# 计算方法 第 5 章习题答案

**5.1** 求下列给定区间上关于权函数 $\omega(x)$ 的正交多项式 $g_0(x), g_1(x), g_2(x)$.

(1) $[0,1],\ \omega(x)=\sqrt{x}$;　　(2) $[-1,1],\ \omega(x)=1+x^2$;

(3) $[0,1],\ \omega(x)=\sqrt{x(1-x)}$;　　(4) $[-1,1],\ \omega(x)=\sqrt{1-x^2}$.

**解** 根据三项递推关系式构造正交多项式. 注意计算内积时需带入权函数.

(1) 取 $g_0(x)=1$, 计算

$$\beta_0=(xg_0,g_0)=\int_0^1 x\sqrt{x}dx=\frac{2}{5},\quad \gamma_0=(g_0,g_0)=\int_0^1\sqrt{x}dx=\frac{2}{3},$$

因此 $g_1(x)=x-\beta_0/\gamma_0=x-\frac{3}{5}$. 进一步计算

$$\beta_1=(xg_1,g_1)=\int_0^1 x\big(x-\frac{3}{5}\big)^2\sqrt{x}dx=\frac{184}{7875},$$

$$\gamma_1=(g_1,g_1)=\int_0^1\big(x-\frac{3}{5}\big)^2\sqrt{x}dx=\frac{8}{175},$$

因此

$$\begin{cases} b_1=\dfrac{\beta_1}{\gamma_1}=\dfrac{23}{45},\\ c_1=\dfrac{\gamma_1}{\gamma_0}=\dfrac{12}{175},\end{cases}\quad\Longrightarrow\quad g_2(x)=(x-b_1)g_1(x)-c_1g_0(x)=x^2-\frac{10}{9}x+\frac{5}{21}.$$

(2) 取 $g_0(x)=1$, 计算

$$\beta_0=(xg_0,g_0)=\int_{-1}^1 x(1+x^2)dx=0,\quad \gamma_0=(g_0,g_0)=\int_{-1}^1(1+x^2)dx=\frac{8}{3},$$

因此 $g_1(x)=x-\beta_0/\gamma_0=x$. 进一步计算

$$\beta_1=(xg_1,g_1)=\int_{-1}^1 x^3(1+x^2)dx=0,\quad \gamma_1=(g_1,g_1)=\int_{-1}^1 x^2(1+x^2)dx=\frac{16}{15},$$

因此

$$\begin{cases} b_1=\dfrac{\beta_1}{\gamma_1}=0,\\ c_1=\dfrac{\gamma_1}{\gamma_0}=\dfrac{2}{5},\end{cases}\quad\Longrightarrow\quad g_2(x)=(x-b_1)g_1(x)-c_1g_0(x)=x^2-\frac{2}{5}.$$

(3) 取 $g_0(x)=1$, 计算

$$\beta_0=(xg_0,g_0)=\int_0^1 x\sqrt{x(1-x)}dx=\frac{\pi}{16},\quad \gamma_0=(g_0,g_0)=\int_0^1\sqrt{x(1-x)}dx=\frac{\pi}{8},$$

因此 $g_1(x)=x-\beta_0/\gamma_0=x-\frac{1}{2}$. 进一步计算

$$\beta_1=(xg_1,g_1)=\int_0^1 x\big(x-\frac{1}{2}\big)^2\sqrt{x(1-x)}dx=\frac{\pi}{256},$$

$$\gamma_1=(g_1,g_1)=\int_0^1\big(x-\frac{1}{2}\big)^2\sqrt{x(1-x)}dx=\frac{\pi}{128},$$

因此

$$\begin{cases} b_1 = \dfrac{\beta_1}{\gamma_1} = \dfrac{1}{2}, \\ c_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{1}{16}, \end{cases} \quad \Longrightarrow \quad g_2(x) = (x - b_1)g_1(x) - c_1 g_0(x) = x^2 - x + \frac{3}{16}.$$

(4) 取 $g_0(x) = 1$, 计算

$$\beta_0 = (xg_0, g_0) = \int_{-1}^{1} x\sqrt{1-x^2}dx = 0, \quad \gamma_0 = (g_0, g_0) = \int_{-1}^{1} \sqrt{1-x^2}dx = \frac{\pi}{2},$$

因此 $g_1(x) = x - \beta_0/\gamma_0 = x$. 进一步计算

$$\beta_1 = (xg_1, g_1) = \int_{-1}^{1} x^3\sqrt{1-x^2}dx = 0, \quad \gamma_1 = (g_1, g_1) = \int_{-1}^{1} x^2\sqrt{1-x^2}dx = \frac{\pi}{8},$$

因此

$$\begin{cases} b_1 = \dfrac{\beta_1}{\gamma_1} = 0, \\ c_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{1}{4}, \end{cases} \quad \Longrightarrow \quad g_2(x) = (x - b_1)g_1(x) - c_1 g_0(x) = x^2 - \frac{1}{4}.$$

**5.3** 求下列函数在指定区间上的最优平方逼近一次多项式:

(1) $y = \sqrt{x},\ [0, 1]$; (2) $y = \mathrm{e}^x,\ [-1, 1]$.

解 (1) 设最优平方逼近一次多项式为 $p_1(x) = c_0 + c_1 x$, 则有 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$(\varphi_i, \varphi_j) = \int_0^1 x^{i+j}dx = \frac{1}{i+j+1}, \quad i, j = 0, 1$$
$$(\varphi_0, f) = \int_0^1 \sqrt{x}dx = \frac{2}{3}, \ (\varphi_1, f) = \int_0^1 x^{\frac{3}{2}}dx = \frac{2}{5}.$$

于是对应的正规方程组为

$$\begin{pmatrix} (\varphi_0, \varphi_0) & (\varphi_0, \varphi_1) \\ (\varphi_1, \varphi_0) & (\varphi_1, \varphi_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} (\varphi_0, f) \\ (\varphi_1, f) \end{pmatrix},$$

即

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{2}{3} \\ \frac{2}{5} \end{pmatrix},$$

解得 $c_0 = \frac{4}{15},\ c_1 = \frac{4}{5}$, 因此有

$$p_1(x) = \frac{4}{15} + \frac{4}{5}x.$$

(2) 设最优平方逼近一次多项式为 $p_1(x) = c_0 + c_1 x$, 则有 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$(\varphi_i, \varphi_j) = \int_{-1}^1 x^{i+j}dx = \frac{1 + (-1)^{i+j}}{i+j+1}, \quad i, j = 0, 1$$
$$(\varphi_0, f) = \int_{-1}^1 \mathrm{e}^x dx = \mathrm{e} - \mathrm{e}^{-1}, \ (\varphi_1, f) = \int_0^1 x\mathrm{e}^x dx = 2\mathrm{e}^{-1}.$$

于是对应的正规方程组为

$$\begin{pmatrix} (\varphi_0,\varphi_0) & (\varphi_0,\varphi_1) \\ (\varphi_1,\varphi_0) & (\varphi_1,\varphi_1) \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} (\varphi_0,f) \\ (\varphi_1,f) \end{pmatrix},$$

即

$$\begin{pmatrix} 2 & 0 \\ 0 & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \mathrm{e}-\mathrm{e}^{-1} \\ 2\mathrm{e}^{-1} \end{pmatrix},$$

解得 $c_0 = \frac{\mathrm{e}-\mathrm{e}^{-1}}{2}$, $c_1 = 3\mathrm{e}^{-1}$, 因此有

$$p_1(x) = \frac{\mathrm{e}-\mathrm{e}^{-1}}{2} + 3\mathrm{e}^{-1}x.$$

**5.4** 用正交多项式求下列函数的最优平方逼近二次多项式:

(1)

| $x_i$ | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|
| $y_i$ | 2 | 7 | 8 | 10 | 11 | 11 | 10 | 9 | 8 |

(2) $y = \arcsin x, \quad [0,1]$.

**解** (1) 设最优平方逼近二次多项式为 $p_2(x) = a_0 + a_1 g_1(x) + a_2 g_2(x)$. 利用三项递推关系构造正交多项式. 取

$$g_0(x) = 1, \quad g_1(x) = x - \frac{\beta_0}{\gamma_0}, \quad g_2(x) = (x-b_1)g_1(x) - c_1 g_0(x),$$

而

$$\gamma_0 = (g_0,g_0) = (1,1) = \sum_{i=1}^{9} 1 = 9, \quad \beta_0 = (xg_0,g_0) = (x,1) = \sum_{i=1}^{9} x_i = 53,$$

故 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{53}{9}$. 又因

$$\gamma_1 = (g_1,g_1) = \sum_{i=1}^{9} \left(x_i - \frac{53}{9}\right)^2 = \frac{620}{9},$$

$$\beta_1 = (xg_1,g_1) = \sum_{i=1}^{9} x_i\left(x_i - \frac{53}{9}\right)^2 = \frac{29780}{81},$$

因此

$$\begin{cases} b_1 = \dfrac{\beta_1}{\gamma_1} = \dfrac{1489}{279}, \\ c_1 = \dfrac{\gamma_1}{\gamma_0} = \dfrac{620}{81}, \end{cases} \Longrightarrow \quad g_2(x) = (x-b_1)g_1(x) - c_1 g_0(x) = x^2 - \frac{348}{31}x + \frac{737}{31}.$$

另一方面, 由于

$$(g_2,g_2) = \sum_{i=1}^{9} \left(x_i^2 - \frac{348}{31}x + \frac{737}{31}\right)^2 = \frac{15708}{31}, \qquad (g_0,f) = \sum_{i=1}^{9} y_i = 76,$$

$$(g_1,f) = \sum_{i=1}^{9} \left(x_i - \frac{53}{9}\right)y_i = \frac{373}{9}, \quad (g_2,f) = \sum_{i=1}^{9} \left(x_i^2 - \frac{348}{31}x + \frac{737}{31}\right)y_i = -\frac{4203}{31},$$

$$a_0 = \frac{(g_0,f)}{(g_0,g_0)} = \frac{76}{9}, \quad a_1 = \frac{(g_1,f)}{(g_1,g_1)} = \frac{373}{620}, \quad a_2 = \frac{(g_2,f)}{(g_2,g_2)} = -\frac{1401}{5236},$$

故

$$p(x)=a_0g_0(x)+a_1g_1(x)+a_2g_2(x)=-\frac{1737}{1190}+\frac{94387}{26180}x-\frac{1401}{5236}x^2.$$

(2) 解法一: 设 $p_2(x)=c_0+c_1x+c_2x^2$, 则 $\varphi_0(x)=1,\varphi_1(x)=x,\varphi_2(x)=x^2$. 先选定多项式（不一定正交），后求解各项系数

$$(\varphi_i,\varphi_j)=\int_0^1 x^{i+j}dx=\frac{1}{i+j+1},\qquad (\varphi_0,f)=\int_0^1\arcsin x dx=\frac{\pi}{2}-1,$$
$$(\varphi_1,f)=\int_0^1 x\arcsin x dx=\frac{\pi}{8},\qquad (\varphi_2,f)=\int_0^1 x^2\arcsin x dx=\frac{\pi}{6}-\frac{2}{9}.$$

由正规方程组有

$$\begin{pmatrix}1&\frac12&\frac13\\ \frac12&\frac13&\frac14\\ \frac13&\frac14&\frac15\end{pmatrix}\begin{pmatrix}c_0\\ c_1\\ c_2\end{pmatrix}=\begin{pmatrix}\frac{\pi}{2}-1\\ \frac{\pi}{8}\\ \frac{\pi}{6}-\frac29\end{pmatrix}$$

解得

$$c_0=5\pi-\frac{47}{3},\quad c_1=76-24\pi,\quad c_2=\frac{45\pi}{2}-70,$$

因此

$$p_2(x)=5\pi-\frac{47}{3}+(76-24\pi)x+(\frac{45\pi}{2}-70)x^2.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x)=1$, 由于

$$\gamma_0=(g_0,g_0)=\int_0^1 1dx=1,\quad \beta_0=(xg_0,g_0)=(x,1)=\int_0^1 xdx=\frac12,$$

因此 $g_1(x)=x-\frac{\beta_0}{\gamma_0}=x-\frac12$. 又因为

$$\gamma_1=(g_1,g_1)=\int_0^1(x-\frac12)^2dx=\frac{1}{12},\quad \beta_1=(xg_1,g_1)=\int_0^1 x(x-\frac12)^2dx=\frac{1}{24},$$

因此

$$g_2(x)=(x-\frac{\beta_1}{\gamma_1})g_1(x)-\frac{\gamma_1}{\gamma_0}g_0(x)=(x-\frac12)^2-\frac{1}{12}=x^2-x+\frac16.$$

另一方面, 由于

$$(g_2,g_2)=\int_0^1(x^2-x+\frac16)dx=\frac{1}{180},\qquad (g_0,f)=\int_0^1\arcsin x dx=\frac{\pi}{2}-1,$$
$$(g_1,f)=\int_0^1(x-\frac12)\arcsin x dx=\frac12-\frac{\pi}{8},$$
$$(g_2,f)=\int_0^1(x^2-x+\frac16)\arcsin x dx=\frac{\pi}{8}-\frac{7}{18},$$

从而得

$$\sum_{i=0}^{n}c_i(\varphi_k,\varphi_i)=(\varphi_k,f),\qquad k=0,1,\cdots,n.$$

$$\begin{aligned}p_2(x)&=c_0g_0(x)+c_1g_1(x)+c_2g_2(x)\\&=\frac{(g_0,f)}{(g_0,g_0)}g_0(x)+\frac{(g_1,f)}{(g_1,g_1)}g_1(x)+\frac{(g_2,f)}{(g_2,g_2)}g_2(x)\\&=5\pi-\frac{47}{3}+(76-24\pi)x+(\frac{45\pi}{2}-70)x^2.\end{aligned}$$

解法三：取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+1}{2}(-1 \leqslant t \leqslant 1)$, 则可令 $g(t) = \arcsin\left(\frac{t+1}{2}\right)$. 前三个 Legendre 正交多项式为

$$p_0(t) = 1, \ \ p_1(t) = t, \quad p_2(t) = \frac{1}{2}(3t^2 - 1).$$

设 $g(t)$ 的最优平方逼近二次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t) + c_2 p_2(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}$, $(k = 0, 1, 2)$. 同时可得

$$(p_0, g) = \int_{-1}^{1} \arcsin\left(\frac{t+1}{2}\right) dt = \pi - 2, \quad (p_1, g) = \int_{-1}^{1} t \arcsin\left(\frac{t+1}{2}\right) dt = 2 - \frac{\pi}{2},$$
$$(p_2, g) = \frac{1}{2} \int_{-1}^{1} (3t^2 - 1) \arcsin\left(\frac{t+1}{2}\right) dt = \frac{3\pi}{2} - \frac{14}{3},$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{\pi}{2} - 1, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = 3 - \frac{3\pi}{4}, \quad c_2 = \frac{(p_2, g)}{(p_2, p_2)} = \frac{15\pi}{4} - \frac{35}{3},$$

故

$$q(t) = \frac{29}{6} - \frac{11\pi}{8} + \left(3 - \frac{3\pi}{4}\right)t + \left(\frac{45\pi}{8} - \frac{35}{2}\right)t^2.$$

再将 $t = 2x - 1$ 带入上式，则得到 $f(x)$ 的最优平方逼近二次多项式

$$p(x) = 5\pi - \frac{47}{3} + (76 - 24\pi)x + \left(\frac{45\pi}{2} - 70\right)x^2.$$

**5.6** 求下列函数在指定区间上的最优平方逼近一次多项式:

(1) $y = \sqrt{x}$, $\left[\frac{1}{4}, 1\right]$; (2) $y = \ln x$, $[1, 2]$;

(3) $y = \mathrm{e}^x$, $[0, 1]$; (4) $y = \sqrt{1 + x^2}$, $[0, 1]$.

**解** (1) 解法一: 设 $p_1(x) = c_0 + c_1 x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$(\varphi_i, \varphi_j) = \int_{\frac{1}{4}}^{1} x^{i+j} dx = \frac{1 - \left(\frac{1}{4}\right)^{i+j+1}}{i + j + 1}, \qquad (\varphi_0, f) = \int_{\frac{1}{4}}^{1} \sqrt{x} dx = \frac{7}{12},$$
$$(\varphi_1, f) = \int_{\frac{1}{4}}^{1} x\sqrt{x} dx = \frac{31}{80},$$

由正规方程组有

$$\begin{pmatrix} \frac{3}{4} & \frac{15}{32} \\ \frac{15}{32} & \frac{21}{64} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{7}{12} \\ \frac{31}{80} \end{pmatrix}$$

解得

$$c_0 = \frac{10}{27}, \quad c_1 = \frac{88}{135},$$

因此

$$p_1(x) = \frac{10}{27} + \frac{88}{135}x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_{\frac{1}{4}}^{1} 1dx = \frac{3}{4}, \ \ \beta_0 = (xg_0, g_0) = (x, 1) = \int_{\frac{1}{4}}^{1} xdx = \frac{15}{32},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{5}{8}$. 另一方面, 由于

$$\begin{aligned}
&(g_0, f) = \int_{\frac{1}{4}}^{1} \sqrt{x}dx = \frac{7}{12}, \quad (g_1, f) = \int_{\frac{1}{4}}^{1} (x - \frac{5}{8})\sqrt{x}dx = \frac{11}{480}, \\
&(g_1, g_1) = \int_{\frac{1}{4}}^{1} (x - \frac{5}{8})^2 dx = \frac{9}{256},
\end{aligned}$$

因此得

$$p_1(x) = c_0 g_0(x) + c_1 g_1(x) = \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) = \frac{10}{27} + \frac{88}{135}x.$$

解法三：取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{3t+5}{8}(-1 \leqslant t \leqslant 1)$, 则可令 $g(t) = \sqrt{\frac{3t+5}{8}}$. 前两个 Legendre 正交多项式为

(1) $y = \sqrt{x}$, $\left[\frac{1}{4}, 1\right]$;

$$p_0(t) = 1, \ \ p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}, (k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^{1} \sqrt{\frac{3t+5}{8}} dt = \frac{14}{9}, \quad (p_1, g) = \int_{-1}^{1} t\sqrt{\frac{3t+5}{8}} dt = \frac{22}{135},$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{7}{9}, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{11}{35},$$

故

$$q(t) = \frac{7}{9} + \frac{11}{35}t.$$

再将 $t = \frac{8x-5}{3}$ 带入上式，则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = \frac{10}{27} + \frac{88}{135}x.$$

(2) 解法一: 设 $p_1(x) = c_0 + c_1 x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$\begin{aligned}
&(\varphi_i, \varphi_j) = \int_1^2 x^{i+j} dx = \frac{2^{i+j+1} - 1}{i + j + 1}, \qquad (\varphi_0, f) = \int_1^2 \ln x dx = 2\ln 2 - 1, \\
&(\varphi_1, f) = \int_1^2 x \ln x dx = 2\ln 2 - \frac{3}{4},
\end{aligned}$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{3}{2} \\ \frac{3}{2} & \frac{7}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} 2\ln 2 - 1 \\ 2\ln 2 - \frac{3}{4} \end{pmatrix}$$

解得

$$c_0 = 20\ln 2 - \frac{29}{2}, \quad c_1 = 9 - 12\ln 2,$$

因此

$$p_1(x) = 20\ln 2 - \frac{29}{2} + \big(9 - 12\ln 2\big)x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_1^2 1dx = 1, \;\; \beta_0 = (xg_0, g_0) = (x, 1) = \int_1^2 xdx = \frac{3}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{3}{2}$. 另一方面, 由于

$$(g_0, f) = \int_1^2 \ln xdx = 2\ln 2 - 1, \quad (g_1, f) = \int_1^2 (x - \frac{3}{2})\ln xdx = \frac{3}{4} - \ln 2,$$
$$(g_1, g_1) = \int_1^2 (x - \frac{3}{2})^2 dx = \frac{1}{12},$$

因此得

$$p_1(x) = c_0g_0(x) + c_1g_1(x) = \frac{(g_0, f)}{(g_0, g_0)}g_0(x) + \frac{(g_1, f)}{(g_1, g_1)}g_1(x) = 20\ln 2 - \frac{29}{2} + \big(9 - 12\ln 2\big)x.$$

解法三：取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+3}{2}(-1 \leqslant t \leqslant 1)$, 则可令 $g(t) = \ln\frac{t+3}{2}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \;\; p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0p_0(t) + c_1p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}, (k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^1 \ln\frac{t+3}{2}dt = 4\ln 2 - 2, \quad (p_1, g) = \int_{-1}^1 t\ln\frac{t+3}{2}dt = 3 - 4\ln 2,$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = 2\ln 2 - 1, \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{9}{2} - 6\ln 2,$$

故

$$q(t) = 2\ln 2 - 1 + \left(\frac{9}{2} - 6\ln 2\right)t.$$

再将 $t = 2x - 3$ 带入上式，则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 20\ln 2 - \frac{29}{2} + \big(9 - 12\ln 2\big)x.$$

(3) 解法一: 设 $p_1(x)=c_0+c_1x$, 则 $\varphi_0(x)=1,\varphi_1(x)=x$.

$$(\varphi_i,\varphi_j)=\int_0^1 x^{i+j}dx=\frac{1}{i+j+1},\qquad (\varphi_0,f)=\int_0^1 \mathrm{e}^x dx=\mathrm{e}-1,$$
$$(\varphi_1,f)=\int_0^1 x\mathrm{e}^x dx=1,$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix}\begin{pmatrix} c_0 \\ c_1 \end{pmatrix}=\begin{pmatrix} \mathrm{e}-1 \\ 1 \end{pmatrix}$$

解得

$$c_0=4\mathrm{e}-10,\quad c_1=18-6\mathrm{e},$$

因此

$$p_1(x)=4\mathrm{e}-10+\big(18-6\mathrm{e}\big)x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x)=1$, 由于

$$\gamma_0=(g_0,g_0)=\int_0^1 1dx=1,\ \ \beta_0=(xg_0,g_0)=(x,1)=\int_0^1 xdx=\frac{1}{2},$$

因此 $g_1(x)=x-\frac{\beta_0}{\gamma_0}=x-\frac{1}{2}$. 另一方面, 由于

$$(g_0,f)=\int_0^1 \mathrm{e}^x dx=\mathrm{e}-1,\quad (g_1,f)=\int_0^1 (x-\frac{1}{2})\mathrm{e}^x dx=\frac{3-\mathrm{e}}{2},$$
$$(g_1,g_1)=\int_0^1 (x-\frac{1}{2})^2 dx=\frac{1}{12},$$

因此得

$$p_1(x)=c_0g_0(x)+c_1g_1(x)=\frac{(g_0,f)}{(g_0,g_0)}g_0(x)+\frac{(g_1,f)}{(g_1,g_1)}g_1(x)=4\mathrm{e}-10+\big(18-6\mathrm{e}\big)x.$$

解法三：取 Legendre 正交多项式作为基函数, 先作变换 $x=\frac{t+1}{2}(-1\leqslant t\leqslant 1)$, 则可令 $g(t)=\mathrm{e}^{\frac{t+1}{2}}$. 前两个 Legendre 正交多项式为

$$p_0(t)=1,\ \ p_1(t)=t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t)=c_0p_0(t)+c_1p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k,p_k)=\frac{2}{2k+1},(k=0,1)$. 同时可得

$$(p_0,g)=\int_{-1}^1 \mathrm{e}^{\frac{t+1}{2}}dt=2\mathrm{e}-2,\quad (p_1,g)=\int_{-1}^1 t\mathrm{e}^{\frac{t+1}{2}}dt=6-2\mathrm{e},$$

从而得

$$c_0=\frac{(p_0,g)}{(p_0,p_0)}=\mathrm{e}-1,\quad c_1=\frac{(p_1,g)}{(p_1,p_1)}=9-3\mathrm{e},$$

故

$$q(t) = \mathrm{e} - 1 + \big(9 - 3\mathrm{e}\big)t.$$

再将 $t = 2x - 1$ 带入上式，则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 4\mathrm{e} - 10 + \big(18 - 6\mathrm{e}\big)x.$$

(4) 解法一: 设 $p_1(x) = c_0 + c_1 x$, 则 $\varphi_0(x) = 1, \varphi_1(x) = x$.

$$(\varphi_i, \varphi_j) = \int_0^1 x^{i+j} dx = \frac{1}{i+j+1}, \qquad (\varphi_0, f) = \int_0^1 \sqrt{1+x^2} dx = \frac{1}{2}\big(\sqrt{2} + \ln(1+\sqrt{2})\big),$$
$$(\varphi_1, f) = \int_0^1 x\sqrt{1+x^2} dx = \frac{1}{3}\big(2\sqrt{2} - 1\big),$$

由正规方程组有

$$\begin{pmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{2}{3} \end{pmatrix} \begin{pmatrix} c_0 \\ c_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{2}\big(\sqrt{2} + \ln(1+\sqrt{2})\big) \\ \frac{1}{3}\big(2\sqrt{2} - 1\big) \end{pmatrix}$$

解得

$$c_0 = 2\big(1 - \sqrt{2} + \ln(1+\sqrt{2})\big), \quad c_1 = 5\sqrt{2} - 4 - 3\ln(1+\sqrt{2}),$$

因此

$$p_1(x) = 2\big(1 - \sqrt{2} + \ln(1+\sqrt{2})\big) + \big(5\sqrt{2} - 4 - 3\ln(1+\sqrt{2})\big)x.$$

解法二: 利用三项递推关系构造正交多项式. 取 $g_0(x) = 1$, 由于

$$\gamma_0 = (g_0, g_0) = \int_0^1 1 dx = 1, \;\; \beta_0 = (xg_0, g_0) = (x, 1) = \int_0^1 x dx = \frac{1}{2},$$

因此 $g_1(x) = x - \frac{\beta_0}{\gamma_0} = x - \frac{1}{2}$. 另一方面, 由于

$$(g_0, f) = \int_0^1 \sqrt{1+x^2} dx = \frac{1}{2}\big(\sqrt{2} + \ln(1+\sqrt{2})\big), \quad (g_1, g_1) = \int_0^1 (x - \frac{1}{2})^2 dx = \frac{1}{12},$$
$$(g_1, f) = \int_0^1 (x - \frac{1}{2})\sqrt{1+x^2} dx = \frac{1}{12}\big(5\sqrt{2} - 4 - 3\ln(1+\sqrt{2})\big),$$

因此得

$$\begin{aligned} p_1(x) = c_0 g_0(x) + c_1 g_1(x) &= \frac{(g_0, f)}{(g_0, g_0)} g_0(x) + \frac{(g_1, f)}{(g_1, g_1)} g_1(x) \\ &= 2\big(1 - \sqrt{2} + \ln(1+\sqrt{2})\big) + \big(5\sqrt{2} - 4 - 3\ln(1+\sqrt{2})\big)x. \end{aligned}$$

解法三：取 Legendre 正交多项式作为基函数, 先作变换 $x = \frac{t+1}{2}(-1 \leqslant t \leqslant 1)$, 则可令 $g(t) = \sqrt{1 + \frac{1}{4}(t+1)^2}$. 前两个 Legendre 正交多项式为

$$p_0(t) = 1, \;\; p_1(t) = t.$$

设 $g(t)$ 的最优平方逼近一次多项式为

$$q(t) = c_0 p_0(t) + c_1 p_1(t).$$

对于 Legendre 正交多项式已知 $(p_k, p_k) = \frac{2}{2k+1}, (k = 0, 1)$. 同时可得

$$(p_0, g) = \int_{-1}^{1} \sqrt{1 + \frac{1}{4}(t+1)^2} dt = \sqrt{2} + \ln(1 + \sqrt{2}),$$
$$(p_1, g) = \int_{-1}^{1} t\sqrt{1 + \frac{1}{4}(t+1)^2} dt = \frac{1}{3}\big(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})\big),$$

从而得

$$c_0 = \frac{(p_0, g)}{(p_0, p_0)} = \frac{1}{2}\big(\sqrt{2} + \ln(1 + \sqrt{2})\big), \quad c_1 = \frac{(p_1, g)}{(p_1, p_1)} = \frac{1}{2}\big(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})\big),$$

故

$$q(t) = \frac{1}{2}\big(\sqrt{2} + \ln(1 + \sqrt{2})\big) + \frac{1}{2}\big(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})\big)t.$$

再将 $t = 2x - 1$ 带入上式，则得到 $f(x)$ 的最优平方逼近一次多项式

$$p(x) = 2\big(1 - \sqrt{2} + \ln(1 + \sqrt{2})\big) + \big(5\sqrt{2} - 4 - 3\ln(1 + \sqrt{2})\big)x.$$