Exercise V.4

Lett $M \in \mathbb{N}\backslash\{0,1\}$. Solve the problem:

$$\text{maximize}_{(x^{(i)})_{1\le i\le M}\in[0,+\infty[^M} \quad \sum_{i=1}^{M} i^2 x^{(i)}, \;\; \text{subject to:} \;\; \sum_{i=1}^{M} x^{(i)} = 1, \;\; \sum_{i=1}^{M} i x^{i} = 2. \tag{1}$$

Solution: We start by writing the problem in the standard form:

$$\text{minimize}_{x\in[0,+\infty[^M} \;\; \langle d|z\rangle, \;\; \text{subject to: } Az = b, \tag{2}$$

where: $d = -\left[1, 2^2, \ldots, M^2\right]^T$,

$$A = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 2 & \cdots & M \end{bmatrix}, \;\; b = \begin{bmatrix} 1 \\ 2 \end{bmatrix}. \tag{3}$$

Based on the theoretical results, we know that if an optimal solution exists, also a basic optimal solution exists. Therefore, we will try to express the basic solutions and find the one(s) that is optimal. Let $\mathbb{I} = \{i, j\}$ represent the column indices of the basic columns corresponding to a speficic basic solution. Clearly, in order to consider all possible solutions, we need to investigate all possible couples $(i, j) \in \{1, \ldots, M\}^2$, $j > i$. The basis corresponding to a given set $\mathbb{I}$ is expressed as:

$$A_{\mathbb{I}} = \begin{bmatrix} 1 & 1 \\ i & j \end{bmatrix}. \tag{4}$$

Provided that $A_{\mathbb{I}}$ is invertible (which is the case here) we can obtain that:

$$A_{\mathbb{I}}^{-1} = \frac{1}{j-i}\begin{bmatrix} j & -1 \\ -i & 1 \end{bmatrix}. \tag{5}$$

As a result, using $A_{\mathbb{I}}$, we obtain a basic solution of the form:

$$z_{\mathbb{I}} = \begin{bmatrix} z^{(i)} \\ z^{(j)} \end{bmatrix} = A_{\mathbb{I}}^{-1} b = \frac{1}{j-i}\begin{bmatrix} j-2 \\ 2-i \end{bmatrix} \tag{6}$$

(with the remaining values of the elements of the basic solution being equal to zero).

Given this form of basic solutions, we see that we obtain basic feasible solutions only for $j \ge 2$ and $i \le 2$. Let us now investigate the possible solutions for $i = 1$ and $i = 2$.

- If $i = 2$ given the form of $z_{\mathbb{I}}$ we obtain the basic solution:

  $$z = \left[0, \frac{j-2}{j-2} = 1, 0, \ldots, 0\right]^T \tag{7}$$

  for all possible $j > 2$, and the corresponding cost function value is equal to:

  $$\langle d|z\rangle = -4 \tag{8}$$

- If $i = 1$, given the form of $z_{\mathbb{I}}$, we obtain that the associated basic feasible solutions have the form:

$$z = \frac{1}{j-1}[j-2, 0, \ldots, 0, \underbrace{1}_{j-th\ \ position}, 0, \ldots, 0]^T, \tag{9}$$

and the correspodning value for $\langle d|z \rangle$ is equal to:

$$\langle d|z \rangle = -\frac{1}{j-1}\left(j - 2 + j^2\right) = -(j+2) \tag{10}$$

As a result, we obtain that for $i = 1$, among the basic feasible solutions, the one that results in the minimum possible value for the cost function corresponds to $j = M$, and the corresponding basic solution is:

$$z = \frac{1}{M-1}[M-2, 0 \ldots, 1] \tag{11}$$

We now note that by investigating the two cases, we obtain that for the specifed values of M, the candidate solution obtained from the second case results in a smaller value for the cost function. Therefore this is an optimal basic feasible solution.