Exercise 4.

Use a branch and bound method for solving the following problem:

$$\begin{aligned}
&\text{maximize} \quad 5x^{(1)} + 4x^{(2)} + 6x^{(3)} \\
&(x^{(1)}, x^{(2)}) \in [0 +\infty[^2 \\
&x^{(3)} \in \mathbb{N} \\
&\text{subject to} \quad 5x^{(1)} + 6x^{(2)} + 3x^{(3)} \leq 20 \\
&\qquad\qquad\quad x^{(1)} + 3x^{(3)} \leq 12
\end{aligned}$$

Iteration 0: We start by setting

$S_{0,1} = [0, +\infty[^3$

and solve the problem by relaxing the constraint $x^{(3)} \in \mathbb{N}$. We then obtain the solution

$$\hat{x} \simeq \begin{bmatrix} 2 \\ 0 \\ 3.33 \end{bmatrix}, \quad \underline{f}_{0,1} = -30$$

By truncating this solution we then obtain the feasible point

$$\tilde{x}_{0,1} = \begin{bmatrix} 2 \\ 0 \\ 3 \end{bmatrix}, \quad \bar{f}_{0,1} = -28$$

As a result we obtain the following upper bound for the optimal value:

$$\bar{f}_0 = -28$$

Iteration 1

We now split the ~~fi~~ search space by branching with respect to $x^{(3)}$ which is the variable which should be discrete. We then obtain the following two ~~subpro~~ subsets

i) $S_{1,1} = \{ [0, +\infty[^3, x^{(3)} \leq 3 \}$

Solving the relaxation of the initial problem (i.e. removing the constraint $x^{(3)} \in N$) in the presence of the additional constraint $x \in S_{1,1}$

we obtain the solution:

$$\hat{x}_{1,1} = \begin{bmatrix} 2.2 \\ 0 \\ 3 \end{bmatrix} \text{ and } p_{1,1} = -29$$

- Since $\hat{x}_{1,1}$ has an integer value for $x^{(3)}$ it is a feasible solution for the initial problem.

ii) $S_{1,2} = \{ [0, +\infty[^3 \mid x^{(3)} \geq 4 \}$

Relaxing the constraint $x^{(3)} \in N$ and adding the constraint $x \in S_{1,2}$ to the initial problem, and solving the resulting problem (using the simplex algorithm) we obtain the solution

$$\hat{x}_{1,2} = \begin{bmatrix} 0 \\ \frac{4}{3} \\ 4 \end{bmatrix}$$

which has an integer value for $x^{(3)}$. As a result it is feasible and results in a value $p_{1,2} = -29.33 < p_{1,1}$.

As a result $\hat{x}_{1,2}$ is the optimal solution.