Exercise 1

A travelling salesperson must visit N cities by departing from one of them and coming back to it.
The travelling time in a direct trip from the i-th city to the j-th city (with $i,j \in \{1,\dots,N\}^N, i \neq j$) is τ_{ij}. The salesperson wants to minimize the duration of his/her whole trip while visiting each city only once. Formulate this problem as a binary linear programming problem. How can we avoid independent subtours within the trip?

Solution:

For every pair $(i,j) \in \{1,\dots,N\}^2$, $i \neq j$, we introduce the decision variable $x_{i,j}$ where

$$x_{i,j} = \begin{cases} 1, & \text{if the trip contains the direct trip from } i \text{ to } j. \\ 0, & \text{otherwise} \end{cases}$$

The problem is then expressed as

$$\underset{x \in \{0,1\}^{N^2-N}}{\text{minimize}} \sum_{i=1}^{N} \sum_{\substack{j=1 \\ j \neq i}}^{N} \tau_{i,j} x_{i,j}$$

subject to:

$$\forall i \in \{1,\dots,N\} \quad \sum_{\substack{j=1 \\ j \neq i}}^{N} x_{i,j} = 1$$

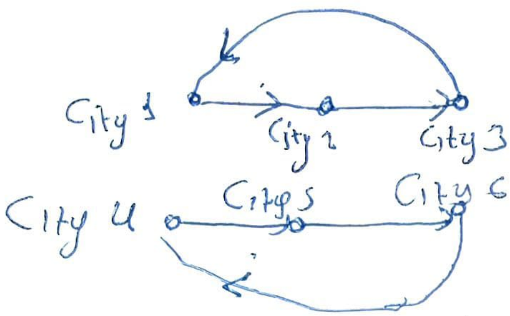
(this constraints represents the fact that we leave from city 1 exactly one time)

$$\forall j \in \{1,\dots,N\} \quad \sum_{\substack{i=1 \\ j \neq i}}^{N} x_{ij} = 1.$$

(this constraint represents the fact that we arrive at each city exactly once)

An Independent subtours appear when we have in our solution subtours which are not interconnected.

e.g.



This solution satisfies the two given constraints however it is not an acceptable solution to our problem.

To avoid such solutions, we note that based on the present example if $C \subset \{1, \dots, N\}$, then a subtour exists in C if

$$\mathrm{card}(C) \notin \{1; N\}$$

and $$\sum_{(i,j) \in C^2,\ i \neq j} x_{i,j} = \mathrm{card}(C).$$

As a result, if we impose the constraint that

$\forall C \subset \{1, \dots, N\}$ such that $\mathrm{card}(C) \notin \{1, N\}$

it should hold that

$$\sum_{\substack{(i,j) \in C^2 \\ i \neq j}} x_{i,j} \leq \mathrm{card}(C) - 1$$

we can avoid subtours.