# Exercise IV.2

Let $L \in \mathbb{R}^{K\times N},\ b \in \mathbb{R}^{K},\ c \in \mathbb{R}^{N}$. We consider the following linear programming problem:

$$\text{minimize}_{x\in[0,+\infty[^N} \langle c|x\rangle,\ \text{s.t. } Lx \geq b \tag{1}$$

.

Show that it is associated with the dual problem:

$$\text{maximize}_{y\in[0,+\infty[^K} \langle b|y\rangle,\ \text{s.t. } L^T y \leq c. \tag{2}$$

Solution: We start by rewriting the given optimization problem in the equivalent form:

$$\text{minimize}_{x\in\mathbb{R}^N} f(x) + g(Lx), \tag{3}$$

where:

$$\begin{aligned} f\left(x\right) &= \langle x|c\rangle + \iota_{[0,+\infty[^N}(x) \\ g\left(z\right) &= \iota_{[0,+\infty[^N}\left(z-b\right). \end{aligned} \tag{4}$$

Let us now find the conjugates of these two functions. There are given as follows:

$$f^*\left(u\right) = \sup_{x\in\mathbb{R}^N} \langle x|u\rangle - \langle x|c\rangle - \iota_{[0,+\infty[^N}(x) = \sup_{x\in\mathbb{R}^N} \langle x|u-c\rangle - \iota_{[0,+\infty[^N}(x), \tag{5}$$

We now notice that in the above equation, the last expression is the conjugate of $\iota_{[0,+\infty[^N}(x)$ evaluated at $u-c$. As a result, using the result of Exercise 1, we have that:

$$f^*\left(u\right) = \iota_{]-\infty,0]^N}(u-c). \tag{6}$$

Concerning $g\left(z\right)$ we have that:

$$g^*\left(v\right) = \sup_{z\in\mathbb{R}^K} \langle z,u\rangle - \iota_{[0,+\infty[^K}(z-b) = \langle b|v\rangle + \iota_{]-\infty,0]^K}(v). \tag{7}$$

As a result, since the dual problem is defined as:

$$\begin{aligned} &\text{minimize}_{v\in\mathbb{R}^K} f^*\left(-L^T v\right) + g^*(v) = \\ &\text{minimize}_{v\in\mathbb{R}^K} \iota_{]-\infty,0]^N}\left(-L^T v - c\right) + \langle b|v\rangle + \iota_{]-\infty,0]^K}(v) = \\ &\text{minimize}_{v\in]-\infty,0]^K} \langle b|v\rangle,\ \text{s.t. } -L^T v - c \leq 0. \end{aligned} \tag{8}$$

after introducing the change of variables $y = -v$, we can rewrite it as:

$$\text{maximize}_{y\in[0,+\infty[^K} \langle b,y\rangle,\ \text{s.t. } L^T y \leq c. \tag{9}$$