Partial Differential Equations

General Introduction

John Cagnol, Pauline Lafitte

The Engineering Program of CentraleSupélec

Lecture 1 – November 26th 2019

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$$\begin{array}{rcl} u \; : \; \mathbb{R}^2 & \to & \mathbb{R} \\ (x, y) & \mapsto & u(x, y) \end{array}$$

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$$\begin{aligned} u \; : \; \mathbb{R}^2 &\to \mathbb{R} \\ (x,y) &\mapsto u(x,y) \end{aligned}$$

The equation

$$\frac{\partial u^2}{\partial x^2}(x,y) + \frac{\partial u^2}{\partial y^2}(x,y) = x^2 + y^2$$

is a PDE

Partial Differential Equations arise naturally when modeling "continuous" phenomena.

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We would like to approximate the solutions of *elliptic* PDEs.

- The Finite Element Method
- The Finite Difference Method
- Linear Algebra (!)

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We will also consider *parabolic* PDEs.

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- To generalize the concept of functions. **Chapter III**. Lecture 4
- The variational formulation. **Chapter IV**. Lecture 5

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- The Finite Element Method **Chapter V**. Lectures 6 and 7
- The Finite Difference Method **Chapter VII**. Lecture 9
- Linear Algebra (!) **Chapter VI**. Lecture 8

We will also consider *parabolic* PDEs. **Chapter VIII**. Lecture 10

Partial Differential Equations

Chapter I - Ordinary Differential Equations

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I.1. Introduction

I.1.1. Definitions

Ordinary Differential Equations

Definition I.1.1

An **ordinary differential equation (ODE)** *is an equation that relates a function with its derivatives (with respect to* **one** *variable).*
The unknown is a function.

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You know (from previous math classes) there is an inifinte number of solutions: $y(x) = \lambda \exp(mx)$ with $\lambda \in \mathbb{R}$.
Specifying $y(0)$ sets λ and provides a unique solution.

Cauchy Problem

Definition I.1.2

A **Cauchy Problem**, *also called an* **Initial Value Problem** *(IVP), is:*

- *An ordinary differential equation and*
- *A specified value of the unknown function at a given point, called the initial condition (IC).*

IVP = ODE + IC

When modeling an evolution phenomenon, the Cauchy Problem specifies how, given initial conditions, the system will evolve with time.

We often consider the initial condition at 0.

Order of an ODE

Definition I.1.3

The **order** *or the ODE is determined by the term with the highest derivatives.*

Example

$y' = my$ *is a first-order ODE.*
$y'' + 2y' + 3 = 0$ *is a second-order ODE*

I.1.2. The Importance of ODEs for Modeling

When modeling, we often have to relate quantities and their variation (derivative).

Subsequently, ODEs arise naturally in many settings:

- mechanics,
- biology,
- chemistry,
- economics,
- etc.

First Example

Let us consider a pendulum.

Credit: Steven B. Troy

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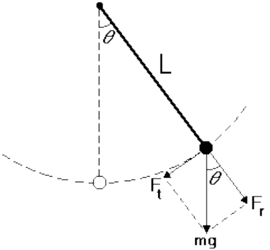

L	Length of the rod (known)
m	Mass of the bob (known)
θ	Angle of the pendulum

Photo credit: Gurjete Ukaj

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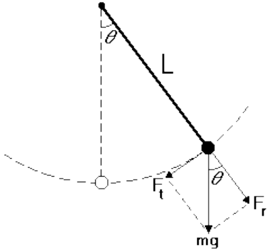

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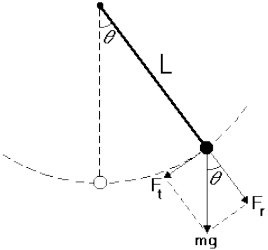

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Model hypothesis: the behavior of the system is marginally impacted by

- The weight of the rod,
- The interaction with the air.

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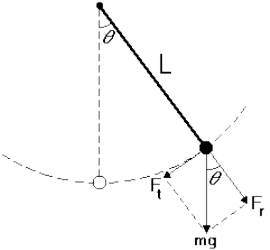

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The speed of the bob is $L\theta'$ (tangential direction)

The acceleration of the bob is $L\theta''$ (tangential direction)

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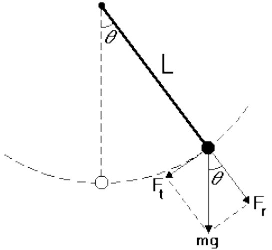

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The gravitational force can be decomposed in:

- F_r the radial component ($\|F_r\| = mg\cos\theta$).
  It is exactly balanced by the force exerted by the string.
- F_t the tangential component ($\|F_r\| = mg\sin\theta$).
  It produces the motion.

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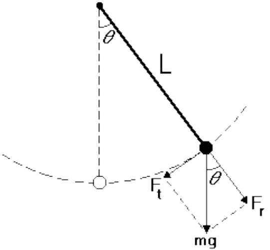

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Newton's second law states that

$$mL\theta'' = -mg\sin\theta$$

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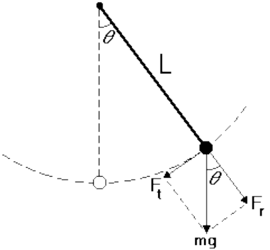

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Newton's second law states that

$$\theta'' = -\frac{g}{L}\sin\theta$$

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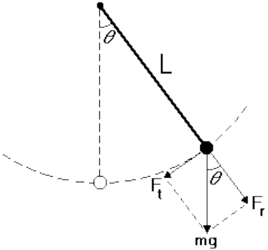

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Newton's second law states that

$$\theta'' = -\frac{g}{L}\sin\theta$$

This is an ODE (second order).

Nondimensionalization

We wrote:

$$\theta'' = -\frac{g}{L} \sin \theta.$$

But these variables have a “dimension”.

For instance, L is a “length”. It is a physical quantity.

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The meter is a unit of length that represents a definite predetermined length: ten-millionth of the length of a quadrant along the Earth’s meridian through Paris (as it was known in 1791).

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Three meters means three times this definite predetermined length.

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In the equation

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θ is an angle (unit: *rad*)

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g is an accelaration (unit: $m.s^{-2}$)

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- θ is an angle (unit: rad)
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- L is a length (unit: m)

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L is a length (unit: m)

It is useful to replace these dimensional variables with adimensional counterparts. This process is known as **nondimensionalization**, sometimes called **scaling**.

Nondimensionalization

Let $\Theta = 1\,rad$ and $\theta = \theta^*\Theta$

Let $T = 1\,s$ and $t = t^* T$. Then $d/dt = T\,d/dt^*$.

Let $G = 1\,m.s^{-2}$ and $g = g^* G$

Let $\Lambda = 1\,m$ and $L = L^*\Lambda$

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin\theta$$

yields

$$\frac{d^2\theta^*}{dt^{*2}} = -\frac{G}{LT^2}\frac{g^*}{L^*}\,\sin\theta^*$$

which is nondimensionalized.

We often drop the $*$ for the sake simplicity.

Second Example

Modeling the evolution of two species: predators and prey.

x: the amount of prey.

y: the amount of predators.

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Assumptions:

- The prey have unlimited food supply.
- The birth rate of the prey is proportional to the amount of prey (coefficient α).
- The rate of predation upon the prey is proportional to the rate at which the predators and the prey meet (coefficient β).

Therefore: the variation of the prey is $\alpha x - \beta xy$

$$x' = \alpha x + \beta xy$$

Second Example

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x: the amount of prey.
y: the amount of predators.

Assumptions:

- The predators have an unlimited appetite.
- The birth rate of the predator is proportional to the amount of predators and the amount of food available (coefficient δ).
- The mortality rate of the predators is proportional to the predators' population (coefficient γ).

Therefore: the variation of the predators is $\delta xy - \gamma y$

$$y' = \delta xy - \gamma y$$

Second Example

Modeling the evolution of two species: predators and prey.

x: the amount of prey.

y: the amount of predators.

$$\begin{cases} x' &= \alpha x - \beta xy \\ y' &= \delta xy - \gamma y \end{cases}$$

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$$\begin{cases} x' & = & \alpha x - \beta xy \\ y' & = & \delta xy - \gamma y \end{cases}$$

This is an ODE (first order) called Lotka-Volterra.
Note the unknown is U

$$\begin{array}{rcl} U : \mathbb{R} & \rightarrow & \mathbb{R}^2 \\ t & \mapsto & \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} \end{array}$$

$$U' = F(U) \ \text{ with } \ F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \alpha x - \beta xy \\ \delta xy - \gamma y \end{bmatrix}$$

Third Example

The Briggs-Rauscher Reaction.

Credit: William Escudier

Third Example

The chemical involved in the reaction are:

- I^-
- I_2
- HOI
- H^+
- H_2O
- HIO_2
- IO_3^-
- O_2
- $C_2H_2O_2$
- C_2H_3IO
- H_2O_2

Third Example

Law of mass action: the rate of a chemical reaction is proportional to the concentrations of the reacting substances.

Using the law of mass action and knowing the reaction between hydrogen peroxide (H_2O_2) and hypoiodous acid (HOI) provides:

$$(d/dt)[H_2O_2] = -k_{D_1}[HOI][H_2O_2]$$

where k_{D_1} is a parameter.

Third Example

We can derive 11 equations... An ODE (first order).

$$
\begin{aligned}
(d/dt)[HOI] &= -k_{I_1}[HOI][I^-][H^+] + k_{I_1}[I_2][H_2O] + k_{I_2}[H^+][HIO_2][I^-] + k_{I_3}[H^+]^2[IO_3^-][I^-] \\
&\quad + k_{I_4}[HIO_2]^2 - k_{D1}[HOI][H_2O_2] \\
(d/dt)[I^-] &= -k_{I_1}[HOI][I^-][H^+] + k_{I_1}[I_2][H_2O] - k_{I_2}[H^+][HIO_2][I^-] - k_{I_3}[H^+]^2[IO_3^-][I^-] \\
&\quad + k_{C_5}[C_2H_2O_2][I_2] + k_{D_1}[HOI][H_2O_2] \\
(d/dt)[H^+] &= -k_{I_1}[HOI][I^-][H^+] + k_{I_1}[I_2][H_2O] - k_{I_2}[H^+][HIO_2][I^-] - k_{I_3}[H^+]^2[IO_3^-][I^-] + k_{I_4}[HIO_2]^2 \\
&\quad k_{I_5}[H^+][IO_3^-][HIO_2] + kc5[C_3H_6O_4][I_2] + k_{D_1}[HOI][H_2O_2] \\
(d/dt)[I_2] &= k_{I_1}[HOI][I^-][H^+] - k_{I_1}[I_2][H_2O] - k_{C_5}[C_3H_6O_4][I_2] \\
(d/dt)[H_2O] &= k_{I_1}[HOI][I^-][H^+] - k_{I_1}[I_2][H_2O] + k_{D_1}[HOI][H_2O_2] \\
(d/dt)[HIO_2] &= -k_{I_2}[H^+][HIO_2][I^-] + k_{I_3}[H^+]^2[IO_3][I^-] - k_{I_4}[HIO_2]^2 + k'_{I_5}[H^+][IO_3^-][HIO_2] \\
(d/dt)[IO_3^-] &= -k_{I_3}[H^+]^2[IO_3^-][I^-] + k_{I_4}[HIO_2]^2 - k_{I_5}[H^+][IO_3^-][HIO_2] \\
(d/dt)[C_3H_6O_4] &= -k_{C_5}[C_3H_6O_4][I_2] \\
(d/dt)[C_3H_5IO_4] &= k_{C_5}[C_3H_6O_4][I_2] \\
(d/dt)[O_2] &= k_{I_5}(1/2)[H^+][IO_3^-][HIO_2] + k_{D_1}[HOI][H_2O_2] \\
(d/dt)[H_2O_2] &= -k_{D_1}[HOI][H_2O_2]
\end{aligned}
$$

Fourth Example

The Solow-Swan model is an economic model attempting to explain long-run economic growth by looking at:

- $K(t)$, the capital available at time t
- δ the depreciation of the capital

- $L(t)$, the labor force available and its efficiency at time t
- n the technological progress leading to a productivity increase

- $Y(t)$, the production at time t, of which
  - a fraction $s \in]0,1[$ is saved (thus reinvested in capital)
  - a fraction $1-s$ is being consumed.

  We usually choose the Cobb-Douglas Model to derive Y from K and L. A parameter $\alpha \in]0,1[$ is being used.

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$$Y(t) = K(t)^{\alpha} L(t)^{1-\alpha}$$

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The model is an ODE:

$$\left\{\begin{array}{rcl} L'(t) & = & nL(t) \\ K'(t) & = & sY(t) - \delta K(t) \\ Y(t) & = & K(t)^{\alpha} L(t)^{1-\alpha} \end{array}\right.$$

I.1.3. Remarks and Notations

Differentiation with respect to one variable

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The term **ordinary** is used in contrast with the term **partial** when there is more than one independent variable.

Later, we will differentiate with respect to several variables.
It will be called a **partial differential equation** or PDE in short.

Notations

The (unknown) function is often denoted y or u.
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The ODE $y'(t) = 3y(t) + t^2$ can also be denoted

$$\frac{dy(t)}{dt} = 3y(t) + t^2$$

$$\dot{y}(t) = 3y(t) + t^2$$

Sometimes (t) is omitted.

A possible false belief...

You may have learned about different types of ODEs:

- $y' = my$ for m in $\mathbb{R}$
- $y'' + ay' + b = f$ for a and b in $\mathbb{R}$ and f a function.
- $y' + P(t)y = Q(t)y^n$. (Bernoulli)
- $a(t)y' + b(t)y + c(t)y^2 + d(t) = 0$ where a, b, c and d are functions and a doesn't have any root. (Ricatti)
- $y = ty' + g(y')$ where g is a differentiable function. (Clairaut)
- etc.

In these cases, we can find an explicit formula to relate y and x.
It is called a **closed form**.

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- etc.

In these cases, we can find an explicit formula to relate y and x. It is called a **closed form**.

However, you might be under the false impression that ODEs can be "solved" by finding y... One needs to find the "trick".

Modus operandi for an ODE

In general, an ODE (IVP) cannot be “solved” by finding a closed form.

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We are interested in finding

1. If y exists around the initial condition
   (in a neighborhood of $y^0 = y(t^0)$)
2. If y exists for large t
   Are there any patterns such as periodicity?
3. If a small change on the initial condition y^0 results in a small change on the solution
4. An estimate of y (computed numerically)

I.1.4. Outline of this chapter

Outline

Today:

- Introduction
- Linear ODEs
- Theoretical resolution

On Thursday:

- Qualitative properties / Stability
- Numerical resolution

I.2. Linear ODEs

I.2.1. Definitions

Definition I.2.1

An ODE is **linear** *if it is linear in the unknown function and its derivatives*

Example

$y' = 3y$ *is a first-order linear ODE.*
$y' = 3y + t^2$ *is a first-order linear ODE.*
$y' = 3y^2 + t$ *is a first-order non-linear ODE.*
The ODEs in the four examples given earlier are all non-linear.

Definition I.2.1

An ODE is **linear** *if it is linear in the unknown function and its derivatives*

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The ODEs in the four examples given earlier are all non-linear.

Remark I.2.2

$\theta'' = -\frac{g}{L}\sin\theta$ *is a second-order non-linear ODE*
$\theta'' = -\frac{g}{L}\theta$ *is a second-order linear ODE*
If θ *is small, we may consider approximating* $\sin\theta$ *by* θ*.*
This is called a **linearization***.*

I.2.2. Dimension 1 and First Order

Finding the Closed Form

Proposition I.2.3

Let a and b be continuous functions on $\mathbb{R}$.

$$\begin{cases} y'(t) = a(t)y(t) + b(t) \\ y(0) = y^0 \in \mathbb{R} \end{cases}$$

has one solution on $\mathbb{R}$:

$$y(t) = e^{A(t)} \left(e^{-A(0)} y^0 + B(t) \right)$$

Where A is an anti-derivative of a and
B is the anti-derivative of $b \exp(-A)$ that vanishes in 0.

Finding the Closed Form

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The closed form of y can be provided if both anti-derivatives can be computed.

Finding the Closed Form

If a is a constant function (equal to m) and $b = 0$

Corollary I.2.4

The ODE

$$\begin{cases} y'(t) = my \\ y(0) = y^0 \in \mathbb{R} \end{cases}$$

has one solution on $\mathbb{R}$*:*

$$y(t) = y^0 e^{mt}$$

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has one solution on $\mathbb{R}$*:*

$$y(t) = y^0 e^{mt}$$

Remark I.2.5

If $y^0 = 0$ *then* $y(t) = 0$ *is the solution.*
It is a **stationary** *solution.*

An interesting question regarding the solutions

Consider

$$\begin{cases} y'(t) = my \\ y(0) = y^0 \in \mathbb{R} \end{cases}$$

with y^0 close to 0 (for instance $y^0 = 10^{-4}$).

Will y be close to the stationary solution 0?

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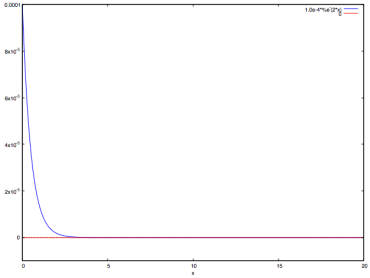
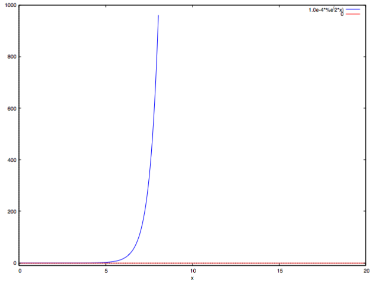


$m < 0$



$m > 0$

An interesting question regarding the solutions

Consider

$$\begin{cases} y'(t) = my \\ y(0) = y^0 \in \mathbb{R} \end{cases}$$

with y^0 close to 0 (for instance $y^0 = 10^{-4}$).

Will y be close to the stationary solution 0?

If $m < 0$ the solution coming from $y^0 = 10^{-4}$ is really close to the solution coming from $y^0 = 0$: a small disturbance of the initial condition leads to a small variation of the solution.
We will later call this **a-stability**.

If $m > 0$ the solution coming from $y^0 = 10^{-4}$ is not close to the solution coming from $y^0 = 0$: a small disturbance of the initial condition leads to potentially large variations of the solution.
We will later call this **instability**.

I.2.3. Dimension 1 and p-th Order

Let us now consider the p-th order linear ODE

$$\sum_{i=0}^{p} a_i y^{(i)} = 0$$

Where a_i are real numbers.

Note that $a_p \neq 0$, otherwise the ODE is not of order p.
We can set $a_p = 1$ with no loss of generality.

The initial condition will be given by specifying all of the $y^{(i)}(0)$.

Example

For $p = 3$

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

The initial condition is given by $y(0)$, $y'(0)$ and $y''(0)$.

Let us now consider:

$$U = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(p-1)} \end{bmatrix} \quad \text{then} \quad U' = \begin{bmatrix} y' \\ y'' \\ \vdots \\ y^{(p)} \end{bmatrix}$$

The ODE is equivalent to $U' = AU$ for

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{p-1} \end{bmatrix}$$

A is called the (transpose) companion matrix to the ODE.

Remark I.2.6

The p-th order linear scalar ODE can be replaced by a first-order linear vectorial ODE where the vector contains p components.

Example

$y'' - 4y' + 3y = 0$ *with* $y : \mathbb{R} \to \mathbb{R}$ *can be replaced by* $u' = Au$ *with* $u : \mathbb{R} \to \mathbb{R}^2$

$$\underbrace{\begin{bmatrix} y' \\ y'' \end{bmatrix}}_{u'} = \underbrace{\begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} y \\ y' \end{bmatrix}}_{u}$$

I.2.4. Dimension d and First-Order

Finding the Closed Form

Definition I.2.7

Let $A \in \mathcal{M}_d(\mathbb{R})$. *Define*

$$\exp(A) = \sum_{k=0}^{+\infty} \frac{1}{k!} A^k$$

Theorem I.2.8 (Duhamel)

For $A \in \mathcal{M}_d(\mathbb{R})$ *and* $u^0 \in \mathbb{R}^d$

$$u'(t) = Au(t)$$

$$u(0) = u^0$$

has a unique solution on $\mathbb{R}$:

$$u(t) = \exp(tA)u^0$$

Finding the Closed Form

Theorem I.2.9 (Duhamel)

For $A \in \mathcal{M}_d(\mathbb{R})$, $t \in I \mapsto b(t)$ and $u^0 \in \mathbb{R}^d$

$$\begin{cases} u'(t) = Au(t) + b(t), \\ u(0) = u^0 \end{cases}$$

has a unique solution on $\mathbb{R}$:

$$u : t \longmapsto \exp(tA)\, u^0 + \int_0^t \exp((t-s)A)\, b(s) \mathrm{d}s.$$

Finding the Closed Form

There exist $P \in \mathrm{GL}_p$ and a Jordan matrix $J = D + N$ such that

$$A = P^{-1} J P$$

- D is a diagonal matrix (with the eigenvalues)
- N is a nilpotent matrix (powers of N will eventually give 0)

Subsequently

$$A^k = (P^{-1}JP)(P^{-1}JP)\cdots(P^{-1}JP) = P^{-1}J^kP$$

$$\exp(tA) = P^{-1}\exp(tJ)P$$

and $\exp(tJ)$ can be computed thanks to its form.

If you are not familiar with this, watch the video by G. Strang at `cagnol.link/expmat` by Thursday.

Example

$y'' - 4y' + 3y = 0$ with $y(0) = 2$ and $y'(0) = 4$
can be replaced by $u' = Au$ with

$$u = \begin{bmatrix} y \\ y' \end{bmatrix}, \text{and} \qquad u(0) = \begin{bmatrix} 2 \\ 4 \end{bmatrix} \qquad \text{and} \qquad A = \begin{bmatrix} 0 & 1 \\ -3 & 4 \end{bmatrix}$$

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$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$
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$\det(A - \lambda I) = \lambda^2 - 4\lambda + 3$
The eigenvalues of A are $\lambda_1 = 1$ and $\lambda_2 = 3$. $(\sigma(A) = \{1, 3\})$

The associated eigenvectors are $v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $v_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$

$$P^{-1} = \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}, \qquad J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad P = \begin{bmatrix} \frac{3}{2} & -\frac{1}{2} \\ -\frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

We have $A = P^{-1}JP$

Example

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$$u(t) = \exp(tA)u(0)$$

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Subsequently $y(t) = e^t + e^{3t}$ is the solution to:

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(which you can verify to make sure there is no mistake in our computations)

Example

If we hadn't specified the initial conditions, we would have had:

$$\begin{bmatrix} y(t) \\ y'(t) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 3e^t - e^{3t} & e^{3t} - e^t \\ 3e^t - 3e^{3t} & 3e^{3t} - e^t \end{bmatrix} \begin{bmatrix} y(0) \\ y'(0) \end{bmatrix}$$

leading to

$$y(t) = \frac{1}{2}(3e^t - e^{3t})y(0) + \frac{1}{2}(e^{3t} - e^t)y'(0)$$

The set of solutions is a 2-dimensional linear space.

Example

$y(t) = \frac{1}{2}(3e^t - e^t)y(0) + \frac{1}{2}(e^t - e^t)y'(0)$

If $(y(0), y'(0)) = (0, 0)$ the solution is $y(t) = 0$.

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The solution would not stay close to 0 because of e^{1t} and e^{3t}.

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When we trace coefficients 1 and 3,

Example

$$e^{tA} = \frac{1}{2}\begin{bmatrix} 3e^{1t} - e^{3t} & e^{3t} - e^{1t} \\ 3e^{1t} - 3e^{3t} & 3e^{3t} - e^{1t} \end{bmatrix}$$

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$$J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \to e^{tA} = \frac{1}{2}\begin{bmatrix} 3e^{1t} - e^{3t} & e^{3t} - e^{1t} \\ 3e^{1t} - 3e^{3t} & 3e^{3t} - e^{1t} \end{bmatrix}$$

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The solution would not stay close to 0 because of e^{1t} and e^{3t}.

When we trace coefficients 1 and 3, we find them in matrix J.

Example

$$\sigma(A) = \{1, 3\} \rightarrow J = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \rightarrow e^{tA} = \frac{1}{2}\begin{bmatrix} 3e^{1t} - e^{3t} & e^{3t} - e^{1t} \\ 3e^{1t} - 3e^{3t} & 3e^{3t} - e^{1t} \end{bmatrix}$$

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They come from the eigenvalues of A.

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When we trace coefficients 1 and 3, we find them in matrix J. They come from the eigenvalues of A.

If the eigenvalues had been negative (or complex with a negative real part) then the solution would have come close to 0 rather than going away from it.

Back to the interesting question regarding the solutions

Corollary I.2.10

Consider $u' = Au$ with the notations of Theorem I.2.8.

If $u(0) = 0$ then $u(t) = 0$ is the solution. It is the **stationary** *one.*

- *If $\sigma(A) \subset \{z \in \mathbb{C}, \mathrm{Re}(z) < 0\}$*
  (all of the eigenvalues of A are in the left half complex plane)
  then $u(t)$ will converge to 0 when $t \to +\infty$.
  Furthermore $\|u(t)\| \leq u(0)e^{-\lambda t}$ where $\lambda = |\max \sigma(A)|$.
- *If $\sigma(A) \cap \{z \in \mathbb{C}, \mathrm{Re}(z) > 0\} \neq \emptyset$*
  (A has an eigenvalue in the right half complex plane)
  then $u(t)$ may diverge for some initial conditions $u(0)$.

Reminder: the set of eigenvalues of A is called the spectrum of A and denoted by $\sigma(A)$.

I.3. Theoretical resolution

I.3.1. Definitions

Formal Definitions

Let $t^0 \in \mathbb{R}$ be given.
Let $I \subset \mathbb{R}$ be an open interval containing t^0.
Let $\mathcal{U}$ be an open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$).
Let $f : I \times \mathcal{U} \to \mathbb{R}^d$
Let $y^0 \in \mathcal{U}$.
Let us consider:

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Formal Definitions

Let $t^0 \in \mathbb{R}$ be given.
Let $I \subset \mathbb{R}$ be an open interval containing t^0.
Let $\mathcal{U}$ be an open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$).
Let $f : I \times \mathcal{U} \to \mathbb{R}^d$
Let $y^0 \in \mathcal{U}$.
Let us consider:

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in I \\ y(t^0) = y^0 \end{cases}$$

Example

For $y'(t) = 3y(t) + t^2, \ \ y(0) = 1$
We will choose: $t^0 = 0, \ \ I = \mathbb{R}, \ \ d = 1, \ \ \mathcal{U} = \mathbb{R}, \ \ y^0 = 1,$
$f(t, x) = 3x + t^2.$

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Example

For $\theta'' + \frac{g}{L} \sin \theta = 0,\ \theta(0) = \frac{\pi}{2},\ \theta'(0) = 0$
We will choose: $t^0 = 0,\ \ I = \mathbb{R},\ \ d = 2,\ \ u(t) = [\theta(t), \theta'(t)]^T,$
$f(t, [x_1, x_2]^T) = [x_2, -\frac{g}{L} \sin x_1]^T.$

Formal Definitions

Definition I.3.1

The **slope field** *associated to the ODE is*

$$\begin{array}{rcl} I \times \mathcal{U} & \to & \mathbb{R}^2 \\ (t, x) & \mapsto & \begin{bmatrix} 1 \\ f(t, x) \end{bmatrix} \end{array}$$

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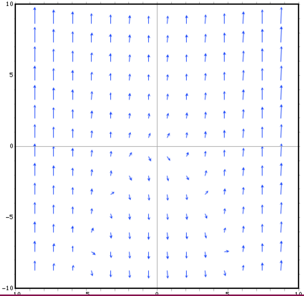
We often normalize the slope field for representation purposes.

Example:
The slope field associated to $y'(t) = 3y(t) + t^2$



Formal Definitions

Definition I.3.2

The curve of a function y solution to the IVP is called an **integral curve** *of the slope field. It is also called an* **orbit**.

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We can represent several orbits on the same graph
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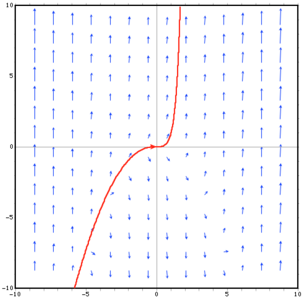
Example for
$y'(t) = 3y(t) + t^2$
$y(0) = 0$



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$y'(t) = 3y(t) + t^2$
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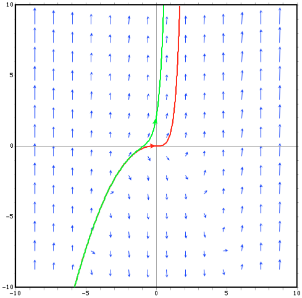

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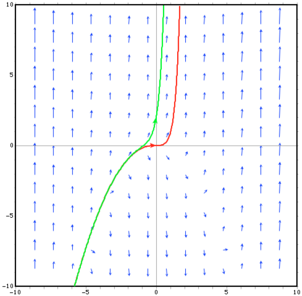

Remark I.3.3

Vectors of the slope field are tangents to the orbits.

I.3.2. The Cauchy-Lipschitz Theorem

Do solutions to the ODE exist?

Let $t^0 \in \mathbb{R}$. Let $I \subset \mathbb{R}$ be an open interval containing t^0. Let $\mathcal{U}$ be an open set of $\mathbb{R}^d$ ($d \in \mathbb{N}^*$). Let $f : I \times \mathcal{U} \to \mathbb{R}^d$. Consider:

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in I \\ y(t^0) = y^0 \end{cases}$$

What hypothesis should we make on f to guarantee existence and uniqueness of a solution y

- In a neighborhood of t^0?
- In the entire interval I?

Statement of the Theorem

Theorem I.3.4 (Local Cauchy-Lipschitz)

Consider the IVP

$$\begin{cases} y'(t) = f(t, y(t)), \quad t \in I \\ y(t^0) = y^0 \end{cases}$$

Suppose

- *For all fixed x, $t \mapsto f(t,x)$ is continuous*
- *For all fixed t, $x \mapsto f(t,x)$ is C^1*

Then, there exists $\varepsilon > 0$ such that the IVP has a unique solution y on $[t^0 - \varepsilon, t^0 + \varepsilon]$.

Sketch of the proof

Let φ_0 to be the constant function $\varphi_0(t) = y^0$.
Then define, for $k \in \mathbb{N}$

$$\varphi_{k+1}(t) = y^0 + \int_{t_0}^{t} f(s, \varphi_k(s))\, ds.$$

This is known as Picard iterations.

We have proved the existence of a fixed point. (Exercise CIP I.9).

Interpretation with an Integral

Remark I.3.5 (Integro-differential form)

Let $y \in C^1([0,T])$,

$$\begin{cases} y'(t) = f(t, y(t)) \\ y(t^0) = y^0 \end{cases}$$

is equivalent to

$$\forall t \in [0,T], \quad y(t) = y^0 + \int_{t^0}^{t} f(s, y(s)) ds.$$

This second expression is called an integro-differential.

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This second expression is called an integro-differential.

If we find a fixed point to the Picard iteration, then it will be a solution to the IVP.

Sketch of the proof

Using the Banach fixed point theorem (Theorem CIP I.2.6), the Picard iterates φ_k is convergent toward a function u.

u is the unique solution of the IVP in a neighborhood of t^0.

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Using the Banach fixed point theorem (Theorem CIP I.2.6), the Picard iterates φ_k is convergent toward a function u.

u is the unique solution of the IVP in a neighborhood of t^0.

Now let us take two solutions u and v to the IVP.
An interesting result, known as the Gronwall's inequality, will show that $u = v$.

Gronwall's inequality

Theorem I.3.6 (Gronwall's inequality)

Let $T > 0$*.*
Suppose ϕ *and* ψ *are continuous functions from* $[0, T]$ *to* $\mathbb{R}^+$ *and there exists a constant* $a \geq 0$ *such that*

$$\forall t \in [0, T], \qquad \phi(t) \leq a + \int_0^t \phi(s)\psi(s)\mathrm{d}s.$$

Then

$$\forall t \in [0, T], \qquad \phi(t) \leq a \exp\left(\int_0^t \psi(s)\mathrm{d}s\right).$$

Global solutions

Theorem I.3.7 (Global Cauchy-Lipschitz)

Consider the IVP

$$\begin{cases} y'(t) = f(t, y(t)), & t \in I \\ y(t^0) = y^0 \end{cases}$$

Suppose

- *For all fixed* x, $t \mapsto f(t, x)$ *is continuous*
- *For all fixed* t, $x \mapsto f(t, x)$ *is globally Lipschitz continuous with a Lipschitz constant independent of* t*:*
  $\exists L > 0, \forall t \in I, \forall x_1, x_2 \in \mathcal{U}, \|f(t, x_1) - f(t, x_2)\| \leq L\|x_1 - x_2\|$

Then, the IVP has a unique solution y *on* I*.*

Well-Posedness

Definition I.3.8

An ODE (and later a PDE) is **well-posed in the sense of Hadamard** *if:*

- *a solution exists,*
- *the solution is unique,*
- *the solution's behavior changes "continuously" with respect to the initial conditions.*

An ODE (or PDE) that is not well-posed is said to be **ill-posed**

The Cauchy-Lipschitz Theorem gives an answer to the first two points. But not to the third one.

Regularity of solutions

Corollary I.3.9 (Regularity)

Let $k \geq 1$ and $f \in C^k(I \times U)$.
The solution to the IVP

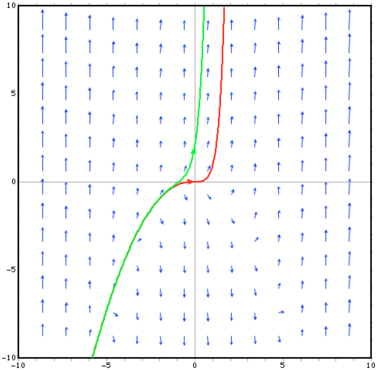
$$\begin{aligned} y'(t) &= f(t, y(t)) \\ y(t^0) &= y^0 \end{aligned}$$

is in $C^{k+1}(J)$.

Geometrical Interpretation on the Slope Field

Under the hypothesis of the Cauchy-Lipschitz theorem,
the orbits of two different initial conditions can never intersect.

(this would violate the uniqueness)



I.3.3. Flow

Flow

Theorem I.3.10

Let $f \in \mathcal{C}^2(I \times \mathcal{U})$.
For all $(t^0, y^0) \in I \times \mathcal{U}$, there exists:

- *A neighborhood $J \times \mathcal{V}$ of (t^0, y^0)*
- $\phi^{t^0} \in \mathcal{C}^1(J \times \mathcal{V}; \mathcal{U})$

such that

$$\phi^{t^0\prime}(t,x) = f(t, \phi^{t^0}(t,x)) \qquad \forall (t,x) \in J \times \mathcal{V}$$
$$\phi^{t^0}(t^0, x) = x \qquad \forall x \in \mathcal{V}$$

Definition I.3.11

ϕ^{t^0} *is called the (local)* **flow** *to the ODE at* t^0