Exercise IV.1

For every $c \in \mathbb{R}^N$, give the expression of the conjugate of:

$$f : \mathbb{R}^N \to]-\infty, +\infty], \quad x \mapsto \langle c|x\rangle. \tag{1}$$

Give also the expression for the function:

$$g : \mathbb{R}^N \to]-\infty, +\infty], \quad x \mapsto \iota_{[0,+\infty[^N}(x - c). \tag{2}$$

Solution: We start by using the definition of the Fenchel conjugate, which gives us:

$$\begin{aligned} f^*(u) &= \sup_{x\in\mathbb{R}^N} \langle x|u\rangle - f(x) = \sup_{x\in\mathbb{R}^N} \langle x|u\rangle - \langle x|c\rangle \\ &= \sup_{x\in\mathbb{R}^N} \langle x|u - c\rangle \end{aligned} \tag{3}$$

We now consider the following two cases:

- $u = c$: In this case we have that $f^*(u) = 0$.
- $u \neq c$: In this case among all vectors of the same value of the norm λ, the inner product $\langle x|u - c\rangle$ is maximized if x is a scalar multiple of $u - c$. In this case, if we let $\|x\| \to \infty$, we obtain that $\sup\langle x|u - c\rangle = +\infty$.

As a result, we obtain that:

$$f^*(u) = \begin{cases} 0, & \text{if } u = c \\ +\infty, & \text{otherwise.} \end{cases} \tag{4}$$

We now move to the second function. Before examining it, we will start by considering the univariate fuction $\iota_{[0,+\infty[}(x)$. For this function, the conjugate is defined as:

$$\iota^*_{[0,+\infty[}(u) = \sup_{x\in\mathbb{R}}\langle u|x\rangle - \iota_{[0,+\infty[}(x). \tag{5}$$

However, we recall from the course slides that for an indicator function $\iota_C(x)$ the conjugate is expressed as:

$$\iota^*_C(u) = \sup_{x\in C}\langle u|x\rangle. \tag{6}$$

In our case, the set C is defined as the set $C = [0, +\infty[$. As a result we have that:

$$\iota^*_C(u) = \sup_{x\in[0,+\infty[} \langle u|x\rangle = \begin{cases} 0, & u \leq 0, \\ +\infty, & u > 0. \end{cases} \tag{7}$$

As a result, we have that:

$$\iota^*_{[0,+\infty[}(u) = \iota_{]-\infty,0]}(u) \tag{8}$$

Let us now consider again $g(x)$. By applying the definition of the conjugate, we have that:

$$
\begin{aligned}
g^\star(u) &= \sup_{x\in\mathbb{R}^N} \langle x|u\rangle - \iota_{[0,+\infty[^N}(x-c) = \sup_{x\in\mathbb{R}^N} \langle x+c|u\rangle - \iota_{[0,+\infty[^N}(x) \\
&= \langle c|u\rangle + \iota^*_{[0,+\infty[^N}(u) = \langle c|u\rangle + \sum_{i=1}^N \iota^*_{[0,+\infty[}\left(u^{(i)}\right) \\
&= \langle c|u\rangle + \sum_{i=1}^N \iota^*_{]-\infty,0]}\left(u^{(i)}\right) = \langle c|u\rangle + \iota_{]-\infty,0]^N}\left(u^{(i)}\right),
\end{aligned}
\tag{9}
$$

where we have used the fact that $\iota_{[0,+\infty[^N}(x)$ is a sum of separable functions and can be written in the form:

$$
\iota_{[0,+\infty[^N}(x) = \sum_{i=1}^N \iota_{[0,+\infty[}(x^{(i)}). \tag{10}
$$