Exercise III.5

1. What is the projection onto the positive orthant $[0,+\infty[^N$ in $\mathbb{R}^N$?
2. Let $\bar{x} \in \mathbb{R}^N$ and $\rho \in]0,+\infty[$. What is the projection onto the closed ball:

$$B(\bar{x},\rho) = \left\{x \in \mathbb{R}^N \mid \|x-\bar{x}\| \leq \rho.\right\} \tag{1}$$

Solution:

1. We start by noting that the positive orthant $[0,+\infty[^N$ is a closed convex non-empty set. We can therefore apply the conditions given in our lecture slides for determining whether a point $\hat{x}$ is a projection of x onto the given set. In particular, for a vector $\hat{x}$ to be the projection of x onto $[0,+\infty[^N$, it must hold that $\hat{x} \in [0,+\infty[^N$ and that:

$$\left(\forall y \in [0,+\infty[^N\right), \ \ \langle x-\hat{x} | y-\hat{x}\rangle \leq 0. \tag{2}$$

Starting now from a vector $x = \left(x^{(i)}\right)_{1\leq i\leq N} \in \mathbb{R}^N$, let us construct the vector $\hat{x} = \left(\hat{x}^{(i)}\right)_{1\leq i\leq N}$ where:

$$\hat{x}^{(i)} = \begin{cases} x^{(i)}, & \text{if } x^{(i)} \geq 0, \\ 0, & \text{otherwise.} \end{cases} \tag{3}$$

Clearly, $\hat{x} \in [0,+\infty[^N$. Moreover, $\forall y = \left(y^{(i)}\right)_{1\leq i\leq N} \in [0,+\infty[^N$, if we define the set:

$$I = \left\{i \in \mathbb{N}^+ | x^{(i)} \geq 0\right\} \tag{4}$$

we can write $\langle x-\hat{x}|y-\hat{x}\rangle$ as:

$$\langle x-\hat{x}|y-\hat{x}\rangle = \sum_{i\notin I} x^{(i)} y^i. \tag{5}$$

We now note that for $i \notin I$, $x^{(i)} < 0$. Morever, since $y \in [0,+\infty[^N$, $y^{(i)} \geq 0$. As a result, we obtain that:

$$\langle x-\hat{x}|y-\hat{x}\rangle \leq 0. \tag{6}$$

We therefore obtain that $\hat{x}$ satisfies the conditions for being the projection of $\hat{x}$ in x.

2. We start once more by noting that the given set $B(\hat{x},\rho)$ is closed, convex and non-empty. As a result the projection onto $B(\hat{x},\rho)$ is well defined and we can apply the conditions given in our lecture slides in order to determine if a vector $\hat{x}$ is the projection of x onto $B(\bar{x},\rho)$.

Let us now consider the vector:

$$\hat{x} = \begin{cases} x, & \text{if } \|x - \bar{x}\| \leq 0 \\ \bar{x} + \rho \frac{x - \bar{x}}{\|x - \bar{x}\|}, & \text{otherwise} \end{cases}, \tag{7}$$

and investigate if it satisfies the conditions for being the projection of x onto $B(\bar{x}, \rho)$. We now consider the following two cases:

- Case 1: $\|x - \bar{x}\| \leq \rho$: In this case $\hat{x} = x$ and $\langle x - \hat{x} | y - \hat{x} \rangle = 0$, $\forall y \in B(\bar{x}, \rho)$.
- Case 2: $\|x - \bar{x}\| \geq \rho$: In this case we obtain that:

$$\begin{aligned} \langle x - \hat{x} | y - \hat{x} \rangle &= \langle x - \bar{x} - \rho \frac{x - \bar{x}}{\|x - \bar{x}\|} | y - \bar{x} - \rho \frac{x - \bar{x}}{\|x - \bar{x}\|} \rangle \\ &= \left(1 - \frac{\rho}{\|x - \bar{x}\|}\right) \left(\langle x - \bar{x} | y - \bar{x} \rangle - \rho \|x - \bar{x}\| .\right) \end{aligned} \tag{8}$$

Using now the Cauchy Schwarz inequality, we obtain that:

$$|\langle x - \bar{x} | y - \bar{x} \rangle| \leq \|x - \bar{x}\| \, \|y - \bar{x}\| . \tag{9}$$

As a result, using this bound, we obtain that:

$$\begin{aligned} \langle x - \hat{x} | y - \hat{x} \rangle \leq & \left(1 - \frac{\rho}{\|x - \bar{x}\|}\right) \left(\|x - \bar{x}\| \, \|y - \bar{x}\| - \rho \|x - \hat{x}\|\right) \\ &= \left(1 - \frac{\rho}{\|x - \bar{x}\|}\right) \|x - \bar{x}\| \left(\|y - \bar{x}\| - \rho\right) \leq 0, \quad \forall y \in B(\bar{x}, \rho) \end{aligned} \tag{10}$$