Exercise V.6

A company produces two items P_1 and P_2 in two factories. For each factory, the number of hours for producing 1 unit of each product as well as the number of working hours vary:

	P_1	P_2	Working hours
Factory 1	4	2	80
Factory 2	2	2	50

The unit price of each product is 500 euros for P_1 and 300 euros for P_2.

1. By using the simplex algorithm, show how to maximize the profit of the company.
2. Evaluate the sensitivity of this strategy with respect to a reduction of the number of working hours possibly caused by a strike.

Solution: What we can actually do in order to maximize the profit, is to decide how much of each product should be produced at each factory. To determine this, we introduce the following variables:

- $x^{(1)}$: This variable corresponds to the amount of product P_1 that should be produced at Factory 1.
- $x^{(2)}$: This variable corresponds to the amount of product P_1 that should be produced at Factory 2.
- $x^{(3)}$: This variable corresponds to the amount of product P_2 that should be produced at Factory 1.
- $x^{(4)}$: This variable corresponds to the amount of product P_2 that should be produced at Factory 2.

The total profit would then be equal to:

$$500\left(x^{(1)}+x^{(2)}\right)+300\left(x^{(3)}+x^{(4)}\right). \tag{1}$$

The constraint of working hours for Factory 1 is expressed as:

$$4x^{(1)}+2x^{(3)}\leq 80 \tag{2}$$

and for Factory 2 as:

$$2x^{(2)}+2x^{(4)}\leq 50. \tag{3}$$

As a result, we can equivalently write down the profit maximization problem as:

$$\begin{aligned}&\text{maximize}_{\left(x^{(i)}\right)_{1\leq i\leq 4}\in[0,+\infty[^4}:\ 5x^{(1)}+5x^{(2)}+3x^{(3)}+3x^{(4)}\\ &\text{subject to: } 2x^{(1)}+x^{(3)}\leq 40\\ &\qquad\qquad\quad x^{(2)}+x^{(4)}\leq 25.\end{aligned} \tag{4}$$

We now proceed and introduce two additional variables $x^{(5)}$ and $x^{(6)}$ such as to convert the inequality constraints to equality constraints and rewrite the problem in its standard form:

$$\begin{aligned} &\text{minimize}_{\left(x^{(i)}\right)_{1\le i\le 6}\in[0,+\infty[^6}: \quad -5x^{(1)} - 5x^{(2)} - 3x^{(3)} - 3x^{(4)} \\ &\text{subject to: } 2x^{(1)} + x^{(3)} + x^{(5)} = 40 \\ &\qquad\qquad\quad x^{(2)} + x^{(4)} + x^{(6)} = 25. \end{aligned} \tag{5}$$

We start now by considering as an inital basis the fifth and sixth column of the constraint matrix (These two columns already have the form of an identity matrix). We then write down the tableau matrix representation for our problem.

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	
2	0	1	0	1	0	40
0	1	0	1	0	1	25
-5	-5	-3	-3	0	0	f

The corresponding reduced cost coefficient vector is the vector:

$$r = \begin{bmatrix} -5 \\ -5 \\ -3 \\ -3 \end{bmatrix}. \tag{6}$$

Based on this reduced cost coefficient vector, using the Bland rule, we pick the first column (i.e., we set $j = 1$ to join the basis), and calculate Δ_j as:

$$\Delta_j = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \end{bmatrix} \tag{7}$$

Since there is only one positive element in Δ_j, (the first one) we select i_j to be the column of the matrix which corresponds to it. This is the fifth column. As a result, we obtain $i_j = 5$, and we set:

$$z^{(j)} = 40/2 = 20. \tag{8}$$

As a result, the new basic solution will be the solution $w = [20, 0, 0, 0, 25]$, $x^{(1)}$ has entered the basis, and $x^{(5)}$ has left the basis. We now start again the tableau as given in the above form and apply the necessary elemenary row operations in order to transform it to the form that follows.

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	20
0	1	0	1	0	1	25
0	-5	$-\frac{1}{2}$	-3	$\frac{5}{2}$	0	f + 100

As a result, we obtain the reduced cost coefficient vector:

$$r = \begin{bmatrix} -5 \\ -\frac{1}{2} \\ -3 \\ \frac{5}{2} \end{bmatrix}. \tag{9}$$

Applying again Bland rule, we pick the first element to join the basis (which corresponds to the second column). As a result, we set $j = 2$. The vector Δ_j will then be equal to

$$\Delta_j = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{10}$$

Since there is only one positive element in Δ_j, we pick it. This corresponds to the second element of the basis, which is the 6-th column of the matrix. This means that the 6-th column leaves the basis and the second one enters the basis. We also obtain:

$$z^{(j)} = \frac{25}{1}. \tag{11}$$

Now, since the basis consists of the first and second column, we apply elementary row operations in order to write it down in the following form (this is an equivalent alternative to derive the experssions given in your slides for calculating r).

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	
1	0	$\frac{1}{2}$	0	$\frac{1}{2}$	0	20
0	1	0	1	0	1	25
0	0	$-\frac{1}{2}$	2	$\frac{5}{2}$	5	f + 225

The corresponding reduced cost vector becomes:

$$r = \begin{bmatrix} -\frac{1}{2} \\ 2 \\ \frac{5}{2} \\ 5 \end{bmatrix}. \tag{12}$$

Since only the first element is negative (and it corresponds to the third column) we pick $j = 3$. We then find Δ_j to be equal to:

$$\Delta_j = \begin{bmatrix} \frac{1}{2} \\ 0 \end{bmatrix}, \tag{13}$$

which leads us to picking $i_j = 1$ and setting $z^{(j)} = 40$. As a result $j = 3$ enters the basis and $i = 1$ leaves the basis. The corresponding basic solution is given by setting $x^{(2)} = 25$ and $x^{(3)} = 40$ and setting the remaining elements equal to zero.

We can then apply elementary row operations which allow us to rewrite the tableau for our problem as follows.

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	$x^{(4)}$	$x^{(5)}$	$x^{(6)}$	
2	0	1	0	1	0	40
0	1	0	1	0	1	25
1	0	0	2	3	5	f + 245,

We see here that the reduced cost vector r has all elements positive. As a result, we obtain that this basic feasible solution is also optimal.

This means that the optimal production strategy is to manifacture product P_1 only at Factory 2 and product P_2 only at Factory 1.

Evaluating the sensitivity of the solution

In order to calculate the sensitivity of our solution to changes in working hours, we need to see for which changes of the vector b, the calculated basic index set still remains optimal. In more detail if b is perturbed (i.e., if the working hours change from b to $b + \delta b$) the basic index set remains optimal if:

$$A_{\mathbb{I}}^{-1}\left(b + \delta_b\right) \geq 0, \tag{14}$$

of equivalently if:

$$A_{\mathbb{I}}^{-1}\delta_b \geq -A_{\mathbb{I}}^{-1} b, \tag{15}$$

with

$$A_{\mathbb{I}} = \begin{bmatrix} 0 & 1 \\ 1 & 0. \end{bmatrix} \tag{16}$$

As a result, we obtain that the basic index set $\{2, 3\}$ remains optimal if:

$$\delta_b \geq -\begin{bmatrix} 80 \\ 50 \end{bmatrix}. \tag{17}$$

This means that for all possible decreases of the working hours, the basic index set remains optimal. Therefore, the basic index set should not change and the production strategy should remain the same.