Lecture X : Parabolic problems

A) Aims of this class

After this class,

- I can recognize a parabolic equation.
- I know the fundamental qualitative properties of a parabolic equation (asymptotic behaviour, maximum principle, regularization).
- I know how to discretize a parabolic problem in time with an Euler scheme and in space by a finite element or finite difference method.
- I know how to code an iterative algorithm that gives me a numerical solution, evolving over time.

B) To become familiar with this class' concepts (to prepare before the examples class)

Question X.1 must be done before the 10th lab. The solution is available online.

Question X.1 (Diagonalizing the Laplacian matrix)

Let $J \in \mathbb{N}^*$.

Q. X.1.1 Recall the matrix $A_{\Delta x}$ corresponding to the discretization of the operator $u \mapsto -u''$ with Dirichlet boundary conditions for a space step $\Delta x = 1/(J+1)$.

Solution de Q. X.1.1 See the corresponding lecture.

Q. X.1.2 Show that the matrix $A_{\Delta x}$ is diagonalizable with real positive eigenvalues.

Solution de Q. X.1.2 The matrix $A_{\Delta x} \in \mathcal{M}_J(\mathbb{R})$ is real, symmetric, so it is diagonalizable with real eigenvalues. Moreover, we showed that it is symmetric positive definite: its spectrum lies in $(0, +\infty)$.

Q. X.1.3 Using Gershgörin-Hadamard's theorem, show that $\mathrm{Sp}(A_{\Delta x}) \subset]0, 4/(\Delta x)^2]$.

Solution de Q. X.1.3 Thanks to the Gershgörin-Hadamard's theorem,

$$\mathrm{Sp}(A_{\Delta x}) \subset \bigcup_{1 \le j \le J} D\left(a_{jj}, \sum_{k \neq j} |a_{jk}|\right),$$

so

$$\mathrm{Sp}((\Delta x)^2 A_{\Delta x}) \subset]0, +\infty[\cap (D(2,2) \cup D(2,1)) =]0, 4].$$

Q. X.1.4 Let $\lambda \in \mathrm{Sp}((\Delta x)^2 A_{\Delta x})$. Let $\Lambda = 2 - 2\cos(\theta)$ for some $\theta \in]0, \pi]$. Let v be an eigenvector associated to Λ. Compute v and deduce the possible values of θ.

Solution de Q. X.1.4 Denote $v = (v_1, \dots, v_J)^T$. Then $(\Delta x)^2 A_{\Delta x} v = \lambda v$ is equivalent to the linear system

$$\begin{cases} 2\cos(\theta) v_1 - v_2 = 0 \\ -v_1 + 2\cos(\theta) v_2 - v_3 = 0 \\ \qquad \vdots \\ -v_{J-2} + 2\cos(\theta) v_{J-1} - v_J = 0 \\ -v_{J-1} + 2\cos(\theta) v_J = 0. \end{cases}$$

Let $v_0 = 0$. Then (v_j) is defined by a linear recursion of order 2 with characteristic polynomial $0 = r^2 - 2\cos(\theta) r + 1 = (r - e^{i\theta})(r - e^{-i\theta})$. Its solutions write

$$v_j = \alpha \cos(j\theta) + \beta \sin(j\theta),$$

with real α, β. Since $v_0 = 0$, $\alpha = 0$. Moreover, the last line of the system reads

$$\begin{aligned} 0 &= \beta(2\cos(\theta)\sin(J\theta) - \sin((J-1)\theta)) \\ &= \beta(2\cos(\theta)\sin(J\theta) - \sin(J\theta)\cos(\theta) + \sin(\theta)\cos(J\theta)) \\ &= \beta(\sin(J+1)\theta). \end{aligned}$$

Note that $2 - 2\cos(\theta) = 4\sin^2(\theta/2)$. The vector v is not zero (and consequently an eigenvector of $A_{\Delta x}$) if and only if $\sin((J+1)\theta) = 0$, that is, if $\theta = k\pi/(J+1)$ for any $k \in \{1, \ldots, J\}$, because $\lambda \neq 0$.

Q. X.1.5 What is the spectrum of $A_{\Delta x}$?

Solution de Q. X.1.5 We just found exactly J different eigenvalues: the spectrum of $A_{\Delta x}$ is

$$\mathrm{Sp}(A_{\Delta x}) = \left\{ \frac{1}{(\Delta x)^2} 4\sin^2\left(\frac{k\pi}{2(J+1)}\right), \quad k \in \{1, \ldots, J\} \right\}.$$

C) Exercises

We consider a homogeneous rod of length 1 whose temperature at time t and position x is $u(t,x)$. Both ends of the rod (positions $x = 0$ and $x = 1$) are kept at zero temperature. A heat source f is applied to every point of the rod. This heat source is constant over time. The evolution of the temperature in the rod is governed by:

$$\text{(H)} \quad \begin{cases} \partial_t u(t,x) - \alpha \partial^2_{xx} u(t,x) = f(x), \ t > 0, \ x \in]0,1[, \\ u(0,x) = u^0(x), \ x \in]0,1[, \\ u(t,0) = 0, \ u(t,1) = 0, \ t \geq 0. \end{cases}$$

We assume $u^0, f \in L^2(0,1)$.

Exercise X.1 (Finite Difference Approximation)
Let us assume $f \in C^2([0,1])$.
Let $T > 0$.
Let $(J,N) \in (\mathbb{N}^*)^2$, $\Delta x = 1/(J+1)$ and $\Delta t = T/N$. We discretize the domain $[0,T] \times [0,1]$ with a grid

$$(t^n, x_j) = (n\Delta t, j\Delta x), \ \ n \in \{0, \ldots, N\}, \ j \in \{0, \ldots, J+1\}$$

Let us consider $y' = a(t,y)$. For $\theta \in [0,1]$., we call *θ-scheme* the method

$$\begin{cases} z^0 \quad \text{given} \\ z^{n+1} = z^n + \Delta t(1-\theta)a(t^n, z^n) + \Delta t\theta a(t^{n+1}, z^{n+1}). \end{cases} \tag{X.1}$$

E. X.1.1 Let $\theta \in [0,1]$. Write the finite difference scheme as a recursive relation in time and space for (H) with the θ-scheme in temps and the three-point stencil in space.

Solution de Q. X.1.1 The approximated problem can be written $(v_j^n)_{j\in\{1,\dots,J\}}$, $n \in \{1,\dots,N\}$ such that

$$\begin{cases} \forall j \in \{1,\dots,J\}, \quad v_j^0 = u^0(x_j), \quad \forall n \geq 0,\ v_0^n = v_{J+1}^n, \\ \forall n \in \{0, N-1\},\ \forall j \in \{1,\dots,J\}, \quad \dfrac{v_j^{n+1}-v_j^n}{\Delta t} = \theta\alpha\dfrac{v_{j+1}^{n+1}-2v_j^{n+1}+v_{j-1}^{n+1}}{\Delta x^2} + (1-\theta)\alpha\dfrac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{\Delta x^2} + f(x_j). \end{cases}$$

let $n \in \{0,\dots,N-1\}$. Let us note $V^n = (v_1^n,\dots,v_J^n)^T$ and $F = (f(x_1),\dots,f(x_J))^T$, then

$$(I_J + \theta\Delta t A_{\Delta x})V^{n+1} = (I_J - \Delta t(1-\theta)\alpha A_{\Delta x})V^n + \Delta t F.$$

For $\theta = 0$, we recognize the Explicit Euler Method. For $\theta = 1$, we recognize the Implicit Euler Method. Thanks for Exercise X.1, we know the spectrum of $A_{\Delta x}$ is positive. Consequently $I_J + \theta\Delta t\alpha A_{\Delta x}$ is non-singular and the θ-scheme is well defined for all $\theta \in [0,1]$.

The scheme for $\theta = 1/2$ is called *Crank-Nicolson*.

E. X.1.2 Compute the consistency error and the order of the θ-scheme.

Solution de Q. X.1.2 Let u be the exact solution to (H). Using Exercise X.2, we have $u \in C^\infty(]0,+\infty[, C^4([0,1]))$. Let us compute the consistency error $\forall n \in \{0, N-1\}$, $\forall j \in \{0,\dots,J\}$,

$$\begin{aligned}
\mathcal{E}_j^n &= \frac{u(t^{n+1},x_j) - u(t^n,x_j)}{\Delta t} - \theta\alpha\frac{u(t^{n+1},x_{j+1}) - 2u(t^{n+1},x_j) + u(t^{n+1},x_{j-1})}{(\Delta x)^2} \\
&\qquad - (1-\theta)\alpha\frac{u(t^n,x_{j+1}) - 2u(t^n,x_j) + u(t^n,x_{j-1})}{(\Delta x)^2} - f(x_j) \\
&= \partial_t u(t^n,x_j) + \frac{\Delta t}{2}\partial_{tt}^2 u(t^n,x_j) \underbrace{-\theta\alpha\partial_{xx}^2 u(t^{n+1},x_j) - (1-\theta)\alpha\partial_{xx}^2 u(t^n,x_j)}_{=-\partial_{xx}^2 u(t^n,x_j) - \theta\Delta t\alpha\partial_{txx}^3 u(t^n,x_j) + O(\Delta t^2)} - f(x_j) + O(\Delta t^2) + O(\Delta x^2) \\
&= \partial_t u(t^n,x_j) - \partial_{xx}^2 u(t^n,x_j) + \frac{\Delta t}{2}\partial_t\left(\partial_t u(t^n,x_j) - 2\theta\underbrace{\alpha\partial_{xx}^2 u(t^n,x_j)}_{=\partial_t u(t^n,x_j) - f(x_j)}\right) - f(x_j) + O(\Delta t^2) + O(\Delta x^2) \\
&= \partial_t u(t^n,x_j) - \partial_{xx}^2 u(t^n,x_j) - f(x_j) + \frac{\Delta t}{2}(1-2\theta)\partial_{tt}^2 u(t^n,x_j) + O(\Delta t^2) + O(\Delta x^2) \\
&= \frac{\Delta t}{2}(1-2\theta)\partial_{tt}^2 u(t^n,x_j) + O(\Delta t^2) + O(\Delta x^2).
\end{aligned}$$

Therefore the θ-scheme is of order 1 for $\theta \neq 1/2$ and or order 2 for $\theta = 1/2$ (Crank-Nicolson).

E. X.1.3 Give a sufficient condition for stability in L^2 of the CFL type for the θ-scheme. Why is Crank-Nicolson so interesting?

Solution de Q. X.1.3 Definition **??** states the scheme is stable in the norm $\|\cdot\|$ if there exists a constant $C > 0$ such that the iteration matrix B satisfies $\|B^n\|_{\mathcal{M}_J(\mathbb{R})} \leq C$. Here

$$B = (I + \theta\Delta t A_{\Delta x})^{-1}(I - (1-\theta)\Delta t A_{\Delta x})$$

It is a rational fraction in $A_{\Delta x}$ of the form $B = R(A_{\Delta x})$ where $R : x \mapsto (1-(1-\theta)x)/(1+\theta x)$. Since R is a decresting function, $\sigma(B) = R(\sigma(A))$. Therefore

$$\sigma(B) = \left\{ \frac{1-(1-\theta)\Delta t\lambda}{1+\theta\Delta t\lambda},\ \lambda \in \mathrm{Sp}(A_{\Delta x}) \right\}.$$

Since B is real and symmetric, tts 2-norm is its spectral radius. Therefore, for all $n \in \mathbb{N}$, $\|B^n\|_2 = \rho(B^n) = \rho(B)^n$. Thus, $(\|B^n\|_2)_{n\geq 0}$ is bounded iff

$$\forall J \in \mathbb{N}^*, \qquad -1 \leq R(\max \mathrm{Sp}(\alpha A_{\Delta x})) \leq R(\min \mathrm{Sp}(\alpha A_{\Delta x})) \leq 1$$

iff

$$\forall J \in \mathbb{N}^*, \qquad (1-2\theta)\alpha\frac{\Delta t}{\Delta x^2}\sin^2(J\pi/(2(J+1))) \leq \frac{1}{2}$$

iff

$$(1-2\theta)\alpha\frac{\Delta t}{\Delta x^2} \leq \frac{1}{2}. \tag{X.2}$$

which happens to be the CFL condition. If $\theta \geq 1/2$, the scheme is unconditionnaly stable. Furthoremore if $\theta = 1/2$, the order of the scheme is 2, which makes it interesting.

E. X.1.4 Prove the θ-scheme converges under the CFL-type condition.

Solution de Q. X.1.4 Let $n \geq 1$. Define (E^n), $n \in \{0, \ldots, N\}$ by

$$E_j^n := u(t^n, x_j) - v_j^n, \text{ for } j \in \{0, \ldots, J\}$$

For $n \in \{0, \ldots, N-1\}$, we have

$$E^{n+1} = \Pi_{\Delta x} u(t^{n+1}, \cdot) - V^{n+1} = BE^n + \Delta t(I_J + \Delta t\theta A_{\Delta x})^{-1}\mathcal{E}^n.$$

thus, using that $A_{\Delta x}$ and B commute:

$$\|E^n\|_2 \leq \|B\|_2^n \|E^0\|_2 + \Delta t\|(I_J + \Delta t\theta A_{\Delta x})^{-1}\|_2 \sum_{k=0}^{n-1} \|B^{n-1-k}\|_2 \|\mathcal{E}^k\|_2$$

Using $\|(I_J + \Delta t\theta A_{\Delta x})^{-1}\|_2 \leq 1$ and the CFL-type condition $\|B\|_2 \leq 1$, if necessary, we have

$$\|E^n\|_2 \leq \|E^0\|_2 + n\Delta t \max_{k\in\{0,\ldots,n-1\}} \|\mathcal{E}^k\|_2.$$

Since $\|\mathcal{E}^k\|_2 \leq \|\mathcal{E}^k\|_\infty$, we get

$$\|E^n\|_2 \leq \|E^0\|_2 + (1-2\theta)O(\Delta t) + O(\Delta t^2) + O(\Delta x^2).$$

Exercise X.2 (Resolution using separation of variables)
For the sake of simplifying the formulas, we will consider $\alpha = 1$ throughout this exercise. A general α does not introduce any mathematical difficulty, though (provided it is positive).

Consider $\mathcal{H}_0$ and $\mathcal{H}$ defined by

$$\mathcal{H}_0 = C^0([0,+\infty[, L^2(0,1)) \cap C^\infty(]0,+\infty[\times[0,1])$$

$$\mathcal{H} = C^0([0,+\infty[, L^2(0,1)) \cap C^\infty(]0,+\infty[, H^2(0,1))$$

E. X.2.1 Prove (H) has at most one solution in $\mathcal{H}_0$ by considering the energy associated to the homogeneous problem $E : t \mapsto \int_{]0,1[} \tilde{u}^2(t,x)dx$.

Solution de Q. X.2.1 Let $\tilde{u} \in \mathcal{H}_0$ be a solution to the homogenous problem associated to (H). The function

$$E : t \mapsto \int_{]0,1[} \tilde{u}^2(t,x)\mathrm{d}x$$

can be differetiated on $]0,+\infty[$ since we meet the criteria of the differentiation of the functions defined by an integral:

- for all $t \in \mathbb{R}^+$, $\tilde{u}^2(t,\cdot)$ is integrable on $]0,1[$,
- For all $x \in [0,1]$, the function $t \mapsto \tilde{u}^2(t,x)$ can be differentiated on $\mathbb{R}^{+*}$. Its derivative is $2\tilde{u}\partial_t\tilde{u}$,
- For all $\varepsilon > 0$, $A > \varepsilon$, and $(t,x) \in [\varepsilon, A] \times [0,1]$, $|\partial_t \tilde{u}^2(t,x)| \leq \|\partial_t \tilde{u}^2\|_{L^\infty([\varepsilon,A]\times[0,1])}$. The constant function can be integrated on $[0,1]$.

Consequently, for all $\varepsilon > 0$ and for $t > 0$,

$$E'(t) = 2\int_0^1 \tilde{u}(t,x)\partial_t\tilde{u}(t,x)\mathrm{d}x = 2\int_0^1 \tilde{u}(t,x)\partial^2_{xx}\tilde{u}(t,x)\mathrm{d}x = -2\int_0^1 (\partial_x\tilde{u}(t,x))^2\mathrm{d}x.$$

Since $\tilde{u}(\cdot,0) = 0$, the Poincaré inequality holds for all $t > 0$,

$$\int_0^1 (\tilde{u}(t,x))^2\mathrm{d}x \leq \int_0^1 (\partial_x\tilde{u}(t,x))^2\mathrm{d}x.$$

Consequently, for all $t > 0$,

$$E'(t) \leq -2E(t),$$

Therefore, for all $\varepsilon > 0$ and for all $t \geq \varepsilon$,

$$E(t) \leq E(\varepsilon)e^{-2(t-\varepsilon)}.$$

Since $\tilde{u} \in C^0([0,+\infty[, L^2(0,1))$, we have $E(\varepsilon) \underset{\varepsilon\to 0}{\longrightarrow} E(0)$ and for all $t \geq 0$, $E(t) \leq E(0)e^{-2t}$. Since $u^0 = 0$ (homogeneous problem), we have $E(0) = 0$. Therefore E vanishes. Thus $\tilde{u}$ is the null function. The uniqueness derives from the linearity of the problem[1].

[1] as usual, one takes two functions u_1 and u_2 solution to the problem and $\tilde{u} = u_1 - u_2$ is a the null function

We now look for a formal solution u to $\partial_t u = \partial^2_{xx} u$ with BC $u(\cdot, 0) = u(\cdot, 1) = 0$ that **does not** vanish. We shall look for the solution in the form $u(t, x) = T(t)X(x)$.

E. X.2.2 Prove that X''/X and T'/T are equal to a non-positive constant. We will note $-\lambda^2$ this constant.

Solution de Q. X.2.2 Assume there exists a non-empty open interval $I \subset \mathbb{R}^{+*}$ and a non-empty open interval $J \subset [0, 1]$ such that T is differentiable on I And X is twice differentiable on J. Furthermore assume that T and X do not vanish on I and J respectively. Then,

$$\forall (t, x) \in I \times J, \quad T'(t)X(x) = T(t)X''(x)$$

which is equivalent to

$$\forall (t, x) \in I \times J, \quad \frac{T'}{T}(t) = \frac{X''}{X}(x).$$

Since t and x are independent variables, this quotient is equal to μ. Let us look at

$$\begin{cases} X'' - \mu X = 0 \text{ in }]0, 1[\\ X(0) = X(1) = 0. \end{cases}$$

Three cases arise:

- If $\mu = 0$, then X is polynomial of degree 1 or smaller, that vanish in two points. Therefore it vanishes everywhere.
- If $\mu < 0$ then X is such that

  $$X : x \mapsto a \cos(\sqrt{-\mu}\, x) + b \sin(\sqrt{-\mu}\, x)$$

  The condition $X(0) = 0$ implies and the condition $X(1) = 0$ implies $b \sinh(\sqrt{\mu}) = 0$, which means $b = 0$. It follows that X is zero.
- If $\mu < 0$, then X can be written

  $$X : x \mapsto a \cos(\sqrt{-\mu}\, x) + b \sin(\sqrt{-\mu}\, x)$$

  The condition $X(0) = 0$ implies $a = 0$ and the condition $X(1) = 0$ implies $b \sin(\sqrt{-\mu}) = 0$, which is equivalent to $b = 0$ or $\sin(\sqrt{-\mu}) = 0$. It follows that $\sqrt{-\mu} = k\pi$, $k \in \mathbb{N}^*$, which is $\mu = -k^2\pi^2$.

In this third case, T can be written $T : t \mapsto a e^{-k^2\pi^2 t}$, $a \in \mathbb{R}$. We verify that $u_k : (t, x) \mapsto e^{-k^2\pi^2 t} \sin(k\pi x)$ satisfies $\partial_t u = \partial^2_{xx} u$ with the BC $u(\cdot, 0) = u(\cdot, 1) = 0$.

E. X.2.3 Assuming there exists a solution in $\mathcal{H}_0$, deduce the general expression of the solutions u with separate variables.

Solution de Q. X.2.3 When $f = 0$, we look for solutions of the form:

$$u : (t, x) \mapsto \sum_{k\geq 1} u_k e^{-k^2\pi^2 t} \sin(k\pi x).$$

It is worth to point out that for all $t > 0$, for all $k \geq 1$, and for all $P \in \mathbb{R}[X]$,

$$P(k)e^{-k^2\pi^2 t} = O(1/k^2)$$

is a converging series.
For all $a \in \mathbb{R}^{+*} \times [0, 1]$, the series and its derivatives converge normally on $[a, +\infty[$. Therefore

$$u \in C^\infty(]0, +\infty[\times[0, 1])$$

If $f \neq 0$, we look for solutions of the form:

$$u : (t, x) \mapsto \sum_{k\geq 1} T_k(t)\sqrt{2}\sin(k\pi x).$$

The, for all $k \geq 1$, T_k must satisfy:

$$T_k' + k^2\pi^2 T_k = (f, \sqrt{2}\sin(k\pi\cdot))_{L^2(0,1)}. \tag{X.3}$$

E. X.2.4 Assuming there exists a solution in $\mathcal{H}$, what is the expression of the solution to (H)?

Solution de Q. X.2.4 On commence par décomposer u^0 et f sur la base hilbertienne des $(\sqrt{2}\sin(k\pi\cdot))_{k\geq 1}$: on prolonge u^0 par imparité à $]-1, 1[$, puis par périodicité de période 2 à $\mathbb{R}$ tout entier et enfin on calcule la série de Fourier sur $]-1, 1[$ de la fonction impaire ainsi obtenue. Le développement est alors en $\sin(k\pi\cdot)$. On obtient :

$$\begin{cases} u^0 = \sqrt{2}\sum_{k\geq 1} a_k \sin(k\pi\cdot), \\ f = \sqrt{2}\sum_{k\geq 1} b_k \sin(k\pi\cdot) \end{cases}$$

avec, pour tout $k \geq 1$, $a_k = \sqrt{2}\int_{]0,1[} u^0 \sin(k\pi\cdot)$ et $b_k = \sqrt{2}\int_{]0,1[} f \sin(k\pi\cdot)$. En résolvant (X.3), on trouve alors pour tout $k \geq 1$,

$$T_k : t \mapsto a_k e^{-k^2\pi^2 t} + \frac{b_k}{k^2\pi^2}(1 - e^{-k^2\pi^2 t}).$$

On pose

$$u : (t, x) \mapsto \sum_{k\geq 1} \left(a_k e^{-k^2\pi^2 t} + \frac{b_k}{k^2\pi^2}(1 - e^{-k^2\pi^2 t}) \right) \sin(k\pi x).$$

E. X.2.5 Prove that $u \in \mathcal{H}$.

Solution de Q. X.2.5 For all $t > 0$, the function $u(t,\cdot)$ is in $H^2(0,1)$ as a consequence of the theorems on differentiation. Furthermore $f = 0$ implies $u \in \mathcal{H}_0$.

E. X.2.6 What is the steady state problem associated to (H)?

Solution de Q. X.2.6

$$\begin{cases} -\partial^2_{xx}\bar{u}(t,x) = f(x), \ x \in]0,1[, \\ \bar{u}(0) = 0, \ \bar{u}(1) = 0. \end{cases}$$

E. X.2.7 What is the limit function $\bar{u}$ (which is independent of time) toward which converges $u(t,\cdot)$ when t goes to $+\infty$? Prove $\|u(t,\cdot) - \bar{u}(\cdot)\|_{L^2(0,1)} \underset{t\to+\infty}{\longrightarrow} 0$.

Solution de Q. X.2.7 The limit is the fonction $\bar{u} \in H^2(0,1)$ solution to the steady state problem. Let $w : (t,x) \mapsto u(t,x) - \bar{u}(x)$. Then w is solution to:

$$\text{(H)} \quad \begin{cases} \partial_t w(t,x) - \partial^2_{xx} w(t,x) = 0, \ t > 0, \ x \in]0,1[, \\ w(0,x) = u^0(x) - \bar{u}(x), \ x \in]0,1[, \\ w(t,0) = 0, \ w(t,1) = 0, \ t > 0. \end{cases}$$

Using the previous quesitons, $w \in \mathcal{H}_0$. Furthermore the estimate on $E : t \mapsto \int_{]0,1[} w^2(t,x)\mathrm{d}x$ from E.X.2.1 yields

$$\forall t \geq 0, \qquad E(t) \leq E(0)e^{-2t}.$$

Exercise X.3
Let us consider these schemes, defined for $j \in \{1,\dots,J\}$ and $n \in \{0,\dots,N-1\}$, by:

- The Six-point scheme:

$$\frac{v^{n+1}_{j+1} - v^n_{j+1}}{12\Delta t} + \frac{5(v^{n+1}_j - v^n_j)}{6\Delta t} + \frac{v^{n+1}_{j-1} - v^n_{j-1}}{12\Delta t} - \alpha\frac{v^n_{j+1} - 2v^n_j + v^n_{j-1}}{2\Delta x^2} - \alpha\frac{v^{n+1}_{j+1} - 2v^{n+1}_j + v^{n+1}_{j-1}}{2\Delta x^2} = f_j,$$

- The DuFort-Frankel scheme:

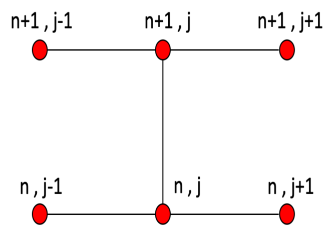
$$\frac{v^{n+1}_j - v^{n-1}_j}{2\Delta t} - \alpha\frac{v^n_{j+1} - v^{n+1}_j - v^{n-1}_j + v^n_{j-1}}{\Delta x^2} = f_j,$$

- The Gear Scheme:
$$\frac{3v_j^{n+1} - 4v_j^n + v_j^{n-1}}{2\Delta t} - \alpha \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{\Delta x^2} = f_j.$$

E. X.3.1 (Consistency) Prove the order of the six-point scheme is 2 in time and 4 in space. Prove the order of the Gear scheme is 2 in time and 2 in space. What are the orders of the DuFort-Frankel scheme in space and time?

Solution de Q. X.3.1

The six-point scheme: The stencil of this scheme is:



We can use the Taylor expansion to express each term of the scheme with respect to the center point (t^n, x_j). There are a lot of computations but they will lead the the result, eventually.

There is a more clever way to go: noticing the six-point scheme is a θ-scheme.
Looking at the three first terms of the scheme, we notice that

$$\begin{aligned}
&\frac{v_{j+1}^{n+1} - v_{j+1}^n}{12\Delta t} + \frac{5(v_j^{n+1} - v_j^n)}{6\Delta t} + \frac{v_{j-1}^{n+1} - v_{j-1}^n}{12\Delta t} \\
&= \frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{(v_{j+1}^{n+1} - v_{j+1}^n) - 2(v_j^{n+1} - v_j^n) + (v_{j-1}^{n+1} - v_{j-1}^n)}{12\Delta t} \\
&= \frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{12\Delta t} - \frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{12\Delta t} \\
&= \frac{v_j^{n+1} - v_j^n}{\Delta t} + \frac{(\Delta x)^2}{12\Delta t}\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1})}{(\Delta x)^2} - \frac{(\Delta x)^2}{12\Delta t}\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2}
\end{aligned}$$

Therefore

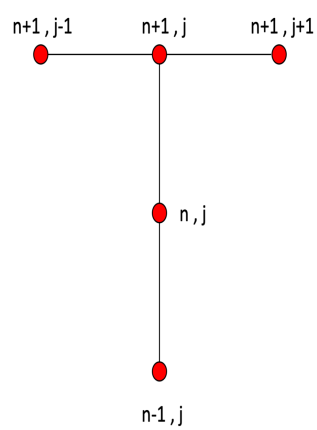
$$\begin{aligned}
&\frac{v_{j+1}^{n+1} - v_{j+1}^n}{12\Delta t} + \frac{5(v_j^{n+1} - v_j^n)}{6\Delta t} + \frac{v_{j-1}^{n+1} - v_{j-1}^n}{12\Delta t} - \alpha\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{2\Delta x^2} - \alpha\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{2\Delta x^2} \\
&= \frac{v_j^{n+1} - v_j^n}{\Delta t} - \left(\frac{\alpha}{2} + \frac{(\Delta x)^2}{12\Delta t}\right)\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} - \left(\frac{\alpha}{2} - \frac{(\Delta x)^2}{12\Delta t}\right)\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{(\Delta x)^2} \\
&= \frac{v_j^{n+1} - v_j^n}{\Delta t} - \alpha(1-\theta)\frac{v_{j+1}^n - 2v_j^n + v_{j-1}^n}{(\Delta x)^2} + \alpha\theta\frac{v_{j+1}^{n+1} - 2v_j^{n+1} + v_{j-1}^{n+1}}{(\Delta x)^2}
\end{aligned}$$

where
$$\theta = \frac{1}{2} - \frac{(\Delta x)^2}{12\alpha\Delta t}$$
This proves the six-point scheme is a θ-scheme[2] for this specific θ.

At this point, we know from E.X.1.2 the scheme is (at least) of order 1 in time and 2 in space. But, it might (and will) be better for this specific choice of θ. To show this, let us go back to the expression in E.X.1.2.
$$\mathcal{E}_j^n = \frac{\Delta t}{2}(1-2\theta)\partial_{tt}^2 u(t^n, x_j) + O(\Delta t^2) + O(\Delta x^2)$$
When substituting θ by $\frac{1}{2} - \frac{(\Delta x)^2}{12\alpha\Delta t}$, we get
$$\mathcal{E}_j^n = \frac{(\Delta x)^2}{24\alpha}\partial_{tt}^2 u(t^n, x_j) + O(\Delta t^2) + O(\Delta x^2)$$
consequently
$$\mathcal{E}_j^n = O(\Delta t^2) + O(\Delta x^2)$$
The scheme is (at least) of order 2 in time and space.

We may actually get an even better convergence. In order to find out, we need to go back to the computations of E.X.1.2 and push the development further. When doing this, we obtain
$$\mathcal{E}_j^n = \left(\frac{\nu}{240}(\Delta x)^4 - \frac{\nu^3}{12}(\Delta t)^2\right)\partial_{xxxxxx}^6 u + O(\Delta t^2) + O(\Delta x^4)$$
The six-point scheme is of order 2 in time and order 4 in space.

The Gear scheme: The stencil of this scheme is:



We start, as usual, by computing Taylor expansions of every term of the scheme around the point of interest (t^n, x_j). After several pages of long computations, we are both worn out and concerned about

[2]It is easy to get confused with the negative signs. Note that the terms are all to the left hand side of equation (X.1).

the number of trees need to carry out rhis proof. We turn back to the stencil and try to see if it suggests a different modus operandi.

The configuration of the stencil suggests we try computing Taylor expansions around the point (t^{n+1}, x_j) instead of the point of interest as we usually do.

$$\begin{aligned}
&-4u(t^n, x_n)\\
&= -4u(t^{n+1} - \Delta t, x_n)\\
&= -4u(t^{n+1}, x_j) + 4\Delta t \partial_t u(t^{n+1}, x_j) - 2(\Delta t)^2 \partial^2_{tt} u(t^{n+1}, x_j) + \frac{2}{3}(\Delta t)^3 \partial^3_{ttt} u(t^{n+1}, x_j) + O((\Delta t)^4)
\end{aligned}$$

$$\begin{aligned}
&u(t^{n-1}, x_n)\\
&= u(t^{n+1} - 2\Delta t, x_n)\\
&= u(t^{n+1}, x_j) - 2\Delta t \partial_t u(t^{n+1}, x_j) + 2(\Delta t)^2 \partial^2_{tt} u(t^{n+1}, x_j) - \frac{4}{3}(\Delta t)^3 \partial^3_{ttt} u(t^{n+1}, x_j) + O((\Delta t)^4)
\end{aligned}$$

Therefore

$$\frac{3u(t^{n+1}, x_j) - 4u(t^n, x_n) + u(t^{n-1}, x_n)}{2\Delta t} = \partial_t u(t^{n+1}, x_j) - \frac{1}{3}(\Delta t)^2 \partial^3_{ttt} u(t^{n+1}, x_j) + O((\Delta t)^3)$$

Similarly, we have:

$$\begin{aligned}
&u(t^{n+1}, x_{j+1})\\
&= u(t^{n+1}, x_n + \Delta x)\\
&= u(t^{n+1}, x_n) + \Delta x \partial_x u(t^{n+1}, x_n) + \frac{1}{2}(\Delta x)^2 \partial^2_{xx} u(t^{n+1}, x_n) + \frac{1}{6}(\Delta x)^3 \partial^3_{xxx} u(t^{n+1}, x_n)\\
&\quad + \frac{1}{24}(\Delta x)^4 \partial^4_{xxxx} u(t^{n+1}, x_n) + \frac{1}{120}(\Delta x)^5 \partial^5_{xxxxx} u(t^{n+1}, x_n) + O((\Delta x)^6)
\end{aligned}$$

$$\begin{aligned}
&u(t^{n+1}, x_{j-1})\\
&= u(t^{n+1}, x_n - \Delta x)\\
&= u(t^{n+1}, x_n) - \Delta x \partial_x u(t^{n+1}, x_n) + \frac{1}{2}(\Delta x)^2 \partial^2_{xx} u(t^{n+1}, x_n) - \frac{1}{6}(\Delta x)^3 \partial^3_{xxx} u(t^{n+1}, x_n)\\
&\quad + \frac{1}{24}(\Delta x)^4 \partial^4_{xxxx} u(t^{n+1}, x_n) - \frac{1}{120}(\Delta x)^5 \partial^5_{xxxxx} u(t^{n+1}, x_n) + O((\Delta x)^6)
\end{aligned}$$

Therefore

$$\frac{u(t^{n+1}, x_{j+1}) - 2u(t^{n+1}, x_j) + u(t^{n+1}, x_{j-1})}{(\Delta x)^2} = \partial^2_{xx} u(t^{n+1}, x_n) + \frac{1}{12}(\Delta x)^2 \partial^4_{xxxx} u(t^{n+1}, x_n) + O((\Delta x)^4)$$

Consequently

$$\begin{aligned}
&\frac{3u(t^{n+1}, x_j) - 4u(t^n, x_n) + u(t^{n-1}, x_n)}{2\Delta t} - \alpha \frac{u(t^{n+1}, x_{j+1}) - 2u(t^{n+1}, x_j) + u(t^{n+1}, x_{j-1})}{(\Delta x)^2}\\
&= \partial_t u(t^{n+1}, x_j) - \frac{1}{3}(\Delta t)^2 \partial^3_{ttt} u(t^{n+1}, x_j) - \alpha \partial^2_{xx} u(t^{n+1}, x_n) - \frac{\alpha}{12}(\Delta x)^2 \partial^4_{xxxx} u(t^{n+1}, x_n) + O((\Delta t)^3) + O((\Delta x)^4)
\end{aligned}$$

Using (H) we have

$$\partial_t u(t^{n+1}, x_j) = \alpha \partial^2_{xx} u(t^{n+1}, x_j) + f_i$$

and

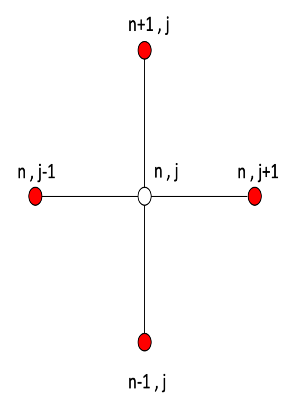
$$\begin{aligned}
&\partial^3_{ttt} u(t^{n+1}, x_j) \\
&\partial^2_{tt}(\partial_t u(t^{n+1}, x_j)) \\
&= \partial^2_{tt}(\alpha \partial^2_{xx} u(t^{n+1}, x_j) + f_i) \\
&= \partial^2_{tt}(\alpha \partial^2_{xx} u(t^{n+1}, x_j)) \\
&= \alpha \partial^2_{xx} \partial_t(\partial_t u(t^{n+1}, x_j)) \\
&= \alpha \partial^2_{xx} \partial_t(\alpha \partial^2_{xx} u(t^{n+1}, x_j) + f_i) \\
&= \alpha \partial^2_{xx} \partial_t(\alpha \partial^2_{xx} u(t^{n+1}, x_j)) \\
&= \alpha^2 \partial^4_{xxxx} \partial_t(u(t^{n+1}, x_j)) \\
&= \alpha^3 \partial^6_{xxxxxx} u(t^{n+1}, x_j) + \alpha^2 \partial^4_{xxxx} f_i
\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{3u(t^{n+1}, x_j) - 4u(t^n, x_n) + u(t^{n-1}, x_n)}{2\Delta t} - f_i - \alpha \frac{u(t^{n+1}, x_{j+1}) - 2u(t^{n+1}, x_j) + u(t^{n+1}, x_{j-1})}{(\Delta x)^2} \\
&= -\frac{1}{3}(\Delta t)^2 \alpha^3 \partial^6_{xxxxxx} u(t^{n+1}, x_j) - \frac{\alpha}{12}(\Delta x)^2 \partial^4_{xxxx} u(t^{n+1}, x_n) - \frac{1}{3}(\Delta t)^2 \alpha^2 \partial^4_{xxxx} f_i + O((\Delta t)^3) + O((\Delta x)^4) \\
&= O((\Delta t)^2) + O((\Delta x)^2)
\end{aligned}$$

Therefore, the Gear scheme is of order 2 in time and space.

The DuFort-Frankel scheme: The stencil of this scheme is:



Let us compute the Taylor expansions around the point of interest (t^n, x_j). We have:

$$
\begin{aligned}
&\frac{u(t^{n+1},x_j)-u(t^{n-1},x_j)}{2\Delta t}\\
&=\frac{u(t^n,x_j)+\Delta t\partial_t u(t^n,x_j)+\frac{1}{2}(\Delta t)^2\partial_{tt}^2 u(t^n,x_j)-u(t^n,x_j)+\Delta t\partial_t u(t^n,x_j)-\frac{1}{2}(\Delta t)^2\partial_{tt}^2 u(t^n,x_j)+O(\Delta t^3)}{2\Delta t}\\
&=\frac{2\Delta t\partial_t u(t^n,x_j)+O(\Delta t^3)}{2\Delta t}\\
&=\partial_t u(t^n,x_j)+O(\Delta t^2)
\end{aligned}
$$

and

$$
\begin{aligned}
&\frac{u(t^n,x_{j+1})-u(t^{n+1},x_j)-u(t^{n-1},x_j)+u(t^n,x_{j-1})}{\Delta x^2}\\
&=\frac{1}{\Delta x^2}(u(t^n,x_j)+\Delta x\partial_x u(t^n,x_j)+\frac{1}{2}(\Delta x)^2\partial_{xx}^2 u(t^n,x_j)+\frac{1}{6}(\Delta x)^3\partial_{xxx}^3 u(t^n,x_j)\\
&\quad+\frac{1}{24}(\Delta x)^4\partial_{xxxx}^4 u(t^n,x_j)+\frac{1}{120}(\Delta x)^5\partial_{xxxxx}^5 u(t^n,x_j)-u(t^n,x_j)\\
&\quad-\Delta t\partial_t u(t^n,x_j)-\frac{1}{2}(\Delta t)^2\partial_{tt}^2 u(t^n,x_j)-\frac{1}{6}(\Delta t)^3\partial_{ttt}^3 u(t^n,x_j)-u(t^n,x_j)\\
&\quad+\Delta t\partial_t u(t^n,x_j)-\frac{1}{2}(\Delta t)^2\partial_{tt}^2 u(t^n,x_j)+\frac{1}{6}(\Delta t)^3\partial_{ttt}^3 u(t^n,x_j)+u(t^n,x_j)\\
&\quad-\Delta x\partial_x u(t^n,x_j)+\frac{1}{2}(\Delta x)^2\partial_{xx}^2 u(t^n,x_j)-\frac{1}{6}(\Delta x)^3\partial_{xxx}^3 u(t^n,x_j)\\
&\quad+\frac{1}{24}(\Delta x)^4\partial_{xxxx}^4 u(t^n,x_j)-\frac{1}{120}(\Delta x)^5\partial_{xxxxx}^5 u(t^n,x_j)+O(\Delta x^6)+O(\Delta t^4))\\
&=\partial_{xx}^2 u(t^n,x_j)-\frac{(\Delta t)^2}{(\Delta x^) 2}\partial_{tt}^2 u(t^n,x_j)+\frac{1}{12}(\Delta x)^2\partial_{xxxx}^4 u(t^n,x_j)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right))
\end{aligned}
$$

It follows:

$$\begin{aligned}
&\frac{u(t^{n+1},x_j)-u(t^{n-1},x_j)}{2\Delta t}-\nu\frac{u(t^n,x_{j+1})-u(t^{n+1},x_j)-u(t^{n-1},x_j)+u(t^n,x_{j-1})}{\Delta x^2}-f_i\\
&=\underbrace{\partial_t u(t^n,x_j)+\alpha\partial^2_{xx}u(t^n,x_j)}_{=f(x_j)}+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)-\alpha\frac{(\Delta t)^2}{(\Delta x)^2}\partial^2_{tt}u(t^n,x_j)-f_i+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=-\alpha\frac{(\Delta t)^2}{(\Delta x)^2}\partial_t(\partial_t u(t^n,x_j))+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=-\alpha\frac{(\Delta t)^2}{(\Delta x)^2}\partial_t(\alpha\partial^2_{xx}u(t^n,x_j)+f(x_j))+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=-\alpha^2\frac{(\Delta t)^2}{(\Delta x)^2}\partial_t(\partial^2_{xx}u(t^n,x_j))+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=-\alpha^2\frac{(\Delta t)^2}{(\Delta x)^2}\partial^2_{xx}(\partial_t u(t^n,x_j))+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=-\alpha^3\frac{(\Delta t)^2}{(\Delta x)^2}\partial^4_{xxxx}u(t^n,x_j)+\alpha\frac{(\Delta t)^2}{(\Delta x)^2}\partial^2_{xx}f(x_j)+\frac{1}{12}(\Delta x)^2\partial^4_{xxxx}u(t^n,x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=\left(\frac{(\Delta x)^2}{12}-\alpha^3\frac{(\Delta t)^2}{(\Delta x)^2}\right)\partial^4_{xxxx}u(t^n,x_j)+\alpha\frac{(\Delta t)^2}{(\Delta x)^2}\partial^2_{xx}f(x_j)+O(\Delta t^2)+O(\Delta x^4)+O\left(\frac{\Delta t^4}{\Delta x^2}\right)\\
&=O(\Delta x^2)+O\left(\frac{\Delta t^2}{\Delta x^2}\right)
\end{aligned}$$

The “legal” answer to the question should be “the scheme is not consistent”: it does not fall into the definition given in class becaure of the term with $\Delta t^2/\Delta x^2$. However, since we can choose the Δt and Δx, it would make sense to choose

$$\Delta t=\Delta x^2$$

then

$$O(\Delta x^2)+O\left(\frac{\Delta t^2}{\Delta x^2}\right)=O(\Delta x^2)$$

the Gear scheme becomes of the second order in space.

E. X.3.2 (Stability in norm L^2) Prove that all three schemes are inconditionnally stable in the norm L^2.

Solution de Q. X.3.2

The six-point scheme: We have noticed the six-point scheme is a θ-scheme for $\theta=\frac{1}{2}-\frac{(\Delta x)^2}{12\alpha\Delta t}$. Since $\frac{1}{12}<\frac{1}{2}$, Equation (X.2) provides the stability of the six-point scheme.

The Gear scheme:

E. X.3.3 (Maximum principle) Prove that, if $2\alpha\Delta t/\Delta x^2 \leq 1$, then the θ-scheme verifies the maximum principle. Prove the same result for the DuFort-Frankel if $\alpha\Delta t/\Delta x^2 \leq 1$. What can we say of the stability in the norm L^∞?

D) Going further

These exercises can be found on edunao as Jupyter notebooks.