Tutorial session 4 - Constraints in continuous optimization

1 Problem

A winner at Euromillions receives a S million prize and decides to live on the investment of her gain. Her welfare at time $n+1$ is supposed to be proportional to $r_n^{1/2}$ where r_n is the money spent between time n and $n+1$. She thus wants to maximize

$$\sum_{n=0}^{N-2} \beta^n \sqrt{r_n}$$

where $N-1 \geq 1$ is her remaining life expectancy and the exponentially decaying term β^n with $\beta \in]0,1[$ accounts for the fact that spending money today is more enjoyable than spending it tomorrow (in other words, the younger, the more fun !). If x_n designates the available amount of money at time $n \in \{0,\dots,N-1\}$, we have

$x_1 = \gamma x_0 - r_0 = \gamma S - r_0.$

$$(\forall n \in \{0,\dots,N-2\}) \quad x_{n+1} = \gamma x_n - r_n$$

$= \gamma^n x_1 - (r_n + \gamma r_{n-1} \cdots + \gamma^{n-1} r_1).$

where $\gamma \in [1,+\infty[$ represents the investment rate.

$= \gamma^{n+1} S - (r_n + \gamma r_{n-1} \cdots + \gamma^n r_0).$

Our purpose is to find the optimal way of spending money.

$\geq 0.$

$\Rightarrow \gamma^{N-2} r_0 + \dots + r_{N-2} \leq \gamma^{N-1} S.$ $\quad 0 \leq r_i \leq \gamma x_i.$

2 Numerical solution

1. What are the equality and inequality constraints to be handled ?
2. Which kind of optimization problem should be solved ? non-linear programming
3. The provided PPXA Matlab function allows us to tackle the related constrained optimization problem. Use it to solve numerically the problem when $N = 100$ semesters, $S = 130$ millions, $\gamma = 1.03$, and $\beta = 0.96$.
4. Plot $(x_n)_{0 \leq n \leq N-1}$ and $(r_n)_{0 \leq n \leq N-2}$. Compute the associated cost value.
5. Compare the optimal strategy with the strategy which would consist in spending half of the money available at each time n. $r_n = \frac{1}{2}\gamma x_n \Rightarrow x_{n+1} = \frac{1}{2}\gamma x_n$
6. The winner might live longer than expected. As a cautious person, she plans to leave a residual amount of money R at duration $N-1$. Solve numerically the modified optimization problem when $R = 10$ millions. $x_{N-1} = R.$

3 Analytic solution

1. Propose a Lagrange formulation of the latter problem.
2. Deduce the closed form expression of the solution.
3. Compare it with the numerical solution.