BAYESIAN INFERENCE

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THANKS

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Research and development at Amazon

FAU FRIEDRICH-ALEXANDER UNIVERSITÄT ERLANGEN-NÜRNBERG

LMS

EARS
Embodied Audition for RobotS

SEVENTH FRAMEWORK PROGRAMME

OUTLINE

- What is Bayesian inference?
  - Overview
  - Classical vs. Bayesian approach
  - Bayes Theorem & Example
  - General Methodology
- Bayesian inference by examples
  - Direct inference
  - The Expectation-Maximization algorithm
  - Variational Bayes methods
- Markov chain Monte Carlo

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What is Bayesian Inference?

Inference

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Ingredients Observations Model

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Goals

Estimation Quantitative deductions on causes or consequences of the observations, i.e., find underlying model parameters.

Example: *I observed a certain amount of rain drops forming on my window in the last minute. What is the current rainfall in milimeters?*

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Example: *I observed a certain amount of rain drops forming on my window in the last minute. What is the current rainfall in milimeters?*

Prediction From the infered model, predict what missing or future observations should be.

Example: *How many more raindrops will form on my window in the next hour?*

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Example: *I observed a certain amount of rain drops forming on my window in the last minute. What is the current rainfall in milimeters?*

Prediction From the infered model, predict what missing or future observations should be.

Example: *How many more raindrops will form on my window in the next hour?*

Decision Take a decision out of a discrete set of choices

Example: *Is it safe to open my window 1 minute to get some fresh air?*

What is Bayesian Inference?

Statistics: Inference from the real world observations of a random phenomenon using probability theory

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Classical/Frequentist Statistics	Bayesian Statistics

Definition

Features

Tools

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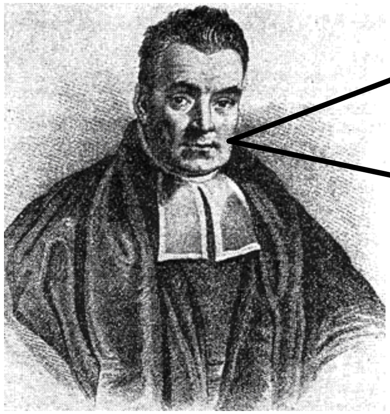
Tools	• Linear estimators • First and second order statistics • A lot of: $\mathbb{E}\{\dots\}$	• **Bayes' Theorem** • Explicit PDFs • A lot of: $\mathcal{N}(\dots)$

What is Bayesian Inference?

- Bayes' Theorem



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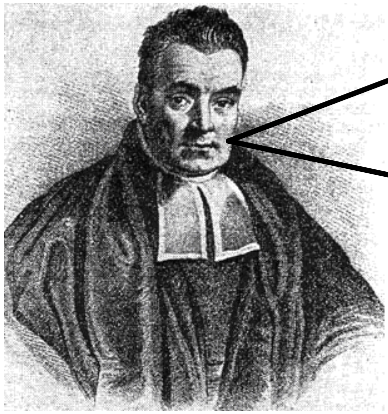

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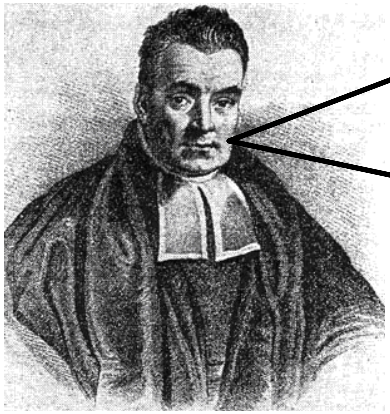

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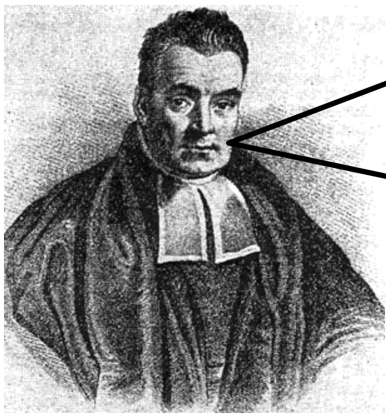

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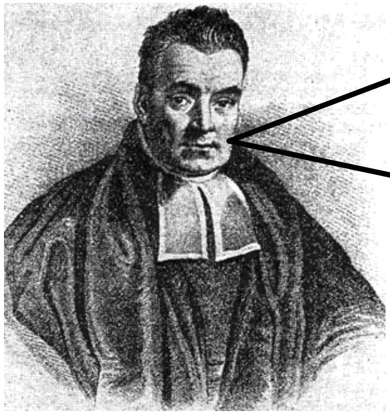

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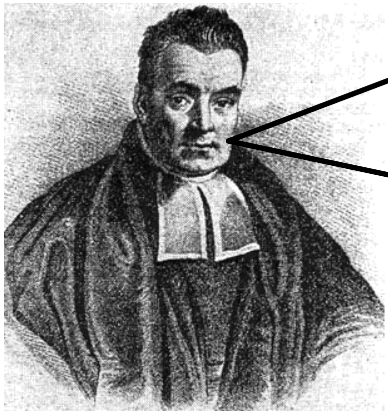

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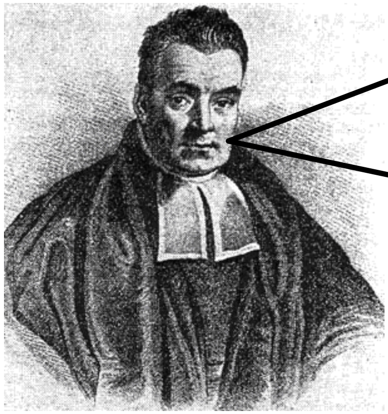

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Remark 1: ***Bayes does not « forbid » model parameters !***

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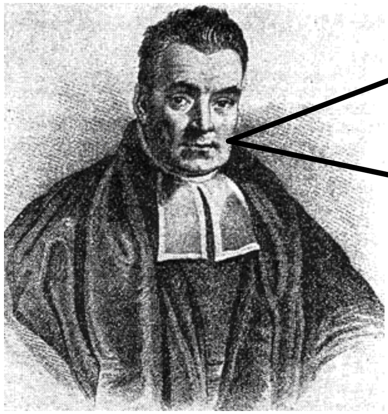

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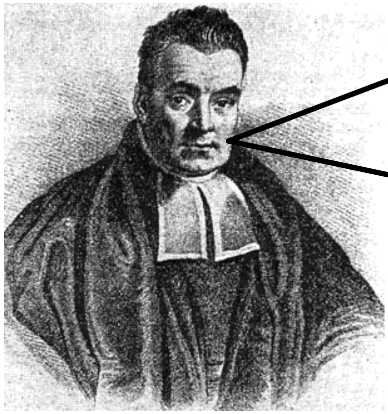

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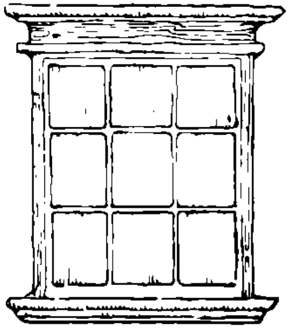
Remark 2: ***Why hidden variables?***
- Formally not needed: $p(\boldsymbol{X})$ can be obtained by marginalizing out hidden variables
- A convenient and powerful **view point** which makes inference possible in complex scenarios through a variety of methods

What is Bayesian Inference?

An Example:

Observations

X_{D}	X_{C}
I see drops on my window	I see a cat or a dog at my window



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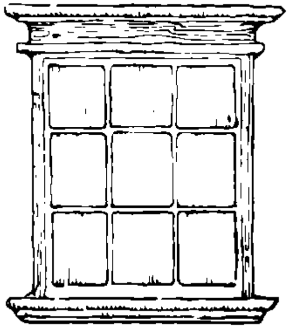

Hypothesis

$Z = 0$	$Z = 1$	$Z = 2$
It's not raining	It's raining rain	It's raining cats and dogs

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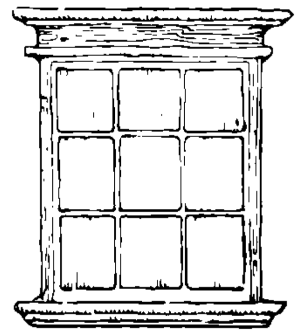

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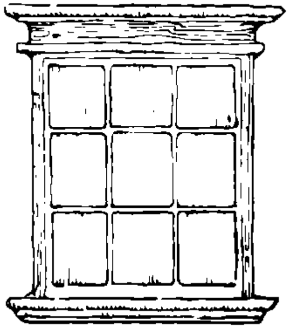

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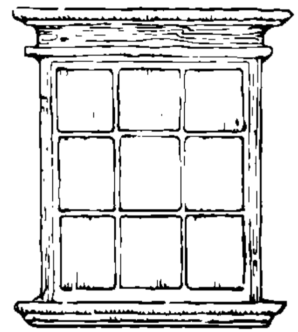

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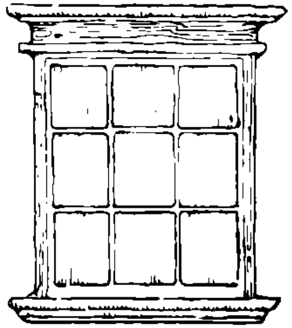

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- **Add priors**: $p(Z=0)=49.995\%$ $p(Z=1)=49.995\%$ $p(Z=2)=0.01\%$
- **Bayes’ theorem:**

$$p(Z=1|X_{\mathrm{D}},X_{\mathrm{C}})\approx 99.98\% \quad p(Z=2|X_{\mathrm{D}},X_{\mathrm{C}})\approx 0.02\%$$

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General Methodology

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Modeling

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- What is hidden?
- What is observed?
- Dependencies?

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Graphical model

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Graphical model + Choice of prior and conditional PDFs

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=

Joint PDF

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Apply Baye's Theorem

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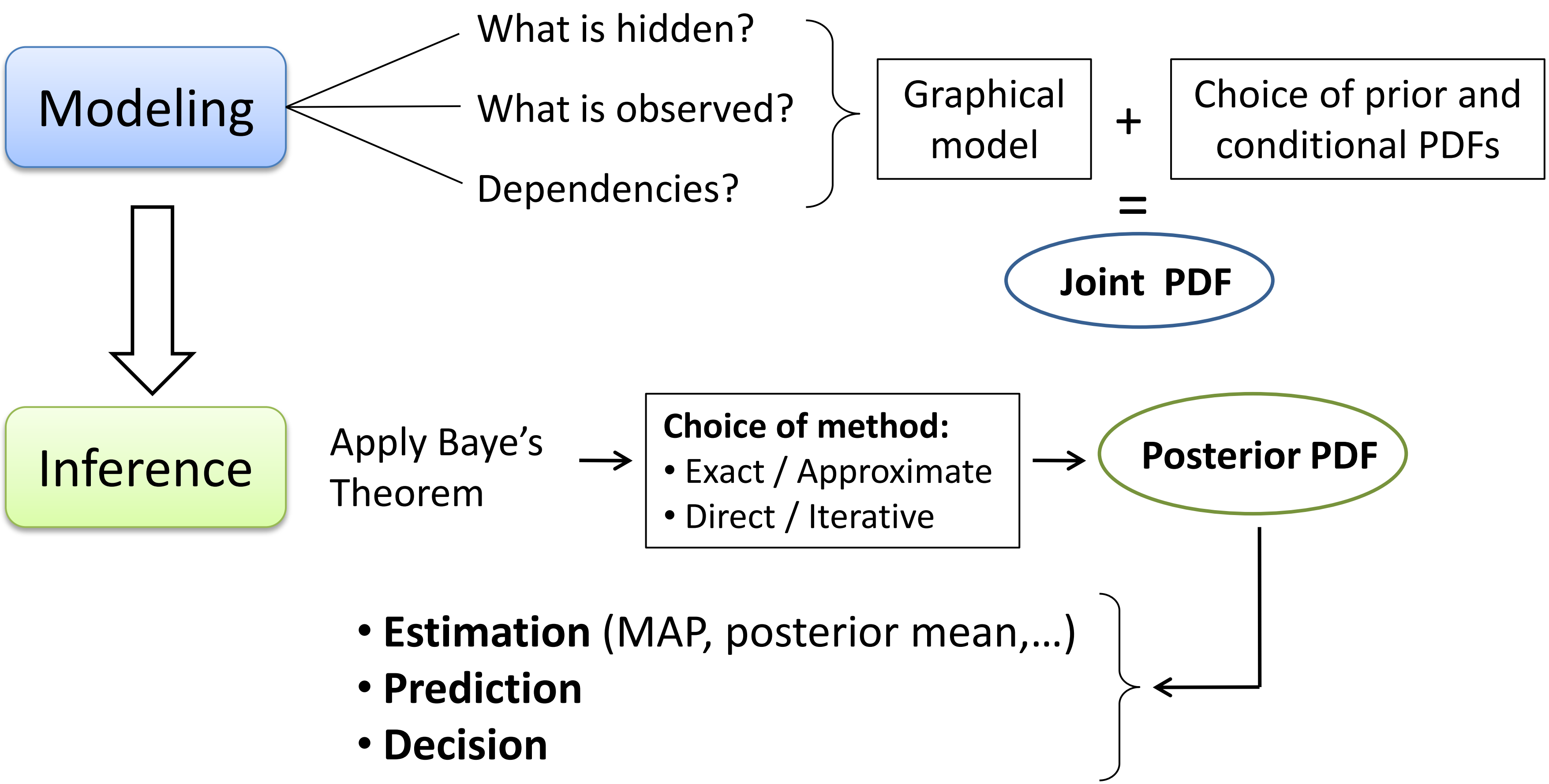
→ **Posterior PDF**

General Methodology



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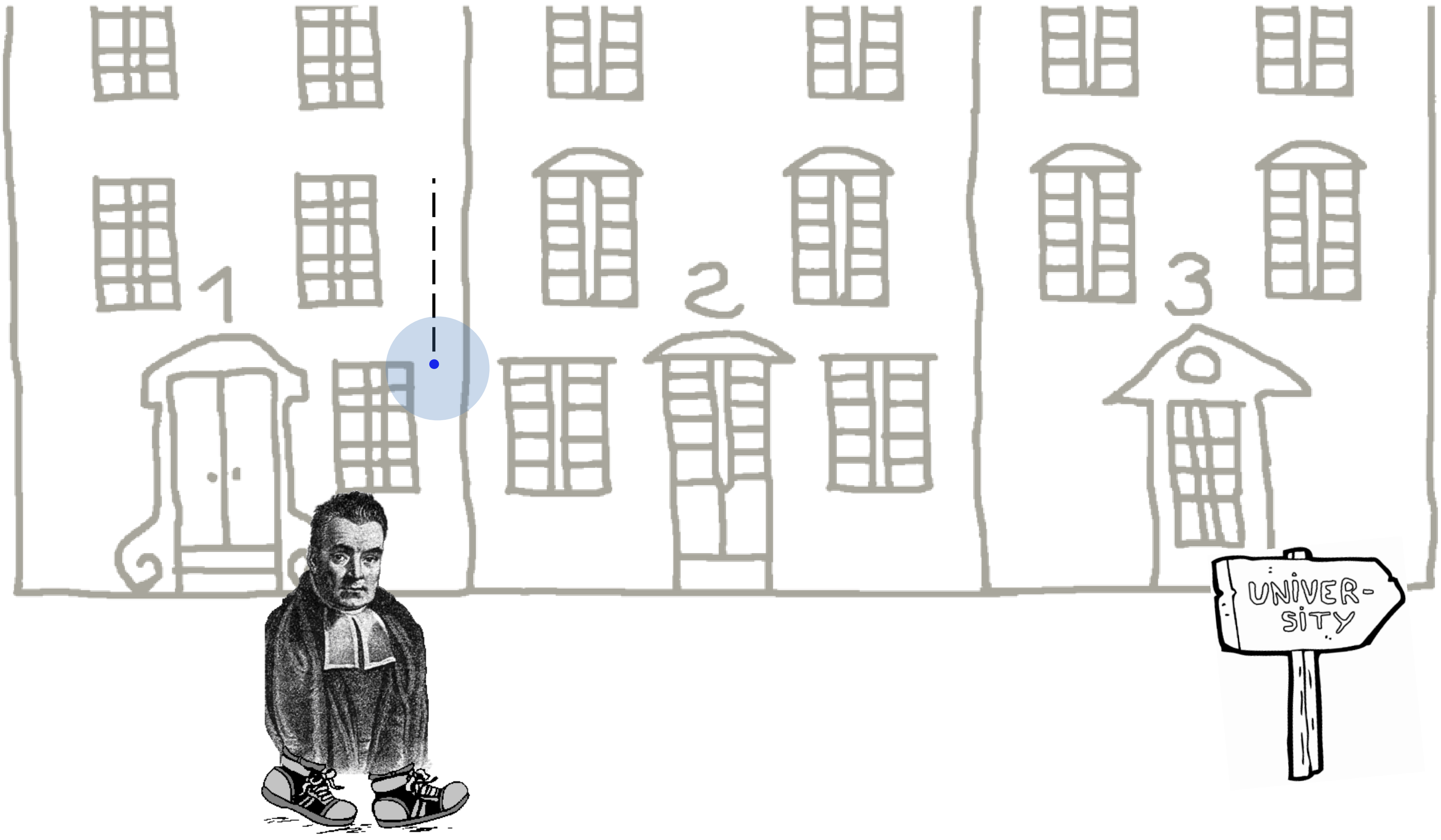

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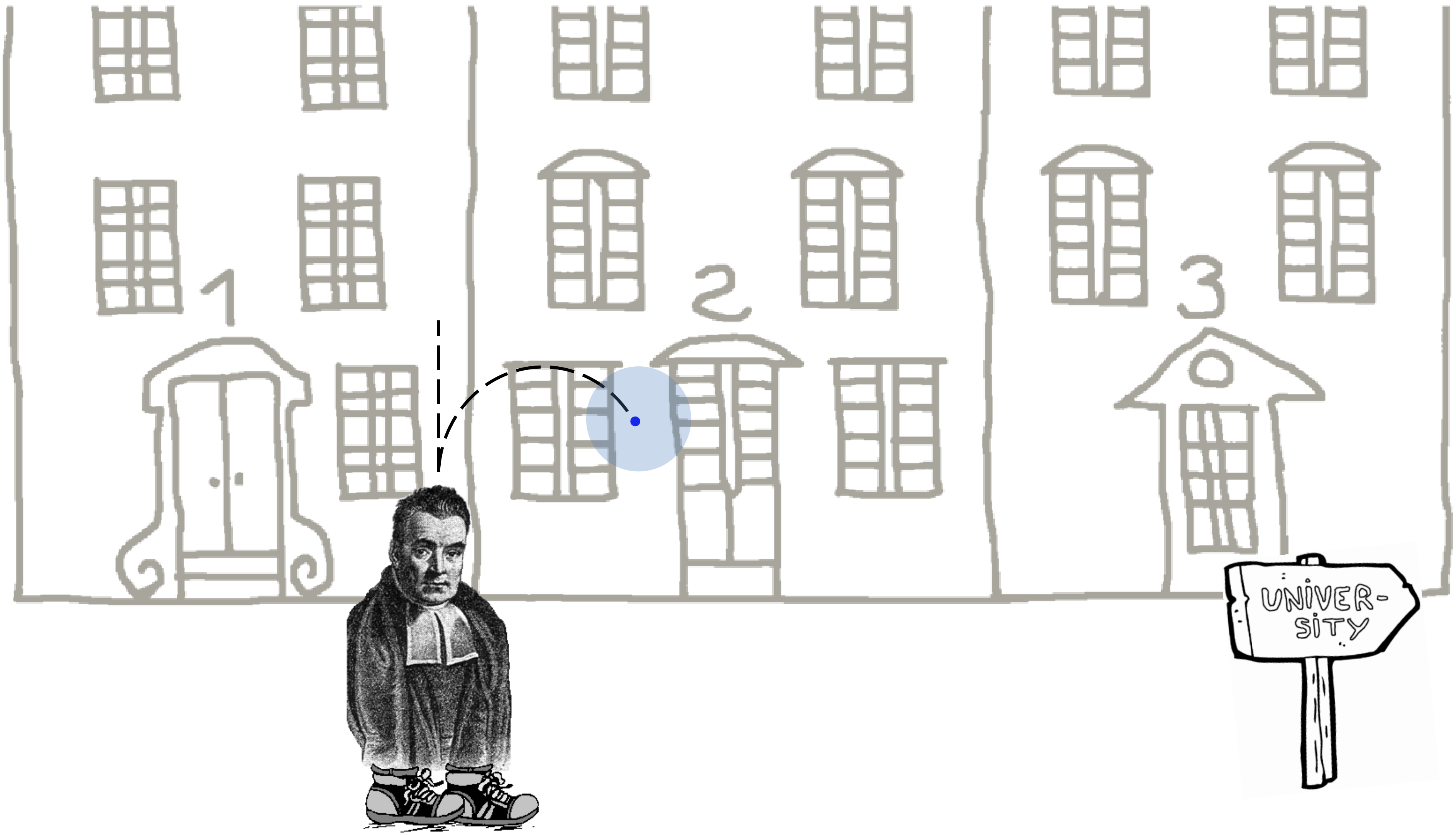

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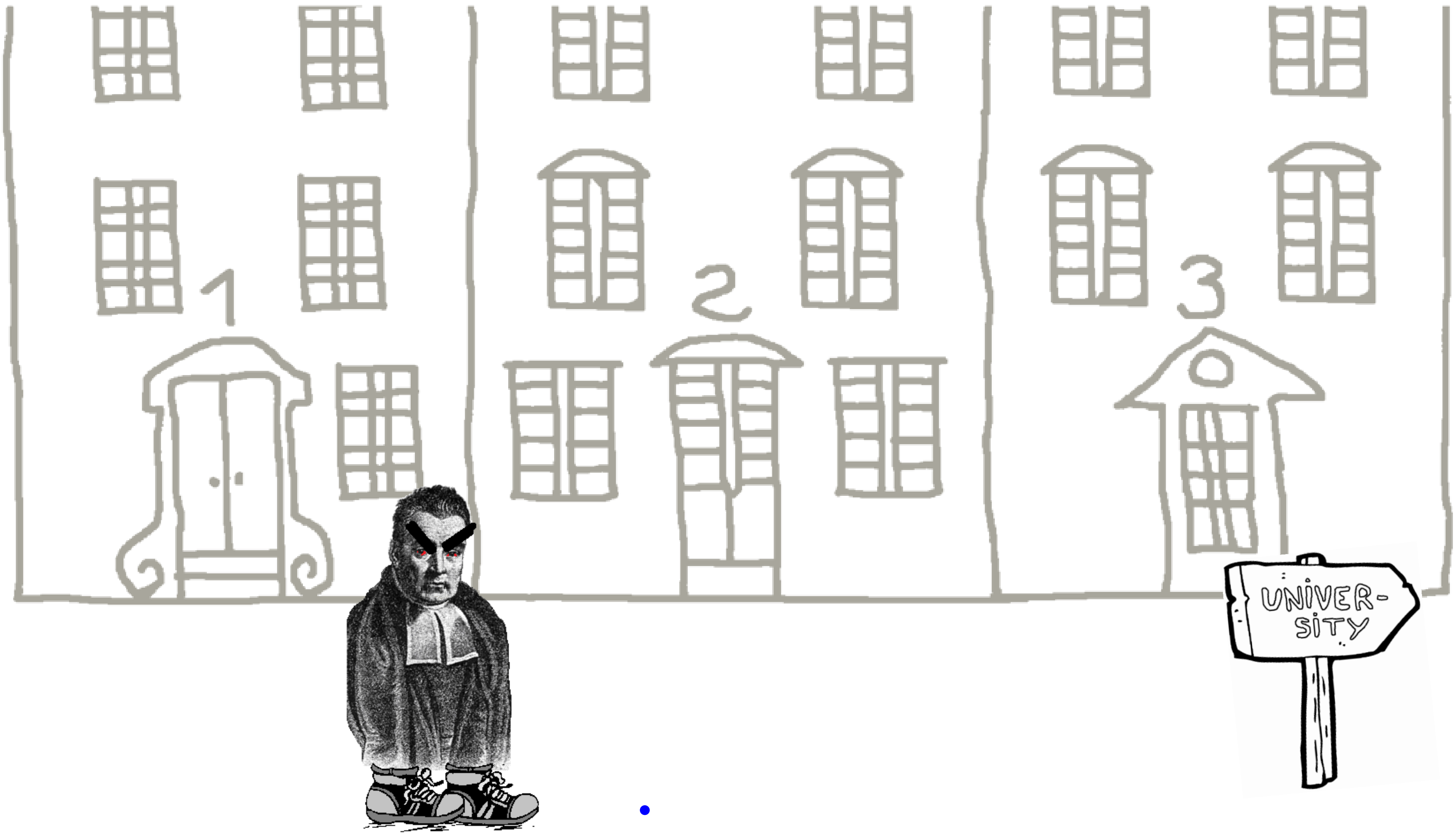

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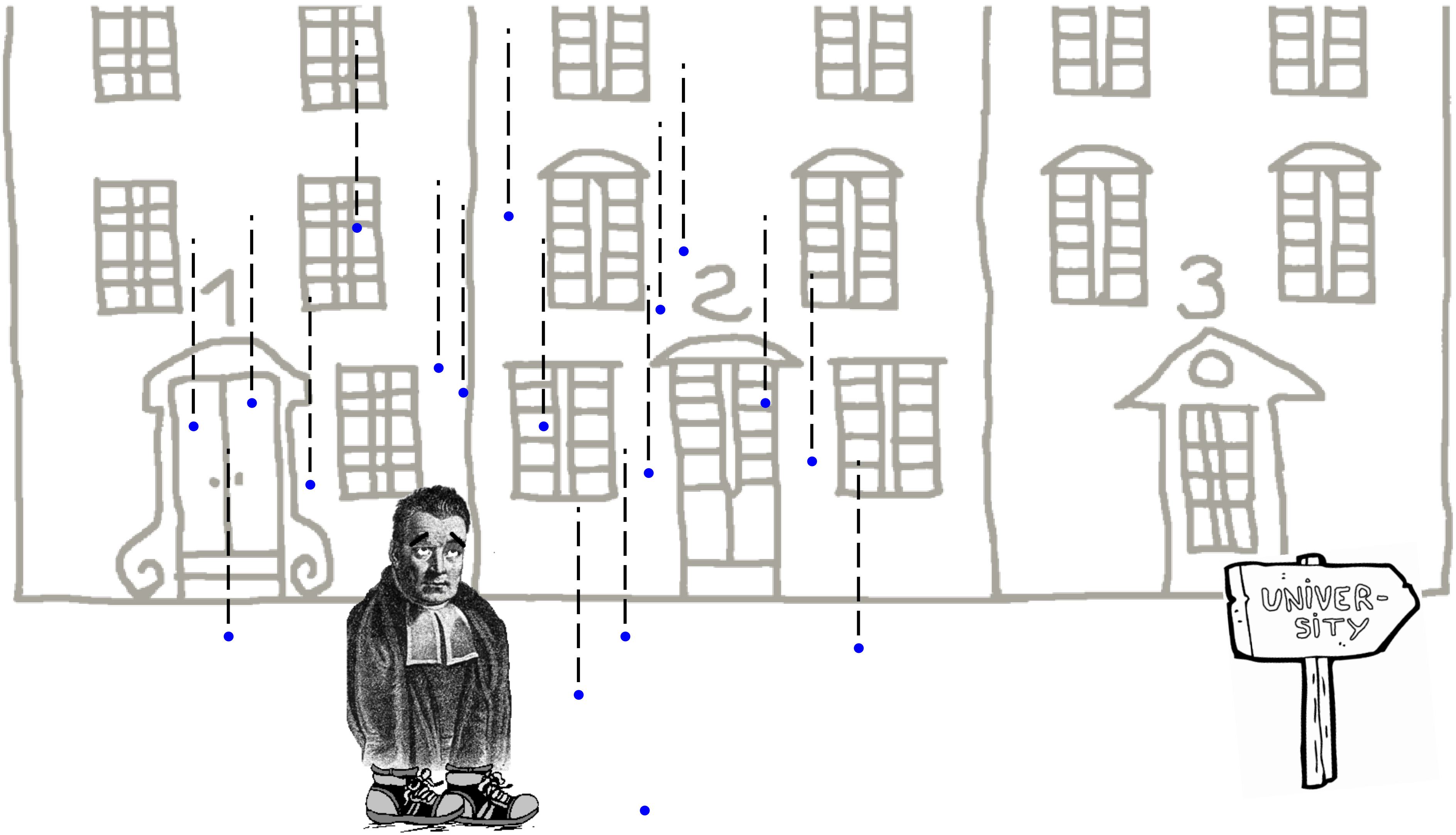

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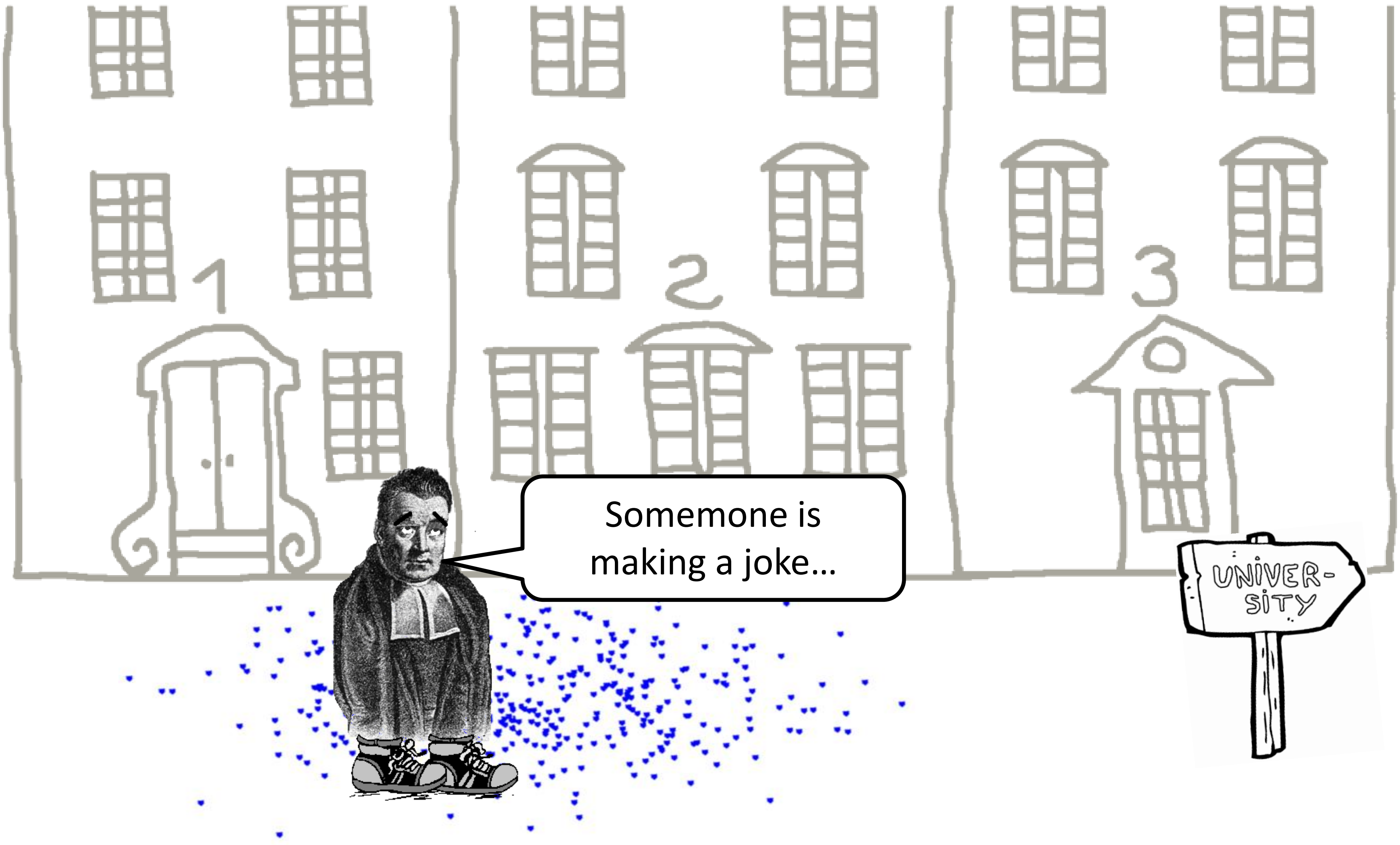

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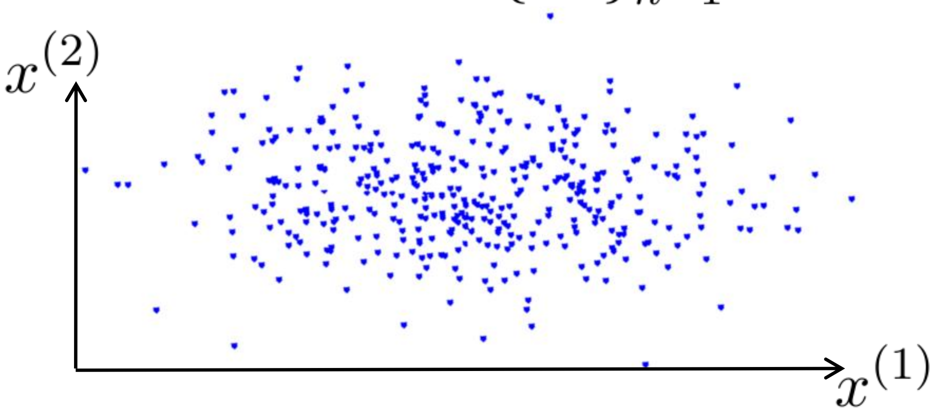
Observed variables: $\{\boldsymbol{x}_n\}_{n=1}^{N} \subset \mathbb{R}^2$



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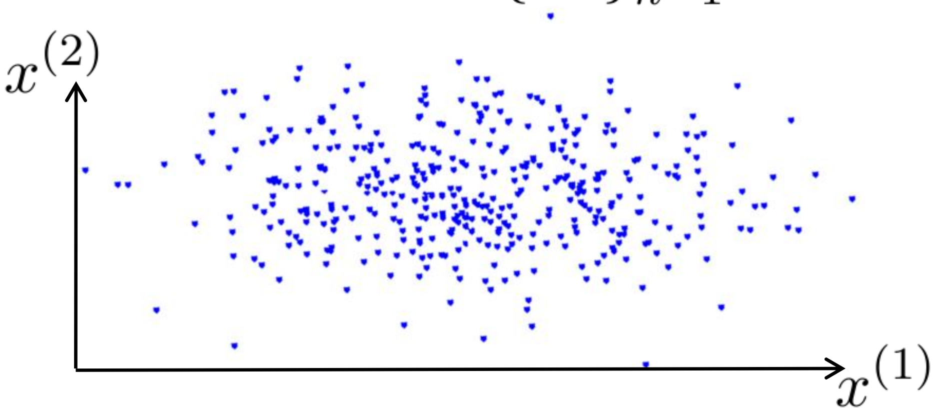

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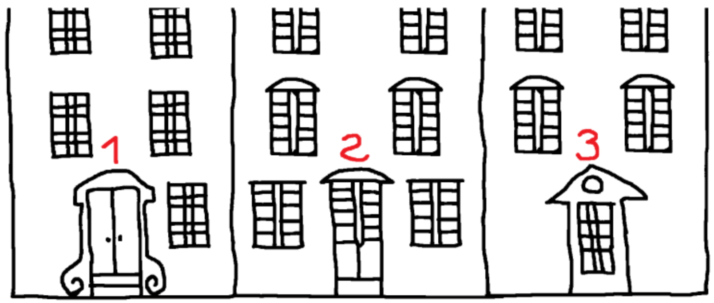

Guilty house number?

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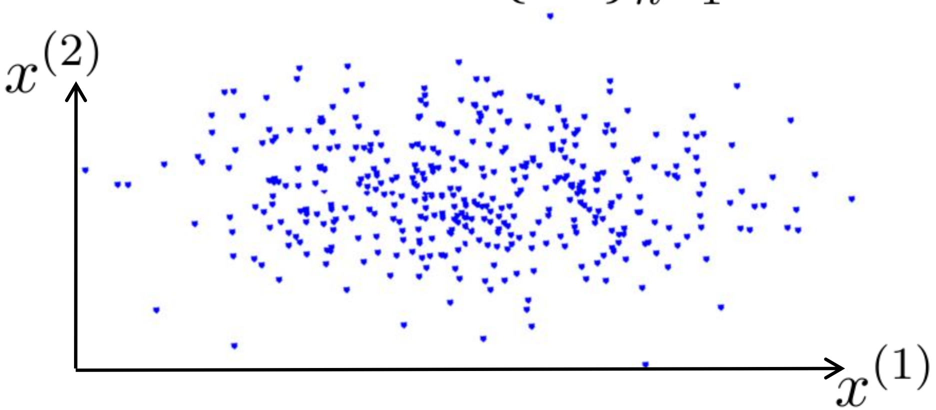

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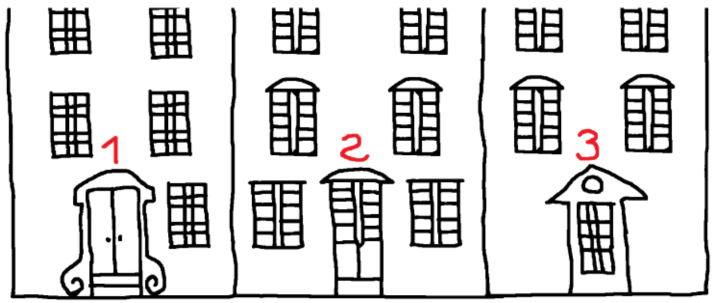

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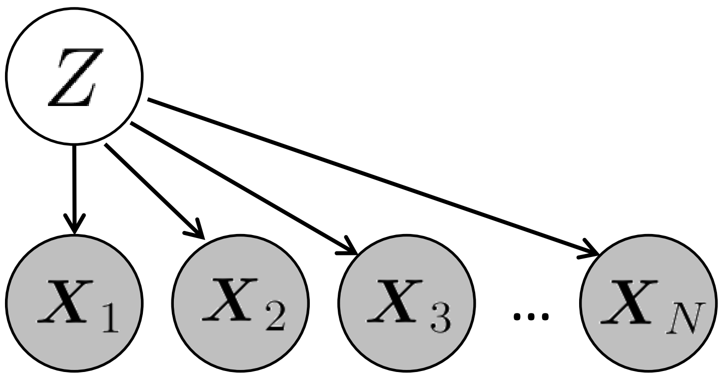

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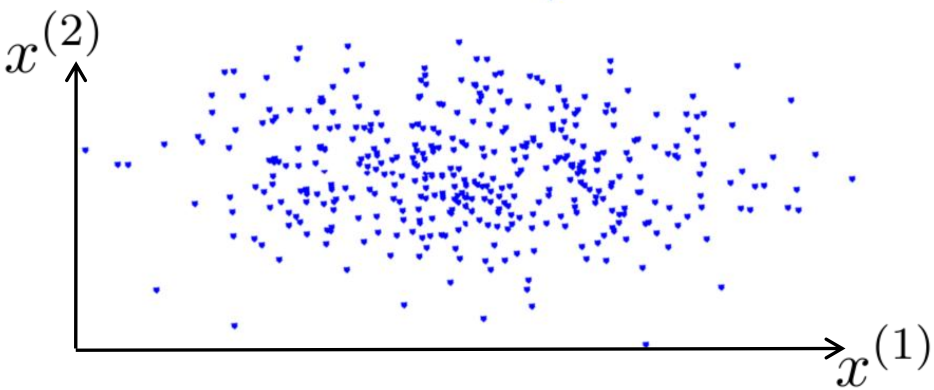

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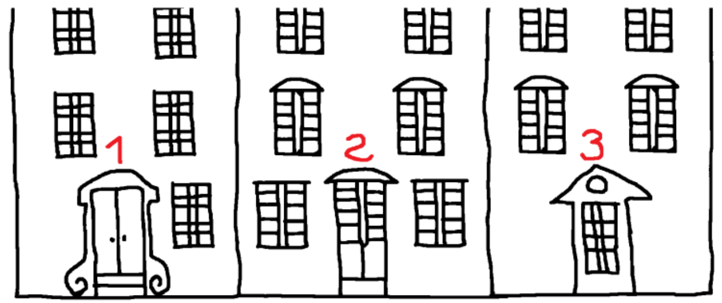

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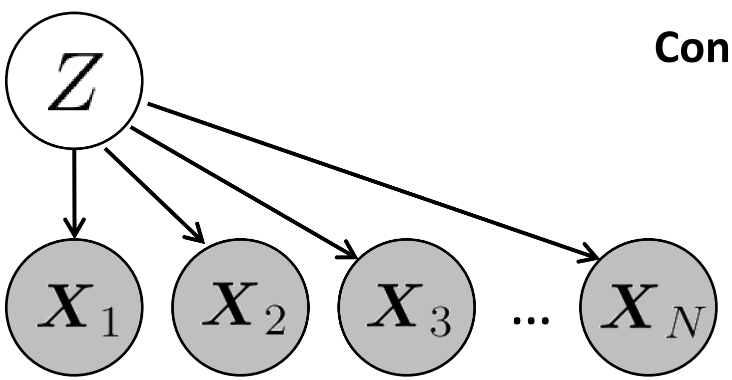


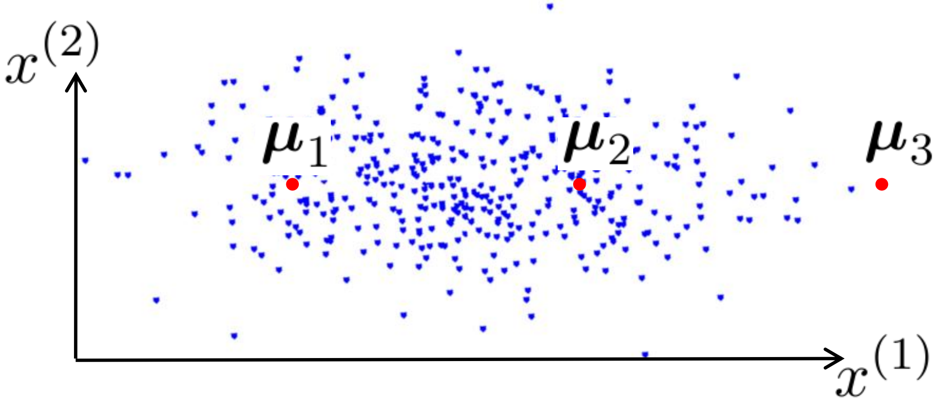
Conditionals

$$\begin{cases} p(\boldsymbol{X}_n = \boldsymbol{x}_n | Z = 1) = \mathcal{N}(\boldsymbol{x}_n; \boldsymbol{\mu}_1, \mathbf{I}) \\ p(\boldsymbol{X}_n = \boldsymbol{x}_n | Z = 2) = \mathcal{N}(\boldsymbol{x}_n; \boldsymbol{\mu}_2, \mathbf{I}) \\ p(\boldsymbol{X}_n = \boldsymbol{x}_n | Z = 3) = \mathcal{N}(\boldsymbol{x}_n; \boldsymbol{\mu}_3, \mathbf{I}) \end{cases}$$

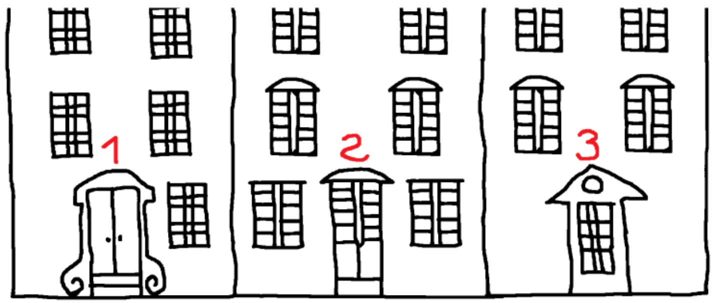
Bayesian Inference: Examples

Modeling

Observed variables: $\{\boldsymbol{x}_n\}_{n=1}^{N} \subset \mathbb{R}^2$

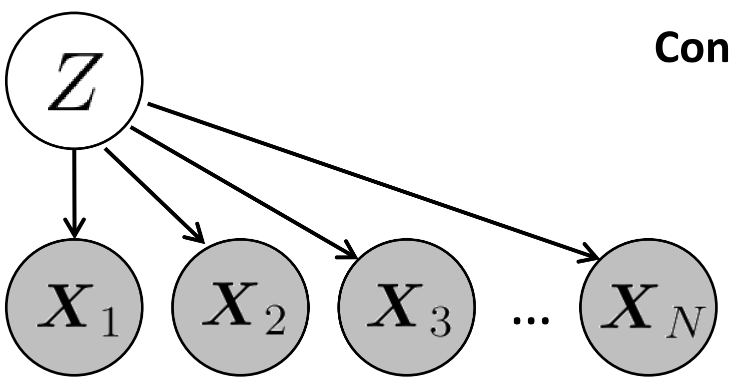


Hidden variable: $Z \in \{1, 2, 3\}$



Guilty house number?

Graphical Model:



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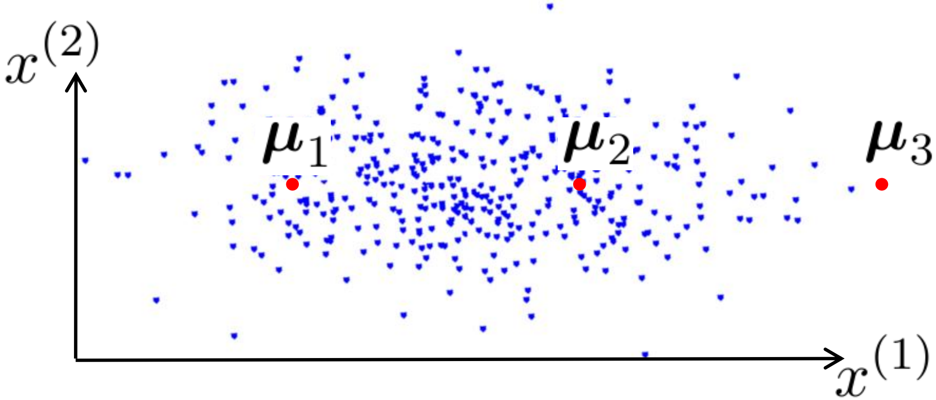

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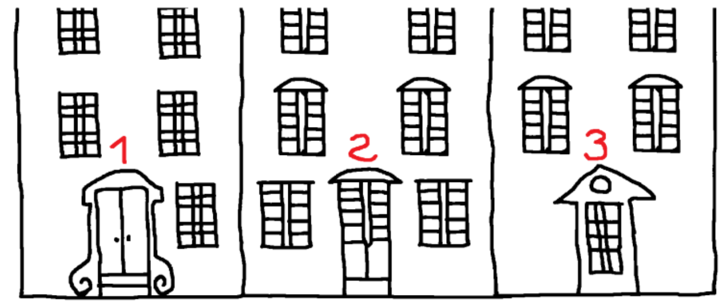

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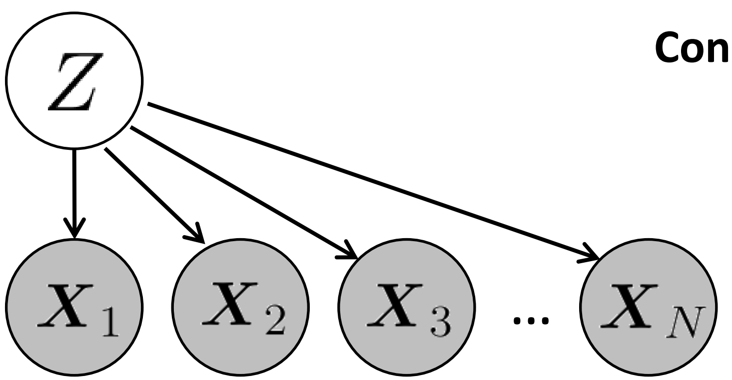

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Priors

$$\begin{cases} p(Z = 1) = 0.1 & \text{(Grandma Jane)} \\ p(Z = 2) = 0.3 & \text{(Student house)} \\ p(Z = 3) = 0.6 & \text{(Family with kids)} \end{cases}$$

Bayesian Inference: Examples

Inference

Bayes' Theorem:

$$p(Z = i | \boldsymbol{x}_1, \dots, \boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N | Z = i) p(Z = i)}{p(\boldsymbol{x}_1, \dots, \boldsymbol{x}_N)}$$

Bayesian Inference: Examples

Inference

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⇨ Direct computation

Bayesian Inference: Examples

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$\Rightarrow$ Direct computation

Estimation: Maximum a Posteriori (MAP)

$$\hat{z} = \underset{i}{\operatorname{argmax}}\Big[p(Z=i|\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\Big]$$

Bayesian Inference: Examples

Inference

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$$p(Z=i|\boldsymbol{x}_1,\dots,\boldsymbol{x}_N) = \frac{p(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N|Z=i)p(Z=i)}{p(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)}$$

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Estimation: Maximum a Posteriori (MAP)

$$\hat{z} = \underset{i}{\operatorname{argmax}}\left[p(Z=i|\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\right] \Rightarrow \hat{z} = 2$$, the student house

Bayesian Inference: Examples

Inference

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$\Rightarrow$ Direct computation

Estimation: Maximum a Posteriori (MAP)

$$\hat{z}=\underset{i}{\operatorname{argmax}}\Big[p(Z=i|\boldsymbol{x}_1,\dots,\boldsymbol{x}_N)\Big]$$

$\Rightarrow$ $\hat{z}=2$, the student house

Decision:

These pranksters will hear from me at the Uni!

OUTLINE

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  - Overview
  - Classical vs. Bayesian approach
  - Bayes Theorem & Example
  - General Methodology
- **Bayesian inference by examples**
  - **Direct inference**
  - **The Expectation-Maximization algorithm**
  - **Variational Bayes methods**
- Markov chain Monte Carlo

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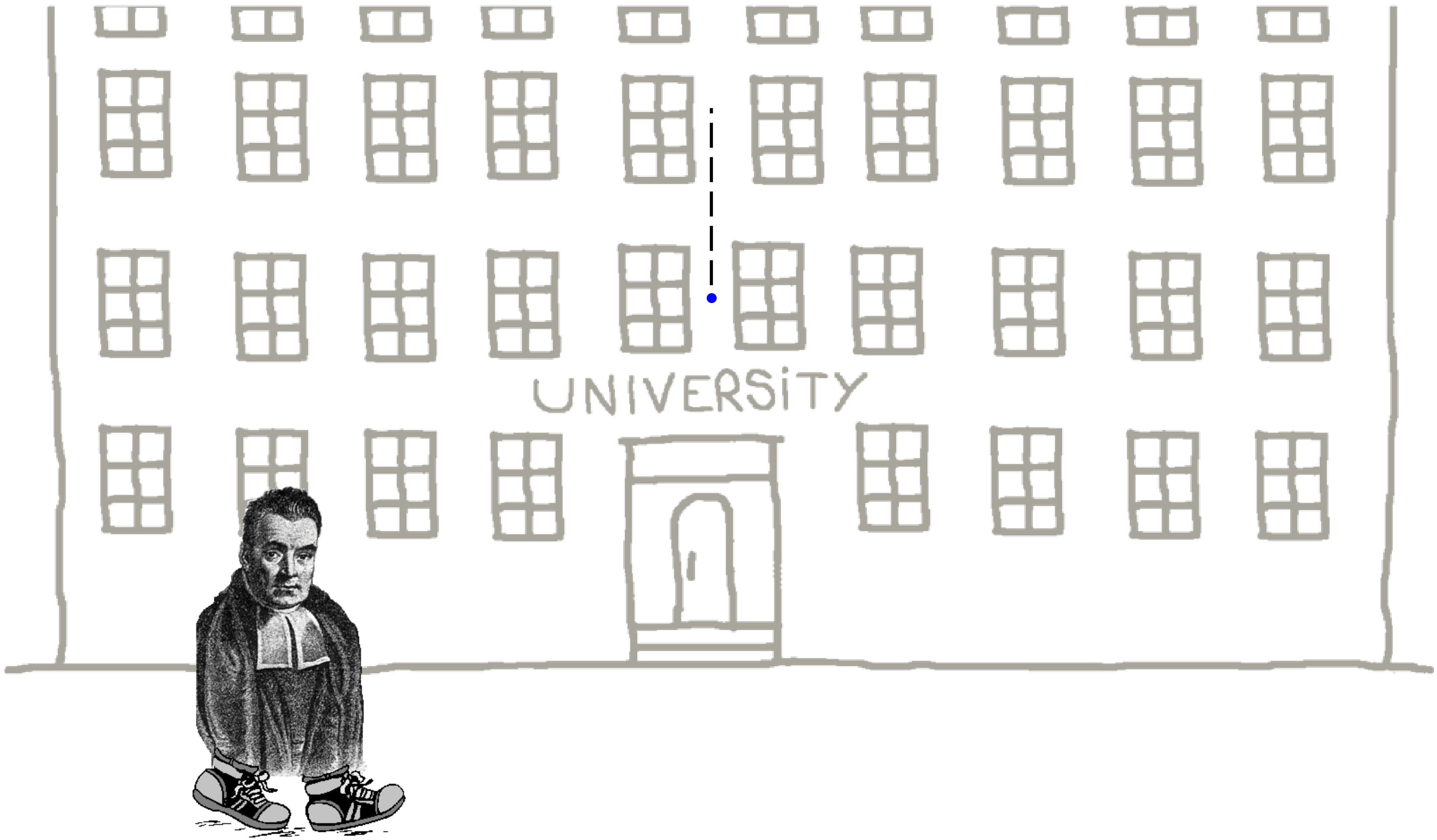
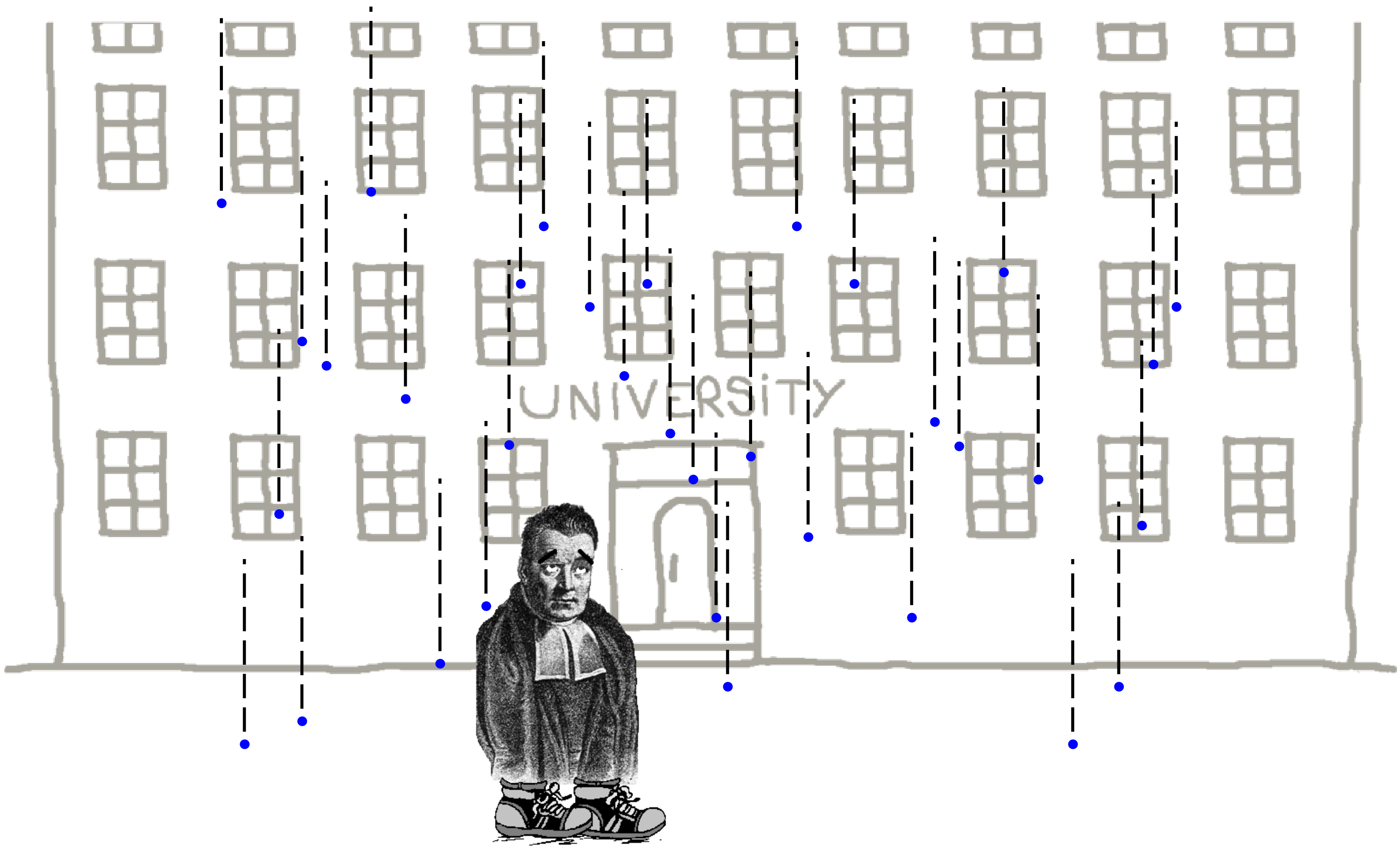
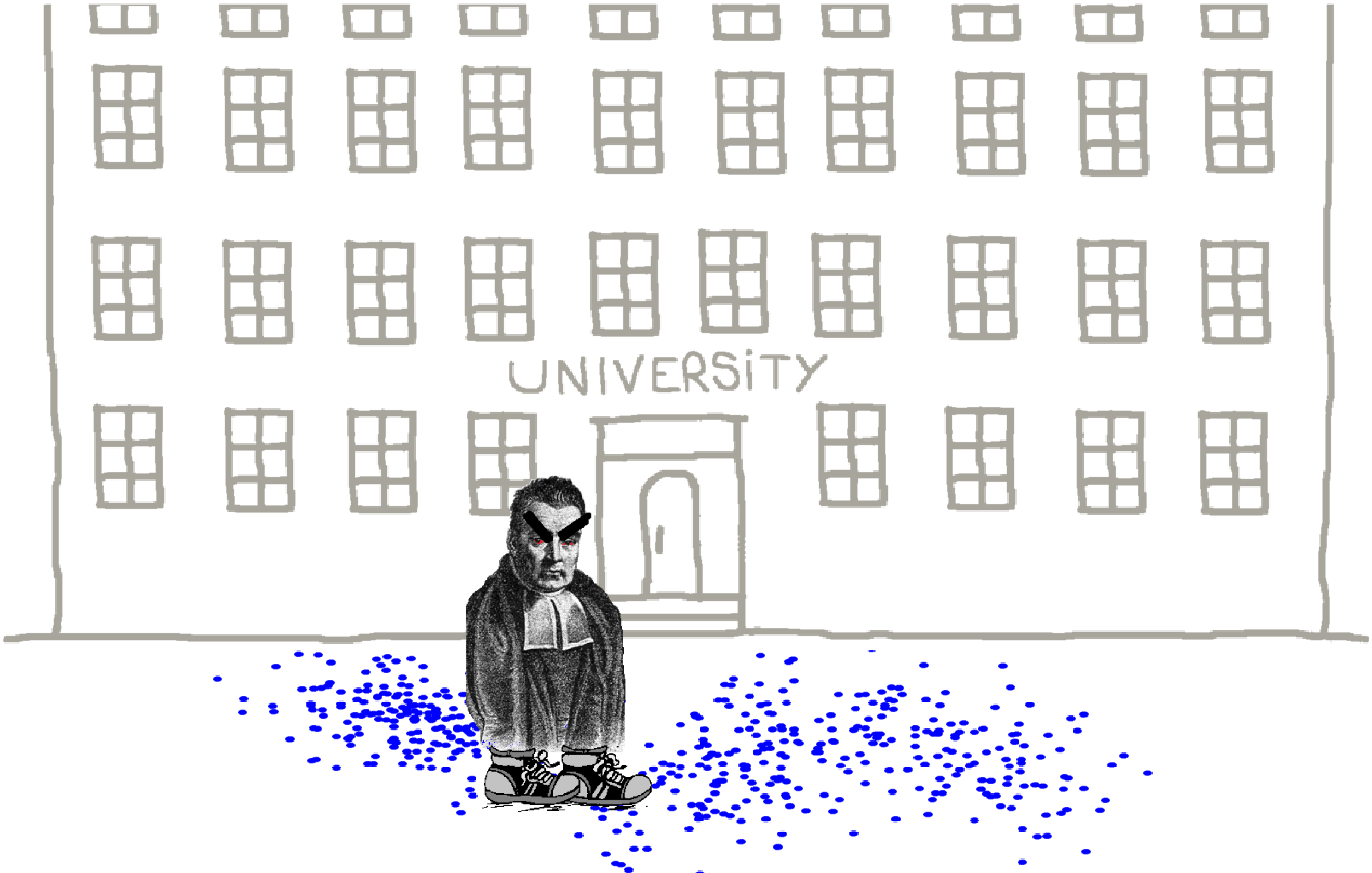
Bayesian Inference: Examples



Bayesian Inference: Examples



Bayesian Inference: Examples



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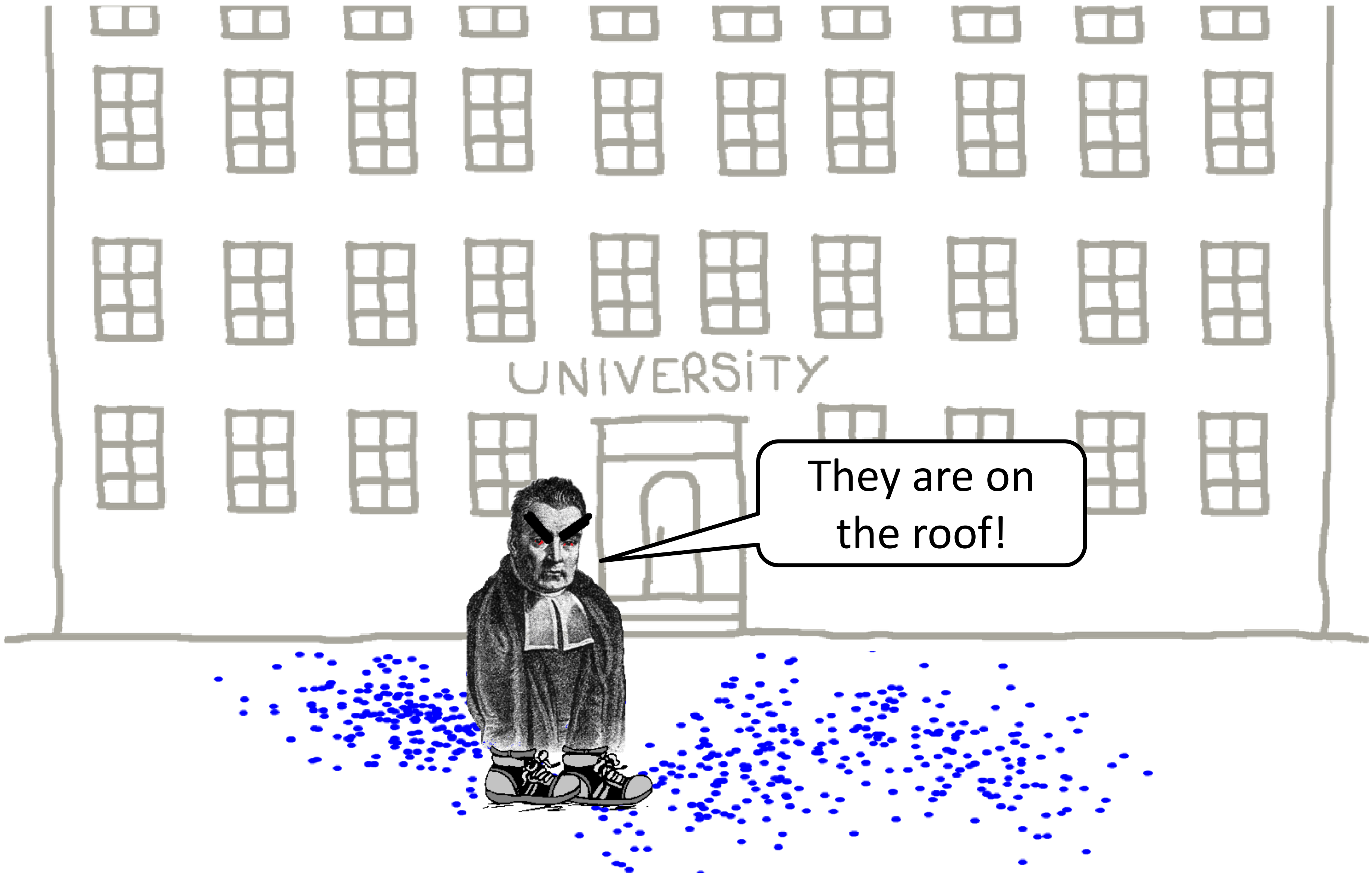


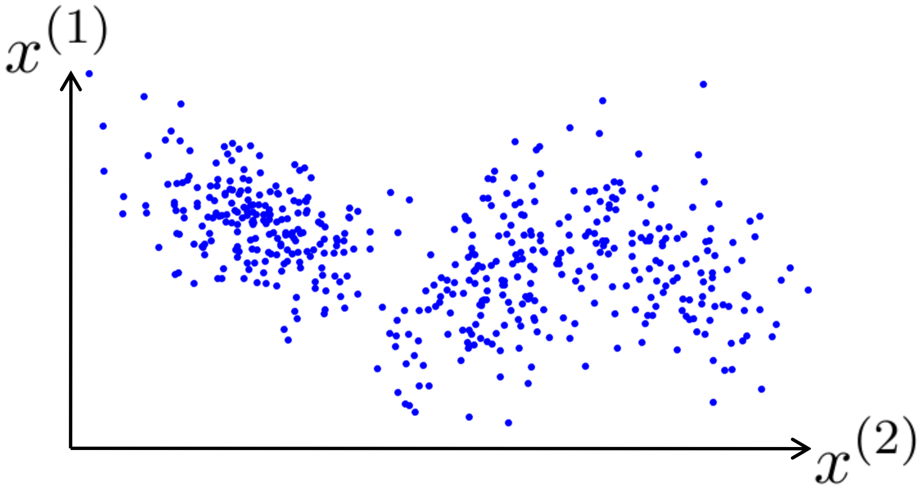
Bayesian Inference: Examples

Modeling

Bayesian Inference: Examples

Modeling

Observed variables: $\{\boldsymbol{x}_n\}_{n=1}^{N} \subset \mathbb{R}^2$



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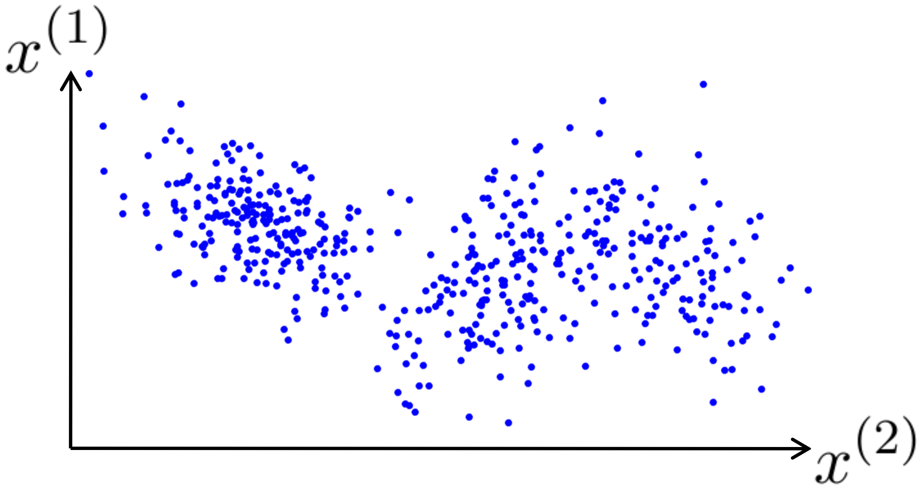

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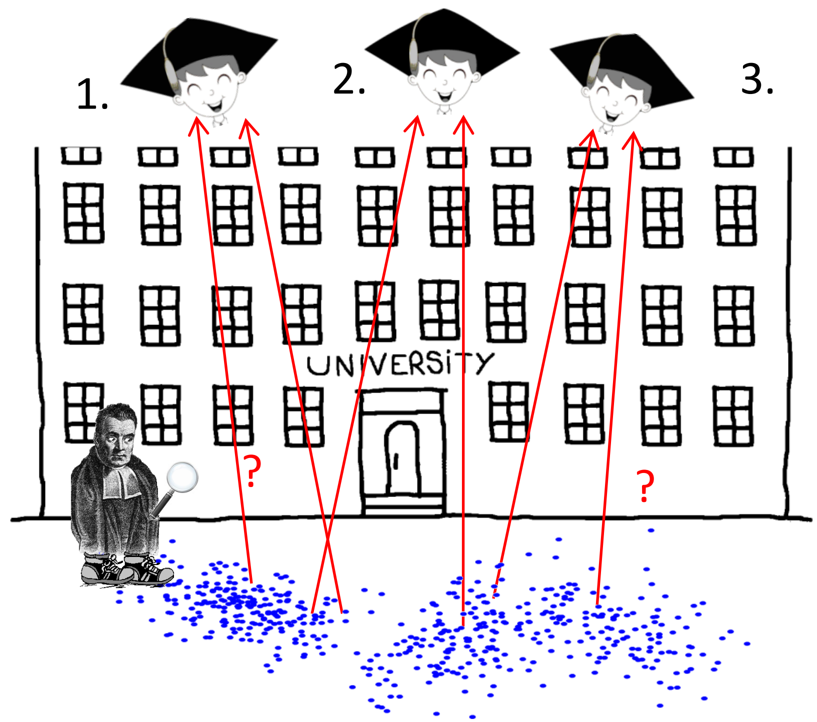

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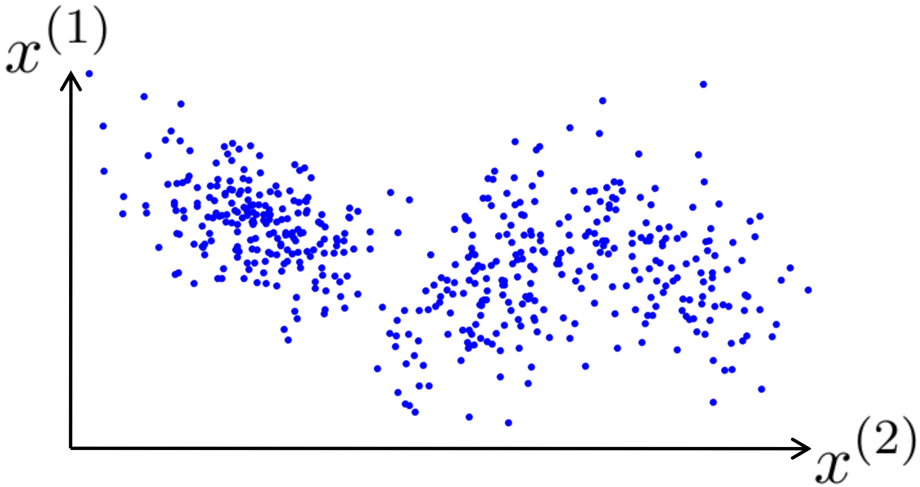

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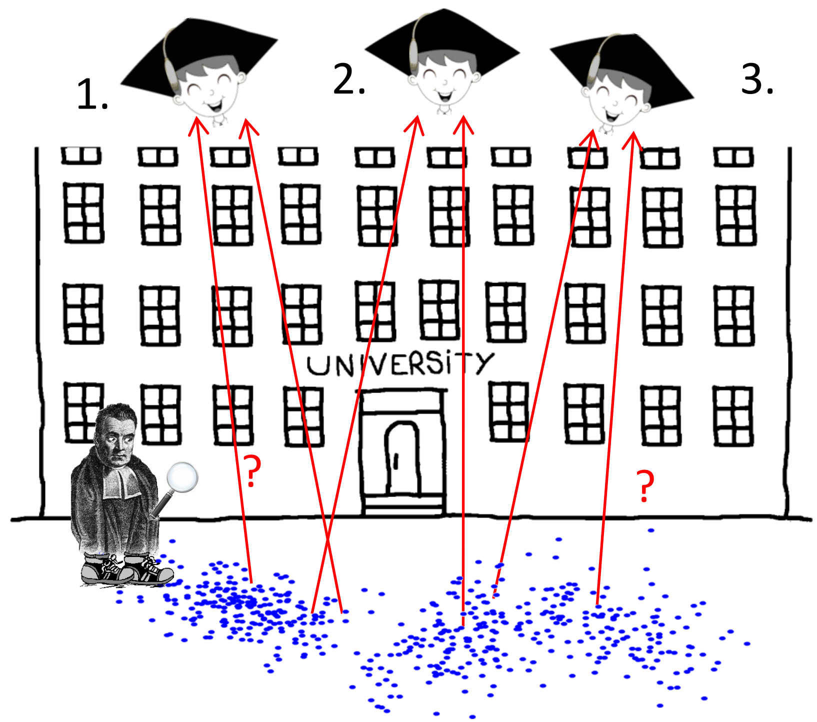
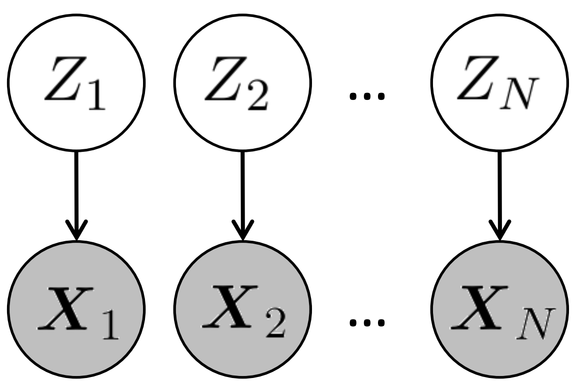


Graphical Model:



Bayesian Inference: Examples

Modeling

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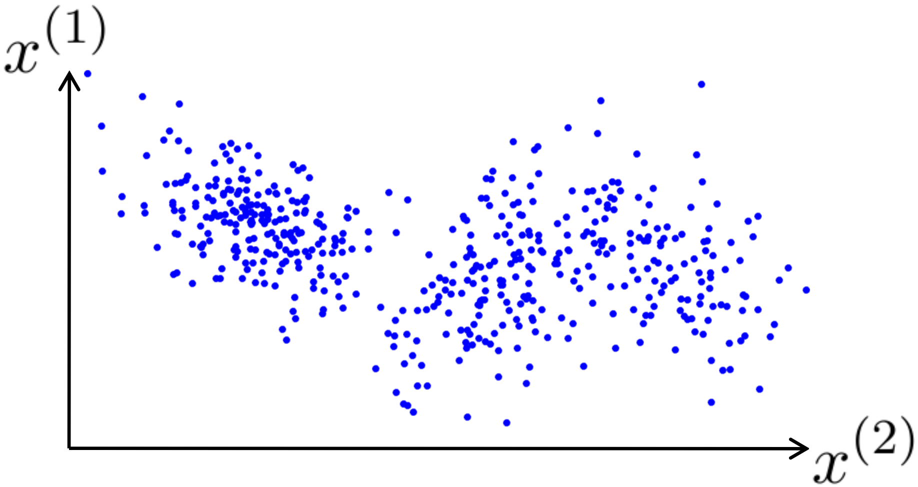

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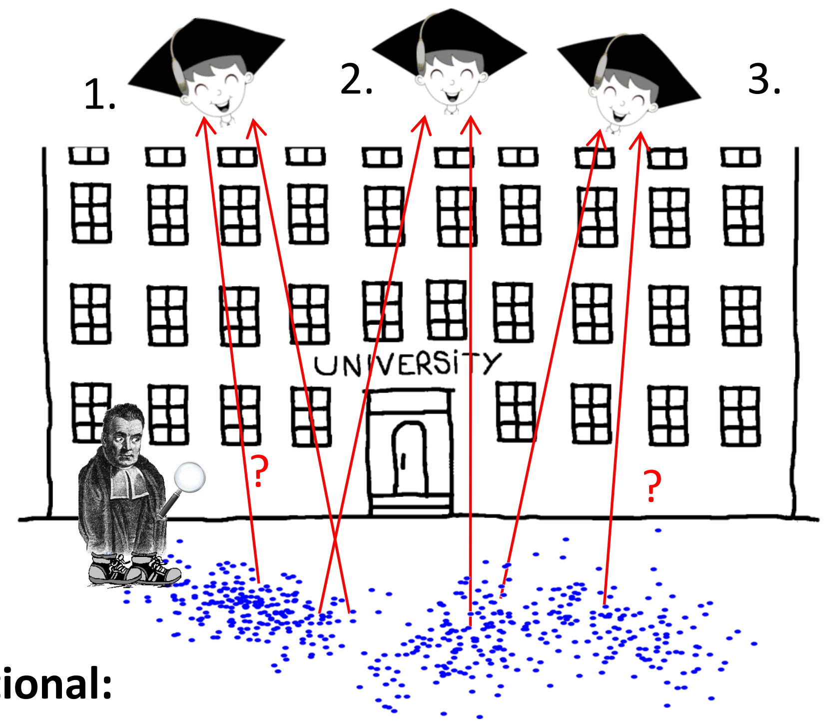

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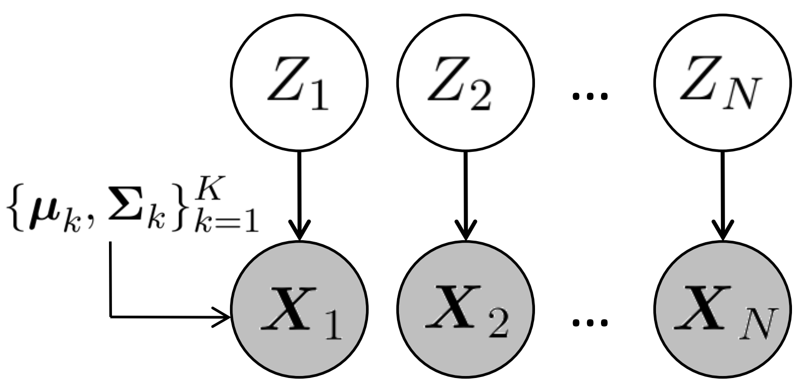

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$$p(\boldsymbol{X}_n = \boldsymbol{x}_n | Z_n = k; \boldsymbol{\theta}) = \mathcal{N}(\boldsymbol{x}_n; \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

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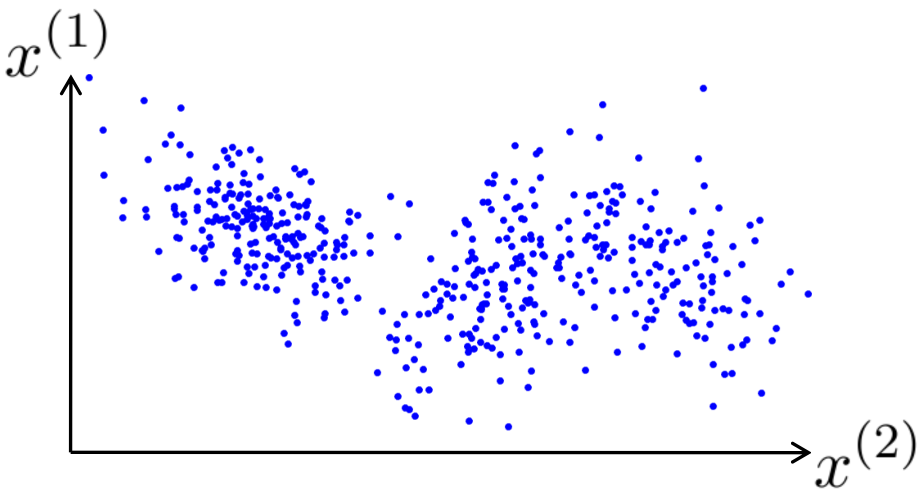

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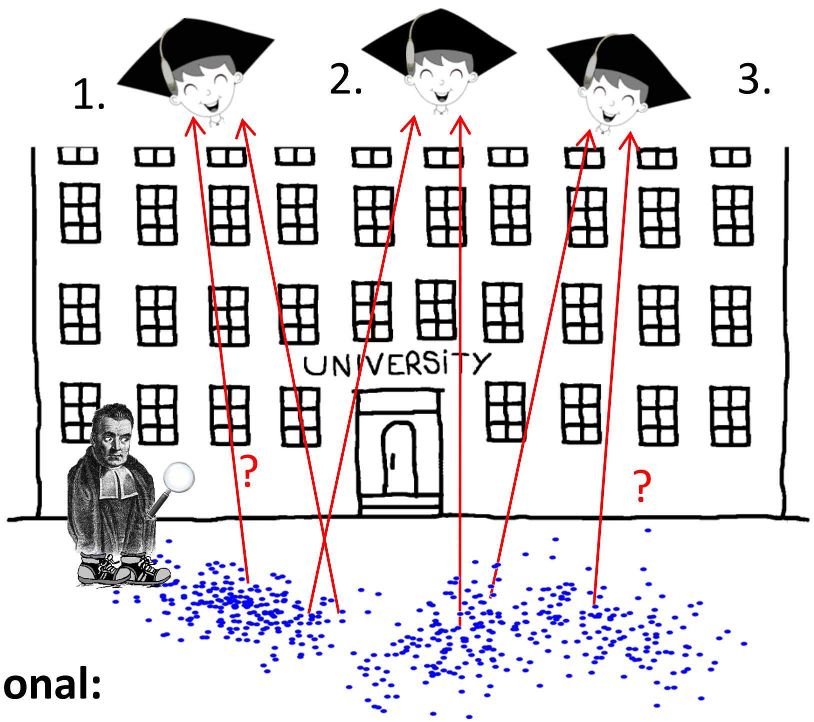

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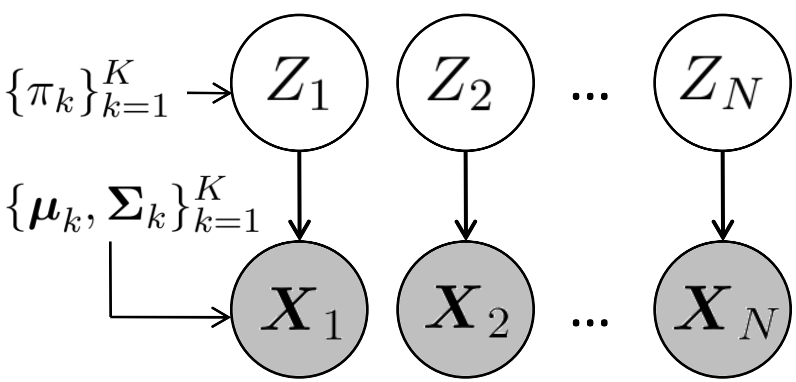

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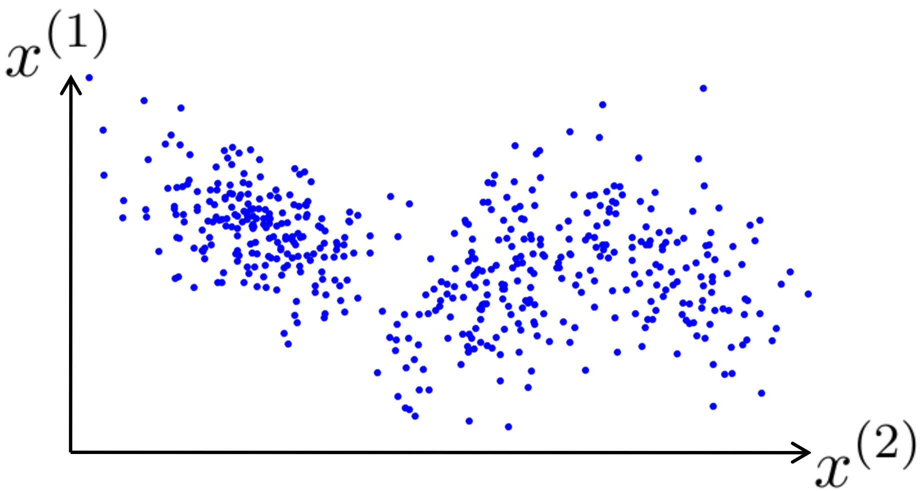

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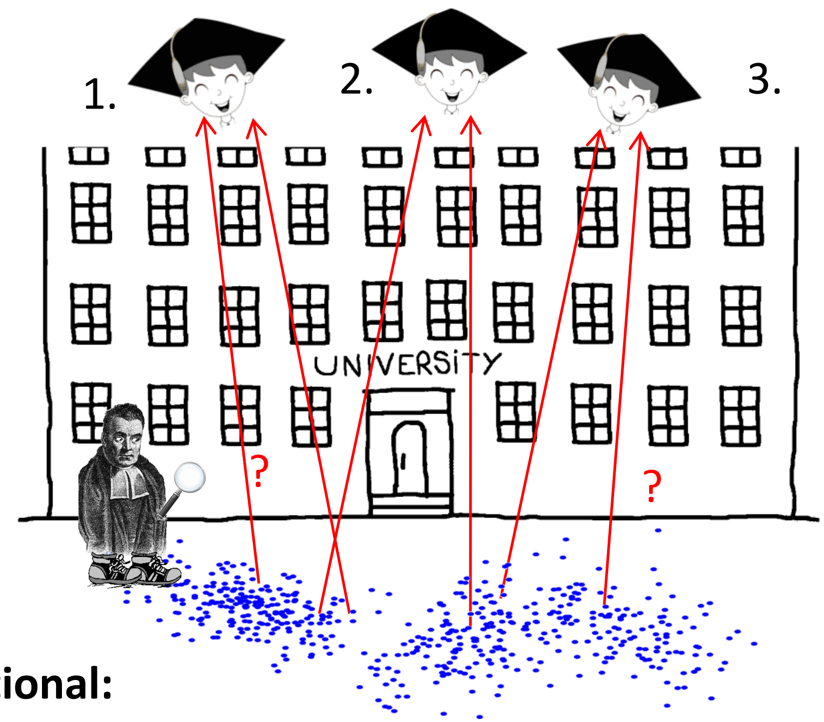

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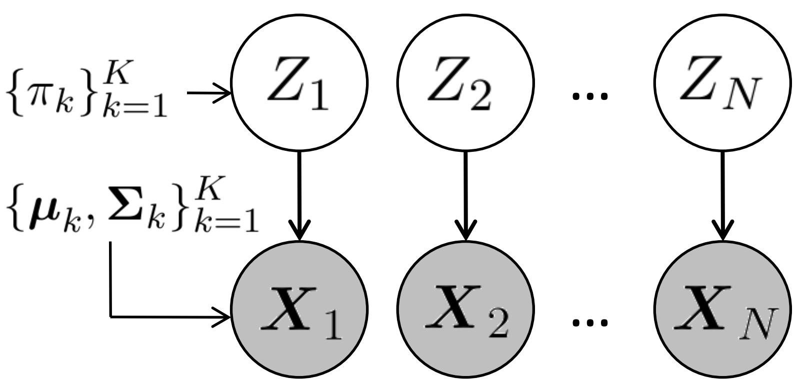

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Bayesian Inference: Examples

Inference

Bayes' Theorem: $p(Z_1 = z_1, \ldots, Z_N = z_N | \boldsymbol{x}_1, \ldots, \boldsymbol{x}_N; \boldsymbol{\theta}) = \prod_{n=1}^{N} p(Z_n = z_n | \boldsymbol{x}_n; \boldsymbol{\theta})$

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Simple form, but $\boldsymbol{\theta}$ is unknown

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- Combinatorial
- Intractable

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Bayesian Inference: Examples

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$\Longrightarrow$ **Expectation-Maximization** (EM) **algorithm**

Bayesian Inference: Examples

Posterior expectation

EM algorithm

- **E-step:** $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)}) = \mathbb{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(i)}}[\log \mathcal{L}(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})]$
- **M-step:** $\boldsymbol{\theta}^{(i+1)} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)})$

Complete-data log-likelihood

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Complete-data log-likelihood

Proof of correctness:

$$p(\mathbf{Z}|\mathbf{X}, \boldsymbol{\theta}) = \frac{p(\mathbf{X}, \mathbf{Z}|\boldsymbol{\theta})}{p(\mathbf{X}|\boldsymbol{\theta})}$$ (Baye's theorem)

Bayesian Inference: Examples

Posterior expectation

EM algorithm

- **E-step:** $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)}) = \mathbb{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(i)}}[\log \mathcal{L}(\boldsymbol{\theta}; \mathbf{X}, \mathbf{Z})]$
- **M-step:** $\boldsymbol{\theta}^{(i+1)} = \underset{\boldsymbol{\theta}}{\operatorname{argmax}}\, Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)})$

Complete-data log-likelihood

Proof of correctness:

$$p(\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}) = \frac{p(\mathbf{X},\mathbf{Z}|\boldsymbol{\theta})}{p(\mathbf{X}|\boldsymbol{\theta})}$$ (Baye's theorem)

$$\log p(\mathbf{X}|\boldsymbol{\theta}) = \log p(\mathbf{X},\mathbf{Z}|\boldsymbol{\theta}) - \log p(\mathbf{Z}|\mathbf{X},\boldsymbol{\theta})$$

Bayesian Inference: Examples

Posterior expectation

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Bayesian Inference: Examples

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According to Gibb's inequality, we have $H(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)}) \geq H(\boldsymbol{\theta}^{(i)}|\boldsymbol{\theta}^{(i)})$.

Bayesian Inference: Examples

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Bayesian Inference: Examples

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The likelihood can only increase at each step!

Bayesian Inference: Examples

Derivations for the Gaussian mixture model

Bayesian Inference: Examples

Derivations for the Gaussian mixture model

- **E-step:** computing the current posterior probabilities

$$r_{n,k}^{(i)} = p(Z_n = k|\boldsymbol{x}_n;\boldsymbol{\theta}^{(i)}) = \frac{p(\boldsymbol{x}_n|Z_n = k;\boldsymbol{\theta}^{(i)})p(Z_n = k;\boldsymbol{\theta}^{(i)})}{p(\boldsymbol{x}_1,\dots,\boldsymbol{x}_N;\boldsymbol{\theta}^{(i)})} \propto \boxed{\pi_k^{(i)}\mathcal{N}(\boldsymbol{x}_n;\boldsymbol{\mu}_k^{(i)},\boldsymbol{\Sigma}_k^{(i)})}$$

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We deduce $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)})$:

$$\begin{aligned}
Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)}) &= \mathbb{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(i)}}\{\log\mathcal{L}(\boldsymbol{\theta};\mathbf{X},\mathbf{Z})\} \\
&= \mathbb{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(i)}}\left\{\log\prod_{n=1}^{N}\sum_{k=1}^{K}\mathbb{I}\{Z_n = k\}\pi_k\mathcal{N}(\boldsymbol{x}_n;\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k)\right\} \\
&= \sum_{k,n=1}^{K,N}\mathbb{E}_{\mathbf{Z}|\mathbf{X},\boldsymbol{\theta}^{(i)}}\{\mathbb{I}\{Z_n = k\}\}\log\big(\pi_k\mathcal{N}(\boldsymbol{x}_n;\boldsymbol{\mu}_k,\boldsymbol{\Sigma}_k)\big) \\
&= \sum_{k,n=1}^{K,N} r_{n,k}^{(i)}\big(\log\pi_k - \frac{1}{2}\log|\boldsymbol{\Sigma}_k| - \frac{1}{2}(\boldsymbol{x}_n-\boldsymbol{\mu}_k)^\top\boldsymbol{\Sigma}_k^{-1}(\boldsymbol{x}_n-\boldsymbol{\mu}_k)\big) + \text{const.}
\end{aligned}$$

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- **M-step:** maximizing $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(i)})$ by finding the zeros of the derivative

Bayesian Inference: Examples

Derivations for the Gaussian mixture model

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$$\boxed{\pi_k^{(i+1)} = \frac{1}{N}\sum_{n=1}^{N} r_{n,k}^{(i)}\,,} \quad \boxed{\boldsymbol{\mu}_k^{(i+1)} = \frac{1}{N}\sum_{n=1}^{N} r_{n,k}^{(i)}\boldsymbol{x}_n\,,} \quad \boxed{\boldsymbol{\Sigma}_k^{(i+1)} = \frac{1}{N}\sum_{n=1}^{N} r_{n,k}^{(i)}(\boldsymbol{x}_n-\boldsymbol{\mu}_k^{(i+1)})(\boldsymbol{x}_n-\boldsymbol{\mu}_k^{(i+1)})^\top}$$

Bayesian Inference: Examples

Inference

EM algorithm

- **Initialization:** Random « guess » for $\boldsymbol{\theta}$
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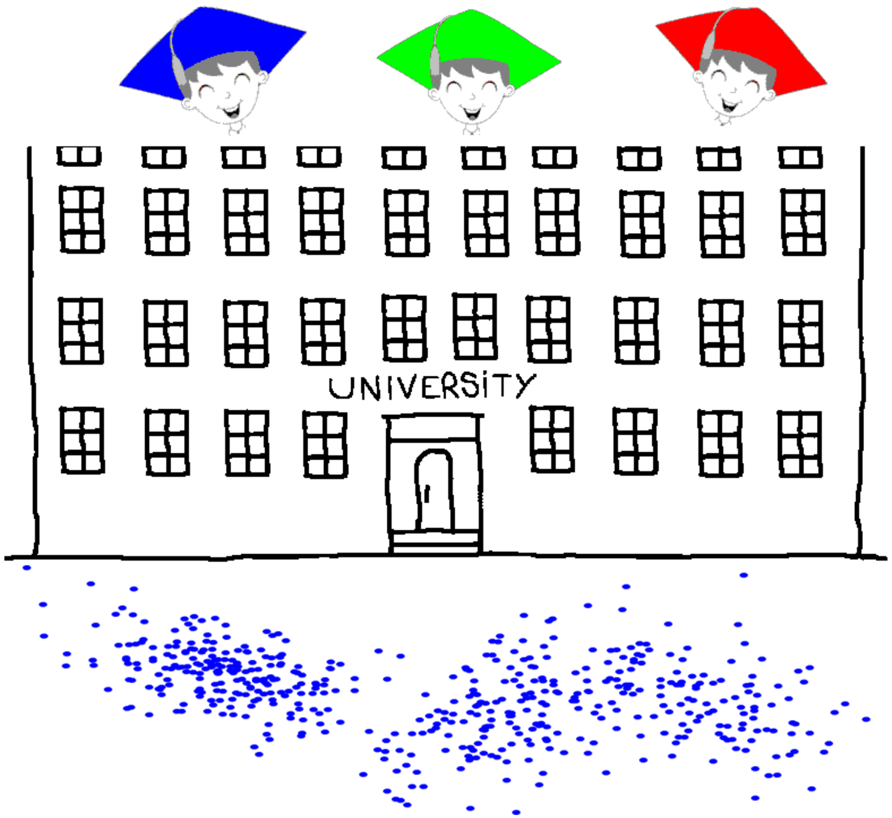
- **Convergence**



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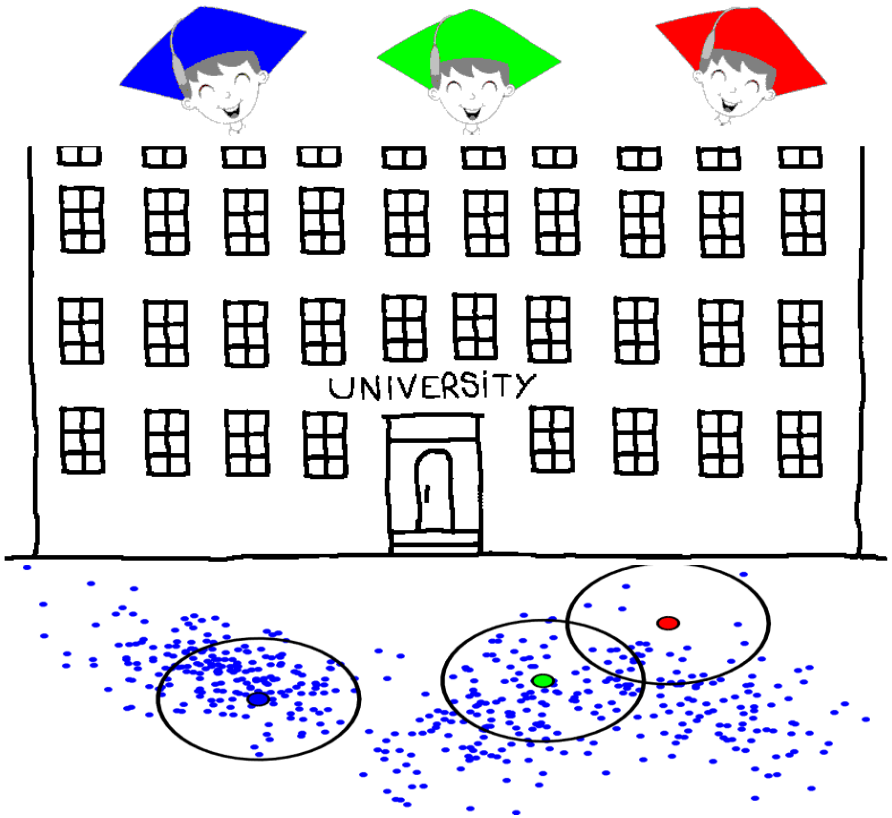

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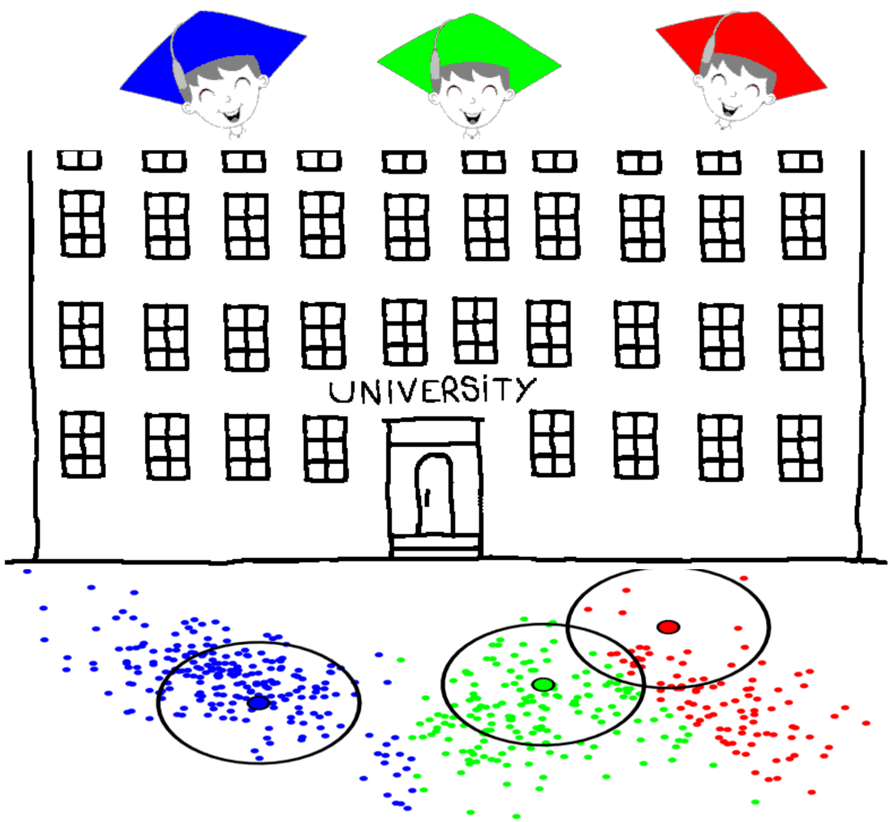

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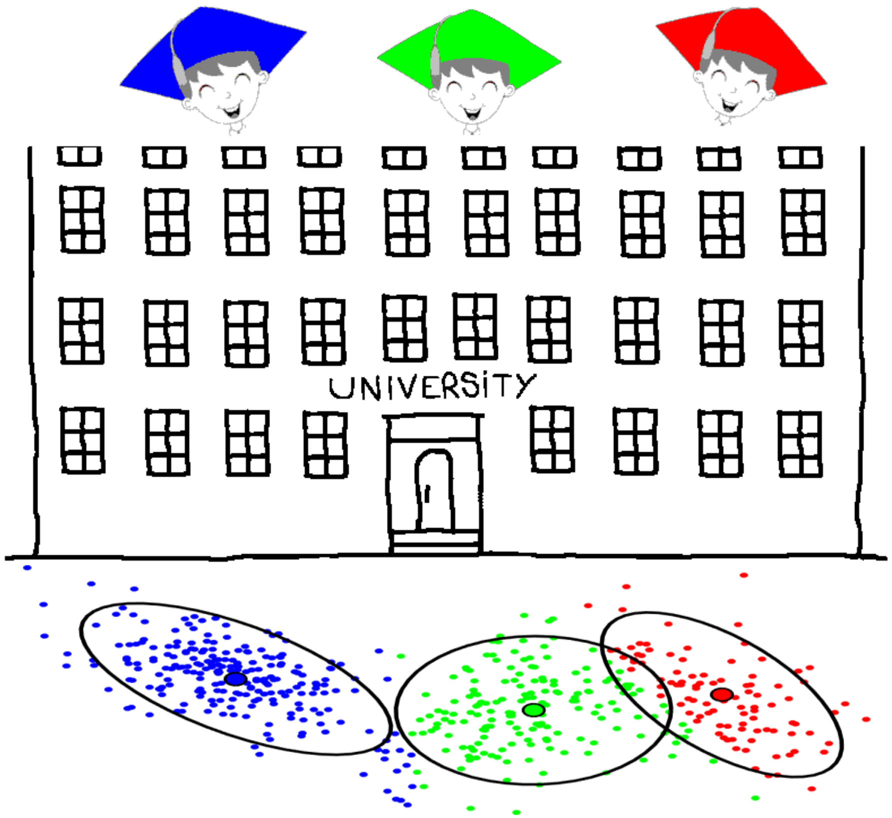

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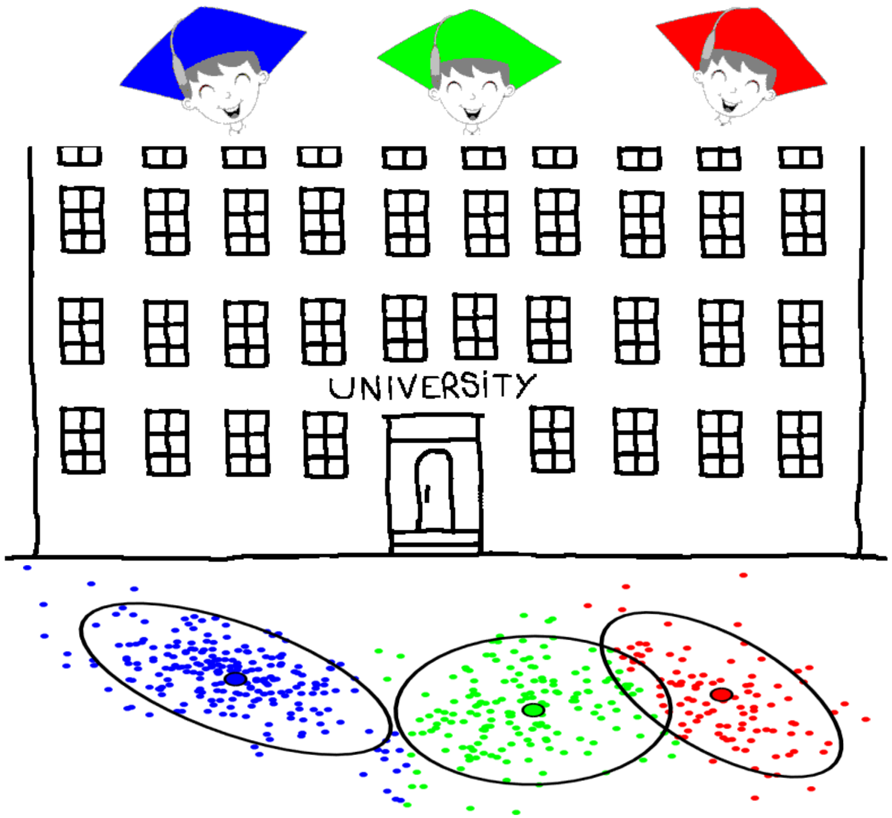

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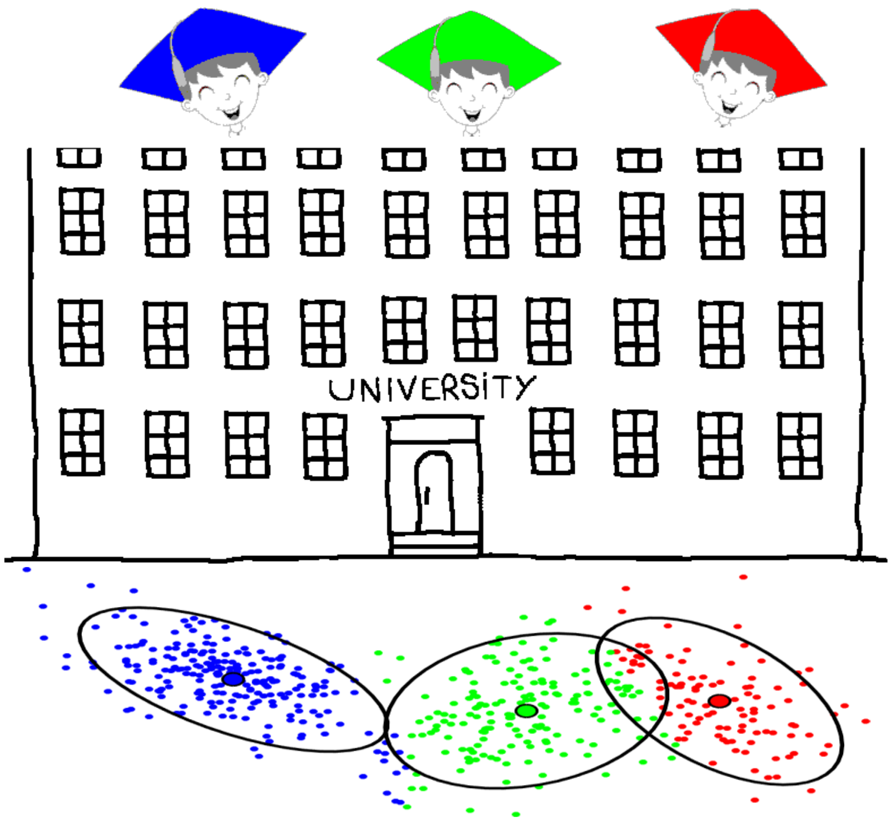

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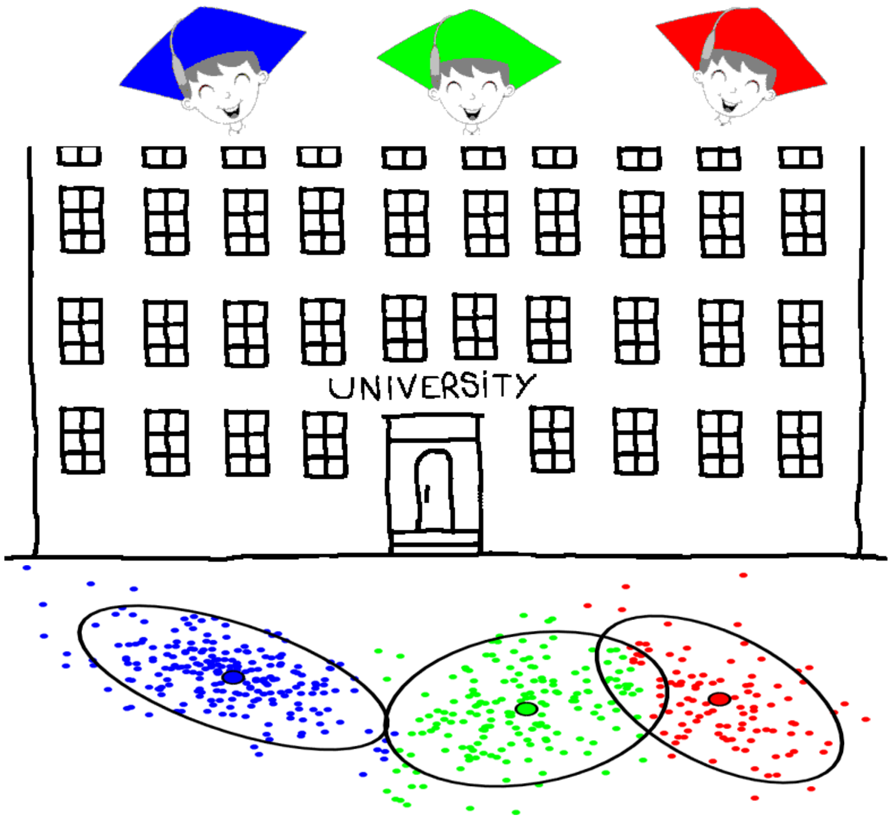

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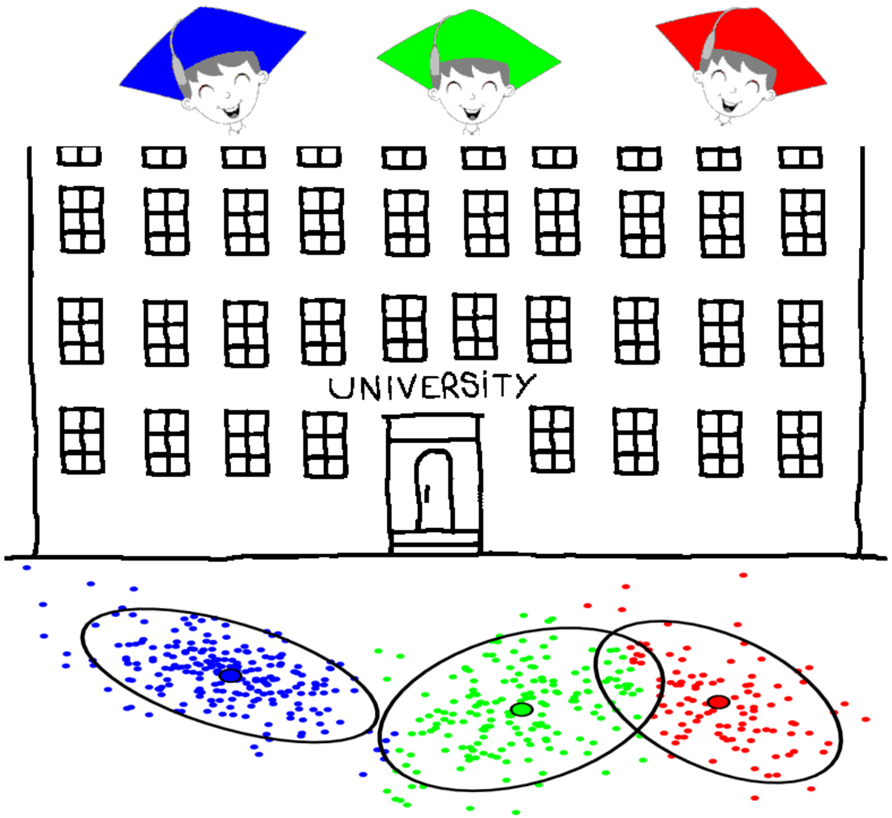

Bayesian Inference: Examples

Inference

EM algorithm

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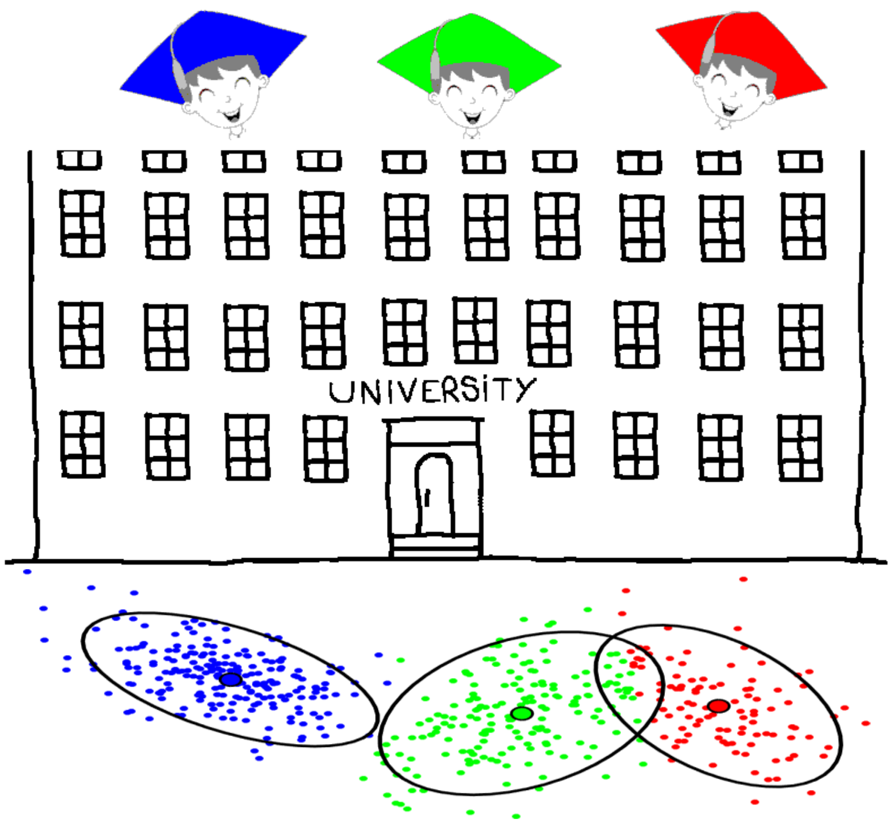

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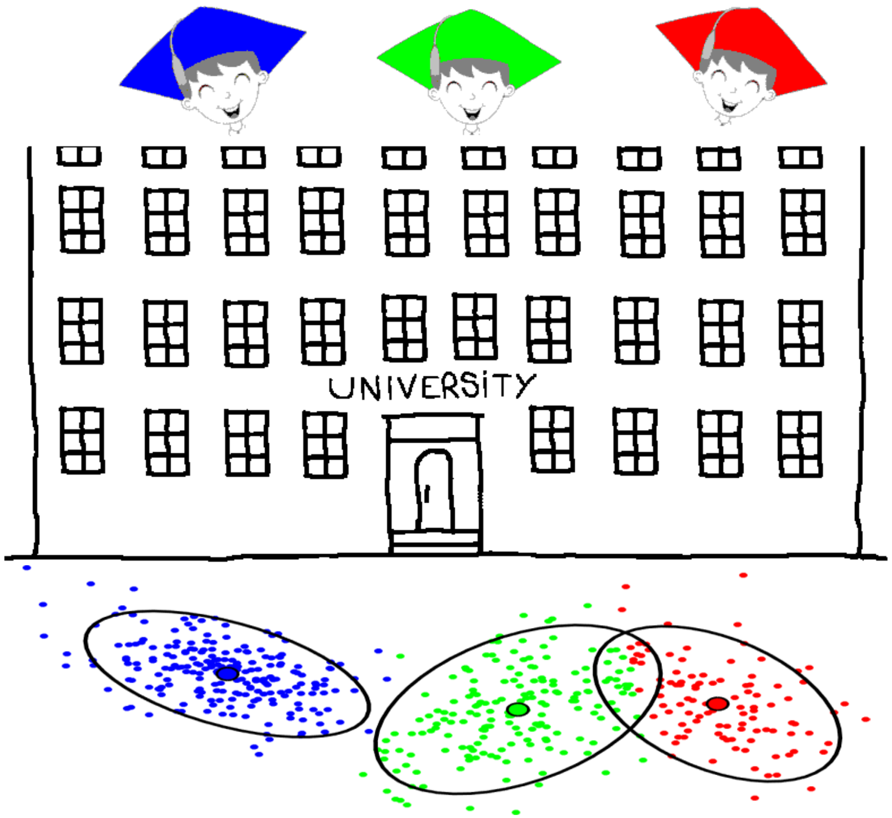

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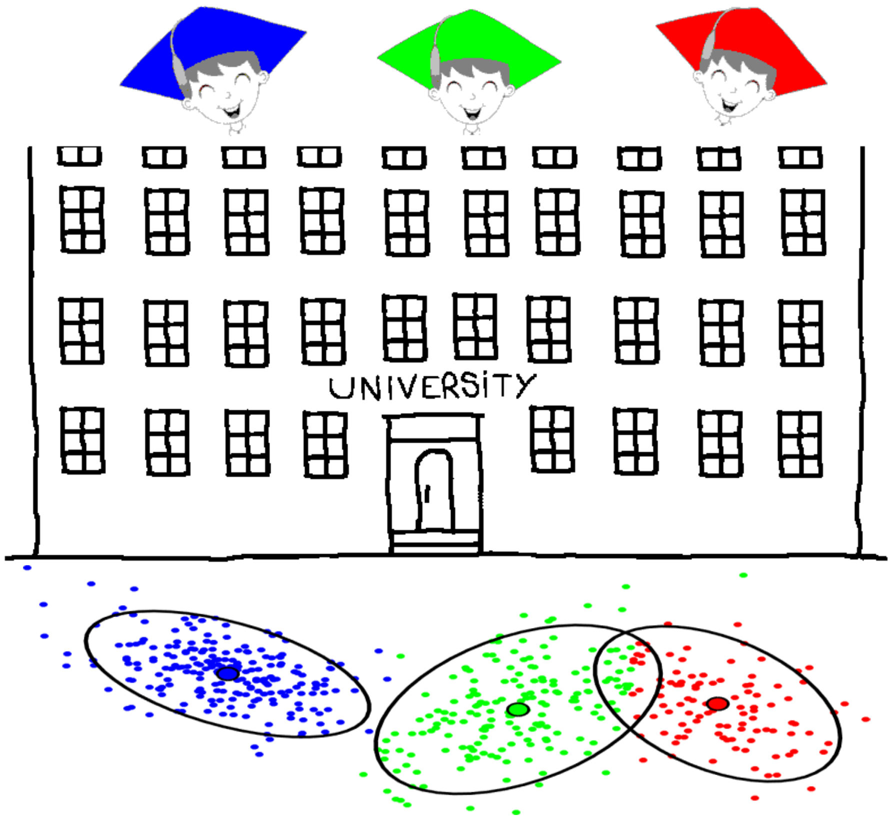

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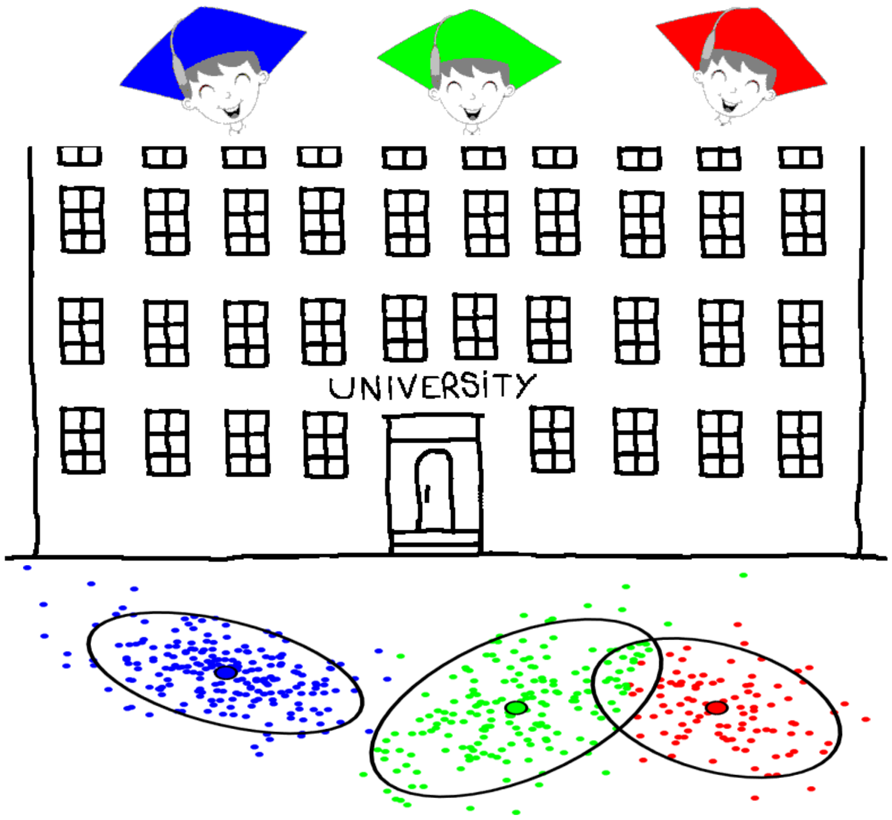

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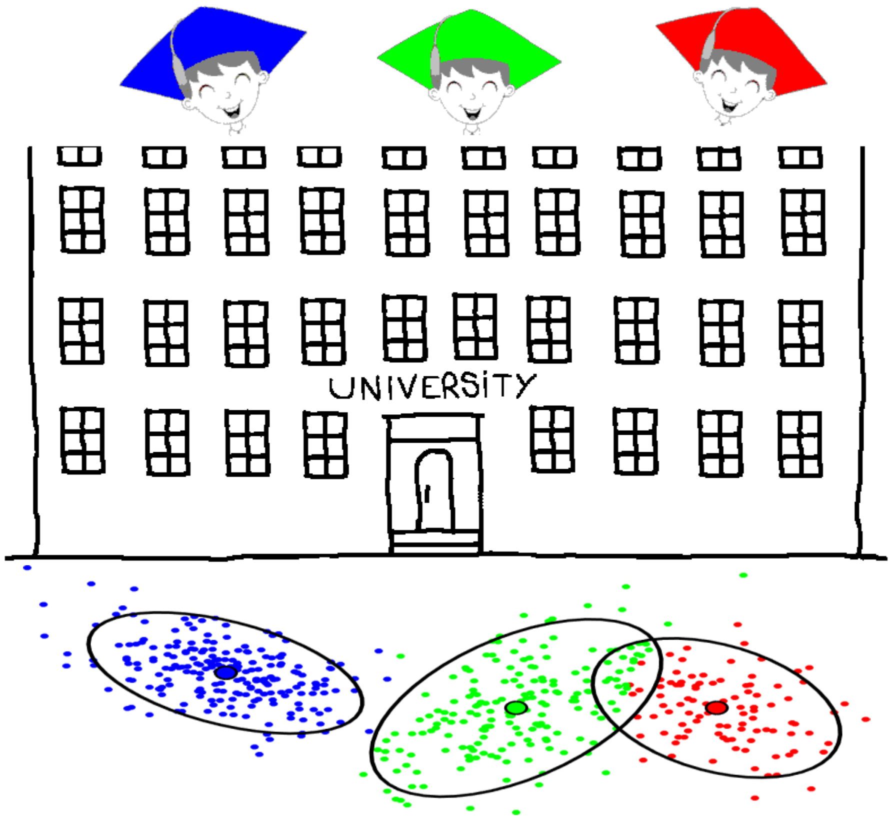
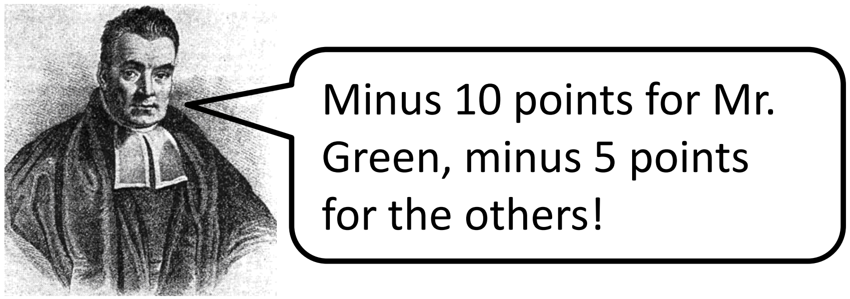


Decision:



OUTLINE

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  - Overview
  - Classical vs. Bayesian approach
  - Bayes Theorem & Example
  - General Methodology
- **Bayesian inference by examples**
  - **Direct inference**
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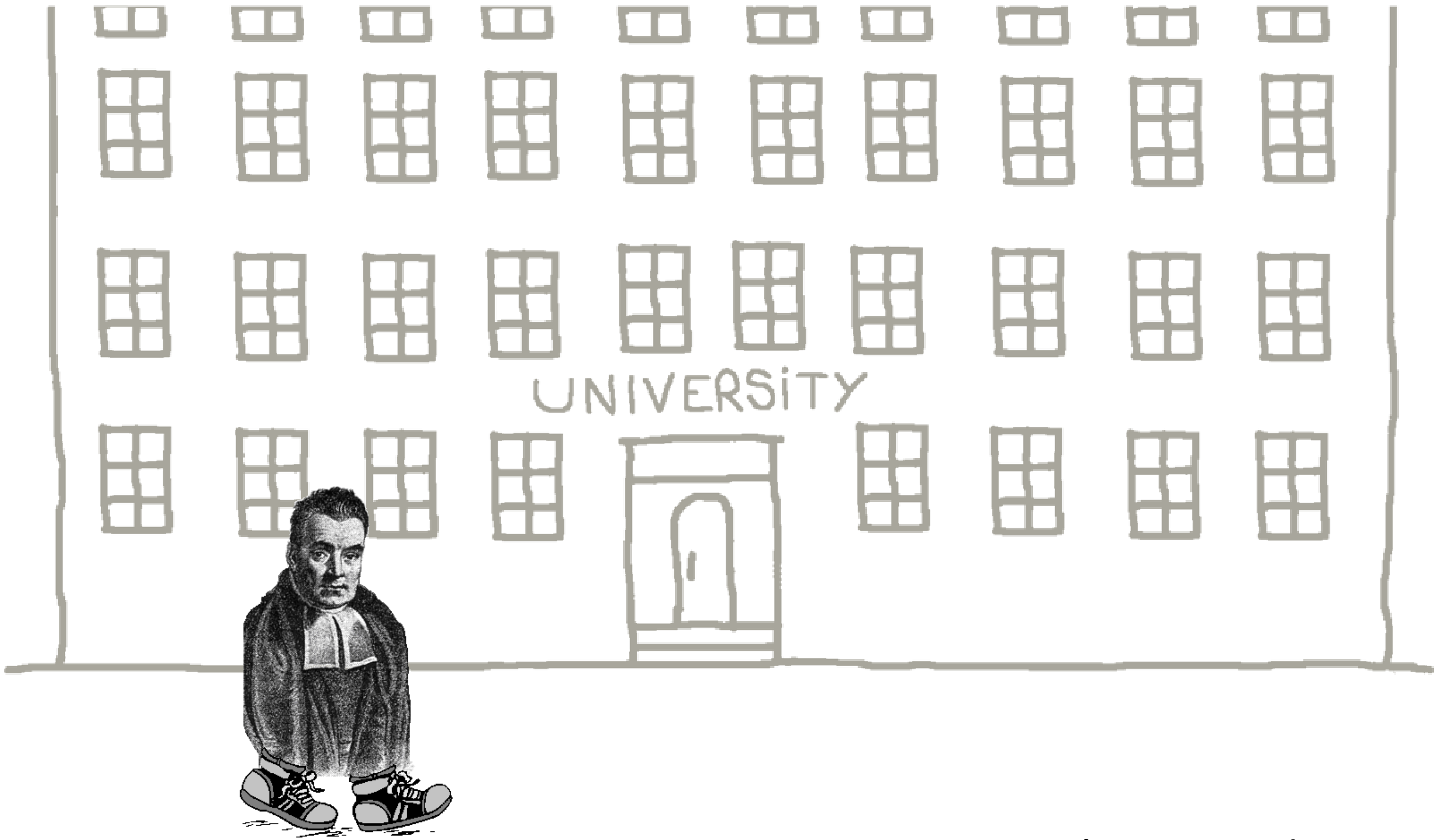
Bayesian Inference: Examples



The next day…

Bayesian Inference: Examples

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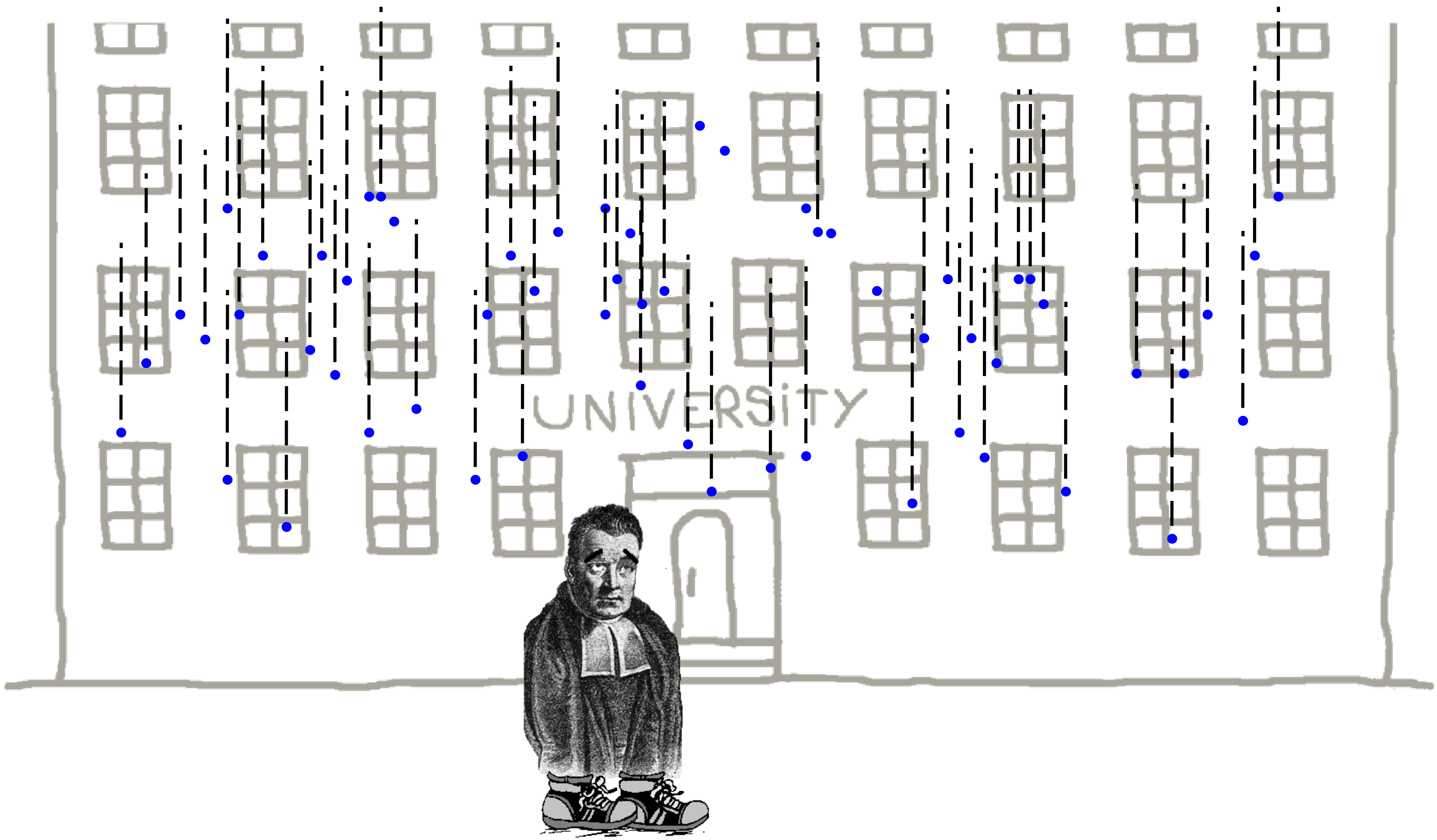

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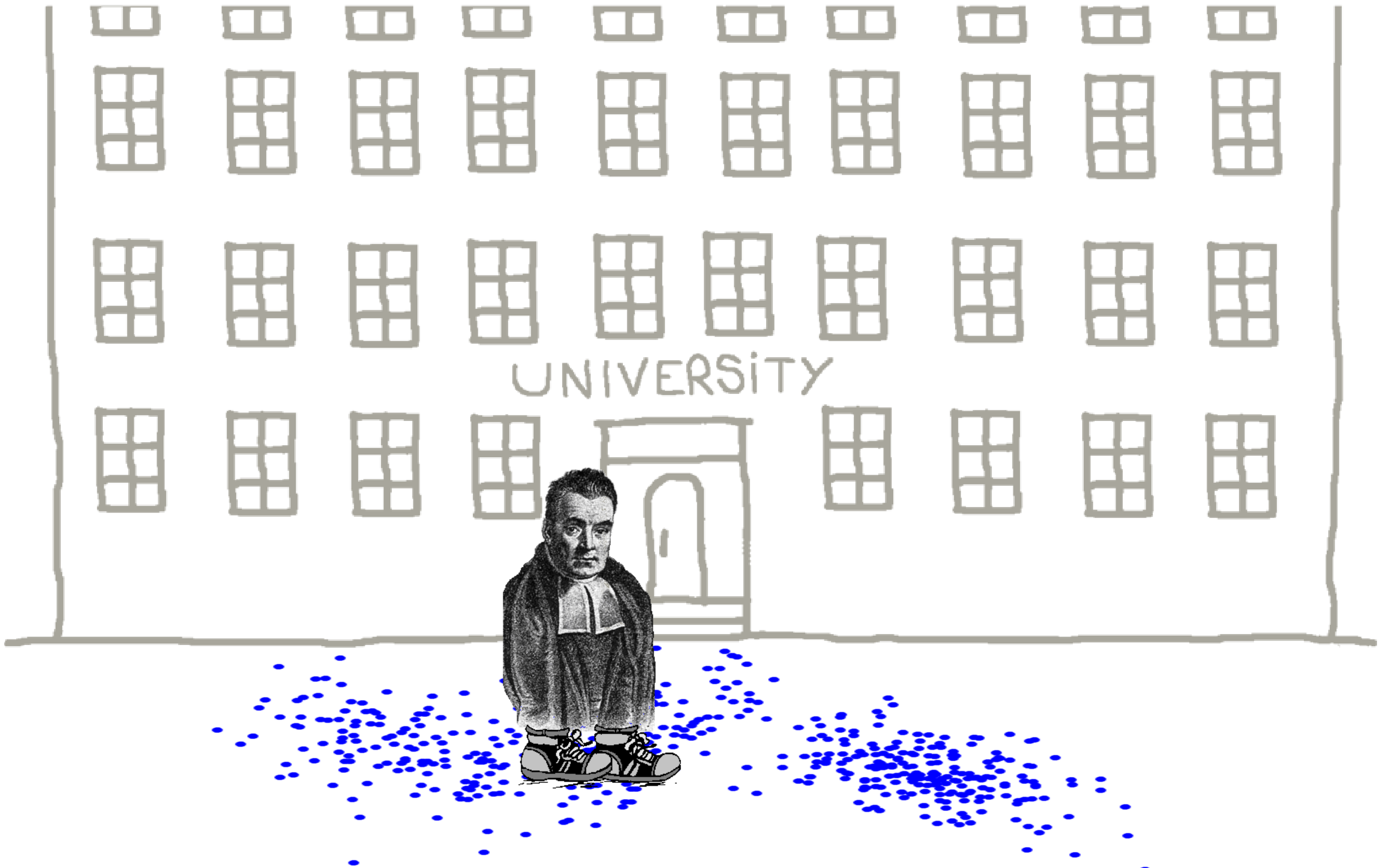
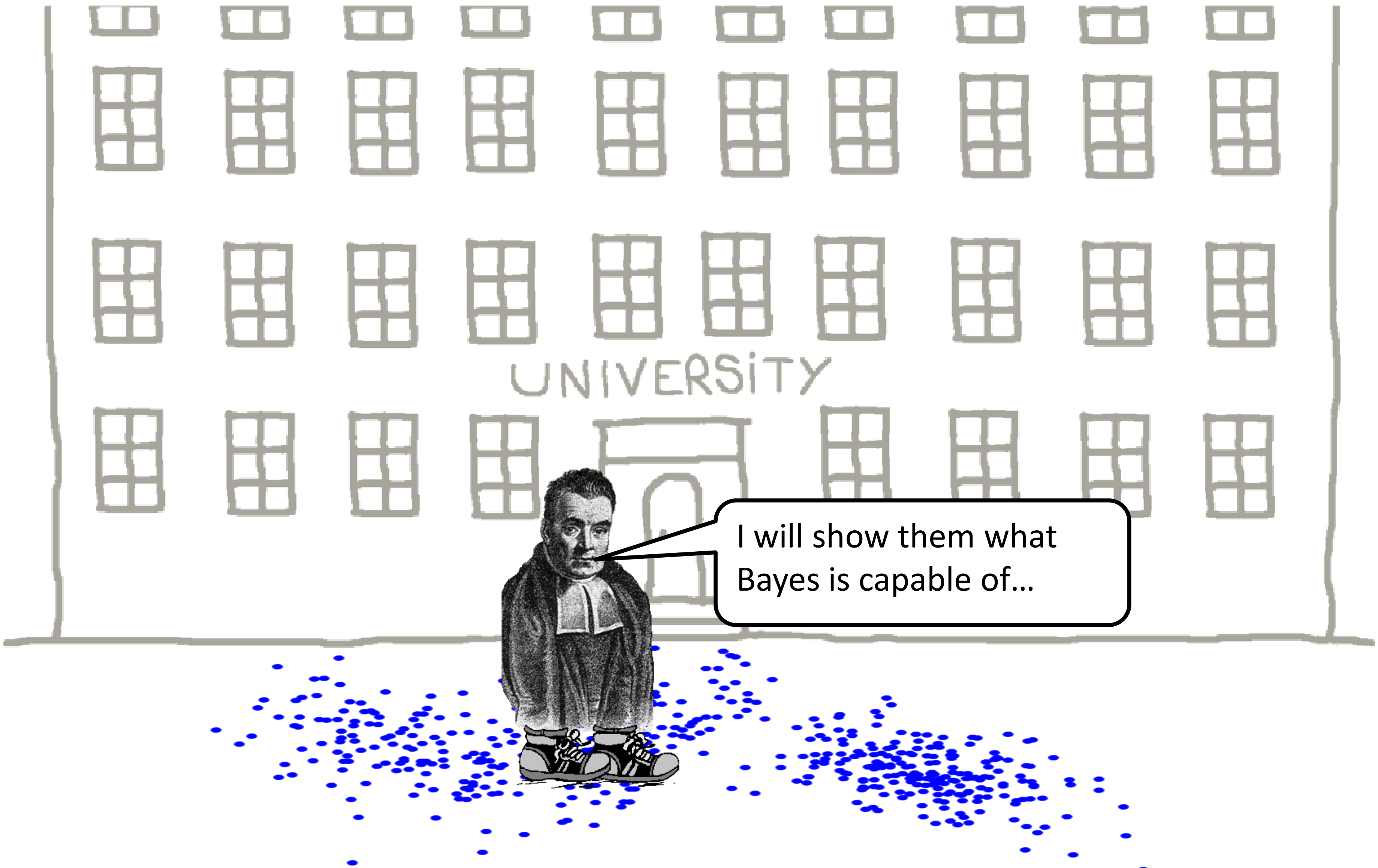
Bayesian Inference: Examples



Bayesian Inference: Examples



Bayesian Inference: Examples

Modeling

A « fully Bayesian » model

$$p(\boldsymbol{x}_n | Z_n = k, \boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k) = \mathcal{N}(\boldsymbol{x}_n; \boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k^{-1})$$
$$p(Z_n = k | \boldsymbol{\pi}) = \pi_k$$

GMM (same as before)

Bayesian Inference: Examples

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$$p(\boldsymbol{\pi}) = \mathrm{SymDir}(\boldsymbol{\pi}|\alpha_0)$$

Priors on all parameters

Note: These are the *conjugate priors* for the normal and the multinomial distributions, i.e., they are such that $p(\boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k|\boldsymbol{x}_n, Z_n = k) \cong p(\boldsymbol{\mu}_k, \boldsymbol{\Lambda}_k)$ and $p(\boldsymbol{\pi}|z_n) \cong p(\boldsymbol{\pi})$

Bayesian Inference: Examples

Modeling A « fully Bayesian » model

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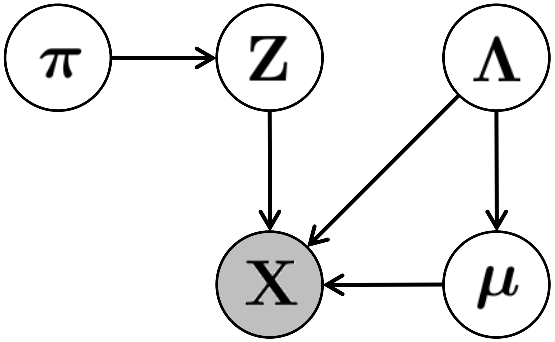

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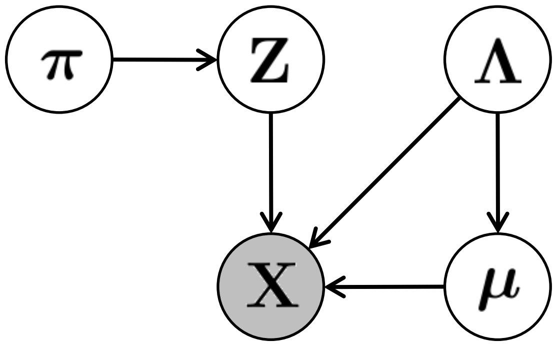

Choice of hyperparameters:

$$\mathbf{W}_0 = \mathbf{I}$$
$$\nu_0 = D = 2$$
$$\boldsymbol{m}_0 = \mathrm{mean}(\mathbf{X})$$

$\alpha_0 > 0$: low values will allow Gaussian weights to be close to 0

Bayesian Inference: Examples

Inference

- The posterior distribution $p(\mathbf{Z}, \mathbf{\Lambda}, \boldsymbol{\mu}, \boldsymbol{\pi} | \mathbf{X})$ is intractable

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- Such procedures are referred to as ***Variational Bayesian EM algorithms.***
  For $p(\mathbf{Z},\mathbf{W}|\mathbf{X}) \approx q_Z(\mathbf{Z})q_W(\mathbf{W})$:

VB-EM
- **E-Z step:** $q_Z^{(i)}(\mathbf{Z}) \propto \exp\left(\mathbb{E}_{q_W^{(i-1)}(\mathbf{W})}\{\log p(\mathbf{Z}|\mathbf{X},\mathbf{W})\}\right)$
- **E-W step:** $q_W^{(i)}(\mathbf{W}) \propto \exp\left(\mathbb{E}_{q_Z^{(i)}(\mathbf{Z})}\{\log p(\mathbf{W}|\mathbf{X},\mathbf{Z})\}\right)$

Bayesian Inference: Examples

Inference

Proof of correctness

• Using a similar reasonning as for EM, we can show that the VB-EM iteratively minimizes the Kulback-Leibler divergence between the true posterior $p(\mathbf{Z},\mathbf{W}|\mathbf{X})$ and its variational approximation $q(\mathbf{Z},\mathbf{W}) = q_Z(\mathbf{Z})q_W(\mathbf{W})$:

$$(q_Z^{(\infty)}, q_W^{(\infty)}) = \operatorname*{argmin}_{q_Z, q_W} \mathrm{KL}(q||p)$$

VB-EM

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Bayesian Inference: Examples

Inference

Derivations for the Bayesian mixture of Gaussian

• **E-Λμπ-Step:**

$$\log q^{(i)}_{\Lambda\mu\pi}(\boldsymbol{\Lambda},\boldsymbol{\mu},\boldsymbol{\pi}) = \log p(\boldsymbol{\pi}) + \sum_{k=1}^{K} \log p(\boldsymbol{\mu}_k,\boldsymbol{\Lambda}_k) + \sum_{n,k=1}^{N,K} q^{(i-1)}_{Z_n}(k) \log \pi_k \mathcal{N}(\boldsymbol{x}_n;\boldsymbol{\mu}_k,\boldsymbol{\Lambda}_k^{-1}) + \mathrm{const.}$$

Using the decomposition $q^{(i-1)}_{Z}(\mathbf{Z}) = \prod_{n=1}^{N} q^{(i-1)}_{Z_n}(k)$ (see E-Z-step).

Bayesian Inference: Examples

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This leads to the factorization: $q^{(i)}_{\Lambda\mu\pi}(\boldsymbol{\Lambda},\boldsymbol{\mu},\boldsymbol{\pi}) = q^{(i)}_{\pi}(\boldsymbol{\pi}) \prod_{k=1}^{K} q^{(i)}_{\Lambda_k\mu_k}(\boldsymbol{\Lambda}_k,\boldsymbol{\mu}_k)$

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with $\begin{cases} q^{(i)}_{\pi}(\boldsymbol{\pi}) = \mathrm{Dir}(\boldsymbol{\pi};\alpha_0 + N^{(i)}_1,\ldots,\alpha_0 + N^{(i)}_K) \\ q^{(i)}_{\Lambda_k\mu_k}(\boldsymbol{\Lambda}_k,\boldsymbol{\mu}_k) = \mathcal{N}\left(\boldsymbol{\mu}_k;\boldsymbol{m}^{(i)}_k,\dfrac{\boldsymbol{\Lambda}_k^{-1}}{1+N^{(i)}_k}\right)\mathcal{W}\left(\boldsymbol{\Lambda}_k;\mathbf{W}^{(i)}_k,\nu^{(i)}_k\right), \end{cases}$

Bayesian Inference: Examples

Inference

Derivations for the Bayesian mixture of Gaussian

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where: $\overline{\boldsymbol{x}}_k^{(i)} = \dfrac{1}{N_k^{(i)}}\sum_{n=1}^{N} q^{(i-1)}_{Z_n}(k)\boldsymbol{x}_n,\qquad \mathbf{S}_k^{(i)} = \dfrac{1}{N_k^{(i)}}\sum_{n=1}^{N} q^{(i-1)}_{Z_n}(k)(\boldsymbol{x}_n-\overline{\boldsymbol{x}}_k^{(i)})(\boldsymbol{x}_n-\overline{\boldsymbol{x}}_k^{(i)})^\top,$

$$N_k^{(i)} = \sum_{n=1}^{N} q^{(i-1)}_{Z_n}(k),\qquad \boldsymbol{m}_k^{(i)} = \frac{\boldsymbol{m}_0 + N_k^{(i)}\overline{\boldsymbol{x}}_k^{(i)}}{1+N_k^{(i)}},$$

$$\mathbf{W}_k^{(i)-1} = \mathbf{W}_0^{-1} + N_k^{(i)}\mathbf{S}_k^{(i)} + \frac{N_k^{(i)}}{N_k^{(i)}+1}(\overline{\boldsymbol{x}}_k^{(i)}-\boldsymbol{m}_0)(\overline{\boldsymbol{x}}_k^{(i)}-\boldsymbol{m}_0)^\top,\qquad \nu_k^{(i)} = \nu_0 + N_k^{(i)}$$

Bayesian Inference: Examples

Inference

Derivations for the Bayesian mixture of Gaussian

- **E-Z-Step:**

Bayesian Inference: Examples

Inference

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$$\log q_Z^{(i)}(\mathbf{Z}) = \mathbb{E}_{q^{(i)}_{\Lambda\mu\pi}}\{\log p(\mathbf{Z}|\mathbf{X},\mathbf{\Lambda},\boldsymbol{\mu},\boldsymbol{\pi})\} + \text{const.}$$

$$= \sum_{n,k=1}^{N,K} \mathbb{I}\{Z_n = k\} \log \rho^{(i)}_{n,k} + \text{const.}$$

where $\rho^{(i)}_{n,k} = \mathbb{E}_{q^{(i)}_{\pi}}\{\log \pi_k\} + \frac{1}{2}\mathbb{E}_{q^{(i)}_{\Lambda_k}}\{\log|\mathbf{\Lambda}_k|\} - \frac{D}{2}\log 2\pi - \frac{1}{2}\mathbb{E}_{q^{(i)}_{\Lambda_k\mu_k}}\{(\boldsymbol{x}_n - \boldsymbol{\mu}_k)^\top \mathbf{\Lambda}_k(\boldsymbol{x}_n - \boldsymbol{\mu}_k)\}$

Bayesian Inference: Examples

Inference

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It follows that $q_Z^{(i)}(\mathbf{Z}) = \prod_{n=1}^{N} q_{Z_n}^{(i)}(k)$ where $q_{Z_n}^{(i)}(k) = \frac{\rho_{n,k}^{(i)}}{\sum_{j=1}^{K}\rho_{n,j}^{(i)}} = r_{n,k}^{(i)}$

Bayesian Inference: Examples

Inference

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It follows that $q_Z^{(i)}(\mathbf{Z}) = \prod_{n=1}^{N} q_{Z_n}^{(i)}(k)$ where $q_{Z_n}^{(i)}(k) = \frac{\rho_{n,k}^{(i)}}{\sum_{j=1}^K \rho_{n,j}^{(i)}} = r_{n,k}^{(i)}$

Finally, we can express $r_{n,k}^{(i)}$ as a function of the parameters calculated in previous step:

$$r_{n,k}^{(i)} \propto |\mathbf{W}_k^{(i)}|^{1/2} \exp\left(\psi(\alpha_0 + N_k^{(i)}) + \sum_{i=1}^{D}\psi\left(\frac{\nu_k^{(i)} + 1 - i}{2}\right) - \frac{D}{2N_k^{(i)} + 2} - \frac{\nu_k^{(i)}}{2}(\boldsymbol{x}_n - \boldsymbol{m}_k^{(i)})^\top \mathbf{W}_k^{(i)}(\boldsymbol{x}_n - \boldsymbol{m}_k^{(i)})\right)$$

where $\psi(.)$ denotes the digamma function.

Bayesian Inference: Examples

Inference

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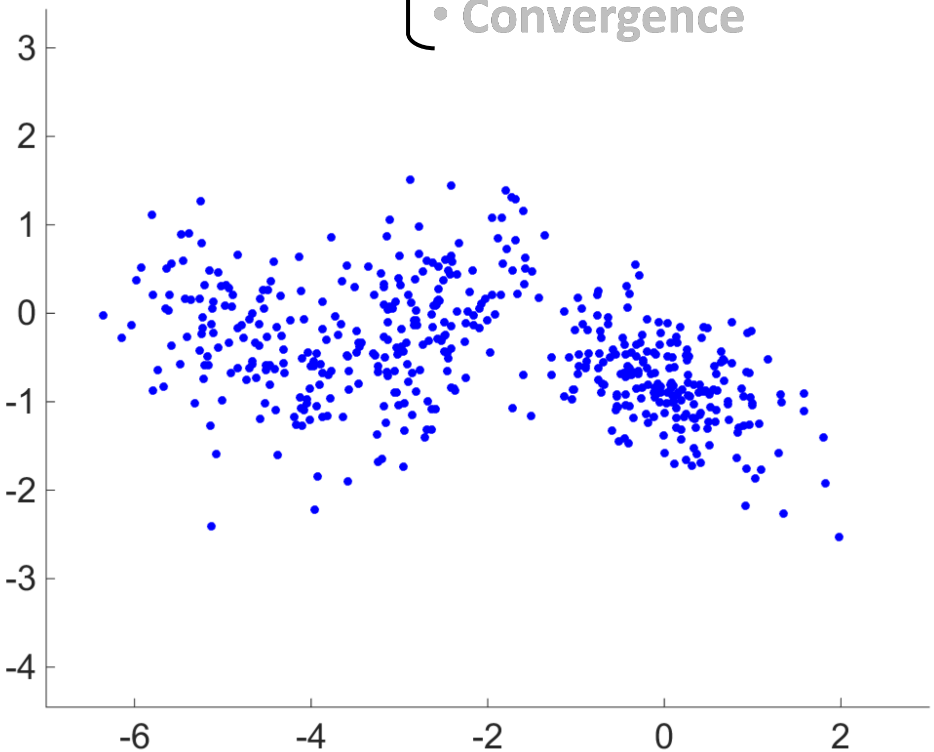

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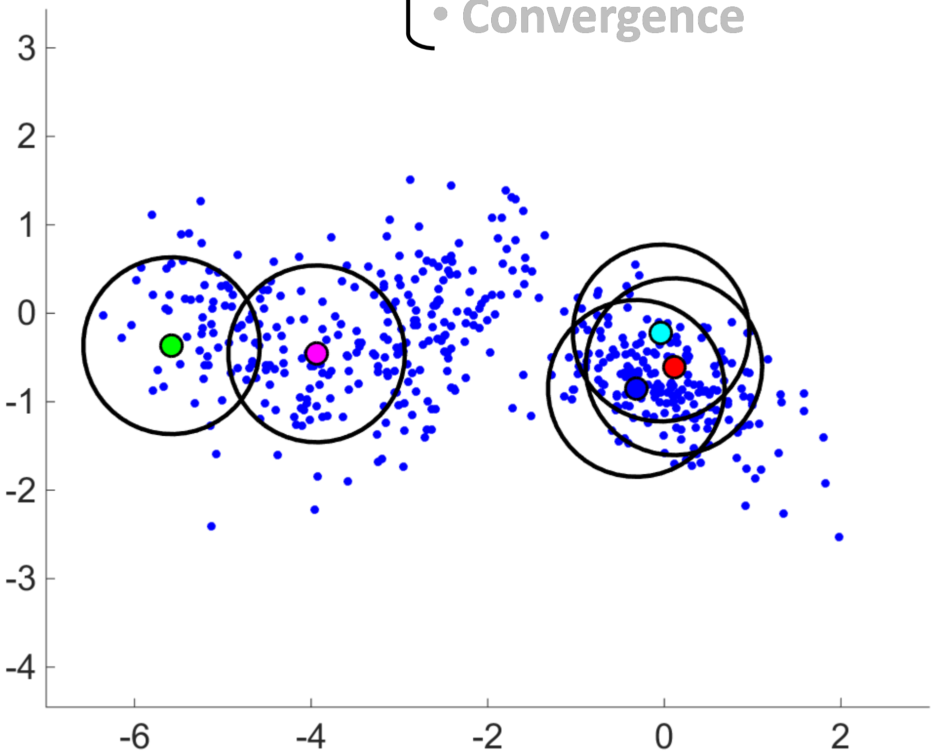

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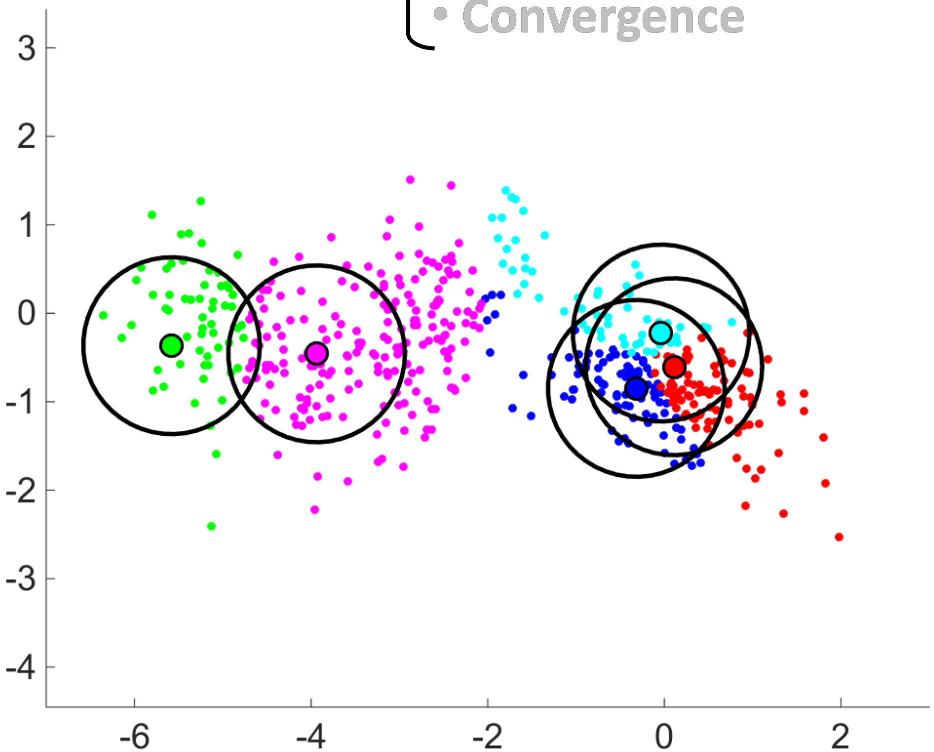

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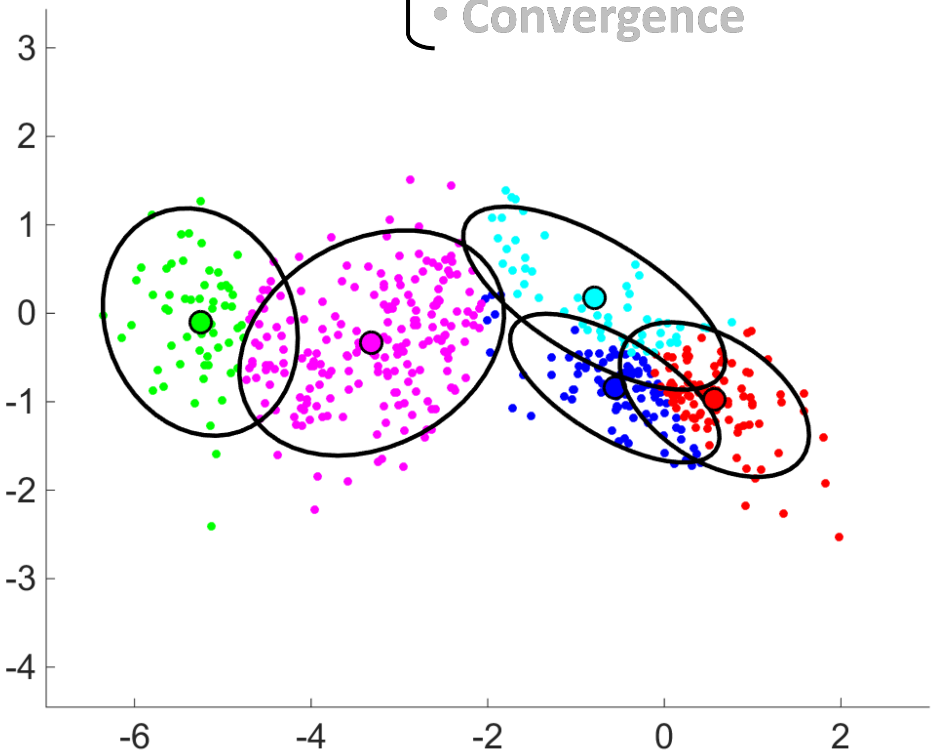

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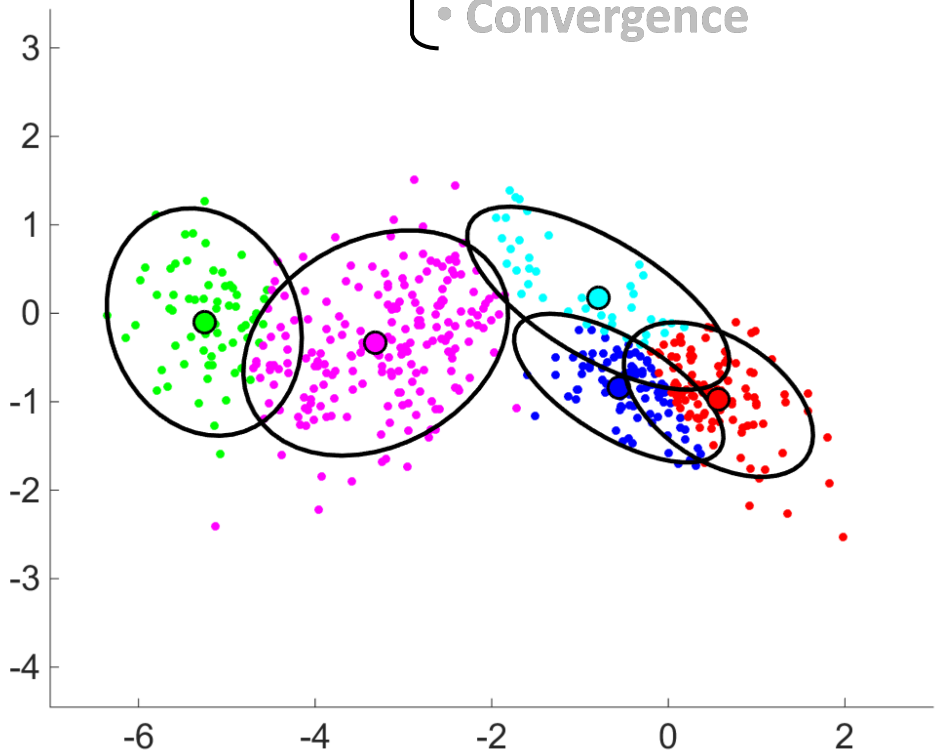

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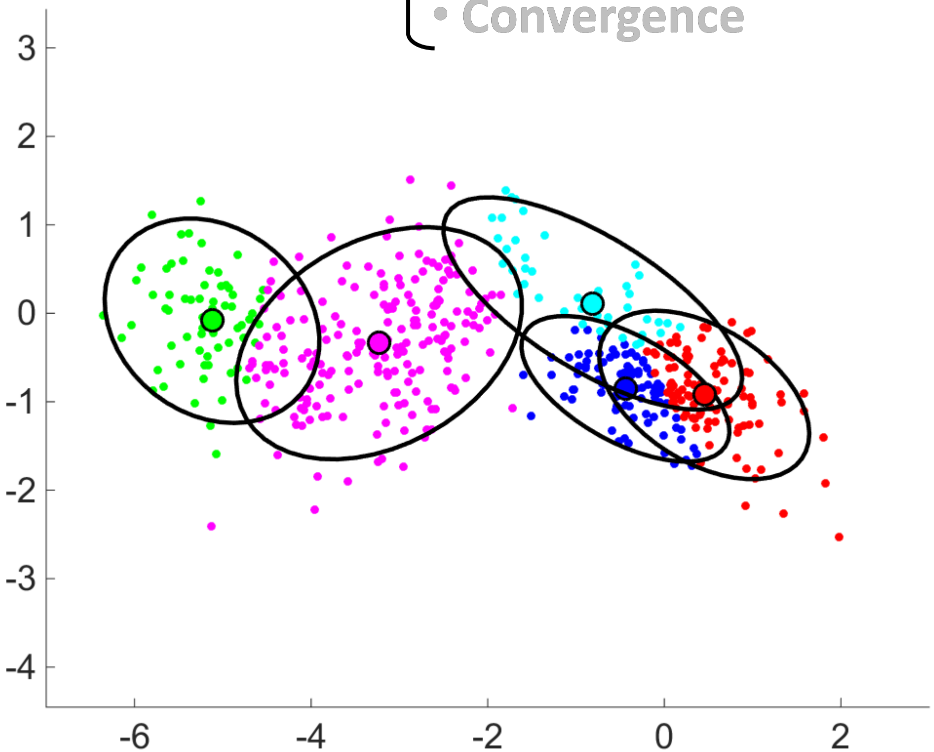

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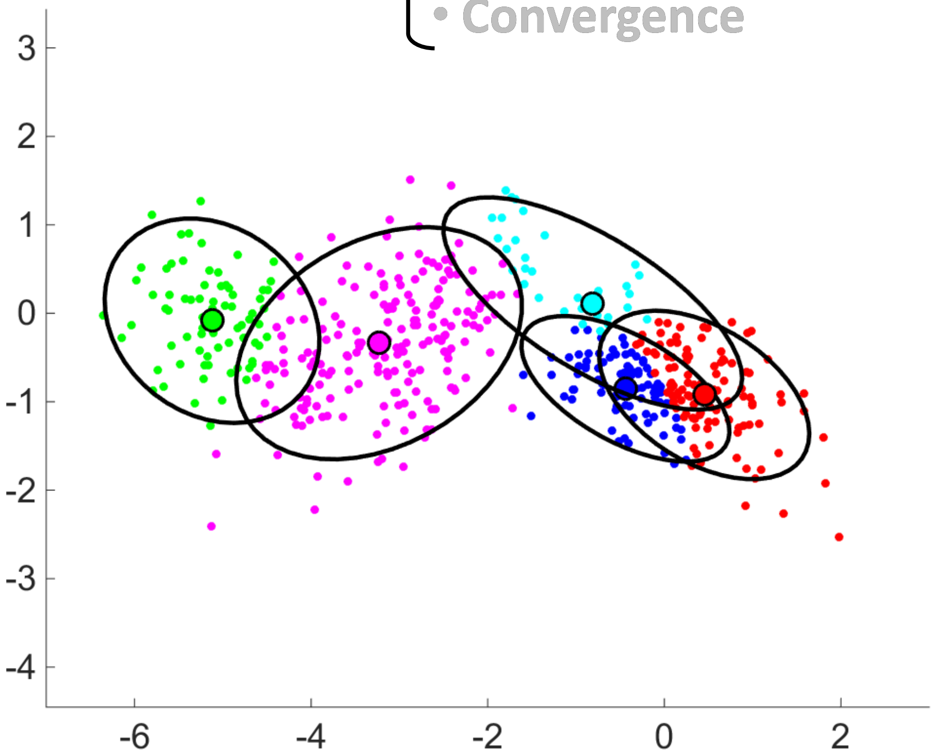

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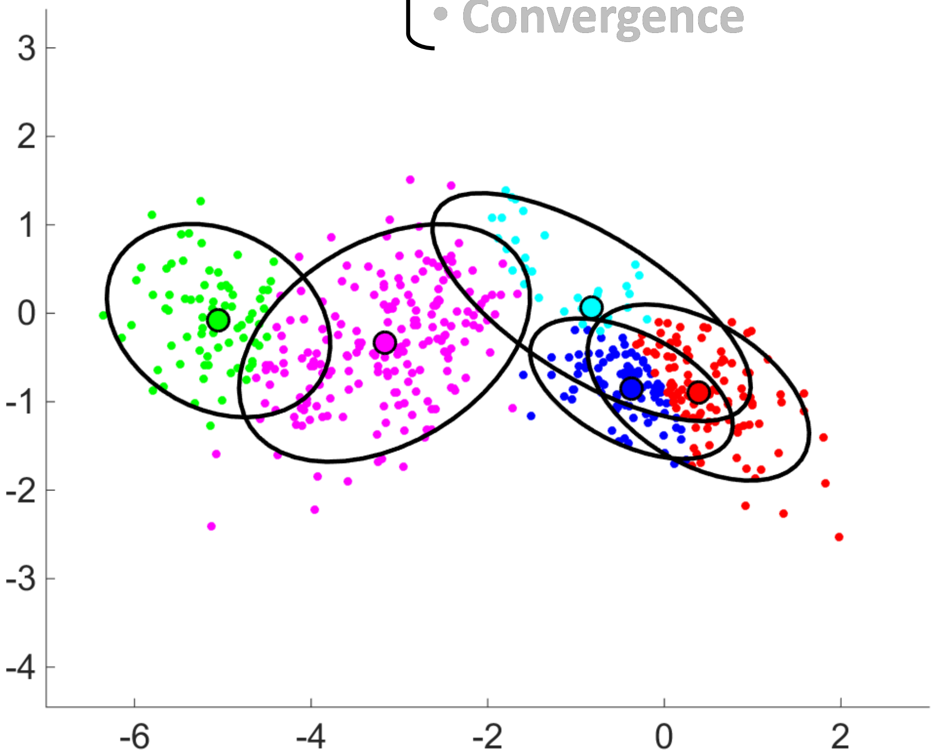

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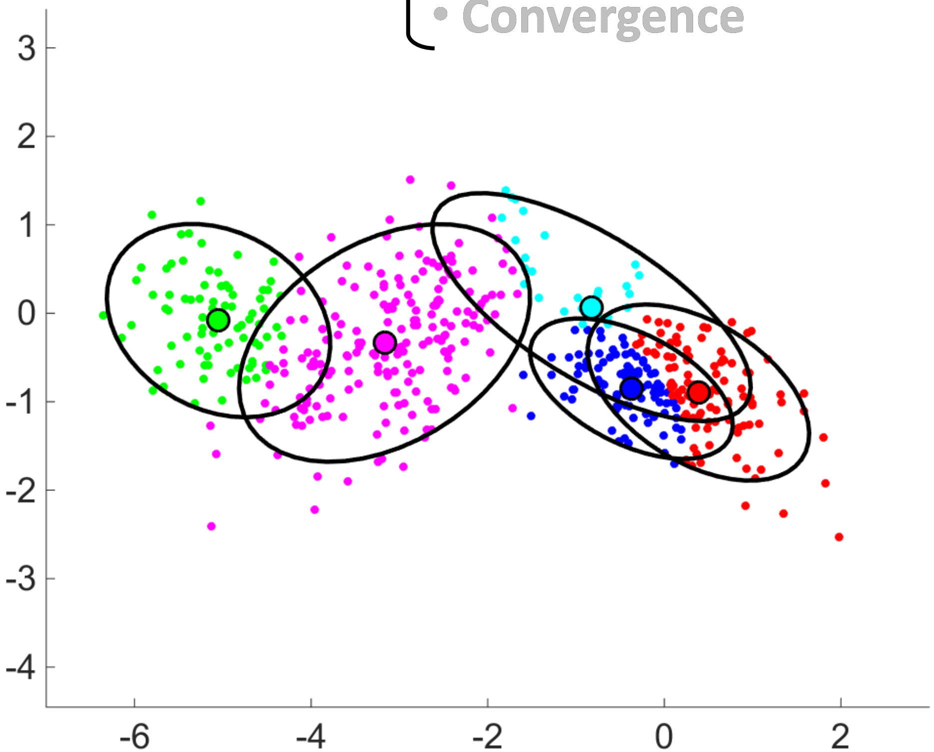

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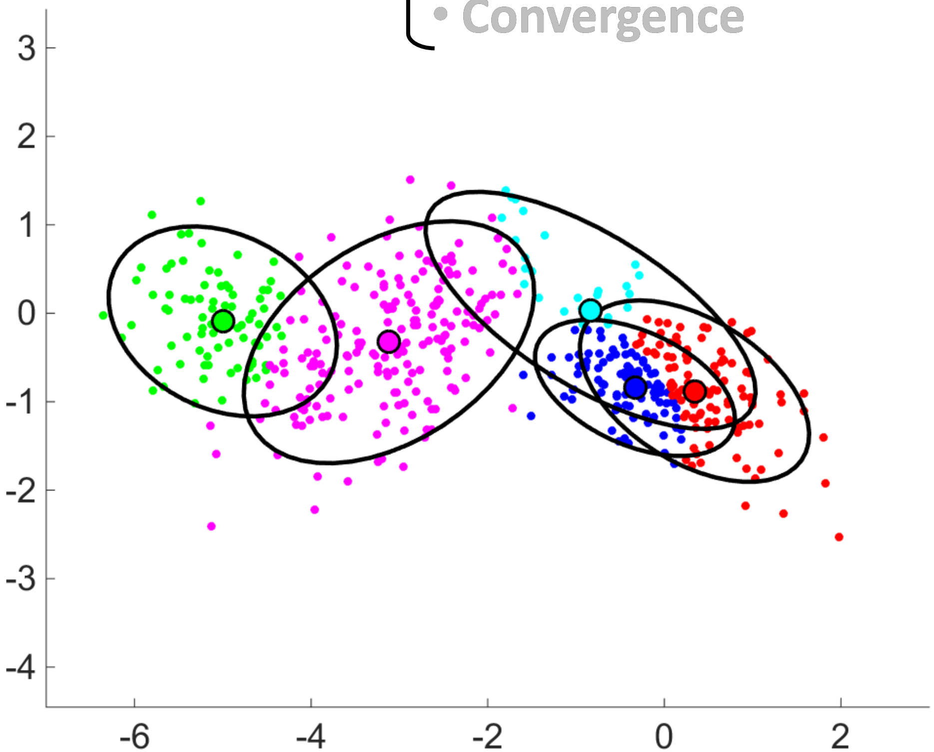

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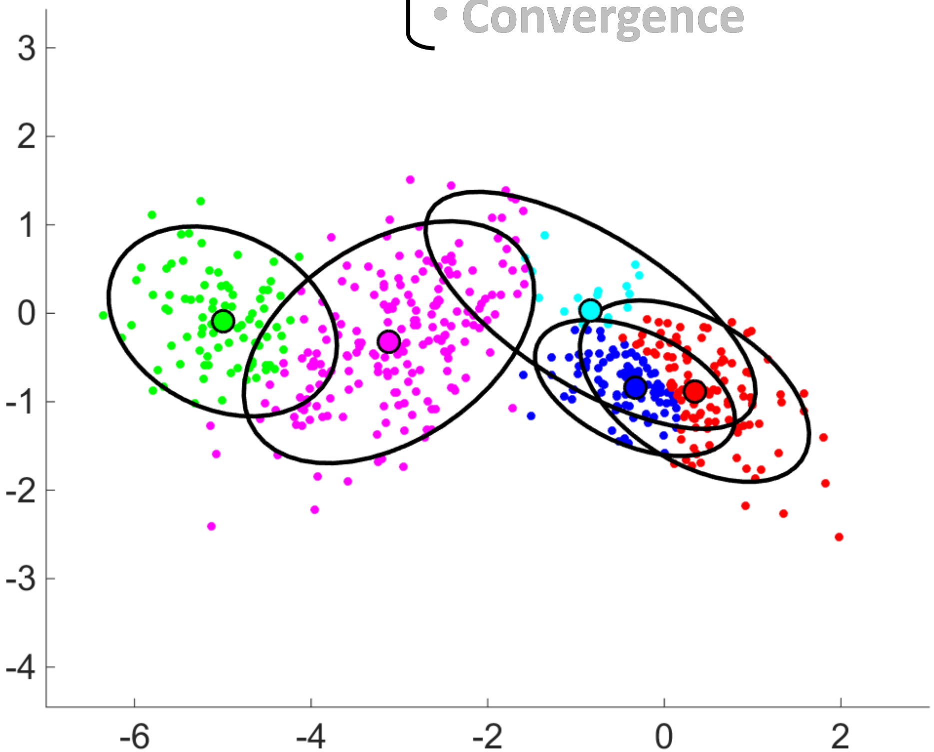

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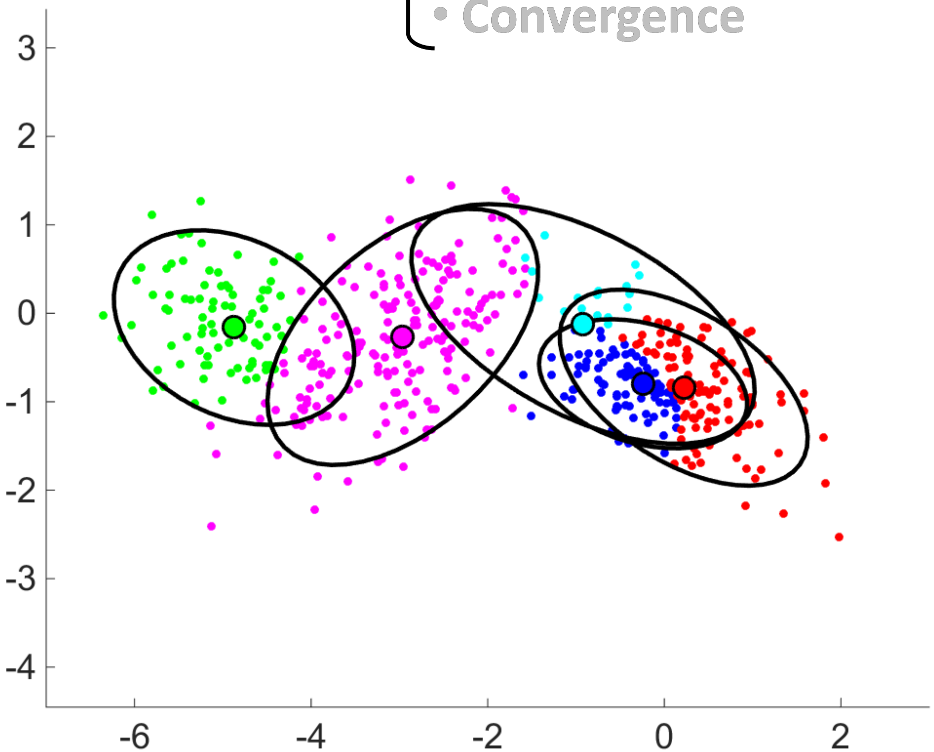

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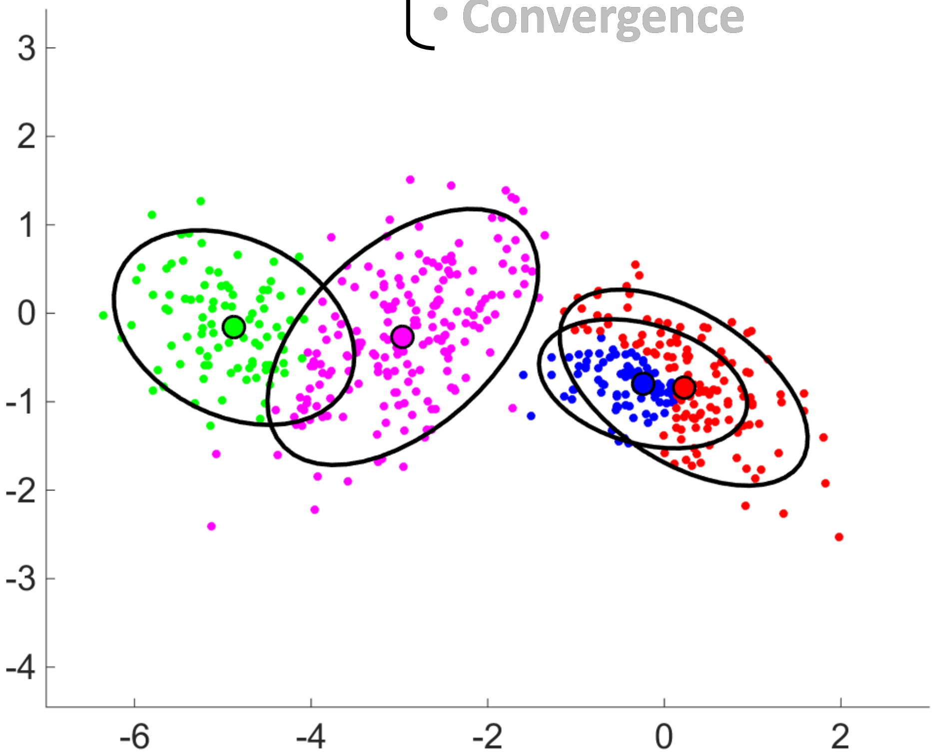

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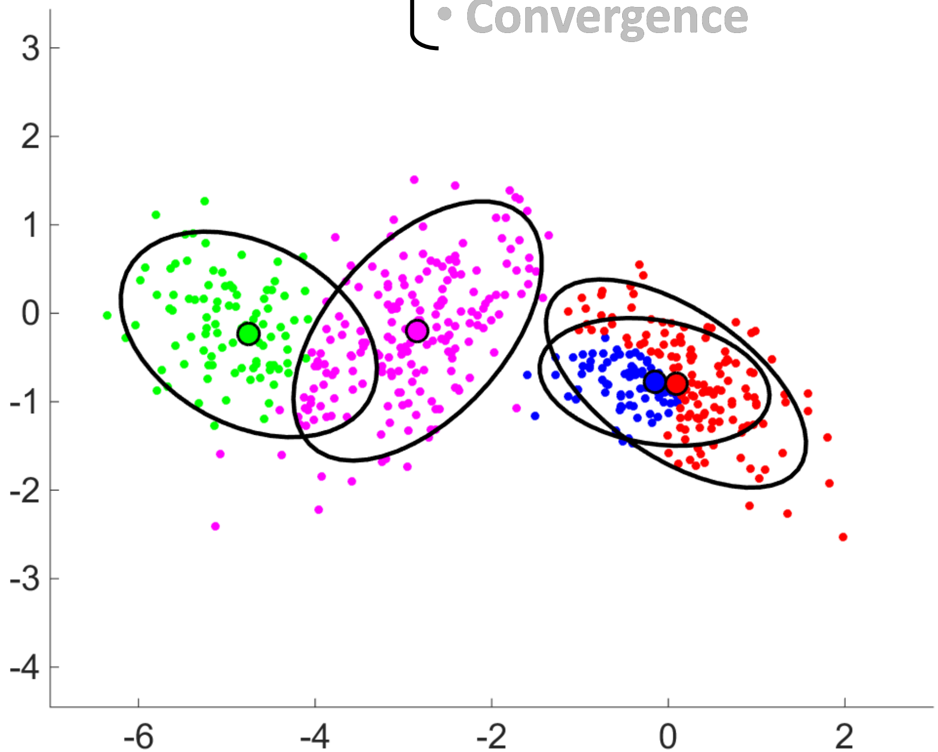

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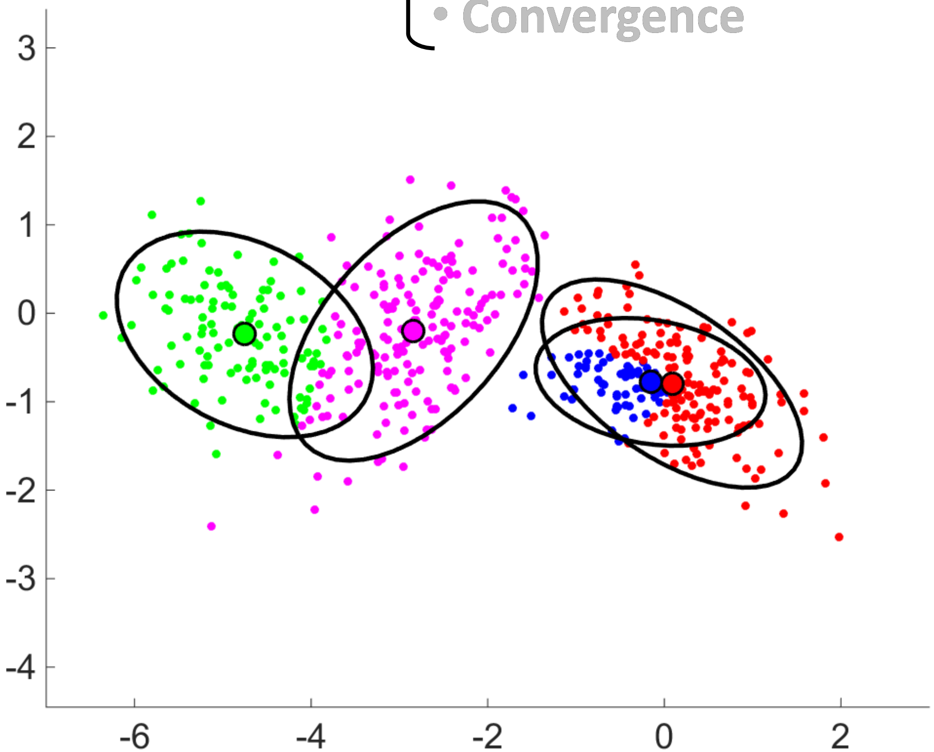

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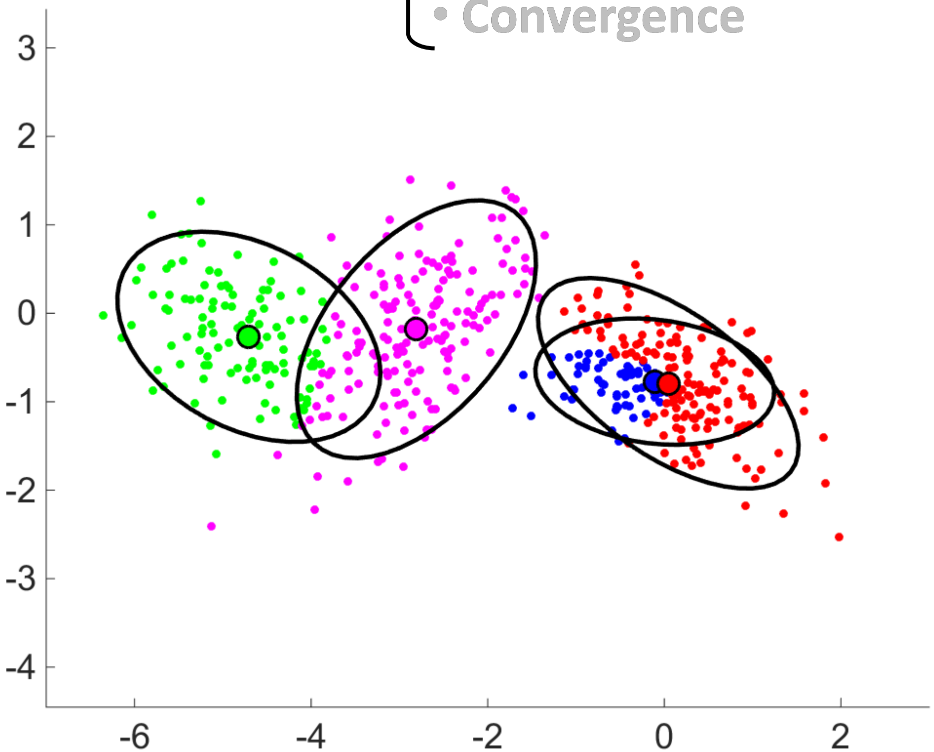

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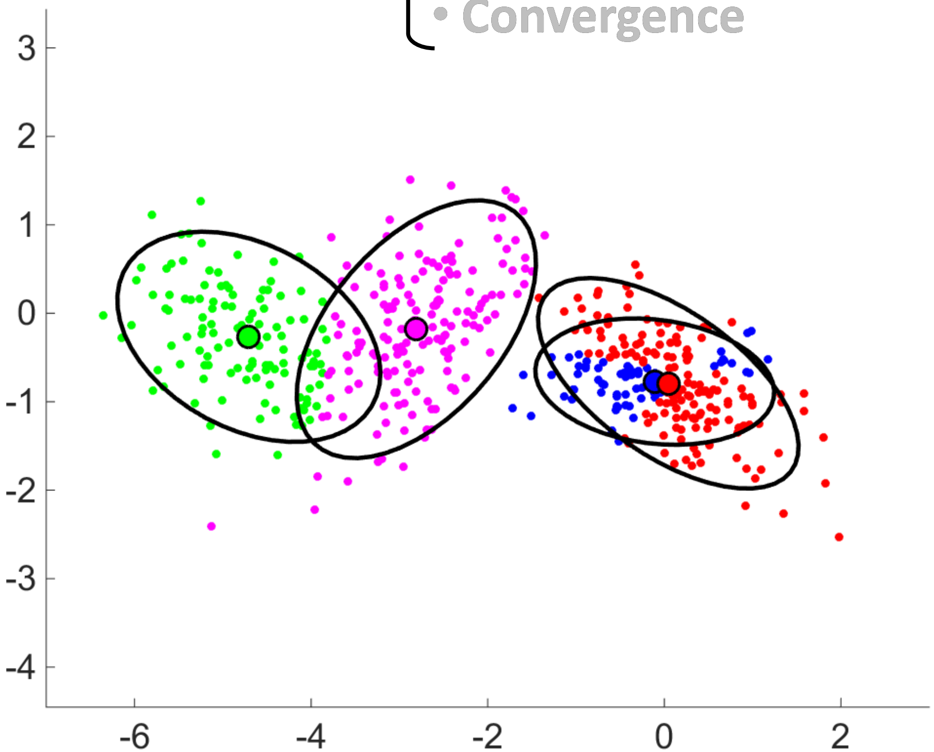

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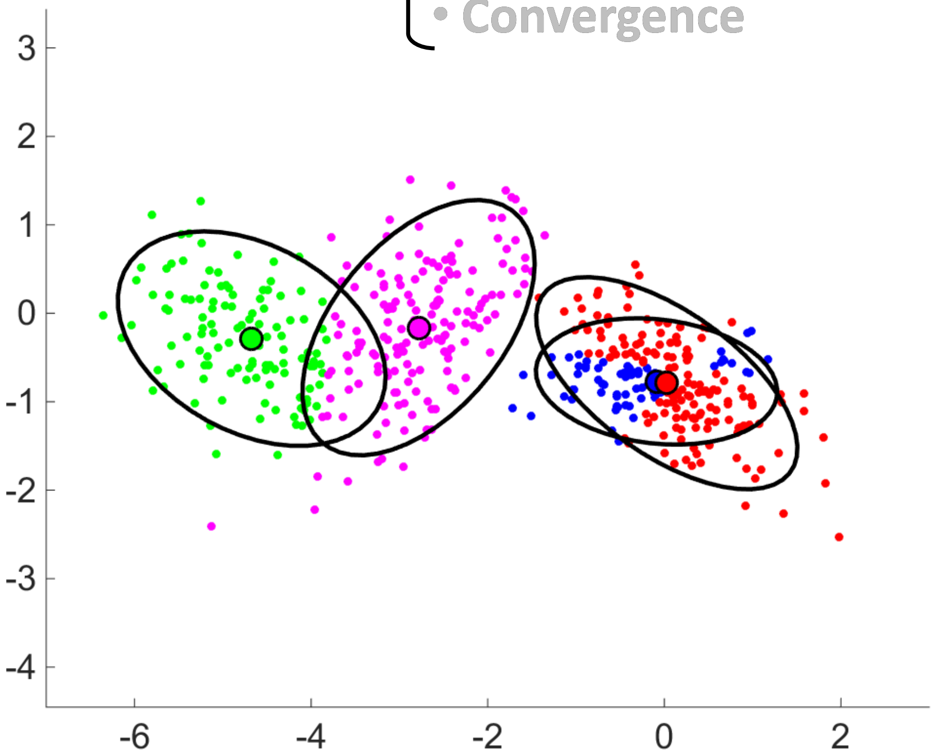

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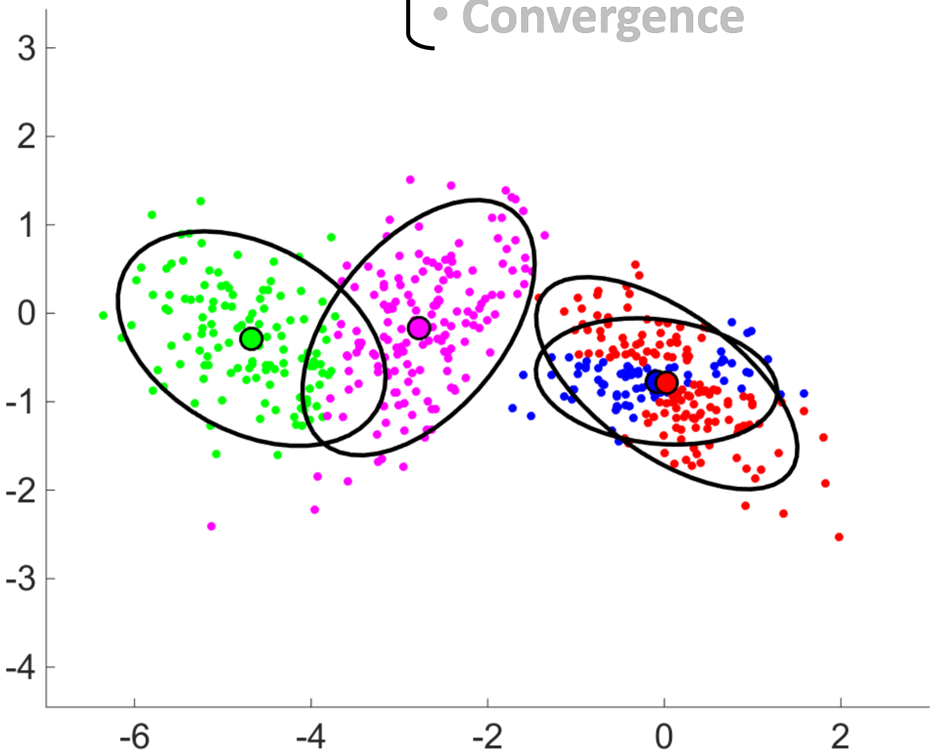

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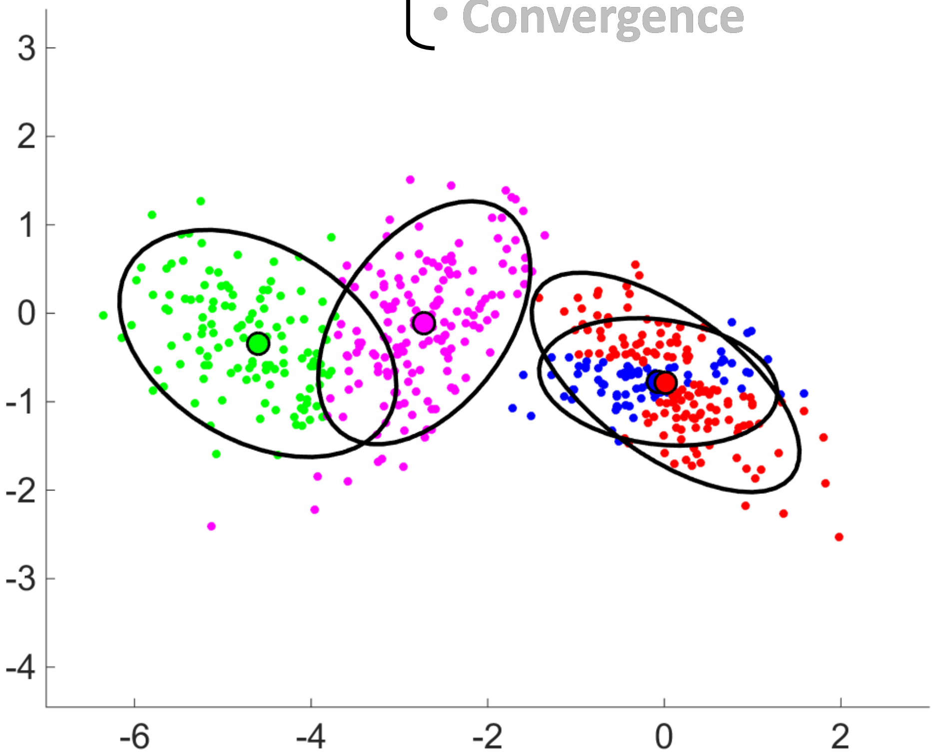

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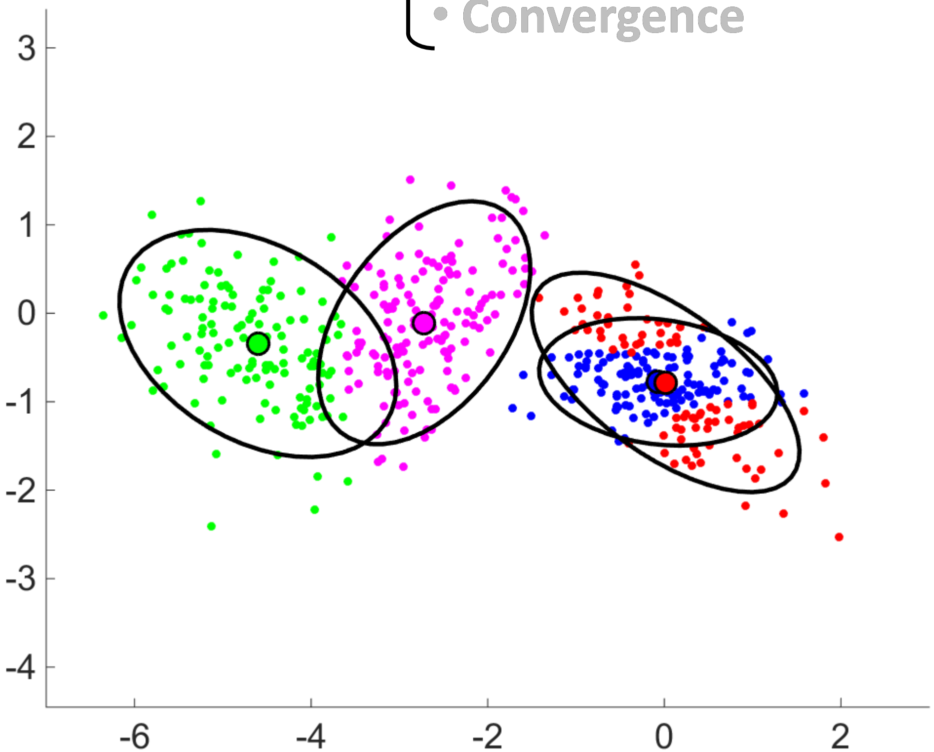

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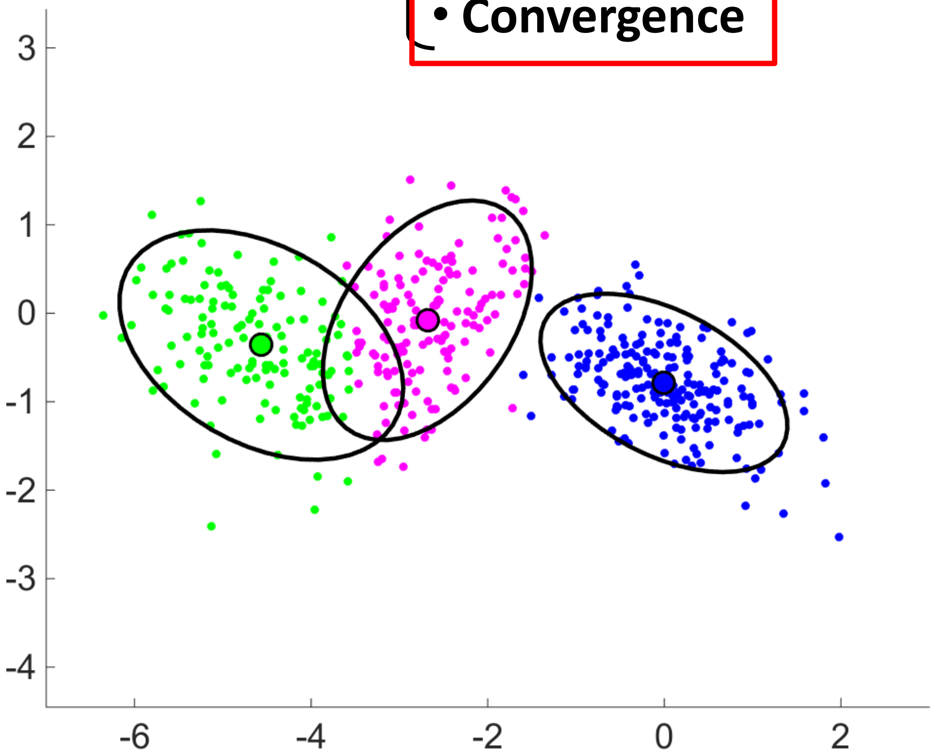

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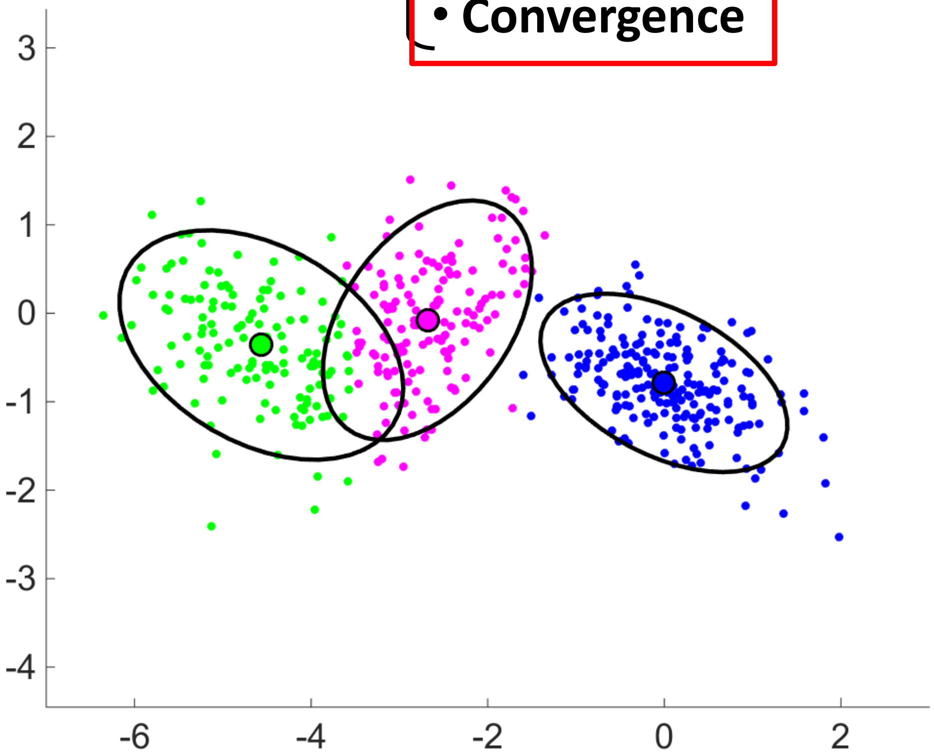

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Conclusions on GMM VB-EM

- Similar computational time as GMM-EM (though slightly more iterations)
- Priors on Gaussian weights handle automatically degenerate or unused clusters
- Determination of K
- Works even for very small data samples

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