

## 计算方法 第6章习题答案

6.1 设函数  $f(x)$  在区间  $[a, b]$  上二阶导数连续, 试导出以下求积公式:

$$(1) \int_a^b f(x)dx = (b-a)f(a) + \frac{(b-a)^2}{2}f'(\eta), \quad a < \eta < b;$$

$$(2) \int_a^b f(x)dx = (b-a)f(a) - \frac{(b-a)^2}{2}f'(\eta), \quad a < \eta < b;$$

$$(3) \int_a^b f(x)dx = (b-a)f\left(\frac{a+b}{2}\right) + \frac{(b-a)^3}{24}f''(\eta), \quad a < \eta < b.$$

解: (1) 左矩形公式:  $f(x) \approx f(a)$ ,  $\int_a^b f(x)dx \approx \int_a^b f(a)dx = f(a)(b-a)$ , 因此

$$\begin{aligned} \int_a^b f(x)dx - f(a)(b-a) &= \int_a^b f(x)dx - \int_a^b f(a)dx = \int_a^b (f(x) - f(a))dx \quad \text{中值定理} \\ &= \int_a^b f'(\xi)(x-a)dx = f'(\eta) \int_a^b (x-a)dx = \frac{1}{2}(b-a)^2 f'(\eta), \quad \xi, \eta \in (a, b) \end{aligned}$$

(2) 右矩形公式:  $f(x) \approx f(b)$ ,  $\int_a^b f(x)dx \approx \int_a^b f(b)dx = f(b)(b-a)$ , 因此

$$\begin{aligned} \int_a^b f(x)dx - f(b)(b-a) &= \int_a^b f(x)dx - \int_a^b f(b)dx = \int_a^b (f(x) - f(b))dx \\ &= \int_a^b f'(\xi)(x-b)dx = f'(\eta) \int_a^b (x-b)dx = -\frac{1}{2}(b-a)^2 f'(\eta), \quad \xi, \eta \in (a, b) \end{aligned}$$

(3) 中矩形公式:  $f(x) \approx f\left(\frac{a+b}{2}\right)$ , 则

$$\int_a^b f(x)dx \approx \int_a^b f\left(\frac{a+b}{2}\right)dx = f\left(\frac{a+b}{2}\right)(b-a)$$

因此

$$\begin{aligned} \int_a^b f(x)dx - f\left(\frac{a+b}{2}\right)(b-a) &= \int_a^b f(x)dx - \int_a^b f\left(\frac{a+b}{2}\right)dx = \int_a^b \left(f(x) - f\left(\frac{a+b}{2}\right)\right)dx \\ &= \int_a^b \left(f'\left(\frac{a+b}{2}\right)\left(x - \frac{a+b}{2}\right) + \frac{1}{2}f''(\xi)\left(x - \frac{a+b}{2}\right)^2\right)dx \quad \text{Taylor公式} \\ &= f'\left(\frac{a+b}{2}\right) \int_a^b \left(x - \frac{a+b}{2}\right)dx + \frac{1}{2}f''(\eta) \int_a^b \left(x - \frac{a+b}{2}\right)^2dx \\ &= \frac{1}{24}f''(\eta)(b-a)^3, \quad \xi, \eta \in (a, b) \end{aligned}$$

6.7 导出下列求积公式及其截断误差估计式

$$(1) \int_0^h f(x)dx \approx A_0 f(0) + B_0 f'(0) + A_1 f(h) + B_1 f'(h);$$

$$(2) \int_0^{2h} f(x)dx \approx A_0 f(0) + A_1 f(h) + A_2 f(2h);$$

$$(3) \int_{-1}^1 x^2 f(x)dx \approx A_0 f(x_0) + A_1 f(x_1);$$

$$(4) \int_0^1 \sqrt{x} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1);$$

$$(5) \int_0^{+\infty} e^{-x} f(x) dx \approx A_0 f(x_0) + A_1 f(x_1).$$

解: (1) 已知该式有四个待定系数, 故令  $f(x) = 1, x, x^2, x^3$ , 代入求积公式使其精确成立, 得到

$$A_0 + A_1 = h, \quad B_0 + A_1 h + B_1 = \frac{1}{2} h^2, \quad A_1 h^2 + 2B_1 h = \frac{1}{3} h^3, \quad A_1 h^3 + 3B_1 h^2 = \frac{1}{4} h^4,$$

解得

$$A_0 = \frac{h}{2}, \quad A_1 = \frac{h}{2}, \quad B_0 = \frac{h^2}{12}, \quad B_1 = -\frac{h^2}{12}.$$

因此

$$\int_0^h f(x) dx \approx Q[f] = \frac{h}{2} f(0) + \frac{h^2}{12} f'(0) + \frac{h}{2} f(h) - \frac{h^2}{12} f'(h)$$

当  $f(x) = x^4$  时,

$$I[f] = \int_0^h x^4 dx = \frac{1}{5} h^5, \quad Q[f] = \frac{1}{6} h^5 \neq I[f],$$

因此该公式具有3次代数精度. 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi) x^2 (x-h)^2,$$

此时有  $e(0) = e'(0) = e(h) = e'(h) = 0$ ,

$$\begin{aligned} R[f] &= R[e(x)] = \int_0^h e(x) dx - \frac{h}{12} [6e(0) + he'(0) + 6e(h) - he'(h)] \\ &= \int_0^h e(x) dx = \int_0^h \frac{1}{4!} f^{(4)}(\xi) x^2 (x-h)^2 dx \\ &= \frac{1}{4!} f^{(4)}(\eta) \int_0^h x^2 (x-h)^2 dx = \frac{h^5}{720} f^{(4)}(\eta) \end{aligned}$$

(2) 令  $f(x) = 1, x, x^2$ , 代入求积公式使其精确成立, 得到

$$A_0 + A_1 + A_2 = 2h, \quad A_1 h + 2A_2 h = 2h^2, \quad A_1 h^2 + 4A_2 h^2 = \frac{8}{3} h^3,$$

解得

$$A_0 = \frac{h}{3}, \quad A_1 = \frac{4h}{3}, \quad A_2 = \frac{h}{3}.$$

因此

$$\int_0^{2h} f(x) dx \approx Q[f] = \frac{h}{3} f(0) + \frac{4h}{3} f(h) + \frac{h}{3} f(2h)$$

当  $f(x) = x^3$  时,

$$I[f] = \int_0^{2h} x^3 dx = 4h^4, \quad Q[f] = \frac{h}{3} [4h^3 + 8h^3] = 4h^4 = I[f].$$

当  $f(x) = x^4$  时,

$$I[f] = \int_0^{2h} x^4 dx = \frac{32}{5}h^5, \quad Q[f] = \frac{h}{3}[4h^4 + 16h^4] = \frac{20}{3}h^5 \neq I[f].$$

因此该求积公式的代数精度为  $m = 3$ . 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{4!}f^{(4)}(\xi)x(x-h)^2(x-2h),$$

此时  $e(0) = e(h) = e(2h) = 0$ , 且

$$\begin{aligned} R[f] &= R[e(x)] = \int_0^{2h} e(x)dx - \frac{h}{3}[e(0) + 4e(h) + e(2h)] \\ &= \int_0^{2h} e(x)dx = \int_0^{2h} \frac{1}{4!}f^{(4)}(\xi)x(x-h)^2(x-2h)dx \\ &= \frac{1}{4!}f^{(4)}(\eta) \int_0^{2h} x(x-h)^2(x-2h)dx = \frac{h^5}{90}f^{(4)}(\eta). \end{aligned}$$

(3) 令  $f(x) = 1, x, x^2, x^3$ , 代入求积公式使其精确成立, 有

$$A_0 - A_1 = \frac{2}{3}, \quad -A_0x_0 + A_1x_1 = 0, \quad A_0x_0^2 + A_1x_1^2 = \frac{2}{5}, \quad A_0x_0^3 + A_1x_1^3 = 0,$$

解得

$$x_{0,1} = \pm \frac{\sqrt{15}}{5}, \quad A_0 = A_1 = \frac{1}{3}.$$

因此求积公式为

$$\int_{-1}^1 x^2 f(x)dx = \frac{1}{3}f\left(\frac{\sqrt{15}}{5}\right) + \frac{1}{3}f\left(-\frac{\sqrt{15}}{5}\right).$$

又  $R[x^4] = \frac{4}{25} \neq 0$ , 因此代数精度为3. 根据广义佩亚诺定理取

$$e(x) = \frac{1}{4!}f^{(4)}(\xi)\left(x + \frac{\sqrt{15}}{5}\right)^2\left(x - \frac{\sqrt{15}}{5}\right)^2,$$

此时  $e(-\frac{\sqrt{15}}{5}) = e(\frac{\sqrt{15}}{5}) = 0$ , 且

$$R[f] = R[e] = \int_{-1}^1 x^2 e(x)dx = \int_{-1}^1 \frac{1}{4!}f^{(4)}(\xi)x^2\left(x + \frac{\sqrt{15}}{5}\right)^2\left(x - \frac{\sqrt{15}}{5}\right)^2dx = \frac{1}{525}f^{(4)}(\eta)$$

(4) 截断误差  $R[f] = \int_0^1 \sqrt{x}f(x)dx - A_0f(x_0) - A_1f(x_1)$ . 令

$$\begin{cases} R[1] = \frac{2}{3} - A_0 - A_1 = 0 \\ R[x] = \frac{2}{5} - A_0x_0 - A_1x_1 = 0 \\ R[x^2] = \frac{2}{7} - A_0x_0^2 - A_1x_1^2 = 0 \\ R[x^3] = \frac{2}{9} - A_0x_0^3 - A_1x_1^3 = 0 \end{cases}$$

解得

$$x_0 = \frac{5}{9} - \frac{\sqrt{280}}{63}, \quad x_1 = \frac{5}{9} + \frac{\sqrt{280}}{63}, \quad A_0 = \frac{1}{3} - \frac{\sqrt{280}}{300}, \quad A_1 = \frac{1}{3} + \frac{\sqrt{280}}{300}.$$

因此求积公式为

$$\int_0^1 \sqrt{x} f(x) dx \approx \left( \frac{1}{3} - \frac{\sqrt{280}}{300} \right) f \left( \frac{5}{9} - \frac{\sqrt{280}}{63} \right) + \left( \frac{1}{3} + \frac{\sqrt{280}}{300} \right) f \left( \frac{5}{9} + \frac{\sqrt{280}}{63} \right)$$

且

$$R[x^4] = \frac{2}{11} - \left( \frac{1}{3} - \frac{\sqrt{280}}{300} \right) \left( \frac{5}{9} - \frac{\sqrt{280}}{63} \right)^4 - \left( \frac{1}{3} + \frac{\sqrt{280}}{300} \right) \left( \frac{5}{9} + \frac{\sqrt{280}}{63} \right)^4 = \frac{128}{43659} \neq 0$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi) \left( x - \frac{5}{9} + \frac{\sqrt{280}}{63} \right)^2 \left( x - \frac{5}{9} - \frac{\sqrt{280}}{63} \right)^2,$$

此时有  $e\left(\frac{5}{9} - \frac{\sqrt{280}}{63}\right) = e\left(\frac{5}{9} + \frac{\sqrt{280}}{63}\right) = 0$ , 则

$$\begin{aligned} R[f] &= R[e(x)] = \int_0^1 \sqrt{x} e(x) dx - \left( \frac{1}{3} - \frac{\sqrt{280}}{300} \right) e \left( \frac{5}{9} - \frac{\sqrt{280}}{63} \right) - \left( \frac{1}{3} + \frac{\sqrt{280}}{300} \right) e \left( \frac{5}{9} + \frac{\sqrt{280}}{63} \right) \\ &= \int_0^1 \sqrt{x} e(x) dx = \int_0^1 \frac{1}{4!} f^{(4)}(\xi) \sqrt{x} \left( x - \frac{5}{9} + \frac{\sqrt{280}}{63} \right)^2 \left( x - \frac{5}{9} - \frac{\sqrt{280}}{63} \right)^2 dx \\ &= \frac{1}{24} f^{(4)}(\eta) \int_0^1 \sqrt{x} \left( x - \frac{5}{9} + \frac{\sqrt{280}}{63} \right)^2 \left( x - \frac{5}{9} - \frac{\sqrt{280}}{63} \right)^2 dx = \frac{16}{130977} f^{(4)}(\eta). \end{aligned}$$

(5) 截断误差  $R[f] = \int_0^{+\infty} e^{-x} f(x) dx - A_0 f(x_0) - A_1 f(x_1)$ . 令

$$\begin{cases} R[1] = 1 - A_0 - A_1 = 0 \\ R[x] = 1 - A_0 x_0 - A_1 x_1 = 0 \\ R[x^2] = 2 - A_0 x_0^2 - A_1 x_1^2 = 0 \\ R[x^3] = 6 - A_0 x_0^3 - A_1 x_1^3 = 0 \end{cases}$$

解得

$$x_0 = 2 - \sqrt{2}, \quad x_1 = 2 + \sqrt{2}, \quad A_0 = \frac{2 + \sqrt{2}}{4}, \quad A_1 = \frac{2 - \sqrt{2}}{4}.$$

因此求积公式为

$$\int_0^{+\infty} e^{-x} f(x) dx \approx \frac{2 + \sqrt{2}}{4} f(2 - \sqrt{2}) + \frac{2 - \sqrt{2}}{4} f(2 + \sqrt{2}).$$

由于

$$R[x^4] = 24 - \frac{2 + \sqrt{2}}{4} (2 - \sqrt{2})^4 - \frac{2 - \sqrt{2}}{4} (2 + \sqrt{2})^4 = 4 \neq 0,$$

故其代数精度为3. 根据广义佩亚诺定理可取

$$e(x) = \frac{1}{4!} f^{(4)}(\xi) (x - 2 + \sqrt{2})^2 (x - 2 - \sqrt{2})^2,$$

此时有  $e(2 - \sqrt{2}) = e(2 + \sqrt{2}) = 0$ , 则

$$\begin{aligned} R[f] &= R[e(x)] = \int_0^{+\infty} e^{-x} e(x) dx \\ &= \int_0^{+\infty} \frac{1}{4!} f^{(4)}(\xi) e^{-x} (x - 2 + \sqrt{2})^2 (x - 2 - \sqrt{2})^2 dx \\ &= \frac{1}{24} f^{(4)}(\eta) \int_0^{+\infty} e^{-x} (x - 2 + \sqrt{2})^2 (x - 2 - \sqrt{2})^2 dx = \frac{1}{6} f^{(4)}(\eta). \end{aligned}$$

**6.10 确定下列数值微分公式的系数, 导出截断误差简单表达式**

$$(1) f'(0) \approx af(-h) + bf(0) + cf(h)$$

$$(2) f'(h) \approx af'(0) + b[f(2h) - f(h)]$$

**解:** (1) 截断误差为

$$R[f] = f'(0) - af(-h) - bf(0) - cf(h).$$

分别取  $f = 1, x, x^2$ , 令  $R[f] = 0$  得到方程组

$$\begin{cases} a + b + c = 0 \\ (c - a)h = 1 \\ (a + c)h^2 = 0 \end{cases}$$

解得

$$a = -\frac{1}{2h}, \quad b = 0, \quad c = \frac{1}{2h}.$$

故数值微分公式为

$$f'(0) \approx -\frac{1}{2h}f(-h) + \frac{1}{2h}f(h).$$

由于  $R[x^3] = h^2 \neq 0$ , 所以代数精度  $m = 2$ . 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{6} f'''(\xi) x(x+h)(x-h),$$

$$R[f] = R[e] = e'(0) = -\frac{h^2}{6} f'''(\xi).$$

(2) 截断误差为

$$R[f] = f'(h) - af'(0) - b[f(2h) - f(h)].$$

分别取  $f = 1, x, x^2$ , 令  $R[f] = 0$  得到方程组

$$\begin{cases} 0 = 0 \\ a + bh = 1 \\ 3bh^2 = 2h \end{cases}$$

解得  $a = \frac{1}{3}$ ,  $b = \frac{2}{3h}$ . 因此

$$f'(h) \approx \frac{1}{3}f'(0) + \frac{2}{3h}[f(2h) - f(h)].$$

由于  $R[x^3] = -\frac{5}{3}h^2 \neq 0$ , 故代数精度  $m = 2$ . 根据广义佩亚诺定理, 取

$$e(x) = \frac{1}{6}f'''(\xi)x^2(x - 2h),$$

注意到  $e(0) = e'(0) = e(2h) = 0$ , 于是

$$R[f] = R[e] = e'(h) - \frac{1}{3}e'(0) - \frac{2}{3h}[e(2h) - e(h)] = e'(h) + \frac{2}{3h}e(h) = -\frac{5}{18}h^2f'''(\xi) - \frac{1}{24}h^3f^{(4)}(\bar{\xi}).$$