Exercise 3

A financial company can invest money in N different assets. The i-th asset requires an investment of q_i M€ and generates a profit of p_i M€.

An overall amount of Q M€ can be invested.

1. Which kind of optimization problem needs to be solved? Formulate it.
2. Solve the problem for $N=4$, $(q_i)_{1\le i\le N} = [3\ 6\ 8\ 7]$ $(p_i)_{1\le i\le N} = [15\ 24\ 14\ 13]$ and $Q=12$.

Solution

1) We introduce one decision variable $x^{(i)}$ per asset.
More strictly, we define $x^{(i)}$ as

$$x^{(i)} = \begin{cases} 1 & \text{if we decide to invest on the i-th placement} \\ 0 & \text{otherwise.} \end{cases}$$

The problem is expressed as: minimize: $\sum_{i=1}^{N} p_i x^{(i)}$, s.t. $\sum_{i=1}^{N} q_i x^{(i)} \le Q$

2) We can ~~then~~ formulate the problem as:

$$\text{minimize} \quad 15x^{(1)} - 25x^{(2)} - 14x^{(3)} - 13x^{(4)}$$

$$x \in \{0,1\}^4$$

$$\text{subjec to:} \quad 3x^{(1)} + 6x^{(3)} + 8x^{(3)} + 7x^{(4)} \le 12$$

which is an integer (binary) linear programming problem and can be solved using the branch and bound method.

We now consider ~~...~~ $S = [0,1]^4$ (8)
(i.e. we relax the integer constraint)
By solving the resulting linear programming problem, we obtain $\hat{x}_{0,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0.4286 \end{bmatrix}$ and a value $\underline{p}_{0,1} = -45.57$

By dicretizing this solution, we can obtain the feasible point $\tilde{x}_{0,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$, which gives an upper bound for p_0 equal to $\bar{p}_{0,1} = \bar{p}_0 = -40$

We now proceed to the next iteration and we split $S_{0,1}$ to:

$$S_{1,1} = \{x \in [0,1]^4 \mid x^{(4)} = 0\}$$

and $S_{1,2} = \{x \in [0,1]^4 \mid x^{(4)} = 1.$

Solving the relaxed version of our problem, for $x \in S_{1,1}$:

$$\underset{x \in S_{1,1}}{\text{minimize}} \quad 15x^{(1)} - 25x^{(2)} - 14x^{(3)} - 13x^{(4)}$$

$$\text{s.t.} \quad 3x^{(1)} + 6x^{(2)} + 8x^{(3)} + 7x^{(4)} \leq 12$$

we obtain $\hat{x}_{1,1} = \begin{bmatrix} 1 \\ 1 \\ 0.375 \\ 0 \end{bmatrix}$ $\underline{p}_{1,1} = -45.25$

and $\tilde{x}_{1,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \bar{p}_{1,1} = \bar{p}_0 = -40.$

Solving the relaxed version of the initial problem, with the additional constraint $x \in S_{1,2}$

gives us

$$\hat{x}_{1,2} = \begin{bmatrix} 1 \\ 1/3 \\ 0 \\ 1 \end{bmatrix}, \quad \mu_{1,2} \simeq -36.33$$

$$\bar{x}_{1,2} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} \quad \bar{\mu}_{1,2} = -28.$$

$\rightsquigarrow$ We now proceed to the next iteration

We separate $S_{1,1}$ (~~it has the lowest upper bound~~)

and create:

$$S_{2,1} = \{x \in [0,1]^4 \mid x^{(4)} = 0, x^{(3)} = 0\}$$
$$S_{2,2} = \{x \in [0,1]^4 \mid x^{(4)} = 0, x^{(3)} = 1\}$$

By solving the relaxed version of the initial problem, subject to the additional constraint $x \in S_{2,1}$

we obtain the integer solution

$$\hat{x}_{2,1} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \tilde{x}_{2,1}, \quad \mu_{2,1} = -40 = \bar{\mu}_{2,1}$$

By solving the relaxed version of the initial problem subject to the additional constraint $x \in S_{22}$

we obtain $\hat{x}_{2,2} = \begin{bmatrix} 1 \\ 1/6 \\ 1 \\ 0 \end{bmatrix}$, $\mu_{2,2} = -33.16 > \bar{\mu}_{2,1}$.

As a result since $\hat{x}_{2,1}$ is integer and $\mu_{21} < \mu_{22}$,

we obtain that we can rule out S_{22}.

We also note now that

$p_{2,1} < p_{1,2}$.

Hence we can also rule out S_{12}.

As a result, the optimal solution is obtained at $\hat{x}_{2,1}$