

计算方法 第4章习题答案

4.1 已知函数 $y = f(x)$ 在若干点处的函数值如下表所示, 求其拉格朗日插值多项式和牛顿插值多项式, 并写截断误差表达式.

(1)	x_i	0	1	2	5
	y_i	2	3	12	127

(2)	x_i	-2	-1	0	1
	y_i	15	4	5	24

解 (1) 由题设可知 $n = 3$, 拉格朗日插值基函数有 4 个, 分别为:

$$\begin{aligned}
 l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{1}{10}(x-1)(x-2)(x-5), \\
 l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{1}{4}x(x-2)(x-5), \\
 l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{6}x(x-1)(x-5), \\
 l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{1}{60}x(x-1)(x-2),
 \end{aligned}$$

因此拉格朗日插值多项式为:

$$L_3(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 + l_3(x)y_3 = x^3 + x^2 - x + 2,$$

截断误差为:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) = \frac{f^{(4)}(\xi)}{24}x(x-1)(x-2)(x-5), \quad \xi \in (0, 5).$$

对于牛顿插值多项式, 先列差商表如下:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
0	2			
1	3	1		
2	12	9	4	
5	147	45	9	1

由此得牛顿插值多项式为:

$$\begin{aligned}
 N_3(x) &= f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) \\
 &\quad + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2) \\
 &= 2 + x + 4x(x-1) + x(x-1)(x-2) = x^3 + x^2 - x + 2,
 \end{aligned}$$

截断误差为:

$$\begin{aligned}
 R_3(x) &= f[x_0, x_1, x_2, x_3, x](x-x_0)(x-x_1)(x-x_2)(x-x_3) \\
 &= f[0, 1, 2, 3, x]x(x-1)(x-2)(x-5).
 \end{aligned}$$

(2) 由题设可知 $n = 3$, 拉格朗日插值基函数有 4 个, 分别为:

$$\begin{aligned}l_0(x) &= \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} = -\frac{1}{6}x(x-1)(x+1), \\l_1(x) &= \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} = \frac{1}{2}x(x+2)(x-1), \\l_2(x) &= \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} = -\frac{1}{2}(x+2)(x+1)(x-1), \\l_3(x) &= \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} = \frac{1}{6}x(x+2)(x+1),\end{aligned}$$

因此拉格朗日插值多项式为:

$$L_3(x) = l_0(x)y_0 + l_1(x)y_1 + l_2(x)y_2 + l_3(x)y_3 = x^3 + 9x^2 + 9x + 5,$$

截断误差为:

$$R_3(x) = \frac{f^{(4)}(\xi)}{4!}(x-x_0)(x-x_1)(x-x_2)(x-x_3) = \frac{f^{(4)}(\xi)}{24}x(x+2)(x+1)(x-1), \quad \xi \in (-2, 1).$$

对于牛顿插值多项式, 先列差商表如下:

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$
-2	15			
-1	4	-11		
0	5	1	6	
1	24	19	9	1

由此得牛顿插值多项式为:

$$\begin{aligned}N_3(x) &= f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) \\&\quad + f[x_0, x_1, x_2, x_3](x-x_0)(x-x_1)(x-x_2) \\&= 15 - 11(x+2) + 6(x+2)(x+1) + x(x+2)(x+1) = x^3 + 9x^2 + 9x + 5,\end{aligned}$$

截断误差为:

$$\begin{aligned}R_3(x) &= f[x_0, x_1, x_2, x_3, x](x-x_0)(x-x_1)(x-x_2)(x-x_3) \\&= f[-2, -1, 0, 1, x]x(x+2)(x+1)(x-1).\end{aligned}$$

4.3 在 $x = -1, 1, 2$ 处, $f(x) = -3, 0, 4$, 求 $f(x)$ 的二次插值多项式.

(1) $p_2(x) = a_0 + a_1x + a_2x^2;$

(2) 拉格朗日插值多项式;

(3) 牛顿插值多项式.

证明上述三种方法求得的插值多项式是相同的.

解 (1) 根据插值条件有

$$\begin{cases} a_0 + a_1 + a_2 = 0 \\ a_0 - a_1 + a_2 = -3 \\ a_0 + 2a_1 + 4a_2 = 4 \end{cases}$$

解得

$$a_0 = -\frac{7}{3}, \quad a_1 = \frac{3}{2}, \quad a_2 = \frac{5}{6}.$$

因此

$$p_2(x) = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

(2) 拉格朗日插值多项式: 由于

$$\begin{aligned} l_0(x) &= \frac{(x-x_1)(x-x_2)}{(x_0-x_1)(x_0-x_2)} = -\frac{1}{2}(x+1)(x+2), \\ l_1(x) &= \frac{(x-x_0)(x-x_2)}{(x_1-x_0)(x_1-x_2)} = \frac{1}{6}(x-1)(x-2), \\ l_2(x) &= \frac{(x-x_0)(x-x_1)}{(x_2-x_0)(x_2-x_1)} = \frac{1}{3}(x-1)(x+1), \end{aligned}$$

故

$$L_2(x) = \sum_{k=0}^2 l_k(x)y_k = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

(3) 牛顿插值多项式: 由于

$$\begin{aligned} f[x_0] &= -3, \quad f[x_0, x_1] = \frac{f[x_1] - f[x_0]}{x_1 - x_0} = \frac{3}{2}, \quad f[x_1, x_2] = \frac{f[x_2] - f[x_1]}{x_2 - x_1} = 4, \\ f[x_0, x_1, x_2] &= \frac{f[x_1, x_2] - f[x_0, x_1]}{x_2 - x_0} = \frac{5}{6}, \end{aligned}$$

因此

$$N_2(x) = f[x_0] + f[x_0, x_1](x-x_0) + f[x_0, x_1, x_2](x-x_0)(x-x_1) = -\frac{7}{3} + \frac{3}{2}x + \frac{5}{6}x^2.$$

根据插值多项式的存在唯一性, 可知上述三种方法得到的插值多项式是相同的.

4.6 证明 k 阶差商的如下性质:

(1) 设 $p(x) = cf(x)$, 则

$$p[x_0, x_1, \dots, x_k] = cf[x_0, x_1, \dots, x_k];$$

(2) 设 $p(x) = f(x) + g(x)$, 则

$$p[x_0, x_1, \dots, x_k] = f[x_0, x_1, \dots, x_k] + g[x_0, x_1, \dots, x_k].$$

证明 (1) 用数学归纳法证明. 当 $k = 0$ 时,

$$p[x_0] = p(x_0) = cf(x_0) = cf[x_0].$$

当 $k = 1$ 时,

$$p[x_0, x_1] = \frac{p[x_1] - p[x_0]}{x_1 - x_0} = \frac{cf[x_1] - cf[x_0]}{x_1 - x_0} = c \frac{f[x_1] - f[x_0]}{x_1 - x_0} = cf[x_0, x_1].$$

假设当 $k = n$ 时结论成立, 即有

$$p[x_0, x_1, \cdots, x_n] = cf[x_0, x_1, \cdots, x_n].$$

则当 $k = n + 1$ 时,

$$\begin{aligned} p[x_0, x_1, \cdots, x_n, x_{n+1}] &= \frac{p[x_1, \cdots, x_n, x_{n+1}] - p[x_0, x_1, \cdots, x_n]}{x_{n+1} - x_0} \\ &= \frac{cf[x_1, \cdots, x_n, x_{n+1}] - cf[x_0, x_1, \cdots, x_n]}{x_{n+1} - x_0} = cf[x_0, x_1, \cdots, x_n, x_{n+1}]. \end{aligned}$$

因此结论对任意的 $k \geqslant 0$ 都成立.

(2) 亦可用数学归纳法证明. 当 $k = 0$ 时,

$$p[x_0] = p(x_0) = f(x_0) + g(x_0) = f[x_0] + g[x_0].$$

当 $k = 1$ 时,

$$\begin{aligned} p[x_0, x_1] &= \frac{p[x_1] - p[x_0]}{x_1 - x_0} = \frac{f[x_1] + g[x_1] - f[x_0] - g[x_0]}{x_1 - x_0} \\ &= \frac{f[x_1] - f[x_0]}{x_1 - x_0} + \frac{g[x_1] - g[x_0]}{x_1 - x_0} = f[x_0, x_1] + g[x_0, x_1]. \end{aligned}$$

假设当 $k = n$ 时结论成立, 即有

$$p[x_0, x_1, \cdots, x_n] = f[x_0, x_1, \cdots, x_n] + g[x_0, x_1, \cdots, x_n].$$

则当 $k = n + 1$ 时,

$$\begin{aligned} p[x_0, x_1, \cdots, x_n, x_{n+1}] &= \frac{p[x_1, \cdots, x_n, x_{n+1}] - p[x_0, x_1, \cdots, x_n]}{x_{n+1} - x_0} \\ &= \frac{f[x_1, \cdots, x_n, x_{n+1}] + g[x_1, \cdots, x_n, x_{n+1}] - f[x_0, x_1, \cdots, x_n] - g[x_0, x_1, \cdots, x_n]}{x_{n+1} - x_0} \\ &= f[x_0, x_1, \cdots, x_n, x_{n+1}] + g[x_0, x_1, \cdots, x_n, x_{n+1}]. \end{aligned}$$

因此结论对任意的 $k \geqslant 0$ 都成立.

上述两条性质说明差商是线性算子.

4.7 $f(x) = x^7 + x^4 + 3x + 1$, 求 $f[2^0, 2^1, \cdots, 2^7]$ 和 $f[2^0, 2^1, \cdots, 2^8]$.

解 根据差商与导数的关系,

$$f[x_i, x_{i+1}, \dots, x_{i+k}] = \frac{f^{(k)}(\xi)}{k!},$$

可知

$$f[2^0, 2^1, \dots, 2^7] = \frac{f^{(7)}(\xi)}{7!} = 1, \quad f[2^0, 2^1, \dots, 2^8] = \frac{f^{(8)}(\xi)}{8!} = 0.$$

4.9 已知函数 $y = f(x)$ 在若干点处的函数值、导数值如下表所示, 求埃尔米特插值多项式和截断误差表达式:

(1)	x_i	-1	0	1
	y_i	-1	0	1
	y'_i	0	0	0
(2)	x_i	0	1	2
	y_i	1	-1	0
	y'_i	0	0	
	y''_i	2		

解 (1) 设拉格朗日型埃尔米特插值多项式的形式为

$$H_5(x) = \sum_{i=0}^2 h_i(x)f(x_i) + \sum_{i=0}^2 \bar{h}_i(x)f'(x_i) = \sum_{i=0}^2 h_i(x)f(x_i),$$

其中

$$h_i(x) = \left[1 - 2(x - x_i) \sum_{\substack{j=0 \\ j \neq i}}^2 \frac{1}{x_i - x_j} \right] l_i^2(x).$$

由于

$$l_0(x) = \frac{x(x-1)}{2}, \quad l_1(x) = (x+1)(x-1), \quad l_2(x) = \frac{x(x+1)}{2},$$

因此

$$\begin{aligned} h_0(x) &= \frac{1}{4}x^2(x-1)^2(3x+4), \\ h_1(x) &= (x+1)^2(x-1)^2, \\ h_2(x) &= \frac{1}{4}x^2(x+1)^2(4-3x), \end{aligned}$$

代入得埃米尔特插值多项式为

$$H_5(x) = -\frac{1}{4}x^2(x-1)^2(3x+4) + \frac{1}{4}x^2(x+1)^2(4-3x) = -\frac{1}{2}x^3(3x^2-5).$$

进一步, 其截断误差估计式为

$$R_5(x) = f(x) - H_5(x) = \frac{f^{(6)}(\xi)}{6!}(x+1)^2x^2(x-1)^2, \quad \xi \in (-1, 1).$$

(2) 由差商与导数的关系可知

$$f[0, 0] = f'(0) = 0, \quad f[0, 0, 0] = \frac{f''(0)}{2!} = 1, \quad f[1, 1] = f'(1) = 0,$$

作差商表

x_i	$f(x_i)$	$f[x_i, x_{i+1}]$	$f[x_i, x_{i+1}, x_{i+2}]$	$f[x_i, x_{i+1}, x_{i+2}, x_{i+3}]$	\cdots	\cdots
0	1					
0	1	0				
0	1	0	1			
1	-1	-2	-2	-3		
1	-1	0	2	4	7	
2	0	1	1	$-\frac{1}{2}$	$-\frac{9}{4}$	$-\frac{37}{8}$

所以

$$\begin{aligned} H_5(x) &= 1 + x^2 - 3x^3 + 7x^3(x-1) - \frac{37}{8}x^3(x-1)^2 \\ &= \frac{1}{8}(-37x^5 + 130x^4 - 117x^3 + 8x^2 + 8), \end{aligned}$$

$$R_5(x) = f[0, 0, 0, 1, 1, 2, x]x^3(x-1)^2(x-2).$$

4.10 设 x_i ($i = 0, 1, \dots, n$) 是互不相同的插值节点, $l_i(x)$ ($i = 0, 1, \dots, n$) 是拉格朗日插值基函数. 证明:

$$(1) \sum_{i=1}^n l_i(x) = 1;$$

$$(2) \sum_{i=1}^n l_i(x)x_i^k = x^k \quad (k = 1, 2, \dots, n);$$

$$(3) \sum_{i=1}^n l_i(x)(x_i - x)^k = 0 \quad (k = 1, 2, \dots, n);$$

$$(4) \sum_{i=1}^n l_i(0)x_i^k = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \dots, n, \\ (-1)^n x_0 x_1 \dots x_n, & k = n + 1. \end{cases}$$

证明 已知对任意给定的 $n + 1$ 个插值节点, n 次拉格朗日插值多项式及其截断误差可以表示为:

$$L_n(x) = \sum_{i=0}^n l_i(x)f(x_i), \quad R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x).$$

(1) 根据题目可知, 被插函数 $f(x) = 1$, 因此有

$$f^{(n+1)}(x) = 0 \implies f(x) - L_n(x) = R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}\pi_{n+1}(x) = 0,$$

从而有

$$L_n(x) = \sum_{i=0}^n l_i(x) = f(x) = 1.$$

(2) 根据题目可知, 被插函数 $f(x) = x^k$ ($k \leq n$), 因此有

$$f^{(n+1)}(x) = 0 \implies f(x) - L_n(x) = R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x) = 0,$$

从而有

$$L_n(x) = \sum_{i=0}^n l_i(x) x_i^k = f(x) = x^k.$$

(3) 解法 1: 先变量替换, 将 x 换成 t , 则等式结果不发生改变. 此时可知被插函数为 $f(x) = (x - t)^k$ ($k \leq n$), 因此有

$$f^{(n+1)}(x) = 0 \implies R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x) = 0,$$

即

$$L_n(x) = \sum_{i=0}^n l_i(x) f(x_i) = \sum_{i=0}^n l_i(x) (x_i - t)^k = f(x),$$

特别地,

$$L_n(t) = \sum_{i=0}^n l_i(t) (x_i - t)^k = f(t) = 0.$$

解法 2: 将 $(x_i - x)^k$ 按二次项展开得

$$(x_i - x)^k = \sum_{j=0}^k C_k^j x_i^j (-x)^{k-j},$$

因此

$$\sum_{i=0}^n l_i(x) (x_i - x)^k = \sum_{i=0}^n \left(\sum_{j=0}^k C_k^j x_i^j (-x)^{k-j} \right) l_i(x) = \sum_{j=0}^k C_k^j (-x)^{k-j} \sum_{i=0}^n x_i^j l_i(x),$$

由 (2) 得 $\sum_{i=0}^n x_i^j l_i(x) = x^j$, 因此

$$\sum_{i=0}^n l_i(x) (x_i - x)^k = \sum_{j=0}^k C_k^j (-x)^{k-j} x^j = x^k \sum_{j=0}^k (-1)^{k-j} C_k^j = 0.$$

(4) 根据题目可知, 被插函数 $f(x) = x^k$, 因此有

$$f^{(n+1)}(x) = \begin{cases} 0, & k = 0, 1, \dots, n, \\ (n+1)!, & k = n+1, \end{cases}$$

从而有

$$R_n(x) = f(x) - L_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!} \pi_{n+1}(x) = \begin{cases} 0, & k = 0, 1, \dots, n, \\ \pi_{n+1}(x), & k = n+1, \end{cases}$$

因此

$$L_n(x) = \sum_{i=0}^n l_i(x)x_i^k = f(x) - R_n(x) = \begin{cases} x^k, & k = 0, 1, \dots, n, \\ x^k - \pi_{n+1}(x), & k = n+1. \end{cases}$$

特别地,

$$L_n(0) = \sum_{i=0}^n l_i(0)x_i^k = f(0) - R_n(0) = \begin{cases} 1, & k = 0, \\ 0, & k = 1, 2, \dots, n, \\ (-1)^n x_0 x_1 \cdots x_n, & k = n+1. \end{cases}$$

4.13 已知函数 $y = f(x)$ 在若干点处的函数值如下表所示, 求满足以下端点条件的三次样条插值函数:

(1) $S'(0) = 0, S'(4) = 48;$

(2) $S''(0) = 0, S''(4) = 24.$

x_i	0	1	2	3	4
y_i	-8	-7	0	19	56

解 由 $h_i = x_i - x_{i-1}$ 得

$$h_1 = h_2 = h_3 = h_4 = 1.$$

设三次样条插值函数 $S(x)$ 在节点 x_i 处的二阶导数值为 $S''(x_i) = M_i$ ($i = 0, 1, \dots, 4$), 则 $S(x)$ 在子区间 $[x_{i-1}, x_i]$ ($i = 1, \dots, 4$) 上的表达式为:

$$S(x) = \frac{(x_i - x)^3}{6h_i} M_{i-1} + \frac{(x - x_{i-1})^3}{6h_i} + \left(y_{i-1} - \frac{h_i^2}{6} M_{i-1} \right) \frac{x_i - x}{h_i} + \left(y_i - \frac{h_i^2}{6} M_i \right) \frac{x - x_{i-1}}{h_i}.$$

根据 $\mu_i = \frac{h_i}{h_i + h_{i+1}}, \lambda_i = 1 - \mu_i$ ($i = 1, 2, 3$) 得

$$\mu_1 = \mu_2 = \mu_3 = \frac{1}{2}, \quad \lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{2}.$$

再根据 $d_i = 6f[x_{i-1}, x_i, x_{i+1}]$ ($i = 1, 2, 3$) 可得

$$d_1 = 18, \quad d_2 = 36, \quad d_3 = 54.$$

(1) 由第二类边界条件得

$$d_0 = 6f[x_0, x_0, x_1] = 6, \quad d_4 = 6f[x_3, x_4, x_4] = 66$$

进一步可以得到关于第二类边界条件的三弯矩方程组

$$\begin{pmatrix} 2 & 1 & 0 & 0 & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 2 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & 0 & 0 & 1 & 2 \end{pmatrix} \begin{pmatrix} M_0 \\ M_1 \\ M_2 \\ M_3 \\ M_4 \end{pmatrix} = \begin{pmatrix} 6 \\ 18 \\ 36 \\ 54 \\ 66 \end{pmatrix}.$$

解得

$$M_0 = 0, \quad M_1 = 6, \quad M_2 = 12, \quad M_3 = 18, \quad M_4 = 24,$$

因此三次样条插值函数 $S(x)$ 为

$$S(x) = \begin{cases} x^3 - 8, & 0 \leq x \leq 1, \\ x^3 - 8, & 1 \leq x \leq 2, \\ x^3 - 8, & 2 \leq x \leq 3, \\ x^3 - 8, & 3 \leq x \leq 4, \end{cases}$$

即

$$S(x) = x^3 - 8, \quad x \in [0, 4].$$

(2) 由第一类边界条件得 $M_0 = 0, M_4 = 24$, 进一步可得其三弯矩方程组为:

$$\begin{pmatrix} 2 & \frac{1}{2} & 0 \\ \frac{1}{2} & 2 & \frac{1}{2} \\ 0 & \frac{1}{2} & 2 \end{pmatrix} \begin{pmatrix} M_1 \\ M_2 \\ M_3 \end{pmatrix} = \begin{pmatrix} 18 \\ 36 \\ 42 \end{pmatrix},$$

解得

$$M_1 = 6, \quad M_2 = 12, \quad M_3 = 18,$$

因此三次样条插值函数 $S(x)$ 为

$$S(x) = \begin{cases} x^3 - 8, & 0 \leq x \leq 1, \\ x^3 - 8, & 1 \leq x \leq 2, \\ x^3 - 8, & 2 \leq x \leq 3, \\ x^3 - 8, & 3 \leq x \leq 4, \end{cases}$$

即有

$$S(x) = x^3 - 8, \quad x \in [0, 4].$$