Partial Differential Equations

Chapter IV - The Variational Formulation

John Cagnol, Pauline Lafitte

The Engineering Program of CentraleSupélec

Lecture 5 – January 14th 2020

IV.1. Lax-Milgram

IV.1.1. Duality in Hilbert Spaces

Notation

$\mathcal{I}$ will be an open interval of $\mathbb{R}$:

$]a, b[$, $]-\infty, b[$ or $]a, +\infty[$

The Dual space of $L^2(\mathcal{I})$

From CIP:

Theorem IV.1.1

The space $L^2(E, \mathcal{E}, \mu)$ endowed with the following inner product

$$\langle f, g \rangle := \int f\, g\ d\mu$$

is a Hilbert space (on the field $\mathbb{R}$).

The Dual space of $L^2(\mathcal{I})$

From CIP:

Theorem IV.1.1

The space $L^2(E, \mathcal{E}, \mu)$ endowed with the following inner product

$$\langle f, g\rangle := \int f\, g\, d\mu$$

is a Hilbert space (on the field $\mathbb{R}$).

Theorem IV.1.2 (Riesz representation theorem)

Let $\mathcal{H}$ be a Hilbert space and $u \in \mathcal{H}'$. Then there exists a unique $x_u \in \mathcal{H}$ such that

$$\forall x \in \mathcal{H}, \quad u(x) = \langle x, x_u\rangle.$$

The Dual space of $L^2(\mathcal{I})$

Corollary IV.1.3

For all $u \in (L^2(\mathcal{I}))'$ there exists a unique $x_u \in L^2(\mathcal{I})$ such that

$$\forall x \in L^2(\mathcal{I}), \quad u(x) = \langle x, x_u \rangle.$$

The Dual space of $L^2(\mathcal{I})$

Corollary IV.1.3

For all $u \in (L^2(\mathcal{I}))'$ there exists a unique $g_u \in L^2(\mathcal{I})$ such that

$$\forall f \in L^2(\mathcal{I}), \quad u(f) = \langle f, g_u \rangle.$$

The Dual space of $L^2(\mathcal{I})$

Corollary IV.1.3

For all $u \in (L^2(\mathcal{I}))'$ there exists a unique $g_u \in L^2(\mathcal{I})$ such that

$$\forall f \in L^2(\mathcal{I}), \quad u(f) = \langle f, g_u \rangle.$$

We will write

$$(L^2(\mathcal{I}))' = L^2(\mathcal{I})$$

The Dual space of $L^2(\mathcal{I})$

Corollary IV.1.3

For all $u \in (L^2(\mathcal{I}))'$ there exists a unique $g_u \in L^2(\mathcal{I})$ such that

$$\forall f \in L^2(\mathcal{I}), \quad u(f) = \langle f, g_u \rangle.$$

We will write

$$(L^2(\mathcal{I}))' = L^2(\mathcal{I})$$

If $V \subset L^2(\mathcal{I})$ then $(L^2(I))' \subset V'$, therefore

$$V \subset L^2(\mathcal{I}) \subset V'$$

We say that $L^2(\mathcal{I})$ is a **pivot space**.

IV.1.2. Coercivity

Coercivity

Definition IV.1.4 (Coercivity)

Let $\mathcal{H}$ be a Hilbert space.
Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form.
We say that a is **coercive** *(or $\mathcal{H}$-elliptic) if*

$$\exists \alpha > 0, \ \forall x \in \mathcal{H}, \ a(x,x) \geq \alpha \|x\|^2$$

Coercivity

Definition IV.1.4 (Coercivity)

Let $\mathcal{H}$ be a Hilbert space.
Let $a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bilinear form.
We say that a is **coercive** *(or $\mathcal{H}$-elliptic) if*

$$\exists\alpha > 0,\ \forall x \in \mathcal{H},\ a(x,x) \geq \alpha\|x\|^2$$

Example

For $\mathcal{H} = \mathbb{R}^2$ $\qquad x = [x_1, x_2]^T,\ y = [y_1, y_2]^T$

- $a(x,y) = 2x_1y_1 + 3x_2y_2.$
- $a(x,y) = 2x_1y_1 - 3x_2y_2.$
- $a(x,y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2.$

Coercivity

Definition IV.1.4 (Coercivity)

Let $\mathcal{H}$ be a Hilbert space.
Let $a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bilinear form.
We say that a is **coercive** *(or $\mathcal{H}$-elliptic) if*

$$\exists\alpha > 0,\ \forall x \in \mathcal{H},\ a(x,x) \geq \alpha\|x\|^2$$

Example

For $\mathcal{H} = \mathbb{R}^2$ $x = [x_1, x_2]^T,\ y = [y_1, y_2]^T$

- $a(x,y) = 2x_1y_1 + 3x_2y_2$. *Coercive*
- $a(x,y) = 2x_1y_1 - 3x_2y_2$.
- $a(x,y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$.

Coercivity

Definition IV.1.4 (Coercivity)

Let $\mathcal{H}$ be a Hilbert space.
Let $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear form.
We say that a is **coercive** *(or $\mathcal{H}$-elliptic) if*

$$\exists \alpha > 0, \ \forall x \in \mathcal{H}, \ a(x,x) \geq \alpha \|x\|^2$$

Example

For $\mathcal{H} = \mathbb{R}^2$ $\qquad x = [x_1, x_2]^T, \ y = [y_1, y_2]^T$

- $a(x,y) = 2x_1y_1 + 3x_2y_2$. *Coercive*
- $a(x,y) = 2x_1y_1 - 3x_2y_2$. *Not coercive*
- $a(x,y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$.

Coercivity

Definition IV.1.4 (Coercivity)

Let $\mathcal{H}$ be a Hilbert space.
Let $a : \mathcal{H}\times\mathcal{H} \to \mathbb{R}$ be a bilinear form.
We say that a is **coercive** *(or $\mathcal{H}$-elliptic) if*

$$\exists\alpha > 0,\ \forall x \in \mathcal{H},\ a(x,x) \geq \alpha\|x\|^2$$

Example

For $\mathcal{H} = \mathbb{R}^2$ $x = [x_1, x_2]^T,\ y = [y_1, y_2]^T$

- $a(x,y) = 2x_1y_1 + 3x_2y_2$. *Coercive*
- $a(x,y) = 2x_1y_1 - 3x_2y_2$. *Not coercive*
- $a(x,y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$. *Coercive*

IV.1.3. Continuity

Continuity

Remark IV.1.5 (Continuity of a bilinear form)

Let $\mathcal{H}$ be a Hilbert space.
The bilinear form $a : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is **continuous** *if*

$$\exists C > 0,\ \forall x, y \in \mathcal{H},\ a(x, y) \leq \alpha \|x\| \|y\|$$

Note that linearity implies continuity in finite-dimensional spaces.
It is not the case in infinite-dimensional spaces.

IV.1.4. Theorem

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

$$\forall u \in \mathcal{H},\ \ a(x,u) = f(u)$$

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.

$$\forall u \in \mathcal{H}, \ a(x, u) = f(u)$$

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.
Let a be a continuous and coercive bilinear form.

$$\forall u \in \mathcal{H}, \; a(x, u) = f(u)$$

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.
Let a be a continuous and coercive bilinear form.
Let $f \in \mathcal{H}'$ (it is a linear and continuous function from $\mathcal{H}$ to $\mathbb{R}$)

$$\forall u \in \mathcal{H}, \; a(x, u) = f(u)$$

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.
Let a be a continuous and coercive bilinear form.
Let $f \in \mathcal{H}'$

$$\forall u \in \mathcal{H},\ a(x, u) = f(u)$$

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.
Let a be a continuous and coercive bilinear form.
Let $f \in \mathcal{H}'$

The equation

$$\forall u \in \mathcal{H},\ a(x,u) = f(u)$$

- *Has one and only one solution x.*
- *The application $f \mapsto x$ is linear and continuous from $\mathcal{H}'$ to $\mathcal{H}$.*

Lax-Milgram

Theorem IV.1.6 (Lax-Milgram)

Let $\mathcal{H}$ be a Hilbert space.
Let a be a continuous and coercive bilinear form.
Let $f \in \mathcal{H}'$

The equation

$$\forall u \in \mathcal{H}, \; a(x, u) = f(u)$$

- *Has one and only one solution x.*
- *The application $f \mapsto x$ is linear and continuous from $\mathcal{H}'$ to $\mathcal{H}$.*

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\; 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1 y_1 + x_1 y_2 + x_2 y_1 + 4x_2 y_2 = y_1 + y_2$$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\; 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
- *Let* $f(y) = y_1 + y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
- *Let* $f(y) = y_1 + y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2, \ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear
- *Let* $f(y) = y_1 + y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive
- *Let* $f(y) = y_1 + y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive and continuous
- *Let* $f(y) = y_1 + y_2$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive and continuous
- *Let* $f(y) = y_1 + y_2$
  f *is linear*

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive and continuous
- *Let* $f(y) = y_1 + y_2$
  f *is linear and continuous*

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive and continuous
- *Let* $f(y) = y_1 + y_2$
  f is linear and continuous

Lax-Milgram applies: $\forall y \in \mathcal{H}, \exists! x \in \mathcal{H}, a(x, y) = f(y)$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

- $\mathcal{H} = \mathbb{R}^2$ *is a Hilbert space*
- *Let* $a(x, y) = 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2$
  a is bilinear, coercive and continuous
- *Let* $f(y) = y_1 + y_2$
  f is linear and continuous

Lax-Milgram applies: $\forall y \in \mathcal{H}, \exists! x \in \mathcal{H}, a(x, y) = f(y)$
Furthermore $f \mapsto x$ *is continuous.*

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\; 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\; [y_1\;\; y_2] \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = [y_1\;\; y_2] \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1 y_1 + x_1 y_2 + x_2 y_1 + 4x_2 y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ y_2]\left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 0$$

.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2, \; 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2, \; [y_1 \;\; y_2] \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0$$

Which is equivalent to

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2] \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0$$

Which is equivalent to $x_1 = 3/7$, $x_2 = 1/7$.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2]\left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0$$

Which is equivalent to $x_1 = 3/7$, $x_2 = 1/7$.

Replacing $f(y_1, y_2) = y_1 + y_2$ by $c_1y_1 + c_2y_2$ leads to replacing $[1\ 1]^T$ by $[c_1\ \ c_2]^T$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2] \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0$$

Which is equivalent to $x_1 = 3/7$, $x_2 = 1/7$.

Replacing $f(y_1, y_2) = y_1 + y_2$ by $c_1y_1 + c_2y_2$ leads to replacing $[1\ 1]^T$ by $[c_1\ \ c_2]^T$, thus $(x_1, x_2) = (\frac{4}{7}c_1 - \frac{1}{7}c_2, \frac{2}{7}c_2 - \frac{1}{7}c_1)$

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2] \left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = 0$$

Which is equivalent to $x_1 = 3/7$, $x_2 = 1/7$.

Replacing $f(y_1, y_2) = y_1 + y_2$ by $c_1y_1 + c_2y_2$ leads to replacing $[1\ 1]^T$ by $[c_1\ \ c_2]^T$, thus $(x_1, x_2) = (\frac{4}{7}c_1 - \frac{1}{7}c_2, \frac{2}{7}c_2 - \frac{1}{7}c_1)$
The mapping $(c_1, c_2) \mapsto (x_1, x_2)$ is continuous.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2]\left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 0$$

Which is equivalent to $x_1 = 3/7$, $x_2 = 1/7$.

Lax-Milgram works but we could have solved the problem with a different method.

Lax-Milgram

Example

Can we find $(x_1, x_2) \in \mathbb{R}^2$ *s.t.*

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ 2x_1y_1 + x_1y_2 + x_2y_1 + 4x_2y_2 = y_1 + y_2$$

We could have solve the problem without Lax-Milgram by noticing the equation can be rewritten

$$\forall (y_1, y_2) \in \mathbb{R}^2,\ [y_1\ \ y_2]\left(\begin{bmatrix} 2 & 1 \\ 1 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix}\right) = 0$$

Which is equivalent to $x_1 = 3/7,\ x_2 = 1/7.$

Lax-Milgram works but we could have solved the problem with a different method. Lax-Milgram will be really useful later.

IV.2. Wellposedness of elliptic problems

IV.2.1. Introduction

Dirichlet Problem

Let $\Omega = [0, 1]$, consider:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0, 1[\\ u(0) = 0, \ u(1) = 0 \end{cases} \tag{1}$$

Dirichlet Problem

Let $\Omega = [0, 1]$, consider:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0, 1[\\ u(0) = 0, \ u(1) = 0 \end{cases} \tag{1}$$

Is the problem well-posed?

Dirichlet Problem

Let $\Omega = [0, 1]$, consider:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0, 1[\\ u(0) = 0, \;\; u(1) = 0 \end{cases} \tag{1}$$

Is the problem well-posed?
The answer will depend on f and u.

Dirichlet Problem

Let $\Omega = [0, 1]$, consider:

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0, 1[\\ u(0) = 0, \ \ u(1) = 0 \end{cases} \qquad (1)$$

Is the problem well-posed?
The answer will depend on f and u.

Can we find three normed vector spaces F, G and E such that

$$\forall f \in F, \ \forall c \in G, \ \exists! u_{f,c} \in E \text{ solution to } (1)$$

$$\exists C_\Omega > 0, \forall f \in F, \forall c \in G, u_{f,c} \in E \text{ defined by } (1) \ : \ \|u_{f,c}\|_E \le C_\Omega \|f\|_F$$

Dirichlet Problem

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1, consider:

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) \quad x \in [0,1] \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

Dirichlet Problem

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1, consider:

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) \quad x \in [0,1] \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

Is the problem well-posed?

Dirichlet Problem

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1, consider:

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) \quad x \in [0,1] \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

Is the problem well-posed?
The answer will depend on f and u.

Dirichlet Problem

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1, consider:

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) & x \in [0,1] \\ u|_{\partial\Omega} = 0 \end{cases} \tag{2}$$

Is the problem well-posed?
The answer will depend on f and u.

Can we find three normed vector spaces F, G and E such that

$$\forall f \in F,\ \forall c \in G,\ \exists! u_{f,c} \in E \text{ solution to } (2)$$

$$\exists C_\Omega > 0, \forall f \in F, \forall c \in G, u_{f,c} \in E \text{ defined by } (2)\ :\ \|u_{f,c}\|_E \leq C_\Omega \|f\|_F$$

Dirichlet Problem

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1, consider:

$$\begin{cases} -\Delta u(x) + c(x)u(x) = f(x) & x \in [0,1] \\ u|_{\partial\Omega} = 0 \end{cases}$$

Definition IV.2.1

If $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$, the solution is called **classical**.
Otherwise, the solution is be called **weak** *or* **variational**.

IV.2.2. Resolution in dimension $d = 1$

Example of an Elliptic PDE with Dirichlet B.C to be solved

Let $f \in L^2(0,1)$.

$$\begin{cases} -u'' = f \ \text{ in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

Example of an Elliptic PDE with Dirichlet B.C to be solved

Let $f \in L^2(0,1)$.

$$\begin{cases} -u'' = f \text{ in }]0,1[, \\ u(0) = u(1) = 0. \end{cases}$$

Remark IV.2.2

1. *It is possible to solve this equation explicitly!*
2. *Steady state transport-diffusion equation (see Labs):*

   $$-u'' + bu' + cu = f \text{ in }]0,1[\text{ with } b,\ c,\ f \text{ given functions}$$
3. *Boundary conditions: Neumann, Dirichlet-Neumann or both.*

Step 1: Find the weak formulation

- Assume $u \in C^2([0,1])$, $u(0)=u(1)=0$ is a solution.

Step 1: Find the weak formulation

- Assume $u \in C^2([0,1])$, $u(0) = u(1) = 0$ is a solution.
- Let $\phi \in C^1_c(]0,1[)$. Then

$$-\int_{]0,1[} u''\phi = \int_{]0,1[} f\,\phi.$$

Step 1: Find the weak formulation

- Assume $u \in C^2([0,1])$, $u(0) = u(1) = 0$ is a solution.
- Let $\phi \in C^1_c(]0,1[)$. Then

$$-\int_{]0,1[} u''\phi = \int_{]0,1[} f\,\phi.$$

Green's Formula (IP1 – Corollary III.3.7) gives:

$$\int_{]0,1[} u'\phi' - [u'(1)\phi(1) - u'(0)\phi(0)] = \int_{]0,1[} f\,\phi.$$

Step 1: Find the weak formulation

- Assume $u \in C^2([0,1])$, $u(0) = u(1) = 0$ is a solution.
- Let $\phi \in C^1_c(]0,1[)$. Then

$$-\int_{]0,1[} u''\phi = \int_{]0,1[} f\,\phi.$$

Green's Formula (IP1 – Corollary III.3.7) gives:

$$\int_{]0,1[} u'\phi' - [u'(1)\phi(1) - u'(0)\phi(0)] = \int_{]0,1[} f\,\phi.$$

Since $\phi(0) = \phi(1) = 0$, we get

$$\int_{]0,1[} u'\phi' = \int_{]0,1[} f\,\phi$$

Step 2: The variational formulation

Actually, the proof starts now.

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_{]0,1[} u'\phi' = \int_{]0,1[} f\,\phi$$

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_{]0,1[} u'\phi' = \int_{]0,1[} f\,\phi$$

1. Define:

$$\text{A bilinear form:} \qquad a : (u, v) \longmapsto \int_{]0,1[} u'v'$$

$$\text{A linear form:} \qquad \ell : v \longmapsto \int_{]0,1[} f\,v$$

2. a, ℓ are defined defined on $H \subset H^1(0,1)$
3. Since u vanishes on the boundary, let us choose $H = H_0^1(0,1)$ endowed with $\|\cdot\|_{H_0^1}$.

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_{]0,1[} u'\phi' = \int_{]0,1[} f\,\phi$$

1. Define:

$$\text{A bilinear form:} \qquad a : (u, v) \longmapsto \int_{]0,1[} u'v'$$

$$\text{A linear form:} \qquad \ell : v \longmapsto \int_{]0,1[} f\,v$$

2. a, ℓ are defined defined on $H \subset H^1(0, 1)$
3. Since u vanishes on the boundary, let us choose $H = H_0^1(0, 1)$ endowed with $\|\cdot\|_{H_0^1}$.

Problem to solve: Find $u \in H$ s.t. $\forall v \in H,\ a(u, v) = \ell(v)$.

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem (IV.1.6).

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem (IV.1.6).

Let $(u, v) \in (H_0^1(0,1))^2$. Using Cauchy-Schwarz, we get:

$$|a(u,v)| = \left| \int_{]0,1[} u'v' \right| \leq \|u'\|_{L^2} \|v'\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem (IV.1.6).

Let $(u, v) \in (H_0^1(0,1))^2$. Using Cauchy-Schwarz, we get:

$$|a(u,v)| = \left| \int_{]0,1[} u' v' \right| \leq \|u'\|_{L^2} \|v'\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Using Poincaré, we get:

$$|\ell(v)| = \left| \int_{]0,1[} f\, v \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C_\Omega \|f\|_{L^2} \|v\|_{H_0^1}$$

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem (IV.1.6).

Let $(u, v) \in (H_0^1(0,1))^2$. Using Cauchy-Schwarz, we get:

$$|a(u,v)| = \left| \int_{]0,1[} u'v' \right| \leq \|u'\|_{L^2} \|v'\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Using Poincaré, we get:

$$|\ell(v)| = \left| \int_{]0,1[} f\, v \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C_\Omega \|f\|_{L^2} \|v\|_{H_0^1}$$

a and ℓ are continuous on $H_0^1(0,1)$.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(0,1)$. Then

$$a(u,u) = \int_0^1 u'^2 = \|u\|_{H_0^1}^2.$$

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(0,1)$. Then

$$a(u,u) = \int_0^1 u'^2 = \|u\|_{H_0^1}^2.$$

Therefore a is coercive on $H_0^1(0,1)$.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(0,1)$. Then

$$a(u,u) = \int_0^1 u'^2 = \|u\|_{H_0^1}^2.$$

Therefore a is coercive on $H_0^1(0,1)$.

Remark IV.2.3

- *The norm we chose on H_0^1 was not by chance!*
- *It is often the tricky part.*

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u, v) = \ell(v)$$

with

1. $H = H_0^1(0, 1)$ a Hilbert space,
2. $a : H \times H \to \mathbb{R}$ bilinear and $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u,v) = \ell(v)$$

with

1. $H = H_0^1(0,1)$ a Hilbert space,
2. $a : H \times H \to \mathbb{R}$ bilinear and $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u,v) = \ell(v)$.

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u, v) = \ell(v)$$

with

1. $H = H_0^1(0, 1)$ a Hilbert space,
2. $a : H \times H \to \mathbb{R}$ bilinear and $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u, v) = \ell(v)$.

Furthermore

$$a(u, u) = \ell(u) \quad \Longrightarrow \quad \|u\|_{H_0^1} \leq C_\Omega \|f\|_{L^2}.$$

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u,v) = \ell(v)$$

with

1. $H = H_0^1(0,1)$ a Hilbert space,
2. $a : H \times H \to \mathbb{R}$ bilinear and $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u,v) = \ell(v)$.

Furthermore

$$a(u,u) = \ell(u) \quad \Longrightarrow \quad \|u\|_{H_0^1} \leq C_\Omega \|f\|_{L^2}.$$

The variational formulation is well-posed.

Step 6: Solving the PDE in $\mathcal{D}'(]0,1[)$

Since $\mathcal{D}(]0,1[) \subset H_0^1(]0,1[)$, we have

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_0^1 u'\phi' = \int_0^1 f\,\phi,$$

Step 6: Solving the PDE in $\mathcal{D}'(]0,1[)$

Since $\mathcal{D}(]0,1[) \subset H_0^1(]0,1[)$, we have

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_0^1 u'\phi' = \int_0^1 f\,\phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \left\langle u', \phi' \right\rangle = \langle f, \phi \rangle\,.$$

Step 6: Solving the PDE in $\mathcal{D}'(]0,1[)$

Since $\mathcal{D}(]0,1[) \subset H_0^1(]0,1[)$, we have

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_0^1 u'\phi' = \int_0^1 f\,\phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \left\langle u', \phi' \right\rangle = \langle f, \phi \rangle \,.$$

which is equivalent to

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \left\langle -u'' - f, \phi \right\rangle = 0.$$

Step 6: Solving the PDE in $\mathcal{D}'(]0,1[)$

Since $\mathcal{D}(]0,1[) \subset H_0^1(]0,1[)$, we have

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \int_0^1 u'\phi' = \int_0^1 f\,\phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \left\langle u', \phi' \right\rangle = \langle f, \phi \rangle .$$

which is equivalent to

$$\forall \phi \in \mathcal{D}(]0,1[), \quad \left\langle -u'' - f, \phi \right\rangle = 0.$$

Therefore:

$$-u'' = f \text{ in } \mathcal{D}'(]0,1[) \quad \text{and} \quad u(0) = u(1) = 0.$$

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(0,1)$:

- $u \in L^2(0,1)$
- $u' \in L^2(0,1)$

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(0,1)$:

- $u \in L^2(0,1)$
- $u' \in L^2(0,1)$

We have $-u'' = f$ and $f \in L^2(0,1)$ thus $u'' \in L^2(0,1)$

.

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(0,1)$:

- $u \in L^2(0,1)$
- $u' \in L^2(0,1)$

We have $-u'' = f$ and $f \in L^2(0,1)$ thus $u'' \in L^2(0,1)$

Therefore $u \in H^2(0,1)$.

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(0,1)$:

- $u \in L^2(0,1)$
- $u' \in L^2(0,1)$

We have $-u'' = f$ and $f \in L^2(0,1)$ thus $u'' \in L^2(0,1)$

Therefore $u \in H^2(0,1)$.

Furthermore, the Poincaré inequality yields

$$\|u\|_{H^2} = \sqrt{\|u\|_{H^1}^2 + \|u''\|_{L^2}^2} \leq \sqrt{(1 + C_\Omega^2)\|u\|_{H_0^1}^2 + \|f\|_{L^2}^2}$$
$$\leq \sqrt{2 + C_\Omega^2}\, C_\Omega \|f\|_{L^2}.$$

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(0,1)$:

- $u \in L^2(0,1)$
- $u' \in L^2(0,1)$

We have $-u'' = f$ and $f \in L^2(0,1)$ thus $u'' \in L^2(0,1)$

Therefore $u \in H^2(0,1)$.

Furthermore, the Poincaré inequality yields

$$\|u\|_{H^2} = \sqrt{\|u\|_{H^1}^2 + \|u''\|_{L^2}^2} \leq \sqrt{(1 + C_\Omega^2)\|u\|_{H_0^1}^2 + \|f\|_{L^2}^2}$$
$$\leq \sqrt{2 + C_\Omega^2}\, C_\Omega \, \|f\|_{L^2}.$$

Remark IV.2.4

If $f \in C^0([0,1]) \cap L^2(0,1)$, then the solution $u \in C^2([0,1])$ is a classical solution to the problem.

Conclusion

Theorem IV.2.5

Let $f \in L^2(0,1)$. The Problem: Find a solution $u \in H^2(0,1) \cap H_0^1(0,1)$ s.t.

$$\begin{cases} -u'' = f \ \text{ in }]0,1[\\ u(0) = u(1) = 0. \end{cases}$$

is well-posed: there exists a constant $\mathcal{C}_\Omega > 0$ s.t. for a given $f \in L^2(\Omega)$ there exists a unique solution $u_f \in H^2(0,1) \cap H_0^1(0,1)$ continuously dependent on f.

$$\|u_f\|_{H^2} \leq \mathcal{C}_\Omega \|f\|_{L^2}.$$

Seven steps

1. Find the **weak formulation**
2. Write the**variational formulation**
3. Prove the **continuity** of a and ℓ.
4. Prove the **coercivity** of a
5. Apply the **Lax-Milgram** theorem
6. Get solution in the sense **distributions**
7. Get the **regularity** of the solution

See the “handout” for details.

Other interesting problems (Labs)

- Non-homogeneous Dirichlet B.C.
  ($a, b \in \mathbb{R}$, $f, c \in C^0([0,1])$, $c \geq 0$)

$$\begin{cases} -u'' + cu = f & \text{in }]0,1[, \\ u(0) = a \quad \text{and} & u(1) = b. \end{cases}$$

- Non-homogeneous Neumann B.C.
  ($\alpha \in \mathbb{R}$, $f, c \in C^0([0,1])$, $c > 0$)

$$\begin{cases} -u'' + cu = f & \text{in }]0,1[, \\ u'(0) = \alpha \quad \text{and} & u'(1) = 0. \end{cases}$$

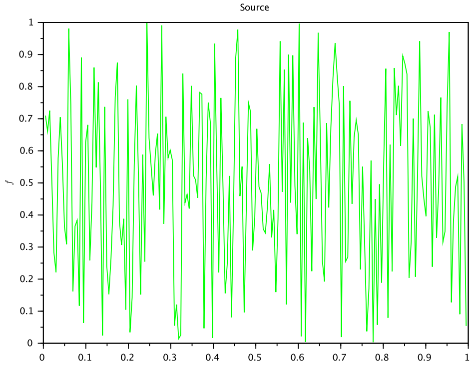
- Dirichlet-Neumann ($f, c \in C^0([0,1])$, $c \geq 0$)

$$\begin{cases} -u'' + cu = f & \text{in }]0,1[, \\ u(0) = 0 \quad \text{and} & u'(1) = 0. \end{cases}$$

Regularity

Solving

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

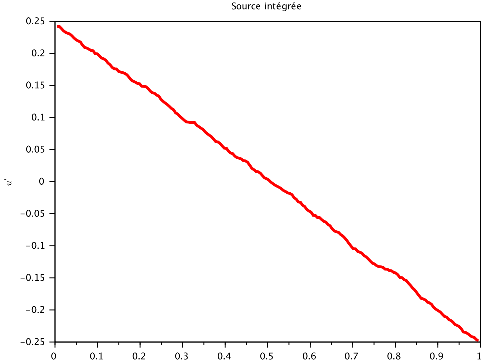


$f \in L^2(0,1)$

Regularity

Solving

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

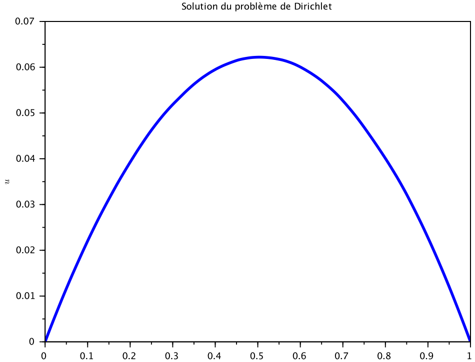


$u' \in H^1(0,1)$

Regularity

Solving

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

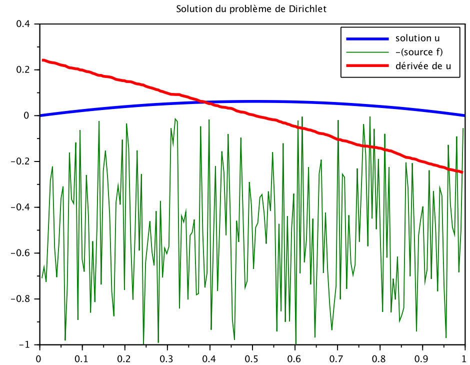


$$u \in H^2(0,1) \cap H^1_0(0,1)$$

Regularity

Solving

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$



$f \in L^2(0,1)$

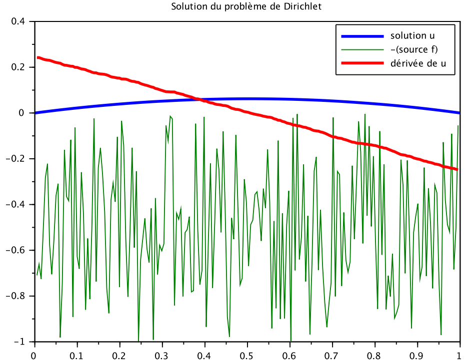
$u' \in H^1(0,1)$

$u \in H^2(0,1)$

Regularity

Solving

$$\begin{cases} -u'' = f \text{ in }]0,1[\\ u(0) = u(1) = 0 \end{cases}$$



$f \geq 0$

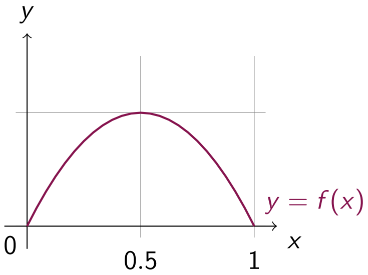
$u' \searrow$

$u(0) = u(1) = 0$ et $u \geq 0$

Qualitative property of the solution

Theorem IV.2.6 (Maximum principle)

Let $f \in L^2(0,1)$, s.t. $f \geq 0$ a.e. then the solution $u \in C^1([0,1])$ of the Dirichlet problem is non-negative on $[0,1]$.



IV.2.3. Resolution in higher dimensions

The method used for $d = 1$ can be adapted for higher dimensions.

It requires defining the space of distributions $\mathcal{D}(\Omega)$ for $\Omega \subset \mathbb{R}^d$ an open set.
It requires defining Sobolev spaces in higher dimensions.

Hereafter, we will consider $d = 2$.

Example of an Elliptic PDE with Dirichlet B.C to be solved

Let $f \in L^2(\Omega)$.

$$\begin{cases} -\Delta u = f \text{ in } \Omega, \\ u|_{\partial\Omega} = 0. \end{cases}$$

Remark IV.2.7

1. *Usually, can't be solved explicitly*
2. *Steady state Transport-diffusion PDE (see Labs) :*

   $-\Delta u + b \cdot \nabla u + cu = f$ *in* Ω *with* b, c, f *given functions.*
3. *Boundary Conditions: Dirichlet, Neumann, both*

Existence and uniqueness

Theorem IV.2.8

Let $\Omega \subset \mathbb{R}^d$ be a open bounded set and $f \in L^2(\Omega)$.

1. *There exists a unique solution $u \in H_0^1(\Omega)$ to the* **variational formulation** *associated to the Dirichlet problem.*
   Furthermore u satisfies:
   $$-\Delta u = f \text{ a.e. in } \Omega \quad \text{and} \quad u \in H_0^1(\Omega).$$
   There exists $\mathcal{C}_\Omega$ independent of f s.t.
   $$\|u\|_{H^1(\Omega)} \leq \mathcal{C}_\Omega \|f\|_{L^2(\Omega)}.$$
2. *If Ω is a regular bounded open set of class C^1, then u is solution to the Dirichlet problem*
   $$-\Delta u = f \text{ a.e. in } \Omega \text{ and } u = 0 \text{ a.e. on } \partial\Omega.$$

Step 1: Find the weak formulation

- Assume $u \in C^2(\overline{\Omega})$, $u|_{\partial\Omega} = 0$ is a solution.

Step 1: Find the weak formulation

- Assume $u \in C^2(\overline{\Omega})$, $u|_{\partial\Omega} = 0$ is a solution.
- Let $\phi \in C_c^1(\Omega)$. Then

$$-\int_\Omega (\Delta u)\phi = \int_\Omega f\,\phi.$$

Green's Formula (IP2 – Corollary III.3.8) gives:

$$\int_\Omega \nabla u \cdot \nabla\phi - \int_{\partial\Omega} \phi\nabla u \cdot n = \int_\Omega f\,\phi.$$

Since $\phi|_{\partial\Omega} = 0$, we get

$$\int_\Omega \nabla u \cdot \nabla\phi = \int_\Omega f\,\phi.$$

Step 2: The variational formulation

Actually, the proof starts now.

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\, \phi.$$

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\, \phi.$$

1. Define:

$$\text{A bilinear form:} \qquad a : (u, v) \longmapsto \int_\Omega \nabla u \cdot \nabla v$$

$$\text{A linear form} \qquad \ell : v \longmapsto \int_\Omega f\, v$$

2. a, ℓ are defined defined $H \subset H^1(\Omega)$.
3. Since u vanishes on the boundary, let us choose $H = H_0^1(\Omega)$ endowed with $\|\cdot\|_{H_0^1}$.

Step 2: The variational formulation

Let us consider (what we found in Step 1):

$$\int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\,\phi.$$

1. Define:

A bilinear form: $\quad a : (u, v) \longmapsto \int_\Omega \nabla u \cdot \nabla v$

A linear form $\quad \ell : v \longmapsto \int_\Omega f\,v$

2. a, ℓ are defined defined $H \subset H^1(\Omega)$.
3. Since u vanishes on the boundary, let us choose $H = H_0^1(\Omega)$ endowed with $\|\cdot\|_{H_0^1}$.

Problem to solve: Find $u \in H$ s.t. $\forall v \in H,\ a(u, v) = \ell(v)$.

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem.

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem.

Let $(u, v) \in (H_0^1(\Omega))^2$. Using Cauchy-Schwarz, we get:

$$|a(u, v)| = \left| \int_\Omega \nabla u \cdot \nabla v \right| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem.

Let $(u, v) \in (H_0^1(\Omega))^2$. Using Cauchy-Schwarz, we get:

$$|a(u, v)| = \left| \int_\Omega \nabla u \cdot \nabla v \right| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Using Poincaré, we get:

$$|\ell(v)| = \left| \int_\Omega f\, v \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C_\Omega \|f\|_{L^2} \|v\|_{H_0^1}$$

Step 3: Continuity of a and ℓ

Our objective is to apply the Lax-Milgram Theorem.

Let $(u, v) \in (H_0^1(\Omega))^2$. Using Cauchy-Schwarz, we get:

$$|a(u,v)| = \left| \int_\Omega \nabla u \cdot \nabla v \right| \leq \|\nabla u\|_{L^2} \|\nabla v\|_{L^2} = \|u\|_{H_0^1} \|v\|_{H_0^1}$$

Using Poincaré, we get:

$$|\ell(v)| = \left| \int_\Omega f\, v \right| \leq \|f\|_{L^2} \|v\|_{L^2} \leq C_\Omega \|f\|_{L^2} \|v\|_{H_0^1}$$

a and ℓ are continuous on $H_0^1(\Omega)$.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(\Omega)$. Then

$$a(u, u) = \int_0^1 \|\nabla u\|^2_{\mathbb{R}^d} = \|u\|^2_{H_0^1}.$$

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(\Omega)$. Then

$$a(u,u) = \int_0^1 \|\nabla u\|_{\mathbb{R}^d}^2 = \|u\|_{H_0^1}^2.$$

Therefore a is coercive on $H_0^1(\Omega)$.

Coercivity of a

We keep working toward our objective to apply Lax-Milgram.

Let $u \in H_0^1(\Omega)$. Then

$$a(u, u) = \int_0^1 \|\nabla u\|_{\mathbb{R}^d}^2 = \|u\|_{H_0^1}^2.$$

Therefore a is coercive on $H_0^1(\Omega)$.

Remark IV.2.9

- *Again, the norm we chose on H_0^1 was not by chance!*
- *It is often the tricky part.*

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u,v) = \ell(v)$$

with

1. H espace de Hilbert,
2. $a : H \times H \to \mathbb{R}$ bilinear et $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u, v) = \ell(v)$$

with

1. H espace de Hilbert,
2. $a : H \times H \to \mathbb{R}$ bilinear et $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u, v) = \ell(v)$.

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u,v) = \ell(v)$$

with

1. H espace de Hilbert,
2. $a : H \times H \to \mathbb{R}$ bilinear et $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u,v) = \ell(v)$.

Furthermore

$$a(u,u) = \ell(u) \quad \Longrightarrow \quad \|u\|_{H_0^1} \leq C_\Omega \|f\|_{L^2}.$$

Step 5: Existence and uniqueness of the weak solution

Let us apply the Lax-Milgram theorem to:

Find $u \in H$ s.t.

$$\forall v \in H, \quad a(u, v) = \ell(v)$$

with

1. H espace de Hilbert,
2. $a : H \times H \to \mathbb{R}$ bilinear et $\ell : H \to \mathbb{R}$ linear and continuous,
3. a coercive.

There exists one and only one $u \in H$ s.t. $\forall v \in H, \quad a(u, v) = \ell(v)$.

Furthermore

$$a(u, u) = \ell(u) \quad \Longrightarrow \quad \|u\|_{H_0^1} \leq C_\Omega \|f\|_{L^2}.$$

The variational formulation is well-posed.

Step 6: Solving the PDE in $\mathcal{D}'(\Omega)$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$, we have

$$\forall \phi \in \mathcal{D}(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\, \phi,$$

Step 6: Solving the PDE in $\mathcal{D}'(\Omega)$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$, we have

$$\forall \phi \in \mathcal{D}(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\, \phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle\,.$$

Step 6: Solving the PDE in $\mathcal{D}'(\Omega)$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$, we have

$$\forall \phi \in \mathcal{D}(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\, \phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle \, .$$

which is equivalent to

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle -\Delta u - f, \phi \rangle = 0.$$

Step 6: Solving the PDE in $\mathcal{D}'(\Omega)$

Since $\mathcal{D}(\Omega) \subset H_0^1(\Omega)$, we have

$$\forall \phi \in \mathcal{D}(\Omega), \quad \int_\Omega \nabla u \cdot \nabla \phi = \int_\Omega f\,\phi,$$

which is the same as

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle \nabla u, \nabla \phi \rangle = \langle f, \phi \rangle\,.$$

which is equivalent to

$$\forall \phi \in \mathcal{D}(\Omega), \quad \langle -\Delta u - f, \phi \rangle = 0.$$

Therefore:

$$-\Delta u = f \qquad \text{in } \mathcal{D}'(\Omega) \quad \text{and} \quad u|_{\partial\Omega} = 0 \quad \text{in} \quad L^2(\partial\Omega).$$

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(\Omega)$:

- $u \in L^2(\Omega)$
- $\nabla u \in L^2(\Omega)$

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(\Omega)$:

- $u \in L^2(\Omega)$
- $\nabla u \in L^2(\Omega)$

We have $-\Delta u = f$ and $f \in L^2(0,1)$ therefore $-\Delta u \in L^2(\Omega)$ a.e.

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(\Omega)$:

- $u \in L^2(\Omega)$
- $\nabla u \in L^2(\Omega)$

We have $-\Delta u = f$ and $f \in L^2(0,1)$ therefore $-\Delta u \in L^2(\Omega)$ a.e.
If Ω is a regular open set of class C^1, then the trace theorem yields $u \overset{L^2}{=} 0$ on $\partial\Omega$.

Step 7: Regularity of the solution to the PDE

We have $u \in H^1(\Omega)$:

- $u \in L^2(\Omega)$
- $\nabla u \in L^2(\Omega)$

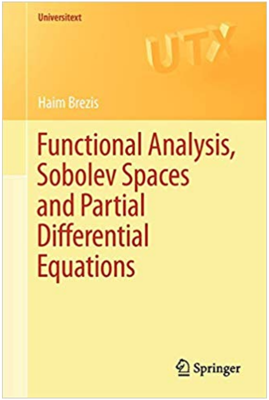
We have $-\Delta u = f$ and $f \in L^2(0,1)$ therefore $-\Delta u \in L^2(\Omega)$ a.e.
If Ω is a regular open set of class C^1, then the trace theorem yields $u \stackrel{L^2}{=} 0$ on $\partial\Omega$.

Remark IV.2.10

- *We cannot conclude immediately that $u \in H^2(\Omega)$ as we did when $d = 1$. It is false in the general case.*
- *$u \in H^2(\Omega)$ if*
  - *Ω is a disk or a square*
  - *Ω is a convex polygon*
  - *Ω is the image of a convex polygon by a diffeomorphism.*

References

To go further on Lax-Milgram.



Haïm Brézis

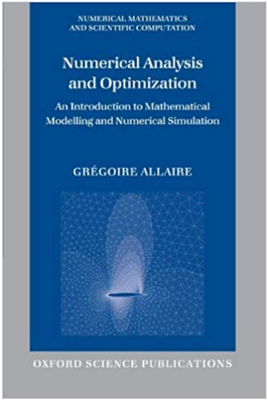
Functional Analysis, Sobolev Spaces and Partial Differential Equations

Chapter 5. Section 5.3.

Publisher: Springer; 2011 edition
ISBN-10: 0-387-70913-4
ISBN-13: 978-0387709130

References

To go further on wellposedness of elliptic equation.



Grégoire Allaire

Numerical Analysis and Optimization: An Introduction to Mathematical Modelling and Numerical Simulation

Chapter 5

Publisher: Oxford U. Pr. USA (2007)
ISBN-10: 0-19-920522-1
ISBN-13: 978-0199205226

Missing proofs for the theorems of this chapter

Some theorems that were not proven are pretty straightforward to prove and you can do so by yourself.

Some are more complicated. You can look in theses references.

Theorem IV.1.6: Brezis. Proposition 5.8
Theorem IV.2.6: Allaire. Proposition 8.4.2