

## 计算方法 第3章习题答案

### 3.1 用雅可比迭代法与高斯-赛德尔迭代法求解方程组

$$\begin{cases} -8x_1 + x_2 + x_3 = 1, \\ x_1 - 5x_2 + x_3 = 16, \\ x_1 + x_2 - 4x_3 = 7, \end{cases}$$

取初始向量  $x^{(0)} = (0, 0, 0)^T$ , 准确到两位数.

解 雅可比迭代法:

$$\begin{cases} x_1^{(k+1)} = (x_2^{(k)} + x_3^{(k)} - 1)/8, \\ x_2^{(k+1)} = (x_1^{(k)} + x_3^{(k)} - 16)/5, \\ x_3^{(k+1)} = (x_1^{(k)} + x_2^{(k)} - 7)/4. \end{cases}$$

高斯-赛德尔迭代法:

$$\begin{cases} x_1^{(k+1)} = (x_2^{(k)} + x_3^{(k)} - 1)/8, \\ x_2^{(k+1)} = (x_1^{(k+1)} + x_3^{(k)} - 16)/5, \\ x_3^{(k+1)} = (x_1^{(k+1)} + x_2^{(k+1)} - 7)/4. \end{cases}$$

注意方程组的精确解为  $x_1 = -1, x_2 = -4, x_3 = -3$ . 下面用迭代法计算, 取初始向量  $x^{(0)} = (0, 0, 0)^T$ , 分别代入以上两种迭代格式, 得计算结果如下表:

$k$	Jacobi			G-S		
	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$	$x_1^{(k)}$	$x_2^{(k)}$	$x_3^{(k)}$
0	0.00	0.00	0.00	0.00	0.00	0.00
1	-0.13	-3.20	-1.75	-0.13	-3.23	-2.59
2	-0.74	-3.58	-2.58	-0.85	-3.89	-2.94
3	-0.90	-3.86	-2.83	-0.98	-3.98	-2.99
4	-0.96	-3.95	-2.94	-1.00	-4.00	-3.00
5	-0.97	-3.98	-2.98			
6	-1.00	-3.99	-2.99			
7	-1.00	-4.00	-3.00			

### 3.2 设有方程组

$$\begin{cases} 10x_1 + 4x_2 + 4x_3 = 13, \\ 4x_1 + 10x_2 + 8x_3 = 11, \\ 4x_1 + 8x_2 + 10x_3 = 25, \end{cases}$$

写出雅可比迭代法、高斯-赛德尔迭代法与  $\omega = 1.2$  的超松弛迭代法的迭代公式, 并说明三种迭代法是否收敛, 为什么?

解 雅可比迭代法:

$$\begin{cases} x_1^{(k+1)} = (13 - 4x_2^{(k)} - 4x_3^{(k)})/10, \\ x_2^{(k+1)} = (11 - 4x_1^{(k)} - 8x_3^{(k)})/10, \\ x_3^{(k+1)} = (25 - 4x_1^{(k)} - 8x_2^{(k)})/10. \end{cases}$$

高斯-赛德尔迭代法:

$$\begin{cases} x_1^{(k+1)} = (13 - 4x_2^{(k)} - 4x_3^{(k)})/10, \\ x_2^{(k+1)} = (11 - 4x_1^{(k+1)} - 8x_3^{(k)})/10, \\ x_3^{(k+1)} = (25 - 4x_1^{(k+1)} - 8x_2^{(k+1)})/10. \end{cases}$$

超松弛迭代法 ( $\omega = 1.2$ ):

$$\begin{cases} x_1^{(k+1)} = (1 - \omega)x_1^{(k)} + \omega(13 - 4x_2^{(k)} - 4x_3^{(k)})/10, \\ x_2^{(k+1)} = (1 - \omega)x_2^{(k)} + \omega(11 - 4x_1^{(k+1)} - 8x_3^{(k)})/10, \\ x_3^{(k+1)} = (1 - \omega)x_3^{(k)} + \omega(25 - 4x_1^{(k+1)} - 8x_2^{(k+1)})/10. \end{cases}$$

观察到方程组的系数矩阵  $A$  对称, 且有

$$\Delta_1 = 10 > 0, \quad \Delta_2 = \begin{vmatrix} 10 & 4 \\ 4 & 10 \end{vmatrix} = 84 > 0, \quad \Delta_3 = |A| = 296 > 0,$$

即 **A 对称正定**. 根据迭代法收敛相关定理可知当  $0 < \omega < 2$  时, 超松弛迭代方法收敛. 因此高斯-赛德尔迭代法 ( $\omega = 1$ ) 收敛,  $\omega = 1.2$  的超松弛迭代法收敛.

又因为

$$2D - A = \begin{pmatrix} 10 & -4 & -4 \\ -4 & 10 & -8 \\ -4 & -8 & 10 \end{pmatrix}, \quad |2D - A| = -216,$$

$2D - A$  不正定, 故雅可比迭代法不收敛.

### 3.3 设有方程组

$$\begin{cases} 2x_1 - 3x_2 + 10x_3 = 3, \\ 5x_1 + 2x_2 + x_3 = -12, \\ -x_1 + 4x_2 + 2x_3 = 20, \end{cases}$$

写出三种收敛的迭代格式并说明收敛的理由.

解 将原方程组进行改写

$$\begin{cases} 5x_1 + 2x_2 + x_3 = -12, \\ -x_1 + 4x_2 + 2x_3 = 20, \\ 2x_1 - 3x_2 + 10x_3 = 3, \end{cases}$$

此时方程组的系数矩阵为

$$\begin{pmatrix} 5 & 2 & 1 \\ -1 & 4 & 2 \\ 2 & -3 & 10 \end{pmatrix}.$$

$A$  为严格对角占优矩阵, 因此雅可比迭代法、高斯-赛德尔迭代法以及  $0 < \omega \leq 1$  的超松弛迭代法收敛. 三种迭代格式分别为:

雅可比迭代法:

$$\begin{cases} x_1^{(k+1)} = (-12 - 2x_2^{(k)} - x_3^{(k)})/5, \\ x_2^{(k+1)} = (20 + x_1^{(k)} - 2x_3^{(k)})/4, \\ x_3^{(k+1)} = (3 - 2x_1^{(k)} + 3x_2^{(k)})/10. \end{cases}$$

高斯-赛德尔迭代法:

$$\begin{cases} x_1^{(k+1)} = (-12 - 2x_2^{(k)} - x_3^{(k)})/5, \\ x_2^{(k+1)} = (20 + x_1^{(k+1)} - 2x_3^{(k)})/4, \\ x_3^{(k+1)} = (3 - 2x_1^{(k+1)} + 3x_2^{(k+1)})/10. \end{cases}$$

超松弛迭代法:

$$\begin{cases} x_1^{(k+1)} = (1 - \omega)x_1^{(k)} + \omega(-12 - 2x_2^{(k)} - x_3^{(k)})/5, \\ x_2^{(k+1)} = (1 - \omega)x_2^{(k)} + \omega(20 + x_1^{(k+1)} - 2x_3^{(k)})/4, \\ x_3^{(k+1)} = (1 - \omega)x_3^{(k)} + \omega(3 - 2x_1^{(k+1)} + 3x_2^{(k+1)})/10. \end{cases}$$

### 3.4 欲用迭代法求解方程组

$$\begin{cases} x_1 - 4x_2 - 2x_4 = 11, \\ 4x_1 + 8x_3 + 3x_4 = 6, \\ 3x_2 + x_3 + 5x_4 = 2, \\ 5x_1 - x_2 + x_3 + 2x_4 = 5, \end{cases}$$

写出收敛的高斯-赛德尔迭代法与  $\omega = 0.8$  的超松弛迭代法的迭代公式. 给定初始向量  $x^{(0)} = (0, 0, 0, 0)^T$ , 用高斯-赛德尔迭代法计算  $x^{(1)}$ .

解 将原方程组改写为

$$\begin{cases} 5x_1 - x_2 + x_3 + 2x_4 = 5, \\ x_1 - 4x_2 - 2x_4 = 11, \\ 4x_1 + 8x_3 + 3x_4 = 6, \\ 3x_2 + x_3 + 5x_4 = 2, \end{cases}$$

此时方程组的系数矩阵为

$$A = \begin{pmatrix} 5 & -1 & 1 & 2 \\ 1 & -4 & 0 & -2 \\ 4 & 0 & 8 & 3 \\ 0 & 3 & 1 & 5 \end{pmatrix}.$$

$A$  为严格对角占优矩阵, 因此雅可比迭代法、高斯-赛德尔迭代法以及  $0 < \omega \leq 1$  的超松弛迭代法收敛.

高斯-赛德尔迭代法:

$$\begin{cases} x_1^{(k+1)} = (5 + x_2^{(k)} - x_3^{(k)} - 2x_4^{(k)})/5, \\ x_2^{(k+1)} = (-11 + x_1^{(k+1)} + 2x_4^{(k)})/4, \\ x_3^{(k+1)} = (6 - 4x_1^{(k+1)} - 3x_4^{(k)})/8, \\ x_4^{(k+1)} = (2 - 3x_2^{(k+1)} - x_3^{(k+1)})/5. \end{cases}$$

$\omega = 0.8$  的超松弛迭代法:

$$\begin{cases} x_1^{(k+1)} = 0.2x_1^{(k)} + 0.8 * (5 + x_2^{(k)} - x_3^{(k)} - 2x_4^{(k)})/5, \\ x_2^{(k+1)} = 0.2x_2^{(k)} + 0.8 * (-11 + x_1^{(k+1)} + 2x_4^{(k)})/4, \\ x_3^{(k+1)} = 0.2x_3^{(k)} + 0.8 * (6 - 4x_1^{(k+1)} - 3x_4^{(k)})/8, \\ x_4^{(k+1)} = 0.2x_4^{(k)} + 0.8 * (2 - 3x_2^{(k+1)} - x_3^{(k+1)})/5. \end{cases}$$

给定初始向量  $x^{(0)} = (0, 0, 0)^T$ , 用高斯-赛德尔迭代法计算得

$$x_1^{(1)} = 1, \quad x_2^{(1)} = -\frac{5}{2}, \quad x_3^{(1)} = \frac{1}{4}, \quad x_4^{(1)} = \frac{37}{20}.$$

因此

$$x^{(1)} = \left(1, -\frac{5}{2}, \frac{1}{4}, \frac{37}{20}\right)^T.$$

### 3.5 设有方程组

$$\begin{cases} 2x_1 - x_2 + x_3 = 3, \\ x_1 + x_2 + x_3 = 2, \\ x_1 + x_2 - 2x_3 = -1, \end{cases}$$

证明雅可比迭代法对任意的初始向量都不收敛, 而高斯-赛德尔迭代法对任意的初始向量都收敛. 迭代矩阵的谱半径

证明 根据  $A = D - E - F$ , 得

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -1 & -1 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

雅可比迭代法的迭代矩阵

$$B = D^{-1}(E + F) = \begin{pmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -\frac{1}{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ -1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ -1 & 0 & -1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{pmatrix},$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & -\frac{1}{2} & \frac{1}{2} \\ 1 & \lambda & 1 \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \lambda^3 + \frac{5}{4}\lambda = 0,$$

解得:

$$\lambda_1 = 0, \quad \lambda_2 = \frac{\sqrt{5}}{2}i, \quad \lambda_3 = -\frac{\sqrt{5}}{2}i.$$

因此  $\rho(B) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = \frac{\sqrt{5}}{2} > 1$ , 雅可比迭代法不收敛.

高斯-赛德尔迭代法的迭代矩阵

$$B = (D - E)^{-1}F = \begin{pmatrix} 2 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & -2 \end{pmatrix}^{-1} \begin{pmatrix} 0 & 1 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & 0 & -\frac{1}{2} \end{pmatrix},$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & -\frac{1}{2} & \frac{1}{2} \\ 0 & \lambda + \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \lambda + \frac{1}{2} \end{vmatrix} = \lambda(\lambda + \frac{1}{2})^2 = 0,$$

解得:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = -\frac{1}{2}.$$

因此  $\rho(B) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = \frac{1}{2} < 1$ , 高斯-赛德尔迭代法收敛.

### 3.6 设有方程组

$$\begin{cases} x_1 + 2x_2 - 2x_3 = -3, \\ x_1 + x_2 + x_3 = 1, \\ 2x_1 + 2x_2 + x_3 = 1, \end{cases}$$

证明雅可比迭代法对任意的初始向量都收敛, 而高斯-赛德尔迭代法对任意的初始向量都不收敛.

证明 根据  $A = D - E - F$ , 得

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad E = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ -2 & -2 & 0 \end{pmatrix}, \quad F = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

雅可比迭代法的迭代矩阵

$$B = D^{-1}(E + F) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ -1 & 0 & -1 \\ -2 & -2 & 0 \end{pmatrix},$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & 2 & -2 \\ 1 & \lambda & 1 \\ 2 & 2 & \lambda \end{vmatrix} = \lambda^3 = 0,$$

解得:

$$\lambda_1 = \lambda_2 = \lambda_3 = 0.$$

因此  $\rho(B) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = 0 < 1$ , 雅可比迭代法收敛.

高斯-赛德尔迭代法的迭代矩阵

$$B = (D - E)^{-1}F = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 2 & 2 & 1 \end{pmatrix}^{-1} \begin{pmatrix} 0 & -2 & 2 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 & 2 \\ 0 & 2 & -3 \\ 0 & 0 & 2 \end{pmatrix},$$

$$|\lambda I - B| = \begin{vmatrix} \lambda & 2 & -2 \\ 0 & \lambda - 2 & 3 \\ 0 & 0 & \lambda - 2 \end{vmatrix} = \lambda(\lambda + 2)^2 = 0,$$

解得:

$$\lambda_1 = 0, \quad \lambda_2 = \lambda_3 = -2.$$

因此  $\rho(B) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} = 2 > 1$ , 高斯-赛德尔迭代法不收敛.

### 3.7 用超松弛迭代法求解方程组

$$\begin{cases} 4x_1 + 3x_2 = 24, \\ 3x_1 + 4x_2 - x_3 = 30, \\ -x_2 + 4x_3 = -24, \end{cases}$$

选择使得收敛速度最快的松弛因子  $\omega$ , 给定初始向量  $x^{(0)} = (0, 0, 0)^T$ , 计算  $x^{(2)}$ .

解 超松弛迭代法:

$$\begin{cases} x_1^{(k+1)} = (1 - \omega)x_1^{(k)} + \omega(24 - 3x_2^{(k)})/4, \\ x_2^{(k+1)} = (1 - \omega)x_2^{(k)} + \omega(30 - 3x_1^{(k+1)} + x_3^{(k)})/4, \\ x_3^{(k+1)} = (1 - \omega)x_3^{(k)} + \omega(-24 + x_2^{(k+1)})/4. \end{cases}$$

迭代矩阵为

$$B = \begin{pmatrix} 4 & 0 & 0 \\ 3\omega & 4 & 0 \\ 0 & -\omega & 4 \end{pmatrix}^{-1} \begin{pmatrix} 4(1-\omega) & -3\omega & 0 \\ 0 & 4(1-\omega) & \omega \\ 0 & 0 & 4(1-\omega) \end{pmatrix}$$

$$= \begin{pmatrix} 1-\omega & -\frac{3}{4}\omega & 0 \\ -\frac{3}{4}\omega(1-\omega) & 1-\omega + \frac{9}{16}\omega^2 & \frac{\omega}{4} \\ -\frac{3}{16}\omega^2(1-\omega) & \frac{1}{4}\omega(1-\omega) + \frac{9}{64}\omega^3 & 1-\omega + \frac{\omega^2}{16} \end{pmatrix},$$

$$|\lambda I - B| = \begin{vmatrix} \lambda - 1 + \omega & \frac{3}{4}\omega & 0 \\ \frac{3}{4}\omega(1-\omega) & \lambda - 1 + \omega - \frac{9}{16}\omega^2 & -\frac{\omega}{4} \\ \frac{3}{16}\omega^2(1-\omega) & -\frac{1}{4}\omega(1-\omega) - \frac{9}{64}\omega^3 & \lambda - 1 + \omega - \frac{\omega^2}{16} \end{vmatrix} = 0,$$

解得:  $\lambda_1 = 1 - \omega$ ,

$$\lambda_2 = 1 - \omega + \frac{5\omega^2 - \sqrt{5}\sqrt{32\omega^2 - 32\omega^3 + 5\omega^4}}{16}, \quad \lambda_3 = 1 - \omega + \frac{5\omega^2 + \sqrt{5}\sqrt{32\omega^2 - 32\omega^3 + 5\omega^4}}{16}.$$

若要迭代法收敛, 则需有

$$\rho(B) = \max\{|\lambda_1|, |\lambda_2|, |\lambda_3|\} < 1 \implies 0 < \omega \leq \frac{1}{5}(16 - 4\sqrt{6}).$$

当  $0 < \omega \leq \frac{1}{5}(16 - 4\sqrt{6})$  时,

$$\rho(B) = 1 - \omega + \frac{5}{16}\omega^2 = \frac{5}{16}\left(\left(\omega - \frac{8}{5}\right)^2 + \frac{16}{25}\right),$$

$\rho(B)$  关于  $\omega$  在区间  $\omega \in (0, \frac{1}{5}(16 - 4\sqrt{6})]$  上单调递减. 当  $\omega = \frac{1}{5}(16 - 4\sqrt{6})$  时,  $\rho(B)$  最小, 迭代法收敛最快.

将初始向量  $x^{(0)} = (0, 0, 0)^T$  代入迭代格式计算得:

$$x_1^{(1)} = \frac{96 - 24\sqrt{6}}{5}, \quad x_2^{(1)} = \frac{6}{25}(-164 + 71\sqrt{6}), \quad x_3^{(1)} = \frac{12}{125}(-741 + 274\sqrt{6});$$

$$x_1^{(2)} = \frac{12}{125}(1143 - 452\sqrt{6}), \quad x_2^{(2)} = \frac{12}{125}(-3288 + 1357\sqrt{6}), \quad x_3^{(2)} = \frac{12}{625}(-7567 + 2988\sqrt{6}).$$

**3.8 (1)** 求解线性方程组  $Ax = b$  (其中  $A$  为对称正定矩阵) 的迭代公式为

$$x^{(k+1)} = x^{(k)} - \omega(Ax^{(k)} - b), \quad k = 0, 1, 2, \dots,$$

证明: 当  $0 < \omega < \frac{2}{\beta}$  时, 上述迭代法收敛 (其中  $0 < \alpha \leq \lambda(A) \leq \beta$ );

(2) 若用 (1) 中的迭代法求解方程组

$$\begin{pmatrix} 3 & -1 \\ -1 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} -1 \\ 3 \end{pmatrix},$$

试确定使得迭代法收敛的  $\omega$  取值范围, 并求出使迭代法收敛最快的最优  $\omega$  值  $\omega_{\text{opt}}$ .

解 (1) 将迭代公式改写为

$$x^{(k+1)} = (I - \omega A)x^{(k)} + \omega b, \quad k = 0, 1, 2, \dots,$$

则迭代矩阵  $B = I - \omega A$ , 其特征值为  $\lambda(B) = 1 - \omega\lambda(A)$ . 若要迭代法收敛, 则需有

$$\rho(B) = |\lambda(B)| = |1 - \omega\lambda(A)| < 1 \implies 0 < \omega < \frac{2}{\lambda(A)}.$$

又因为  $0 < \alpha \leq \lambda(A) \leq \beta$ , 因此有

A对称正定

$$0 < \omega < \frac{2}{\beta}.$$

(2) 由(1)知

$$B = I - \omega A = \begin{pmatrix} 1 - 3\omega & \omega \\ \omega & 1 - 3\omega \end{pmatrix}, \quad |\lambda I - B| = (\lambda - 1 + 3\omega)^2 - \omega^2 = 0,$$

得  $B$  的特征值为

$$\lambda_1 = 1 - 4\omega, \quad \lambda_2 = 1 - 2\omega.$$

迭代法收敛等价于  $\rho(B) < 1$ , 即有

$$\begin{cases} |1 - 4\omega| < 1, \\ |1 - 2\omega| < 1, \end{cases} \implies 0 < \omega < \frac{1}{2}.$$

另外, 易得

$$\rho(B) = \begin{cases} 1 - 2\omega, & \text{if } 0 < \omega \leq \frac{1}{3}, \\ 4\omega - 1, & \text{if } \omega > \frac{1}{3}, \end{cases}$$

由此可知当  $\omega = \frac{1}{3}$  时,  $\rho(B)$  取到最小值  $\frac{1}{3}$ , 此时迭代法收敛最快.

3.10 给定初始向量  $x^{(0)} = (0, 0, 0)^T$ , 用共轭梯度法求解以下方程组

$$(1) \begin{cases} 2x_1 - x_2 - x_3 = 2, \\ -x_1 + 2x_2 + x_3 = 0, \\ -x_1 + x_2 + 2x_3 = -2; \end{cases} \quad (2) \begin{cases} 4x_1 - x_2 + 2x_3 = 12, \\ -x_1 + 5x_2 + 3x_3 = 10, \\ 2x_1 + 3x_2 + 6x_3 = 18. \end{cases}$$

解 方程组 (1) 的精确解为  $x^* = (1, 1, -1)^T$ . 用共轭梯度法求解. 已知

$$A = \begin{pmatrix} 2 & -1 & -1 \\ -1 & 2 & 1 \\ -1 & 1 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 2 \\ 0 \\ -2 \end{pmatrix}.$$



第 1 次迭代取搜索方向为  $d^{(0)} = r^{(0)} = b - Ax^{(0)} = b = (2, 0, -2)^T$ , 则

$$r^{(0)T}d^{(0)} = 8, \quad d^{(0)T}Ad^{(0)} = 24, \quad \alpha_0 = \frac{r^{(0)T}d^{(0)}}{d^{(0)T}Ad^{(0)}} = \frac{1}{3},$$

于是得

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)} = \frac{1}{3}(0, 2, -2)^T, \quad r^{(1)} = b - Ax^{(1)} = \frac{1}{3}(0, 4, 0)^T.$$

第 2 次迭代取搜索方向为  $d^{(1)} = r^{(1)} + \beta_0 d^{(0)}$ . 由于

$$\begin{aligned} r^{(1)T}Ad^{(0)} &= -\frac{16}{3}, \quad \beta_0 = -\frac{r^{(1)T}Ad^{(0)}}{d^{(0)T}Ad^{(0)}} = \frac{2}{9}, \quad d^{(1)} = r^{(1)} + \beta_0 d^{(0)} = \frac{4}{9}(1, 3, -1)^T, \\ r^{(1)T}d^{(1)} &= \frac{16}{9}, \quad d^{(1)T}Ad^{(1)} = \frac{64}{27}, \quad \alpha_1 = \frac{r^{(1)T}d^{(1)}}{d^{(1)T}Ad^{(1)}} = \frac{3}{4}, \end{aligned}$$

因此得

$$x^{(2)} = x^{(1)} + \alpha_1 d^{(1)} = (1, 1, -1)^T, \quad r^{(2)} = b - Ax^{(2)} = (0, 0, 0)^T,$$

即迭代两次求得方程组的精确解  $x^* = x^{(2)} = (1, 1, -1)^T$ .

方程组 (2) 的精确解为  $x^* = (3, 2, 1)^T$ . 用共轭梯度法求解. 已知

$$A = \begin{pmatrix} 4 & -1 & 2 \\ -1 & 5 & 3 \\ 2 & 3 & 6 \end{pmatrix}, \quad b = \begin{pmatrix} 12 \\ 10 \\ 18 \end{pmatrix}.$$

第 1 次迭代取搜索方向为  $d^{(0)} = r^{(0)} = b - Ax^{(0)} = b = (12, 10, 18)^T$ , 则

$$r^{(0)T}d^{(0)} = 568, \quad d^{(0)T}Ad^{(0)} = 4724, \quad \alpha_0 = \frac{r^{(0)T}d^{(0)}}{d^{(0)T}Ad^{(0)}} = \frac{142}{1181},$$

得到

$$x^{(1)} = x^{(0)} + \alpha_0 d^{(0)} = \frac{1}{1181}(1704, 1420, 2556)^T, \quad r^{(1)} = b - Ax^{(1)} = \frac{1}{1181}(3664, -1254, -1746)^T.$$

第 2 次迭代取搜索方向为  $d^{(1)} = r^{(1)} + \beta_0 d^{(0)}$ . 由于

$$\begin{aligned} r^{(1)T}Ad^{(0)} &= -\frac{127084}{1181}, \quad \beta_0 = -\frac{r^{(1)T}Ad^{(0)}}{d^{(0)T}Ad^{(0)}} = \frac{31771}{1394761}, \\ d^{(1)} &= r^{(1)} + \beta_0 d^{(0)} = \frac{1}{1394761}(4708436, -1163264, -1490148)^T, \\ r^{(1)T}d^{(1)} &= \frac{18045928}{1394761}, \quad d^{(1)T}Ad^{(1)} = \frac{86415322336}{1647212741}, \quad \alpha_1 = \frac{r^{(1)T}d^{(1)}}{d^{(1)T}Ad^{(1)}} = \frac{37521551}{152139652}, \end{aligned}$$

于是得

$$\begin{aligned} x^{(2)} &= x^{(1)} + \alpha_1 d^{(1)} = \frac{1}{535703}(1218941, 533924, 1018251)^T, \\ r^{(2)} &= b - Ax^{(2)} = \left( \frac{50094}{535703}, \frac{851598}{535703}, -\frac{72358}{76529} \right)^T. \end{aligned}$$

第 3 次迭代取搜索方向为  $d^{(2)} = r^{(2)} + \beta_1 d^{(1)}$ . 由于

$$\begin{aligned} r^{(2)\text{T}} A d^{(1)} &= -\frac{8798421104}{632665243}, \quad \beta_1 = -\frac{r^{(2)\text{T}} A d^{(1)}}{d^{(1)\text{T}} A d^{(1)}} = \frac{10802548078729}{40750833997678}, \\ d^{(2)} &= r^{(2)} + \beta_1 d^{(1)} = \frac{1}{286977704209} (283647167144, 392755834306, -352613747684)^{\text{T}}, \\ r^{(2)\text{T}} d^{(2)} &= \frac{984276890476}{286977704209}, \quad d^{(2)\text{T}} A d^{(2)} = \frac{719243605008198908}{153734817077873927}, \quad \alpha_2 = \frac{r^{(2)\text{T}} d^{(2)}}{d^{(2)\text{T}} A d^{(2)}} = \frac{535703}{730733}, \end{aligned}$$

因此得到

$$x^{(3)} = x^{(2)} + \alpha_2 d^{(2)} = (3, 2, 1)^{\text{T}}, \quad r^{(3)} = b - A x^{(3)} = (0, 0, 0).$$

故迭代 3 次求得方程组的精确解  $x^* = x^{(3)} = (3, 2, 1)^{\text{T}}$ .