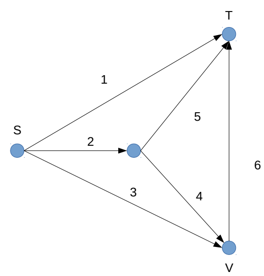
Exercise V.3

We want to maximize the flow of some fluid between nodes S and T on the given oriented network.



The links have maximum capacities $(\zeta_i)_{1 \le i \le 6} \in [0, +\infty[^6$

1. Express the problem as a linear programming one.
2. Is it feasible? Is it bounded?
3. What is the dual of the problem?

Solution: We start by introducing a vector of decision variables :

$$y = \left[y^{(1)}, \ldots, y^{(6)}\right]^T \in [0, +\infty[^6, \tag{1}$$

where $y^{(i)}$ represents the flow on the i-th edge of the network.

Since we want to maximize the flow on the network, we need to maximize the flow that departs form S, or equivalently maximize the quantity:

$$y^{(1)} + y^{(2)} + y^{(3)}. \tag{2}$$

Let us now see the constraints of our problem. These are the following:

- the flow which arrives at U is equal to the one that departs from U, or equivalently:
$$y^{(2)} = y^{(4)} + y^{(5)} \tag{3}$$
- the flow which arrives at V is equal to the one that departs from V, or equilvalently:
$$y^{(3)} + y^{(4)} = y^{(6)} \tag{4}$$
- the flow on each one of the links is upper bounded by the link capacity, i.e.,
$$y^{(i)} \le \zeta^{(i)},\ 1 \le i \le 6. \tag{5}$$

As a result, we can write the problem in the form:

$$\begin{aligned} &\text{maximize}_{y \in [0,+\infty[^6} \ \ \langle b | y \rangle \\ &\text{subject to: } L_1^T y \leq c_1, \ \text{ and } \ L_2^T y = c_2 \end{aligned} \tag{6}$$

where:

$$b = [1,1,1,0,0,0]^T, \ \ c_1 = [\zeta^{(1)}, \zeta^{(2)}, \zeta^{(3)}, \zeta^{(4)}, \zeta^{(5)}, \zeta^{(6)}]^T, \ \ c_2 = [0,0,]^T, \tag{7}$$

$L_1 = I_{6\times 6}$ (i.e., the 6×6 identity matrix) and

$$L_2^T = \begin{bmatrix} 0 & 1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 1 & 1 & 0 & -1 \end{bmatrix}. \tag{8}$$

We now note that the vector $y = [0,0,0,0,0,0]^T$ is a feasible vector. As a result, since we can find at least one feasilbe point, the problem is feasible. Moreover, since the link capacities are upper bounded, the objective function: $y^{(1)} + y^{(2)} + y^{(3)}$ is upper bounded by the quantity $\zeta^{(1)} + \zeta^{(2)} + \zeta^{(3)}$. Hence, the problem is bounded.

Using now the result of Exercise 2, we can write the dual of this problem as:

$$\text{minimize:}_{z_1 \in [0,+\infty[^6, z_2 \in \mathbb{R}^2} \ \ \langle c_1 | z_1 \rangle + \langle c_2 | z_2 \rangle, \ \text{ subject to} L_1 z_1 + L_2 z_2 \geq b. \tag{9}$$

Finally, we note that since the given problem is feasible and bounded strong duality holds and the the maximal flow value (i.e. the value obtained for the solution to the maximal flow problem) is equal to the optimal value obtained for its dual.