Lecture V : Finite Element Methods II

A) Aims of this class

After this class,

- I know how to get a mesh of a domain with a software in dimension 2.
- I can define the finite element method $\mathbb{P}_1$ in dimension 2.
- I can assemble the stiffness matrix.
- I can program in FEniCS the solution of an elliptic problem in dimension 2.

B) To become familiar with this class' concepts (to prepare before the examples class)

Question V.1 must be done before the 7th lab. The solution is available online.

Warning ! You must attend the class with your computer.

Question V.1
Let Ω be an open polyhedral bounded subset of $\mathbb{R}^2$. We want to solve approximately the Dirichlet problem over Ω.

Q. V.1.1 Recall the statement of the Dirichlet problem in dimension 2.

Q. V.1.2 For a given triangular mesh $\mathcal{T}$ of Ω, describe the finite element method $\mathbb{P}_1$: type of problem, polynomials to use, and so on.

Consider a given triangle K and let a_1, a_2, a_3 be its vertices. For $j \in \{1,2,3\}$, denote λ_j the barycentric coordinate associated with the vertex a_j. We already know that these 3 functions $(\lambda_j)_{j\in\{1,2,3\}}$ is a basis of the polynomial space $\mathbb{P}_1$.

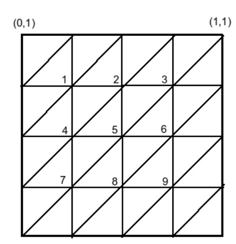
Q. V.1.3 Let $M \in K$ with coordinates (x, y). For $j \in \{a, b, c\}$, give the expression of the barycentric coordinate λ_j depending on x and on y.

Define the elementary stiffness matrix $\mathcal{A}$ associated with the triangle K as the symmetric 9-element matrix: $\quad a_{ij} = \int_K \nabla\lambda_i \cdot \nabla\lambda_j \mathrm{d}x\mathrm{d}y.$

Q. V.1.4 Let $h > 0$. Compute $\mathcal{A}$ for the triangle of vertices $(0,0)$, $(0,-h)$, $(h,0)$.

Q. V.1.5 Let $h > 0$. Compute $\mathcal{A}$ for the triangle of vertices $(0,0)$, $(-h,-h)$, $(0,-h)$.

Q. V.1.6 Compute the stiffness matrix for the Dirichlet problem in the square $]0,1[\times]0,1[$ equipped with the following mesh.



C) Exercises

The exercises can be found on edunao as Jupyter notebooks.

Chapter VII: Solutions

Solution de Q. V.1.3 The barycentric coordinates $(\lambda_j)_{j\in\{a,b,c\}}$ verify

$$\begin{pmatrix} x \\ y \\ 1 \end{pmatrix} = \lambda_a \begin{pmatrix} x_a \\ y_a \\ 1 \end{pmatrix} + \lambda_b \begin{pmatrix} x_b \\ y_b \\ 1 \end{pmatrix} + \lambda_c \begin{pmatrix} x_c \\ y_c \\ 1 \end{pmatrix}$$

which is the linear system

$$\begin{pmatrix} x_a & x_b & x_c \\ y_a & y_b & y_c \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \lambda_a \\ \lambda_b \\ \lambda_c \end{pmatrix} = \begin{pmatrix} x \\ y \\ 1 \end{pmatrix}.$$

On trouve alors

$$\begin{aligned} \lambda_a &= \frac{(x-x_c)(y_b-y_c)-(x_b-x_c)(y-y_c)}{(x_a-x_c)(y_b-y_c)-(x_b-x_c)(y_a-y_c)} \\ \lambda_b &= \frac{(x_a-x_c)(y-y_c)-(x-x_c)(y_a-y_c)}{(x_a-x_c)(y_b-y_c)-(x_b-x_c)(y_a-y_c)} \\ \lambda_c &= 1-\lambda_a-\lambda_b. \end{aligned}$$

Solution de Q. V.1.4 Let us number the vertices $S_1=(0,0)$, $S_4=(0,-h)$, $S_2=(h,0)$ and let us note $T^I=(S_1,S_4,S_2)$,

$$\begin{aligned} \lambda_1^I &= 1-(x-y)/h, & \nabla\lambda_1^I &= (1/h)(-1,1)^T \\ \lambda_4^I &= -y/h, & \nabla\lambda_4^I &= (1/h)(0,-1)^T \\ \lambda_2^I &= x/h, & \nabla\lambda_2^I &= (1/h)(1,0)^T \end{aligned}$$

The matrix can be written

$$\mathcal{A}^I = \frac{1}{h^2}\frac{h^2}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 2 & -1 & -1 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Solution de Q. V.1.5 Let us number the vertices $S_2=(0,0)$, $S_4=(-h,-h)$, $S_5=(0,-h)$ and note $T^{II}=(S_2,S_4,S_5)$,

$$\begin{aligned} \lambda_2^{II} &= 1+y/h, & \nabla\lambda_2^{II} &= (1/h)(0,1)^T \\ \lambda_4^{II} &= -x/h, & \nabla\lambda_4^{II} &= (1/h)(-1,0)^T \\ \lambda_5^{II} &= (x-y)/h, & \nabla\lambda_5^{II} &= (1/h)(1,-1)^T \end{aligned}$$

The matrix can be written

$$\mathcal{A}^{II} = \frac{1}{h^2}\frac{h^2}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{pmatrix}$$

Solution de Q. V.1.6 Let us characterize the basis functions at the interior vertices:

$$\Phi_1 = \lambda_1^{(142)}, \nabla\Phi_1 = (1/h)(-1,1)^T\mathbf{1}_{(142)}$$

$$\Phi_2 = \lambda_2^{(142)}\mathbf{1}_{(142)} + \lambda_2^{(245)}\mathbf{1}_{(245)} + \lambda_2^{(253)}\mathbf{1}_{(253)}, \nabla\Phi_2 = (1/h)((1,0)^T\mathbf{1}_{(142)} + (0,1)^T\mathbf{1}_{(245)} + (-1,1)^T\mathbf{1}_{(253)})$$

$$etc.$$

We work a triangle at a time, then we tally up. Eventually, the stiffness matrix is

$$\mathcal{A} = \begin{pmatrix} 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 4 & -1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 4 & 0 & 0 & -1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 4 & -1 & 0 & -1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 4 & -1 & 0 & -1 & 0 \\ 0 & 0 & -1 & 0 & -1 & 4 & 0 & 0 & -1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 4 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 & -1 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 4 \end{pmatrix}$$