Exercise IV.3

By using Fenchel-Rockafeller duality, solve the followng optimization problem:

$$\text{minimize}_{x\in\mathbb{R}^N} \quad \sum_{i=1}^{N} \exp\left(x^{(i)}\right), \quad \text{s.t.} \quad \sum_{i=1}^{N} x^{(i)} \geq 1. \tag{1}$$

Solution: We start by an indicator function in order to write this constrained optimization problem in the unconstrained form where the cost function appears as a sum. This can be done by writing the problem as follows

$$\text{minimize}_{x\in\mathbb{R}^N} \quad f(x) + g\left(\mathbb{1}^T x\right), \tag{2}$$

where:

$$f\,(x) = \sum_{i=1}^{N} \phi\left(x^{(i)}\right) \tag{3}$$

$$g\,(y) = \iota_{[1,+\infty[}\,(y)\,, \quad y \in \mathbb{R}, \tag{4}$$

and $\mathbb{1} = (1, 1, \ldots, 1)$, a vector of N elements, all of them being equal to 1.

We now notice that the function f is a sum of continuous, convex separable functions. As a result f is also convex and lower semicontinuous and in fact $f \in \Gamma_0\left(\mathbb{R}^N\right)$.

We now investigate the set $g\,(y)$. Since the set $[1, +\infty[$ is nonempty, closed and convex, we have that $\iota_{[1,\infty[}(y) \in \Gamma_0\,(\mathbb{R})$. As a result, both f and g are proper functions, which means that a necessary condition for applying Rockafellar duality results is satisfied.

As a result, we see that in this formulation we have that the cost function has a structure similar to the one necessary in order to apply Fenchel-Rockafellar duality results. What we now need to verify in order to use Fenchel-Rochafellar duality is that:

$$0 \in \text{int}\,(\text{dom}g - L\,(\text{dom}f)) \tag{5}$$

or equivalently if:

$$\text{int}\,(\text{dom}(g)) \cap \mathbb{1}^T\,(\text{dom}(f)) \neq \emptyset, \quad \text{or} \quad \text{dom}(g) \cap \text{int}(\mathbb{1}^T\,(\text{dom}f)) \neq \emptyset. \tag{6}$$

Let us now study the given functions. For f, we have that $\text{dom}(f) = \mathbb{R}^N$, which means that $\mathbb{1}^T(\text{dom}(f)) = \mathbb{R}$. On the other hand, for g, it holds that $\text{dom}(g) = [1, +\infty[$. As a result, the intersection $\text{int}\,(\text{dom}(g)) \cap \mathbb{1}^T\,(\text{dom}(f))$ is non empty. This practically means that strong duality holds and that by solving the dual problem we can obtain the optimal objective value function for the primal problem.

In order to calculcate the dual, we start by finding the conjugate of f. Since f is a sum of separable functions, we first focus on functions:

$$\phi\,(z) = \exp\,(z)\,, \quad z \in \mathbb{R}. \tag{7}$$

Its conjugate is given as:

$$\phi^*(v) = \sup_{z\in\mathbb{R}} vz - \phi(z). \tag{8}$$

This conjugate can be further refined by studying the monotonicity of the function:

$$\psi(z) = vz - \phi(z), \quad z \in \mathbb{R}, \tag{9}$$

and discriminating the following three cases:

1. Case 1: $v < 0$. In this case the derivative of $\psi(\cdot)$ is always negative. As a result the function is monotonically decreasing. Therefore, in this case:

$$\phi^*(v) = \sup_{z\in\mathbb{R}} \psi(z) = \lim_{z\to-\infty} \psi(z) = +\infty \tag{10}$$

2. Case 2: $v = 0$. In this case $\psi(z)$ becomes:

$$\psi(z) = -\phi(z) \tag{11}$$

   and therefore $\phi^*(0) = \sup \psi(z) = 0$.

3. Case 3: $v > 0$. In this case, the equation

$$\frac{d\psi}{dz} = 0, \tag{12}$$

   attains a root for $z = \ln v$. Based on the sign of $\frac{d\psi}{dz}$ we can prove that this zero corresponds to a maximizer. As a result, we have that:

$$\phi^*(v) = \psi(\ln v) = v \ln v - v. \tag{13}$$

Therefore, adopting the convention $0 \ln 0 = 0$ we can compactly write the $\phi^*(v)$ as:

$$\phi^*(v) = \begin{cases} v\ln v - v, & v \geq 0 \\ +\infty, & v < 0. \end{cases} \tag{14}$$

and f^* becomes:

$$f^*(\nu) = \sum_{i=1}^{N} \phi^*\left(\nu^{(i)}\right), \quad \nu \in \mathbb{R}^n \tag{15}$$

On the other hand, for $g(z)$ it holds that:

$$g^*(v) = \sup_{z\in[1,+\infty[} vz = \begin{cases} v, & v \leq 0 \\ +\infty, & v > 0. \end{cases} \tag{16}$$

Using Fenchel Rockafellar duality, we can now write the dual problem as:

$$\text{minimize}_{v\in\mathbb{R}} \;\; f^*(-\mathbb{1}v) + g^*(v), \tag{17}$$

or equivalently as:

$$\text{minimize}_{v\in]-\infty,0]} \;\; N((-v)\ln(-v)+v)+v. \tag{18}$$

By introducing the change of variables $w = -v$, this problem is equivalently written as:

$$\text{minimize}_{w\in[0,\infty[} \;\; Nw\ln w-(N+1)w. \tag{19}$$

Furthermore, by finding the zero of the derivative of the cost function, it can be proven that thisproblem has a minimizer at $w = \exp(1/N)$. The optimal value of the dual problem will therefore be equal to

$$\mu^* = \left. Nw\ln w-(N+1)w\right|_{w=e^{1/N}} = -Ne^{1/N}. \tag{20}$$

As a result, due to strong Rockafellar duality, we will have that $\mu = -\mu^* = Ne^{1/N}$. Therefore, in order to solve the primal problem, it suffices to find a feasible point $x^\star$ for which:

$$f(x^\star) = Ne^{1/N}. \tag{21}$$

Let us now consider the point $x^\star = (\exp(1/N),\ldots,\exp(1/N))$. Clearly for this point:

$$f\,(x^\star) = N\exp(1/N). \tag{22}$$

Therefore this point is a global minimizer.

Question: Is the point a unique minimizer?