



Linearly spaced coordinates on the unit square are transformed through the inverse cumulative distribution function of the Gaussian to produce values of the latent variables which are then mapped through the decoder network.

Practical session