Partial Differential Equations

Chapter V - FEM

John Cagnol, Pauline Lafitte

The Engineering Program of CentraleSupélec

Lecture 6 – January 17th 2020

V.1. Introduction

V.1.1. Goal

Solving PDEs explicitly is often difficult or impossible to do.

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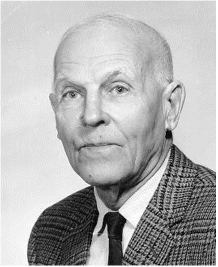
The **Finite Element Method** (FEM) is a numerical method for approximating the solutions of a PDE. It is based on

- The variational formulation of the PDE on Ω
- A discretization of the domain Ω
- Replacing the PDE by its discrete counterpart
- Solving the discretized problem by solving a system of linear equations

V.1.2. History

1940's

Alexander Hrennikoff and Richard Courant developed mesh discretization methods for solving elasticity and structural analysis problems in civil and aeronautical engineering.



credits: (l) Konrad Jacobs, (r) University of British Columbia

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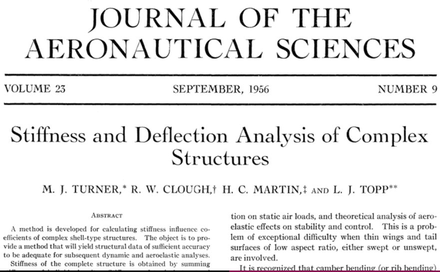

JOURNAL OF THE AERONAUTICAL SCIENCES

VOLUME 23 SEPTEMBER, 1956 NUMBER 9

Stiffness and Deflection Analysis of Complex Structures

M. J. TURNER,* R. W. CLOUGH,† H. C. MARTIN,‡ AND L. J. TOPP**

ABSTRACT

A method is developed for calculating stiffness influence coefficients of complex shell-type structures. The object is to provide a method that will yield structural data of sufficient accuracy to be adequate for subsequent dynamic and aeroelastic analyses. Stiffness of the complete structure is obtained by summing

tion on static air loads, and theoretical analysis of aeroelastic effects on stability and control. This is a problem of exceptional difficulty when thin wings and tail surfaces of low aspect ratio, either swept or unswept, are involved.

It is recognized that camber bending (or rib bending)

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IBM and General Motors built the DAC-1, a computer system to develop automobiles making use of FEM.

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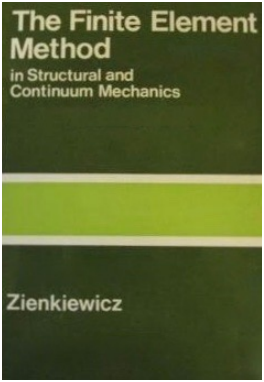

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FEM has expanded rapidly as:
Theoretical mathematical tools for FEM have become available
Computing power have doubled every 18 months

Today

FEM is widely used for numerical approximation of PDEs and numerical simulation.

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(FEM is one of the methods used)

credit: Dassault Aviation

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Multi-physics (coupling)
Multi-scale
Complex systems
Robust solvers
Computing time

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There are still many challenges including:
Multi-physics (coupling)
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Complex systems
Robust solvers
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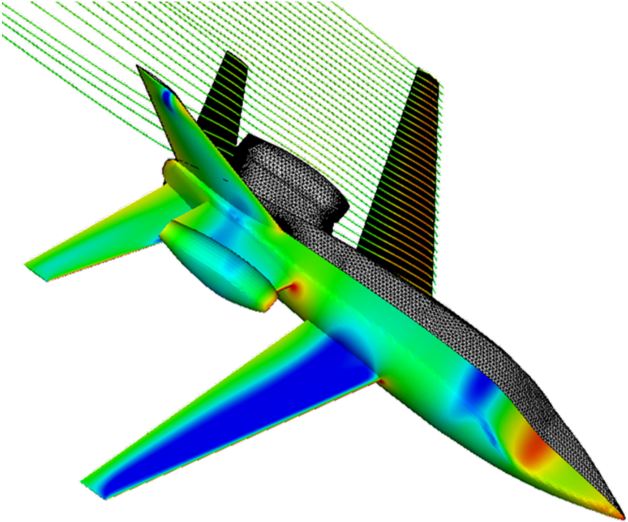
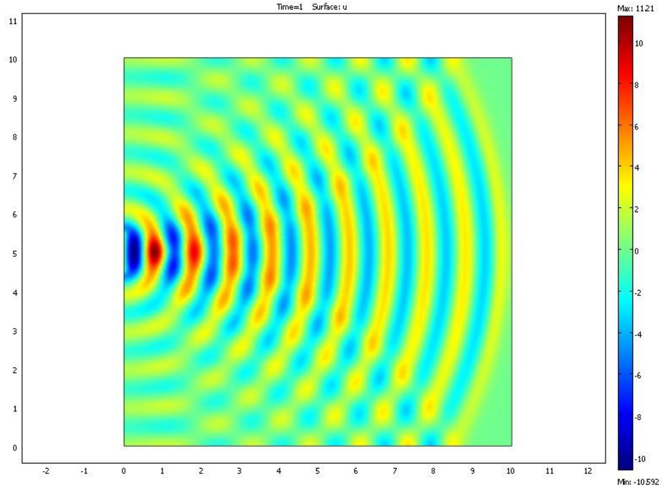
Each CentraleSupélec graduate ought to master the underlying foundations of FEM and be able to use them to approximate the solutions to PDEs while understanding the limitations.

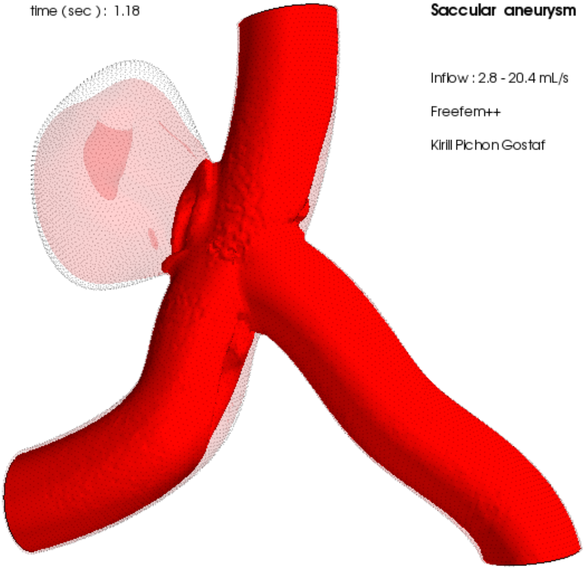
V.1.3. Examples



credit: Aircraft Research Association Ltd.



credit: John Bamonte, Reza Malek-Madani

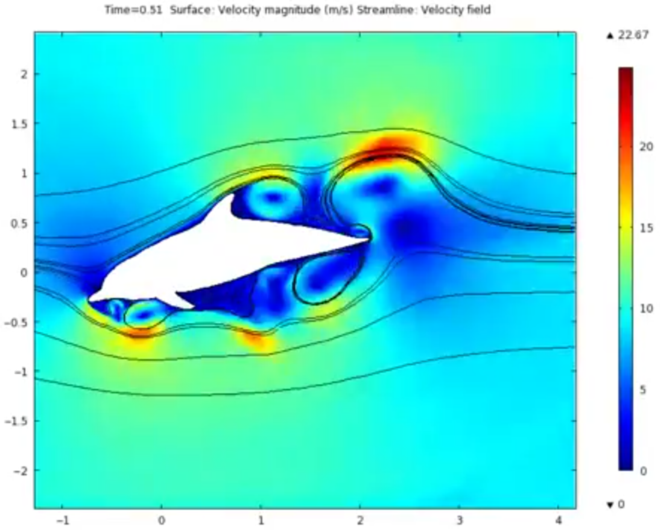


Saccular aneurysm

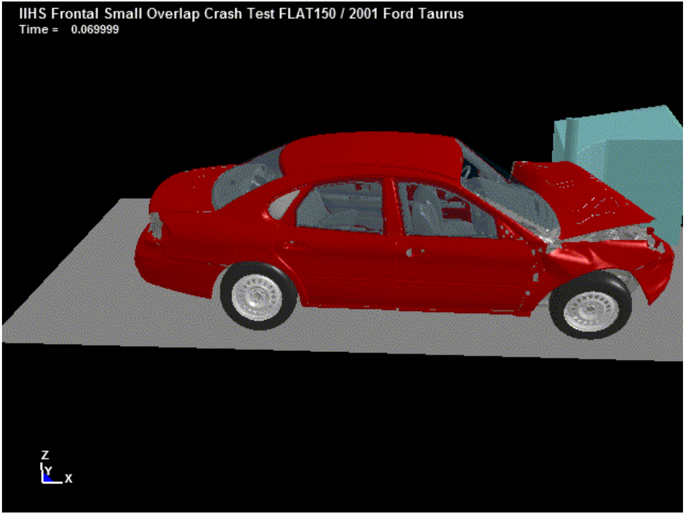
Inflow : 2.8 - 20.4 mL/s

Freefem++

Kirill Pichon Gostaf



credit: Comsol



credit: Lancemore Co.

V.2. Discretization

V.2.1. Generating a Mesh

Definition V.2.1

A **mesh** *of* $\Omega \subset \mathbb{R}^d$ *is a decomposition of* Ω *in sub-domains (the elements) that don't overlap and cover* Ω.
To be useful a mesh needs to meet some quality criteria.

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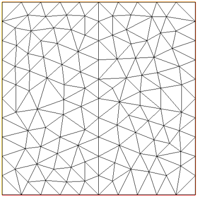
Example

Let $\Omega \subset \mathbb{R}^2$ be a square.



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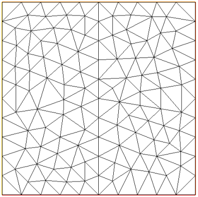

Example

Let $\Omega \subset \mathbb{R}^1$ be an interval

Meshing a domain $\Omega \subset \mathbb{R}$

Let $\Omega = [a, b]$ with $a < b$.

For the sake of simplicity, we have chosen $a = 0$ and $b = 1$ but what follows adapts easily.

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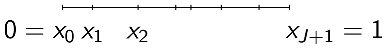
Definition V.2.2

Meshing $[0, 1]$ *consists of giving:*

- $J \in \mathbb{N}^*$
- *A* $(J + 2)$*-tuple* $(x_j)_{j \in \{0,\dots,J+1\}}$ *such that* $x_0 = 0$ *and* $x_{J+1} = 1$.

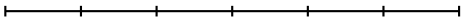


The discretization step is defined by $h = \max_{j \in \{0,\dots,J\}} |x_{j+1} - x_j|$.

Meshing a domain $\Omega \subset \mathbb{R}$

Definition V.2.3

If the $x_{j+1} - x_j$ is constant for all $j \in \{0, \dots, J\}$, then the mesh is said to be **uniform**.



$$0 = x_0 \quad x_1 \quad x_2 \qquad\qquad\qquad\qquad x_{J+1} = 1$$

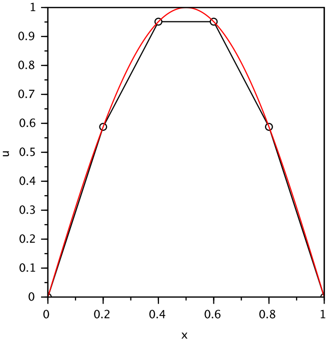
We have

$$h = \frac{1}{J+1}$$
$$x_j = j\,h$$

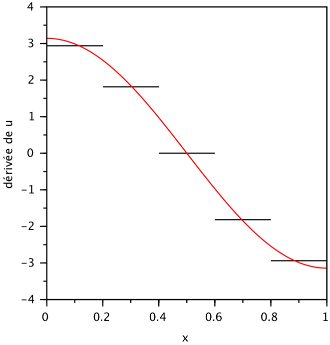
Using a mesh on a 1D-domain to interpolate a function

Let us consider a uniform mesh on $[0,1]$ with a step h.

We can use the mesh to interpolate $u : x \mapsto \sin(\pi x)$.



(a) u (red) and u_{approx} (black)



(b) u' (red) and u'_{approx} (black)

$J = 4$ et $h = 1/5$

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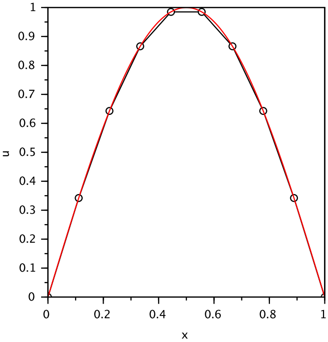
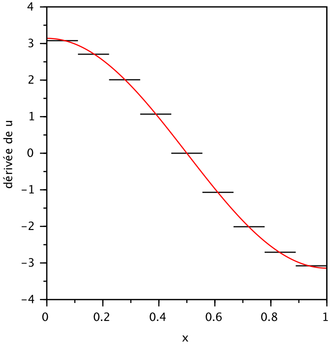


(c) u (red) and u_{approx} (black)



(d) u' (red) and u'_{approx} (black)

$J = 8$ et $h = 1/9$

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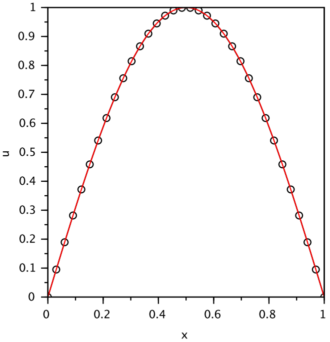
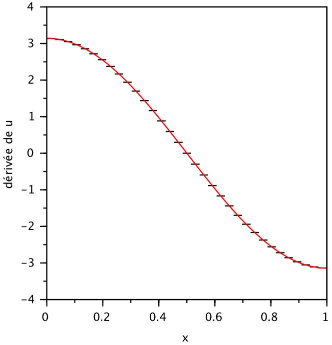


(e) u (red) and u_{approx} (black)



(f) u' (red) and u'_{approx} (black)

$$J = 32 \text{ et } h = 1/33$$

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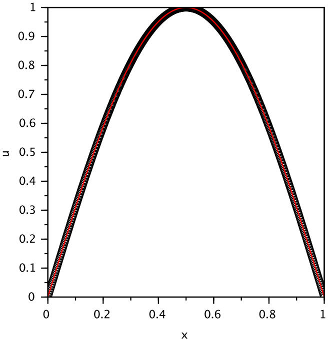
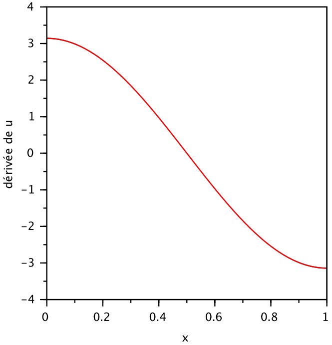


(g) u (red) and u_{approx} (black)



(h) u' (red) and u'_{approx} (black)

$$J = 512 \text{ et } h = 1/513$$

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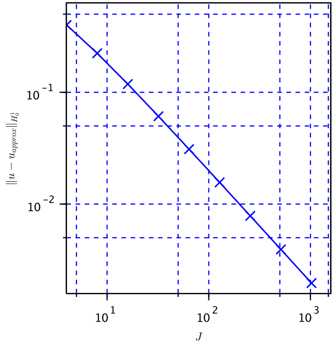

Error $\|u - u_{approx}\|_{H_0^1}$ (log-scale)

Meshing a 2D domain

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There are several ways to mesh Ω.
We will develop meshing with triangles so called triangulation.

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Definition V.2.4

A **triangulation** *of* Ω *is given by a set* $\mathcal{T}$ *of* J_e *triangles* $(K_i)_{i\in\{1,\ldots,J_e\}}$ *such that*

- $\forall i \in \{1, \ldots, J_e\},\ K_i \subset \bar{\Omega}$
- $\bar{\Omega} = \cup_{i\in\{1,\ldots,J_e\}} K_i$
- $\forall i, j \in \{1, \ldots, J_e\},\ K_i \cap K_j$ *is either empty, a vertex or an edge.*

The vertices of the triangles in $\mathcal{T}$ *are called* **nodes**.

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In order to be suitable, the triangles need to meet some additional properties depending on the finite element simulation.

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For instance, some methods will require that all triangles acute, forming non-obtuse meshes.

Meshing a 2D domain

Definition V.2.5

The diameter of a triangle K is define by

$$diam(K) = \sup_{x,y \in K} |x - y|$$

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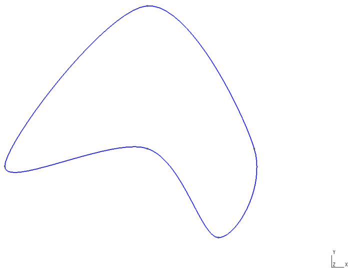
Furthermore, we note

- J_e the number of triangles
- J_v the number of vertices (nodes)
- J_v^D the number of vertices (nodes) not on $\partial\Omega$

Meshing a domain in $\Omega \subset \mathbb{R}^2$



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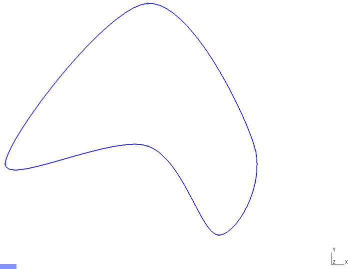

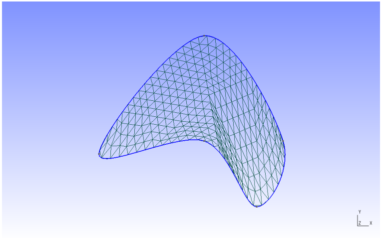

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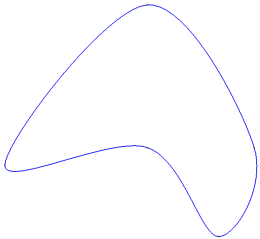
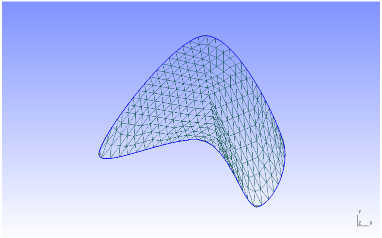
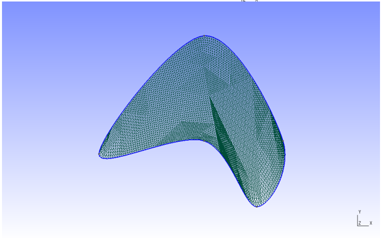
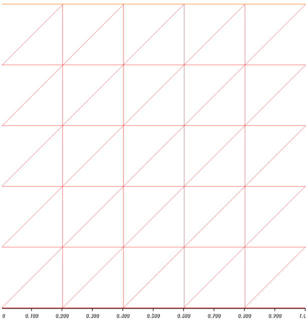




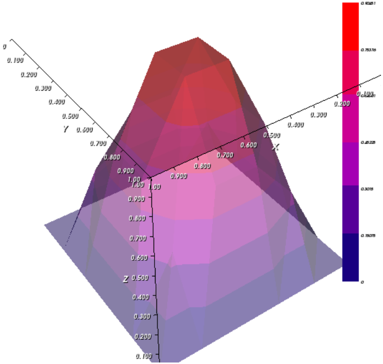


Using a mesh on a 2D-domain to interpolate a function

Consider $u : (x, y) \mapsto \sin(\pi x)\sin(\pi y)$



(i) mesh



(j) u

$$J_v^D = 16 \text{ degrees of freedom (dof)}$$

Using a mesh on a 2D-domain to interpolate a function

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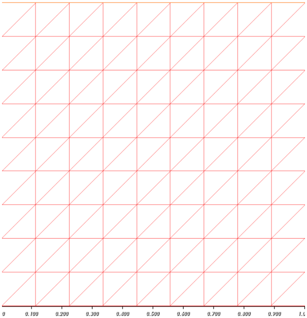
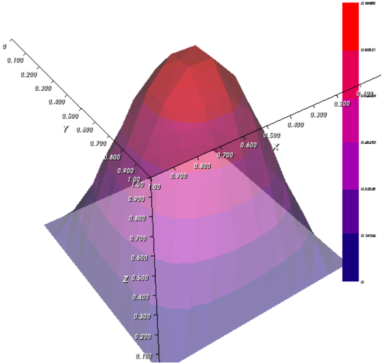


(k) mesh



(l) u

$$J_v^D = 64 \text{ degrees of freedom (dof)}$$

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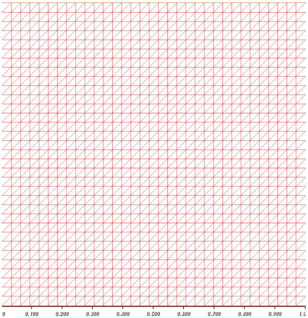
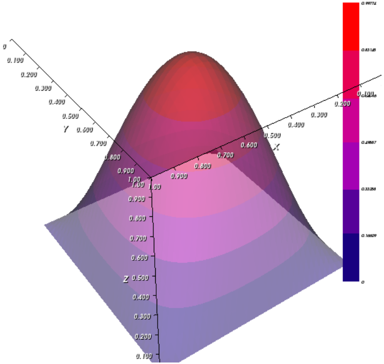


(m) mesh



(n) u

$$J_v^D = 1024 \text{ degrees of freedom (dof)}$$

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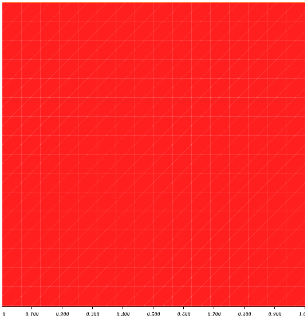
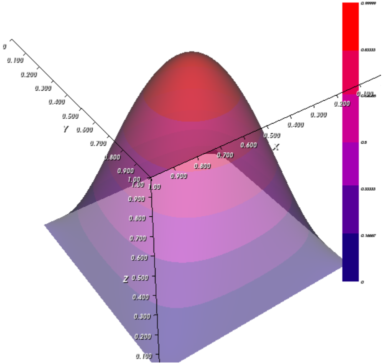


(o) mesh



(p) u

$$J_v^D = 262144 \text{ degrees of freedom (dof)}$$

Meshing a domain $\Omega \subset \mathbb{R}^d$

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The meshing performed in 2D with triangles can be generalized with simplices.

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- $\forall i, j \in \{1, \dots, J_e\},\ K_i \cap K_j$ *is either empty or a simplex of dimension equal or smaller than* $d - 1$

The vertices of the triangles in $\mathcal{T}$ are called **nodes**.

V.2.2. Internal Approximation

Consider a variational formulation

$$(VF) \quad \text{Find } u \in H \text{ s.t. } \forall v \in H \; a(u,v) = \ell(v)$$

where H is a Hilbert space, a is a continuous and coercive bilinear form on $H \times H$ and ℓ is a continuous linear form on H.

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Let us consider a collection of linear subspace $H_h \subset H$ of finite dimension N_h.

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where H is a Hilbert space, a is a continuous and coercive bilinear form on $H \times H$ and ℓ is a continuous linear form on H.

Let us consider a collection of linear subspace $H_h \subset H$ of finite dimension N_h. The parameter h has no significance at this point. Later on, we will choose h to be smaller and smaller.

Consider a variational formulation

$$(VF) \quad \text{Find } u \in H \text{ s.t. } \forall v \in H \; a(u,v) = \ell(v)$$

where H is a Hilbert space, a is a continuous and coercive bilinear form on $H \times H$ and ℓ is a continuous linear form on H.

Let us consider a collection of linear subspace $H_h \subset H$ of finite dimension N_h. The parameter h has no significance at this point. Later on, we will choose h to be smaller and smaller.

Remark V.2.7

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Lemma V.2.9

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$$(VF_h) \quad \textit{Find } u_h \in H_h \textit{ s.t. } \forall v_h \in H_h,\ a(u_h, v_h) = \ell(v_h)$$

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Proof: Lax-Milgram (IV.1.6)

Definition V.2.10

Let $\Phi = (\phi_i)_{i\in\{1,\ldots,N_h\}}$ *be a basis of* H_h*. Define*

- *a* $N_h \times N_h$ *matrix*

$$A_h = [a(\phi_i, \phi_j)]_{i\in\{1,\ldots,N_h\},j\in\{1,\ldots,N_h\}}$$

  It is called the **rigidity matrix**.

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Lemma V.2.12

The solution u_h *in Lemma V.2.9 is the (unique) solution to* $A_h u_h = b_h$.

Let u be the solution to (VF) and u_h be the solution to (VF_h).
Let w_h be any element in H_h

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(Using the coercivity of a)

Let u be the solution to (VF) and u_h be the solution to (VF_h).
Let w_h be any element in H_h

$$a(u - u_h, w_h) = 0$$

Let $v_h \in H_h$

$$\alpha \|u - u_h\|_H^2 \leq a(u - u_h, u - u_h) = a(u - u_h, u - v_h) + a(u - u_h, v_h - u_h)$$

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Let w_h be any element in H_h

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Thus

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Lemma V.2.13 (Céa)

$$\|u - u_h\|_H \leq \frac{M}{\alpha} \inf_{v_h \in H_h} \|u - v_h\|_H.$$

Proposition V.2.14

If a is symmetric then $\|\cdot\|_E = \sqrt{a(\cdot,\cdot)}$ *is a norm.*
It is equivalent to $\|\cdot\|_H$*, the norm of the Hilbert space:*

$$\sqrt{\alpha}\|\cdot\|_H \leq \|\cdot\|_E \leq M\|\cdot\|_H$$

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Definition V.2.15

In this case, $\|\cdot\|_E$ *is called the* **energy norm***.*

Remark V.2.16

if a is symmetric, u_h *is the* **orthogonal projection** *of* u *on* H_h *and*

$$\|u - u_h\|_H \leq \sqrt{\frac{M}{\alpha}} \inf_{v_h \in H_h} \|u - v_h\|.$$

The Céa Lemma provides an upper-bound to the error made when replacing
u (the actual solution in H) by
u_h (the solution in the finite-dimensional space H_h).

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- α: the coercivity constant of a
- M: the continuity constant of a
- How "far away" is u from the points of H_h

If H_h is big (but still finite-dimensional), we can expect u to be closer to the points of H_h because there are more of them.

Theorem V.2.17

Let (H_h) be a decreasing sequence of N_h-dimensional linear subspaces of H such that $\lim_{h\to 0} N_h = +\infty$. $(k < h \;\Rightarrow\; H_h \subset H_k)$

Assume there exists $\mathcal{H}$ a dense linear subspace of H and a linear application $r_h : \mathcal{H} \longrightarrow H_h$ s.t.

$$\forall v \in \mathcal{H}, \quad \lim_{h\to 0} \|v - r_h(v)\|_H = 0.$$

Then, the internal approximation method **converges**, *i.e.*

$$\lim_{h\to 0} \|u - u_h\|_H = 0$$

Furthermore, if $\|u - u_h\|_H = O(h^p)$ the method is of **order** *p.*

Definition V.2.18

r_h is called the **interpolation operator**.

V.3. Putting together the numerical method

V.3.1. Outline of the method

Based on the internal approximation, the outline of the method will be:

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Finding the "good" spaces (H_h) requires some attention.

- Building the interpolation operator $r_h : \mathcal{H} \to H_h$ s.t.

  $$\forall v \in \mathcal{H},\ \lim_{h\to 0} \|v - r_h(v)\|_H = 0$$

  For instance if $H = H^1(\Omega)$ we will need regular functions in $\mathcal{H}$
- We would like the rigidity matrix to be sparse (with many 0) and behave well when solving the linear system.

The Finite Element Method (FEM) consists of putting together spaces composed of piecewise polynomial continuous functions.

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Let r_h be the orthogonal projection from H to H_h.

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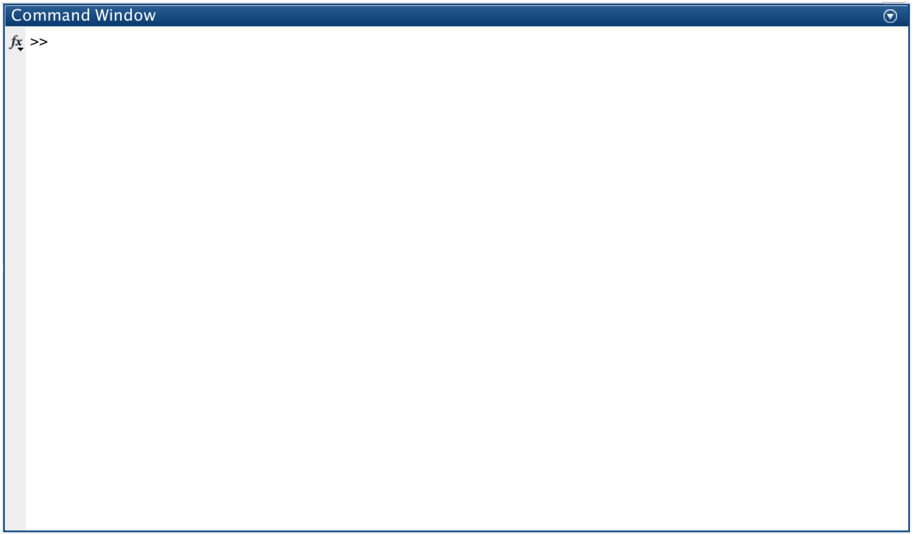
Finding such an Hilbertian basis needs to be done carefully so that system $A_h u_h = b_h$ can be solved easily. We want:

- A_h to be sparse so we have to store less data, and the computation will be faster.
- $\kappa_h = \|A_h\|\|A_h^{-1}\|$ needs to be as small as possible and should not diverge. The real number κ_h is called the condition number. It measures the sensitivity of the solution u_h of $A_h u_h = b_h$ to small errors in b_h.


Command Window
fx >>

Command Window

```
>> A = [4.1 2.8; 9.7 6.6]

A =

    4.1000    2.8000
    9.7000    6.6000

>> 
```

Command Window

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>> A = [4.1 2.8; 9.7 6.6]

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    1.0000
   -0.0000

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    1.0000
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b2 =

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    9.7000

>> 
```

Command Window

```
    4.1000
    9.7000

>> A\b1

ans =

    1.0000
   -0.0000

>> b2 = [4.11; 9.7]

b2 =

    4.1100
    9.7000

>> A\b2

ans =

    0.3400
    0.9700

>>
```

Command Window

```
ans =

    1.0000
   -0.0000

>> b2 = [4.11; 9.7]

b2 =

    4.1100
    9.7000

>> A\b2

ans =

    0.3400
    0.9700

>> cond(A)

ans =

   1.6230e+03

fx >>
```

Command Window

```
b2 =

    4.1100
    9.7000

>> A\b2

ans =

    0.3400
    0.9700

>> cond(A)

ans =

   1.6230e+03

>> eig(A)

ans =

   -0.0093
   10.7093

>>
```

Definition V.3.1

Define

$$P_q = \mathbb{R}_q[X_1, \ldots, X_d] = \text{vect}\left(X_1^{\alpha_1} \ldots X_d^{\alpha_d} \; : \; \sum_{k=1}^{d} \alpha_k \leq q \right)$$

the space of polynomials of degree smaller than q in the variables $X_1, \ldots, X_d$.

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the space of polynomials of degree smaller than q in the variables $X_1, \ldots, X_d$.

Example

When $d = 2$, P_1 *is the linear space of dimension 3 made of polynomial of degree smaller than 1 in* x *and* y:

$$P_1 = \{a + bX + cY,\ (a, b, c) \in \mathbb{R}^3\}$$

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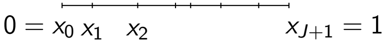
We will use functions of P_q on each cell.

V.3.2. Finite Elements P_1 in Dimension 1

Definition

Mesh:



Definition V.3.2 (Finite Elements P_1)

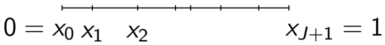
We define

$$H_h = \{v \in C^0([0,1]) \ : \ v|_{[x_j, x_{j+1}]} \in P_1, \ j \in \{0, \ldots, J\}\}$$

and its subspace:

$$H_{0,h} = \{v \in H_h, \ v(0) = v(1) = 0\}$$

Definition

Mesh: 

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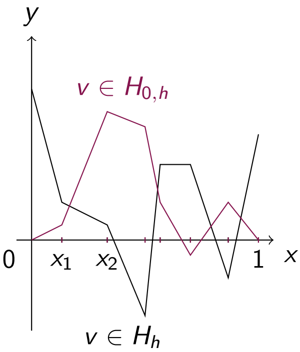
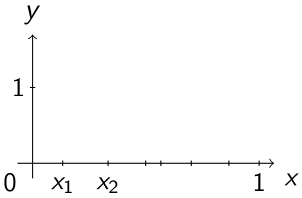


Basis

Lemma V.3.3

H_h is a linear subspace of $H^1(0,1)$ of dimension $J+2$.
$H_{0,h}$ is a linear subspace of $H^1_0(0,1)$ of dimension J.



Basis

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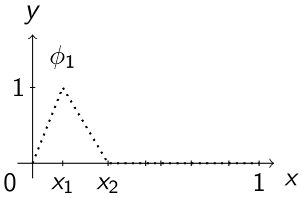

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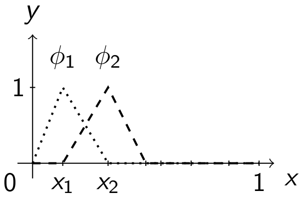

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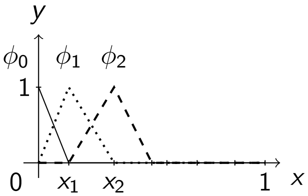

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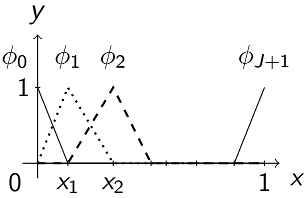

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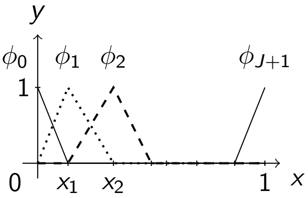

Define $(\phi_j)_{j\in\{0,\dots,J+1\}}$ s.t.
$\forall j \in \{0,\dots,J+1\},\ \phi_j \in H_h$
$\forall i,j \in \{0,\dots,J+1\},\ \phi_j(x_i) = \delta_{ij}$.

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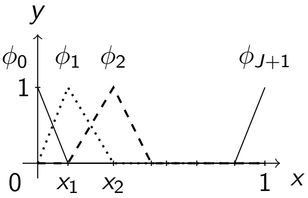

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Lemma V.3.4

$(\phi_j)_{j\in\{0,\dots,J+1\}}$ is a basis of H_h
$(\phi_j)_{j\in\{1,\dots,J\}}$ is a basis of $H_{0,h}$.

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Consider

$$\begin{cases} -u'' = f \text{ in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

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$$(VF) \quad \forall \phi \in C_c^1(0,1), \ \int_{]0,1[} u'\phi' = \int_{]0,1[} f\,\phi$$

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We defined

$$a : (u,v) \mapsto \int_{]0,1[} u'v'$$

that was proven to be a coercive (with $\alpha = 1$) and continuous (with $M = C_\Omega$ the Poincaré constant) bilinear form on $H_0^1(0,1)$

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If $|i-j| = 1$ then

$$[A_h]_{ij} = \int_0^1 -\frac{1}{h^2} 1_{[x_{i-1},x_{i+1}]} 1_{[x_{j-1},x_{j+1}]}$$

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Therefore

$$A_h = \frac{1}{h} \begin{bmatrix} 2 & -1 & 0 & \cdots & \cdots & 0 \\ -1 & 2 & -1 & \ddots & & \vdots \\ 0 & -1 & 2 & -1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & -1 & 2 & -1 \\ 0 & \cdots & \cdots & 0 & -1 & 2 \end{bmatrix}$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Let

$$b_h = \left[\int_0^1 f \phi_i \right]_{i \in \{1,\ldots,J\}}$$

This integral is computed by quadrature.

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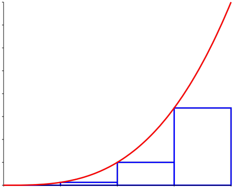
Left Riemann sum

$$\int_{x_j}^{x_{j+1}} f \, \phi = (x_{j+1} - x_j) f(x_j) \phi(x_j) + O((x_{j+1} - x_j)^2)$$



credit: Geoff Richards

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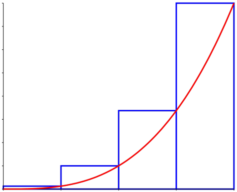

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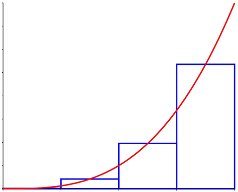
This integral is computed by quadrature.

Midpoint rule

$$\int_{x_j}^{x_{j+1}} f\,\phi = (x_{j+1} - x_j) f\left(\frac{x_j + x_{j+1}}{2}\right) \phi\left(\frac{x_j + x_{j+1}}{2}\right) + O((x_{j+1} - x_j)^3)$$



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Solving $-u'' = f$ with Homogeneous Dirichlet BC

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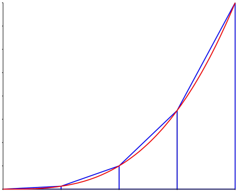
This integral is computed by quadrature.

Trapezoidal rule

$$\int_{x_j}^{x_{j+1}} f\,\phi = (x_{j+1}-x_j)\frac{f(x_j)\phi(x_j)+f(x_{j+1})\phi(x_{j+1})}{2}+O((x_{j+1}-x_j)^3)$$



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Solving $-u'' = f$ with Homogeneous Dirichlet BC

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This integral is computed by quadrature.

Simpson rule

$$\int_{x_j}^{x_{j+1}} f\,\phi = \frac{1}{6}(x_{j+1}-x_j)$$

$$\frac{f(x_j)\phi(x_j) + 4f\left(\frac{x_j+x_{j+1}}{2}\right)\phi\left(\frac{x_j+x_{j+1}}{2}\right) + f(x_{j+1})\phi(x_{j+1})}{6} + O((x_{j+1}-x_j)^5)$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Let

$$b_h = \left[\int_0^1 f\phi_i\right]_{i\in\{1,\ldots,J\}}$$

This integral is computed by quadrature.

See also the Gauß-Kronrod quadrature formula, which is very efficient.

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Definition V.3.5

The **interpolation operator** P_1*, is the linear mapping*

$$\begin{array}{rcl} r_h : H^1(0,1) & \to & H_h \\ v & \longmapsto & \displaystyle\sum_{j=0}^{J+1} v(x_j)\phi_j \end{array}$$

Theorem V.3.6

Let $u \in H_0^1(0,1)$ and $u_h \in H_{0,h}$ solutions to (VF) et (VF_h) respectively. Then, the FEM P_1 converge, i.e.

$$\lim_{h \to 0} \|u - u_h\|_{H^1} = 0.$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

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$$\lim_{h \to 0} \|u - u_h\|_{H^1} = 0.$$

Furthermore,

$$\exists C > 0, \; u \in H^2(0,1) \Rightarrow \|u - u_h\|_{H^1} \leq Ch\|f\|_{L^2}.$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

The proof relies on:

Lemma V.3.7

There exists a constant $C > 0$ s.t.

$$\forall v \in H^2(0,1), \qquad \begin{aligned} \|v - r_h v\|_{L^2} &\leq Ch^2 \|v''\|_{L^2}, \\ \|(v - r_h v)'\|_{L^2} &\leq Ch \|v''\|_{L^2}. \end{aligned}$$

Lemma V.3.8

There exists a constant $C > 0$ s.t.

$$\forall v \in H^1(0,1), \qquad \begin{aligned} \|r_h v\|_{H^1} &\leq C \|v\|_{H^1}, \\ \|v - r_h v\|_{L^2} &\leq Ch \|v'\|_{L^2}, \\ \|(v - r_h v)'\|_{L^2} &\xrightarrow[h\to 0]{} 0. \end{aligned}$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Let us try the method with $f(x) = 1$.

Solving $-u'' = f$ with Homogeneous Dirichlet BC

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Solving

$$\begin{cases} -u'' = 1 \text{ in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

is easy since it can be done "by hand".

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The solution is:

$$u(x) = -\frac{1}{2}x^2 + \frac{1}{2}x$$

Solving $-u'' = f$ with Homogeneous Dirichlet BC

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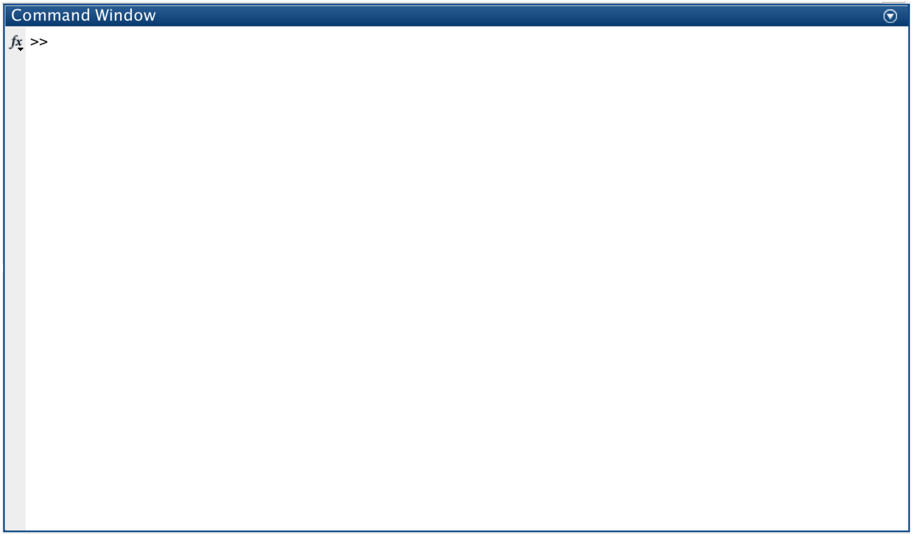
Let's try the FEM P_1. Take $J = 10$.

Solving $-u'' = f$ with Homogeneous Dirichlet BC



Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
>> J=10

J =

    10

>>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
>> J=10

J =

    10

>> h=1/(J+1)

h =

    0.0909

>> 
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
>> Ah =1/h * [2 -1 0 0 0 0 0 0 0 0
   -1 2 -1 0 0 0 0 0 0 0
   0 -1 2 -1 0 0 0 0 0 0
   0 0 -1 2 -1 0 0 0 0 0
   0 0 0 -1 2 -1 0 0 0 0
   0 0 0 0 -1 2 -1 0 0 0
   0 0 0 0 0 -1 2 -1 0 0
   0 0 0 0 0 0 -1 2 -1 0
   0 0 0 0 0 0 0 -1 2 -1
   0 0 0 0 0 0 0 0 -1 2]

Ah =

    22   -11     0     0     0     0     0     0     0     0
   -11    22   -11     0     0     0     0     0     0     0
     0   -11    22   -11     0     0     0     0     0     0
     0     0   -11    22   -11     0     0     0     0     0
     0     0     0   -11    22   -11     0     0     0     0
     0     0     0     0   -11    22   -11     0     0     0
     0     0     0     0     0   -11    22   -11     0     0
     0     0     0     0     0     0   -11    22   -11     0
     0     0     0     0     0     0     0   -11    22   -11
     0     0     0     0     0     0     0     0   -11    22

fx >>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
   -11    22   -11     0     0     0     0     0     0     0
     0   -11    22   -11     0     0     0     0     0     0
     0     0   -11    22   -11     0     0     0     0     0
     0     0     0   -11    22   -11     0     0     0     0
     0     0     0     0   -11    22   -11     0     0     0
     0     0     0     0     0   -11    22   -11     0     0
     0     0     0     0     0     0   -11    22   -11     0
     0     0     0     0     0     0     0   -11    22   -11
     0     0     0     0     0     0     0     0   -11    22

>> bh = h * [ 1 ; 1 ; 1 ; 1 ; 1 ; 1 ; 1 ; 1 ; 1 ; 1 ]

bh =

    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909

>>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909
    0.0909

>> uh = Ah\bh

uh =

    0.0413
    0.0744
    0.0992
    0.1157
    0.1240
    0.1240
    0.1157
    0.0992
    0.0744
    0.0413

>>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
    0.1157
    0.1240
    0.1240
    0.1157
    0.0992
    0.0744
    0.0413

>> uh = [0 ; uh ; 0]

uh =

         0
    0.0413
    0.0744
    0.0992
    0.1157
    0.1240
    0.1240
    0.1157
    0.0992
    0.0744
    0.0413
         0

fx >>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
         0
    0.0413
    0.0744
    0.0992
    0.1157
    0.1240
    0.1240
    0.1157
    0.0992
    0.0744
    0.0413
         0

>> x = h*[0:J+1]

x =

  Columns 1 through 9

         0    0.0909    0.1818    0.2727    0.3636    0.4545    0.5455    0.6364    0.7273

  Columns 10 through 12

    0.8182    0.9091    1.0000

>>
```

Solving $-u'' = f$ with Homogeneous Dirichlet BC

Command Window

```
    0.0413
    0.0744
    0.0992
    0.1157
    0.1240
    0.1240
    0.1157
    0.0992
    0.0744
    0.0413
         0

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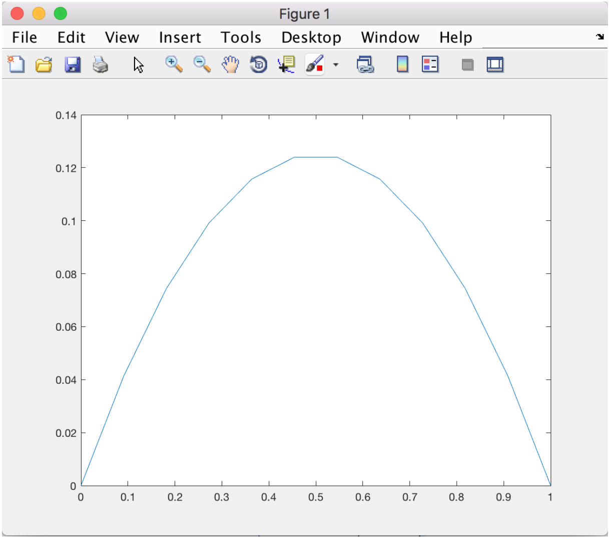
         0    0.0909    0.1818    0.2727    0.3636    0.4545    0.5455    0.6364    0.7273

  Columns 10 through 12

    0.8182    0.9091    1.0000

>> plot(x,uh)
>>
```

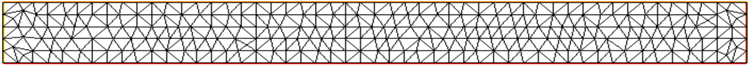
Solving $-u'' = f$ with Homogeneous Dirichlet BC



V.3.3. Finite Elements P_1 in Dimension 2

Definition

Mesh: Triangulation $\mathcal{T}$.



Definition V.3.9 (Finite Elements P_1)

We define

$$H_h = \{v \in C^0(\Omega) \ : \\ v|_{K_j} \in P_1, \ j \in \{0, \dots, J\}\}$$

and its subspace:

$$H_{0,h} = \{v \in H_h, \ v|_{\partial\Omega} = 0\}$$

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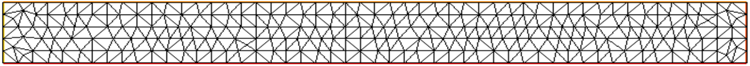

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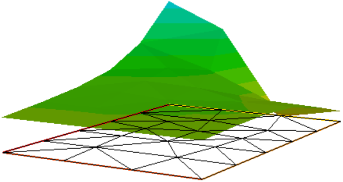
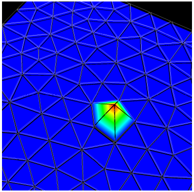


Basis

Lemma V.3.10

H_h is a linear subspace of $H^1(\Omega)$ of dimension J_v
$H_{0,h}$ is a linear subspace of $H_0^1(\Omega)$ of dimension J_v^D.



credit: Michel Kern

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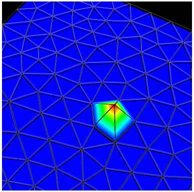

credit: Michel Kern

Define $(\phi_j)_{j\in\{0,\ldots,J+1\}}$ s.t.
$\forall j \in \{1,\ldots,J_v\},\ \phi_j \in H_h$
$\forall i,j \in \{1,\ldots,J_v\},\ \phi_j(x_i) = \delta_{ij}$.

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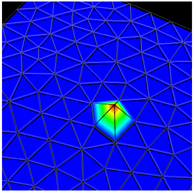

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Lemma V.3.11

$(\phi_j)_{j\in\{1,\ldots,J_v\}}$ is a basis of H_h and $(\phi_j)_{j\in\{1,\ldots,J_v^D\}}$ is a basis of $H_{0,h}$.

Solving $-\Delta u = f$ with Homogeneous Dirichlet BC

Consider

$$\begin{cases} -\Delta u = f \ \text{ on } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

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We defined

$$a : (u, v) \mapsto \int_\Omega \nabla u \, \nabla v$$

that was proven to be a coercive (with $\alpha = 1$) and continuous (with $M = C_\Omega$ the Poincaré constant) bilinear form on $H_0^1(\Omega)$

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Let

$$A_h = [a(\phi_i, \phi_j)]_{i\in\{1,\ldots,J\},j\in\{1,\ldots,J\}}$$

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Let

$$A_h = [a(\phi_i, \phi_j)]_{i\in\{1,\ldots,J\},j\in\{1,\ldots,J\}} = \left[\int_\Omega \nabla\phi_i \,\nabla\phi_j\right]_{i\in\{1,\ldots,J_v^D\},j\in\{1,\ldots,J_v^D\}}$$

$$b_h = [\ell(\phi_i)]_{i\in\{1,\ldots,J_v^D\}}$$

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We end up with A_h and b_h and we solve, for u_h, the linear system

$$A_h u_h = b_h$$

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$$A_h u_h = b_h$$

As before, $\lim_{h\to 0} u_h$ is the solution we are looking for.

A useful tool

Let $K = (A_1 A_2 A_3)$ be a non-degenerate triangle in $\mathbb{R}^2$.

Proposition V.3.12

*The set $\{A_1, A_2, A_3\}$ is P_1-**unisolvent**, i.e. given $(f_1, f_2, f_3) \in \mathbb{R}^3$, there exists a unique $p \in P_1$ such that $p(A_i) = f_i$ for $i \in \{1, 2, 3\}$.*

Definition V.3.13

$$\forall M = (x, y) \in \mathbb{R}^2, \quad \exists!(\lambda_1^K, \lambda_2^K, \lambda_3^K) \in \mathbb{R}^3, \quad \begin{cases} \sum_{i=1}^3 \lambda_i^K = 1 \\ M = \sum_{i=1}^3 \lambda_i^K A_i. \end{cases}$$

*The triplet $(\lambda_1^K, \lambda_2^K, \lambda_3^K)$ is called the **barycentric coordinates** of M. They are linear in (x, y)*

It is a natural basis of P_1 on the triangle K. The barycentric coordinates system depends on the triangle.

V.4. FEM Software and Libraries

A standard situation in the industry

CAD $\rightarrow$ Mesh $\rightarrow$ FEM Computation $\rightarrow$ Visualization

Examples of free software and libraries

- FreeFem++ (Paris VI)

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These are examples. Many others exist.

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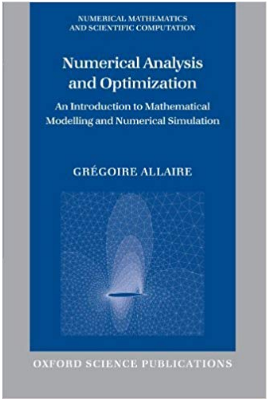
References

To go further on wellposedness of elliptic equation.



Grégoire Allaire

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Chapter 6

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