III.3. Sobolev Spaces

III.3.1. Definitions and Basic Properties

Definition of $H^1(\mathcal{I})$

Definition III.3.1

The Sobolev space of order 1 on $\mathcal{I}$ is defined by

$$\begin{aligned} H^1(\mathcal{I}) &:= \left\{ v \in L^2(\mathcal{I}) \ : \quad (\mathcal{T}_v)' \in L^2(\mathcal{I}) \right\} \\ &:= \left\{ v \in L^2(\mathcal{I}) \ : \quad v' \in L^2(\mathcal{I}) \right\} \end{aligned}$$

where v' is the distributional derivative of v.

If $\mathcal{I} =]a, b[$ is a bounded interval, we note $H^1(\mathcal{I}) = H^1(a, b)$.

The $H^1(\mathcal{I})$ Hilbert space

Theorem III.3.2

The space $H^1(\mathcal{I})$ endowed with the inner product

$$(\cdot,\cdot)_{H^1} : \langle u, v\rangle \mapsto (u,v)_{L^2} + \langle u', v'\rangle_{L^2} \, .$$

is complete for the norm

$$\|\cdot\|_{H^1(\mathcal{I})} : v \mapsto \sqrt{\|v\|^2_{L^2(\mathcal{I})} + \|v'\|^2_{L^2(\mathcal{I})}}$$

It is a Hilbert Space.

Sobolev spaces of higher order

Theorem III.3.3

Let $k \in \mathbb{N}$. The space

$$H^k(\mathcal{I}) := \left\{ u \in L^2(\mathcal{I}) : \quad u^{(m)} \in L^2(\mathcal{I}),\ 0 \leq m \leq k \right\}.$$

endowed with the inner product

$$(u, v) \mapsto \sum_{0 \leq m \leq k} \left\langle u^{(m)}, v^{(m)} \right\rangle_{L^2(\mathcal{I})}$$

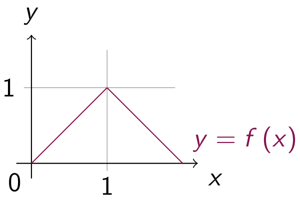
is a Hilbert Space.

We note $H^0(\mathcal{I}) = L^2(\mathcal{I})$.

Example

Let $\mathcal{I} =]0,2[$. Consider the **basis** function:

$$f : x \mapsto \begin{cases} x & \text{if } x \in]0,1[, \\ 2-x & \text{if } x \in [1,2[. \end{cases}$$



- Is f in $H^1(0,2)$?
- Is f in $H^2(0,2)$?

III.3.2. Regularity

Regularity of Sobolev spaces

Theorem III.3.4 (Rellich)

Let $\mathcal{I} =]a, b[$ be a bounded open interval of $\mathbb{R}$.
For any function u in $H^1(]a, b[)$, there exists $\tilde{u}$ in the class of u that is continuous on $[a, b]$ and that is an anti-derivative of u', i.e.

$$\forall x, y \in [a, b], \quad \tilde{u}(x) - \tilde{u}(y) = \int_{[y,x]} u'(t)\,\mathrm{d}t.$$

Furthermore

- *There exists a constant $C > 0$, depending only on $b - a$, s.t.*

$$\forall u \in H^1(]a, b[), \quad \|\tilde{u}\|_\infty \leq C\,\|u\|_{H^1}$$

- *For any bounded sequence of elements of $H^1(]a, b[)$, one can extract a subsequence that converges in $C^0([a, b])$.*

III.3.3. Trace

Trace theorem

Since $H^1(]a,b[) \subset C^0([a,b])$, one can evaluate u at a and b.

Definition III.3.5

Let $u \in H^1(]a,b[)$, the couple $(u(a), u(b))$ is called the **trace** *of u and the linear mapping $\gamma_0 : u \mapsto (u(a), u(b))$ is called the* **trace operator**.

Theorem III.3.6

There exists a constant C, depending only on $b-a$, s.t.

$$\forall x \in [a,b], \forall u \in H^1(]a,b[), \quad |u(x)| \le C \|u\|_{H^1}.$$

Subsequently, the linear mapping γ_0 is continuous from $H^1(]a,b[)$ to $\mathbb{R}^2$.

Application: Integration by parts

Theorem III.3.7 (Integration by parts)

Let $u, v \in H^1(a,b)$. Then

$$\int_{]a,b[} u\, v' = -\int_{]a,b[} v\, u' + u(b)v(b) - u(a)v(a).$$

Theorem III.3.8 (Integration by parts 2)

Let $u \in H^2(a,b)$ and $v \in H^1(a,b)$. Then

$$\int_{]a,b[} u'\, v' + \int_{]a,b[} u''\, v = u'(b)\, v(b) - u'(a)\, v(a)$$

III.3.4. $H_0^1(\mathcal{I})$

The space $H_0^1(\mathcal{I})$

Let $\mathcal{I}$ be an open bounded set of $\mathbb{R}$.
$\mathcal{D}(\mathcal{I})$ is dense in $L^2(\mathcal{I})$ but not in $H^1(\mathcal{I})$!

Definition III.3.9

$H_0^1(\mathcal{I}) := \gamma_0^{-1}(\{(0,0)\})$.

Proposition III.3.10 (Properties)

1. $H_0^1(\mathcal{I}) = \overline{\mathcal{D}(\mathcal{I})}$ *for the norm* H^1.
2. $H_0^1(\mathcal{I}) \subset H^1(\mathcal{I})$ *and* $H_0^1(\mathcal{I}) \neq H^1(\mathcal{I})$.
3. $H_0^1(\mathcal{I})$ *endowed with the norm of* H^1 *is a Hilbert space*

The Poincaré Inéquality

Theorem III.3.11 (Poincaré or Friedrichs)

Let $\mathcal{I}$ be a bounded interval of $\mathbb{R}$. There exists a constant $C_{\mathcal{I}}$ depending only on $\mathcal{I}$ s.t.

$$\forall v \in H_0^1(\mathcal{I}), \quad \|v\|_{L^2(\mathcal{I})} \leq C_{\mathcal{I}} \|v'\|_{L^2(\mathcal{I})}.$$

The semi-norm $v \mapsto \|v'\|_{L^2(\mathcal{I})}$ defined on H^1 by

$$v \mapsto |v|_{H^1(\mathcal{I})} := \|v'\|_{L^2(\mathcal{I})}$$

satisfies, in $H_0^1(\mathcal{I})$:

$$\forall v \in H_0^1(\mathcal{I}), \quad |v|_{H^1(\mathcal{I})} \leq \|v\|_{H^1(\mathcal{I})} \leq \sqrt{1 + C_{\mathcal{I}}^2}\, |v|_{H^1(\mathcal{I})}.$$

Consequence for $H_0^1(\mathcal{I})$

Definition III.3.12

Let us define $\|\cdot\|_{H_0^1} := |\cdot|_{H^1}$ *and*

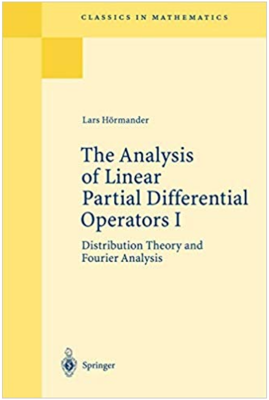
$$\langle \cdot, \cdot \rangle_{H_0^1} : (u, v) \in H_0^1 \times H_0^1 \mapsto \int_{\mathcal{I}} u' v'$$

Theorem III.3.13

The space $H_0^1(\mathcal{I})$ *endowed with the inner product* $\langle \cdot, \cdot \rangle_{H_0^1}$ *is a Hilbert Space.*

References

To go further on distributions.
To generalize to the case of dimension $d > 1$.



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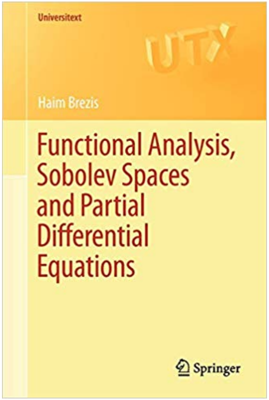
Springer; 2nd ed. 2003 edition
ISBN-10: 3-540-00662-1
ISBN-13: 978-3540006626

References

To go further on Sobolev spaces.



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Publisher: Springer; 2011 edition
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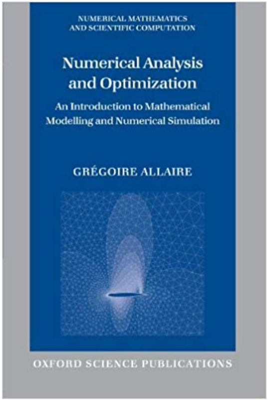
To go further on Sobolev spaces.



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Publisher: Oxford University Press, USA (2007)
ISBN-10: 0-19-920522-1
ISBN-13: 978-0199205226

Missing proofs for the theorems of this chapter

Some theorems that were not proven are pretty straightforward to prove and you can do so by yourself.

Some are more complicated. You can look in theses references.

Theorem III.3.2: Brezis. Proposition 8.1
Theorem III.3.4: Brezis. Theorem 8.2 and Theorem 8.8.
Theorem III.3.6: Lab session. Exercise IV.4.
Theorem III.3.11: Brezis. Corollary 9.12