CentraleSupélec engineering curriculum

CIPPDE - 1st year

Midterm Exam #2

11 January 2019

The exam lasts 1h30. (2 pages)

Instructions

- Documents and draft papers brought with you are prohibited.
- Electronic devices (cell phones, calculators, smart watches ...) are prohibited.
- Do not use correction fluids.
- Please use a blue or black pen (do not use a light blue fountain pen).
- Please fill in the title block very carefully and in block letters.
- Please write the sheet number on each sheet.
- The sheets must be handed out flat and in the same way (the cut corner should be top right).
- Since the duration of the exam is 1½hour, you are not allowed to exit the room.
- *Any of your answers should be justified by an appropriate proof.*
- *The two exercises are independent.*

Exercise 1

Let m be a bob with a mass of $500g$, suspended by an inelastic, massless string of length $10cm$. Assume that there is no friction. The gravity of Earth will be approximated by $10.m/s^{-2}$. Denote by θ the angle between the string and the downward vertical axis, measured counter clockwise. Let T be the observation time.

Q. 1.1 Show that the dimensionless equation that is satisfied by the angular movement of the gravity center of the mass is $\theta'' + 100\,T^2 \sin(\theta) = 0$. Drawing a picture is strongly recommended.

Q. 1.2 Recall the definition of an Initial Value Problem.

Q. 1.3 Assume that the initial angle θ^0 is $0.01rad$ and that the initial velocity vanishes. Write the (dimensionless) Initial Value Problem that is satisfied by the vector $\Theta = (\theta, \theta')^T$. The associated vector field will be denoted by $f : \Theta' = f(\Theta)$.

Q. 1.4 Show that f is globally Lipschitz continuous.

Q. 1.5 Recall the definition of a global solution.

Q. 1.6 Does the problem established at Question Q.1.3 admit a global solution ? What can be said about the uniqueness ? Justify precisely your answer using the results stated in the lecture.

The angle θ^0 being small, the problem of Question Q.1.3 is linearized.

Q.1.7 Give the solution in closed form of this linearized problem.

Exercise 2

Let $\Omega =]0,1[$. Consider the problem

$$\begin{cases} -\nu(x)u''(x) + q(x)u(x) = f(x), & x \in]0,1[, \\ u'(0) = u'(1) = 0, \end{cases} \tag{1}$$

with $\nu, q \in C^0([0,1], \mathbb{R}^{+*}),\ f \in C^0([0,1])$.

Q.2.1 What is the type of this problem ? How are its boundary conditions called ?

Q.2.2 Show that the variational formulation

« *Find $u \in H^1(0,1)$ that solves*

$$\forall v \in H^1(0,1), \qquad \int_{]0,1[} u'v' + \int_{]0,1[} \tilde{q}uv = \int_{]0,1[} \tilde{f}v$$

»

can be associated with Problem (1) for some functions $\tilde{q}$ and $\tilde{f}$ to be given.

Q.2.3 Show that the variational formulation admits one and only one solution in $H^1(0,1)$.

Q.2.4 Show that Problem (1) is well posed in the sense of Hadamard in $H^2(0,1)$.

Q.2.5 Show that the solution of (1) is classical.

Q.2.6 Describe the $\mathbb{P}_1$ finite element method on a uniform mesh of J points uniformly distributed for $J \geq 3$. You will give the explicit expressions of

- — the step h,
- — the approximation subspace $H_h \subset H^1(0,1)$,
- — the chosen basis of H_h,
- — the rigidity matrix A_h of the linear system and the righthand side..

You will give the matrix A_h for $J = 4$ and $q = \nu$.

Q.2.7 What can be said of the matrix A_h if q is the zero function ? If q is only assumed to be nonnegative ? Does the previous work fail, and, if so, where ?

Q.2.8 Propose a phenomenon that can be modeled by Problem (1).