Partial Differential Equations

Chapter III - Distributions

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The Engineering Program of CentraleSupélec

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III.1. Introduction

Notation

Today, $\mathcal{I}$ will be an open interval of $\mathbb{R}$:

$]a, b[$, $]-\infty, b[$ or $]a, +\infty[$

Our goal

We are about to define a superset to the set of functions.
(the set of functions will be a subset of the set of distributions)

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We will work in 1D (generalizing the concept of locally integrable function from $\mathbb{R}$ to $\mathbb{R}$).
But we could define distributions in higher dimensions.

Our goal – example

Note $H_0 = 1_{[0,+\infty[}$. Consider the function $f = 2H_0 - 1$:

$$\begin{array}{rcl} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} -1 & \text{if} \quad x < 0 \\ 1 & \text{if} \quad x \geq 0 \end{cases} \end{array}$$

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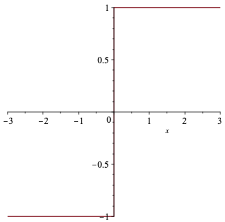

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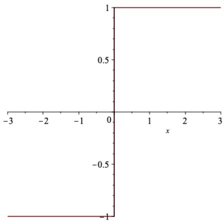

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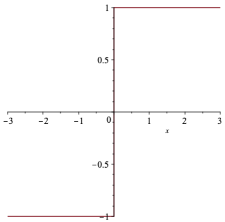
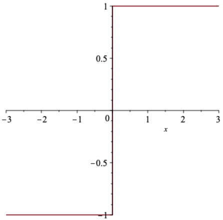


f can be differentiated on $]-\infty, 0[$ and on $]0, +\infty[$
and its derivative is 0. Obviously, f is not differentiable in 0.

Our goal – example

Note $H_0 = 1_{[0,+\infty[}$. Consider the functions

$$\begin{array}{rccc} f : & \mathbb{R} & \to & \mathbb{R} \\ & x & \mapsto & 2H_0(x) - 1 \end{array}$$



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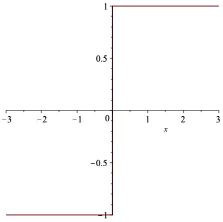
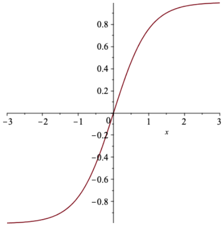
$$f : \mathbb{R} \to \mathbb{R}, \quad x \mapsto 2H_0(x) - 1 \qquad g : \mathbb{R} \to \mathbb{R}, \quad x \mapsto \tanh(x)$$





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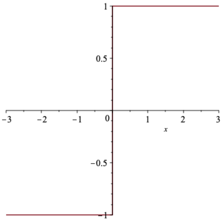

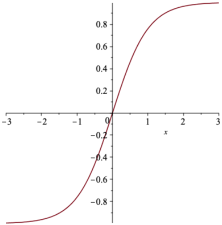

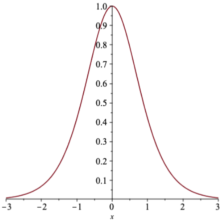

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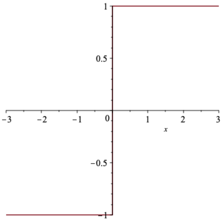
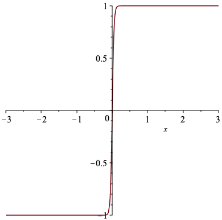
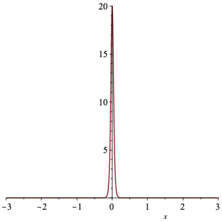
$$\begin{array}{rcrcl} f & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & 2H_0(x)-1 \end{array} \qquad \begin{array}{rcrcl} g_n & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & \tanh(nx) \end{array} \qquad \begin{array}{rcrcl} {g_n}' & : & \mathbb{R} & \to & \mathbb{R} \\ & & x & \mapsto & n(1-\tanh^2(nx)) \end{array}$$







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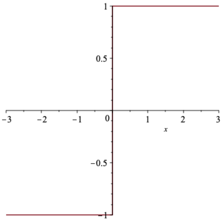

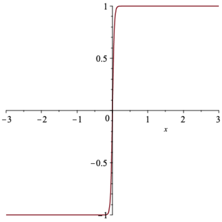

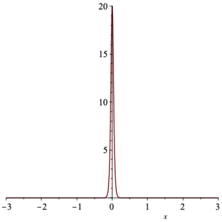

g_n can be differentiated on $]-\infty, \infty[$.
g_n' does not seem to converge in the space of functions.

The concept of functions

(From Middle School)

A function relates one or several variables to one value.

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Example

$$\begin{array}{rcl} f \ : \ \mathbb{R}^4 & \rightarrow & \mathbb{R} \\ (x, y, z, t) & \mapsto & f(x, y, z, t) \end{array}$$

Provides the temperature at (x, y, z) in space and a time t.

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Provides the temperature at (x, y, z) *in space.*

$$\Theta = \Bigg\{$$

$$\text{located at } (x, y, z) \mid (x, y, z) \in \mathbb{R}^3 \Bigg\}$$

Consider

$$\begin{array}{rcll} T & : & \Theta & \to & \mathbb{R} \\ & & \varphi & \mapsto & \text{temperature measured} \end{array}$$

For a thermometer φ in the set Θ, we look at its location (x, y, z) (the location of its tip) and we have $T(\varphi) = f(x, y, z)$.

$\Theta = \Bigg\{$  ,  ,  , ... located everywhere in $\mathbb{R}^3$ $\Bigg\}$

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$$\begin{aligned} T \ : \ \Theta &\ \rightarrow\ \mathbb{R} \\ \varphi &\ \mapsto\ \text{temperature measured} \end{aligned}$$

For a thermometer φ in the set Θ, we note $T(\varphi)$ the temperature shown on φ.

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III.2. Distributions

III.2.1. The Test Functions Space $\mathcal{D}(\mathcal{I})$

Definition III.2.1

The set real-valued continuous functions on $\mathcal{I}$ with **compact support** *is denoted $C_0^0(\mathcal{I})$ or $C_c^0(\mathcal{I})$.*

It is the set of functions whose support is included in a closed bounded interval included in $\mathcal{I}$.

Definition III.2.2

Define the set of **bump functions***:*

$$C_0^\infty(\mathcal{I}) = C_0^0(\mathcal{I}) \cap C^\infty(\mathcal{I})$$

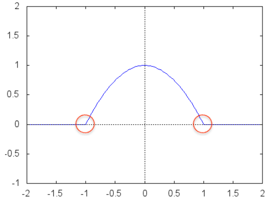
This set is also denoted $C_c^\infty(\mathcal{I})$ or $\mathcal{D}(\mathcal{I})$.
In this course, we will use $\mathcal{D}(\mathcal{I})$ from now on.

It is the set of all functions whose support is bounded and that can be differentiated for all degrees of differentiation.

Example

$$\begin{array}{rcl} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \left\{ \begin{array}{lll} 1 - x^2 & \textit{if} & |x| < 1 \\ 0 & \textit{if} & |x| \geq 1 \end{array} \right. \end{array}$$

$$f \notin \mathcal{D}(\mathcal{I})$$



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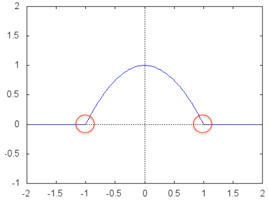

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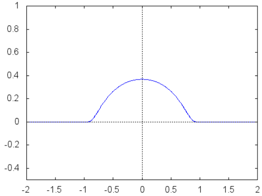
Example

$$\begin{array}{rcl} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} \exp\left(-\frac{1}{1-x^2}\right) & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \geq 1 \end{cases} \end{array}$$



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Definition III.2.3

Let $(\phi_n)_{n\in\mathbb{N}}$ be a sequence of functions in $\mathcal{D}(\mathcal{I})$ and $\phi \in \mathcal{D}(\mathcal{I})$. ϕ_n converges toward ϕ if

- $\exists K \subset \mathcal{I},\ \mathit{compact},\ \forall n \in \mathbb{N},\ \operatorname{supp}(\phi_n) \subset K$
- $\forall m \in \mathbb{N},\ \phi_n^{(m)} \longrightarrow \phi^{(m)}$ *uniformly in $\mathcal{I}$: i.e.:*
  $\forall m \in \mathbb{N}, \forall \varepsilon > 0, \exists N \in \mathbb{N} : \forall n \geq N,\ \|\phi_n^{(m)} - \phi^{(m)}\|_\infty \leq \varepsilon.$

We will note $\phi_n \xrightarrow{\mathcal{D}(\mathcal{I})} \phi$ or $\varphi = \lim^{\mathcal{D}(\mathcal{I})}_{n\to\infty} \varphi_n$.

III.2.2. Definition of a Distribution

Distribution space

Definition III.2.4

The space of **distributions** *is the topological dual of* $\mathcal{D}(\mathcal{I})$.
It is denoted $\mathcal{D}'(\mathcal{I})$.
Its elements are called **distributions** *or* **generalized functions**.

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Linearity: $T(\varphi + \lambda\psi) = T(\varphi) + \lambda T(\psi)$

Continuity: $\lim_{n\to\infty} T(\varphi_n) = T\left(\lim^{\mathcal{D}(\mathcal{I})}_{n\to\infty} \varphi_n\right)$

$\forall \varphi, \psi \in \mathcal{D}(\mathcal{I}),\ \forall \lambda \in \mathbb{R},\ \forall (\varphi_n)_{n\in\mathbb{N}} \in \mathcal{D}(\mathcal{I})^{\mathbb{N}}$

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Continuity: $\phi_n \xrightarrow{\mathcal{D}(\mathcal{I})} \phi \Rightarrow \langle T, \phi_n \rangle \xrightarrow{\mathbb{R}} \langle T, \phi \rangle$

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A first example

Let $\mathcal{I}$ contain 0 and define

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$\lim \delta_0(\varphi_n) = \delta_0(\varphi)$

Therefore δ_0 is a distribution.

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Linearity: For all φ, ψ in $\mathcal{D}(\mathcal{I})$ and for all real λ:

$$\langle \delta_0, \varphi + \lambda\psi \rangle = \varphi(0) + \lambda\psi(0) = \langle \delta_0, \varphi \rangle + \lambda \langle \delta_0, \psi \rangle$$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.
$\langle \delta_0, \varphi_n \rangle = \varphi_n(0)$
$\lim_{n\to+\infty} \varphi_n(0) = \varphi(0)$ (uniform convergence)
$\lim \langle \delta_0, \varphi_n \rangle = \langle \delta_0, \varphi \rangle$

Therefore δ_0 is a distribution.

The Dirac Distribution

More generally, we can define a distribution that returns the value of the test function in a point a.

Definition III.2.5

Define the **Dirac distribution** *in $a \in \mathcal{I}$ by*

$$\forall \phi \in \mathcal{D}(\mathcal{I}),\ \langle \delta_a, \phi \rangle = \phi(a).$$

It is denoted δ_a.

A second example

Let $K \subset \mathcal{I}$ be compact. Define T such that

$$T : \phi \in \mathcal{D}(\mathcal{I}) \longmapsto \int_K \phi(x)\mathrm{d}x$$

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Linearity: Obvious

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.

$$|\langle T, \varphi_n - \varphi\rangle| = \left|\int_K \varphi_n(x) - \varphi(x)\mathrm{d}x\right| \leq \mathrm{mes}(K)\|\varphi_n - \varphi\|_\infty$$

Thus $\lim_{n\to+\infty} \langle T, \varphi_n\rangle = \langle T, \varphi\rangle$

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Thus $\lim_{n\to+\infty} \langle T, \varphi_n\rangle = \langle T, \varphi\rangle$

Therefore T is a distribution.

A variation on the second example

Let $f \in L^1_{loc}(\mathcal{I})$. Define $\mathcal{T}_f$ by

$$\mathcal{T}_f : \phi \in \mathcal{D}(\mathcal{I}) \longmapsto \int_I f(x)\phi(x)\mathrm{d}x$$

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Linearity: Obvious

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.

$$|\langle \mathcal{T}_f, \varphi_n - \varphi \rangle| \leq \|\varphi_n - \varphi\|_\infty \int_I f(x)\mathrm{d}x$$

Thus $\lim_{n\to+\infty} \langle \mathcal{T}_f, \varphi_n \rangle = \langle \mathcal{T}_f, \varphi \rangle$

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$$|\langle \mathcal{T}_f, \varphi_n - \varphi\rangle| \leq \|\varphi_n - \varphi\|_\infty \int_I f(x)\mathrm{d}x$$

Thus $\lim_{n\to+\infty} \langle \mathcal{T}_f, \varphi_n\rangle = \langle \mathcal{T}_f, \varphi\rangle$

Therefore $\mathcal{T}_f$ is a distribution.

III.2.3. L^1_{loc} as a subset of $\mathcal{D}'$

From functions to Distributions

Definition III.2.6

The **regular distribution** *associated to* $f \in L^1_{loc}(\mathcal{I})$ *is defined by:*

$$\mathcal{T}_f : \phi \in \mathcal{D}(\mathcal{I}) \longmapsto \int_I f(x)\phi(x)\mathrm{d}x$$

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Not all distributions are regular distributions. (*e.g.* Dirac)

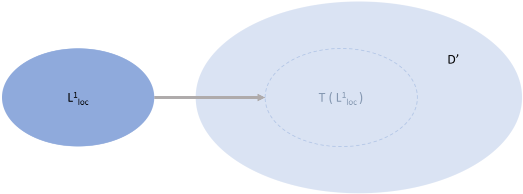
Theorem III.2.7

This mapping is injective

$$\begin{array}{rcl} \mathcal{T}_\cdot \; : \; L^1_{loc}(\mathcal{I}) & \to & \mathcal{D}'(\mathcal{I}) \\ f & \mapsto & \mathcal{T}_f \end{array}$$

All functions of $L^1_{loc}(\mathcal{I})$ can be represented by their associated regular distribution $\mathcal{T}_f$.

We can identify $L^1_{loc}(\mathcal{I})$ as a subset of $\mathcal{D}'(\mathcal{I})$.
(This subset being the set of regular distributions)



We can identify $L^1_{loc}(\mathcal{I})$ as a subset of $\mathcal{D}'(\mathcal{I})$.
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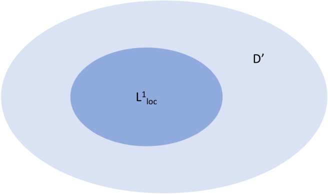
Thus:

$$L^1_{loc}(\mathcal{I}) \subset \mathcal{D}'(\mathcal{I})$$

We will note $\langle f, \varphi \rangle$ instead of $\langle \mathcal{T}_f, \varphi \rangle$.



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Thus:

$$L^1_{loc}(\mathcal{I}) \subset \mathcal{D}'(\mathcal{I})$$

We will note $\langle f, \varphi \rangle$ instead of $\langle \mathcal{T}_f, \varphi \rangle$.

Note:

$$\langle \underset{\substack{\pitchfork \\ L^1_{loc}(\mathcal{I})}}{f} , \underset{\substack{\pitchfork \\ \mathcal{D}(\mathcal{I})}}{\varphi} \rangle = \int_I f\varphi$$

We can identify $L^1_{loc}(\mathcal{I})$ as a subset of $\mathcal{D}'(\mathcal{I})$.
(This subset being the set of regular distributions)

Thus:

$$L^1_{loc}(\mathcal{I}) \subset \mathcal{D}'(\mathcal{I})$$

We will note $\langle f, \varphi \rangle$ instead of $\langle \mathcal{T}_f, \varphi \rangle$.

Note:

$$\begin{array}{ccccccc} \langle & f & , & \varphi & \rangle & = & \int_I f\varphi \\ & \pitchfork & & \pitchfork & & & \\ & L^1_{loc}(\mathcal{I}) & & \mathcal{D}(\mathcal{I}) & & & \\ & & & \cap & & & \\ & & & L^2(\mathcal{I}) & & & \end{array}$$

We can identify $L^1_{loc}(\mathcal{I})$ as a subset of $\mathcal{D}'(\mathcal{I})$.
(This subset being the set of regular distributions)

Thus:

$$L^1_{loc}(\mathcal{I}) \subset \mathcal{D}'(\mathcal{I})$$

We will note $\langle f, \varphi \rangle$ instead of $\langle \mathcal{T}_f, \varphi \rangle$.

Note: If $f \in L^2(\mathcal{I})$

$$\begin{array}{ccccccc} \langle & f & , & \varphi & \rangle & = & \int_I f\varphi \\ & \cap & & \cap & & & \\ & L^2(\mathcal{I}) & & \mathcal{D}(\mathcal{I}) & & & \\ & \cap & & \cap & & & \\ & L^1_{loc}(\mathcal{I}) & & L^2(\mathcal{I}) & & & \end{array}$$

The is the inner product in $L^2(\mathcal{I})$ and the notations are consistent.

III.2.4. Operations on Distributions

Sum of two distributions

Given two distributions T_1 and T_2. Define Z by

$$\langle Z, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$$

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Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle T_1, \varphi + \lambda\psi \rangle + \langle T_2, \varphi + \lambda\psi \rangle$

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Given two distributions T_1 and T_2. Define Z by

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Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle + \lambda(\langle T_1, \psi \rangle + \langle T_2, \psi \rangle)$

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Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle Z, \varphi \rangle + \lambda \langle Z, \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.
$\langle Z, \varphi_n \rangle = \langle T_1, \varphi_n \rangle + \langle T_2, \varphi_n \rangle$

Since: $\lim_{n\to+\infty} \langle T_1, \varphi_n \rangle = \langle T_1, \varphi \rangle$ and
$\lim_{n\to+\infty} \langle T_2, \varphi_n \rangle = \langle T_2, \varphi \rangle$

We have: $\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$
$\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle Z, \varphi \rangle$

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$\langle Z, \varphi_n \rangle = \langle T_1, \varphi_n \rangle + \langle T_2, \varphi_n \rangle$

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We have: $\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$
$\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle Z, \varphi \rangle$

Therefore Z is a distribution.

Sum of two distributions

Given two distributions T_1 and T_2. Define $T_1 + T_2$ by

$$\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$$

Linearity: $\langle T_1 + T_2, \varphi + \lambda\psi \rangle = \langle T_1 + T_2, \varphi \rangle + \lambda \langle T_1 + T_2, \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.

$\langle T_1 + T_2, \varphi_n \rangle = \langle T_1, \varphi_n \rangle + \langle T_2, \varphi_n \rangle$

Since: $\lim_{n\to+\infty} \langle T_1, \varphi_n \rangle = \langle T_1, \varphi \rangle$ and $\lim_{n\to+\infty} \langle T_2, \varphi_n \rangle = \langle T_2, \varphi \rangle$

We have:

$\lim_{n\to+\infty} \langle T_1 + T_2, \varphi_n \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$

$\lim_{n\to+\infty} \langle T_1 + T_2, \varphi_n \rangle = \langle T_1 + T_2, \varphi \rangle$

Therefore $T_1 + T_2$ is a distribution.

It extends the sum of L^1_{loc} since $\mathcal{T}_{f_1+f_2} = \mathcal{T}_{f_1} + \mathcal{T}_{f_2}$.

Product of two distributions

Given two distributions T_1 and T_2. Can we define $T_1 \times T_2$ by

$$\langle T_1 \times T_2, \varphi \rangle = \langle T_1, \varphi \rangle \ \langle T_2, \varphi \rangle$$

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Linearity: $\langle T_1 \times T_2, \varphi + \lambda\psi\rangle$
$= \langle T_1, \varphi + \lambda\psi\rangle \langle T_2, \varphi + \lambda\psi\rangle$

Product of two distributions

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Linearity: $\langle T_1 \times T_2, \varphi + \lambda\psi \rangle$

$$\begin{aligned} &= \langle T_1, \varphi + \lambda\psi \rangle \langle T_2, \varphi + \lambda\psi \rangle \\ &= \langle T_1, \varphi \rangle \langle T_2, \varphi \rangle \\ &\quad + \lambda \langle T_1, \varphi \rangle \langle T_2, \psi \rangle \, \lambda \langle T_1, \psi \rangle \langle T_2, \varphi \rangle \\ &\quad + \lambda^2 \langle T_2, \psi \rangle \langle T_2, \psi \rangle \end{aligned}$$

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$$\begin{aligned}
&= \langle T_1, \varphi + \lambda\psi \rangle \langle T_2, \varphi + \lambda\psi \rangle \\
&= \langle T_1, \varphi \rangle \langle T_2, \varphi \rangle \\
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Ooops! It does not work!

We can't define the product of two distributions this way!

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&= \langle T_1, \varphi\rangle \langle T_2, \varphi\rangle \\
&\quad + \lambda \langle T_1, \varphi\rangle \langle T_2, \psi\rangle \, \lambda \langle T_1, \psi\rangle \langle T_2, \varphi\rangle \\
&\quad + \lambda^2 \langle T_2, \psi\rangle \langle T_2, \psi\rangle
\end{aligned}$$

Ooops! It does not work!

We can't define the product of two distributions this way!
... Which is too bad because it was the extension of the product of L^1_{loc}.

Product of a distributions and a smooth function

Given a distributions T and a function $h \in C^\infty(I)$. Define Z by

$$\langle Z, \varphi \rangle = \langle T, h\varphi \rangle$$

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$$\langle Z, \varphi \rangle = \langle T, h\varphi \rangle$$

Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle T, h(\varphi + \lambda\psi) \rangle$

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Given a distributions T and a function $h \in C^\infty(I)$. Define Z by

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Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle Z, \varphi \rangle + \lambda \langle Z, \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.

$\langle Z, \varphi_n - \varphi \rangle = \langle T, h(\varphi_n - \varphi) \rangle$

We have $h\varphi \in \mathcal{D}(\mathcal{I})$

Thus $(h(\varphi^n - \varphi))^{(m)} = \sum_{i=0}^{m} \binom{i}{m} h^{(m-i)}(\varphi_n^{(i)} - \varphi^{(i)})$

Therefore $\lim_{n\to+\infty}^{\mathcal{D}(\mathcal{I})} h(\varphi^n - \varphi) = 0$

Therefore $\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle Z, \varphi \rangle$

Product of a distributions and a smooth function

Given a distributions T and a function $h \in C^\infty(I)$. Define Z by

$$\langle Z, \varphi \rangle = \langle T, h\varphi \rangle$$

Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle Z, \varphi \rangle + \lambda \langle Z, \psi \rangle$

Continuity: Let $(\varphi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.
$\langle Z, \varphi_n - \varphi \rangle = \langle T, h(\varphi_n - \varphi) \rangle$
We have $h\varphi \in \mathcal{D}(\mathcal{I})$
Thus $(h(\varphi^n - \varphi))^{(m)} = \sum_{i=0}^{m} \binom{i}{m} h^{(m-i)}(\varphi_n^{(i)} - \varphi^{(i)})$
Therefore $\lim^{\mathcal{D}(\mathcal{I})}_{n \to +\infty} h(\varphi^n - \varphi) = 0$
Therefore $\lim_{n \to +\infty} \langle Z, \varphi_n \rangle = \langle Z, \varphi \rangle$

Therefore Z is a distribution.

Product of a distributions and a smooth function

Given a distributions T and a function $h \in C^\infty(I)$. Define hT by

$$\langle hT, \varphi \rangle = \langle T, h\varphi \rangle$$

Linearity: $\langle hT, \varphi + \lambda\psi \rangle = \langle hT, \varphi \rangle + \lambda \langle hT, \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.
$\langle hT, \varphi_n - \varphi \rangle = \langle T, h(\varphi_n - \varphi) \rangle$
We have $h\varphi \in \mathcal{D}(\mathcal{I})$
Thus $(h(\varphi^n - \varphi))^{(m)} = \sum_{i=0}^m \binom{i}{m} h^{(m-i)}(\varphi_n^{(i)} - \varphi^{(i)})$
Therefore $\lim_{n\to+\infty}^{\mathcal{D}(\mathcal{I})} h(\varphi^n - \varphi) = 0$
Therefore $\lim_{n\to+\infty} \langle hT, \varphi_n \rangle = \langle hT, \varphi \rangle$

Therefore hT is a distribution.
It extends the product in C^∞ with L^1_{loc} since $\mathcal{T}_{hf} = h\mathcal{T}_f$.

Differentiation

Given a distributions T, define Z by

$$\langle Z, \varphi \rangle = -\left\langle T, \varphi' \right\rangle$$

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Differentiation

Given a distributions T, define Z by

$$\langle Z, \varphi \rangle = -\left\langle T, \varphi' \right\rangle$$

Linearity: $\langle Z, \varphi + \lambda\psi \rangle = -\langle T, \varphi' \rangle - \lambda \langle T, \psi' \rangle$

Differentiation

Given a distributions T, define Z by

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Linearity: $\langle Z, \varphi + \lambda\psi \rangle = \langle Z, \varphi \rangle + \lambda \langle Z, \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ. We have $\lim_{n\to+\infty} \langle Z, \varphi_n \rangle = \langle Z, \varphi \rangle$

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Therefore Z is a distribution.

Differentiation

Given a distributions T, define T' by

$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$$

Linearity: $\langle T', \varphi + \lambda\psi \rangle = \langle T', \varphi \rangle + \lambda \langle T', \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ.
We have $\lim_{n\to+\infty} \langle T', \varphi_n \rangle = \langle T', \varphi \rangle$

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Using the integration by parts, this is an extension of the differentiation of the differentiable functions of L^1_{loc}

$$\langle \mathcal{T}_{f'}, \varphi \rangle = \int_I f'\varphi$$

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Using the integration by parts, this is an extension of the differentiation of the differentiable functions of L^1_{loc}

$$\langle T_{f'}, \varphi \rangle = \int_I f'\varphi = 0 - \int_I f\varphi'$$

Differentiation

Given a distributions T, define T' by

$$\left\langle T', \varphi \right\rangle = -\left\langle T, \varphi' \right\rangle$$

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Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ. We have $\lim_{n\to+\infty} \langle T', \varphi_n \rangle = \langle T', \varphi \rangle$

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$$\langle T', \varphi \rangle = -\langle T, \varphi' \rangle$$

Linearity: $\langle T', \varphi + \lambda\psi \rangle = \langle T', \varphi \rangle + \lambda \langle T', \psi \rangle$

Continuity: Let $(\varphi_n)_{n\in\mathbb{N}}$ be a sequence in $\mathcal{D}(\mathcal{I})$ converging to φ. We have $\lim_{n\to+\infty} \langle T', \varphi_n \rangle = \langle T', \varphi \rangle$

Therefore T' is a distribution.

Using the integration by parts, this is an extension of the differentiation of the differentiable functions of L^1_{loc}

$$\langle \mathcal{T}_{f'}, \varphi \rangle = \langle \mathcal{T}'_f, \varphi \rangle$$

Differentiation (example)

Consider the Heaviside function $H_0 = \mathbf{1}_{\mathbb{R}^+} \in L^1_{loc}(\mathbb{R})$.

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Conclusion: $T'_{H_0} = \delta_0$.
Remark: $T'_{2H_0-1} = 2\delta_0$.

Differentiation

Proposition III.2.8

Let $a, b \in \mathbb{R} \cup \{\pm\infty\}$, $a < b$. If $f \in L^1_{loc}(a, b)$ is differentiable $]a, b[$ and $f\,' \in L^1_{loc}(a, b)$ then $\mathcal{T}_{f\,'} = (\mathcal{T}_f)\,'$.

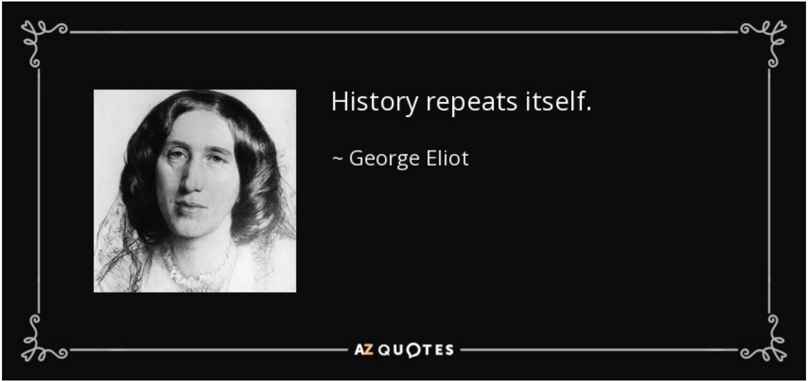
Theorem III.2.9 (Jumps formula)

Let $\mathcal{I} =]a_0, a_{k+1}[$ and $f \in C^{\,1}_{piecewise}(\mathcal{I})$.
Let $a_1 < \ldots < a_k$ be the points where f is not continuous. Then

$$(\mathcal{T}_f)\,' = \mathcal{T}_{f'} + \sum_{i=1}^{k} (f\,(a_i^+) - f\,(a_i^-))\delta_{a_i},$$

where f' is the derivative of the restriction of f to each sub-interval $]a_i, a_{i+1}[$, $0 \leq i \leq k$.

III.2.5. History repeats itself


History repeats itself.
~ George Eliot
AZ QUOTES

Remember when...?

Let us consider $Z = \mathbb{R}^2$.

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Define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
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$$J(x_1 + x_2) = J(x_1) + J(x_2)$$
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The operations $+$ and $\times$ in Z extend the operations $+$ and $\times$ if for the elements of Z that are (represented) in $\mathbb{R}$.

Remember when...?

Let us consider $\mathbb{C} = \mathbb{R}^2$.

For any $x \in \mathbb{R}$ define $J(x) = (x, 0)$. It is an injection. We say

$$\mathbb{R} \subset \mathbb{C}$$

Define

$$(x_1, y_1) + (x_2, y_2) = (x_1 + x_2, y_1 + y_2)$$
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The operations $+$ and $\times$ in $\mathbb{C}$ extend the operations $+$ and $\times$ if for the elements of $\mathbb{C}$ that are (represented) in $\mathbb{R}$.

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The operations $+$ and $\times$ in $\mathbb{C}$ extend the operations $+$ and $\times$ if for the elements of $\mathbb{C}$ that are (represented) in $\mathbb{R}$. From now on, we will use the **notation** $x + iy$ for (x, y).

What we did today

Let us consider $D'(\mathcal{I})$.

For any $f \in L^1_{loc}(\mathcal{I})$ define $\mathcal{T}.(f) = \mathcal{T}_f$. It is an injection. We say

$$L^1_{loc}(\mathcal{I}) \subset D'(\mathcal{I})$$

Define

$T_1 + T_2$ by $\langle T_1 + T_2, \varphi \rangle = \langle T_1, \varphi \rangle + \langle T_2, \varphi \rangle$
T' by $\langle T', \varphi \rangle = - \langle T, \varphi' \rangle$

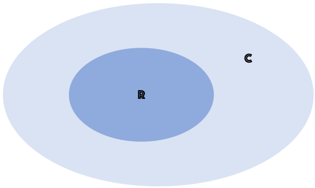
For a function f in $L^1_{loc}(\mathcal{I})$ we have

$\mathcal{T}.(f_1 + f_2) = \mathcal{T}.(f_1) + \mathcal{T}.(f_2)$.
$\mathcal{T}.(f') = \mathcal{T}.(f)'$.

The operations $+$ and the differentiation in $D'(\mathcal{I})$ extend the operations $+$ and the differentiation for the elements of $D'(\mathcal{I})$ that are (represented) in $L^1_{loc}(\mathcal{I})$.

$\mathbb{R} \subset \mathbb{C}$ and $L^1_{loc} \subset D'(\mathcal{I})$

$\mathbb{R} \subset \mathbb{C}$

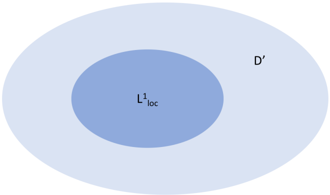


Every element now has square roots.

Extended: $+$ and $\times$.

Lost: the order relation $\leq$ compatible with $+$ and $\times$.

$L^1_{loc} \subset D'(\mathcal{I})$



Every element can now be differentiated.

Extended: $+$, the product with a C^∞ function, the differentiation.

Lost: $\times$ and possibly other operations.

Already lost: the value of T at a point.

III.3. Sobolev Spaces

III.3.1. Definitions and Basic Properties

Definition of $H^1(\mathcal{I})$

Definition III.3.1

The Sobolev space of order 1 on $\mathcal{I}$ is defined by

$$\begin{aligned} H^1(\mathcal{I}) &:= \left\{ v \in L^2(\mathcal{I}) \ : \quad (\mathcal{T}_v)' \in L^2(\mathcal{I}) \right\} \\ &:= \left\{ v \in L^2(\mathcal{I}) \ : \quad v' \in L^2(\mathcal{I}) \right\} \end{aligned}$$

where v' is the distributional derivative of v.

If $\mathcal{I} =]a, b[$ is a bounded interval, we note $H^1(\mathcal{I}) = H^1(a, b)$.

The $H^1(\mathcal{I})$ Hilbert space

Theorem III.3.2

The space $H^1(\mathcal{I})$ endowed with the inner product

$$(\cdot,\cdot)_{H^1} : \langle u, v\rangle \mapsto (u, v)_{L^2} + \langle u', v'\rangle_{L^2} .$$

is complete for the norm

$$\|\cdot\|_{H^1(\mathcal{I})} : v \mapsto \sqrt{\|v\|^2_{L^2(\mathcal{I})} + \|v'\|^2_{L^2(\mathcal{I})}}$$

It is a Hilbert Space.

Sobolev spaces of higher order

Theorem III.3.3

Let $k \in \mathbb{N}$. The space

$$H^k(\mathcal{I}) := \left\{ u \in L^2(\mathcal{I}) : \quad u^{(m)} \in L^2(\mathcal{I}),\ 0 \leq m \leq k \right\}.$$

endowed with the inner product

$$(u, v) \mapsto \sum_{0 \leq m \leq k} \left\langle u^{(m)}, v^{(m)} \right\rangle_{L^2(\mathcal{I})}$$

is a Hilbert Space.

We note $H^0(\mathcal{I}) = L^2(\mathcal{I})$.

To be continued on Tuesday, January 14th