Exercise III.2

1. Let $\mathcal{H}$ be a Hilbert space. Show that $x \mapsto \|x\|^2$ is strictly convex.

2. A function $f : \mathcal{H} \to]-\infty, +\infty]$ is strongly convex with modulus $\beta \in]0, +\infty[$ if there exists a convex function $g : \mathcal{H} \to]-\infty, +\infty]$ such that:
$$f = g + \frac{\beta}{2} \|\cdot\|^2 . \tag{1}$$
Show that every strongly convex function is strictly convex.

3. Show that a function $f : \mathcal{H} \to]-\infty, +\infty]$ is strongly convex with modulus $\beta \in]0, +\infty[$ if and only if:
$$\begin{aligned} &\left(\forall (x, y) \in \mathcal{H}^2\right) (\forall \alpha \in]0, 1[) \\ &f(\alpha x + (1-\alpha) y) + \alpha (1-\alpha) \frac{\beta}{2} \|x - y\|^2 \leq \alpha f(x) + (1-\alpha) f(y). \end{aligned} \tag{2}$$

Solution:

1. We already now that (due to triangular inequality) it holds that:
$$\forall \alpha \in]0, 1[, \text{ and } \forall (x, y) \in \mathcal{H}^2, \ \|\alpha x + (1-\alpha) y\| \leq \alpha \|x\| + (1-\alpha) \|y\| . \tag{3}$$
Moreover, it is easy to show that the function $(\cdot)^2$ is an increasing and convex function in $[0, +\infty[$. Therefore, we can square both sides of the above inequality and obtain the inequality.
$$(\|\alpha x + (1-\alpha) y\|)^2 \leq (\alpha \|x\| + (1-\alpha) |y\|)^2 . \tag{4}$$
Furthermore, function $(\cdot)^2$ is convex, which allows us to write the inequality:
$$(\alpha \|x\| + (1-\alpha) |y\|)^2 \leq \alpha \|x\|^2 + (1-\alpha) \|y\|^2 . \tag{5}$$
As a result, by combining the above inequalities we obtain that:
$$(\|\alpha x + (1-\alpha) y\|)^2 \leq (\alpha \|x\| + (1-\alpha) |y\|)^2 \leq \alpha \|x\|^2 + (1-\alpha) \|y\|^2 , \tag{6}$$
which finally proves that $\|\cdot\|^2$ is convex.
In order now to determine if it also strictly convex we need to determine under which conditions, equality:
$$(\|\alpha x + (1-\alpha) y\|)^2 = \alpha \|x\|^2 + (1-\alpha) \|y\|^2 \tag{7}$$
holds. For this purpose, we start by expanding $(\|\alpha x + (1-\alpha) y\|)^2$ as:
$$\begin{aligned} \|\alpha x + (1-\alpha) y\|^2 &= \langle \alpha x + (1-\alpha) y | \alpha x + (1-\alpha) y \rangle \\ &= \alpha^2 \|x\|^2 + 2\alpha (1-\alpha) \langle x | y \rangle + (1-\alpha)^2 \|y\|^2 . \end{aligned} \tag{8}$$

As a result, equality in (7) is satisfied when:

$$\alpha^2 \|x\|^2 + 2\alpha(1-\alpha)\langle x|y\rangle + (1-\alpha)^2\|y\|^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \quad (9)$$

However, (9) can be written as:

$$\alpha(\alpha-1)\left(\|x\|^2 - 2\langle x|y\rangle + \|y\|^2\right) = 0, \quad (10)$$

or equivalently as:

$$\alpha(a-1)\|x-y\|^2 = 0. \quad (11)$$

Given the fact that $\alpha \in]0,1[$, this last equation is equivalent to $\|x-y\| = 0$, i.e., to $x = y$.

Summarizing, we have that $\|\cdot\|^2$ is convex and that:

$$(\|\alpha x + (1-\alpha)y\|)^2 = \alpha\|x\|^2 + (1-\alpha)\|y\|^2 \quad (12)$$

if and only if $x = y$. As a result, the function is strictly convex.

2. Let f be strongly convex. We then have that:

$$f(x) = g(x) + \frac{\beta}{2}\|x\|^2, \quad (13)$$

for some convex function $g(x)$. Moreover, due to convexity of $g(\cdot)$ we have that $\forall\ (x,y) \in \mathcal{H}^2$ and $\forall \alpha \in]0,1[$:

$$g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha)g(y). \quad (14)$$

Moreover, due to strict convexity of $\|\cdot\|^2$ we also have that:

$$\frac{\beta}{2}\|\alpha x + (1-\alpha)y\|^2 < \frac{\beta}{2}\left(\alpha\|x\|^2 + (1-\alpha)\|y\|^2\right) \quad (15)$$

By combinnig (14) and (15) we now obtain that:

$$\forall \alpha \in]0,1[,\ \ \forall (x,y) \in \mathcal{H}^2,\ \ f(\alpha x + (1-\alpha)y) < \alpha f(x) + (1-\alpha) f(y), \quad (16)$$

which proves that $f(\cdot)$ is strictly convex.

3. Using the definition of strong convexity, we have that a function f is strongly convex with modulus β if and only if:

$$g(\alpha x + (1-\alpha)y) \le \alpha g(x) + (1-\alpha) g(y), \quad (17)$$

where $g(x) = f(x) - \frac{\beta}{2}\|x\|^2$ (i.e., if and only if $g(\cdot)$ is convex). Using this expression for $g(x)$, we obtain that f is strongly convex with modulus β if and only if:

$$\begin{aligned} f(\alpha x + (1-\alpha)y) - \frac{\beta}{2}\|\alpha x + (1-\alpha)y\|^2 \le{}& \alpha\left(f(x) - \frac{\beta}{2}\|x\|^2\right) \\ &+ (1-\alpha)\left(f(y) - \frac{\beta}{2}\|y\|^2\right) \end{aligned} \quad (18)$$

As a result, by using the above inequality and the expansion:

$$\begin{aligned} \|\alpha x + (1-\alpha)y\|^2 &= \langle \alpha x + (1-\alpha)y | \alpha x + (1-\alpha)y \rangle \\ &= \alpha^2 \|x\|^2 + 2\alpha (1-\alpha) \langle x, y \rangle + (1-\alpha)^2 \|y\|^2 \end{aligned} \tag{19}$$

we can derive the derived result.