Exercise III.4

Let $f : \mathbb{R}^2 \to \mathbb{R}$:

$$(x, y) \mapsto \ln\left(\exp(x) + \exp(x)\right). \tag{1}$$

Show that f is convex. Is it strictly convex?

Solution: We start by calculating the gradient and the Hessian matrix for the given problem. These are expressed as follows:

$$\nabla f\left(x, y\right) = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} \frac{e^x}{e^x + e^y} \\ \frac{e^y}{e^x + e^y} \end{bmatrix} \tag{2}$$

and

$$\nabla^2 f\left(x, y\right) = \frac{e^{x+y}}{\left(e^x + e^y\right)^2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{e^{x+y}}{\left(e^x + e^y\right)^2} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \left[1 - 1\right]. \tag{3}$$

Due to this last form of the Hessian matrix, it is evident that the Hessian matrix is positive semidefinite. As, a result, for any $z \in \mathbb{R}^2$ we obtain that:

$$\langle z | \nabla^2 f(x, y) z \rangle \geq 0, \forall \ \ (x, y) \in \mathbb{R}^2. \tag{4}$$

As a result, $f\left(x, y\right)$ is convex.

We now focus on points of the form (x, x). For such points, we have that:

$$f(x, x) = x + \ln 2. \tag{5}$$

As a result we see that when moving on the the direction of the form (x, x), the function becomes affine. As a result, it cannot be strictly convex.

Remark: In this example we see that when we are given a function defined over a subest of $\mathbb{R}^N$ (or over $\mathbb{R}^N$), proving convexity is equivalent to proving that the Hessian is positive semidefinite over the considered subset of $\mathbb{R}^N$ (or over $\mathbb{R}^N$). Revise your linear algebra notes in order to remember how to prove that a matrix is positive semidefinite!!