CentraleSupélec engineering curriculum | **CIPPDE - 1st year**
Final Exam | April 5th 2019

Final exam : PDE part (2 pages)

Instructions

- Documents and draft papers brought with you are prohibited as well as electronic devices (cell phones, calculators, smart watches ...).
- Do not use correction fluids.
- Please use a blue or black pen (do not use a light blue fountain pen).
- Please fill in the title block very carefully and in block letters.
- Please write the sheet number on each sheet.
- The sheets must be handed out flat and in the same way (the cut corner should be top right).
- *Any of your answers should be justified by an appropriate proof.*
- *The exercises 1 and 2 are independent.*

Exercise 1

On $[0,1]$, we study the following problem : find $w : [0,1] \to \mathbb{R}$ such that

$$\begin{cases} -\left(x \mapsto \dfrac{w'(x)}{1+x}\right)' + w = 0, \\ w(0) = 1, \quad w'(1) = 0. \end{cases} \tag{Q}$$

Q.1.1 How do you reduce (Q) to a problem with a homogeneous (zero) condition at $x = 0$?

Q.1.2 Show that $H = \{v \in H^1(0,1) \,:\, v(0) = 0\}$ equipped with the inner product of $H^1(0,1)$ is a Hilbert space.

Q.1.3 Show that, on H, $v \mapsto \|v'\|_{L^2(0,1)}$ is a norm that we will denote $\|\cdot\|_H$, and that, still on H, this norm is equivalent to the classical $H^1(0,1)$ norm.

Q.1.4 Show the existence and uniqueness of the solution of class $\mathcal{C}^\infty([0,1])$ of the problem (Q). **You will prove rigorously that this solution satisfies the boundary conditions of the problem (Q).**

Exercise 2

Recall that a square matrix B is monotone if for all vector y with non negative coefficients such that there exists x satisfying $Bx = y$, the coefficients of x are all non negative.

Q.2.1 (Basic question) Show that a monotone matrix B is invertible.

Q.2.2 (Basic question) Show that all the coefficients of the inverse of a monotone matrix B are non negative.

Let $J \geq 2$. Let $h = 1/J$ be the discretization step and $x_j = jh$ for $j \in \{0, \ldots, x_{J+1}\}$. We consider the following numerical scheme : we look for $V = (v_j)_{1 \leq j \leq J} \in \mathbb{R}^J$, knowing that $v_0 = 1$ is fixed, such that

$$\begin{cases} -\dfrac{1}{h}\left(\dfrac{v_{j+1}-v_j}{(1+h(j+1/2))h} - \dfrac{v_j - v_{j-1}}{(1+h(j-1/2))h}\right) + v_j = 0,\ j \in \{1,\dots,J\}, \\ \\ v_{J+1} = v_{J-1}. \end{cases} \tag{S}$$

Q.2.3 Write the scheme (S) as a linear system $A_h u_h = b_h$ of size J, detailing the matrix A_h and the vector b_h.
INDICATION : You can use the notations $\beta : x \mapsto (1+x)^{-1}$ and $\beta_{j-1/2} = \beta(h(j-1/2))/h^2$ for $j \in \{1,\dots,J\}$.

Q.2.4 Write the matrix A_h and the vector b_h for $J = 4$.

Q.2.5 Show that the matrix A_h is monotone.

Q.2.6 Show that, for all $x \in]h/2, 1-h/2[$ and $f \in C^2([0,1])$,

$$f(x+h/2) - f(x-h/2) = f'(x) + O(h^2).$$

Q.2.7 Show that the scheme (S) is consistent at the order (at least) 1 with the equation

$$-\left(x \mapsto \frac{w'(x)}{1+x}\right)' + w = 0 \text{ sur }]0,1[. \tag{E}$$

Q.2.8 Find the solution of class $C^\infty([0,1])$ of the following problem and show it is unique :

$$\begin{cases} -\left(x \mapsto \dfrac{w_1'(x)}{1+x}\right)' = 1, \\ w_1(0) = 1, \quad w_1'(1) = 0. \end{cases} \tag{Q1}$$

Q.2.9 Show that the matrix $A_h - I$ is monotone.

Q.2.10 (More difficult question) Show that the scheme (S) is stable for the $\|\cdot\|_\infty$ norm.
INDICATION : You can use the properties of the function w_1 and the techniques that are used to show that the scheme is consistent with (E).

Q.2.11 Using the previous questions, show that the numerical scheme converges to the solution of (Q).