# Plancherel theorem

In mathematics, the **Plancherel theorem** (sometimes called the Parseval-Plancherel identity[1]) is a result in harmonic analysis, proven by Michel Plancherel in 1910. It states that the integral of a function's squared modulus is equal to the integral of the squared modulus of its frequency spectrum. That is, if $f(x)$ is a function on the real line, and $\widehat{f}(\xi)$ is its frequency spectrum, then

$$\int_{-\infty}^{\infty} |f(x)|^2 \, dx = \int_{-\infty}^{\infty} |\widehat{f}(\xi)|^2 \, d\xi \qquad\qquad \textbf{(Eq.1)}$$

A more precise formulation is that if a function is in both Lp spaces $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, then its Fourier transform is in $L^2(\mathbb{R})$, and the Fourier transform map is an isometry with respect to the $L^2$ norm. This implies that the Fourier transform map restricted to $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ has a unique extension to a linear isometric map $L^2(\mathbb{R}) \mapsto L^2(\mathbb{R})$, sometimes called the Plancherel transform. This isometry is actually a unitary map. In effect, this makes it possible to speak of Fourier transforms of quadratically integrable functions.

Plancherel's theorem remains valid as stated on $n$-dimensional Euclidean space $\mathbb{R}^n$. The theorem also holds more generally in locally compact abelian groups. There is also a version of the Plancherel theorem which makes sense for non-commutative locally compact groups satisfying certain technical assumptions. This is the subject of non-commutative harmonic analysis.

The unitarity of the Fourier transform is often called Parseval's theorem in science and engineering fields, based on an earlier (but less general) result that was used to prove the unitarity of the Fourier series.

Due to the polarization identity, one can also apply Plancherel's theorem to the $L^2(\mathbb{R})$ inner product of two functions. That is, if $f(x)$ and $g(x)$ are two $L^2(\mathbb{R})$ functions, and $\mathcal{P}$ denotes the Plancherel transform, then

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx = \int_{-\infty}^{\infty} (\mathcal{P}f)(\xi)\overline{(\mathcal{P}g)(\xi)} \, d\xi,$$

and if $f(x)$ and $g(x)$ are furthermore $L^1(\mathbb{R})$ functions, then

$$(\mathcal{P}f)(\xi) = \widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x)e^{-2\pi i\xi x} \, dx,$$

and

$$(\mathcal{P}g)(\xi) = \hat{g}(\xi) = \int_{-\infty}^{\infty} g(x)e^{-2\pi i\xi x} \, dx,$$

so

$$\int_{-\infty}^{\infty} f(x)\overline{g(x)} \, dx = \int_{-\infty}^{\infty} \widehat{f}(\xi)\overline{\hat{g}(\xi)} \, d\xi. \qquad\qquad \textbf{(Eq.2)}$$

## See also

- Plancherel theorem for spherical functions

## References

1. Cohen-Tannoudji, Claude; Dupont-Roc, Jacques; Grynberg, Gilbert. *Photons and Atoms : Introduction to Quantum Electrodynamics*. Wiley. p. 11. ISBN 0-471-18433-0.

- Plancherel, Michel; Mittag-Leffler (1910), "Contribution à l'étude de la représentation d'une fonction arbitraire par les intégrales définies", *Rendiconti del Circolo Matematico di Palermo*, **30** (1): 289–335, doi:10.1007/BF03014877 (https://doi.org/10.1007%2FBF03014877).
- Dixmier, J. (1969), *Les C*-algèbres et leurs Représentations*, Gauthier Villars.
- Yosida, K. (1968), *Functional Analysis*, Springer Verlag.

## External links

- Hazewinkel, Michiel, ed. (2001) [1994], "Plancherel theorem" (https://www.encyclopediaofmath.org/index.php?title=p/p072770), *Encyclopedia of Mathematics*, Springer Science+Business Media B.V. / Kluwer Academic Publishers, ISBN 978-1-55608-010-4
- Plancherel's Theorem (http://mathworld.wolfram.com/PlancherelsTheorem.html) on Mathworld