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Role of Twice Fréchet-Differentiable Mappings in General Auxiliary Problem Principle

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Abstract—The approximation-solvability of the following class of nonlinear variational inequality problems (NVIP) based on the general auxiliary problem principle (GAPP) is discussed. Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0, \qquad \text{for all } x \in K,$$

where $T : K \to E^*$ is a nonlinear mapping from a nonempty closed convex subset K of a reflexive Banach space E into its dual E^*, and $f : K \to R$ is a continuous convex functional on K.

Keywords—General auxiliary problem principle, Approximation-solvability, Approximate solutions.

1. INTRODUCTION

In some recent work, Argyros and Verma [1,2] established a new global Kantorovich-type convergence theorem for Newton's method for approximating a solution of a nonlinear equation in a Banach space setting where a solution was assumed to exist and the mapping involved as m^{th} Fréchet-differentiable and Lipschitz continuous. The class of m-times continuously Fréchet-differentiable mappings seem to have some nice applications to the general auxiliary problem principle—a suitable framework to describe and analyze iterative optimization algorithms such as gradient or subgradient as well as decomposition/coordination algorithm.

Cohen [3] introduced the auxiliary problem principle and applied it to approximate solutions of a class of nonlinear variational inequalities involving strongly monotone mappings in a reflexive Banach space setting, but convergence analysis requires the mapping involved to not just be strongly monotone but Lipschitz continuous as well. When Zhu and Marcotte [4] computed the solutions of a similar class of nonlinear variational inequalities involving cocoercive mappings, because of convergence concerns in a reflexive Banach space, they investigated in R^n instead. In that situation, any bounded sequence can have a strongly convergent subsequence leading to the existence of a cluster point, and the Lipschitz continuity is derived from the cocoercivity since any β-cocoercive mapping is $(1/\beta)$-Lipschitz continuous.

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We consider here based on [1,2] a general version of the auxiliary problem principle [3] and apply it to the approximation-solvability of a general class of nonlinear variational inequalities. This framework involves a mapping $h : E \to F$ from a Banach space E into F, which may be m-times continuously Fréchet-differentiable, while in the case of the auxiliary problem principle proposed by Cohen [3], $h : E \to R$ is just a functional with first continuous derivative. In the present approximation solvability, unlike the case of Cohen [3], the mapping involved only needs to be partially relaxed monotone.

Let E and F be any two real Banach spaces with norms for both denoted by $\|\cdot\|$ and $L(E,F)$ denote the space of all bounded linear mappings from E into F. Let $T : E \to L(E,F)$ and $h : E \to F$ be two mappings, and let $\langle , \rangle$ denote the pairing between $L(E,F)$ and E. Then the norm $\|\cdot\|$ on $L(E,F)$ is defined by

$$\|Q\| = \sup\{\|Q(x)\| : \|x\| \leq 1\}, \qquad \text{for } Q \in L(E,F).$$

It turns out that $L(E,F)$ is a Banach space. Now, we consider a class of nonlinear variational inequality problems (abbreviated as NVIP): find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0, \qquad \text{for all } x \in K, \tag{1.1}$$

where $T(x^*) \in L(E,F)$ and $f : E \to F$ is continuous convex.

For $F = R$, the NVIP (1.1) reduces to: determine an element $x^* \in K$ such that

$$\langle T(x^*), x - x^* \rangle + f(x) - f(x^*) \geq 0, \qquad \text{for all } x \in K, \tag{1.2}$$

where E is a Banach space with the duality pairing $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. Here, $T : K \to E^*$ is any mapping from K, a closed convex subset of E, to E^* (dual of E).

Now, we need to recall the following definitions crucial to the work on hand, especially to the approximation-solvability of nonlinear variational inequality problems.

A mapping $T : E \to L(E,F)$ is said to be γ-μ-partially relaxed monotone if for each $x, y, z \in E$, we have

$$\langle T(x) - T(y), z - y \rangle \geq (-\gamma)\|z - x\|^2 + \mu\|x - y\|^2, \qquad \text{for constants } \gamma,\ \mu > 0.$$

It follows that the γ-μ-partial relaxed monotonicity implies the γ-partial relaxed monotonicity; that is,

$$\langle T(x) - T(y), z - y \rangle \geq (-\gamma)\|z - x\|^2, \qquad \text{for a constant } \gamma > 0.$$

When $F = R$, we find the following. A mapping $T : E \to E^*$ is called γ-μ-partially relaxed monotone if for each $x, y, z \in F$, we have

$$\langle T(x) - T(y), z - y \rangle \geq (-\gamma)\|z - x\|^2 + \mu\|x - y\|^2, \qquad \text{for constants } \gamma, \mu > 0.$$

When $\mu = 0$, the γ-μ-partial relaxed monotonicity implies the γ-partial relaxed monotonicity; that is,

$$\langle T(x) - T(y), z - y \rangle \geq (-\gamma)\|z - x\|^2, \qquad \text{for a constant } \gamma > 0.$$

The γ-partial relaxed monotonicity is more general than the notions of strong monotonicity and cocoercivity. Let us consider an example of a γ-partially relaxed monotone mapping [5].

EXAMPLE 1.1. Let $T : R^n \to R^n$ be defined by

$$T(x) = dI(x) + v,$$

where $d > 0$, $x, v \in R^n$ with v fixed, and I is the $n \times n$ identity matrix. Then, T is γ-partially relaxed monotone for $d = \gamma$, and if T is γ-partially relaxed monotone, then $d < 4\gamma$.

To see that T is γ-partially relaxed monotone, let $x, y, z \in R^n$. Then, we have

$$\|y-z\|^2 + \|y-x\|^2 + \|x-z\|^2 \geq 0.$$

It follows that

$$\langle y-z, y-z\rangle + \langle y-x, y-x\rangle + \langle x-z, x-z\rangle \geq 0,$$

or

$$-\langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle + \langle z, z\rangle - \langle z, x\rangle + \langle x, x\rangle \geq 0,$$

or

$$d[\langle x-y, z-y\rangle + \langle z-x, z-x\rangle] \geq 0,$$

or

$$\langle dx - dy, z-y\rangle + d\|z-x\|^2 \geq 0,$$

or

$$\langle T(x) - T(y), z-y\rangle + d\|z-x\|^2 \geq 0,$$

which clearly shows that T is γ-partially relaxed monotone for $d = \gamma$. For a more detailed account on the partial relaxed monotonicity, we recommend [5,6].

2. GENERAL AUXILIARY PROBLEM PRINCIPLE

In this section, we discuss the approximation-solvability of the NVIP (1.2), based on a general version of the existing auxiliary problem principle (APP) initiated by Cohen [3] and applied and studied by others. Let E and F be any two Banach spaces with norms denoted $\|\cdot\|$ and $L(E,F)$ be the space of all bounded linear mappings from E into F. If $T : E \to L(E,F)$ and $h : E \to F$ are two mappings and $\langle , \rangle$ denotes the pairing between $L(E,F)$ and E, then the general auxiliary problem principle (GAPP) is described as follows.

GAPP 2.1. *For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)*

$$\left\langle \rho T\left(x^k\right) + h'\left(x^{k+1}\right) - h'\left(x^k\right), x - x^{k+1}\right\rangle + \rho\left[f(x) - f\left(x^{k+1}\right)\right] \geq 0, \qquad \text{for all } x \in K, \quad (2.1)$$

where $h : E \to F$ is two-times continuously Fréchet-differentiable, $f : E \to F$ is proper continuous convex, $\rho > 0$ a constant, and $h''(x) \in L(E^2, F)$ denotes the second Fréchet-derivative of h.

For $F = R$ in (2.1), we have the following problem.

GAPP 2.2. *For a given iterate x^k, determine an x^{k+1} such that (for $k \geq 0$)*

$$\left\langle \rho T\left(x^k\right) + h'\left(x^{k+1}\right) - h'\left(x^k\right), x - x^{k+1}\right\rangle + \rho\left[f(x) - f\left(x^{k+1}\right)\right] \geq 0, \qquad \text{for all } x \in K, \quad (2.2)$$

where $h : E \to R$ is two-times continuously Fréchet-differentiable, $f : E \to R$ is proper convex lower semicontinuous convex, and ρ is a positive constant.

The GAPP (2.2) is an extension to the auxiliary problem principle proposed and studied by Cohen [3].

Next, we recall some auxiliary results crucial to the approximation-solvability of the NVIP (1.1).

LEMMA 2.1. *Let E_1 and E_2 be two Banach spaces and $h : E_1 \to E_2$ be a nonlinear mapping such that h is two-times continuously Fréchet-differentiable on E_1. Suppose that there exist an $x^* \in E_1$ and nonnegative numbers α and α_2 such that*

$$\left\langle h''(x) - h''(x^*), (x - x^*)^2\right\rangle \geq \alpha\|x - x^*\|^3$$

and

$$\|h''(x^*)\| \geq \alpha_2.$$

Then we have

$$h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle \geq \left[\left(\frac{\alpha_2}{2!} \right) + \left(\frac{\alpha}{3!} \right) \|x - x^*\| \right] \|x - x^*\|^2.$$

PROOF. The proof follows using the above hypotheses and the following identity [1]:

$$\begin{aligned} h(x) - h(x^*) - \langle h'(x^*), x - x^* \rangle &= \int_0^1 \langle h'(x^* + \theta_1(x - x^*)) - h'(x^*), x - x^* \rangle \, d\theta_1 \\ &= \int_0^1 \int_0^1 \left\langle h''(x^* + \theta_2\theta_1(x - x^*)), \theta_1(x - x^*)^2 \right\rangle \, d\theta_1 \, d\theta_2 \\ &= \int_0^1 \int_0^1 \left\langle h''(x^* + \theta_2\theta_1(x - x^*)) - h''(x^*), \theta_1(x - x^*)^2 \right\rangle \, d\theta_1 \, d\theta_2 \\ &\quad + \int_0^1 \int_0^1 \left\langle h''(x^*), \theta_1(x - x^*)^2 \right\rangle \, d\theta_1 \, d\theta_2. \end{aligned}$$

Now, we are just about ready to present, based on the GAPP (2.2), the approximation-solvability of the NVIP (1.2).

THEOREM 2.1. *Let E be a reflexive Banach space and $T : K \to E^*$ a γ-μ-partially relaxed monotone mapping from a nonempty closed convex subset K of E into E^*, the dual of E. Let $f : K \to R$ be proper, convex, and lower semicontinuous (lsc) on K and $h : K \to R$ be two-times continuously Fréchet-differentiable on K. Suppose that there exist an $x' \in K$ and nonnegative numbers α_2 and α such that*

$$\left\langle h''(x) - h''(x'), (x - x')^2 \right\rangle \geq \alpha \|x - x'\|^3, \tag{2.3}$$

and

$$\|h''(x')\| \geq \alpha_2. \tag{2.4}$$

Then, for any fixed solution $x^* \in K$ of the NVIP (1.2) and

$$0 < \rho < \frac{\alpha_2}{2\gamma},$$

we find that the sequence $\{x^k\}$ converges (strongly) to x^*.

PROOF. To show the sequence $\{x^k\}$ converges to x^*, a solution of the NVIP (1.2), we need to compute the estimates. Let us define a function Λ^* by

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x \rangle.$$

Then, by Lemma 2.1 we have

$$\Lambda^*(x) := h(x^*) - h(x) - \langle h'(x), x^* - x \rangle \geq \left\{ \left[\frac{\alpha_2}{2!} \right] + \left[\frac{\alpha}{3!} \right] \|x - x^*\| \right\} \|x - x^*\|^2,$$

where x^* is any fixed solution of the NVIP (1.2). It follows that

$$\Lambda^* \left(x^{k+1}\right) = h(x^*) - h\left(x^{k+1}\right) - \left\langle h'\left(x^{k+1}\right), x^* - x^{k+1} \right\rangle.$$

Now, we can write

$$\begin{aligned} \Lambda^*\left(x^k\right) - \Lambda^*\left(x^{k+1}\right) &= h\left(x^{k+1}\right) - h\left(x^k\right) - \left\langle h'\left(x^k\right), x^{k+1} - x^k \right\rangle \\ &\quad + \left\langle h'\left(x^{k+1}\right) - h'\left(x^k\right), x^* - x^{k+1} \right\rangle \\ &\geq \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha}{3!} \right) \left\|x^{k+1} - x^k\right\| \right] \left\|x^{k+1} - x^k\right\|^2 \\ &\quad + \left\langle h'\left(x^{k+1}\right) - h'\left(x^k\right), x^* - x^{k+1} \right\rangle \\ &\geq \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha}{3!} \right) \left\|x^{k+1} - x^k\right\| \right] \left\|x^{k+1} - x^k\right\|^2 \\ &\quad + \rho \left\langle T\left(x^k\right), x^{k+1} - x^* \right\rangle + \rho \left(f\left(x^{k+1}\right) - f(x^*) \right), \end{aligned} \tag{2.5}$$

for $x = x^*$ in (2.2).

If we replace x by x^{k+1} in (1.2) and combine with (2.5), we obtain

$$\begin{aligned}\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right) &\geq \left[\frac{\alpha_2}{2!}+\left(\frac{\alpha}{3!}\right)\left\|x^{k+1}-x^k\right\|\right]\left\|x^{k+1}-x^k\right\|^2 \\ &\quad+\rho\left\langle T\left(x^k\right), x^{k+1}-x^*\right\rangle-\rho\left\langle T(x^*), x^{k+1}-x^*\right\rangle \\ &= \left[\frac{\alpha_2}{2!}+\left(\frac{\alpha}{3!}\right)\left\|x^{k+1}-x^k\right\|\right]\left\|x^{k+1}-x^k\right\|^2 \\ &\quad+\rho\left\langle T\left(x^k\right)-T(x^*), x^{k+1}-x^*\right\rangle .\end{aligned}$$

Since T is γ-μ-partially relaxed monotone, it implies that

$$\begin{aligned}\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right) &\geq \left[\frac{\alpha_2}{2!}-\left(\frac{\alpha}{3!}\right)\left\|x^{k+1}-x^k\right\|\right]\left\|x^{k+1}-x^k\right\|^2 \\ &\quad-\rho\gamma\left\|x^k-x^{k+1}\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2 \\ &\geq \left[\frac{\alpha_2}{2!}\right]\left\|x^{k+1}-x^k\right\|^2-\rho\gamma\left\|x^k-x^{k+1}\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2 \\ &= \left(\frac{1}{2}\right)\left[\alpha_2-2\rho\gamma\right]\left\|x^k-x^{k+1}\right\|^2+\rho\mu\left\|x^k-x^*\right\|^2 \\ &\geq \rho\mu\left\|x^k-x^*\right\|^2, \qquad \text{for } \alpha_2-2\rho\gamma>0;\end{aligned}$$

that is,

$$\Lambda^*\left(x^k\right)-\Lambda^*\left(x^{k+1}\right) \geq \rho\mu\left\|x^k-x^*\right\|^2, \qquad \text{for } \alpha_2-2\rho\gamma>0. \tag{2.6}$$

It follows from (2.6) that (except for $x^k = x^*$) $\Lambda^*(x^k) - \Lambda^*(x^{k+1})$ is nonnegative and as a result, the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing for

$$0<\rho<\frac{\alpha_2}{2\gamma},$$

and nonnegative by its definition. Thus, it converges to some number, and therefore, the difference of two successive terms tends to zero, which implies in turn that $x^k \to x^*$ (strongly) as $k \to \infty$. This concludes the proof.

When we consider the case of γ-partially relaxed monotone mappings, the convergence analysis changes dramatically in the sense that we have to restrict even involved Banach spaces to be finite-dimensional so that a bounded sequence can have a cluster point. Otherwise, any bounded sequence in a reflexive (infinite-dimensional) Banach space converges only weakly.

THEOREM 2.2. *Let E be a reflexive (finite-dimensional) Banach space and $T : K \to E^*$ be γ-partially relaxed monotone from a nonempty closed subset K of E into E^*, the dual of E. Let $f : K \to R$ be proper, convex, and lower semicontinuous (lsc) on K and $h : K \to R$ be two-times continuously Fréchet-differentiable on K. Suppose that there exist an $x' \in K$ and nonnegative numbers α_2 and α such that*

$$\left\langle h''(x)-h''(x'),(x-x')^2\right\rangle \geq \alpha\|x-x'\|^3, \tag{2.7}$$

and

$$\|h''(x')\| \geq \alpha_2, \tag{2.8}$$

$$\left\langle h''(x)-h''(x'),(x-x')^2\right\rangle \leq \beta\|x-x'\|^3, \tag{2.9}$$

and

$$\|h''(x')\| \leq \beta_2. \tag{2.10}$$

Then, if $x^* \in K$ is any fixed solution of the NVIP (1.2) and

$$0 < \rho < \frac{\alpha_2}{2\gamma},$$

it follows that the sequence $\{x^k\}$ converges (strongly) to x^*.

PROOF. Based on the proof of Theorem 2.1, using the γ-partial relaxed monotonicity of the mapping T, we can have

$$\begin{aligned}\Lambda^*\left(x^k\right) - \Lambda^*\left(x^{k+1}\right) &\geq \left[\frac{\alpha_2}{2!} + \left(\frac{\alpha}{3!}\right)\left\|x^{k+1} - x^k\right\|\right]\left\|x^{k+1} - x^k\right\|^2 - \rho\gamma\left\|x^k - x^{k+1}\right\|^2 \\ &\geq \left[\frac{\alpha_2}{2!}\right]\left\|x^{k+1} - x^k\right\|^2 - \rho\gamma\left\|x^k - x^{k+1}\right\|^2 \\ &= \left(\frac{1}{2}\right)[\alpha - 2\rho\gamma]\left\|x^k - x^{k+1}\right\|^2.\end{aligned} \tag{2.11}$$

It follows from similar arguments as that of the proof of Theorem 2.1 that the sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing for

$$0 < \rho < \frac{\alpha_2}{2\gamma},$$

and nonnegative by its definition. As a result, it converges to some number, which implies that the difference of two successive terms tends to zero. It follows in turn that

$$\lim_{k\to\infty}\left\|x^k - x^{k+1}\right\|^2 = 0.$$

Since (applying (2.7)) $\|x^k - x^*\|^3 \leq (1/\alpha)\Lambda^*(x^k)$ and sequence $\{\Lambda^*(x^k)\}$ is strictly decreasing, we claim that the sequence $\{x^k\}$ is bounded. Thus, if we assume c is any cluster point of the sequence $\{x^k\}$ and take the limit in (2.2), then c is a solution of the NVIP (1.2).

Now, if we replace x^* by c, the above convergence analysis can still hold for the corresponding function Λ. That means the sequence $\{\Lambda(x^k)\}$ is strictly decreasing. It follows from (2.9) that

$$\Lambda\left(x^k\right) \leq \beta\left\|x^k - c\right\|^3.$$

This implies that the sequence $\{\Lambda(x^k)\}$ tends to zero. On the other hand, using (2.7), we have

$$\Lambda\left(x^k\right) \geq \alpha\left\|x^k - c\right\|^3.$$

Based on the above arguments, we conclude that the entire sequence $\{x^k\}$ converges to c.

3. APPLICATIONS

This section deals with some applications to Theorem 2.2 on partially relaxed monotone mappings in R^n, that is, for $E = R^n$. For $E = R^n$ and $F = R$, the NVIP (1.1) reduces to the following. Find an element $x^* \in K$ such that

$$\langle T(x^*), x - x^*\rangle + f(x) - f(x^*) \geq 0, \qquad \text{for all } x \in K, \tag{3.1}$$

where $T : K \to R^n$ is any mapping from K, a closed convex subset of R^n, into R^n.

THEOREM 3.1. *Let $T : K \to R^n$ be γ-partially relaxed monotone from a nonempty closed convex subset K of R^n into R^n, $f : K \to R$ proper convex continuous on K, and $h : K \to R$ two-times continuously Fréchet-differentiable on K. Suppose that there exist an $x' \in K$ and nonnegative numbers α_2 and α such that*

$$\left\langle h''(x) - h''(x'), (x - x')^2\right\rangle \geq \alpha\|x - x^*\|^3,$$

and

$$\|h''(x')\| \geq \alpha_2,$$

and

$$\langle h''(x) - h''(x'), (x - x')^2 \rangle \leq \beta \|x - x^*\|^3,$$

and

$$\|h''(x')\| \leq \beta_2.$$

Then, if $x^* \in K$ *is any fixed solution of the NVIP (3.1) and*

$$0 < \rho < \frac{\alpha_2}{2\gamma},$$

the sequence $\{x^k\}$ *converges (strongly) to* x^*.

When $T = dI(x) + v$, as in Example 1.1, and $E = R^n$ in Theorem 2.2, we have the following theorem.

THEOREM 3.2. *Let* $T : K \to R^n$ *be a mapping as in Example 1.1,* $f : K \to R$ *be proper convex continuous on* K*, and* $h : K \to R$ *be two-times continuously Fréchet-differentiable. Suppose that there exist an* $x' \in K$ *and nonnegative numbers* α_2 *and* α *such that*

$$\langle h''(x) - h''(x'), (x - x')^2 \rangle \geq \alpha \|x - x^*\|^3,$$

and

$$\|h''(x')\| \geq \alpha_2,$$

$$\langle h''(x) - h''(x'), (x - x')^2 \rangle \leq \beta \|x - x^*\|^3,$$

and

$$\|h''(x')\| \leq \beta_2.$$

Then, if $x^* \in K$ *is any fixed solution of the NVIP (3.1) and*

$$0 < \rho < \frac{\alpha_2}{2d},$$

it follows that the sequence $\{x^k\}$ *converges (strongly) to* x^*.

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