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On some consequences of Mazur–Orlicz theorem to Hahn–Banach–Lagrange theorem

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ABSTRACT
The paper examines different kinds of p-convexity of a function g which are sufficient for the existence of a linear functional $l \leqslant p$ such that $\inf_A[f + l \circ g] = \inf_A[f + p \circ g]$ in Theorem 1.13 of Simons in his monograph 'From Hahn–Banach to monotonicity', published in Springer lecture notes (2008). We replace sublinearity of p with convexity, the field $\mathbb{R}$ with Dedekind vector lattice and present p_f-convexity which is also necessary. In Theorem 4.7 we also generalize a result of MM. Neumann from 1991 published in Czech. Mathem. Journal Vol 41 on the Mazur–Orlicz theorem.

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1. Introduction

Simons, using the concept of p-convexity, proved a version of the Hahn–Banach theorem (Theorem 1.13 in [1]), which is a generalization of Hahn–Banach–Lagrange theorem (Theorem 1.11 in [1]) and which finds applications in optimization theory, minimax theory and convex analysis (see [1–4]).

Throughout the paper by X we will denote a nontrivial vector space over the field of real numbers. By $(F, \leqslant)$ we denote a Dedekind complete real vector lattice and for $w \in F$ we put $(\leftarrow, w] = \{t \in F | t \leqslant w\}$. Here *vector lattice* means that the partial order in F satisfies the following property: $x \leqslant y \Longrightarrow \lambda x + z \leqslant \lambda y + z$ for all $x, y, z \in F$ and $\lambda \in \mathbb{R}, \lambda \geqslant 0$ and that for every $x, y \in F$ there exists the infimum or the greatest lower bound $x \wedge y$. By *Dedekind complete* we mean that every nonempty bounded from below subset of F has its infimum. For the sake of convenience we write that the infimum of a nonempty unbounded from below subset of F is equal to $-\infty$.

We say that a function $p : X \longrightarrow F$ is *convex* if $p(\lambda x + (1 - \lambda)y) \leqslant \lambda p(x) + (1 - \lambda)p(y)$ for all $x, y \in X$ and $\lambda \in [0, 1] \subset \mathbb{R}$. A function p is *positively homogenous* if $p(\lambda x) = \lambda p(x)$ for all $x \in X$ and $\lambda \in \mathbb{R}_+ = [0, \infty)$. A function p is *subadditive* if $p(x + y) \leqslant p(x) + p(y)$ for all $x, y \in X$. We call p *sublinear* if it is positively homogenous and subadditive. Notice that p is sublinear if and only if it is positively homogenous and convex.

Throughout the paper we denote $F_+ = \{t \in F | t \geqslant 0\}$. For $A \subset X$ by conv$A$ we denote a convex hull of A and cone$A = \bigcup_{\lambda \in \mathbb{R}_+} \lambda A$.

2. Application of Mazur and Mazur–Orlicz theorems

The following lemma (cf. Lemma 1 in [5]) is essential in the proof of Theorem 2.2 and is also applied in the proof of Lemma 2.5.

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Lemma 2.1: *Let $p\colon X \to F$ be a convex function and $y \in X$. For all $x \in X$, let*

$$p_y(x) := \inf_{\lambda>0} \frac{p(y+\lambda x) - p(y)}{\lambda}, \quad p_\circ(x) := \inf_{\lambda>0} \frac{p(\lambda x)}{\lambda}. \tag{1}$$

Then:

(a) $p_y\colon X \to F$ *is sublinear and* $p(y) - p(2y) \leqslant p_y(-y) \leqslant p(0) - p(y)$.
(b) *If* $p(0) \geqslant 0$ *then* $p_\circ : X \longrightarrow F$ *is the greatest sublinear operator less than* p.
(c) *If* p *is sublinear then* $p_y \leqslant p$ *and* $p_y(-y) = -p(y)$.

Lemma 2.1 is a generalization of Lemma 1 in [5] from the case of real valued functions to the case of Dedekind lattice. The proof of Lemma 2.1 is analogous. Lemma 2.1 enables us to give a short proof to the following version of Mazur theorem.

Theorem 2.2 (Mazur): *Let $p\colon X \to F$ be a convex function, $p(0) \geqslant 0$. Then there exists a linear operator l on X such that $l \leqslant p$.*

***Proof*:** Denote by $\mathcal{C}_p(X)$ the set of all convex functions q on X such that $p \geqslant q$ and $q(0) \geqslant 0$. For every $q \in \mathcal{C}_p(X)$ and $x \in X$, we have $q(x) \geqslant -q(-x) + 2q(0) \geqslant -p(-x)$. Then, by Kuratowski–Zorn's lemma, there exists a minimal element l, of $\mathcal{C}_p(X)$. Now, by Lemma 2.1, $l_\circ$ is sublinear and $l_\circ \leqslant l$. Hence $l_\circ = l$ and l is sublinear. For a fixed y and any x, we have $l_y(x) := \inf_{\lambda>0} \frac{l(y+\lambda x)-l(y)}{\lambda} \leqslant l(y+x) - l(y) \leqslant l(x)$. Hence, by l_y being sublinear, we obtain $l_y = l$. Now, from Lemma 2.1 (c), we have $l(-y) = l_y(-y) = -l(y)$ for $y \in X$. Thus l is linear on X. □

Classical Hahn–Banach theorem (see Lemma 1.2 in [1]) follows from Theorem 2.2. This theorem will be used in the proof of Lemma 2.5.

A convex subset V of a vector space X is called a *convex cone* if $\alpha V \subset V$ for all $\alpha \geqslant 0$.

Proposition 2.3: *Let $p\colon X \to F$ be a convex function, V be a convex cone in X. Then $p \geqslant 0$ on V if and only if there exists a linear operator $l : X \to F$ such that $l \leqslant p$ on X and $l \geqslant 0$ on V.*

***Proof*:** Suppose that $p \geqslant 0$ on V. We define $q(x) := \inf_{y\in V} p(x+y)$. Since $p \geqslant 0$ on V, $-p(-x) \leqslant p(x+y)$ for $y \in V$. Hence $q(x) > -\infty$. Moreover, we have $q \leqslant p$ and $q(0) \geqslant 0$. Since for every $x_1, x_2 \in X, \alpha, \beta \geqslant 0, \alpha + \beta = 1$ and $y_1, y_2 \in V$ we obtain

$$q(\alpha x_1 + \beta x_2) \leqslant p\left(\alpha(x_1 + y_1) + \beta(x_2 + y_2)\right) \leqslant \alpha p(x_1 + y_1) + \beta p(x_2 + y_2),$$

we deduce that $q(\alpha x_1 + \beta x_2) \leqslant \alpha q(x_1) + \beta q(x_2)$ so the function q is convex. From Mazur theorem there exists a linear operator $l : X \to F$ such that $l \leqslant q$ on X. From the definition of the function q we get $q(-y) \leqslant p(0)$ for all $y \in V$. Hence, taking any $y \in V$ and a natural number n we obtain $l(-ny) \leqslant q(-ny) \leqslant p(0)$, and $l(y) \geqslant -\frac{1}{n}p(0)$. Thus $l \geqslant 0$ on V. Converse implication is obvious. □

Proposition 2.3 will appear equivalent to Mazur–Orlicz theorem.

Remark 1: If $p\colon X \to F$ is sublinear, V is a convex subset of X. Then $p \geqslant 0$ on cone V if and only if $p \geqslant 0$ on V.

Mazur and Orlicz [6] gave certain generalization of the Hahn–Banach theorem. A short proof of the Mazur–Orlicz theorem for sublinear functionals is given by Pták [7]. We presented a version of the Mazur–Orlicz theorem [8] for convex functionals in [5]. Theorem 2.4 is a generalization of this version from the case of real valued functions to the case of Dedekind lattice. Here we provide a different proof based on Proposition 2.3.

Theorem 2.4 (Mazur–Orlicz): *Let $p\colon X \to F$ be a convex function. Moreover textup, let $g\colon A \to X$ and $f\colon A \to F$ be functions defined on a nonempty subset A of X. Then the following statements are equivalent:*

(a) *There exists a linear operator $l : X \to F$ such that $l \leqslant p$ on X and $f \leqslant l \circ g$ on A.*
(b) *For every finite sequence $a_1, \dots, a_n \in A$, and for arbitrary non-negative real numbers $\lambda_1, \dots, \lambda_n$ the inequality*

$$\sum_{i=1}^{n} \lambda_i f(a_i) \leqslant p\left(\sum_{i=1}^{n} \lambda_i g(a_i)\right) \tag{*}$$

holds.

***Proof*:** Assume that $p(0) \geqslant 0$. Let $\hat{X} := X \times F$, $\hat{p}\colon \hat{X} \to F$ be defined by $\hat{p}(x,\lambda) := p(x) - \lambda$. Then $\hat{p}$ is convex on $\hat{X}$. Now let $\hat{l}$ be a linear operator on $\hat{X}$ and $\hat{l} \leqslant \hat{p}$. From the inequality $\hat{l}(0,\lambda) \leqslant p(0) - \lambda$ we get $\hat{l}(0,\lambda) = -\lambda$. Hence $l(x) = \hat{l}(x,0)$ is a linear operator on X, $l \leqslant p$ and $\hat{l}(x,\lambda) = l(x) - \lambda$. Conversely if l is a linear operator $l : X \to F$ and $l \leqslant p$. Then $\hat{l}(x,\lambda) := l(x) - \lambda$ is linear on $\hat{X}$ and $\hat{l} \leqslant \hat{p}$. Now we consider $\hat{A} := \{(g(a), f(a)) \mid a \in A\}$ and a convex cone $\hat{V} := \{\sum_{i=1}^{n} \lambda_i b_i \mid b_i, \in \hat{A}, \lambda_i \geqslant 0; i = 1, \dots, n; n \in \mathbb{N}\}$. Then the condition (a) is equivalent to $\hat{l} \leqslant \hat{p}$ and $\hat{l} \geqslant 0$ on $\hat{V}$. Moreover the condition (b) is equivalent $\hat{p} \geqslant 0$ on $\hat{V}$. Hence from Proposition 2.3 the condition (a) and (b) are equivalent. □

If $A = V$ is a convex cone, $g(a) = a$, $f(a) = 0$ for $a \in A$, then from Mazur–Orlicz theorem we get Proposition 2.3, hence Mazur–Orlicz theorem and Proposition 2.3 are equivalent.

Moreover from Theorem 2.4 follows also Simons' version of Mazur–Orlicz theorem (Lemma 1.6 in [1]), classical Mazur–Orlicz theorem [6] and Mazur theorem [8,9] which is a generalization of the Hahn–Banach theorem [10,11].

The following two lemmas, i.e. Lemmas 2.5 and 3.2, are essential in the proof of Theorem 4.2. They are also used in the proof of Theorem 4.7.

Lemma 2.5: *Let $p\colon X \to F$ be a convex function and A be a nonempty subset of X. Then the following statements are equivalent:*

(a) *There exists a linear operator $l : X \to F$ such that $l \leqslant p$ and $\inf_A l = \inf_A p$.*
(b) *We have $p(0) \geqslant 0$ and $\inf_A p = \inf_{convA} p_\circ$.*

***Proof*:** From (a) and Lemma 2.1 we get $p(0) \geqslant 0$ and $l \leqslant p_\circ$. Moreover

$$\inf_{\mathrm{conv}A} p_\circ \geqslant \inf_{\mathrm{conv}A} l = \inf_A l = \inf_A p \geqslant \inf_{\mathrm{conv}A} p_\circ.$$

Hence (a) implies (b). Now let $\varrho := \inf_A p$. If $\varrho = -\infty$, then by Mazur theorem (Theorem 2.2) there exists a linear operator $l : X \to F$ such that $l \leqslant p$. Thus (a) holds. Now let $\varrho \in F$. From (b) it follows that for every finite sequence $a_1, \dots, a_n \in A$ and for arbitrary non-negative real numbers $\lambda_1, \dots, \lambda_n$, where $\lambda = \sum_{i=1}^{n} \lambda_i > 0$ the following inequality

$$\varrho \leqslant p_\circ\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} a_i\right)$$

holds. Then

$$\sum_{i=1}^{n} \lambda_i \varrho \leqslant \lambda p_\circ\left(\sum_{i=1}^{n} \frac{\lambda_i}{\lambda} a_i\right) = p_\circ\left(\sum_{i=1}^{n} \lambda_i a_i\right) \leqslant p\left(\sum_{i=1}^{n} \lambda_i a_i\right).$$

Hence for every finite sequence $a_1, \dots, a_n \in A$ and for arbitrary non-negative real numbers $\lambda_1, \dots, \lambda_n$ (possibly $\sum_{i=1}^n \lambda_i = 0$) the inequality

$$\sum_{i=1}^{n} \lambda_i \varrho \leqslant p\left(\sum_{i=1}^{n} \lambda_i a_i\right)$$

holds. Applying Mazur–Orlicz theorem (Theorem 2.4) to $f(a) = \varrho$ and $g(a) = a$ we get (a). □

3. Generalized set convexity

This section is dedicated to study and generalization of the following notion of p-convexity with respect to a function $f : A \longrightarrow \mathbb{R}$ of a function $g : A \longrightarrow X$, where X is a vector space, a function $p : X \longrightarrow \mathbb{R}$ is sublinear and A is a subset of X. The function g is p_f-convex if for every $a_1, a_2 \in A$ there exists $a \in A$ such that

$$p\left(g(a) - \left(\frac{1}{2}g(a_1) + \frac{1}{2}g(a_2)\right)\right) \leqslant 0 \quad \text{and} \quad f(a) - \left(\frac{1}{2}f(a_1) + \frac{1}{2}f(a_2)\right) \leqslant 0.$$

This convexity is an assumption of Theorem 1.13 of Simons [1]. Our Theorem 4.2 is a generalization of Simons theorem.

In the following definition we introduce different kinds of set convexity dependent on convex function.

Definition 3.1: Let a function $p \colon X \to F$ be convex, a set A be a nonempty subset of X and $\alpha, \beta > 0$.

The set A is $p^{\alpha,\beta}$*-convex* if $p(0) \geqslant 0$, and there exists $v \in F_+$ such that for all $a_1, a_2 \in A$, $\epsilon > 0$ there exists $a \in A$ such that $p\left(a - \alpha a_1 - \beta a_2\right) \leqslant \epsilon v$.

Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$. Then the set A is p^{α}*-convex* if A is $p^{\alpha,\beta}$-convex.

The set A is p*-convex* if $p(0) \geqslant 0$ and there exists $v \in F_+$ such that for every $u = \sum_{i=1}^n \lambda_i a_i \in$ conv$A$, $\epsilon > 0$ there exists $a \in A$ such that $p(a) \leqslant p_\circ(u) + \epsilon v$.

Let us notice that the p-convexity of g with respect to f, which is mentioned in the first paragraph of this section is simply a $\hat{p}^{\frac{1}{2}}$-convexity of a set $\hat{A}$, where $v = (0,0)$, $\hat{p} : X \times \mathbb{R} \longrightarrow \mathbb{R} \times \mathbb{R}$, $\hat{p}(x,t) = (p(x), t)$, $\hat{A} = \{(g(a), f(a)) | a \in A\}$ and $\mathbb{R} \times \mathbb{R}$ is a Dedekind lattice with the ordering $(t_1, t_2) \leqslant (t_1', t_2') \Longleftrightarrow t_1 \leqslant t_1', t_2 \leqslant t_2'$.
If a set A is p-convex then $p(0) \geqslant 0$ and there exists $v \in F_+$ such that for every $u = \sum_{i=1}^n \lambda_i a_i \in$ cone$A$, $\epsilon > 0$ there exists $a \in A$ such that $p(a) \leqslant \frac{p(u)}{\sum_{i=1}^n \lambda_i} + \epsilon v$. The opposite statement holds true if $F = \mathbb{R}$.

Lemma 3.2: *Let $p \colon X \to F$ be a convex function and A be a nonempty subset of X. If the set A is p-convex then $p(0) \geqslant 0$ and $\inf_A p = \inf_{convA} p_\circ$. Moreover, if $F = \mathbb{R}$, $p(0) \geqslant 0$ and $\inf_A p = \inf_{convA} p_\circ$ then the set A is p-convex.*

***Proof*:** If A is p-convex then, by definition, $p(0) \geqslant 0$ and for some $v \in F_+$ and for all $\varepsilon > 0$ the following statement holds true: for all $u \in$ conv A there exists $a \in A$ such that $p(a) \leqslant p_\circ(u) + \varepsilon v$. Then for some $v \in F_+$ and for all $\varepsilon > 0$ we obtain $\inf_A p = \inf_{\mathrm{conv}A} p_\circ + \varepsilon v$. Hence $\inf_A p = \inf_{\mathrm{conv}A} p_\circ$

On the other hand, if $F = \mathbb{R}$, $p(0) \geqslant 0$ and $\inf_A p = \inf_{\mathrm{conv}A} p_\circ$ then for all $u \in$ conv A we have $\inf_A p \leqslant p_\circ(u)$. Hence for all $\varepsilon > 0$ there exists $a \in A$ such that $p(a) \leqslant p_\circ(u) + \varepsilon$. □

If $p \colon X \to F$ is sublinear then $p_\circ = p$.

Remark 2: Let us notice that if p is a linear function, $A = \{a_1, a_2\}$ and elements $p(a_1)$, $p(a_2)$ are not comparable in F_+ then $\inf_A p = p(a_1) \wedge p(a_2) = \inf \mathrm{conv}p(A) = \inf_{\mathrm{conv}A} p = \inf_{\mathrm{conv}A} p_\circ$. However, the p-convexity of the set A would imply that for $u = \frac{1}{2}(a_1 + a_2)$ one of inequalities $p(a_1) \leqslant p(u)$, $p(a_2) \leqslant p(u)$ holds true. But since $p(a_1)$, $p(a_2)$ are not comparable, the condition the set A is not p-convex. Therefore, if $F \neq \mathbb{R}$ then F contains noncomparable elements and even for some functions p as simple as linear functions the equality $\inf_{\mathrm{conv}A} p = \inf_{\mathrm{conv}A} p_\circ$ does not imply the p-convexity of the set A.

Definition 3.3: We say that an interval $(\leftarrow, v]$, $v \in F_+$ *absorbs* a subset G if $G \subset \bigcup_{n=1}^{\infty} n(\leftarrow, v]$.

Remark 3: If for some $w \in F_+$ the interval $(\leftarrow, w]$ absorbs F then in the definition of p-convexity we can replace v with this w.

Let $\mathbb{D} := \{\frac{k}{2^n} \mid 0 \leqslant k \leqslant 2^n, n \in \mathbb{N} \cup \{0\}\}$ be the set of *dyadic numbers* of the interval $[0, 1]$. For $A \subset X$ we define a *dyadic convex hull* of A, $d\text{-conv}A := \{\sum_{i=1}^{n} \lambda_i a_i \mid a_i \in A, \lambda_i \in \mathbb{D}, \sum_{i=1}^{n} \lambda_i = 1\}$.

For $p\colon X \to F$, $\epsilon > 0$ and $v \in F_+$ we define $V_{p,\epsilon v} := \{x \mid p(x) \leqslant \epsilon v\}$. If $p : X \to F$ is a convex function, then $V_{p,\epsilon v}$ is a convex set. Moreover if $p(0) \geqslant 0$ then $V_{p_\circ,\epsilon v}$ is convex and $\lambda V_{p_\circ,\epsilon v} = V_{p_\circ,\lambda\epsilon v}$ for $\lambda > 0$.

Lemma 3.4: *Let $p\colon X \to F$ be a convex function and A be a nonempty subset of X. Let $\alpha, \beta > 0$. Then the following statements are equivalent:*

(a) *A is $p^{\alpha,\beta}$-convex.*
(b) *$\alpha A + \beta A \subset \bigcap_{\epsilon>0} (A - V_{p,\epsilon v})$ for some $v \in F_+$.*

The proof of the lemma is straightforward and follows from the definitions.

Lemma 3.5: *Let $p\colon X \to F$ be a sublinear function and A be a nonempty subset of X. Then the following statements are equivalent:*

(a) *A is $p^{\frac{1}{2}}$-convex.*
(b) *There exists $v \in F_+$ such that for every $n \in \mathbb{N}$ we have*

$$\overbrace{\frac{1}{2^n}A + \ldots + \frac{1}{2^n}A}^{2^n} \subset \bigcap_{\epsilon>0} (A - V_{p,\epsilon v}).$$

(c) *There exists $v \in F_+$ such that for every $n \in \mathbb{N}$, $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \in \mathbb{D}$ we have*

$$\sum_{i=1}^{n} \lambda_i A \subset \bigcap_{\epsilon>0} (A - V_{p,\epsilon v}).$$

***Proof*:** By Lemma 3.4 the statement (a) is equivalent to the inclusion $\frac{1}{2}A + \frac{1}{2}A \subset \bigcap_{\epsilon>0} (A - V_{p,\epsilon v})$. Obviously (b) implies (a). Now let (a) hold true and let

$$\overbrace{\frac{1}{2^n}A + \ldots + \frac{1}{2^n}A}^{2^n} \subset \bigcap_{\epsilon>0} (A - V_{p,\epsilon v})$$

for a fixed $n \geqslant 1$. Then from (a) we get

$$\begin{aligned}\sum_{i=1}^{2^{n+1}} \frac{1}{2^{n+1}}A &= \frac{1}{2}\left(\sum_{i=1}^{2^n} \frac{1}{2^n}A\right) + \frac{1}{2}\left(\sum_{i=1}^{2^n} \frac{1}{2^n}A\right) \\ &\subset \frac{1}{2}A - V_{p,\frac{\epsilon v}{3}} + \frac{1}{2}A - V_{p,\frac{\epsilon v}{3}} \subset A - 3V_{p,\frac{\epsilon v}{3}} = A - V_{p,\epsilon v}.\end{aligned}$$

Hence the conditions (a) and (b) are equivalent.

Now let $\sum_{i=1}^{n} \lambda_i = 1$, $\lambda_i \in \mathbb{D}$. We can assume that all λ_i are positive. Then there exists $m \in \mathbb{N}$ such that $\lambda_i = \frac{k_i}{2^m}$. Hence from (b) we have

$$\sum_{i=1}^{n} \lambda_i A \subset \sum_{i=1}^{2^m} \frac{1}{2^m} A \subset A - V_{p,\epsilon v}.$$

Then the statements (b) and (c) are equivalent. □

Lemma 3.6: *Let $p\colon X \to F$ be a sublinear function and A be a nonempty subset of X. If A is $p^{\frac{1}{2}}$-convex then*

$$\inf_A p = \inf_{d\text{-conv}A} p.$$

***Proof*:** By Lemma 3.5 from $p^{\frac{1}{2}}$-convexity of A there exists $v \in F_+$ such that for every $n \in \mathbb{N}$, $\sum_{i=1}^{n} \lambda_i = 1, \lambda_i \in \mathbb{D}$ we have

$$\sum_{i=1}^{n} \lambda_i A \subset \bigcap_{\epsilon > 0} (A - V_{p,\epsilon v}).$$

Take any $x \in d$-conv$A$. Then for any $\epsilon > 0$ we have $a - x \in V_{p,\epsilon v}$ for some $a \in A$. Hence

$$\inf_A p \leqslant p(a) \leqslant p(a - x) + p(x) \leqslant p(x) + \epsilon v.$$

Since ϵ is arbitrary, we obtain $\inf_A p \leqslant p(x)$ for all $x \in d$-conv$A$. □

Proposition 3.7: *Let $p\colon X \to F$ be a sublinear function, A be a nonempty subset of X and an interval $(\leftarrow, v]$ absorbs the set $p(A - A)$. If A is p^{α}-convex for some $\alpha \in (0, 1)$, then A is $p^{\frac{1}{2}}$-convex.*

***Proof*:** Let $\beta = 1 - \alpha$. For $n \in \mathbb{N}$ denote

$$\gamma_n = \sum_{i=0}^{[\frac{n}{2}]} \binom{n}{2i} \alpha^{2i} \beta^{n-2i} \quad \text{and} \quad \delta_n = \sum_{i=0}^{[\frac{n-1}{2}]} \binom{n}{2i+1} \alpha^{2i+1} \beta^{n-2i-1}.$$

Since $\alpha\gamma_n + \beta\delta_n = \gamma_{n+1}$ and $\beta\gamma_n + \alpha\delta_n = \delta_{n+1}$, it is easy to observe that if $\alpha A + \beta A \subset A - V_{p,w}$ then $\gamma_n A + \delta_n A \subset A - V_{p,nw}$, $n \in \mathbb{N}$.

Hence by Lemma 3.4 there exists $w \in F_+$ such that for any $a_1, a_2 \in A$ and $\varepsilon > 0$ we find $a \in A$ satisfying the inclusion $\alpha A + \beta A \subset A - V_{p,w}$, and the following inequalities hold true:

$$p\left(a - \frac{1}{2}a_1 - \frac{1}{2}a_2\right) \leqslant p(a - \gamma_n a_1 - \delta_n a_2) + p\left(\left(\gamma_n - \frac{1}{2}\right) a_1 + \left(\delta_n - \frac{1}{2}\right) a_2\right)$$

$\leqslant n\varepsilon w + |\gamma_n - \frac{1}{2}|\, p(a_1 - a_2)$ for all $n \in \mathbb{N}$.

Since for fixed $a_1, a_2 \in A$ and ε' we have $M > 0$ such that $p(a_1 - a_2) \leqslant Mv$, we can choose $m \in \mathbb{N}$ such that $M|\gamma_m - \frac{1}{2}| < \frac{1}{2}\varepsilon'$ and then choose

$$\varepsilon = \min\left(\frac{\varepsilon'}{2m}, \frac{\varepsilon'}{2M|\gamma_m - \frac{1}{2}|}\right).$$

Then

$p(a - \frac{1}{2}a_1 - \frac{1}{2}a_2) \leqslant m\varepsilon w + |\gamma_m - \frac{1}{2}|\, p(a_1 - a_2) \leqslant \varepsilon' \sup(w, v)$. □

In order to prove the next proposition we need two lemmas.

Lemma 3.8: *Let $I = [x_0 - \eta, x_0 + \eta] \subset \mathbb{R}$ and $p\colon I \to F$ be a convex function. Denote $v = \eta^{-1}[\sup\{p(x_0 - \eta), p(x_0 + \eta)\} - p(x_0)]$. Then*

$$-|x - x_0|v \leqslant p(x) - p(x_0) \leqslant |x - x_0|v$$

for all $x \in I$.

The proof is analogous to the proof in the case of real valued function. This lemma can be extended to n-dimensional case.

Lemma 3.9: *Let $\mathbb{B} = \mathbb{B}(x_0, \eta) \subset \mathbb{R}^n$ be a closed ball with center x_0 and radius η in the norm from ℓ_1, and let $p\colon \mathbb{B} \to F$ be a convex function. Then for some $v \in F_+$ we have*

$$-\|x - x_0\|_1 v \leqslant p(x) - p(x_0) \leqslant \|x - x_0\|_1 v$$

for all $x \in \mathbb{B}$.

***Proof*:** Let w be the supremum of all values of p at $2n$ extreme points of the ball $\mathbb{B}$. Then w is the supremum of p on $\mathbb{B}$. Take $x \in \mathbb{B}$ and let k be the line passing through x_0 and x. Applying the last lemma to the function p restricted to $k \cap \mathbb{B}$ we obtain

$$-\|x - x_0\|_1 v_x \leqslant p(x) - p(x_0) \leqslant \|x - x_0\|_1 v_x,$$

where $v_x = \frac{\sup\{p(x_0 - \eta(x - x_0)\|x - x_0\|_1^{-1}), p(x_0 + \eta(x - x_0)\|x - x_0\|_1^{-1})\} - p(x_0)}{\eta}$. Notice that for $v = \frac{w - p(x_0)}{\eta}$ we obtain $v_x \leqslant v$ for all x. Then

$$-\|x - x_0\|_1 v \leqslant p(x) - p(x_0) \leqslant \|x - x_0\|_1 v,$$

for all $x \in \mathbb{B}$. □

Proposition 3.10: *Let $p\colon X \to F$ be a sublinear operator, the interval $(\leftarrow, v]$ absorbs F and A be a nonempty subset of X. If A is $p^{\frac{1}{2}}$-convex, then A is p-convex.*

***Proof*:** Let us fix $b \in \operatorname{conv} A$, $b = \sum_{i=1}^n \lambda_i a_i$, $\lambda_i \geqslant 0$, $\epsilon > 0$. Since p is sublinear, by the last lemma applied to some finite dimensional ball in a subspace spanned by $a_1, \dots, a_n$ we can find some dyadic numbers $\lambda_i' \geqslant 0, i = 1, \dots, n$ which are sufficiently close to $\lambda_i, i = 1, \dots, n$, $\sum_{i=1}^n \lambda_i' = 1$ such that $p(\sum_{i=1}^n \lambda_i' a_i) \leqslant p(b) + \frac{\epsilon}{2} v$. Now from Lemma 3.5 (c) for $b' = \sum_{i=1}^n \lambda_i' a_i$ we can find $a \in A$ such that $p(a) \leqslant p(b') + \frac{\epsilon}{2} v$. □

Proposition 3.11: *Let $p\colon X \to F$ be a sublinear operator and A be a nonempty subset of X. If the subset A is $p^{\alpha,\beta}$-convex for some $\alpha, \beta > 0$, then $A_1 = \bigcup_{n \in \mathbb{Z}} (\alpha + \beta)^n A$ is $p^{\frac{\alpha}{\alpha+\beta}}$-convex.*

***Proof*:** At first we observe that $p^{\alpha,\beta}$-convexity of the set A is equivalent to the following inclusion $\alpha A + \beta A \subset \bigcap_{\epsilon > 0} (A - V_{p,\epsilon v})$. Since $\lambda V_{p,\epsilon v} = V_{p,\lambda \epsilon v}$ for $\lambda > 0$, we have $(\alpha + \beta)^k A \subset \bigcap_{\epsilon > 0} (A - V_{p,\epsilon v})$ for $k \in \mathbb{N} \cup \{0\}$. Hence for $k, l \in \mathbb{Z}$ we get $\alpha(\alpha + \beta)^k A + \beta(\alpha + \beta)^l A \subset \bigcap_{\epsilon > 0} (A_1 - V_{p,\epsilon v})$. Since for $n \in \mathbb{Z}$, $(\alpha + \beta)^n A_1 = A_1$, we obtain $\frac{\alpha}{\alpha+\beta} A_1 + \frac{\beta}{\alpha+\beta} A_1 \subset \bigcap_{\epsilon > 0} (A_1 - V_{p,\epsilon v})$. □

Remark 4:

(a) If $p\colon X \to F$ is a sublinear operator, $A \subset X$ and the set A is $p^{\alpha,\beta}$-convex then $\alpha A + \beta A \subset \bigcap_{\epsilon > 0} (A - V_{p,\epsilon v})$ and we have $\operatorname{conv}(\alpha A + \beta A) = \alpha \operatorname{conv} A + \beta \operatorname{conv} A = (\alpha + \beta) \operatorname{conv} A \subset \bigcap_{\epsilon > 0} (\operatorname{conv} A - V_{p,\epsilon v})$. Hence

$$\inf_A p \leqslant \inf_{\alpha A + \beta A} p \leqslant (\alpha + \beta) \inf_A p \quad \text{and} \quad \inf_{\operatorname{conv} A} p \leqslant (\alpha + \beta) \inf_{\operatorname{conv} A} p. \tag{2}$$

If $\alpha+\beta>1$ and $\inf_{\mathrm{conv}A} p \in F$, then from (2) we get $\inf_{\mathrm{conv}\,A} p \geqslant 0$. Now from Lemma 2.5, there exists a linear functional l on X such that $l \leqslant p$ and $\inf_{\mathrm{conv}\,A} l \geqslant 0$. Hence $l \geqslant 0$ on A.
If $\alpha+\beta<1$, then from (2) we get $\inf_A p \leqslant 0$.

(b) Let X be a normed vector space, $p(x)=\|x\|$. Then $A \subset X$ is $p^{\alpha,\beta}$-convex if and only if $\alpha A+\beta A \subset \bigcap_{\epsilon>0}(A+\epsilon\mathbb{B}) = \bar{A}$, where $\mathbb{B}$ is the closed unit ball in X. Hence in the case of $A \neq \{0\}$ and $\alpha+\beta \neq 1$ the set A is infinite. Similarly if A has at least two elements. If $\alpha+\beta>1$ the set A is unbounded and for $\alpha+\beta<1$ we get $0 \in \bar{A}$.

4. Generalized function convexity. The generalization of Simons' result on the Hahn–Banach theorem

Definition 4.1: Let $p\colon X \to F$ be a convex operator, A be a nonempty subset of X and $f\colon A \to F$. A function $g\colon A \to X$ is said to be:

p-convex if A is convex, $p(0) \geqslant 0$ and for all $a_1, a_2 \in A$ and $\lambda_1, \lambda_2 > 0$ such that $\lambda_1+\lambda_2=1$ we have $p(g(\lambda_1 a_1+\lambda_2 a_2)) - \lambda_1 g(a_1) - \lambda_2 g(a_2)) \leqslant 0$.

$p_f^{\alpha,\beta}$*-convex*, $\alpha, \beta > 0$ if $p(0) \geqslant 0$ and for some $v \in F_+$ and all $a_1, a_2 \in A$, $\varepsilon > 0$ there exists $a \in A$ such that $p\left(g(a) - \alpha g(a_1) - \beta g(a_2)\right) + f(a) \leqslant \alpha f(a_1) + \beta f(a_2) + \epsilon v$.

p_f^{α}*-convex* if g is $p_f^{\alpha,\beta}$-convex with $\alpha+\beta=1$.

p_f*-convex* if $p(0) \geqslant 0$ and for some $v \in F_+$ and for every $u=\sum_{i=1}^n \lambda_i g(a_i) \in \mathrm{conv} g(A)$, $\epsilon > 0$ there exists $a \in A$ such that $(p \circ g)(a) + f(a) \leqslant p_\circ(u) + \sum_{i=1}^n \lambda_i f(a_i) + \epsilon v$.

Now we are ready to prove a generalization of Theorem 12 in [5], which is itself a generalization of Theorem 1.13 of Simons [1].

Theorem 4.2: *Let $p\colon X \to F$ be a convex operator. Moreover textup, let $g\colon A \to X$ and $f\colon A \to F$ be functions defined on a nonempty subset A of X. Then for p_f-convexity of g it is necessary (and in the case of $F=\mathbb{R}$ it is also sufficient) that there exists a linear operator l on X such that $l \leqslant p$ and $\inf_A[l \circ g + f] = \inf_A[p \circ g + f]$.*

***Proof*:** Let $\hat{X} := X \times F$, $\hat{p}(x,w) := p(x)+w$, $\hat{A} := \{(g(a), f(a)) \mid a \in A\}$. Then $\hat{p}$ is a convex functional on $\hat{X}$ and g is p_f convex if and only if $\hat{p}(0) \geqslant 0$ and for some $v \in F_+$ and for any $u=\sum_{i=1}^n \lambda_i b_i \in \mathrm{conv}\hat{A}$, $\varepsilon > 0$ there exists $b \in \hat{A}$ such that $\hat{p}(b) \leqslant \hat{p}_\circ(u) + \varepsilon v$. From Lemmas 2.5 and 3.2 it follows (in the case of $F=\mathbb{R}$ it is equivalent to) the existence of a linear functional $\hat{l}$ on $\hat{X}$ such that $\hat{l} \leqslant \hat{p}$ and $\inf_{\hat{A}} \hat{l} = \inf_{\hat{A}} \hat{p}$. This, in turn, is equivalent to the fact that

$$\inf_A[p \circ g + f] \geqslant \inf_A[l \circ g + f] \geqslant \inf_{a \in A}[l(g(a)) + \hat{l}(0, f(a))] = \inf_A[p \circ g + f],$$

where $l(x) := \hat{l}(x,0)$ and $\hat{l} \leqslant \hat{p}$ implies $\hat{l}(0,y) \leqslant p(0)+y$, and by F being Dedekind lattice $\hat{l}(0,y) \leqslant y$. $\square$

Theorem 4.2 is a generalization of Theorem 12 in [5]. The assumption of Theorem 12 in [5] is that $p\colon X \to \mathbb{R}$, A be a nonempty subset of X and $f\colon A \to \mathbb{R}$, for every $b \in \mathrm{conv}\, g(A)$, $b=\sum_{i=1}^n \lambda_i g(a_i)$, $\epsilon > 0$ there exists $a \in A$ such that for every $\lambda \geqslant 0$

$$\lambda p \circ g(a) \leqslant p(\lambda b) + \epsilon \quad \text{and} \quad f(a) \leqslant \sum_{i=1}^n \lambda_i f(a_i) + \epsilon.$$

Theorem 4.2 shows that we can weaken the assumption and replace either $\mathbb{R}$ with Dedekind lattice F or implication with equivalence leaving $F=\mathbb{R}$.

Applying in turn Theorem 4.2 to $g=\mathrm{id}_A$, $f \equiv 0$ and Lemma 2.5 we obtain Lemma 3.2.

Example 4.3: Let $X=F:=\mathbb{R}$, $p(x) := 2|x|$, $A := \{0,1\}$, $g(x) := x$ and $f(x) := \mathrm{sign}\, x$ on A. Then for $l(x) := x$ we get $\inf_A[l \circ g + f] = \inf_A[p \circ g + f] = 0$. Hence from Theorem 4.2, g is

p_f-convex. In fact the function g also satisfies the stronger assumption of Theorem 12 in [5]. Now let $a_1 = 0, a_2 = 1$, $0 < \epsilon < \frac{1}{2}$. Suppose that $p(g(a) - \frac{1}{2}) + \operatorname{sign} a \leqslant \frac{1}{2} + \epsilon$ for a some $a \in A$. Then $\operatorname{sign} a + \frac{1}{2} \leqslant \epsilon$ and we have $\frac{1}{2} \leqslant \epsilon$. Hence g is not $p_f^{\frac{1}{2}}$-convex. This property of g is weaker than the assumption of Theorem 1.13 of Simons [1]. Since A is finite, g is not $p_f^{\alpha,\beta}$-convex for all $\alpha, \beta > 0$.

In order to show that the assumption of Theorem 12 in [5] is stronger than p_f-convexity of g we present the following examples.

Example 4.4: Let $X = \mathbb{R}^2$, $F = \mathbb{R}$, $p(x) := \sqrt{x_1^2 + x_2^2}$, $A := \{(0,0),(1,0),(0,1)\}$, $g(x) := x$ and $f(0,0) := \frac{\sqrt{2}}{2}$, $f(1,0) = f(0,1) := 0$. Since $p(0) \geqslant 0$ and for every $u = \sum_{i=1}^n \lambda_i a_i \in \operatorname{conv} A$, we have $p(0,0) + f(0) = \frac{\sqrt{2}}{2} \leqslant p(u) + \sum_{i=1}^n \lambda_i f(a_i)$, the function g is p_f-convex. However, the function g does not satisfy the assumption of Theorem 12 in [5] because the inequalities in the assumption cannot simultaneously hold true for $b = (\frac{1}{4}, \frac{1}{4})$, $\varepsilon = \frac{\sqrt{2}}{8v}$ and any $a \in A$.

Example 4.5: Let $X = F := \mathbb{R}$, $p(x) := |x|$, $A := \{-1, 1\}$, $g(x) := x$ and $f(x) := -1$ on A. Let $\varrho := \inf_A[p \circ g + f]$. Then $\varrho = 0$. If $l \leqslant p$ on X then $l(x) := \lambda x$ for some $\lambda \in [-1, 1]$ and $\inf_A[l \circ g + f] = \inf_A[\lambda x - 1] = -1 - |\lambda| \leqslant -1$. Hence there is no linear functional l on X such that $\inf_A[l \circ g + f] = \varrho$. Then from Theorem 4.2, g is not p_f-convex. However, if $\sigma := \inf_{\operatorname{conv} A}[p \circ g + f] = \inf\{p(u) + \sum_{i=1}^n \lambda_i f(a_i) \mid u = \sum_{i=1}^n \lambda_i g(a_i) \in \operatorname{conv} g(A)\}$ then for $l(x) = 0$, $\inf_A[l \circ g + f] = \sigma = -1$.

Proposition 4.6: *Let F be absorbed by an interval $(\leftarrow, v]$ for some $v \in F_+$, $p\colon X \to F$ be a sublinear operator, A be a nonempty subset of X and $f\colon A \to F$. If a function $g\colon A \to X$ is $p_f^{\frac{1}{2}}$-convex, then g is p_f-convex.*

Proof: Let $\hat{X} := X \times F$, $\hat{p}\colon X \to F$ be defined by $\hat{p}(x, w) := p(x) + w$ and $\hat{A} := \{(g(a), f(a)) \mid a \in A\}$. Then $\hat{p}$ is a sublinear on $\hat{X}$ and $\hat{A}$ is $\hat{p}^{\frac{1}{2}}$-convex. From Proposition 3.10, $\hat{A}$ is $\hat{p}$-convex. Hence g is p_f-convex. □

Theorem 4.7: *Let F be absorbed by an interval $(\leftarrow, v]$ for some $v \in F_+$ and $p\colon X \to F$ be a sublinear operator. Moreover textup, let $g\colon A \to X$ and $f\colon A \to F$ be functions defined on a nonempty subset A of X such that $p \circ g + f \geqslant 0$ on A and g be $p_f^{\alpha,\beta}$-convex. Then there exists a linear operator l on X such that $l \leqslant p$ on X and $l \circ g + f \geqslant 0$ on A.*

Proof: Let $\hat{X} := X \times F$, $\hat{p}(x, w) := p(x) + w$ and $\hat{A} := \{(g(a), f(a)) \mid a \in A\}$. Then $\hat{p}$ is a sublinear operator on $\hat{X}$ and $\hat{A}$ is $\hat{p}^{\alpha,\beta}$-convex. Now by Proposition 3.11, $\hat{A}_1 = \bigcup_{n \in \mathbb{Z}} (\alpha + \beta)^n \hat{A}$ is $\hat{p}^{\frac{\alpha}{\alpha+\beta}}$-convex. Hence by Proposition 3.7 the set $\hat{A}_1$ is $\hat{p}^{\frac{1}{2}}$-convex, and by Proposition 3.10 the set $\hat{A}_1$ is $\hat{p}$-convex. Now, by Lemmas 2.5 and 3.2, there exists a linear operator $\hat{l}$ on $\hat{X}$ such that $\hat{l} \leqslant \hat{p}$ on $\hat{X}$ and $\inf_{\hat{A}_1} \hat{l} = \inf_{\hat{A}_1} \hat{p}$. Since $\hat{p} \geqslant 0$ on $\hat{A}$, we have $\hat{p} \geqslant 0$ on $\hat{A}_1$. Therefore $\hat{l} \geqslant 0$ on $\hat{A}$, but $\hat{l}(x, w) = l(x) + w$ for some linear operator l on X. Hence $l \leqslant p$ on X and $l \circ g + f \geqslant 0$ on A. □

If in Theorem 4.7 we take $g(a) = a$ and $-f$ instead of f, then we obtain Neumann theorem (main Theorem in [12]):

Theorem 4.8: *Let $p : X \longrightarrow F$ be a sublinear operator, and consider an arbitrary mapping $f : A \longrightarrow F$ on a nonempty subset A of X such that $f \leqslant p$ on A. Moreover, assume that for some pair of real numbers $\alpha, \beta > 0$ and some $u \in F_+$ the following condition is fulfilled:*

For all $x, y \in A$ and all $\varepsilon > 0$ there exists some $z \in A$ such that $p(z - \alpha x - \beta y) \leqslant f(z) - \alpha f(x) - \beta f(y) + \varepsilon u$.

Then there exists a linear operator $l : X \longrightarrow F$ such that $f \leqslant l$ on A and $l \leqslant p$ on X.

Let $f\colon A \to (-\infty, \infty]$. The set $\operatorname{dom} f := \{x \in A \mid f(x) \in \mathbb{R}\}$ is called the *effective domain* of f. We say that f is *proper* if $\operatorname{dom} f \neq \emptyset$.

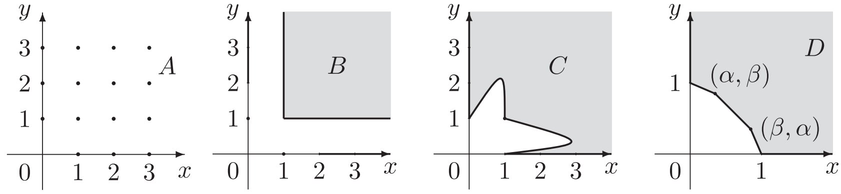

Figure 1. Sets from Example 4.10 which are $p^{1,1}$-convex.

Corollary 4.9: *Let $p\colon X \to \mathbb{R}$ be a sublinear functional and let A be a nonempty subset of X, $f\colon A \to (-\infty, \infty]$ be proper, $g\colon A \to X$ be $p_f^{\alpha,\beta}$-convex and $\alpha + \beta \leqslant 1$. Then there exists a linear functional l on X such that $l \leqslant p$ on X and*

$$\inf_A [l \circ g + f] = \inf_A [p \circ g + f].$$

***Proof*:** Let $\varrho = \inf_A [p \circ g + f]$. If $\varrho = -\infty$ then every linear functional $l\colon X \to \mathbb{R}$ with $l \leqslant p$ on X has the desired property. Now let $\varrho \in \mathbb{R}$. If $\alpha + \beta < 1$ then from Remark 2 in section 3 we have $\varrho \leqslant 0$. Hence g is $p_{f-\varrho}^{\alpha,\beta}$-convex for $\alpha + \beta \leqslant 1$ and $p \circ g + f - \varrho \geqslant 0$ on A. Hence by Theorem 4.7, there exists a linear functional l on X such that $l \leqslant p$ on X and $\inf_A [l \circ g + f] \geqslant \varrho$. □

Remark 5: Let $(X, \|\cdot\|)$ be a normed space, $p(x) := \|x\|$ and $A + A \subset \bar{A}$. Then the set A is $p^{1,1}$-convex. Moreover, A is $p^{\alpha,\beta}$-convex, for all integer $\alpha, \beta \geqslant 1$. In particular, it happens when $(A, +)$ is a semigroup.

Example 4.10: Let $X := \mathbb{R}^2$ and p be Euclidean norm $p(x) := ||x||_2$. Let $A := \mathbb{Z}_+ \times \mathbb{Z}_+ \setminus \{(0,0)\}$, $B := \{(x,y) \in \mathbb{R}^2 |\, x, y, x + y - 1 \in \{0\} \cup [1, \infty)\}$ (for A, B, C and D, see Figure 1). The sets A, B, C, D are semigroups. Moreover, the sets B and C are $p^{\alpha,\beta}$-convex, for all real $\alpha, \beta \geqslant 1$.
Notice that $\inf_A p = 1 > \inf_{\mathrm{conv}A} p = \frac{\sqrt{2}}{2}$ for all four sets from Figure 1. By Lemma 2.5 there is no linear functional l on X such that, $l \leqslant p$ and $\inf_A l = \inf_A p$. The set D is $p^{\alpha,\beta}$-convex for some $\alpha, \beta < 1$ such that $\alpha + \beta > 1$. Hence Corollary 4.9 is not true for any $\alpha + \beta > 1$.

Remark 6: Though Corrolary 4.9 is not true for $\alpha + \beta > 1$, in the case of $F = \mathbb{R}$ Theorem 4.7 follows from Corrolary 4.9. Suppose that assumptions of Theorem 4.7 are satisfied. Then $\hat{p}(x, \lambda) := p(x) + \lambda$ is a sublinear functional on $\hat{X} := X \times \mathbb{R}$ and the set $\hat{A}_1 = \bigcup_{n \in \mathbb{Z}} (\alpha + \beta)^n \hat{A}$ is $\hat{p}^{\frac{\alpha}{\alpha+\beta}}$-convex. Hence by Corrolary 4.9 there exists a linear functional $\hat{l}$ on $\hat{X}$ such that $\hat{l} \leqslant \hat{p}$ and $\inf_{\hat{A}_1} \hat{l} = \inf_{\hat{A}_1} \hat{p}$. Since $p \circ g + f \geqslant 0$ on A, we have $\hat{p} \geqslant 0$ on $\hat{A}_1$. But $\hat{l}(x, \lambda) = l(x) + \lambda$ for a some linear functional l on X. Therefore $l \leqslant p$ on X and $l \circ g + f \geqslant 0$ on A. Hence in the case of $F = \mathbb{R}$ Corrolary 4.9 and Theorem 4.7 are equivalent.

Disclosure statement

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