CentraleSupélec

ST7 – Optimization
Part VI.1: Integer linear programming

整数线性规划.

NP-complete

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0-1 整数线性规划, binary.

NP-hardness.

Motivation example in drug logistics

- **Problem**: A health organization wants to distribute g_t vaccine doses against some disease at week $t \in \{1, \ldots, T\}$. Vaccines can be bought from P suppliers with unit cost $(c_{p,t})_{1 \leq p \leq P, 1 \leq t \leq T}$. Supplier $p \in \{1, \ldots, P\}$ can only deliver a minimum quantity $m_{p,t}$ and a maximum quantity $M_{p,t}$ of doses. The health organization has a storage facility allowing to keep at most s doses from one week to the other (any further excess is wasted).
  What is the minimal cost strategy?

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  What is the minimal cost strategy?
- **Variables**: For every $p \in \{1, \ldots, P\}$ and $t \in \{1, \ldots, T\}$,
  - $z_{p,t}$: binary decision variable indicating if doses are bought or not from supplier p at week t
  - $n_{p,t}$: number of doses bought from supplier p at week t
  - q_t: number of stored doses at the end of week t.

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  $\rightsquigarrow$ vector of variables x of dimension $N = (2P + 1)T$.
- **Cost function**

$$f(x) = \sum_{t=1}^{T} c_{p,t} n_{p,t}.$$

Motivation example in drug logistics

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  - q_t: number of stored doses at the end of week t.
- **Constraints**: By setting $q_0 = 0$,

$$(\forall t \in \{1, \ldots, T\}) \quad \begin{cases} q_{t-1} + \sum_{p=1}^{P} n_{p,t} \geq q_t + g_t \\ q_t \leq s \\ (\forall p \in \{1, \ldots, P\}) \;\; m_{p,t} z_{p,t} \leq n_{p,t} \leq M_{p,t} z_{p,t} \end{cases}$$

$\rightsquigarrow$ linear inequality constraints.

Motivation example in drug logistics

- **Optimization formulation** 优化公式

$$
\begin{aligned}
&\underset{x \in \mathbb{R}^{T(2P+1)}}{\text{minimize}} \quad \sum_{t=1}^{T} c_{p,t} n_{p,t} \\
&\text{s.t} \quad (\forall t \in \{1, \ldots, T\}) \\
&\quad \begin{cases}
q_{t-1} + \sum_{p=1}^{P} n_{p,t} \geq q_t + g_t \\
q_t \leq s \\
\color{blue}{q_t \in \mathbb{N}} \\
(\forall p \in \{1, \ldots, P\}) \quad \begin{cases}
m_{p,t} z_{p,t} \leq n_{p,t} \leq M_{p,t} z_{p,t} \\
\color{blue}{n_{p,t} \in \mathbb{N}} \\
\color{red}{z_{p,t} \in \{0, 1\}.}
\end{cases}
\end{cases}
\end{aligned}
$$

Motivation example in drug logistics

药品物流

- **Optimization formulation**

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad \sum_{t=1}^{T} c_{p,t} n_{p,t}$$

$$\text{s.t} \quad (\forall t \in \{1, \dots, T\})$$

$$\begin{cases} q_{t-1} + \sum_{p=1}^{P} n_{p,t} \geq q_t + g_t \\ q_t \leq s \\ (\forall p \in \{1, \dots, P\}) \quad \left\{ m_{p,t} z_{p,t} \leq n_{p,t} \leq M_{p,t} z_{p,t} \right. \end{cases}$$

with $\mathcal{N} = \{0,1\}^{PT} \times \mathbb{N}^{PT} \times \mathbb{N}^{T}$.

Canonical problem 经典问题.

Let $L \in \mathbb{R}^{K\times N}$, $b \in \mathbb{R}^K$, and $c \in \mathbb{R}^N$.

Let $\mathcal{N}$ be a nonempty subset of $\mathbb{N}^N$.

We consider the following integer linear programming problem:

$$\underset{x\in\mathcal{N}}{\text{minimize}} \quad \langle c \mid x\rangle \quad \text{s.t.} \quad Lx \geq b.$$

- The feasibility set is 可行集.

$$\mathcal{A}_{\mathcal{N}} = \{x \in \mathcal{N} \mid Lx \geq b\}.$$

- The problem is feasible if $\mathcal{A}_{\mathcal{N}} \neq \varnothing$.
- The problem is (lower) bounded if $\mu_{\mathcal{N}} = \inf_{x\in\mathcal{A}_{\mathcal{N}}} \langle c \mid x\rangle > -\infty$.

<u>Remark</u>: Often $\mathcal{N}$ is finite and the problem is thus bounded.
If, in addition, it is feasible, then it admits a solution.

Canonical problem

Let $L \in \mathbb{R}^{K \times N}$, $b \in \mathbb{R}^K$, and $c \in \mathbb{R}^N$.
Let $\mathcal{N}$ be a nonempty subset of $\mathbb{N}^N$.
We consider the following integer linear programming problem:

$$\underset{x \in \mathcal{N}}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \geq b.$$

Remarks:

N is a nonempty

- $\mathcal{N}$ is nonconvex (when it does not reduce to a singleton)
  $\Rightarrow$ an ILP problem is nonconvex. 非凸.
- If $\mathcal{N} = \{0, 1\}^N$, we obtain a binary linear programming problem.
  二进制线性规划.

Binary linear programming

Any integer linear programming problem over a finite set $\mathcal{N}$ is equivalent to a binary linear programming problem

等价.

Binary linear programming

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Proof: If $\mathcal{N}$ is finite, $\mathcal{N} = \{e_1, \ldots, e_B\}$ where $(\forall b \in \{1, \ldots, B\})\ e_b \in \mathbb{N}^N$. Therefore, $x \in \mathcal{N}$ if and only if there exists $\widetilde{x} = (\widetilde{x}^{(b)})_{1 \leq b \leq B} \in \{0,1\}^B$ such that

$$\underset{x \in \mathcal{N}}{\text{minimize}}\ \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \geq b.$$

$$x = \sum_{b=1}^{B} \widetilde{x}^{(b)} e_b$$

$$1^\top \widetilde{x} = 1.$$

$$\sum_{b=1}^{B} \widetilde{x}^{(b)} = 1 \quad \Leftrightarrow \quad \langle 1 \mid \widetilde{x} \rangle = 1$$

where $1 = [1, \ldots, 1]^\top \in \mathbb{R}^B$. Thus, the problem can be recast as

$$\underset{\widetilde{x} \in \{0,1\}^B}{\text{minimize}}\ \langle \widetilde{c} \mid \widetilde{x} \rangle \quad \text{s.t.} \quad \widetilde{L}\widetilde{x} \geq \widetilde{b}$$

where $\widetilde{L} = \begin{bmatrix} Le_1 & \cdots & Le_B \\ 1 & \cdots & 1 \\ -1 & \cdots & -1 \end{bmatrix}$, $\widetilde{b} = \begin{bmatrix} b \\ 1 \\ -1 \end{bmatrix}$, and $\widetilde{c} = \begin{bmatrix} \langle c \mid e_1 \rangle \\ \vdots \\ \langle c \mid e_B \rangle \end{bmatrix}$.

Knapsack problem 背包问题.

A knapsack problem is a binary linear programming problem with $K = 1$, $L \leq 0$, $b \leq 0$, and $c \leq 0$.

Knapsack problem

A knapsack problem is a binary linear programming problem with $K = 1$, $L \leq 0$, $b \leq 0$, and $c \leq 0$.

Interpretation: Let $\overline{\ell} = -L^T$, $\overline{b} = -b$, and $\overline{c} = -c$.
The problem can be rewritten as

$$\underset{x \in \{0,1\}^N}{\text{maximize}} \sum_{i=1}^{N} \overline{c}^{(i)} x^{(i)} \quad \text{s.t.} \quad \sum_{i=1}^{N} \overline{\ell}^{(i)} x^{(i)} \leq \overline{b}.$$

We want to fill a knapsack with N possible objects by maximizing the value of its contents. The i-th component $x^{(i)}$ of x indicates whether the i-th object is present ($x^{(i)} = 1$) or not ($x^{(i)} = 0$). The components $(\overline{c}^{(i)})_{1 \leq i \leq N}$ of $\overline{c}$ correspond to the value of each possible object.
In addition, we have a limitation $\overline{b}$ on the global weight of the contents of the knapsack. The components $(\overline{\ell}^{(i)})_{1 \leq i \leq N}$ of $\overline{\ell}$ correspond to the weights of each object.

Knapsack problem

Any feasible binary linear programming problem with $K = 1$ is equivalent to a Knapsack problem.

NP问题 (*).

Knapsack problem

Any feasible binary linear programming problem with $K = 1$ is equivalent to a Knapsack problem.

The binary LP reads

$$\underset{(x^{(i)})_{1\leq i\leq N}\in\{0,1\}^N}{\text{minimize}} \sum_{i=1}^{N} c^{(i)} x^{(i)} \quad \text{s.t.} \quad \sum_{i=1}^{N} \ell^{(i)} x^{(i)} \geq b.$$

For every $j \in \{1, \ldots, N\}$,

- if $c^{(j)} \leq 0$ and $\ell^{(j)} \geq 0$, setting $x^{(j)} = 1$ leads to a lower value of the cost while giving more freedom in the choice of the $(x^{(i)})_{i\neq j}$ So it is the best choice for the j-th optimized variable.
- If $c^{(j)} > 0$ and $\ell^{(j)} \leq 0$, by symmetry, $x^{(j)} = 0$ is the optimal choice.

Let $\mathbb{I}$ be the indices of the remaining components to be optimized.

Knapsack problem

> Any feasible binary linear programming problem with $K = 1$ is equivalent to a Knapsack problem.

If $\mathbb{I} \neq \varnothing$, the binary LP reduces to

$$\operatorname*{minimize}_{(x^{(i)})_{i\in\mathbb{I}}\in\{0,1\}^{|\mathbb{I}|}} \sum_{i\in\mathbb{I}} c^{(i)}x^{(i)} \quad \text{s.t.} \quad \sum_{i\in\mathbb{I}} \ell^{(i)}x^{(i)} \geq b' = b - \sum_{i\in\{1,\dots,N\}\setminus\mathbb{I}} \ell^{(i)}x^{(i)}.$$

We have $\mathbb{I} = \mathbb{I}_+ \cup \mathbb{I}_-$ where $\mathbb{I}_+ = \{i \in \mathbb{I} \mid \ell^{(i)} > 0, c^{(i)} > 0\}$ and $\mathbb{I}_- = \{i \in \mathbb{I} \mid \ell^{(i)} < 0, c^{(i)} \leq 0\}$. So, we have to

$$\begin{aligned}
&\operatorname*{minimize}_{(x^{(i)})_{i\in\mathbb{I}}\in\{0,1\}^{|\mathbb{I}|}} \sum_{i\in\mathbb{I}_-} c^{(i)}x^{(i)} + \sum_{i\in\mathbb{I}_+}(-c^{(i)})(1-x^{(i)})\\
&\text{s.t.} \quad \sum_{i\in\mathbb{I}_-} \ell^{(i)}x^{(i)} + \sum_{i\in\mathbb{I}_+}(-\ell^{(i)})(1-x^{(i)}) \geq b' - \sum_{i\in\mathbb{I}_+} \ell^{(i)} = \widetilde{b}.
\end{aligned}$$

Since the problem is feasible, $\widetilde{b} \leq 0$, and we thus obtain a Knapsack problem with respect to $\widetilde{x} = (\widetilde{x}^{(i)})_{1\leq i\leq N} \in \{0,1\}^{|\mathbb{I}|}$ where

$$(\forall i \in \mathbb{I}) \quad \widetilde{x}^{(i)} = \begin{cases} x^{(i)} & \text{if } i \in \mathbb{I}_- \\ 1 - x^{(i)} & \text{if } i \in \mathbb{I}_+. \end{cases}$$

Some combinatorial problems

Let $(S_i)_{1\leq i\leq N}$ be nonempty subsets of a set E and let $\mathbb{I} \subset \{1, \ldots, N\}$.

- $(S_i)_{i\in\mathbb{I}}$ is a cover of E if $\cup_{i\in\mathbb{I}} S_i = E$
- $(S_i)_{i\in\mathbb{I}}$ is a packing of E if $(\forall (i,j) \in \mathbb{I}^2)\ i \neq j \Rightarrow S_i \cap S_j = \varnothing$
- $(S_i)_{i\in\mathbb{I}}$ is a partition of E if it is a coverage and a packing of E.

组合问题.

partition.

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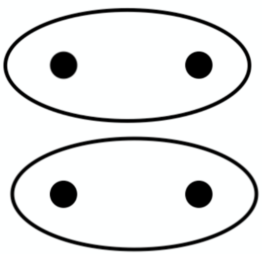
$N = 4$



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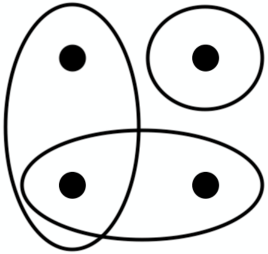

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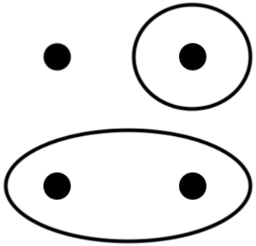

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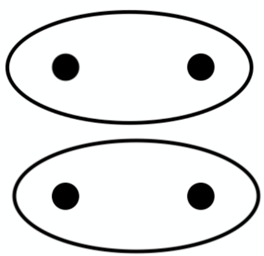
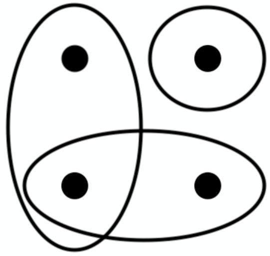
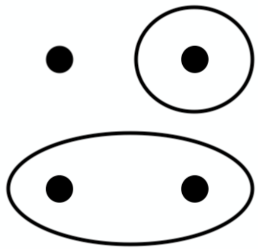


partition



cover



packing

组合优化.

Some combinatorial problems

Let $(S_i)_{1\leq i\leq N}$ be nonempty subsets of a set E and let $\mathbb{I}\subset\{1,\ldots,N\}$.

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- $(S_i)_{i\in\mathbb{I}}$ is a partition of E if it is a coverage and a packing of E.

Assume that card $E = K$ and let $L=(L_{j,i})_{1\leq j\leq K, 1\leq i\leq N}\in\{0,1\}^{K\times N}$ be such that, for every $(i,j)\in\{1,\ldots,N\}\times\{1,\ldots,K\}$,

$$L_{j,i}=\begin{cases}1 & \text{if the } j\text{-th element of } E \text{ belongs to } S_i\\ 0 & \text{otherwise.}\end{cases}$$

Let $x=(x^{(i)})_{1\leq i\leq N}\in\{0,1\}^N$ be such that

$$(\forall i\in\{1,\ldots,N\})\quad x^{(i)}=\begin{cases}1 & \text{if } i\in\mathbb{I}\\ 0 & \text{otherwise.}\end{cases}$$

Then

- $(S_i)_{i\in\mathbb{I}}$ is a cover of E if $Lx\geq 1$
- $(S_i)_{i\in\mathbb{I}}$ is a packing of E if $Lx\leq 1$
- $(S_i)_{i\in\mathbb{I}}$ is a partition of E if $Lx=1$.

Some combinatorial problems

Optimal cover

Assume that, for every $i \in \{1, \dots, N\}$, selecting S_i has a cost $c^{(i)} > 0$.
We want to find a cover with minimum global cost.

Reformulation:

$$\operatorname*{minimize}_{x=(x^{(i)})_{1\leq i\leq N}\in\{0,1\}^N} \sum_{i=1}^{N} c^{(i)} x^{(i)} \quad \text{s.t.} \quad Lx \geq 1$$

that is

$$\operatorname*{minimize}_{x\in\{0,1\}^N} \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \geq 1$$

where $c = (c^{(i)})_{1\leq i\leq N}$.

Some combinatorial problems

Optimal packing

Assume that, for every $i \in \{1, \ldots, N\}$, selecting S_i has a cost $c^{(i)} < 0$.
We want to find a packing with minimum global cost.

Some combinatorial problems

Optimal packing

Assume that, for every $i \in \{1, \ldots, N\}$, selecting S_i has a cost $c^{(i)} < 0$.
We want to find a packing with minimum global cost.

Reformulation:

$$\underset{x \in \{0,1\}^N}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \leq 1.$$

Some combinatorial problems

Optimal packing

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Reformulation:

$$\underset{x \in \{0,1\}^N}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \leq 1.$$

Optimal partition

Assume that, for every $i \in \{1, \dots, N\}$, selecting S_i has a cost $c^{(i)}$.
We want to find a partition with minimum global cost.

Some combinatorial problems

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We want to find a partition with minimum global cost.

Reformulation:

$$\underset{x \in \{0,1\}^N}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx = 1.$$

Continuous relaxation 松弛.

ILP problem

Let $L \in \mathbb{R}^{K\times N}$, $b \in \mathbb{R}^K$, and $c \in \mathbb{R}^N$.
Let $\mathcal{N}$ be a nonempty subset of $\mathbb{N}^N$.
We consider the following integer linear programming problem:

$$\underset{x\in\mathcal{N}}{\text{minimize}} \quad \langle c \mid x\rangle \quad \text{s.t.} \quad Lx \geq b.$$

Convex relaxation

Let $L \in \mathbb{R}^{K\times N}$, $b \in \mathbb{R}^K$, and $c \in \mathbb{R}^N$.
Let S be a nonempty closed convex such that $\mathcal{N} \subset S$.
We consider the following convex problem:

$$\underset{x\in S}{\text{minimize}} \quad \langle c \mid x\rangle \quad \text{s.t.} \quad Lx \geq b.$$

Continuous relaxation

Convex relaxation

Let $L \in \mathbb{R}^{K\times N}$, $b \in \mathbb{R}^{K}$, and $c \in \mathbb{R}^{N}$.
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Basic properties

Let

$$\mathcal{A}_{\mathcal{N}} = \left\{ x \in \mathcal{N} \;\middle|\; Lx \geq b \right\}, \quad \mu_{\mathcal{N}} = \inf_{x\in\mathcal{A}_{\mathcal{N}}} \langle c \mid x \rangle$$
$$\mathcal{A}_{S} = \left\{ x \in S \;\middle|\; Lx \geq b \right\}, \quad \mu_{S} = \inf_{x\in\mathcal{A}_{S}} \langle c \mid x \rangle .$$

Then,

- $\mathcal{A}_{\mathcal{N}} \subset \mathcal{A}_{S}$
- if $\mathcal{A}_{S} = \varnothing$, the ILP problem is unfeasible
- $\mu_S \leq \mu_{\mathcal{N}}$
- if $c \in \mathbb{Z}^N$, $\lceil \mu_S \rceil \leq \mu_{\mathcal{N}}$ ($\lceil\cdot\rceil$: *round up*).

Continuous relaxation

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Let S be a nonempty closed convex such that $\mathcal{N} \subset S$.
We consider the following convex problem:

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Basic properties

Let $\widehat{x}$ be a solution to the convex relaxation.
$\widehat{x} \in \mathcal{N}$ if and only if $\widehat{x}$ is a solution to the ILP problem.

Continuous relaxation

Convex relaxation

Let $L \in \mathbb{R}^{K \times N}$, $b \in \mathbb{R}^K$, and $c \in \mathbb{R}^N$.
Let S be a nonempty closed convex such that $\mathcal{N} \subset S$.
We consider the following convex problem:

$$\underset{x \in S}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \geq b.$$

Remark: Often $S = [0, +\infty[^N$ (or, for binary problems, $S = [0,1]^N$).

Total unimodularity 单位模，幺模.

- A square matrix is unimodular if its determinant is either equal to 1 or -1.
- A matrix $L \in \mathbb{R}^{K \times N}$ is totally unimodular (TUM) if every non singular square matrix extracted from L is unimodular.

Total unimodularity

- A square matrix is **unimodular** if its determinant is either equal to 1 or -1.
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Basic properties

Let L be a $K \times N$ TUM matrix. Then the following hold.

- $L \in \{-1, 0, 1\}^{K \times N}$.
- Any submatrix extracted from L is TUM.
- Adding a row/column with at most one nonzero component equal to 1 or -1 results in a TUM matrix.
- $L^\top$ is TUM.
- Changing the sign of a row/column of L results in a TUM matrix.

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \le j \le K, 1 \le i \le N} \in \{-1, 0, 1\}^{K \times N}$.
Assume that

(i) each column of L contains at most 2 nonzero elements;
(ii) there exists two disjoint subsets of row indices $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $\{1, \ldots, K\} = \mathbb{K}_1 \cup \mathbb{K}_2$ and, for every column $i \in \{1, \ldots, N\}$ having two nonzero elements,

$$\sum_{j \in \mathbb{K}_1} L_{j,i} = \sum_{j \in \mathbb{K}_2} L_{j,i}$$

(with $\sum_{j \in \varnothing}(\cdot) = 0$).

Then L is TUM.

Total unimodularity

Sufficient condition

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$$\sum_{j\in\mathbb{K}_1} L_{j,i} = \sum_{j\in\mathbb{K}_2} L_{j,i}$$

(with $\sum_{j\in\varnothing}(\cdot) = 0$).

Then L is TUM.

Remark: Condition (ii) means that, for each column having two nonzero elements,

- if they are of the same sign, then the row index of one of them should be in $\mathbb{K}_1$, and the other in $\mathbb{K}_2$
- if they have different signs, then their row indices should be both in $\mathbb{K}_1$ or both in $\mathbb{K}_2$.

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \leq j \leq K, 1 \leq i \leq N} \in \{-1, 0, 1\}^{K \times N}$.
Assume that

(i) each column of L contains at most 2 nonzero elements;
(ii) there exists two disjoint subsets of row indices $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $\{1, \ldots, K\} = \mathbb{K}_1 \cup \mathbb{K}_2$ and, for every column $i \in \{1, \ldots, N\}$ having two nonzero elements,

$$\sum_{j \in \mathbb{K}_1} L_{j,i} = \sum_{j \in \mathbb{K}_2} L_{j,i}$$

(with $\sum_{j \in \varnothing} (\cdot) = 0$).

Then L is TUM.

Proof: Assume that (i) and (ii) hold and that $L \in \{-1, 0, 1\}^{K \times N}$ is not TUM. Let T be a square submatrix of L with minimum size such that $\det T \notin \{-1, 0, 1\}$.
Any column of T is non null (otherwise, $\det T = 0$).
Any column of T contains 2 nonzero elements (otherwise, T would not be of minimum size).

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1\le j\le K, 1\le i\le N} \in \{-1,0,1\}^{K\times N}$.
Assume that

(i) each column of L contains at most 2 nonzero elements;
(ii) there exists two disjoint subsets of row indices $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $\{1,\ldots,K\} = \mathbb{K}_1 \cup \mathbb{K}_2$ and, for every column $i \in \{1,\ldots,N\}$ having two nonzero elements,

$$\sum_{j\in\mathbb{K}_1} L_{j,i} = \sum_{j\in\mathbb{K}_2} L_{j,i}$$

(with $\sum_{j\in\varnothing}(\cdot) = 0$).

Then L is TUM.

Proof: We have $T = (L_{j,i})_{j\in\mathbb{J}, i\in\mathbb{I}}$ where $\mathbb{I} \subset \{1,\ldots,N\}$ and $\mathbb{J} \subset \{1,\ldots,K\}$.
For every $i \in \mathbb{I}$,

$$\sum_{j\in\mathbb{K}_1} L_{j,i} = \sum_{j\in\mathbb{K}_2} L_{j,i}.$$

Since the i-th column of T contains two nonzero components and (i) holds, for every $j \notin \mathbb{J}$, $L_{j,i} = 0$, and consequently,

$$\sum_{j\in\mathbb{K}_1\cap\mathbb{J}} L_{j,i} = \sum_{j\in\mathbb{K}_2\cap\mathbb{J}} L_{j,i}.$$

This shows that $\sum_{j\in\mathbb{K}_1\cap\mathbb{J}} \ell_j = \sum_{j\in\mathbb{K}_2\cap\mathbb{J}} \ell_j$, where $(\ell_j)_{j\in\mathbb{J}}$ are the columns of matrix T. Due to this linear dependence, $\det T = 0$, which contradicts our assumption.

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \le j \le K, 1 \le i \le N} \in \{-1, 0, 1\}^{K \times N}$.
Assume that

(i) each column of L contains at most 2 nonzero elements;
(ii) there exists two disjoint subsets of row indices $\mathbb{K}_1$ and $\mathbb{K}_2$ such that $\{1, \dots, K\} = \mathbb{K}_1 \cup \mathbb{K}_2$ and, for every column $i \in \{1, \dots, N\}$ having two nonzero elements,

$$\sum_{j \in \mathbb{K}_1} L_{j,i} = \sum_{j \in \mathbb{K}_2} L_{j,i}$$

(with $\sum_{j \in \varnothing} (\cdot) = 0$).

Then L is TUM.

Example: L totally unimodular ?

$$L = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \leq j \leq K, 1 \leq i \leq N} \in \{-1, 0, 1\}^{K \times N}$.
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(with $\sum_{j \in \varnothing}(\cdot) = 0$).

Then L is TUM.

Example: L totally unimodular ?

$$L = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

No: $\det(L) = 2$!

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \le j \le K, 1 \le i \le N} \in \{-1, 0, 1\}^{K \times N}$.
Assume that

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(with $\sum_{j \in \varnothing} (\cdot) = 0$).

Then L is TUM.

Example: L totally unimodular ?

$$L = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1\le j\le K, 1\le i\le N} \in \{-1,0,1\}^{K\times N}$.
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(with $\sum_{j\in\varnothing}(\cdot) = 0$).

Then L is TUM.

Example: L totally unimodular ?

$$L = \begin{bmatrix} 1 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

Yes: $\mathbb{K}_1 = \{1,2,3,4\}$ and $\mathbb{K}_2 = \varnothing$.

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1\le j\le K, 1\le i\le N} \in \{-1,0,1\}^{K\times N}$.
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(with $\sum_{j\in\varnothing}(\cdot) = 0$).

Then L is TUM.

Example: L totally unimodular ?

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Total unimodularity

Sufficient condition

Let $L = (L_{j,i})_{1 \le j \le K, 1 \le i \le N} \in \{-1, 0, 1\}^{K \times N}$.
Assume that

(i) each column of L contains at most 2 nonzero elements;
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Then L is TUM.

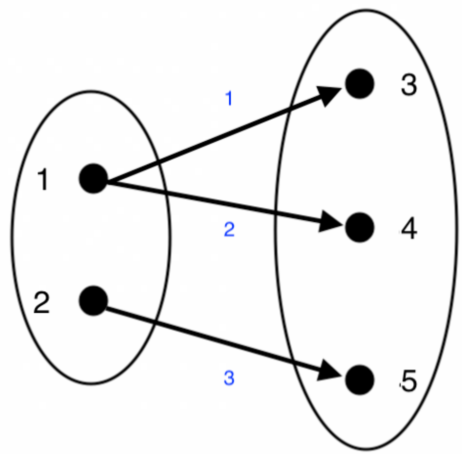
Example: L totally unimodular ?

$$L = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & -1 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Yes: $\mathbb{K}_1 = \{1, 3\}$ and $\mathbb{K}_2 = \{2, 4\}$.

Bipartite graphs 二分图.

A graph is bipartite if its set of nodes can be divided into two sets $\mathbb{V}_1$ and $\mathbb{V}_2$ such that any edge of the graph links a node in $\mathbb{V}_1$ to a node in $\mathbb{V}_2$.

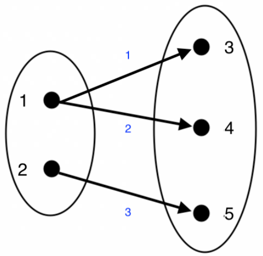


Bipartite graphs

A graph is bipartite if its set of nodes can be divided into two sets $\mathbb{V}_1$ and $\mathbb{V}_2$ such that any edge of the graph links a node in $\mathbb{V}_1$ to a node in $\mathbb{V}_2$.

Let N be the number of edges and K the number of nodes.
$L = (L_{j,i})_{1\leq j\leq K, 1\leq i\leq N}$ is the undirected incidence matrix of the graph if, for every $i \in \{1, \ldots, N\}$ and $j \in \{1, \ldots, K\}$,

$$L_{j,i} = \begin{cases} 1 & \text{if the } i\text{-th edge connects the } j\text{-th node} \\ 0 & \text{otherwise.} \end{cases}$$



$$L = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Bipartite graphs

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The undirected incidence matrix of a bipartite graph is TUM.

Bipartite graphs

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The undirected incidence matrix of a bipartite graph is TUM.

Proof: Choose $\mathbb{K}_1$ (resp. $\mathbb{K}_2$) as the index set of the nodes in $\mathbb{V}_1$ (resp. $\mathbb{V}_2$).

Bipartite graphs

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Application: Assume that the graph models compatible relations between two sets of individuals we would like to pair. Let $x = (x^{(i)})_{1\le i\le N}$ be such that $x^{(i)} = 1$ if the individuals linked by the i-th edge are paired, 0 otherwise. Assume that the selection of the i-th link results in a satisfaction score $\overline{c}^{(i)}$. Let $\overline{c} = (\overline{c}^{(i)})_{1\le i\le N}$.
An optimal matching is a solution to

$$\underset{x\in\{0,1\}^N}{\text{maximize}} \quad \langle \overline{c} \mid x \rangle \quad \text{s.t.} \quad Lx \le 1.$$

Exact LP relaxation

Let L be a $K \times N$ TUM matrix, let $b \in \mathbb{Z}^K$, and $c \in \mathbb{R}^N$.
Any solution $\widehat{x}$ delivered by the simplex to the LP problem

$$\underset{x \in [0,+\infty[^N}{\text{minimize}} \quad \langle c \mid x \rangle \quad \text{s.t.} \quad Lx \geq b.$$

is such that $\widehat{x} \in \mathbb{N}^N$.

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is such that $\widehat{x} \in \mathbb{N}^N$.

Proof: The standard form of the LP problem reads

$$\underset{z \in [0,+\infty[^M}{\text{minimize}} \quad \langle d \mid z \rangle \quad \text{s.t.} \quad Az = b.$$

where $M = N + K$, $A = [L \quad -\mathrm{Id}]$ is TUM, $z = \begin{bmatrix} x \\ s \end{bmatrix} \in \mathbb{R}^M$, and $d = \begin{bmatrix} c \\ 0 \end{bmatrix} \in \mathbb{R}^M$.

If a solution exists, the optimal basic index set $\mathbb{I}$ returned by the simplex is associated with a solution $\widehat{z}$ such that

$$\widehat{z}_{\mathbb{I}} = A_{\mathbb{I}}^{-1} b \quad \text{and} \quad \widehat{z}_{\mathbb{J}} = 0 \quad \text{with} \quad \mathbb{J} = \{1, \ldots, M\} \setminus \mathbb{I}.$$

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$A_{\mathbb{I}}$ is a matrix composed of the columns of A with indices in $\mathbb{I}$.
Since A is TUM, $A_{\mathbb{I}}$ is TUM.
According to Cramer's rule,

$$A_{\mathbb{I}}^{-1} = \frac{1}{\det A_{\mathbb{I}}} \operatorname{co}(A_{\mathbb{I}})^\top$$

where $\operatorname{co}(A_{\mathbb{I}})$ is the matrix of cofactors of $A_{\mathbb{I}}$ which are up to a sign change equal to the determinants of minors of matrix $A_{\mathbb{I}}$.
Hence, the elements of $A_{\mathbb{I}}^{-1}$ are in $\{-1, 0, 1\}$ and $\widehat{z} \in \mathbb{Z}^M \cap [0, +\infty[^N$.

Exact LP relaxation

Let L be a $K \times N$ TUM matrix, let $b \in \mathbb{Z}^K$, and $c \in \mathbb{R}^N$.
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Exercise 1

A travelling salesperson must visit N cities by departing from one of them and coming back to it. The travelling time in a direct trip from the i-th city to the j-th one, with $(i, j) \in \{1, \ldots, N\}^2$ $i \neq j$, is $\tau_{i,j}$. The salesperson wants to minimize the duration of his/her whole trip while visiting each city only once.

Formulate this problem as a binary linear programming problem.

How to avoid any independent subtours within the trip ?

Exercise 2

A medical doctor is only allowed to prescribe 3 possible drugs out of 6 available ones to a patient. The expected benefit of the i-th drug with $i \in \{1, \ldots, 6\}$ is quantified by a value β_i (for example, related to the viral load after some given period). It is assumed that the benefits of different drugs can be added.

Furthermore, the following rules must be applied:

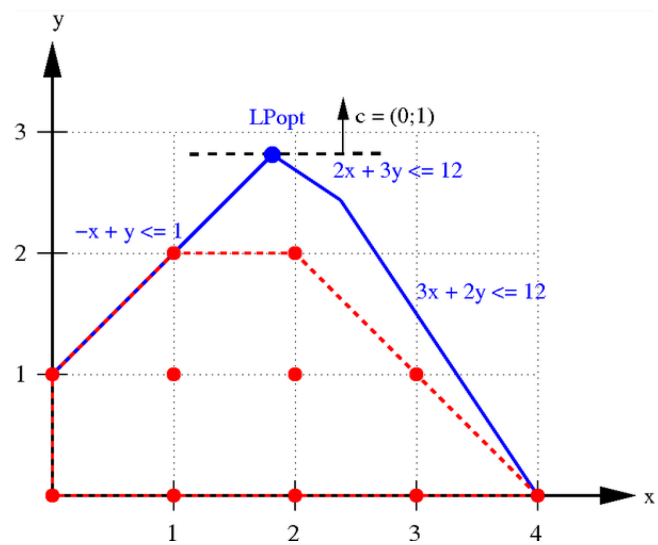
- drugs 1 and 2 are incompatible;
- If drug 2 is prescribed, then drug 3 must be prescribed;
- If drugs 3 and 4 are prescribed, then drug 5 cannot be prescribed;
- if drug 4 or 5 is prescribed, then drug 6 cannot be prescribed.

Formulate this problem as a binary linear programming problem.

线性规划的松弛

维基百科，自由的百科全书

在数学中，0-1整数规划的**线性规划的松弛**是这样的问题：把每个变量必须为0或1的约束，替换为较弱的每个变量属于区间[0,1]的约束。



A (general) integer program and its LP–relaxation

也就是说，对于原整数规划的每个下列形式的约束：

$$x_i \in \{0,1\}$$

我们转而使用一对线性约束来代替：

$$0 \le x_i \le 1.$$

这样产生的松弛是线性规划，因此得名线性规划的松弛。这种松弛技术把NP难的最优化问题（整数规划）转化为一个相关的多项式时间可解的问题（线性规划）。我们可以用松弛后的线性规划的解来获得关于原整数规划的解的信息。

目录

例子

考虑集覆盖问题，该问题的线性规划松弛最先由Lovász (1975)详细研究。在该问题中，给定输入为一族集合$F = \{S_0, S_1, \ldots\}$；任务是找到其中的一个集合数量尽可能少的子族，其并集也是F。

若想把该问题形式化为0-1整数规划，对每个集合S_i构造一个指示变量x_i，它取值为1当S_i属于所选子族时，取0当其不属于。那么一个有效的覆盖可由一个满足下列约束的对指示变量的赋值来描述：

$$x_i \in \{0,1\}$$

（即只允许指定的指示变量值）并且，对于每个并集F的元素e_j：

$$\sum_{\{i \mid e_j \in S_i\}} x_i \ge 1$$

（即覆盖每个元素）。最小的对应指示变量赋值的集覆盖满足这些约束且最小化线性目标函数：

$$\min \sum_i x_i.$$

这个集覆盖问题的线性规划松弛描述了一个*分数覆盖*，其中输入集被赋予权值，使得包含每个元素的这些集合的总权值至少是1，且所有集合的总权值最小。

对精确解的分支定界

在近似理论中，线性规划的松弛也有应用。线性规划在计算困难的最优化问题的最优解时的分支定界算法中也扮演着重要角色。

割平面方法

两种有着相同的目标函数和相同的可行解集因而等价的整数规划，可能有着非常不一样的线性规划松弛：一种线性规划松弛可从几何上视为包含了所有可行解并排除了所有其余0-1向量的凸多面体，而且有无穷多的多面体都具有这种性质。理想情况下，我们想把可行解的凸包作为松弛来使用，因为这种多面体上的线性规划将自动产生原整数规划的正确解。尽管如此，一般情况下，这种多面体有指数多的面且难以构造。典型的松弛，比如我们前面讨论过的集覆盖问题的松弛，构造了一个严格包含可行解的凸包且排除可解非松弛问题的0-1向量的多面体。

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