Lecture VI : Finite Element Methods I

A) Aims of this class

After this class,

- I know the principle of an internal variational approximation (the Céa lemma, the interpolation operator).
- I can define the finite element method $\mathbb{P}_1$ in dimension 1.
- I can calculate the stiffness matrix.
- I installed and tested the Python platform FEniCS.
- I can program in Python or Matlab or FEniCS the approximate resolution of an elliptic problem in dimension 1.
- I know how to determine the order of the method numerically.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions VI.1, VI.2 and VI.3 must be done before the 6th lab. The solutions are available online.

Warning ! The platform FEniCS must be installed and tested on your computer before the 6th lab. The instructions to install it are on edunao. You must attend the class with your computer.

Question VI.1
Let $f \in L^2(0,1)$. Apply the $\mathbb{P}_1$ method to the problem

$$\text{(E)} \quad \begin{cases} -u'' = f & \text{in }]0,1[, \\ u(0) = a \quad \text{and} \quad u(1) = b. \end{cases}$$

with $a, b \in \mathbb{R}$. Check that the nonhomogeneous Dirichlet boundary conditions appear in the right-hand side of the linear system that we got.

Question VI.2

Consider the problem $$\text{(E)} \quad \begin{cases} -u'' + bu' + cu = f & \text{in }]0,1[, \\ u(0) = 0 \quad \text{and} \quad u(1) = 0. \end{cases}$$

The functions f and (b, c) are given respectively in $L^2(0,1)$ and in $C^0([0,1])^2$. We discretize $[0,1]$ uniformly in $J+1$ intervals, $J \geq 1$.

Q. VI.2.1 Write (E) as a symmetric variational problem (PV). Give a sufficient condition on $c(\cdot)$ so that there exists a unique solution in a well-chosen Hilbert space H.

Let $H_h \subset H$, the vector subspace of finite dimension J defined as

$$H_h = \{v \in H \ / \ v_{|[x_j, x_{j+1}]} \in \mathbb{P}_1, j \in \{0, \cdots, J\}, \quad \text{and} \quad v(0) = v(1) = 0\}.$$

Q. VI.2.2 Give the linear system associated with the approximated variational problem in the basis of "hat-functions" :

$$(\text{PV}_h) \quad \text{Find } u_h \in H_h \text{ such that } \quad \forall\, v_h \in H, \quad a(u_h, v_h) = \ell(v_h),$$

where $a(\cdot, \cdot)$ (resp. $\ell(\cdot)$) is the bilinear (resp. linear) form that appears in (PV).

Question VI.3
Consider the equation of Problem (E) with Neumann boundary conditions:

$$u'(0) = 0 \quad \text{and} \quad u'(1) = 0.$$

Q. VI.3.1 Define H and give a sufficient condition on c to ensure that the new problem admits a unique solution.

Q. VI.3.2 Define H_h and the associated linear system.

C) Exercises

The exercises can be found on edunao as Jupyter notebooks.

Chapter V: Solutions

Solution of Q. VI.1 Following the lifting technique, it suffices to choose a function u_0 that takes the value a at 0 and b at 1, and then to look for a solution to the homogeneous problem

$$(\mathrm{E}_0)\quad \begin{cases} -\tilde{u}'' = g & \text{in }]0,1[, \\ \tilde{u}(0) = 0 \quad \text{and} \quad \tilde{u}(1) = 0. \end{cases}$$

with $g : x \mapsto f(x) + u_0''(x)$.
Let us implement the $\mathbb{P}_1$ method to the problem (E_0) (we omit the ˜= to make the notations simpler). We already established the well posedness of the problem in $H^2(0,1) \cap H_0^1(0,1)$ and the associated variational formulation:

Find $u \in H_0^1(0,1)$ such that

$$\forall v \in H_0^1(0,1), \qquad \int_{]0,1[} u'v' = \int_{]0,1[} gv.$$

Here are the steps **in practice** that compose the $\mathbb{P}_1$ method:

(i) Let $J \geq 1$. We define the following discretization $(x_j)_{j\in\{0,\dots,J+1\}}$, with $x_0 = 0$ and $x_{J+1} = 1$, of $[0,1]$. Let $h = \max_{j\in\{0,\dots,J\}}(x_{j+1} - x_j)$.

(ii) We choose the approximation space $H_{0,h} \subset H_0^1(0,1)$, with basis $(\phi_j)_{j\in\{1,\dots,J\}}$.

(iii) We write the variational approximation:

Find $u_h \in H_{0,h}$ such that

$$\forall v_h \in H_{0,h}, \qquad \int_{]0,1[} u_h'v_h' = \int_{]0,1[} gv_h.$$

(iv) We write this variational approximation in the basis $(\phi_j)_{j\in\{1,\dots,J\}}$ (since the problem is linear, it is equivalent) and the problem becomes:

Find $u_h = \sum_{j=1}^J \mathbf{u}_j\phi_j \in H_{0,h}$ such that

$$\forall i \in \{1,\dots,J\}, \qquad \sum_{j=1}^J \int_{]0,1[} \mathbf{u}_j\phi_j'\phi_i' = \int_{]0,1[} g\phi_i.$$

(v) it is a linear system with stiffness matrix $\mathbf{A}_h \in \mathcal{M}_J(\mathbb{R})$

$$\forall (i,j) \in \{1,\dots,J\}^2, \qquad [\mathbf{A}_h](i,j) = \int_{]0,1[} \phi_i'\phi_j'.$$

(vi) We need to evaluate the right-hand side:

$$\forall i \in \{1,\dots,J\}, \qquad [\mathbf{b}_h](i) = \int_{]0,1[} g\phi_i = \int_{]x_{i-1},x_{i+1}[} g\phi_i = \int_{]x_{i-1},x_{i+1}[} f\phi_i + \int_{]x_{i-1},x_{i+1}[} u_0''\phi_i.$$

Indeed, the boundary conditions appear as u_0''.

So we find the vector $U_h = (\mathbf{u}_1, \dots, \mathbf{u}_J)^T \in \mathbb{R}^J$. The Finite Element solution that we sought is continuous, piecewise linear

$$a\phi_0 + \sum_{j=1}^{J} (\mathbf{u}_j + u_0(x_j))\phi_j + b\phi_{J+1}.$$

Solution of Q. VI.2.1 Multiply the equation by the nonvanishing function $\zeta : x \mapsto \exp(-\int_0^x b)$: we get

$$-(\zeta u')' + c\zeta u = \zeta f.$$

Using again the resolution techniques, we get the variational formulation: *find* $u \in H_0^1(0,1)$ *such that*

$$\forall v \in H_0^1(0,1), \qquad \int_{]0,1[} \zeta u' v' + \int_{]0,1[} c\zeta u v = \int_{]0,1[} \zeta f v,$$

the associated bilinear and linear forms being

$$\begin{cases} a : (u,v) \mapsto \int_{]0,1[} \zeta u' v' + \int_{]0,1[} c\zeta u v \\ \\ \ell : v \mapsto \int_{]0,1[} \zeta f v. \end{cases}$$

A sufficient condition to apply the Lax-Milgram theorem is that the function c has nonnegative values (in that case, ζf also has and, indeed, one shows that, thanks to the Poincaré inequality, a is coercive in $H_0^1(0,1)$). One deduces that there exists a unique solution $u \in H_0^1(0,1)$. Applying the resolution techniques, one shows that $u \in H^2(0,1)$ and that, if $f \in C^0([0,1])$, $u \in C^2([0,1])$.

Solution of Q. VI.2.2 Recall that H_h is of dimension the number of degrees of freedom, which is the number of interior points, that is, J.
The benefit of having symmetrized the variational formulation resides in the fact that the linear system is symmetric positive definite. Because of the small size of the support of the hat-functions, the matrix is tridiagonal.

We look for $u_h = \sum_{j=1}^{J} u_j \phi_j \in H_h$ solution of

$$\forall i \in \{1, \dots, J\}, \qquad a\left(\sum_{j=1}^{J} u_j \phi_j, \phi_i\right) = \ell(\phi_i).$$

So the matrix is

$$\forall (i,j) \in \{1,\ldots,J\}^2, \qquad [A_h]_{ij} = \int_{]0,1[} \zeta \phi_i' \phi_j' + \int_{]0,1[} \zeta c \phi_i \phi_j$$
$$= \int_{]x_{i-1},x_{i+1}[\cap]x_{j-1},x_{j+1}[} \zeta \phi_i' \phi_j' + \int_{]x_{i-1},x_{i+1}[\cap]x_{j-1},x_{j+1}[} \zeta c \phi_i \phi_j$$
$$= \begin{cases} 0 & \text{if } |i-j| \geq 2 \\ -\dfrac{1}{h^2} \int_{]x_{i-1},x_i[} \zeta + \int_{]x_{i-1},x_i[} \zeta c \phi_i \phi_{i-1} & \text{if } i = j+1 \\ \dfrac{1}{h^2} \int_{]x_{i-1},x_{i+1}[} \zeta + \int_{]x_{i-1},x_{i+1}[} \zeta c \phi_i^2 & \text{if } i = j. \end{cases}$$

The righthand side is then

$$\forall i \in \{1,\ldots,J\}, \quad [F_h]_i = \int_{]0,1[} \zeta f \phi_i = \int_{]x_{i-1},x_{i+1}[} \zeta f \phi_i.$$

These terms will be computed by quadrature (rectangle formulas, trapezoid formula, ...).

Solution of Q. VI.3.1 Now, we need to consider $H = H^1(0,1)$.

The bilinear form is the one from question VI.2.1. A sufficient condition to guarantee the coercivity of a on $H^1(0,1)^2$ is that c be positive.

Since the Neumann boundary condition is homogeneous, we can take the linear form that appears in question VI.2.1.

Solution of Q. VI.3.2 For H_h, we will let the boundary points to be "free":

$$H_h := \{v \in H \;:\; v_{|[x_j,x_{j+1}]} \in \mathbb{P}_1, j \in \{0,\cdots,J\}\}.$$

The dimension of this space is $J+2$. It corresponds to the number of nodes in the mesh. The matrix of the linear system is

$$\forall (i,j) \in \{0,\ldots,J+1\}^2, \qquad [A_h]_{ij} = \begin{cases} 0 & \text{if } |i-j| \geq 2, \quad i,j \in \{1,\ldots,J\} \\ -\dfrac{1}{h^2} \int_{]x_{i-1},x_i[} \zeta + \int_{]x_{i-1},x_i[} \zeta c \phi_i \phi_{i-1} & \text{if } i = j+1, \quad j \in \{0,\ldots,J\} \\ \dfrac{1}{h^2} \int_{]x_{i-1},x_{i+1}[} \zeta + \int_{]x_{i-1},x_{i+1}[} \zeta c \phi_i^2 & \text{if } i = j, \quad i,j \in \{1,\ldots,J\} \\ \dfrac{1}{h^2} \int_{]0,x_1[} \zeta + \int_{]0,x_1[} \zeta c \phi_0^2 & \text{if } i = j = 0 \\ \dfrac{1}{h^2} \int_{]x_J,1[} \zeta + \int_{]x_J,1[} \zeta c \phi_J^2 & \text{if } i = j = J+1 \end{cases}$$

The right hand side is

$$\forall i \in \{0, \ldots, J+1\}, \quad [F_h]_i = \begin{cases} \int_{]x_{i-1}, x_{i+1}[} \zeta f \phi_i, & \text{if } i \in \{1, \ldots, J\} \\ \int_{]0, x_1[} \zeta f \phi_0, & \text{if } i = 0 \\ \int_{]x_J, 1[} \zeta f \phi_J. & \text{if } i = J+1 \end{cases}$$