Exercise III.3

Let $\mathcal{H}$ be a Hilbert space and $f : \mathcal{H} \to \mathbb{R}$ be Gâteaux differentiable and concave. Let $C \subset \mathcal{H}$ and let $\hat{x} \in \mathrm{int}\,(C)$. Show that f admits a global minimizer on C at $\hat{x}$ if and only if f is constant.

Solution: Let us assume that f is concave and it has a minimizer $\hat{x} \in \mathrm{int}(C)$. Due to the fact that f is concave (or equivalently that $-f$ is convex) it holds that $\forall x \in \mathcal{H}$:

$$-f\,(x) \geq -f\,(\hat{x}) - \langle \nabla f\,(\hat{x})\,| x - \hat{x} \rangle, \tag{1}$$

or equivalently that $\forall x \in C$:

$$f\,(x) \leq f\,(\hat{x}) + \langle \nabla f\,(\hat{x})\,| x - \hat{x} \rangle. \tag{2}$$

On the other hand, due to the fact that $\hat{x}$ is a global minimizer, we obtain that $\forall x \in C$:

$$f\,(x) \geq f\,(\hat{x})\,. \tag{3}$$

Moreover, since $\hat{x} \in \mathrm{int}(C)$ the necessary conditions for a (local or global) minimizer impose that

$$\nabla f\,(\hat{x}) = 0. \tag{4}$$

As a result, we obtain that the following conditions need to be concurrently satisfied:

$$f\,(x) \geq f\,(\hat{x})\,, \text{ and } f\,(x) \leq f\,(\hat{x})\,, \;\; \forall x \in C, \tag{5}$$

which can only be (concurrently) satisfied if $f\,(x)$ is constant valued.

Concerning the inverse part of the proof, we note that if f is constant, then obviously it has global minimizers in $\mathrm{int}C$.