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Partial Differential Equations

Chapter 1 - Ordinary Differential Equations Part 2

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The Engineering Program of CentraleSupélec

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Outline of this chapter

On Tuesday:

- Introduction
- Linear ODEs
- Theoretical resolution

Today:

- Qualitative properties / Stability
- Numerical resolution

I.4. Qualitative properties. Stability

I.4.1. Equilibrium of an autonomous ODE

Autonomous ODE

Let $t^0 \in \mathbb{R}$ be given and $I \subset \mathbb{R}$ be an open interval containing t^0.
Let $\mathcal{U}$ be an open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$). Let $y^0 \in \mathcal{U}$.
Let $f : I \times \mathcal{U} \to \mathbb{R}^d$.

We consider:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Definition I.4.1

The ODE is said to be **autonomous** *if f is constant with respect to its first variable.*

Example

$y'(t) = 3y(t)$ *is autonomous* ($f(t,x) = 3x$).
$y'(t) = 3y(t) + t$ *is not* ($f(t,x) = 3x + t$).

Equilibria

Let us consider an autonomous ODE

$$\begin{cases} y'(t) = f(y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Definition I.4.2

The ODE has an equilibrium at x^* if $f(x^*) = 0$.
When $y^0 = x^*$, the solution $y(t)$ is stationary.

Example

The ODE $y'(t) = 3y(t)$ has an equilibrium at $x^* = 0$.

I.4.2. Stability of equilibria. Lyapunov Theorem

Lyapunov Stability

Definition I.4.3

The equilibrium x^* is said to be **Lyapunov stable***, if for any neighborhood W of x^*, there exists another neighborhood V of x^* such that*

$$y^0 \in V \Rightarrow \forall t > 0, \ y(t) \in W$$

Lyapunov stability of x^* means that solutions starting "close enough" to x^* will remain "close" to x^* forever.

Definition I.4.4

An equilibrium x^* which is not Lyapunov stable is called **unstable***.*

Asymptotic Stability

Definition I.4.5

The equilibrium x^* is said to be **asymptotically stable**, *if there exists a neighborhood V of x^* such that*

$$y^0 \in V \Rightarrow \lim_{t \to +\infty} y(t) = x^*$$

Asymptotic stability of equilibrium x^* means that solutions starting "close enough" to x^* not only remain close to x^* but will also converge to x^*.

Exponential Stability

Definition I.4.6

The equilibrium x^* is said to be **exponentially stable** *if it is asymptotically stable and there exist $\alpha > 0$, $\beta > 0$ and a neighborhood V of x^* such that*

$$y^0 \in V \Rightarrow \forall t > 0, \|y(t) - x^*\| \leq \alpha \|y(0) - x^*\| e^{-\beta t}$$

Exponential stability of equilibrium x^* means that solutions not only converge, they converge rapidly.

Example

$y'(t) = -3y(t)$ has an equilibrium $x^* = 0$. It is exponentially stable because $y(t) = y(0)\exp(-3t)$. We have $\alpha = 1$ and $\beta = -3$

Types of Stability

Remark I.4.7

Exponential stability $\Rightarrow$ *Asymptotic stability* $\Rightarrow$ *Lyapunov stability*

Stability of Linear ODEs

With these definitions in mind we can rewrite corollary I.2.10 seen on Tuesday.

Corollary I.2.10

Consider $y' = Ay$ *where* $A \in \mathcal{M}_d(\mathbb{R})$, $d \in \mathbb{N}^*$.
$x^* = 0$ *is the only equilibrium.*

- *If* $\sigma(A) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) < 0\}$ *then 0 is exponentially stable.*
- *If* $\sigma(A) \cap \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\} \neq \emptyset$ *then 0 is unstable*

Stability in the General Case

Theorem I.4.8 (Lyapunov)

Assume f is C^2

If x^* is an equilibrium and all the eigenvalues of the Jacobian matrix $Df(x^*)$ have negative real parts, then x^* is exponentially stable.

If at least one eigenvalue has a positive real part, then x^* is unstable.

This theorem can be proven by investigating the behavior of the ODE in the neighborhood of the equilibrium.

I.4.3. Linearization of an ODE

Linearization

Consider $y'(t) = f(y(t))$ with an equilibrium x^*.

Define $\xi(t) = y(t) - x^*$.

$$\begin{aligned}
\xi'(t) &= y'(t) \\
&= f(y(t)) \\
&= f(x^* + \xi(t)) \\
&= f(x^*) + Df(x^*)\xi(t) + O(\|\xi(t)\|^2)
\end{aligned}$$

$Df(x^*) = [\partial f_i/\partial x_j]_{ij}$ denotes the $d \times d$ Jacobian matrix evaluated at the equilibrium x^*. And $O(\|\xi(t)\|^2)$ denotes terms of quadratic and higher order in the components $\xi_i(t)$.

Taking into account that $f(x^*) = 0$ and ignoring the small term $O(\|\xi(t)\|^2)$, we get the linear system:

$$\xi'(t) = Df(x^*)\xi(t)$$

Sketch of the Lyapunov Theorem proof

Definition I.4.9

Consider the ODE $y'(t) = f(y(t))$ *with an equilibrium* x^* *and* $\xi(t) = y(t) - x^*$.

$$\xi'(t) = Df(x^*)\xi(t)$$

is called the **linearization** *of the ODE around* x^*.

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Provided f is C^1, the behavior of the ODE around x^* is “close” to the behavior of its linear counterpart.

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Note: this statement is here to understand what is happening but cannot be used in a proof. One needs to choose a $\varepsilon > 0$, etc.

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Provided f is C^1, the behavior of the ODE around x^* is "close" to the behavior of its linear counterpart.

Note: this statement is here to understand what is happening but cannot be used in a proof. One needs to choose a $\varepsilon > 0$, etc. For this reason what follows is a "sketch of a proof" and not a "proof".

Sketch of the Lyapunov Theorem proof

We apply Corollary I.2.10 to the linearization of the ODE around the equilibrium point x^*

If $\sigma(Df(x^*)) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) < 0\}$ then x^* is exponentially stable.
If $\sigma(Df(x^*)) \cap \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\} \neq \emptyset$ then x^* is unstable.

Remark I.4.10

We assumed that all of the eigenvalues are with positive or negative real parts. (no eigenvalue has its real part equal to zero).

Definition I.4.11

x^* is called a **hyperbolic** *equilibrium if all the eigenvalues of $Df(x^*)$ have non-zero real parts. Otherwise it is called* **degenerate**.

I.4.4. Chaos

We have addressed the question of an initial condition “close” to the equilibrium.

Regardless of equilibria and more generally, given an ODE and two initial conditions that are “close”, it is possible that the solutions will not remain “close”.

This is a major problem for the engineer because measurements are never exact and because numerical errors will happen.

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Well-Posedness requires existence of a solution, uniqueness, **and** continuity of the solutions with respect to the initial conditions.

The equations for the pendulum are well-posed.

What about a double-pendulum?

Credit: Steven B. Troy

Credit: Brian Weinstein

In the left animation both penduli begin horizontally.

In the right animation the red pendulum begins horizontally and the blue one is rotated by 0.1 rad.

The behaviors are drastically different!

Definition I.4.12

The behavior of the system is **chaotic** *when small differences in the initial conditions produce very different outcomes.*

Consequently, long-term prediction of these systems is impossible in general.

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This is also known as the "butterfly effect": the small change in air pressure due to a butterfly flapping its wings can lead to a storm. There would not have been a storm without this butterfly.

"Chaos is when the present determines the future, but the approximate present does not approximately determine the future."
Edward Lorenz (1917–2008)

I.5. Approximation of solutions

I.5.1. Introduction. The Euler Method.

Let $d = 1$. We consider:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Let $h > 0$. The Taylor expansion yields

$$y(t^0 + h) = y(t^0) + y'(t^0)h + O(h^2)$$

Since $y'(t^0) = f(t^0, y(t^0))$, we get

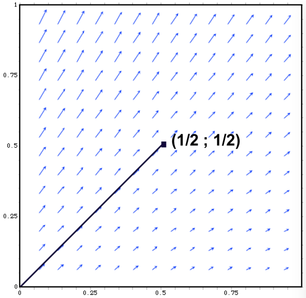
$$y(t^0 + h) = y(t^0) + h\, f(t^0, y(t^0)) + O(h^2)$$

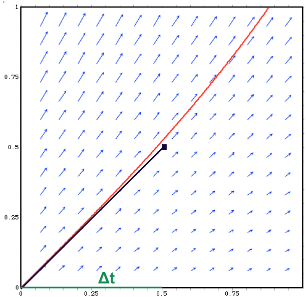
$O(h^2)$ can be made smaller than any chosen $\varepsilon > 0$, provided h is small enough therefore we can approximate $y(t^0 + h)$ by $y(t^0) + h\, f(t^0, y(t^0))$ with an error smaller than ε. Both terms can be computed.

We can now repeat this process to compute $y(t^0 + 2h)$ then $y(t^0 + 3h)$, etc.

This way to approximate the solution is called the **Euler Forward Method**, or in short, the **Euler Method**.

Graphically, it means we use the slope field to approximate the solution.





Definition I.5.1

The Euler forward method is given by

$$\begin{cases} z^0 = y^0 \\ z^{n+1} = z^n + h\, f(t^n, z^n) \;\; \textit{for } n \in \{0, \ldots, N\} \\ t^{n+1} = t^n + h \;\; \textit{for } n \in \{0, \ldots, N\} \end{cases}$$

Remark I.5.2

It means we substitute $y'(t^n)$ by

$$\frac{z^{n+1} - z^n}{h}$$

in the IVP.

Example

$y'(t) = \exp(-y(t)) + t$ *with the initial condition* $y(0) = 0$.
We have $f(t,x) = \exp(-x) + t$.

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$$z^0 = 0$$

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$$\begin{aligned} z^0 &= 0 \\ z^1 &= z^0 + h\, f(t^0, z^0) \;=\; 0.5 \end{aligned}$$

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$$
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z^0 & = & 0 \\
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\end{array}
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z^4 & = & z^3 + h\, f(t^3, z^3) & \simeq & 2.567
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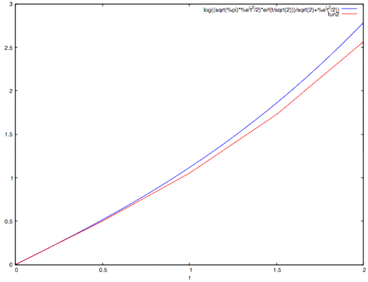
$$
\begin{array}{lllllll}
z^0 & = & 0 & & & \leftarrow & y(0) \\
z^1 & = & z^0 + h\, f(t^0, z^0) & = & 0.5 & \leftarrow & y(0.5) \\
z^2 & = & z^1 + h\, f(t^1, z^1) & \simeq & 1.053 & \leftarrow & y(1) \\
z^3 & = & z^2 + h\, f(t^2, z^2) & \simeq & 1.728 & \leftarrow & y(1.5) \\
z^4 & = & z^3 + h\, f(t^3, z^3) & \simeq & 2.567 & \leftarrow & y(2)
\end{array}
$$

Example

With some work, one could actually find the closed form for y.

Compare the approximate solution with the exact solution.

t	Euler	Exact
0.0	0.000	0.000
0.5	0.500	0.517
1.0	1.053	1.118
1.5	1.728	1.860
2.0	2.567	2.787



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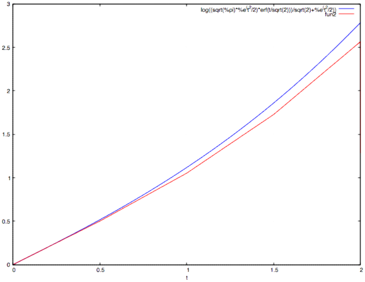
$\uparrow$	$\uparrow$	$\uparrow$
t^n	z^n	$y(t^n)$



Example

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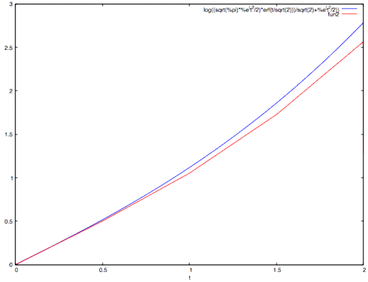
Compare the approximate solution with the exact solution.

t	Euler	Exact	Error
0.0	0.000	0.000	0.000
0.5	0.500	0.517	0.017
1.0	1.053	1.118	0.065
1.5	1.728	1.860	0.132
2.0	2.567	2.787	0.220

$\uparrow$ t^n $\quad$ $\uparrow$ z^n $\quad$ $\uparrow$ $y(t^n)$ $\quad$ $\uparrow$ $y(t^n) - z^n$



Question:
Can we make the approximate solution (z^n) as close as we want to the real solution $(y(t^n))$, provided we choose a small enough Δt?

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Can we make the approximate solution (z^n) as close as we want to the real solution $(y(t^n))$, provided we choose a small enough Δt?

Let

$$e^n = y(t^n) - z^n \text{ for } n \in \{0, \dots, N-1\}$$

$$E^N = \max_{0 \le n \le N} \|e^n\|$$

Do we have

$$\lim_{\Delta t \to 0} E^N = 0$$

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Spoiler Alert! The Euler method is convergent (provided f is Lipschitz continuous)

I.5.2. Discretization and k-step methods

Discretization

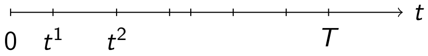
Definition I.5.3 (mesh)

Let $N \in \mathbb{N}^*$.

The $(N+1)$-tuple $\mathcal{T} = (t^0, t^1, \ldots, t^N)$ where $0 = t^0 < t^1 < \ldots < t^N = T$ is called a **mesh** *of $[0, T]$.*

The positive number $\Delta t^n = t^{n+1} - t^n$ is called the **step**.

If Δt^n is constant (thus equal to T/N), the mesh is called **regular**. *It is often denoted h.*



k-step Methods

Let $d \in \mathbb{N}^*$. We note $\|\cdot\|$ the norm on $\mathbb{R}^d$.
Let $t^0 = 0$ and $T > 0$. Let $\mathcal{T} = (t^0, \dots, t^N)$ be a mesh on $[0, T]$.
Let $f \in C^0([0, T] \times \mathbb{R}^d)$, Lipschitz continuous with respect to its second variable, with a Lipschitz constant L. We consider:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Definition I.5.4

A sequence $(z^n)_{n\in\{0,\dots,N\}}$, defined using the relation

$$z^{n+1} = F_{\mathcal{T}}(t^n, z^{n+1}, z^n, \dots, z^{n-k+1})$$

*such that z^n should approximate $y(t^n)$ is called a k-**step method**.*

Explicit vs. Implicit Methods

Definition I.5.5 (Explicit Methods)

When the relation

$$z^{n+1} = F_{\mathcal{T}}(t^n, z^{n+1}, z^n, \dots, z^{n-k+1})$$

can be expressed

$$z^{n+1} = F_{\mathcal{T}}(t^n, \cancel{z^{n+1}}, z^n, \dots, z^{n-k+1})$$

the method will be called **explicit**.

The value of z^{n+1} can simply be computed by evaluating $F_{\mathcal{T}}$ on the previous values of the sequence.

Explicit vs. Implicit Methods

Definition I.5.6 (Implicit Methods)

When the relation

$$z^{n+1} = F_{\mathcal{T}}(t^n, z^{n+1}, z^n, \dots, z^{n-k+1})$$

requires solving for z^{n+1}, the method is called **implicit**.

Remark I.5.7

While implicit methods are more complicated to put in place and each iteration takes more time, they will have an advantage over explicit methods in terms of "stability" (term to be defined later).

Example

What kind of a method is the Euler Forward Method?

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$$\begin{cases} z^0 = y^0 \\ z^{n+1} = z^n + h\, f(t^n, z^n) \ \text{ for } n \in \{0, \dots, N\} \end{cases}$$

Example

What kind of a method is the Euler Forward Method?

$$\begin{cases} z^0 = y^0 \\ z^{n+1} = z^n + h\, f(t^n, z^n) \ \text{ for } n \in \{0, \dots, N\} \end{cases}$$

It is a one-step explicit method.

$$z^{n+1} = F_{\mathcal{T}}(t^n, z^n)$$

with $F_{\mathcal{T}}(t, x) = x + h\, f(t, x)$.

One-step methods

When $k = 1$,

- Implicit methods are of the form $z^{n+1} = F_{\mathcal{T}}(t^n, z^{n+1}, z^n)$.

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One-step methods

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  When the mesh is regular with a step $h = \Delta t$, we will usually write $F_{\mathcal{T}}(t, x) = x + h\, \Phi(t, h, x)$, so one-step explicit methods are written:
  $$z^{n+1} = z^n + h\, \Phi(t^n, h, z^n)$$
  The function Φ, defined on $[0, T] \times [0, \Delta t] \times \mathcal{U}$, is called the increment function.

One-step methods

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  $$z^{n+1} = z^n + h\,\Phi(t^n, h, z^n)$$
  The function Φ, defined on $[0, T] \times [0, \Delta t] \times \mathcal{U}$, is called the increment function.

  We often choose Φ continuous with respect to the two first variables (t and Δt) and C^1 with respect to the third one y.

From now on, we will consider one-step methods.

However, everything can be adapted to k-step methods rather easily.

I.5.3. Convergence

Consistency

Definition I.5.8 (Local Truncation Error)

Consider the IVP

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t^0) = y^0 \end{cases}$$

and a one-step method given by $z^0 = y^0$ and the recurrence relation $z^{n+1} = F_{\mathcal{T}}(t^n, z^{n+1}, z^n)$.
We define the local truncation error

$$\varepsilon^n = y(t^{n+1}) - F_{\mathcal{T}}(t^n, y(t^{n+1}), y(t^n))$$

The local truncation error is the difference between a true solution y to the IVP at t^{n+1} and what happens to the true solution y to the IVP when we plugged it at t^n into the method.

Consistency

Example

The local truncation error for a one-step explicit method is

$$\varepsilon^n = y(t^{n+1}) - y(t^n) - \Delta t\ \Phi(t^n, \Delta t, y(t^n))$$

Consistency

Definition I.5.9 (Consistency)

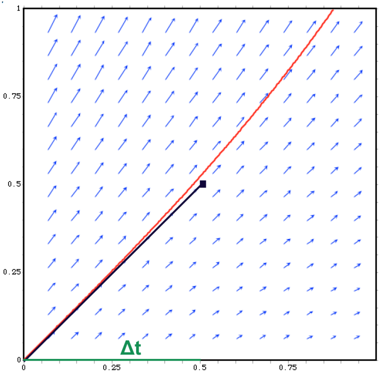
The numerical method is **consistent** *if*

$$\lim_{\Delta t \to 0} \sum_{n=0}^{N-1} \|\varepsilon^n\| = 0$$

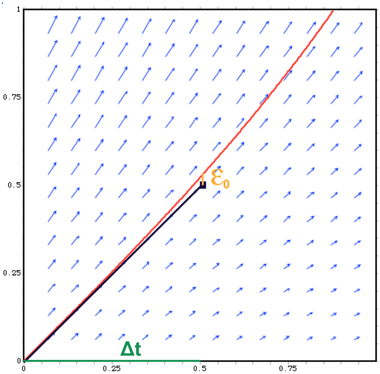
Furthermore, the consistency is of order p if $\varepsilon^n = O((\Delta t)^{p+1})$

The local truncation errors will add up. Consistency means we can render the sum as small as we want, provided we chose a small enough step Δt.

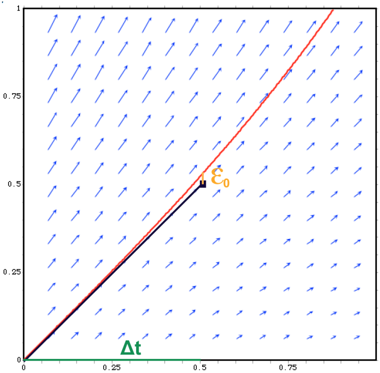
Consistency



Consistency

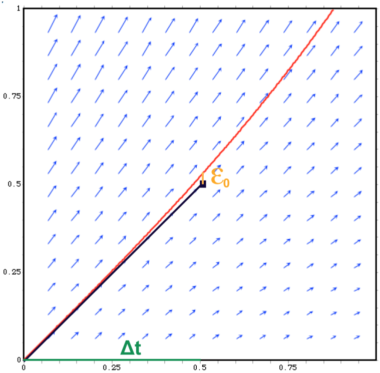


Consistency



Consistency means the sum of all of the length of the orange lines needs to tend to 0 when $\Delta t \to 0$.

Consistency



Consistency means the sum of all of the length of the orange lines needs to tend to 0 when $\Delta t \to 0$. The number of such lines will go up as Δt decreases. So their length really needs to go down.

Consistency

Example

Consider the Euler forward method ($\Phi(t, h, x) = f(t, x)$).
We have $y(t^{n+1}) = y(t^n + \Delta t)$, hence

$$y(t^{n+1}) = y(t^n) + \Delta t\, f(t^n, y(t^n)) + O(\Delta t^2)$$

Consistency

Example

Consider the Euler forward method ($\Phi(t,h,x) = f(t,x)$).
We have $y(t^{n+1}) = y(t^n + \Delta t)$, *hence*

$$y(t^{n+1}) - y(t^n) - \Delta t\, f(t^n, y(t^n)) = O(\Delta t^2)$$

Consistency

Example

Consider the Euler forward method ($\Phi(t, h, x) = f(t, x)$).
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We have $y(t^{n+1}) = y(t^n + \Delta t)$, hence

$$\varepsilon^n = y(t^{n+1}) - y(t^n) - \Delta t\, f(t^n, y(t^n)) = O(\Delta t^2)$$

$$\sum_{n=0}^{N-1} \|\varepsilon^n\| = \sum_{n=0}^{N-1} O(\Delta t^2) = N\, O(\Delta t^2) = \frac{T}{\Delta t}\, O(\Delta t^2) = O(\Delta t)$$

Consistency

Example

Consider the Euler forward method ($\Phi(t, h, x) = f(t, x)$).
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$$\lim_{\Delta t \to 0} \sum_{n=0}^{N-1} \|\varepsilon^n\| = 0$$

The Euler forward method is consistent of order 1.

Consistency

Proposition I.5.10 (NSC for consistency)

A one-step explicit method given by Φ is consistent iff
$\forall t \in [0, T],\ \forall x \in \mathcal{U},\ \Phi(t, 0, x) = f(t, x)$

Stability

Definition I.5.11 (Stability)

Let $(\eta^n)_{n\in\{0,\ldots,N\}}$ be a "perturbing" sequence.

For z^0 and w^0 in $\mathcal{U}$, define for all $n \in \{0, \ldots, N-1\}$:

$$\begin{aligned} z^{n+1} &= F_{\mathcal{T}}(t^n, z^{n+1}, z^n) \\ w^{n+1} &= F_{\mathcal{T}}(t^n, w^{n+1}, w^n) + \eta^n \end{aligned}$$

The numerical method is **stable** *if there exists a constant K s.t. for all z^0, w^0, $(\eta^n)_{n\in\{0,\ldots,N\}}$ we have*

$$\forall n \in \{0, \ldots, N\}, \quad \|z^n - w^n\| \leq K \left(\|z^0 - w^0\| + \sum_{n=0}^{N-1} \|\eta^n\| \right)$$

Stability means the method does not amplify small perturbations.

Stability

Definition I.5.12 (Stability for a one-step explicit method)

Let $(\eta^n)_{n\in\{0,\dots,N\}}$ be a "perturbing" sequence.

For z^0 and w^0 in $\mathcal{U}$, define for all $n \in \{0,\dots,N-1\}$:

$$\begin{aligned} z^{n+1} &= z^n + \Delta t\,\Phi(t^n, \Delta t, z^n) \\ w^{n+1} &= w^n + \Delta t\,\Phi(t^n, \Delta t, w^n) + \eta^n \end{aligned}$$

The numerical method is **stable** *if there exists a constant K s.t. for all z^0, w^0, $(\eta^n)_{n\in\{0,\dots,N\}}$ we have*

$$\forall n \in \{0,\dots,N\}, \quad \|z^n - w^n\| \le K\left(\|z^0 - w^0\| + \sum_{n=0}^{N-1} \|\eta^n\|\right)$$

Stability means the method does not amplify small perturbations.

Stability

From

$$\begin{aligned} z^{n+1} &= z^n + \Delta t\, f(t^n, z^n) \\ w^{n+1} &= w^n + \Delta t\, f(t^n, w^n) + \eta^n \end{aligned}$$

We derive

$$z^{n+1} - w^{n+1} = z^n - w^n + \Delta t\, (f(t^n, z^n) - f(t^n, w^n)) - \eta^n$$

Therefore

$$\begin{aligned} \|z^{n+1} - w^{n+1}\| &\leq \|z^n - w^n\| + \Delta t\, \|f(t^n, z^n) - f(t^n, w^n)\| + \|\eta^n\| \\ &\leq \|z^n - w^n\| + \Delta t\, L\, \|z^n - w^n\| + \|\eta^n\| \\ &\leq \underbrace{(1 + \Delta t\, L)}_{\leq \exp(\Delta t\, L)} \|z^n - w^n\| + \|\eta^n\| \end{aligned}$$

Stability

Lemma I.5.13

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be non-negative sequences.
Let $q \in \mathbb{R}^+$ and assume $\forall n \in \mathbb{N}^*,\ \alpha^{n+1} \leq q\alpha^n + \beta^n$ then

$$\alpha^N \leq q^N \alpha^0 + \sum_{n=0}^{N-1} q^{N-1-n} \beta^n$$

Stability

Lemma I.5.13

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be non-negative sequences.
Let $q \in \mathbb{R}^+$ and assume $\forall n \in \mathbb{N}^*,\ \alpha^{n+1} \leq q\alpha^n + \beta^n$ then

$$\alpha^N \leq q^N \alpha^0 + \sum_{n=0}^{N-1} q^{N-1-n} \beta^n$$

With $\alpha^n = \|z^n - w^n\|$, $\beta^n = \|\eta^n\|$, $q = \exp(\Delta t\, L)$, we get

$$\|z^N - w^N\| \leq \exp(\Delta t\, L)^N \|z^0 - w^0\| + \sum_{n=0}^{N-1} \exp(\Delta t\, L)^{N-1-n} \|\eta^n\|$$

Stability

Lemma I.5.13

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be non-negative sequences.
Let $q \in \mathbb{R}^+$ and assume $\forall n \in \mathbb{N}^*,\ \alpha^{n+1} \leq q\alpha^n + \beta^n$ then

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$$\|z^N - w^N\| \leq \exp(\Delta t\, L)^N \|z^0 - w^0\| + \sum_{n=0}^{N-1} \exp(\Delta t\, L)^{N-1-n} \|\eta^n\|$$

$$\|z^N - w^N\| \leq \exp(LT)\|z^0 - w^0\| + \sum_{n=0}^{N-1} \exp(LT)\|\eta^n\|$$

Stability

Lemma I.5.13

Let $(\alpha_n)_{n\in\mathbb{N}}$ and $(\beta_n)_{n\in\mathbb{N}}$ be non-negative sequences.
Let $q \in \mathbb{R}^+$ and assume $\forall n \in \mathbb{N}^*,\ \alpha^{n+1} \leq q\alpha^n + \beta^n$ then

$$\alpha^N \leq q^N\alpha^0 + \sum_{n=0}^{N-1} q^{N-1-n}\beta^n$$

With $\alpha^n = \|z^n - w^n\|$, $\beta^n = \|\eta^n\|$, $q = \exp(\Delta t\, L)$, we get

$$\|z^N - w^N\| \leq \exp(\Delta t\, L)^N \|z^0 - w^0\| + \sum_{n=0}^{N-1} \exp(\Delta t\, L)^{N-1-n} \|\eta^n\|$$

$$\|z^N - w^N\| \leq \exp(LT)\left(\|z^0 - w^0\| + \sum_{n=0}^{N-1} \|\eta^n\|\right)$$

Stability

This proves:

Proposition I.5.14

The one-step method is stable provided Φ is Lipschitz-continuous in its second variable.

Remark I.5.15

It is the case under the hypothesis we chose for the Euler numerical method.

Remark I.5.16

However, $\exp(LT)$ is bad news if one wants to see what is going on for large values of T.

Convergence

Definition I.5.17

The numerical method to approximate

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(0) = y^0 \end{cases}$$

converges if

$$\forall y^0 \in \mathcal{U}, \quad \underbrace{\lim_{\Delta t \to 0} \max_{0 \le n \le N} \| \underbrace{y(t^n) - z^n}_{\mathbf{e}^n} \|}_{E^N} = 0.$$

Furthermore, if $E^N = O((\Delta t)^p)$ the method is of **order** *p.*

Convergence

Convergence means the numerical solution approaches the exact solution of the ODE and converges to it as the step tends to zero.

Convergence

Theorem I.5.18 (Lax Equivalence)

A consistent numerical scheme for which the initial-value problem is well posed is convergent if and only if it is stable.

Since the Euler Forward method is both consistent and stable, we have:

Corollary I.5.19

The Euler Forward method is convergent.

Convergence. Proof of the Lax Equivalence Theorem

Define

$$e^n = y(t^n) - z^n \text{ for } n \in \{0, \dots, N-1\}$$
$$E^N = \max_{0 \leq n \leq N} \|e^n\|$$

Provided the numerical method (z^n) is stable and consistent (of order p), we want to prove

$$\lim E^N = 0 \quad (\text{and } E^N = O(\Delta t^p))$$

Let m be a parameter. Define the sequence $(Z_m^n)_{n \geq m}$ by

$$Z_m^n = \begin{cases} y(t^m) & \text{if } n = m \\ F_{\mathcal{T}}(t^{n-1}, Z_m^n, Z_m^{n-1}) & \text{if } n > m \end{cases}$$

Note that m is a parameter and the index of the sequence is superscript n.

We have $(z^n)_{n \in \mathbb{N}} = (Z_0^n)_{n \in \mathbb{N}}$.

Convergence. Proof of the Lax Equivalence Theorem

$$e^n = y(t^n) - z^n$$

Convergence. Proof of the Lax Equivalence Theorem

$$\begin{aligned} e^n &= y(t^n) - z^n \\ &= Z_n^n - Z_0^n \end{aligned}$$

Convergence. Proof of the Lax Equivalence Theorem

$$
\begin{aligned}
e^n &= y(t^n) - z^n \\
&= Z_n^n - Z_0^n \\
&= Z_n^n - Z_{n-1}^n + Z_{n-1}^n - \ldots - Z_1^n + Z_1^n - Z_0^n
\end{aligned}
$$

Convergence. Proof of the Lax Equivalence Theorem

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&= Z_n^n - Z_{n-1}^n + Z_{n-1}^n - \ldots - Z_1^n + Z_1^n - Z_0^n
\end{aligned}
$$

$$
\|e^n\| \leq \|Z_n^n - Z_{n-1}^n\| + \ldots + \|Z_k^n - Z_{k-1}^n\| + \ldots + \|Z_1^n - Z_0^n\|
$$

Convergence. Proof of the Lax Equivalence Theorem

$$\begin{aligned} e^n &= y(t^n) - z^n \\ &= Z_n^n - Z_0^n \\ &= Z_n^n - Z_{n-1}^n + Z_{n-1}^n - \ldots - Z_1^n + Z_1^n - Z_0^n \end{aligned}$$

$$\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\|$$

Convergence. Proof of the Lax Equivalence Theorem

$$
\begin{aligned}
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\end{aligned}
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$$
\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\|
$$

The sequences $(Z_k^n)_{n \geq k}$ and $(Z_{k-1}^n)_{n \geq k}$ both satisfy the same recurrence relation but differ by their initialization

$$
\begin{aligned}
Z_k^k &= y(t^k) \\
Z_{k-1}^k &= F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1}))
\end{aligned}
$$

Convergence. Proof of the Lax Equivalence Theorem

$$\begin{aligned} e^n &= y(t^n) - z^n \\ &= Z_n^n - Z_0^n \\ &= Z_n^n - Z_{n-1}^n + Z_{n-1}^n - \ldots - Z_1^n + Z_1^n - Z_0^n \end{aligned}$$

$$\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\|$$

The sequences $(Z_k^n)_{n\geq k}$ and $(Z_{k-1}^n)_{n\geq k}$ both satisfy the same recurrence relation but differ by their initialization

$$\begin{aligned} Z_k^k &= y(t^k) \\ Z_{k-1}^k &= F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1})) \end{aligned}$$

We use the stability of the numerical method.

Convergence. Proof of the Lax Equivalence Theorem

Stability of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| \leq K \, \|y(t^k) - F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1}))\|$$

Convergence. Proof of the Lax Equivalence Theorem

Stability of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| \leq K \, \|y(t^k) - F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1}))\|$$

Consistency of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| = K' \Delta t^{p+1}$$

Convergence. Proof of the Lax Equivalence Theorem

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Consistency of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| = K' \Delta t^{p+1}$$

Subsequently:

$$\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\| \leq K'' \Delta t^p$$

Convergence. Proof of the Lax Equivalence Theorem

Stability of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| \leq K \left\|y(t^k) - F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1}))\right\|$$

Consistency of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| = K' \Delta t^{p+1}$$

Subsequently:

$$\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\| \leq K'' \Delta t^p$$

$$\|E^n\| \leq K''' \Delta t^p$$

Convergence. Proof of the Lax Equivalence Theorem

Stability of the numerical method yields:

$$\|Z_k^n - Z_{k-1}^n\| \leq K \, \|y(t^k) - F_{\mathcal{T}}(t^k, Z_{k-1}^k, y(t^{k-1}))\|$$

Consistency of the numerical method yields:

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Subsequently:

$$\|e^n\| \leq \sum_{k=1}^{n} \|Z_k^n - Z_{k-1}^n\| \leq K'' \Delta t^p$$

$$\|E^n\| \leq K''' \Delta t^p$$

Thus

$$\lim E^N = 0 \;\; (\text{and } E^N = O(\Delta t^p))$$

QED

Asymptotic behavior

We have investigated the situation on $[0, T]$.
Convergence means convergence on $[0, T]$.
What are the consequences of choosing a large T?

We have been given a hint that things can go astray with the Euler Forward method because the stability bound is $\exp(LT)$.

Asymptotic behavior

Let's investigate this IVP (known as the Dahlquist's test equation):

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t^0) = y^0 \end{cases}$$

Let's take $\lambda = -10$ and $y^0 = 1$.
The solution is known: $y(t) = \exp(-10t)$ and $\lim_{t\to+\infty} y(t) = 0$.

The Euler Forward Method is given by

$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t\ \lambda\ z^n \quad \text{for } n \in \{0, \dots, N\} \end{cases}$$

Thus

$$z^n = (1 + \lambda\ \Delta t)^n$$

(with $\lambda = -10$).

Asymptotic behavior

$z^n = (1 - 10\,\Delta t)^n$ will diverge if $\Delta t > \frac{2}{10}$.

Obviously, with the Euler Forward Method Δt needs to be small enough otherwise the method will not approximate the solution in the long run.

Asymptotic behavior

Definition I.5.20 (A-stability)

A method is called **A-stable**, *if* $\lim(z^n) = 0$ *when the method is applied with fixed positive h to any Dahlquist's test equation*

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t^0) = 1 \end{cases}$$

where $\mathrm{Re}(\lambda) < 0$.

Example

The Euler Forward Method is not A-stable.

I.5.4. A few good methods

The Euler Forward Method

We have already discussed the Euler Forward Method.

$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t \, \lambda \, z^n \quad \text{for } n \in \{0, \dots, N\} \end{cases}$$

The Euler Forward Method

We have already discussed the Euler Forward Method.

$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t \, \lambda \, z^n \quad \text{for } n \in \{0, \ldots, N\} \end{cases}$$

It is a one-step explicit method.

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It is a one-step explicit method.

It is convergent of order one.

The Euler Forward Method

We have already discussed the Euler Forward Method.

$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t \, \lambda \, z^n \quad \text{for } n \in \{0, \ldots, N\} \end{cases}$$

It is a one-step explicit method.

It is convergent of order one.

It is not A-stable. We can have issues for large T.

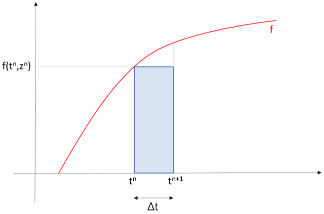
Recalling Remark I.3.5: the IVP is equivalent to

$$\forall t \in [0, T], \quad y(t) = y^0 + \int_0^t f(s, y(s))ds. \tag{1}$$

Starting with $z^0 = y^0$ and using the Euler Forward method

$$z^{n+1} = z^n + \Delta t \, f(t^n, z^n)$$

is a way to compute (1) with the left-point rectangle method.



If you are not familiar with this method, watch the video by C. Breiner at `cagnol.link/int1`.

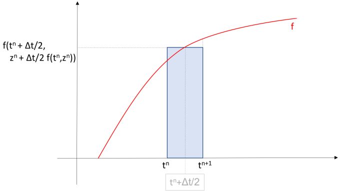
Consider a one-step explicit method with the increment function

$$\Phi(t, h, x) = f\left(t + \frac{h}{2}, x + \frac{h}{2} f(x, t)\right)$$

Starting with $z^0 = y^0$ and using

$$z^{n+1} = z^n + \Delta t\, f\left(t^n + \frac{1}{2}\Delta t, z^n + \frac{1}{2}\Delta t\, f(t^n, z^n)\right)$$

is a way to compute (1) with the mid-point rectangle method.



The value $y(t^n + \Delta t/2)$ is guessed:

$z^n + \frac{1}{2}\Delta t\, f(t^n, z^n)$.

This numerical method is called RK2.

We could be tempted to choose the increment function:

$$\Phi(t, h, x) = f\left(t + h, x + h\, f(x, t)\right)$$

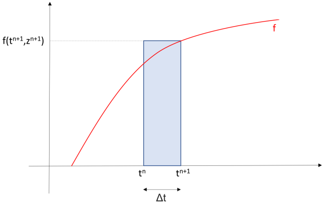
as a way to compute (1) with the right-point rectangle method.

We could be tempted to choose the increment function:

$$\Phi(t,h,x) = f\,(t+h, x+h\,f(x,t))$$

as a way to compute (1) with the right-point rectangle method.

Instead of "guessing" $y(t^n+\Delta t)$ we could note that it is y^{n+1}.



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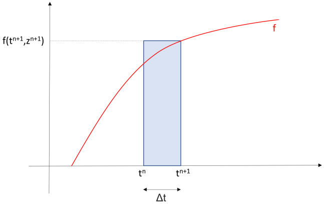

We would have $z^{n+1} = z^n + \Delta t\; f(t^{n+1}, z^{n+1})$.

We need to solve for z^{n+1}.

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$$\Phi(t, h, x) = f\left(t + h, x + h\, f(x, t)\right)$$

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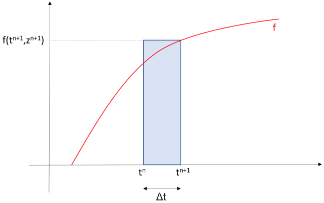

We would have $z^{n+1} = z^n + \Delta t\ f(t^{n+1}, z^{n+1})$.

We need to solve for z^{n+1}.

This requires solving for z^{n+1}. It is **not** an explicit method. It is an implicit method.

The Euler backward method

Definition I.5.21

The Euler backward method is the method obtained with

$$\begin{cases} z^0 = y^0 \\ z^{n+1} = z^n + \Delta t\, f(t^{n+1}, z^{n+1}) \quad \textit{for } n \in \{0, \dots, N\} \end{cases}$$

It is an implicit method.

The Euler backward method

Example

$y'(t) = \exp(-y(t)) + t$ with the initial condition $y(0) = 0$.
The sequence for the Euler backward method:

$$\begin{cases} z^0 = 0 \\ z^{n+1} = z^n + \Delta t \ \exp(-z^{n+1}) + t^{n+1} \quad \textit{for } n \in \{0, \dots, N\} \end{cases}$$

The Euler backward method

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Consider a regular mesh on $[0, 2]$ with step $\Delta t = h = 0.5$.

The Euler backward method

$z^0 = 0$
We need to solve $z^1 = 0.5 \exp(-z^1) + 0.5$.

The Euler backward method

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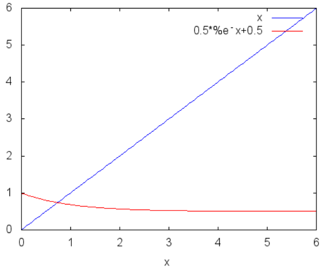
We need to solve $z^1 = 0.5\exp(-z^1) + 0.5$.



The Euler backward method

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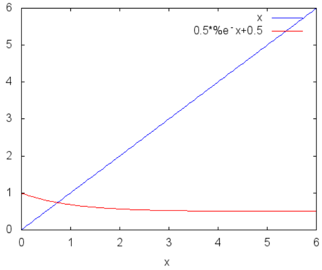

We find (numerically):
$z^1 \simeq 0.738$.

The Euler backward method

$$z^0 = 0$$
$$z^1 \simeq 0.738.$$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
We need to solve $z^2 = 0.738 + 0.5\ \exp(-z^2) + 1.$

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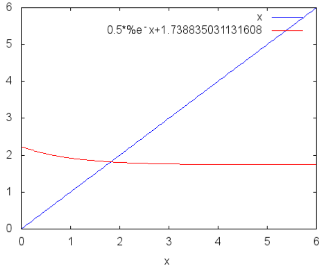

The Euler backward method

$z^0 = 0$
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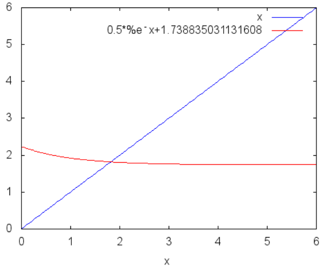
We need to solve $z^2 = 0.738 + 0.5\ \exp(-z^2) + 1.$



We find (numerically):
$z^2 \simeq 1.819.$

The Euler backward method

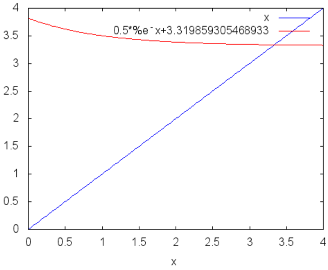
$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$
We need to solve $z^3 = 1.819 + 0.5\ \exp(-z^3) + 1.5.$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$
We need to solve $z^3 = 1.819 + 0.5\ \exp(-z^3) + 1.5.$



The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
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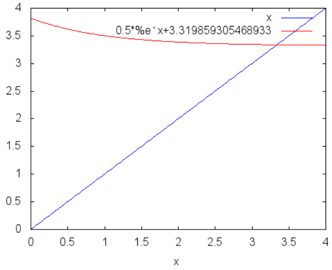
We need to solve $z^3 = 1.819 + 0.5\ \exp(-z^3) + 1.5.$



We find (numerically):
$z^3 \simeq 3.337.$

The Euler backward method

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$$z^1 \simeq 0.738.$$
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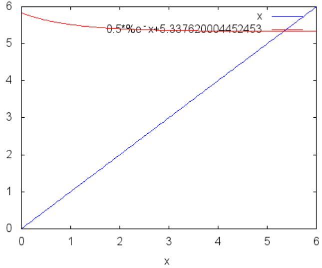
$$z^3 \simeq 3.337.$$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$
$z^3 \simeq 3.337.$
We need to solve $z^4 = 3.337 + 0.5\ \exp(-z^4) + 2.$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$
$z^3 \simeq 3.337.$
We need to solve $z^4 = 3.337 + 0.5\ \exp(-z^4) + 2.$



The Euler backward method

$z^0 = 0$
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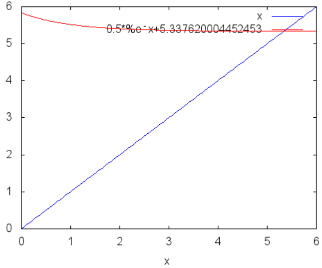
We need to solve $z^4 = 3.337 + 0.5\ \exp(-z^4) + 2.$



We find (numerically): $z^4 \simeq 5.340.$

The Euler backward method

$$
\begin{aligned}
z^0 &= 0 \\
z^1 &\simeq 0.738. \\
z^2 &\simeq 1.819. \\
z^3 &\simeq 3.337. \\
z^4 &\simeq 5.340.
\end{aligned}
$$

The Euler backward method

$z^0 = 0$
$z^1 \simeq 0.738.$
$z^2 \simeq 1.819.$
$z^3 \simeq 3.337.$
$z^4 \simeq 5.340.$

We have approximated y on $[0, 2]$.

$y(0) = z^0$
$y(0.5)$ is approximated by z^1.
$y(1)$ is approximated by z^2.
$y(1.5)$ is approximated by z^3.
$y(2)$ is approximated by z^4.

The Euler backward method

What is the behavior of the Euler Backward method on the Dahlquist test equation with $\mathrm{Re}\lambda < 0$

$$\begin{cases} y'(t) = \lambda y(t) \\ y(t^0) = 1 \end{cases}$$

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$$\begin{cases} y'(t) = \lambda y(t) \\ y(t^0) = 1 \end{cases}$$

The Euler Backward Method is given by

$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t \; \lambda \; z^{n+1} \quad \text{for } n \in \{0, \dots, N\} \end{cases}$$

The Euler backward method

What is the behavior of the Euler Backward method on the Dahlquist test equation with $\mathrm{Re}\lambda < 0$

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Thus

$$(1 - \Delta t \; \lambda) z^{n+1} = z^n$$

The Euler backward method

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Thus

$$z^{n+1} = \frac{1}{1 - \Delta t \; \lambda} z^n$$

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$$\begin{cases} z^0 = 1 \\ z^{n+1} = z^n + \Delta t \, \lambda \, z^{n+1} \quad \text{for } n \in \{0, \ldots, N\} \end{cases}$$

Thus

$$z^{n+1} = \frac{1}{1 - \Delta t \, \lambda} z^n$$

$$z^n = \left(\frac{1}{1 - \Delta t \, \lambda} \right)^n$$

The Euler backward method

Since

$$\left|\frac{1}{1-\Delta t\ \lambda}\right| < 1$$

We have $\lim z^n = 0$.

Remark I.5.22

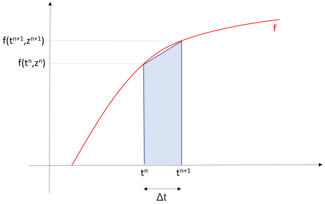
The Euler backward method is A-stable.

The Crank-Nicolson Method

Using Remark I.3.5,

- The Euler forward method corresponds to computing an integral with the left-point rectangle method.
- The Euler backward method corresponds to computing an integral with the right-point rectangle method.

What method would correspond to computing the integral with the trapezoidal rule?



The Crank-Nicolson Method

Definition I.5.23

The Crank-Nicolson method is given by

$$\begin{cases} z^0 = y^0 \\ z^{n+1} = z^n + \frac{\Delta t}{2} f(t^n, z^n) + \frac{\Delta t}{2} f(t^{n+1}, z^{n+1}) \quad \textit{for } n \in \{0, \dots, N\} \end{cases}$$

The Crank-Nicolson Method

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The Crank-Nicolas method is a second-order method.

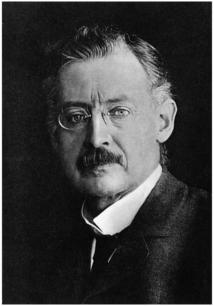
The Crank-Nicolson Method

This method was developed in the middle of the XXth century by John Crank and Phyllis Nicolson.

source: wikipedia images

The Runge-Kutta Family of Methods

The Runge-Kutta family of methods was developed at the beginning of the XXth century by Carl Runge and Martin W. Kutta.

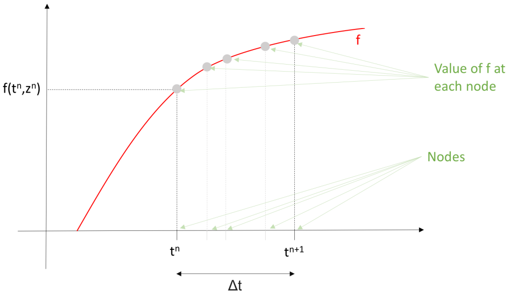


source: (l) Voit collection, (r) TUM

The Runge-Kutta Family of Methods

Idea: (based on Remark I.3.5)

- Subdivide $[t^n, t^{n+1}]$ in $s+1$ nodes.
- Guess the value of f at each note using a linear combination of the values at the previous nodes.
- Give a weight to the value of f at each node and add them up.



The Runge-Kutta Family of Methods

The family of explicit Runge-Kutta methods is given by

$$z^{n+1} = z^n + h \sum_{i=1}^{s} b_i k_i(t^n, z^n)$$

where

The Runge-Kutta Family of Methods

The family of explicit Runge-Kutta methods is given by

$$z^{n+1} = z^n + h \sum_{i=1}^{s} b_i k_i(t^n, z^n)$$

where

$$\begin{aligned}
k_1(t,x) &= f(t,x),\\
k_2(t,x) &= f(t + c_2 h, x + h(a_{21} k_1(t,x))),\\
k_3(t,x) &= f(t + c_3 h, x + h(a_{31} k_1(t,x) + a_{32} k_2(t,x))),\\
&\ \vdots\\
k_s(t,x) &= f(t + c_s h,\\
&\qquad x + h(a_{s1} k_1(t,x) + a_{s2} k_2(t,x) + \cdots + a_{s,s-1} k_{s-1}(t,x))).
\end{aligned}$$

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where

$$k_i(t,x) = f\left(t + c_i h, x + h \sum_{j=1}^{i-1} a_{ij} k_j(t,x)\right), \quad \text{for } i \in \{1, \dots, s\}$$

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where

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Parameters:

$$\begin{array}{rll} \text{Number of stages} & s & \\ \text{Runge-Kutta matrix} & a_{ij} & i \in \{1,\dots,s\},\ j \in \{1,\dots,i-1\} \\ \text{weights} & b_i & i \in \{1,\dots,s\} \\ \text{nodes} & c_i & i \in \{1,\dots,s\} \end{array}$$

The Runge-Kutta Family of Methods

The parameters of the Runge-Kutta method are often represented this way:

$$
\begin{array}{c|cccccc}
0 & & & & & \\
c_2 & a_{21} & & & & \\
c_3 & a_{21} & a_{32} & & & \\
\vdots & \vdots & & \ddots & & \\
c_s & a_{s1} & a_{s2} & \cdots & a_{s,s-1} & \\
\hline
 & b_1 & b_2 & \cdots & b_{s-1} & b_s
\end{array}
$$

This representation is called the **Butcher tableau** or simply the **tableau**.

The Runge-Kutta Family of Methods

The Euler Forward Method is in the Runge-Kutta family.

The Runge-Kutta Family of Methods

The Euler Forward Method is in the Runge-Kutta family.

Its tableau is:

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

The Runge-Kutta Family of Methods

The Euler Forward Method is in the Runge-Kutta family.

Its tableau is:

$$\begin{array}{c|c} 0 & \\ \hline & 1 \end{array}$$

Parameters: $s = 1$, $b_1 = 1$, $c_1 = 0$, a is not applicable.

$$z^{n+1} = z^n + h \sum_{i=1}^{s} b_i k_i(t^n, z^n)$$

$$k_1(t, x) = f(t, x)$$

The Runge-Kutta Family of Methods

A widely used method “RK4” is given by this tableau:

The Runge-Kutta Family of Methods

A widely used method "RK4" is given by this tableau:

$$
\begin{array}{c|cccc}
0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}
$$

The Runge-Kutta Family of Methods

A widely used method "RK4" is given by this tableau:

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0 & & & & \\
\frac{1}{2} & \frac{1}{2} & & & \\
\frac{1}{2} & 0 & \frac{1}{2} & & \\
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\hline
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The RK4 method is a fourth-order method.

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1 & 0 & 0 & 1 & \\
\hline
 & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6}
\end{array}$$

The RK4 method is a fourth-order method.
i.e. The local truncation error is of the order of $O(\Delta t^5)$
and the total accumulated error is of the order of $O(\Delta t^4)$.