计算方法　第 9 章习题答案

9.1 取 $h=0.1$, 用欧拉法、后退欧拉法、中点法和梯形法求解下列初值问题:

(1) $y'=1-y,\ \ y(0)=0,\ \ 0\leqslant x\leqslant 1;$

(2) $y'=xy^2,\ \ y(0)=1,\ \ 0\leqslant x\leqslant 1;$

(3) $y'=y-2x/y,\ \ y(0)=1,\ \ 0\leqslant x\leqslant 1;$

(4) $y'=x-y+1,\ \ y(0)=1,\ \ 0\leqslant x\leqslant 1.$

解 欧拉法、后退欧拉法、中点法和梯形法计算公式分别为：

欧拉法　$y_{i+1}=y_i+hf(x_i,y_i)$

后退欧拉法　$y_{i+1}=y_i+hf(x_{i+1},y_{i+1})$

中点法　$y_{i+1}=y_{i-1}+2hf(x_i,y_i)$

梯形法　$y_{i+1}=y_i+\dfrac{h}{2}\big[f(x_i,y_i)+f(x_{i+1},y_{i+1})\big]$

(1) 由题知 $f(x,y)=1-y$, 因此

欧拉法　$y_{i+1}=y_i+h(1-y_i),\ \ i=0,\cdots 9$

后退欧拉法　$y_{i+1}=y_i+h(1-y_{i+1})\ \Longrightarrow\ y_{i+1}=\dfrac{y_i+h}{1+h},\ \ i=0,\cdots 9$

中点法　$y_1=\dfrac{y_0+h}{1+h},\ y_{i+1}=y_{i-1}+2h(1-y_i),\ \ i=1,\cdots 9$

梯形法　$y_{i+1}=y_i+\dfrac{h}{2}(2-y_i-y_{i+1})\ \Longrightarrow\ y_{i+1}=\dfrac{y_i(1-h/2)+h}{1+h/2},\ \ i=0,\cdots,9$

取 $h=0.1$, 计算结果如下

x_i	欧拉法	后退欧拉法	中点法	梯形法
0.0	0.0000000000	0.0000000000	0.0000000000	0.0000000000
0.1	0.1000000000	0.0909090909	0.0909090909	0.0952380952
0.2	0.1900000000	0.1735537190	0.1818181818	0.1814058956
0.3	0.2710000000	0.2486851990	0.2545454545	0.2593672389
0.4	0.3439000000	0.3169865446	0.3309090909	0.3299036923
0.5	0.4095100000	0.3790786769	0.3883636363	0.3937223883
0.6	0.4685590000	0.4355260699	0.4532363636	0.4514631132
0.7	0.5217031000	0.4868418817	0.4977163636	0.5037047215
0.8	0.5695327900	0.5334926197	0.5536930909	0.5509709385
0.9	0.6125795110	0.5759023816	0.5869777454	0.5937356110
1.0	0.6513215599	0.6144567105	0.6362975418	0.6324274576

(2) 由题知 $f(x,y)=xy^2$, 因此

欧拉法 $y_{i+1}=y_i+hx_iy_i^2,\ \ i=0,\cdots 9$

简单迭代法求解

后退欧拉法 $y_{i+1}=y_i+hx_{i+1}y_{i+1}^2 \implies y_{i+1}\approx y_{i+1}^{(k+1)}=y_i+hx_{i+1}\big(y_{i+1}^{(k)}\big)^2,\ \ i=0,\cdots 9$

中点法 y_1 由后退欧拉法计算, $y_{i+1}=y_{i-1}+2hx_iy_i^2,\ \ i=1,\cdots 9$

梯形法 $y_{i+1}=y_i+\frac{h}{2}(x_iy_i^2+x_{i+1}y_{i+1}^2)$

$$\implies y_{i+1}\approx y_{i+1}^{(k+1)}=y_i+\frac{h}{2}\big(x_iy_i^2+x_{i+1}(y_{i+1}^{(k)})^2\big),\ \ i=0,\cdots,9$$

取 $h=0.1$, 计算结果如下

x_i	欧拉法	后退欧拉法	中点法	梯形法
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0050506338	1.0102051443	1.0102051443	1.0050506338
0.2	1.0205157925	1.0314843433	1.0204102886	1.0205157925
0.3	1.0473855652	1.0655459911	1.0518546306	1.0473855652
0.4	1.0874936624	1.1153019266	1.0867941785	1.0874936624
0.5	1.1438567164	1.1855821822	1.1463443575	1.1438567164
0.6	1.2213152493	1.2845929204	1.2182047171	1.2213152493
0.7	1.3277673965	1.4271698938	1.3244270855	1.3277673965
0.8	1.4766965200	1.6431706861	1.4637797118	1.4766965200
0.9	1.6928855988	2.0049576534	1.6672512526	1.6928855988
1.0	2.0273584968	2.7750452891	1.9641305249	2.0273584968

(3) 由题知 $f(x,y)=y-2x/y$, 因此

欧拉法 $y_{i+1}=y_i+h(y_i-2x_i/y_i),\ \ i=0,\cdots 9$

后退欧拉法 $y_{i+1}=y_i+h(y_{i+1}-2x_{i+1}/y_{i+1})$

$$\implies y_{i+1}\approx y_{i+1}^{(k+1)}=y_i+h(y_{i+1}^{(k)}-2x_{i+1}/y_{i+1}^{(k)}),\ \ i=0,\cdots 9$$

中点法 y_1 由后退欧拉法计算, $y_{i+1}=y_{i-1}+2h(y_i-2x_i/y_i),\ \ i=1,\cdots 9$

梯形法 $y_{i+1}=y_i+\frac{h}{2}(y_i-2x_i/y_i+y_{i+1}-2x_{i+1}/y_{i+1})$

$$\implies y_{i+1}\approx y_{i+1}^{(k+1)}=y_i+\frac{h}{2}\big(y_i-2x_i/y_i+y_{i+1}^{(k)}-2x_{i+1}/y_{i+1}^{(k)}\big),\ \ i=0,\cdots,9$$

取 $h=0.1$, 计算结果如下

x_i	欧拉法	后退欧拉法	中点法	梯形法
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0956558383	1.0907375368	1.0907375368	1.0956558383
0.2	1.1835936692	1.1740757613	1.1814750737	1.1835936692
0.3	1.2654405290	1.2512485068	1.2593205856	1.2654405290
0.4	1.3423224171	1.3230934978	1.3380497138	1.3423224171
0.5	1.4150581051	1.3901780746	1.4073535064	1.4150581051
0.6	1.4842660555	1.4528699233	1.4774097093	1.4842660555
0.7	1.5504279081	1.5113768372	1.5403889728	1.5504279081
0.8	1.6139284038	1.5657672355	1.6037152341	1.6139284038
0.9	1.6750816920	1.6159772545	1.6615953481	1.6750816920
1.0	1.7341493621	1.6618070426	1.7193750548	1.7341493621

(4) 由题知 $f(x,y) = x - y + 1$, 因此

欧拉法 $\quad y_{i+1} = y_i + h(x_i - y_i + 1), \quad i = 0, \cdots 9$

后退欧拉法 $\quad y_{i+1} = y_i + h(x_{i+1} - y_{i+1} + 1)$

$$\Longrightarrow \quad y_{i+1} \approx y_{i+1}^{(k+1)} = y_i + hh(x_{i+1} - y_{i+1}^{(k)} + 1), \quad i = 0, \cdots 9$$

中点法 $\quad y_1$ 由后退欧拉法计算, $y_{i+1} = y_{i-1} + 2h(x_i - y_i + 1), \quad i = 1, \cdots 9$

梯形法 $\quad y_{i+1} = y_i + \dfrac{h}{2}(x_i - y_i + 2 + x_{i+1} - y_{i+1})$

$$\Longrightarrow y_{i+1} \approx y_{i+1}^{(k+1)} = y_i + \frac{h}{2}\big(x_i - y_i + 2 + x_{i+1} - y_{i+1}^{(k)}\big), \quad i = 0, \cdots, 9$$

取 $h = 0.1$, 计算结果如下

x_i	欧拉法	后退欧拉法	中点法	梯形法
0.0	1.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	1.0047619048	1.0090909091	1.0090909091	1.0047619048
0.2	1.0185941043	1.0264462810	1.0181818182	1.0185941043
0.3	1.0406327610	1.0513148009	1.0454545455	1.0406327610
0.4	1.0700963076	1.0830134554	1.0690909091	1.0700963076
0.5	1.1062776116	1.1209213231	1.1116363636	1.1062776116
0.6	1.1485368867	1.1644739301	1.1467636364	1.1485368867
0.7	1.1962952785	1.2131581182	1.2022836364	1.1962952785
0.8	1.2490290615	1.2665073802	1.2463069091	1.2490290615
0.9	1.3062643889	1.3240976184	1.3130222545	1.3062643889
1.0	1.3675725424	1.3855432894	1.3637024582	1.3675725424

9.5 利用标准的四级四阶 R-K 法求解习题 9.1.

解 标准的四级四阶 R-K 法为：

$$\begin{cases} y_{i+1} = y_i + \dfrac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4), \\ K_1 = hf(x_i, y_i), \\ K_2 = hf\big(x_i + \dfrac{1}{2}h, y_i + \dfrac{1}{2}K_1\big), \\ K_3 = hf\big(x_i + \dfrac{1}{2}h, y_i + \dfrac{1}{2}K_2\big), \\ K_4 = hf(x_i + h, y_i + K_3). \end{cases}$$

取 $h = 0.1$, 计算结果见下表:

x_i	题目(1)	题目(2)	题目(3)	题目(4)
0.0	0.0000000000	1.0000000000	1.0000000000	1.0000000000
0.1	0.0951625000	1.0050251359	1.0954455317	1.0048375000
0.2	0.1812690985	1.0204082056	1.1832167455	1.0187309014
0.3	0.2591815779	1.0471205219	1.2649122283	1.0408184220
0.4	0.3296797110	1.0869567284	1.3416423538	1.0703202889
0.5	0.3934690655	1.1428575201	1.4142155779	1.1065309344
0.6	0.4511880656	1.2195128436	1.4832422228	1.1488119344
0.7	0.5034143813	1.3245043524	1.5491964523	1.1965856187
0.8	0.5506707102	1.4705896531	1.6124553497	1.2493292897
0.9	0.5934300087	1.6806729081	1.6733246590	1.3065699912
1.0	0.6321202255	1.9999911976	1.7320563652	1.3678797744

9.7 利用待定系数法确定下列求解公式中的系数，使其阶数尽可能高，并导出截断误差表示式：

(1) $y_{i+1} = \alpha_0 y_i + \alpha_1 y_{i-1} + \beta h f_{i+1}$;

(2) $y_{i+1} = y_i + h(\beta_0 f_i + \beta_1 f_{i-1})$;

(3) $y_{i+1} = y_{i-3} + h(\beta_0 f_i + \beta_1 f_{i-1} + \beta_2 f_{i-2})$.

解 (1) 局部截断误差为

$$R[y] = y(x_{i+1}) - y_{i+1} = y(x_i + h) - \alpha_0 y(x_i) - \alpha_1 y(x_i - h) - \beta h y'(x_i + h).$$

取 $x_i=0$, 令 $R[x^k]=0\ (k=0,1,2)$ 得

$$\begin{cases} 1-\alpha_0-\alpha_1=0, \\ 1+\alpha_1-\beta=0, \\ 1-\alpha_1-2\beta=0, \end{cases}$$

解得

$$\alpha_0=\frac{4}{3},\ \ \alpha_1=-\frac{1}{3},\ \ \beta=\frac{2}{3},$$

即有

$$y_{i+1}=\frac{4}{3}y_i-\frac{1}{3}y_{i-1}+\frac{2}{3}hf_{i+1}.$$

令 $y=x^3$, 则 $R[y]=-\frac{4}{3}h^2\neq 0$. 所以上述公式的代数精度 $m=2$. 取

$$e(x)=\frac{1}{6}y'''(\xi)(x-x_{i+1})^2(x-x_{i-1}),$$

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注意到 $e(x_{i-1})=e(x_{i+1})=e'(x_{i+1})=0$, 于是有

$$R[y]=R[e]=e(x_{i+1})-\frac{4}{3}e(x_i)+\frac{1}{3}e(x_{i-1})-\frac{2}{3}he'(x_{i+1})=-\frac{4}{3}e(x_i)=-\frac{2}{9}h^3y'''(\xi).$$

(2) 局部截断误差为

$$R[y]=y(x_{i+1})-y_{i+1}=y(x_i+h)-y(x_i)-h(\beta_0y'(x_i)+\beta_1y'(x_i-h)).$$

取 $x_i=0$, 令 $R[x^k]=0\ (k=0,1)$ 得
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$$\begin{cases} 1-\beta_0-\beta_1=0, \\ 1+2\beta_1=0, \end{cases}$$

解得

$$\beta_0=\frac{3}{2},\ \ \beta_1=-\frac{1}{2},$$

即有

$$y_{i+1}=y_i+\frac{1}{2}h(3f_i-f_{i-1}).$$

令 $y=x^2$, 则 $R[y]=0$. 再令 $y=x^3$, 则 $R[y]=\frac{5}{2}h^3\neq 0$. 所以上述公式的代数精度 $m=2$. 取

$$e(x)=\frac{1}{6}y'''(\xi)(x-x_i)(x-x_{i-1})^2,$$

注意到 $e(x_i)=e(x_{i-1})=e'(x_{i-1})=0$, 于是有

$$R[y]=R[e]=e(x_{i+1})-e(x_i)-\frac{1}{2}h\big(3e'(x_i)-e'(x_{i-1})\big)=e(x_{i+1})-\frac{3}{2}he'(x_i)=\frac{5}{12}h^3y'''(\xi).$$

(3) 局部截断误差为

$$R[y]=y(x_{i+1})-y_{i+1}=y(x_i+h)-y(x_i-3h)-h(\beta_0y'(x_i)+\beta_1y'(x_i-h)+\beta_2y'(x_i-2h)).$$

取 $x_i=0$, 令 $R[x^k]=0\ (k=0,1,2,3)$ 得

$$\begin{cases} 1-1=0, \\ 4-\beta_0-\beta_1-\beta_2=0, \\ 4-\beta_1-2\beta_2=0, \\ 28-3(\beta_1+4\beta_2)=0, \end{cases}$$

解得

$$\beta_0=\frac{8}{3},\ \ \beta_1=-\frac{4}{3},\ \ \beta_2=\frac{8}{3},$$

即有

$$y_{i+1}=y_{i-3}+\frac{4}{3}h(2f_i-f_{i-1}+2f_{i-2}).$$

令 $y=x^4$, 则 $R[y]=0$. 再令 $y=x^5$, 则 $R[y]=\frac{112}{3}h^5\neq 0$. 所以上述公式的代数精度 $m=4$. 取

$$e(x)=\frac{1}{720}y^{(5)}(\xi)(x-x_i)(x-x_{i-1})^2(x-x_{i-2})^2,$$

注意到 $e(x_i)=e(x_{i-1})=e'(x_{i-1})=e(x_{i-2})=e'(x_{i-2})=0$, 于是有

$$\begin{aligned} R[y]=R[e]&=e(x_{i+1})-e(x_{i-3})-\frac{4}{3}h\big(2e'(x_i)-e'(x_{i-1})+2e'(x_{i-2})\big) \\ &=e(x_{i+1})-e(x_{i-3})-\frac{8}{3}he'(x_i) \\ &=\frac{7}{135}h^5y^{(5)}(\xi). \end{aligned}$$

9.11 用欧拉法和标准四级四阶 R-K 法求解下列初值问题：

$$(1)\ \begin{cases} y''=y'(1-y^2)-y, \\ y(x_0)=y_0,\ y'(x_0)=y_0'; \end{cases}$$

$$(2)\ \begin{cases} y'''=y''-2y'+y-x+1, \\ y(x_0)=y_0',\ y'(x_0)=y_0',\ y''(x_0)=y_0''. \end{cases}$$

解 (1) 令 $y_1=y,\ y_2=y'$, 则二阶初值问题转化为

$$\begin{cases} y_1'=y_2, \\ y_2'=y_2(1-y_1^2)-y_1, \\ y_1(0)=y_0,\ y_2(0)=y_0'. \end{cases}$$

记

$$y=\begin{pmatrix} y_1 \\ y_2 \end{pmatrix},\ y(0)=\begin{pmatrix} y_1(0) \\ y_2(0) \end{pmatrix}=\begin{pmatrix} y_0 \\ y_0' \end{pmatrix},\ f(x,y)=\begin{pmatrix} f_1(x,y) \\ f_2(x,y) \end{pmatrix}=\begin{pmatrix} y_2 \\ y_2(1-y_1^2)-y_1 \end{pmatrix}.$$

则欧拉法的计算公式为 $y_{i+1} = y_i + hf(x_i, y_i)$, 即

$$\boxed{\begin{pmatrix} y_{1,i+1} \\ y_{2,i+1} \end{pmatrix} = \begin{pmatrix} y_{1i} \\ y_{2i} \end{pmatrix} + h \begin{pmatrix} y_{2i} \\ y_{2i}(1 - y_{1i}^2) - y_{1i} \end{pmatrix}.}$$

若用标准四级四阶 R-K 方法求解, 则

$$y_{i+1} = y_i + \frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4),$$

其中

$$K_1 = hf(x_i, y_i) = h \begin{pmatrix} y_{2i} \\ y_{2i}(1 - y_{1i}^2) - y_{1i} \end{pmatrix} = \begin{pmatrix} K_{11} \\ K_{12} \end{pmatrix},$$

$$K_2 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\right) = \begin{pmatrix} K_{21} \\ K_{22} \end{pmatrix}$$

$$= h \begin{pmatrix} y_{2i} + \frac{1}{2}K_{12} \\ \left(y_{2i} + \frac{1}{2}K_{12}\right)\left[1 - \left(y_{1i} + \frac{1}{2}K_{11}\right)^2\right] - y_{1i} - \frac{1}{2}K_{11} \end{pmatrix},$$

$$K_3 = hf\left(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\right) = \begin{pmatrix} K_{31} \\ K_{32} \end{pmatrix}$$

$$= h \begin{pmatrix} y_{2i} + \frac{1}{2}K_{22} \\ \left(y_{2i} + \frac{1}{2}K_{22}\right)\left[1 - \left(y_{1i} + \frac{1}{2}K_{21}\right)^2\right] - y_{1i} - \frac{1}{2}K_{21} \end{pmatrix},$$

$$K_4 = hf\left(x_i + h, y_i + K_3\right) = \begin{pmatrix} K_{41} \\ K_{42} \end{pmatrix}$$

$$= h \begin{pmatrix} y_{2i} + K_{32} \\ \left(y_{2i} + K_{32}\right)\left[1 - (y_{1i} + K_{31})^2\right] - y_{1i} - K_{31} \end{pmatrix}.$$

(2) 令 $y_1 = y$, $y_2 = y'$, $y_3 = y''$, 则三阶初值问题转化为

$$\begin{cases} y_1' = y_2, \\ y_2' = y_3, \\ y_3' = y_3 - 2y_2 + y_1 - x + 1, \\ y_1(0) = y_0,\ y_2(0) = y_0',\ y_3(0) = y_0''. \end{cases}$$

记

$$y = \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix},\ y(0) = \begin{pmatrix} y_1(0) \\ y_2(0) \\ y_3(0) \end{pmatrix} = \begin{pmatrix} y_0 \\ y_0' \\ y_0'' \end{pmatrix},\ f(x, y) = \begin{pmatrix} f_1(x, y) \\ f_2(x, y) \\ f_3(x, y) \end{pmatrix} = \begin{pmatrix} y_2 \\ y_3 \\ y_3 - 2y_2 + y_1 - x + 1 \end{pmatrix}.$$

则欧拉法的计算公式为 $y_{i+1} = y_i + hf(x_i, y_i)$, 即

$$\begin{pmatrix} y_{1,i+1} \\ y_{2,i+1} \\ y_{3,i+1} \end{pmatrix} = \begin{pmatrix} y_{1i} \\ y_{2i} \\ y_{3i} \end{pmatrix} + h \begin{pmatrix} y_{2i} \\ y_{3i} \\ y_{3i} - 2y_{2i} + y_{1i} - x_i + 1 \end{pmatrix}.$$

若用标准四级四阶 R-K 方法求解, 则

$$\boxed{y_{i+1} = y_i + \frac{1}{6}\big(K_1 + 2K_2 + 2K_3 + K_4\big),}$$

其中

$$K_1 = hf(x_i, y_i) = h\begin{pmatrix} y_{2i} \\ y_{3i} \\ y_{3i} - 2y_{2i} + y_{1i} - x_i + 1 \end{pmatrix} = \begin{pmatrix} K_{11} \\ K_{12} \\ K_{13} \end{pmatrix},$$

$$K_2 = hf\big(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_1\big) = \begin{pmatrix} K_{21} \\ K_{22} \\ K_{23} \end{pmatrix}$$

$$= h\begin{pmatrix} y_{2i} + \frac{1}{2}K_{12} \\ y_{3i} + \frac{1}{2}K_{13} \\ y_{3i} + \frac{1}{2}K_{13} - 2\big(y_{2i} + \frac{1}{2}K_{12}\big) + y_{1i} + \frac{1}{2}K_{11} - x_i - \frac{h}{2} + 1 \end{pmatrix},$$

$$K_3 = hf\big(x_i + \frac{h}{2}, y_i + \frac{1}{2}K_2\big) = \begin{pmatrix} K_{31} \\ K_{32} \\ K_{33} \end{pmatrix}$$

$$= h\begin{pmatrix} y_{2i} + \frac{1}{2}K_{22} \\ y_{3i} + \frac{1}{2}K_{23} \\ y_{3i} + \frac{1}{2}K_{23} - 2\big(y_{2i} + \frac{1}{2}K_{22}\big) + y_{1i} + \frac{1}{2}K_{21} - x_i - \frac{h}{2} + 1 \end{pmatrix},$$

$$K_4 = hf\big(x_i + h, y_i + K_3\big) = \begin{pmatrix} K_{41} \\ K_{42} \\ K_{43} \end{pmatrix}$$

$$= h\begin{pmatrix} y_{2i} + K_{32} \\ y_{3i} + K_{33} \\ y_{3i} + K_{33} - 2\big(y_{2i} + K_{32}\big) + y_{1i} + K_{31} - x_i - h + 1 \end{pmatrix}.$$