Lecture V : Theoretical resolution of elliptic problems

A) Aims of this class

After this class,

- I know how to define a partial derivative in the sense of distributions.
- I know how to define the trace of a H^1 function on the boundary of the domain.
- I know the formulas that extend the integration by parts.
- I can find a variational formulation based on a linear elliptic problem, taking into account correctly the conditions at the boundary.
- I can solve a variational formulation.
- I know how to go back to the initial elliptic problem and to solve it theoretically.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions V.1 and V.2 must be done before the 5th lab. The solutions are available online.

Question V.1 (A variational problem)
Let $f \in L^2(0,1)$ and

$$a : (u,v) \mapsto \int_{]0,1[} (u'v' + uv) - \frac{u(0)v(0)}{4}.$$

Q. V.1.1 Prove a is properly defined, bilinear, continuous and coercive on $H^1(0,1) \times H^1(0,1)$.

Q. V.1.2 Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = \int_{]0,1[} fv$.

Q. V.1.3 Prove there exists a unique $u \in H^1(0,1)$ such that $\forall v \in H^1(0,1), a(u,v) = v(0)$.

Q. V.1.4 What is the regularity of the solution u in Question V.1.2?

Q. V.1.5 What is the regularity of the solution u in question V.1.3? What is the elliptic problem satisfied by u?

Question V.2 (A convection-diffusion problem)
We solve here, theoretically, the 1D-stationary convection-diffusion problem:

$$\textbf{(CD)} \qquad \begin{cases} -\kappa u''(x) + bu'(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases}$$

with $\kappa \in \mathbb{R}^{+*}$, $b \in \mathbb{R}$, $c \in C^0([0,1], \mathbb{R}^+)$ and $f \in C^0([0,1], \mathbb{R})$.
Variable u (the unknown) represents the temperature if the problem of interest is heat transfer or the concentration if the problem of interest is mass transfer.

Q. V.2.1 Show that, if $b = 0$, then (CD) has one and only one classical solution, that is, of class $C^2([0,1])$.

Q. V.2.2 Deduce that, in the general case, (CD) admits one and only one classical solution.
Hint: Change unknowns $v : x \mapsto \exp(-\delta x)u(x)$, with δ to be determined.

Q. V.2.3 We assume that κ, c are constant and $f : x \mapsto \exp(bx/(2\kappa))$. Solve (CD).

C) Exercises

Exercise V.1 (Lifting)
Let a and $b \in \mathbb{R}$, $c \in C^0([0,1], \mathbb{R}^+)$ and $f \in C^0([0,1])$. Consider the problem

$$\begin{cases} -u'' + cu = f \ \text{ on }]0,1[, \\ u(0) = a \quad \text{and} \quad u(1) = b. \end{cases}.$$

E. V.1.1 Let u_0 and u_1 be defined from $[0,1]$ to $\mathbb{R}$ by $u_0 : x \mapsto a + (b-a)x$ and $u_1 : x \mapsto a + (b-a)x^2$.

E.V.1.1.1 Show that there exists a unique $\tilde{u}$ (resp. $\bar{u}$) in $C^2([0,1])$ such that $u = u_0 + \tilde{u}$ (resp. $v = u_1 + \bar{u}$) is a solution of the problem.

E.V.1.1.2 Show that $u = v$. Is the problem well-posed in $C^2([0,1])$?

E. V.1.2 Same question for u_0 and u_1 functions of $C^2([0,1])$ such that $u_0(0) = u_1(0) = a$ and $u_0(1) = u_1(1) = b$.

Exercise V.2
Let the problem:

$$\begin{cases} -\kappa u'' + cu = f \text{ in }]0,1[, \\ u'(0) = \alpha \quad \text{and} \quad u'(1) = 0, \end{cases}$$

where $\alpha \in \mathbb{R}$, $\kappa, c \in C^0([0,1], \mathbb{R}^{+*})$ and $f \in L^2(]0,1[)$.

E. V.2.1 Of what type is this problem?

E. V.2.2 Write the variational formulation of this problem.

E. V.2.3 Show that the variational problem has a unique solution.

E. V.2.4 Give a functional space that makes the initial problem well-posed.

E. V.2.5 Rewrite the initial problem as a minimization problem.

Exercise V.3 (Dirichlet-Neumann conditions)
We now turn to the following problem:

$$\textbf{(P)} \qquad \begin{cases} -u''(x) + c(x)u(x) = f(x), \ \ x \in]0,1[, \\ u(0) = 0 \text{ and } u'(1) = 0, \end{cases}$$

with $f \in \mathcal{C}^0([0,1], \mathbb{R})$ and $c \in \mathcal{C}^0([0,1], \mathbb{R}^+)$. Prove the existence and uniqueness of the classical solution.

Exercise V.4
Let $\Omega =]a,b[\times]c,d[$, with $a,b,c,d \in \mathbb{R}$, $a < b$ and $c < d$. Consider the problem

$$\begin{cases} -\Delta u = 1 \text{ in } \Omega, \\ u(a,y) = 0, \quad \partial_x u(b,y) = 0, \ c < y < d, \\ \partial_y u(x,c) = 1, \quad \partial_y u(x,d) = x, \ a < x < b. \end{cases}$$

E. V.4.1 Write the associated variational formulation.

E. V.4.2 Study this variational formulation.

D) Going further

Exercise V.5

Consider the problem (F) $\begin{cases} -u'' + qu = f \text{ in }]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0, \end{cases}$

where $f \in L^2(]0,1[)$ and q is a nonnegative constant.

E. V.5.1 Give a variational formulation (FV) of (F) in a Hilbert space H to be precised. Denote respectively $a(\cdot,\cdot)$ and $\ell(\cdot)$ the bilinear (resp. linear) form associated with this variational problem.

E. V.5.2 Check that for all $q \geq 0$, (F) admits a unique solution u in a Sobolev space to be precised.

Let $m \geq 1$. Introduce the finite dimensional vector space H_m generated by the functions

$$\phi_k : x \mapsto \sin(k\pi x), \;\; k = 1, \cdots, m.$$

E. V.5.3 Show that $H_m \subset H, \forall m \in \mathbb{N}^*$. Give the dimension of H_m.

We approximate the solution of (FV) by $u_m = \sum_{k=1}^{m} \mathbf{u}_k \phi_k$ solution of

$$(\mathrm{FV}'_m) \quad \text{Find } u_m \in H_m \text{ such that } \forall v_m \in H_m, \quad a(u_m, v_m) = \ell(v_m).$$

E. V.5.4 Write the associated linear system. What can be said about its matrix ?

E. V.5.5 Deduce the expression of the coefficients $\mathbf{u}_k$, $k = 1, \ldots, m$, and those of u_m.

E. V.5.6 Justify the existence of a Hilbert basis $L^2(]0,1[)$, denoted by $(w_k)_{k\geq 1}$, such that, $\forall k \geq 1$,

$$w_k \in H_0^1(]0,1[) \quad \text{and} \quad \forall v \in H_0^1(]0,1[), \quad \int_0^1 w_k' v' \mathrm{d}x = \lambda_k \int_0^1 w_k v \mathrm{d}x.$$

E. V.5.7 Establish a link between $(w_k)_{k\geq 1}$ and $(\phi_k)_{k\geq 1}$.

E. V.5.8 Show that $\forall m \in \mathbb{N}^*$, $\|u - u_m\|^2_{L^2(]0,1[)} \leq \dfrac{1}{(\pi^2(m+1)^2 + q)^2} \displaystyle\sum_{k=m+1}^{+\infty} \left(\int_0^1 f(x) \sin(k\pi x) \mathrm{d}x \right)^2$,

then that $\|u - u_m\|^2_{L^2(0,1)} \to 0$ as $m \to +\infty$.

Chapter V: Solutions

Solution of Q. V.1.1 Let us consider

$$a : (u,v) \mapsto (u,v)_{H^1(0,1)} - \frac{1}{4}u(0)v(0)$$

This function is

- Properly defined:
  1. u and v belong to $H^1(0,1)$ therefore $(u,v)_{H^1(0,1)}$ is properly defined.
  2. The Rellich Theorem **??** states that $H^1(0,1) \subset C^0([0,1])$. Since $u, v \in H^1(0,1)$, the functions u and v are continuous therefore $u(0)$ and $v(0)$ are properly defined[1].

  So a is well-defined.
- Bilinearity: obvious.
- Continuity: $\forall (u,v) \in H^1(0,1)^2$,

$$|a(u,v)| \leq |(u,v)_{H^1(0,1)}| + \frac{1}{4}|u(0)v(0)| \leq \|u\|_{H^1}\|v\|_{H^1} + \frac{1}{2}\|u\|_{H^1}\|v\|_{H^1} \leq \frac{3}{2}\|u\|_{H^1}\|v\|_{H^1},$$

  where we have used the Cauchy-Schwarz inequality to bound $|(u,v)_{H^1(0,1)}|$ and the exercise **??** to bound $|u(0)v(0)|$.
- Coercivity: For all $u \in H^1(0,1)$,

$$a(u,u) = \|u\|_{H^1}^2 - \frac{1}{4}u(0)^2 \geq \left(1 - \frac{2}{4}\right)\|u\|_{H^1}^2 = \frac{1}{2}\|u\|_{H^1}^2$$

  where we have used the exercise **??** to get $-u(0)^2 \geq -2\|u\|_{H^1}^2$.

Solution of Q. V.1.2 The Lax-Milgram Theorem **??** applies because:

- The space $H^1(0,1)$ endowed with its usual inner product is a Hilbert Space.
- The bilinear form a is continuous and coercive on $H^1(0,1)$.
- The function $v \mapsto \int_{]0,1[} fv$ is a continuous linear form on $H^1(0,1)$ (consequence of the Cauchy-Schwarz inequality).

[1] As a reminder, $u(0)$ makes no sense for a general $u \in L^2(0,1)$ because we are dealing with a class of function.

Hence, there exists a unique $u \in H^1(0,1)$ such that

$$\forall v \in H^1(0,1), \quad a(u,v) = \int_{]0,1[} fv.$$

Solution of Q. V.1.3 Exercise **??** yields the linear form $v \mapsto v(0)$ is continuous on $H^1(0,1)$. The Lax-Milgram Theorem **??** applies once again.

Solution of Q. V.1.4 Let $\phi \in \mathcal{D}(]0,1[)$. Since $\mathcal{D}(]0,1[\subset H^(1,0)$, the following equality holds:

$$a(u,\phi) = \int_{]0,1[} u'\phi' + u\phi = \int_{]0,1[} f\phi$$

Since $u' \in L^2(0,1) \subset L^1_{loc}(0,1)$, the regular distriution $T_{u'}$ satisfies $\int_{]0,1[} u'\varphi' = \langle T_{u'}, \varphi' \rangle$. From now on[2], we will note $u' = T_{u'}$, as we usually do. Subsequently[3] $\int_{]0,1[} u'\varphi' = -\langle u'', \varphi \rangle$. We deduce:

$$\langle -u'' + u - f, \phi \rangle = 0.$$

So $u'' = u - f \in L^2(0,1)$ and[4] $u \in H^2(0,1)$.

Solution of Q. V.1.5 Let $\phi \in \mathcal{D}(]0,1[)$. Once again, $\phi \in H^1(0,1)$ so we have:

$$a(u,\phi) = \int_{]0,1[} (u'\phi' + u\phi) = \phi(0) = 0.$$

As before, since $T_{u'}$ is a regular distribution (now denote u') we get $\int_{]0,1[} u'\phi' = -\langle u'', \phi \rangle$ and

$$\langle -u'' + u, \phi \rangle = 0.$$

So $u'' = u \in H^1(0,1)$ and $u \in H^3(0,1)$. Note we can iterate this argument to obtain $u \in H^s(0,1)$ for all $s \in \mathbb{N}$.

Now that we have established $-u'' + u = 0$ in $]0,1[$, let us go back to the variational formulation:

$$\forall v \in H^1(0,1),\ a(u,v) = v(0)$$

That is

$$\forall v \in H^1(0,1), \int_{]0,1[} (u'v' + uv) - \frac{1}{4}u(0)v(0) - v(0) = 0$$

[2] As we usually do. We explained in the lecture why we can do so

[3] At this point, the right hand side is the duality bracket: u'' is a distribution and φ a test function. We do not know *yet* if u'' is a regular distrbution.

[4] At this point, we know that u'' is a regular distribution and can be identified to a function in L^1_{loc} that happens to be in $H^2(0,1)$.

Applying the Green's theorem[5], we get

$$-\int_{]0,1[}(-u''+u)v+\big(u'(1)v(1)-u'(0)v(0)\big)-\frac{1}{4}u(0)v(0)-v(0)=0$$

Since $-u''+u=0$ in $]0,1[$ we get

$$u'(1)v(1)-u'(0)v(0)-\frac{1}{4}u(0)v(0)-v(0)=0$$

Since this is true for any $v\in H^1(0,1)$, we can[6]

- choose v such that $v(0)=0$ and $v(1)=1$, which gives $u'(1)=0$.
- then choose a different v such that $v(0)=1$ and $v(1)=0$, which gives $u'(0)+1=0$.

Thus u satisfies the variational forumlation

$$\begin{cases}-u''+u=0 \text{ in }]0,1[\\ u'(0)=-1 \text{ and } u'(1)=0.\end{cases}$$

Solution of Q. V.2.1 We apply the outline of the seven-step process exposed during the lecture. For a given $g,c\in C^0([0,1])$, we will show that there exists a unique solution $v\in C^2([0,1])$ such that

$$\begin{cases} -v''+cv=g \text{ sur }]0,1[,\\ v(0)=0 \quad\text{and}\quad v(1)=0.\end{cases} \tag{V.1}$$

E1: Weak formulation

Assume that the solution v is of class $C^2([0,1])$. Let $\phi\in\mathcal{D}(]0,1[)$. Let us multiply the equation by ϕ and then integrate of $]0,1[$. An integration by parts yields:

$$\int_{]0,1[}(v'\phi'+cv\phi)=\int_{]0,1[}g\phi.$$

Note that this step is not actually part of the proof but it helps us identify the weak formulation we are going to work on.

E2: Variational Formulation

We look for a solution v that vanishes at the boundary. The functional space we choose is $H_0^1(0,1)$. Equipped with the norm $\|\cdot\|_{H_0^1}: x\mapsto\|u'\|_{L^2}$, it is a Hilbert space. The variational problem is: Find $v\in H_0^1(0,1)$ such that

$$\forall w\in H_0^1(0,1),\quad a(v,w)=\ell(w)$$

with

$$\begin{cases}a:(v,w)\in(H_0^1)^2\mapsto\int_{]0,1[}(v'w'+cvw),\\ \ell: w\in H_0^1\mapsto\int_{]0,1[}gw.\end{cases}$$

[5] Also know as the Integration by Parts since we are in dimension 1.

[6] The Rellich Theorem that embeds $H^1(0,1)$ in $C^0(0,1)$ allows us to evaluate v in 0 and 1 and we can select such functions $v\in H^1(0,1)$.

E3: Continuity of a and ℓ

Let $(v,w) \in (H^1_0(0,1))^2$. Then, thanks to the Cauchy-Schwarz inequality, we get

$$\begin{aligned}|a(v,w)| &\leq \|v'\|_{L^2}\|w'\|_{L^2} + \|c\|_\infty \|v\|_{L^2}\|w\|_{L^2}\\ &\leq \|v\|_{H^1_0}\|w\|_{H^1_0} + \|c\|_\infty C_\Omega^2 \|v'\|_{L^2}\|w'\|_{L^2}\\ &\leq (1+\|c\|_\infty C_\Omega^2)\|v\|_{H^1_0}\|w\|_{H^1_0}\end{aligned}$$

where C_Ω is the constant appearing in the Poincaré inequality. So the bilinear form a is continuous on $(H^1_0(0,1))^2$.

We also get, thanks to the Cauchy-Schwarz and Poincaré inequalities,

$$|\ell(w)| \leq \|g\|_{L^2}\|w\|_{L^2} \leq C_\Omega \|g\|_{L^2}\|w\|_{H^1_0}.$$

So the linear form ℓ is continuous on $H^1_0(0,1)$.

E4: Coercivity

Let $v \in H^1_0(0,1)$. Then

$$a(v,v) = \int_{]0,1[}((v')^2 + cv^2) \geq \|v\|^2_{H^1_0} + \min_{[0,1]} c \|v\|^2_{L^2}.$$

But c is nonnegative. So

$$a(v,v) \geq \|v\|^2_{H^1_0}$$

and the bilinear form a is coercive on $(H^1_0(0,1))^2$.

Remark 1

(a) The assumption: c is nonnegative is of prime importance.

(b) If c had been positive (rather than nonnegtive) we could have chosen the norm H^1. Here, we have to chose H^1_0 to insure coercivity because c could be equal to zero.

E5: Existence and uniqueness of the solution of the variational formulation

We apply the Lax-Milgram theorem: there exists a unique $v \in H^1_0(0,1)$ such that

$$\forall w \in H^1_0(0,1), \quad a(v,w) = \ell(w).$$

Furthermore

$$\|v\|_{H^1_0} \leq C_\Omega \|g\|_{L^2}.$$

E6: Solution of the PDE in the distributional sense

We know that[7] $\mathcal{D}(]0,1[) \subset H^1_0(0,1)$. So we get

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_{]0,1[} v'\phi' = \langle v', \phi' \rangle = \int_{]0,1[} (-cv+g)\phi = \langle -cv+g, \phi \rangle .$$

[7]We even have density of $\mathcal{D}(]0,1[)$ in $H^1_0(0,1)$ for the H^1 norm.

We conclude
$$\forall \phi \in \mathcal{D}(]0,1[), \quad -\langle v'', \phi\rangle = \langle -cv + g, \phi\rangle ,$$
that is, in the distributional sense, $v'' = g - cv$. Moreover, thanks to the Minkowsky inequality, we get the estimate
$$\|v''\|_{L^2} \leq \|g\|_{L^2} + \|c\|_\infty \|v\|_{L^2} \leq (1 + \|c\|_\infty C_\Omega)\|g\|_{L^2}.$$

E7: Regularity of the solution
Thanks to the Rellich theorem in 1D (the functions $H^1(0,1)$ are continuous in $[0,1]$), and so are the functions $v \in C^0([0,1]$, and $g - cv$, as well as v'': so the function v is of class $C^2([0,1])$.

Solution of Q. V.2.2 Let $\delta \in \mathbb{R}$. Then, if $u = e_\delta v$ with $e_\delta : x \mapsto \exp(\delta x)$, we get (in the distributional sense)
$$\begin{cases} u' = e_\delta(\delta v + v') \\ u'' = e_\delta(\delta^2 v + 2\delta v' + v'') \end{cases}$$
applying the Leibniz formula of Exercice **??**. So, if u is a solution to (CD) then v is a solution to
$$\begin{cases} -\kappa v'' + (b - 2\kappa\delta)v' + (c - \kappa\delta^2 + b\delta)v = fe_{-\delta} \\ v(0) = 0 \text{ et } v(1) = 0. \end{cases}$$

Letting $\delta = b/(2\kappa)$, the first-order derivative disappears and v is a solution of
$$\begin{cases} -\kappa v'' + \left(c + \dfrac{b^2}{4\kappa}\right) v = fe_{-\delta} \\ v(0) = 1 \text{ et } v(1) = 0. \end{cases}$$

We can apply the result of the previous question since $c + \dfrac{b^2}{4\kappa}$ is a nonnegative continuous function.

Solution of Q. V.2.3 Letting $\omega = \sqrt{4c\kappa + b^2}/(2\kappa)$, we get
$$\begin{cases} v'' - \omega^2 v = -1/\kappa \\ v(0) = 0 \text{ and } v(1) = 0. \end{cases}$$

The solutions of this 2nd order ODE are linear combinations of sinh and cosh added to the particular solution $x \mapsto 1/(\kappa\omega^2)$: there exist real numbers α, β such that
$$v : x \mapsto \alpha \sinh(\omega x) + \beta \cosh(\omega x) + 1/(\kappa\omega^2).$$

But $v(0) = 0$ and $v(1) = 0$ implies
$$\begin{cases} \beta + 1/(\kappa\omega^2) = 0 \\ \alpha \sinh(\omega) + \beta\cosh(\omega) + 1/(\kappa\omega^2) = 0 \end{cases}$$

so the unique solution is $\alpha = -(1-\cosh(\omega))/(\kappa\omega^2 \sinh(\omega))$ and $\beta = -1/(\kappa\omega^2)$. Consequently, as

$$v : x \mapsto -\frac{1}{\kappa\omega^2}\left(\frac{1-\cosh(\omega)}{\sinh(\omega)}\sinh(\omega x) + \cosh(\omega x) - 1\right).$$

the solution to (CD) is

$$u : x \mapsto -\exp(bx/(2\kappa))\frac{1}{\kappa\omega^2}\left(\frac{1-\cosh(\omega)}{\sinh(\omega)}\sinh(\omega x) + \cosh(\omega x) - 1\right).$$

Solution of Q. V.1.1 We see that $u = u_0 + \tilde{u}$ is a solution to our initial problem if and only if $\tilde{u}$ is a solution to

$$\begin{cases} -(u_0+\tilde{u})'' + c(u_0+\tilde{u}) = f \text{ on }]0,1[, \\ (u_0+\tilde{u})(0) = a \quad \text{and} \quad (u_0+\tilde{u})(1) = b, \end{cases}$$

which is equivalent to

$$\begin{cases} -\tilde{u}'' + c\tilde{u} = f - cu_0 \text{ sur }]0,1[, \\ \tilde{u}(0) = 0 \quad \text{and} \quad \tilde{u}(1) = 0. \end{cases}$$

Likewise, u is a solution to the initial problem iff

$$\begin{cases} -\bar{u}'' + c\bar{u} = f + 2(b-a) - cu_1 \text{ on }]0,1[, \\ \bar{u}(0) = 0 \quad \text{and} \quad \bar{u}(1) = 0. \end{cases}$$

We still have to prove these two **Homogeneous Dirichlet** problems have one unique solution in $C^2([0,1])$. We carry out the seven-step method given during the lecture. For a given $g \in C^0([0,1])$, prove there exists a unique solution $v \in C^2([0,1])$ such that

$$\begin{cases} -v'' + cv = g \text{ on }]0,1[, \\ v(0) = 0 \quad \text{and} \quad v(1) = 0. \end{cases} \tag{V.2}$$

E1 Weak Formulation
Assume the solution v is in $C^2([0,1])$. Let $\phi \in \mathcal{D}(]0,1[)$. Multiplying and integrating over $]0,1[$ and performing an IPP yields

$$\int_{]0,1[}(v'\phi' + cv\phi) = \int_{]0,1[} g\phi.$$

E2 Variational Formulation
We are looking for a solution v that vanishes on the boundary. We choose the space $H_0^1(0,1)$ and the norm $\|\cdot\|_{H_0^1} : x \mapsto \|u'\|_{L^2}$. Endowed with this norm $H_0^1(0,1)$ is a Hilbert space. The variation problem is: Find $v \in H_0^1(0,1)$ such that

$$\forall w \in H_0^1(0,1), \quad a(v,w) = \ell(w)$$

with

$$\begin{cases} a : (v,w) \in (H_0^1)^2 \mapsto \int_{]0,1[}(v'w' + cvw), \\ \ell : w \in H_0^1 \mapsto \int_{]0,1[} gw. \end{cases}$$

E3 Continuity of a and ℓ
Let $(v,w) \in (H_0^1(0,1))^2$. Cauchy-Schwarz yields

$$\begin{aligned}|a(u,v)| &\le \|v'\|_{L^2}\|w'\|_{L^2} + \|c\|_\infty \|v\|_{L^2}\|w\|_{L^2} \\ &\le \|v\|_{H_0^1}\|w\|_{H_0^1} + \|c\|_\infty C_\Omega^2 \|v'\|_{L^2}\|w'\|_{L^2} \\ &\le (1+\|c\|_\infty C_\Omega^2)\|v\|_{H_0^1}\|w\|_{H_0^1}\end{aligned}$$

where C_Ω is the Poincaré constant. The bilnear form a is continuous on $(H_0^1(0,1))^2$.
Furthermore, Cauchy-Schwarz and Poincaré yield:

$$|\ell(w)| \le \|g\|_{L^2}\|w\|_{L^2} \le C_\Omega \|g\|_{L^2}\|w\|_{H_0^1}.$$

The linear form ℓ is continuous on $H_0^1(0,1)$.

E4 Coercivité
Let $v \in H_0^1(0,1)$. Then

$$a(v,v) = \int_{]0,1[} ((v')^2 + cv^2) \ge \|v\|_{H_0^1}^2 + \min_{[0,1]} c \|v\|_{L^2}^2.$$

Since c is non-negative, we have

$$a(v,v) \ge \|v\|_{H_0^1}^2$$

The bilinear form a is coercive on $(H_0^1(0,1))^2$.

Remark 2 (a) The non-negativeness of c is crucial!
(b) The coercivity of a requires[8] to endow $H_0^1(0,1)$ with the norm $\|\cdot\|_{H_0^1}$ and not $\|\cdot\|_{H^1}$. However, if c is positive, choosing the latter norm would have been possible.

E5 Existence and uniqueness of the solution to the vairational formulation
The Lax-Milgram Theorem **??** yields the existence of a unique $v \in H_0^1(0,1)$ such that

$$\forall w \in H_0^1(0,1), \quad a(v,w) = \ell(w).$$

Furthermore, $\ell(v) = a(v,v) \ge \|v\|_{H_0^1(0,1)}^2$, which leads to

$$\|v\|_{H_0^1} \le C_\Omega \|g\|_{L^2}.$$

E6 Solution to the PDE in the sense of distributions
Since $\mathcal{D}(]0,1[) \subset H_0^1(0,1)$, we have

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_{]0,1[} v'\phi' = \langle v', \phi' \rangle = \int_{]0,1[} (-cv+g)\phi = \langle -cv+g, \phi \rangle .$$

[8]Make sure you understand why.

It follows

$$\forall \phi \in \mathcal{D}(]0,1[), \quad -\langle v'', \phi \rangle = \langle -cv + g, \phi \rangle \,,$$

In the sense of distributions, we have $v'' = g - cv$. Moreover, the Minkowsky inequality yields

$$\|v''\|_{L^2} \leq \|g\|_{L^2} + \|c\|_\infty \|v\|_{L^2} \leq (1 + \|c\|_\infty C_\Omega)\|g\|_{L^2}.$$

E7 Regularity of the solution
The Rellich Theorem **??** in dimension 1 states that functions in $H^1(0,1)$ are in $C^0([0,1])$. It follows that $v \in C^0([0,1]$. In turn $g - cv$ belongs to $C^0([0,1]$, so does v''. Consequently, $v \in C^2([0,1])$.

Regarding the uniqueness: let us note that if $g = 0$ then any solution $v \in C^2([0,1])$ to (V.2) satisfies $\|v'\|^2_{L^2} + \|\sqrt{c}\, v\|^2_{L^2} = 0$. Therefore v is constant. Since v vanishes on the boundary, it vanishes everywhere.

Applying the earlier result to $g = f - cu_0$ and $g = f + 2(b-a) - cu_1$ proves the existence and uniqueness of $\tilde{u}$ and $\bar{u}$. Note that $(u_0 + \tilde{u}) - (u_1 - \bar{u})$ is a solution to the homogeneous Dirichlet problem with a zero source term. The uniqueness of the solution provides $(u_0 + \tilde{u}) = (u_1 - \bar{u})$.

Solution of Q. V.1.2 The proof is the same. Any functions for which we impose the values on the boundary will allow us to get a Homogeneous Dirichlet condition.

Remark 3
We could have applied the Stampacchia Theorem on the closed convex set $K = \{v \in H^1(0,1),\ v(0) = a \text{ and } v(1) = b\}$ and the bilnear form a defined above.

Solution of Q. V.2.1 The PDE is elliptic. The boundary condition is of non-homoneneous Neumann type. (It is useful to model a flux through the boundary)

Solution of Q. V.2.2 Since $\tilde{c} = c/\kappa$ and $\tilde{f} = f/\kappa$ are in the same sets as c and f, we can assume[9] $\kappa = 1$ without loss of generality.
We are looking for the weak formulation:
Let us assume[10] $u \in C^2([0,1])$. Let[11] $\phi \in C^1([0,1])$. We multiply both sides of the PDE by ϕ, integrate over $]0,1[$ and perform an integration by parts. We get:

$$-(u'(1)\phi(1) - u'(0)\phi(0)) + \int_0^1 (u'\phi' + cu\phi) = \int_0^1 f\phi.$$

[9] Technically, we divide the equation by κ and work with $-u'' + \tilde{c}u = \tilde{f}$. To simplify the notations we drop the tildes.

[10] We make this assumption to figure out the weak formulation. Once we have found it, all we will need is enough reularity on u so the weak formulation is defined. The way we derived it will not matter.

[11] Note that we do not prescribe values of ϕ on the boundary $\{0;1\}$. Try to see what happens if you take $\phi \in C^1_0([0,1])$. You will get stuck later on in the proof. Try to see where.

Since $u'(0) = \alpha$ and $u'(1) = 0$, the weak formulation is:

$$\int_0^1 (u'\phi' + cu\phi) = \int_0^1 f\phi - \alpha\phi(0).$$

Let $H = H^1(0,1)$, endowed with its usual[12] inner product. Define a on $H \times H$ and ℓ on H by:

$$\begin{cases} a : (u,v) \mapsto \int_{]0,1[}(u'v' + cuv) \\ \ell : v \mapsto \int_{]0,1[} fv - \alpha v(0). \end{cases}$$

When u and v are in $H^1(0,1)$ we have

- $u' \in L^2(0,1)$ and $v' \in L^2(0,1)$ thus $u'v' \in L^1(0,1)$
- $u \in H^1(0,1) \subset L^2(0,1)$ and $v' \in H^1(0,1) \subset L^2(0,1)$ thus $uv \in L^1(0,1)$

consequently $u'v' + uv \in L^1(0,1)$ so a is properly defined.

Similarly, when $v \in H^1(0,1)$, we have $fv \in L^1(0,1)$ since f is continuous. Therefore ℓ is properly defined.

Solution of Q. V.2.3 Our goal is to apply the Lax-Milgram Theorem **??**. Let us check that a, ℓ and H meet the criteria to apply the theorem.

- Continuity: Let $(u,v) \in (H^1(0,1))^2$. Then

  $$|a(u,v)| \leq (1 + \|c\|_\infty)\|u\|_{H^1}\|v\|_{H^1}.$$

  thus a is continuous on $H = H^1(0,1)$.
  Morevoer, Exercice **??** yields

  $$\forall x \in [0,1], \forall w \in H^1(0,1), \quad |w(x)| \leq \sqrt{2}\|w\|_{H^1}.$$

  Thus

  $$|\ell(v)| \leq (\|f\|_{L^2} + \sqrt{2}|\alpha|)\|v\|_{H^1}$$

  which proves the continuity of ℓ in $H = H^1(0,1)$.

- Coercivity: Let $u \in H^1(0,1)$.

  $$a(u,u) = \int_{]0,1[}((u')^2 + cu^2) \geq \|u'\|_{L^2}^2 + \left(\min_{[0,1]} c\right)\|u\|_{L^2}^2 \geq \min(1, \min_{[0,1]} c)\|u\|_{H^1}^2.$$

  Since $c \in C^0([0,1], \mathbb{R}^{+*})$, we have $\min_{[0,1]} c > 0$. Therefore a is coercive.

- H is a Hilbert space.

[12] *i.e.* the inner product defined in Theorem **??**.

The Lax-Milgram Theorem **??** proves there exists a unique $u \in H^1(0,1)$ such that

$$\forall v \in H^1(0,1), \quad \int_{]0,1[} (u'v' + cuv) = \int_{]0,1[} fv.$$

Furthermore

$$\|u\|_{H^1} \leq \frac{\|f\|_{L^2} + \sqrt{2}|\alpha|}{\min(1, \min c)}$$

with $\alpha = 1$.

Solution of Q. V.2.4 We are now going to work with the function u, whose existence and uniqueness have been established in the previous question. We are going to work in the space of distributions $\mathcal{D}'(0,1)$ because we want to have the latitude to differentiate u twice[13]. Let[14] $\phi \in \mathcal{D}(]0,1[)$. We can plug ϕ in lieu of v in the previous equality since $\mathcal{D}(]0,1[) \subset H^1(0,1)$.

$$\int_{]0,1[} (u'\phi' + cu\phi) = \int_{]0,1[} f\phi - \alpha\phi(0) = \int_{]0,1[} f\phi$$

Therefore

$$\left\langle -u'' + cu, \phi \right\rangle = \langle f, \phi \rangle$$

Plugging this in the variational formulation and using the Green's Formula yields:

$$\begin{aligned}
\forall v \in H^1(0,1), \quad 0 &= \int_{]0,1[} (u'v' + cuv - fv) + \alpha v(0) \\
&= [u'(1)v(1) - u'(0)v(0)] + \int_{]0,1[} (-u'' + cu - f)v + \alpha v(0) \\
&= u'(1)v(1) - (u'(0) - \alpha)v(0),
\end{aligned}$$

Since $u \in H^2(0,1)$, the Rellich Theorem **??** gives u' is continuous on $[0,1]$, thus u' is pointwise defined on $[0,1]$. Since the equality above is true for all $v \in H^1(0,1)$ and since $x \mapsto x$ and $x \mapsto 1-x$ both belong to $H^1(0,1)$, we must have $u'(1) = 0$ and $u'(0) = \alpha$.
We proved the solution $u \in H^2(0,1)$ satisfies our iniatial problem. Furthermore, the Minkowsky inequality yields

$$\|u''\|_{L^2} \leq \|c\|_\infty \|u\|_{L^2} + \|f\|_{L^2} \leq \|c\|_\infty \|u\|_{H^1} + \|f\|_{L^2} \leq \frac{\|c\|_\infty}{\min(1, \min_{[0,1]} c)} \left(\|f\|_{L^2} + \sqrt{2}|\alpha| \right) + \|f\|_{L^2}.$$

This proves that u is continuous with respect to f and α (the data).

Note that if f and α are equal to zero then u will be equal to zero as well.

If u_1 and u_2 are two solutions to the initial problem, then $u_1 - u_2$ will be solution to the initial problem where $f = 0$ and $\alpha = 0$. Following our earlier comment, it implies that $u_1 - u_2$ is zero, therefore we

[13] At this point all we know is $u \in H^1(0,1)$ so we have no garantee that u'' is a function. Therefore we **must** work in the space of distributions. If things go well, we will show that u'' is a regular distribution and will conclude that u'' was a function to begin with. Stay tuned!

[14] Since we want to establish an equality in $D'(]0,1[)$, we take ϕ in $\mathcal{D}(]0,1[)$ whose dual space is $\mathcal{D}'(]0,1[)$

have uniqueness in $H^2(0,1)$. We have proved the problem is well-posed in $H^2(0,1)$ and the boundary conditions are met.

Solution of Q. V.2.5 The Lax-Milgram Theorem has an additional part that was not discussed in the lecture: If a is symmetric then u is characterized by

$$\begin{cases} u \in H \\ \dfrac{1}{2}a(u,u) - \langle \phi, u\rangle = \min\limits_{v\in H}\left\{\dfrac{1}{2}a(v,v) - \langle \phi, v\rangle\right\}. \end{cases}$$

In this exercise, a is symmetric, therefore:

$$\begin{cases} u \in H^1(0,1) \\ \dfrac{1}{2}\displaystyle\int_{]0,1[}(u'^2 + cu^2) - \int_{]0,1[} fu + \alpha u(0) = \min_{v\in H^1(0,1)} \dfrac{1}{2}\int_{]0,1[}(v'^2 + cv^2) - \int_{]0,1[} fv + \alpha v(0) \end{cases}$$

Solution of Q. V.3 We proceed as we did before to obtain the variational formulation. It is worth noting that prescribing the boundary condition at $x = 0$ requires to work in the space $H = H^1(0,1) \cap \{v \in C^0([0,1]) \,:\, v(0) = 0\}$.
The point of this exercise is to prove that H is a Hilbert space. This is done by proving that H endowed with the norm $\|\cdot\|_H : v \mapsto \sqrt{(v,v)_H}$ is a Hilbert space because it is a **closed** subspace of $H^1(0,1)$:

- Inner product
  The form $(\cdot,\cdot)_H$ is defined, bilinear, symmetric on H. Furthermore $(v,v)_H = 0$ implies $v' = 0$ in $L^2(\Omega)$. It follows that v is a constant. Since $v(0) = 0$ we conclude that $v = 0$. This proves that $\|\cdot\|_H$ is a norm.
- Equivalence of the norms $\|\cdot\|_H$ and $\|\cdot\|_{H^1}$ on H:
  Obviously $\|\cdot\|_H \le \|\cdot\|_{H^1}$. The counterpart of the Poincaré inéquality is true on H. To prove it, it suffices to adapt the proof of Poincaré, which is easy.
- H is a closed supspace of $H^1(\Omega)$:
  Let us note $\Psi : v \in H^1(0,1) \mapsto v(0)$, we have $H = \Psi^{-1}\langle\{0\}\rangle$. According to **??** and V.2.3, Ψ is a linear continuous form on $H^1(0,1)$. H is a closed subspace of $H^1(0,1)$ since it is the pre-image of a closed set by a linear continuous form.

We then proceed as in the previous exercises, with the variational form

$$\forall v \in H, \quad \int_{]0,1[}(u'v' + cuv) = \int_{]0,1[} fv.$$

Solution of Q. V.4.1 This is an elliptic problem with mixed boundary conditions (Homogeneous Dirichlet and Non-homogenerous Neumann) on a rectangle Ω in dimension $d = 2$, whose oriented boundary $\partial\Omega = \Omega_D \sqcup \Omega_N$ is described by

$$\Omega_N = \underbrace{\{(x,c), \quad a < x < b\}}_{=:S_1} \sqcup \underbrace{\{(b,y), \quad c < y < d\}}_{=:S_2} \sqcup \underbrace{\{(x,d), \quad a < x < b\}}_{=:S_3},$$
$$\Omega_D = \{(a,y), \quad c < y < d\} =: S_4$$

The outward unit normal vector field is defined on $(x,y) \in \partial\Omega \setminus \{(a,c),(b,c),(b,d),(a,d)\}$ and its value is:

$$n = \begin{cases} (0,-1) \text{ on } S_1 \setminus \{(a,c),(b,c)\} \\ (1,0) \text{ on } S_2 \setminus \{(b,c),(b,d)\} \\ (0,1) \text{ on } S_3 \setminus \{(b,d),(a,d)\} \\ (-1,0) \text{ on } S_4 \setminus \{(a,d),(a,c)\}. \end{cases}$$

To find the weak formulation, we proceed as we usually do in dimension $d = 1$. Assume $u \in C^2(\overline{\Omega})$. Let $\phi \in C^1(\overline{\Omega})$ such that $\forall y \in [c,d]$, $\phi(a,y) = 0$ ($\phi|_{\partial\Omega_D} = 0$). We multiply by ϕ and integrate over Ω. The Green's formula yields:

$$-\int_\Omega (\Delta u)\phi = -\int_{\partial\Omega} \phi \nabla u \cdot n + \int_\Omega \nabla u \cdot \nabla \phi,$$

with

$$\begin{aligned}\int_{\partial\Omega} \phi \nabla u \cdot n &= \sum_{i=1}^{4} \int_{S_i} \phi \nabla u \cdot n \\ &= -\int_a^b \phi(x,c)\partial_y u(x,c)\mathrm{d}x + \int_c^d \phi(b,y)\partial_x u(b,y)\mathrm{d}y \\ &\quad + \int_b^a \phi(x,d)\partial_y u(x,d)\mathrm{d}x - \int_d^c \phi(a,y)\partial_x u(a,y)\mathrm{d}y\end{aligned}$$

Using the boundary conditions and the properties of ϕ, we get

$$\int_{\partial\Omega} \phi \nabla u \cdot n = -\int_a^b \phi(x,c)\mathrm{d}x + \int_b^a \phi(x,d)x\mathrm{d}x.$$

Therefore, the weak formulation is

$$\int_\Omega \nabla u \cdot \nabla \phi + \int_a^b \phi(x,c)\mathrm{d}x - \int_b^a x\phi(x,d)\mathrm{d}x = \int_\Omega \phi.$$

The bilinear form to consider is:

$$a : (u,v) \mapsto \int_\Omega \nabla u \cdot \nabla v$$

and the linear form

$$\ell : v \mapsto \int_\Omega v - \int_{S_1} \gamma(v) + \int_{S_3} x\gamma(v),$$

where $\gamma : H^1(\Omega) \to L^2(\partial\Omega)$ is the trace operator. We admit that, for a rectangle, this application is continuous. We will work on the space $H = \{v \in H^1(\Omega) \,:\, \gamma(v)|_{S_4} = 0 \text{ in the } L^2 \text{ sense}\}$.

Solution of Q. V.4.2 Our intent is to use the Lax-Milgram Theorem **??**. We need to verify three points:

(i) H is a Hilbert space:
The trace operator is continuous therefore H is the preimage of a closed set: it is a Hilbert space for the norm $\|\cdot\|_{H^1}$. The Poincaré inéquality is true on H because the traces of functions in H vanish on the boundary. We can endow H with the inner product $(\cdot,\cdot)_H : (u,v) \mapsto (\nabla u, \nabla v)_{L^2}$ whose associated norm is equivalent on H to the H^1-norm. Let us note C the positive constant such that $\forall v \in H$, $\|v\|_{H^1} \leq C\|v\|_H$.

(ii) a and ℓ are continuous :
Let $(u,v) \in H$. Using the Cauchy-Schwarz inequality, we have these two inequalities

$$|a(u,v)| \leq \|\nabla u\|_2 \|\nabla v\|_2 \leq \|\nabla u\|_H \|\nabla v\|_H$$

$$|\ell(v)| \leq \sqrt{(b-a)(d-c)}\|v\|_{L^2} + C_\gamma \|v\|_H \leq (C\sqrt{(b-a)(d-c)} + C_\gamma)\|v\|_H$$

where we have used the continuity of γ (constant C_γ).

(iii) a is coercive: Let $u \in H$. Then
$$a(u,u) = \|\nabla u\|_{L^2}^2 = \|u\|_H^2.$$

The Lax-Milgram Theorem **??** applies: there exists a unique $u \in \mathcal{H}$ such that

$$\forall v \in H, \quad \int_\Omega \nabla u \cdot \nabla v = \int_\Omega v - \int_{S_1} \gamma(v) + \int_{S_3} x\gamma(v).$$

Let $\phi \in \mathcal{D}(\Omega) \subset H$. Then
$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega \phi$$
since the support ϕ is compact, the integral terms on the boundary vanish. Therefore

$$\int_\Omega \nabla u \cdot \nabla \phi = \langle \nabla u, \nabla \phi \rangle = -\langle \Delta u, \phi \rangle = \langle 1, \phi \rangle$$

that is
$$-\Delta u = 1 \quad \text{dans } \mathcal{D}'(\Omega).$$

We have[15] $u \in H^2(\Omega)$. See Remark **??**. The Green's formula allows us to conclde the Neumann conditions are met. The Dirichlet boundary condition is guarenteed from $u \in H$.

Solution of Q. V.5.1 The space is $H = H_0^1(0,1)$ and the bilinear and linear forms are respectively

$$\begin{cases} a : (u,v) \mapsto \int_{]0,1[} (u'v' + quv) \\ \\ \ell : v \mapsto \int_{]0,1[} fv. \end{cases}$$

Solution of Q. V.5.2 We use the Lax-Milgram Theorem **??**.

[15]Note that we can't derive $u \in H^2(\Omega)$ from $-\Delta u \in L^2(\Omega)$ as we would in dimension 1.

Solution of Q. V.5.3 The fonctions $(\phi_k)_{k\geq 1}$ belong to $C^\infty(\mathbb{R})$ and vanish on 0 and 1. Therefore, they are in $H^1_0(0,1)$. To show that $(\phi_k)_{k\in\{1,\dots,m\}}$ is a basis of H_m, it is sufficient to prove that such two functions are orthogonals for the usual inner product of $L^2(0,1)$:

$$\begin{aligned}
\forall (k,l) \in \{1,\dots,m\}, \quad (\phi_k,\phi_l)_{L^2} &= \int_0^1 \sin(k\pi x)\sin(l\pi x)\mathrm{d}x \\
&= \frac{1}{2}\int_0^1 \left(\cos((k-l)\pi x) - \cos((k+l)\pi x)\right)\mathrm{d}x \\
&= \frac{1}{2}\left(\int_0^1 \cos((k-l)\pi x)\mathrm{d}x - \frac{1}{\pi(k+l)}\underbrace{[\sin((k+l)\pi x)]_0^1}_{=0}\right) \\
&= \begin{cases} \dfrac{1}{2}\dfrac{1}{\pi(k-l)}\underbrace{[\sin((k-l)\pi x)]_0^1}_{=0} & \text{if } k\neq l \\ \\ \dfrac{1}{2} & \text{if } k = l \end{cases}
\end{aligned}$$

Therefore $(\phi_k,\phi_l)_{L^2} = 2^{-1}\delta_{kl}$. This set of vectors is linearly independant and it's span is H_m (which is of dimension m).

Solution of Q. V.5.4 The solution u_n is a solution to this linear system

$$\forall i \in \{1,\dots,m\}, \qquad a\left(\sum_{j=1}^J \mathbf{u}_j\phi_j,\phi_i\right) = \sum_{j=1}^J \mathbf{u}_j a\left(\phi_j,\phi_i\right) = \ell(\phi_i).$$

The matrix we are looking for is

$$\begin{aligned}
\forall (i,j) \in \{1,\dots,m\}^2, \qquad [A_m]_{ij} &= \int_0^1 \phi_i'\phi_j' + q\int_0^1 \phi_i\phi_j \\
&= -\int_0^1 \phi_i''\phi_j + \frac{q}{2}\delta_{ij} = \frac{(i\pi)^2+q}{2}\delta_{ij}.
\end{aligned}$$

Subsequently, A_m is a diagonal matrix.

Solution of Q. V.5.5 The conclusion is

$$\begin{aligned}
\forall k \in \{1,\dots,m\}, \qquad \mathbf{u}_k &= \frac{\ell(\phi_k)}{[A_m]_{kk}} \\
&= 2\frac{\int_0^1 f\phi_k}{(k\pi)^2+q}.
\end{aligned}$$

Solution of Q. V.5.6

If $(w_k)_{k\geq 1}$ exists, then $-w_k'' = \lambda_k w$ in the sense of distributions, $w_k(0) = w_k(1) = 0$ and $w_k \in H^2(0,1) \cap H_0^1(0,1)$. Now, if $\lambda_k \leq 0$, the problem $-w_k'' = \lambda_k w$, $w(0) = w(1)$ can only have zero as a solution. This imposes $\lambda_k > 0$. Moreover, the solutions are linear combinations of $\sin(\sqrt{\lambda_k}\cdot)$ and $\cos(\sqrt{\lambda_k}\cdot)$: there exist $(\alpha, \beta) \in \mathbb{R}^2$ such that $w = \alpha \sin(\sqrt{\lambda_k}\cdot) + \beta \cos(\sqrt{\lambda_k}\cdot)$. The boundary conditions impose that $\beta = 0$. If we look for a solution that is not zero, we need $\sin(\sqrt{\lambda_k}) = 0$, which is equivalent to $\sqrt{\lambda_k} \in \pi\mathbb{N}$. The set $w_k = \sqrt{2}\phi_k$, $k \geq 1$ is a Hilbertian basis provided it is total. It is total thanks to the Weierstraßtheorem.

Solution of Q. V.5.7 In the previous question, we proved we can take $w_k = \pm\sqrt{2}\phi_k$ for all $k \geq 1$. From now on, we will choose $w_k = \sqrt{2}\phi_k$.

Solution of Q. V.5.8 Let $m \geq 1$. We use that

$$u = \frac{1}{\sqrt{2}} \sum_{k\geq 1} \mathbf{u}_k w_k.$$

Since $(w_k)_{k\geq 1}$ is a Hilbertian basis

$$\|u - u_m\|^2_{L^2(0,1)} = \frac{1}{2} \sum_{k\geq m+1} \mathbf{u}_k^2 \leq \frac{2}{(((m+1)\pi)^2 + q)^2} \left(\sum_{k\geq m+1} \int_0^1 f \sin(k\pi\cdot) \right)^2.$$

According to the Parseval theorem, if f is being extended as an odd function on $[-1, 1]$ and then using the 2-periodicity, its Fourier's series holds only sin and we have:

$$\|f\|^2_{L^2} = \frac{1}{2} \sum_{k\geq 1} (f, \sin(k\pi\cdot))^2.$$

It follows that

$$\|u - u_m\|^2_{L^2(0,1)} \leq \frac{1}{(((m+1)\pi)^2 + q)^2} \|f\|^2_{L^2(0,1)^2}.$$

We obtain $\|u - u_m\|^2_{L^2} \underset{m\to+\infty}{\longrightarrow} 0$.