Lecture V : Finite Difference Method

A) Aims of this class

After this class,

- I can write a discretization of a stationary partial differential equation.
- I can write the corresponding linear system.
- I can compute the order of consistency of the method.
- I can prove its stability.
- I can prove the convergence of the method.

B) To become familiar with this class' concepts (to prepare before the examples class)

Questions V.1 and V.2 must be done before the 9th lab. The solutions are available online.

From now on, $J \geq 1$ is fixed and the interval $[0, 1]$ is split into $J + 1$ subintervals of length h. For $j \in \{0, \ldots, J + 1\}$, let $x_j = jh$ and denote the solution to the approximated problem by $V_h = (v_j)_{j\in\{1,\ldots,J\}}$. As well, let $F = (f(x_j))_{j\in\{1,\ldots,J\}}$ and $C = (c(x_j))_{j\in\{1,\ldots,J\}}$.

Question V.1
Let $u \in C^6(\mathbb{R})$, $(x, h) \in \mathbb{R}^2$.

Q. V.1.1 Show the following equality

$$u^{(4)}(x) = \frac{u(x-2h) - 4u(x-h) + 6u(x) - 4u(x+h) + u(x+2h)}{h^4} + \mathcal{O}(h^2)$$

Q. V.1.2 Let us consider the finite difference formula

$$u'(x) = \frac{u(x-2h) - 8u(x-h) + 8u(x+h) - u(x+2h)}{12h} + \mathcal{O}(h^k)$$

What is the order of approximation of this method?

Question V.2
Let us establish some results on the approximate resolution of

$$\textbf{(CD)} \qquad \begin{cases} -\nu u''(x) + bu'(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 1 \text{ and } u(1) = 0, \end{cases}$$

with $\nu \in \mathbb{R}^{+*}$, $b \in \mathbb{R}^+$, $c \in C^2([0, 1], \mathbb{R}^+)$ and $f \in C^2([0, 1], \mathbb{R})$.
We introduce the three following discretizations of u' $h \in \mathbb{R}$:

$$\text{downwind: } \frac{u(\cdot+h) - u(\cdot)}{h}, \text{ upwind: } \frac{u(\cdot) - u(\cdot-h)}{h}, \text{ centered: } \frac{u(\cdot+h) - u(\cdot-h)}{2h}$$

Q. V.2.1 Give the consistency errors of these discretizations. What are their orders?

Q. V.2.2 Recall the formula of the 3-point centered scheme used to discretize u''.

Q. V.2.3 Give the finite difference schemes of (CD) corresponding to the discretizations of the first-order derivative that follow.

Q. V.2.4 Give the orders of these schemes.

C) Exercises

Exercise V.1
Let $J \geq 1$, $(\alpha_j)_{j\in\{1,\dots,J\}} \in \mathbb{R}^J$, β and $\gamma \in \mathbb{R}_*^+$ such that $\alpha_j - \beta - \gamma \geq 0$ for any $j \in \{1,\dots,J\}$. Let A the square matrix of size $J \geq 1$ with coefficients, for all $(i,j) \in \{1,\dots,J\}^2$,

$$a_{ij} = \begin{cases} \alpha_i & \text{if } i = j, \\ -\beta & \text{if } i = j+1,\ j \leq J-1, \\ -\gamma & \text{if } i = j-1,\ j \geq 2, \\ 0 & \text{otherwise.} \end{cases}.$$

E. V.1.1 Let $G \in (\mathbb{R}^+)^J$ such that $G \in \mathrm{Im}(A)$, that is, there exists $V \in \mathbb{R}^J$ such that $G = AV$. Show that $V \in (\mathbb{R}^+)^J$.

E. V.1.2 Deduce that A is invertible, and that the coefficients of A^{-1} are nonnegative.

Exercise V.2
This exercise follows Question Q.V.2, and uses the results of Exercise E.V.1.

E. V.2.1 Write the schemes in a matrix form. Let A_h^+ (resp. A_h^-, A_h^0) the matrix corresponding to the downwind discretization (resp. to the upwind discretization, to the centered scheme). What is the shape of the right-hand side in the different cases ?

E. V.2.2 We want to define a scheme that satisfies the property of the discrete maximum priciple. Which scheme should be used, with respect to the values of b? Sum up the results in a table.

E. V.2.3 Let $V \in \mathbb{R}^J$. Let $v_0 = v_{J+1} = 0$. Show the Discrete Poincaré Inequality:

$$\|V\|_2^2 := h\sum_{i=0}^{J} v_i^2 \leq \sum_{i=0}^{J} \frac{(v_{i+1} - v_i)^2}{h}.$$

Denote by B_h the matrix corresponding to $b = 0$, $c = 0$.

E. V.2.4 Let $V \in \mathbb{R}^J$. Set $v_0 = v_{J+1} = 0$. Prove that $(B_h V, V) = \dfrac{\nu}{h^2}\displaystyle\sum_{i=0}^{J}(v_{i+1} - v_i)^2$.

Assume from now on that $b > 2\nu/h$. Moreover, assume $v_0 = v_{J+1} = 0$: note that, according to the previous table, an upwind discretization gives a monotonous matrix.

E. V.2.5 Explain why considering homogeneous boundary conditions does not contradict (CD).

E. V.2.6 Show that $(A_h V, V) \geq \left(\dfrac{\nu}{h} + \dfrac{b}{2}\right)\|V\|_2^2$, then that $(A_h V_h = G \Rightarrow \|V_h\|_2 \leq \dfrac{1}{(\nu + bh/2)}\|G\|_2)$.

E. V.2.7 Give a L^2 estimate of the error.

Exercise V.3 (Boundary conditions of Dirichlet-Neumann type)
Consider the following problem:

$$\textbf{(P)} \qquad \begin{cases} -u''(x) + c(x)u(x) = f(x), & x \in]0,1[, \\ u(0) = 0 \text{ and } u'(1) = 0, \end{cases}$$

with $f \in C^2([0,1],\mathbb{R})$ and $c \in C^2([0,1],\mathbb{R}^+)$. At first, discretize u'' with the classical centered scheme and assume that at the fictive point x_{J+2}, $v_{J+2} = v_{J+1}$.

E. V.3.1 Give the discretization matrix A_h.

E. V.3.2 Show that the matrix A_h satisfies the maximum principle.

E. V.3.3 *A priori*, is the scheme consistent with (P)?

E. V.3.4 Let $Y \in \mathbb{R}^{J+1}$ such that $y_{J+1} = 0$. Show that (V_h solution of $A_h V_h = Y \Rightarrow \|V_h\|_\infty \leq \frac{1}{2}\|Y\|_\infty$).

E. V.3.5 Let $Y \in \mathbb{R}^{J+1}$ such that $y_j = 0$ if $j \in \{1,\ldots,J\}$. Show that (V_h such that $A_h V_h = Y \Rightarrow \|V_h\|_\infty \leq h\|Y\|_\infty = h|y_J|$).

E. V.3.6 Deduce that the scheme is convergent. What is its order ?

Now set $v_{J+2} = v_J$.

E. V.3.7 What discretisation does this correspond to? Compute the order of consistency.

D) Going further

These exercises can be found on edunao as Jupyter notebooks.

Chapter IX: Solutions

Solution de Q. V.1.1 Let us write the Taylor expansion

$$u(x \pm h) = u(x) \pm hu'(x) + \frac{h^2}{2}u''(x) + \pm\frac{h^3}{6}u^{(3)}(x) + \frac{h^4}{24}u^{(4)}(x) + \pm\frac{h^5}{120}u^{(5)}(x) + O(h^6)$$

and

$$u(x \pm 2h) = u(x) \pm 2hu'(x) + \frac{4h^2}{2}u''(x) + \pm\frac{8h^3}{6}u^{(3)}(x) + \frac{16h^4}{24}u^{(4)}(x) + \pm\frac{32h^5}{120}u^{(5)}(x) + O(h^6).$$

A well-chosen linear combination gives, the odd terms vanishing by symmetry,

$$\begin{aligned}\frac{u(x-2h) - 4u(x-h) + 6u(x) - 4u(x+h) + u(x+2h)}{h^4} &= \underbrace{(1-4+6-4+1)}_{=0}\frac{u(x)}{h^4} + \underbrace{(4h^2 - 4h^2 - 4h^2 + 4h^2)}_{=0}\frac{u''(x)}{2h^4} \\ &+ \underbrace{(16h^4 - 4h^4 - 4h^4 + 16h^4)}_{=24h^4}\frac{u^{(4)}(x)}{24h^4} + O(h^2)\end{aligned}$$

Solution de Q. V.1.2 Using again the Taylor expansions of the previous question, a well-chosen linear combination gives, the odd terms vanishing by symmetry

$$\begin{aligned}\frac{u(x-2h) - 8u(x-h) + 8u(x+h) - u(x+2h)}{12h} &= \underbrace{1-8+8-1}_{=0}\frac{u(x)}{12h} + \underbrace{(-2h + 8h + 8h - 2h)}_{=12h}\frac{u'(x)}{12h} \\ &+ \underbrace{(-8h^3 + 8h^3 + 8h^3 - 8h^3)}_{=0}\frac{u^{(3)}(x)}{72h} \\ &+ \underbrace{(-32h^5 + 8h^5 + 8h^5 - 32h^5)}_{=-48}\frac{u^{(5)}(x)}{1440h} + O(h^6) = u'(x) + O(h^4).\end{aligned}$$

This formula is an approximation of fourth order.

Solution de Q. V.2.1 We indicate the assumptions related to (o and O) between parentheses.

- Downwind discretization: let $u \in D^1(]0,1[)$ ($u \in C^1(]0,1[)$), $x \in]h, 1-h[$, as $u(x+h) = u(x) + hu'(x) + o(h)(+O(h^2))$,

$$u'(x) - \frac{u(x+h) - u(x)}{h} = o(1)(= O(h))$$

- Upwind discretization: let $u \in D^1(]0,1[)$ ($u \in C^1(]0,1[)$), $x \in]h, 1-h[$, as $u(x-h) = u(x) - hu'(x) + o(h)(+O(h^2))$,

$$u'(x) - \frac{u(x) - u(x-h)}{h} = o(1)(= O(h))$$

- Centered discretization: let $u \in D^2(]0,1[)$ ($u \in C^2(]0,1[)$), $x \in]h, 1-h[$, as $u(x+h) - u(x-h) = hu'(x) + o(h^2)(+O(h^3))$,

$$u'(x) - \frac{u(x+h) - u(x-h)}{2h} = o(h)(= O(h^2))$$

Solution de Q. V.2.3 Denote

- for the downwind scheme for u' + centered discretization for u'':

$$\begin{cases} -\nu \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + b\dfrac{v_{j+1} - v_j}{h} + c_j v_j = f_j, \quad 1 \le j \le J \\ v_0 = 1; \quad v_{J+1} = 0 \end{cases}$$

- upwind scheme for u' + centered discretization for u''::

$$\begin{cases} -\nu \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + b\dfrac{v_j - v_{j-1}}{h} + c_j v_j = f_j, \quad 1 \le j \le J \\ v_0 = 1; \quad v_{J+1} = 0 \end{cases}$$

- centered scheme for u' + centered discretization for u'':

$$\begin{cases} -\nu \dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + b\dfrac{v_{j+1} - v_{j-1}}{2h} + c_j v_j = f_j, \quad 1 \le j \le J \\ v_0 = 1; \quad v_{J+1} = 0 \end{cases}$$

Solution de Q. V.2.4 To compute the consistency error : let u be the exact solution to (CD), of class $C^4([0,1])$. Then, $\forall j \in \{1, \dots, J\}$,

- for the downwind scheme for u' + centered discretization for u'':

$$\begin{aligned} \mathcal{E}_j &= -\nu \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + b\frac{u(x_{j+1}) - u(x_j)}{h} + c_j u(x_j) - f(x_j) \\ &= -\nu \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + b\frac{u(x_{j+1}) - u(x_j)}{h} + c_j u(x_j) + \nu u''(x_j) - bu'(x_j) - c_j u(x_j) \\ &= -\nu u''(x_j) + O(h^2) + \nu u''(x_j) + bu'(x_j) + O(h) - bu'(x_j) = O(h). \end{aligned}$$

- as well, for the upwind scheme for u' + centered discretization for u'':

$$\mathcal{E}_j = O(h),$$

- as well, for the centered scheme for u' + centered discretization for u'':

$$\mathcal{E}_j = -\nu u''(x_j) + O(h^2) + \nu u''(x_j) + bu'(x_j) + O(h^2) - bu'(x_j) = O(h^2).$$

Solution de Q. V.1.1 We have

$$\begin{cases} \alpha_1 v_1 - \beta v_2 \geq 0 \\ \forall j \in \{2, \ldots, J-1\},\ -\beta v_{j-1} + \alpha_j v_j - \gamma v_{j+1} \geq 0 \\ -\gamma v_{J-1} + \alpha_J v_J \geq 0. \end{cases}$$

Let $m \in \{1, \ldots, J\}$ such that $v_m = \min_{1 \leq j \leq J} v_j$.
Let us consider three cases:

- If $m = 1$, then $\alpha_1 v_1 - \gamma v_2 \geq 0$, which implies $(\alpha_1 - \gamma) v_1 + \gamma(v_1 - v_2) \geq 0$. Using $\alpha_1 - \gamma \geq \beta > 0$ we obtain $v_1 \geq \gamma(v_2 - v_1)/(\alpha - \gamma) \geq 0$.
  Therefore, since $v_m = v_1 \geq 0$, $\forall j \in \{1, \ldots, J\}$, $v_j \geq v_m \geq 0$.
- If $m = J$, we do the same thing replacing γ by β.
- If $2 \leq m \leq J-1$ then $-\beta v_{m-1} + \alpha_m v_m - \gamma v_{m+1} \geq 0$ therefore

$$\underbrace{\beta(v_m - v_{m-1})}_{\leq 0 \text{ definition of } v_m} + (\alpha_m - \beta - \gamma) v_m + \underbrace{\gamma(v_m - v_{m+1})}_{\leq 0 \text{ definition of } v_m} \geq 0$$

  - If $\alpha_m - \beta - \gamma > 0$ then $v_m \geq 0$
  - If $\alpha_m - \beta - \gamma = 0$ then $v_m - v_{m-1} \geq 0$ and $v_m - v_{m-1} \leq 0$, this implies that $v_m = v_{m-1}$. We conclude that $v_m = v_{m+1}$. We are in the situation where $v_{m-1} = \min_j v_j$, which leads by induction, to the case $m = 1$.

This proves A is monotone.

Solution de Q. V.1.2 We apply Lemmae **??** and **??**.

Solution de Q. V.2.1 In Exercise V.2, we substituted the finite difference quotients in **(CD)** and obtained equations relating the v_j, f_j and c_j. We can write these relations in the matrix form

$$A_h V_h = F_h$$

where

$$V_h = \begin{bmatrix} v_1 \\ \vdots \\ v_J \end{bmatrix}, \quad F_h = \begin{bmatrix} f_1 \\ \vdots \\ f_J \end{bmatrix}$$

and A_h is one of these three matrices[1] depending on the scheme:

[1] The diagonal is written in red to improve readiblity.

- Forward (upwind)

$$A_h^+ = \begin{bmatrix} \frac{2\nu}{h^2} - \frac{b}{h} + c_1 & -\frac{\nu}{h^2} + \frac{b}{h} & & & \mathbf{0} \\ -\frac{\nu}{h^2} & \frac{2\nu}{h^2} - \frac{b}{h} + c_2 & -\frac{\nu}{h^2} + \frac{b}{h} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{\nu}{h^2} & \frac{2\nu}{h^2} - \frac{b}{h} + c_{J-1} & -\frac{\nu}{h^2} + \frac{b}{h} \\ \mathbf{0} & & & -\frac{\nu}{h^2} & \frac{2\nu}{h^2} - \frac{b}{h} + c_J \end{bmatrix}$$

- Backward (downwind)

$$A_h^- = \begin{bmatrix} \frac{2\nu}{h^2} - \frac{b}{h} + c_1 & -\frac{\nu}{h^2} & & & \mathbf{0} \\ -\frac{\nu}{h^2} - \frac{b}{h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_2 & -\frac{\nu}{h^2} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{\nu}{h^2} - \frac{b}{h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_{J-1} & -\frac{\nu}{h^2} \\ \mathbf{0} & & & -\frac{\nu}{h^2} - \frac{b}{h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_J \end{bmatrix}$$

- Centered

$$A_h^0 = \begin{bmatrix} \frac{2\nu}{h^2} - \frac{b}{h} + c_1 & -\frac{\nu}{h^2} + \frac{b}{2h} & & & \mathbf{0} \\ -\frac{\nu}{h^2} - \frac{b}{2h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_2 & -\frac{\nu}{h^2} + \frac{b}{2h} & & \\ & \ddots & \ddots & \ddots & \\ & & -\frac{\nu}{h^2} - \frac{b}{2h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_{J-1} & -\frac{\nu}{h^2} + \frac{b}{2h} \\ \mathbf{0} & & & -\frac{\nu}{h^2} - \frac{b}{2h} & \frac{2\nu}{h^2} - \frac{b}{h} + c_J \end{bmatrix}$$

As a reminded $v_0 = 0$ and $v_{J+1} = 0$.

Should you want to enter these matrices in Matlab, you can use this (once h, b and C have been defined):

- $A_h^+ = \frac{\nu}{h^2}(2\text{diag}(ones(J,1)) - \text{diag}(ones(J-1,1),-1) - \text{diag}(ones(J-1,1),1)) + \frac{b}{h}(\text{diag}(ones(J-1,1),1) - \text{diag}(ones(J,1)) + diag(C)$,
- $A_h^- = \frac{\nu}{h^2}(2\text{diag}(ones(J,1)) - \text{diag}(ones(J-1,1),-1) - \text{diag}(ones(J-1,1),1)) + \frac{b}{h}(\text{diag}(ones(J,1)) - \text{diag}(ones(J-1,-1)) + diag(C)$,
- $A_h^0 = \frac{\nu}{h^2}(2\text{diag}(ones(J,1)) - \text{diag}(ones(J-1,1),-1) - \text{diag}(ones(J-1,1),1)) + \frac{b}{2h}(\text{diag}(ones(J-1,1),1) - \text{diag}(ones(J-1,1)) + diag(C)$.

Solution de Q. V.2.2 E.V.1.1 provides a sufficient condition for the matrix to be monotone.

- For A_h^+ :

$$\begin{cases} h^2\alpha_i &= 2\nu - bh + c_i h^2, \quad i \in \{1,\dots,J\} \\ h^2\beta &= \nu - bh \\ h^2\gamma &= \nu \end{cases}$$

  The sufficient condition is satisfied iff

$$\nu - bh > 0 \qquad i.e. \qquad \frac{bh}{\nu} < 1.$$

- For A_h^- :

$$\begin{cases} h^2\alpha_i &= 2\nu + bh + c_i h^2, \quad i \in \{1,\dots,J\} \\ h^2\beta &= \nu \\ h^2\gamma &= \nu + bh \end{cases}$$

  The sufficient condition is satisfied iff

$$\nu + bh > 0 \qquad i.e. \qquad \frac{bh}{\nu} > -1.$$

- For A_h^0 :

$$\begin{cases} h^2\alpha_i &= 2\nu + c_i h^2, \quad i \in \{1,\dots,J\} \\ h^2\beta &= \nu - \frac{bh}{2} \\ h^2\gamma &= \nu + \frac{bh}{2} \end{cases}$$

  The sufficient condition is satisfied iff

$$|b|h < 2\nu \qquad i.e. \qquad \frac{|b|h}{\nu} < 2.$$

For a given $b \in \mathbb{R}$, $\nu > 0$ and $h > 0$. Let $\lambda = \dfrac{bh}{\nu}$.

λ	$-\infty$		-2		-1		0		1		2		$+\infty$
A_h^+										?		?	
A_h^-		?		?									
A_h^0		?										?	

Solution de Q. V.2.3 We take our inspiration from the proof in the continuous case. The ingredients are the same (Cauchy-Schwarz, etc.)
Let $j \in \{0, \dots, J\}$.

$$\begin{aligned}
|v_j| &= \left| v_0 + \sum_{k=0}^{j-1} (v_{k+1} - v_k) \right| \\
&\le \sqrt{\sum_{k=0}^{j-1} 1} \sqrt{\sum_{k=0}^{j-1} (v_{k+1} - v_k)^2} \\
&\le \sqrt{(J+1)} \sqrt{\sum_{k=0}^{J} (v_{k+1} - v_k)^2} \\
&\le \sqrt{\sum_{k=0}^{J} \frac{(v_{k+1} - v_k)^2}{h}}
\end{aligned}$$

Since $h(J+1) = 1$. We derive

$$\begin{aligned}
h \sum_{i=0}^{J} v_i^2 &\le h \sum_{j=0}^{J} \sum_{k=0}^{J} \frac{(v_{k+1} - v_k)^2}{h} \\
&\le \sum_{k=0}^{J} \frac{(v_{k+1} - v_k)^2}{h}.
\end{aligned}$$

Solution de Q. V.2.4 For $b = 0$ and $c = 0$, we have

$$B_h = \frac{\nu}{h^2} \begin{pmatrix} 2 & -1 & & & 0 \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ 0 & & & -1 & 2 \end{pmatrix}$$

which is the matrix of Exercise **??** with $c_i = 0$. We obtain

$$\frac{h^2}{\nu} v^T B_h v = v_1^2 + v_{J+1}^2 + \sum_{i=2}^{J} (v_i - v_{i-1})^2 = \sum_{i=1}^{J+1} (v_i - v_{i-1})^2 = \sum_{i=0}^{J} (v_{i+1} - v_i)^2$$

Which leads to

$$(B_h v, v) = \frac{\nu}{h^2} \sum_{i=0}^{J} (v_{i+1} - v_i)^2$$

Solution de Q. V.2.5 We transfer the (non-homogeneous) boundary conditions onto the right hand side of the linear system. Let $G = \left(f(x_1) + \frac{\nu}{h^2} u(0) + \frac{b}{h} u(0), f(x_2), \dots, f(x_J) \right)^T$.

Solution de Q. V.2.6 We can decompose A^- as the sum of three matrices $A = B_h + D + \mathrm{diag}(C)$ where

$$D = \frac{b}{h}\begin{pmatrix} 1 & 0 & & & \\ -1 & 1 & 0 & & \\ & \ddots & \ddots & & \\ & 0 & -1 & 1 & 0 \\ & & 0 & -1 & 1 \end{pmatrix}$$

and $\mathrm{diag}(C)$ is the diagonal matrix composed of the c_i.

We have $(B_h V, V) \geq \frac{\nu}{h}\|V\|_2^2$, and $(CV, V) = \sum_{j=1}^{J} c_j v_j^2 \geq 0$. Moreover

$$\begin{aligned}
(DV, V) &= \frac{b}{h}\sum_{j=1}^{J}(v_j - v_{j-1})v_j \\
&= \frac{b}{h}\sum_{j=1}^{J}(v_j - v_{j-1})(v_j - v_{j-1}) + \frac{b}{h}\sum_{j=1}^{J}(v_j - v_{j-1})v_{j-1} \\
&= \frac{b}{2h}\sum_{j=1}^{J}(v_j - v_{j-1})^2 + \frac{b}{2h}\sum_{j=1}^{J}(v_j - v_{j-1})(v_j - v_{j-1} + 2v_{j-1}) \\
&= \frac{b}{2h}\left(\sum_{j=1}^{J}(v_j - v_{j-1})^2 + v_J^2\right) \\
&\geq \frac{b}{2h}\sum_{j=1}^{J}(v_j - v_{j-1})^2 \\
&\geq \frac{b}{2}\|V\|_2^2
\end{aligned}$$

Adding up these estimates leads to

$$\left(\frac{\nu}{h} + \frac{b}{2}\right)\|V\|_2^2 \leq (A_h V_h, V_h) \leq \sqrt{\sum_{j=1}^{J} f_i^2}\sqrt{\sum_{j=1}^{J} v_i^2} = \frac{1}{h}\|G\|_2\|V_h\|_2.$$

The results derives.

Solution de Q. V.2.7 Let u be the regular solution to **(CD)**. Let $\Pi_h u = (u(x_1, \ldots, u(x_J))^T$ and let V_h lbe the solution to the linear system $A_h V_h = G$, where G is the right hand side appearing in Question V.2.5. We are trying to estimate $\|E\|_2$ where $E = V_h - \Pi_h u$.

From the previous question, we have $\|A_h^{-1}\|_2 \leq (\nu + bh/2)^{-1}$

$$\begin{aligned}
(A_hE)_j &= -\frac{\nu}{h^2}(v_{j+1} - 2v_j + v_{j-1}) + \frac{b}{h}(v_j - v_{j-1}) + c_jv_j \\
&\qquad + \frac{\nu}{h^2}(u(x_{j+1}) - 2u(x_j) + u(x_{j-1})) - \frac{b}{h}(u(x_j) - u(x_{j-1})) - c(x_j)u(x_j) \\
&= G_j - (f(x_j) + \mathcal{E}_j) \\
&= f(x_j) - f(x_j) - \mathcal{E}_j \\
&= -\mathcal{E}_j
\end{aligned}$$

where $\mathcal{E}_j$ is the consistency error computed in V.2.4.
For $j = 1$,

$$\begin{aligned}
(A_hE)_1 &= -\frac{\nu}{h^2}(-2v_1 + v_2) + \frac{b}{h}v_1 + c_1v_1 \\
&\qquad + \frac{\nu}{h^2}(-2u(x_1) + u(x_2)) - \frac{b}{h}u(x_1) - c(x_1)u(x_1) \\
&= G_1 + \frac{\nu}{h^2}(-2u(x_1) + u(x_2)) - \frac{b}{h}u(x_1) - c(x_1)u(x_1) \\
&= f(x_1) + \frac{\nu}{h^2}u(0) + \frac{b}{h}u(0) + \frac{\nu}{h^2}(-2u(x_1) + u(x_2)) - \frac{b}{h}u(x_1) - c(x_1)u(x_1) \\
&= f(x_1) + \frac{\nu}{h^2}(u(x_0) - 2u(x_1) + u(x_2)) - \frac{b}{h}(u(x_1) - u(x_0)) - c(x_1)u(x_1) \\
&= f(x_1) - f(x_1) - \mathcal{E}_1 \\
&= -\mathcal{E}_1
\end{aligned}$$

For $j = J$, the same mathod applied and we get $(A_hE)_J = -\mathcal{E}_J$ using $u(1) = v_{J+1} = 0$.

Let $\mathcal{E}_0 = 0$. We know $\|\mathcal{E}\|_\infty \leq Ch$, where C is a constant depending only on u , b and ν. Since

$$\|\mathcal{E}\|_2^2 = h\sum_{j=0}^{J} \mathcal{E}_j^2 \leq h(J+1)\|\mathcal{E}\|_\infty^2 = \|\mathcal{E}\|_\infty^2.$$

we have

$$\|E\|_2 \leq \|A_h^{-1}\|_2\|\mathcal{E}\|_2 \leq \frac{Ch}{\nu + bh/2} \leq \frac{Ch}{\nu}.$$

The scheme converges and is of order 1.

Solution de Q. V.3.1 Le schéma s'écrit

$$\begin{cases}
-\dfrac{v_{j+1} - 2v_j + v_{j-1}}{h^2} + c_jv_j = f_j, \quad 1 \leq j \leq J \\
-\dfrac{-v_{J+1} + v_J}{h^2} + c_{J+1}v_{J+1} = f_{J+1}, \\
v_0 = 1
\end{cases}$$

La matrice $A_h \in \mathcal{M}_{J+1}(\mathbb{R})$ s'écrit donc

$$A_h = \frac{1}{h^2}\begin{pmatrix} 2+h^2c_1 & -1 & 0 & & \\ -1 & 2+h^2c_2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -1 & 2+h^2c_J & -1 \\ & & 0 & -1 & 1+h^2c_{J+1} \end{pmatrix}$$

Solution de Q. V.3.2 On applique la question V.1.1. La matrice A_h est donc inversible.

Solution de Q. V.3.3 Calculons l'erreur de consistance. Let $u \in C^4([0,1])$ la solution de (P). Alors, pour tout $j \in \{1,\dots,J\}$,

$$\begin{aligned}\mathcal{E}_j &= -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + c_j u(x_j) - f(x_j) \\ &= -\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} + c_j u(x_j) + u''(x_j) - c_j u(x_j) \\ &= -\nu u''(x_j) + O(h^2) + \nu u''(x_j) = O(h^2).\end{aligned}$$

Calculons $\mathcal{E}_{J+1}$, sachant que $u'(1) = 0$:

$$\begin{aligned}\mathcal{E}_{J+1} &= -\frac{-u(x_{J+1}) + u(x_J)}{h^2} + c_{J+1}u(x_{J+1}) - f(x_{J+1}) \\ &= -\frac{-u(1) + u(1-h)}{h^2} + c(1)u(1) - f(1) \\ &= \frac{hu'(1) - (h^2/2)u''(1) + o(h^2)}{h^2} + c(1)u(1) - f(1) \\ &= \frac{u''(1)}{2} + o(1).\end{aligned}$$

Il n'y a pas de raison *a priori* pour que l'erreur de consistance soit nulle en x_{J+1} (considérer le cas $c = 0$). Le schéma n'est donc pas consistant !

Solution de Q. V.3.4 Notons $\mathbf{e} := (1,\dots,1,0)^T$ Notons $A_h^0 = A_h - \operatorname{diag}([C, c(1)])$.
Traitons tout d'abord le cas $c = 0$ ($A_h^0 = A_h$). On note V_h^0 la solution de $A_h^0 V_h^0 = Y$.
Considérons le problème

$$\begin{cases} -u'' = 1 & \text{in} \quad]0,1[\\ u(0) = 1 & \text{and} \quad u(1) = u(1-h). \end{cases}$$

L'unique solution du problème est un polynôme de degré 2 :

$$\bar{u} : x \mapsto -\frac{x^2}{2} + \left(1 - \frac{h}{2}\right)x.$$

On vérifie immédiatement que $A_h^0(\Pi_h \bar{u}) = \mathbf{e}$. En effet, le schéma est exact pour ce problème, puisque la solution est polynomiale de degré inférieur ou égal à 3 !
Remarquons que $\|\bar{u}\|_\infty = ((1-h/2)^2)/2 \leq 1/2$.
Considérons à présent le vecteur $W_- = Y - \|Y\|_\infty \mathbf{e}$, qui est **à coefficients négatifs**. Comme A_h^0 est monotone, $(A_h^0)^{-1}W_-$ est également à coefficients négatifs.
Or on a

$$(A_h^0)^{-1}W_- = (A_h^0)^{-1}(Y - \|Y\|_\infty \mathbf{e}) = V_h^0 - \|Y\|_\infty \Pi_h \bar{u}.$$

On a donc prouvé que

$$\forall j \in \{1, \dots, J+1\}, \qquad v_j^0 - \|Y\|_\infty \bar{u}(x_j) \leq 0.$$

De même, en considérant $W_+ = Y + \|Y\|_\infty \mathbf{e}$, qui est à coefficients positifs, on prouve que

$$\forall j \in \{1, \dots, J+1\}, \qquad v_j^0 + \|Y\|_\infty \bar{u}(x_j) \geq 0.$$

On a donc

$$\|V_h^0\|_\infty \leq \|\bar{u}\|_\infty \|Y\|_\infty \leq \frac{\|Y\|_\infty}{2}.$$

Cas général : c est une fonction positive. Let V_h la solution de $A_h V_h = Y$.
Rappelons que A_h^{-1} and $(A_h^0)^{-1}$ sont à coefficients positifs ou nuls. Remarquons que

$$(A_h)^{-1} - (A_h^0)^{-1} = A_h^{-1}(A_h^0 - A_h)(A_h^0)^{-1} = -\mathrm{diag}([C, c(1)]).$$

La matrice $(A_h)^{-1} - (A_h^0)^{-1}$ est donc à coefficients négatifs ! On en déduit que

$$\forall j \in \{1, \dots, J+1\}, \qquad 0 \leq ((A_h)^{-1}\mathbf{e})_j \leq ((A_h^0)^{-1}\mathbf{e})_j.$$

En reprenant la démonstration dans le cas $c = 0$, on écrit

$$(A_h)^{-1}W_- = V_h - \|Y\|_\infty (A_h)^{-1}\mathbf{e}.$$

Donc, pour tout $j \in \{1, \dots, J+1\}$, on a

$$v_j \leq \|Y\|_\infty (A_h^{-1}\mathbf{e}) \leq \|Y\|_\infty ((A_h^0)^{-1}\mathbf{e}) \leq \frac{\|Y\|_\infty}{2}.$$

De même, on montre que

$$v_j \geq -\frac{\|Y\|_\infty}{2}.$$

Solution de Q. V.3.5 On raisonne comme à la question précédente.
On traite d'abord le cas $c = 0$.
Considérons le problème

$$\begin{cases} -u'' = 0 \quad \text{in} \quad]0,1[\\ u(0) = 1 \quad \text{and} \quad u(1) = u(1-h) + h^2. \end{cases}$$

L'unique solution du problème est un polynôme de degré 1 :

$$\bar{u} : x \mapsto hx.$$

En appliquant le même raisonnement qu'à la question précédente en prenant $W_{\pm} = Y \pm |y_J| e_{J+1}$, on montre l'inégalité demandée.

Solution de Q. V.3.6 Let u la solution régulière de (P).
Alors l'erreur $E = V_h - \Pi_h u$ est évaluée grâce à

$$E = A_h^{-1} A_h E = A_h^{-1} A_h (V_h - \Pi_h u) = A_h^{-1}(F - A_h \Pi_h) = A_h^{-1} \mathcal{E}$$

qui donne

$$\|E\|_\infty \leq \left\| A_h^{-1} \begin{pmatrix} \mathcal{E}_1 \\ \vdots \\ \mathcal{E}_J \\ 0 \end{pmatrix} \right\|_\infty + \left\| A_h^{-1} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ \mathcal{E}_{J+1} \end{pmatrix} \right\|_\infty \leq \frac{1}{2} C h^2 + h |\mathcal{E}_{J+1}| = O(h).$$

Le schéma converge à l'ordre 1.

Solution de Q. V.3.7 Ceci est une discrétisation centrée de la dérivée en x_J : la matrice A_h devient

$$A_h = \frac{1}{h^2} \begin{pmatrix} 2 + h^2 c_1 & -1 & 0 & & \\ -1 & 2 + h^2 c_2 & -1 & 0 & \\ & \ddots & \ddots & \ddots & \\ & 0 & -1 & 2 + h^2 c_J & -1 \\ & & 0 & -2 & 2 + h^2 c_{J+1}. \end{pmatrix}$$

Le dernier coefficient de l'erreur de consistance est donc

$$\begin{aligned} \mathcal{E}_{J+1} &= -\frac{-2u(x_{J+1}) + 2u(x_J)}{h^2} + c_{J+1} u(x_{J+1}) - f(x_{J+1}) \\ &= -\frac{-2u(1) + 2u(1-h)}{h^2} + c(1)u(1) - f(1) \\ &= 2\frac{hu'(1) - (h^2/2)u''(1) + O(h^3)}{h^2} + c(1)u(1) - f(1) \\ &= O(h). \end{aligned}$$

Le nouveau schéma est donc consistant à l'ordre 1.