Exercise V.5

Use the simplex algorithm to solve the problem:

$$\begin{aligned} &\text{maximize:}_{(x^{(1)},x^{(2)})\in[0,+\infty[^2} \ \ 2x^{(1)} - x^{(2)} \\ &\text{subject to: } x^{(1)} - x^{(2)} \leq 1, \ \ 2x^{(1)} + x^{(2)} \geq 6. \end{aligned} \tag{1}$$

Solution: We start by introducing two slack variables $x^{(3)}$ and $x^{(4)}$ (one for each constraint), in order to express the problem in the standard form:

$$\begin{aligned} &\text{minimize:}_{(x^{(1)},x^{(2)},x^{(3)},x^{(4)})\in[0,+\infty[^4} \ \ -2x^{(1)} + x^{(2)} \\ &\text{subject to: } x^{(1)} - x^{(2)} + x^{(3)} = 1, \\ &\qquad\qquad 2x^{(1)} + x^{(2)} - x^{(4)} = 6. \end{aligned} \tag{2}$$

In order to apply the simplex algorithm we now need to first find a basic feasible solution. To do so, let as start by setting $x^{(3)} = x^{(4)} = 0$ and find the resulting basic solution, by solving the linear system:

$$\begin{aligned} x^{(1)} - x^{(2)} &= 1 \\ 2x^{(1)} + x^{(2)} &= 6, \end{aligned} \tag{3}$$

which leads to the solution $x^{(1)} = 7/3$ and $x^{(2)} = 4/3$ which is feasible. As a result, we can start the simplex algorihtm by selecting as basic columns the first two columns (i.e., setting as basic index set the set $\mathbb{I} = \{1, 2\}$). The corresponding basic solution will then be $x = (7/3, 4/3, 0, 0)$

We now write down the tableau representation of our problem as:

$x^{(1)}$	$x^{(2)}$	$x^{(3)}$	x^4	
1	- 1	1	0	1
2	1	0	- 1	6
-2	1	0	0	f.

Given that the first columns are the basic ones, and the above tableau representation, we now can write down the corresponding reduced cost vector as:

$$r = d_{\mathbb{J}} - A_{\mathbb{J}}^T \left(A_{\mathbb{I}}^{-1}\right)^T d_{\mathbb{I}} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}^T \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 4 \\ -1 \end{bmatrix} \tag{4}$$

Since the reduced cost vector has only one negative element (which correponds to $j = 4$) we pick to investigate the fourth column, and calculate:

$$\Delta_4 = \mathbb{A}_{\mathbb{I}}^{-1} a_4 = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}^{-1} \begin{bmatrix} 0 \\ -1 \end{bmatrix} = -\frac{1}{3} \begin{bmatrix} 1 \\ 1. \end{bmatrix} \tag{5}$$

Since both elements are negative, we obtain that the problem is unbounded.