Partial Differential Equations

Chapter VII - Finite Difference Method

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The Engineering Program of CentraleSupélec

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VII.1. Introduction

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The core idea underlying the FEM is:

- To mesh the domain and build a finite-dimensional subspace of elementary functions.
- To project u on that subspace of functions.
- To end up with a finite number of unknowns and to solve a linear system.

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There is another way...

As you know...

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Higher order derivatives can be approximated by iteration of this process.

The core idea underlying the FDM is:

- To mesh the domain thus obtaining a grid
- To approximate the derivatives of u using the value of u on the grid
- To have a finite numbers of unknowns (the values of u on the grid) and to solve a linear system.

VII.2. FDM in dimension $d = 1$

VII.2.1. Approximating the derivative

Forward, backward and central differences

Let $b > a$, and f be a real-valued function defined on $[a, b]$.
Let $J \in \mathbb{N}^*$ and

$$(x_j)_{j\in\{0,\dots,J+1\}}$$

be a mesh of $[a, b]$ where $x_0 = a$ and $x_{J+1} = b$.
Note $h = (b - a)/(J + 1)$

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Definition VII.2.3

The **central difference** *at $x = x_j$ is defined by*
$f(x + h/2) - f(x - h/2)$

Forward, backward and central differences

Remark VII.2.4

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The problem is often remedied by taking the average on the two points nearby on the grid:

- *we replace $f(x + \frac{h}{2})$ by $(f(x + h) + f(x))/2$*
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This leads to

$$f\left(x + \frac{h}{2}\right) - f\left(x - \frac{h}{2}\right) \quad \textit{replaced by} \quad \frac{f(x_{j+1}) + f(x_j)}{2} - \frac{f(x_{j-1}) + f(x_j)}{2}$$

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Subsequently, the **central difference** *at $x = x_j$ is often defined by*

$$\frac{f(x_{j+1}) - f(x_{j-1})}{2}$$

Forward, backward and central difference quotient

Definition VII.2.5

The **forward difference quotient** *is given by*

$$\frac{f(x+h)-f(x)}{h}$$

Forward, backward and central difference quotient

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The **forward difference quotient** *is given by*

$$\frac{f(x+h)-f(x)}{h} = \frac{f(x_{j+1})-f(x_j)}{x_{j+1}-x_j}$$

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Definition VII.2.7

The **central difference quotient** *is given by*

$$\frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h}$$

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Relation between difference quotients and derivatives

Provided f is twice differentiable, the Taylor expansion gives

$$f(x+h) = f(x) + h\,f'(x) + O(h^2)$$

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The **local truncation error** *or* **discretization error** *is the difference between the difference quotient and the derivative.*
If the local truncation error is $O(h^k)$, the method is of **order k**.

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Proposition VII.2.9

The forward difference quotient approximates the derivative.
The method is of order 1.

Relation between difference quotients and derivatives

Provided f is twice differentiable, the Taylor expansion gives

$$f(x-h) = f(x) - h\,f'(x) + O(h^2)$$

Therefore

$$\frac{f(x) - f(x-h)}{h} - f'(x) = O(h) \xrightarrow[h \to 0]{} 0$$

Proposition VII.2.10

The backward difference quotient approximates the derivative. The method is of order 1.

Relation between different quotients and derivatives

Provided f is three times differentiable, the Taylor expansion gives

$$f\left(x + \frac{h}{2}\right) = f(x) + \frac{h}{2} f'(x) + \frac{1}{2}\left(\frac{h}{2}\right)^2 f''(x) + O(h^3)$$

$$f\left(x - \frac{h}{2}\right) = f(x) - \frac{h}{2} f'(x) + \frac{1}{2}\left(\frac{h}{2}\right)^2 f''(x) + O(h^3)$$

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Therefore

$$f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right) = h\, f'(x) + O(h^3)$$

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$$\frac{f\left(x+\frac{h}{2}\right) - f\left(x-\frac{h}{2}\right)}{h} - f'(x) = O(h^2) \xrightarrow[h\to 0]{} 0$$

Proposition VII.2.11

The central difference quotient approximates the derivative.
The method is of order 2.

Relation between different quotients and derivatives

Question:
Is there a way to approximate the derivative with a method of order 4?

Relation between different quotients and derivatives

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Answer:
Will be given in Exercise Q.V.1 during the next lab session.

Stencils

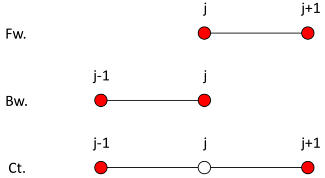
Definition VII.2.12

The **stencil** *of a numerical approximation method is the geometric arrangement of the nodes that relate to the point of interest.*

In dimension 1, the stencil is "flat".
We can have more complex arrangements in dimension $d > 1$.

Example

Stencils for the forward, backward and central difference quotients.



Remarks regarding the notations

Remark VII.2.13

Difference quotients *are also called*

- **Newton quotient**
- **Fermat difference quotient**
- **Finite difference approximation of the derivative**

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Remark VII.2.14

In some books, forward/backward/central **finite difference approximation of the derivative** *is shortened by forward/backward/central* **difference**.

This can be confusing.
In case of doubt, make sure you look at the definitions.

Remarks regarding the notations

Remark VII.2.15

In some books,

- *Forward finite differences are denoted* Δ
- *Backward finite differences are denoted* ∇
- *Central finite differences are denoted* δ.

We will **not** *use these notations in this class to avoid confusion with the Laplace and Del operators.*

VII.2.2. Approximating higher-order derivatives

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$$f''(x) = \frac{f'(x + \frac{h}{2}) - f'(x - \frac{h}{2})}{h} + O(h^2)$$

(second-order central)

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One can obtain finite difference approximations to higher order derivatives by applying the finite difference approximation to f'.

For example, consider f differentiable four times,

$$f''(x) = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} + O(h) + O(h^2)$$

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We can do better than $O(h)$

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For example, consider f differentiable four times, the Taylor expansion gives

$$f(x+h) = f(x) + h\,f'(x) + \frac{1}{2}h^2 f''(x) + \frac{1}{6}h^3\, f'''(x) + O(h^4)$$
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$$f(x+h) + f(x-h) = 2f(x) + h^2 f''(x) + O(h^4)$$

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Proposition VII.2.16

The second-order central difference quotient

$$\frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

approximates the second derivative. The method is of order 2.

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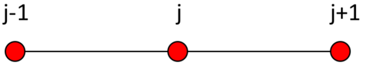
The stencil is:



More generally,

Proposition VII.2.17

The nth-order central difference quotient

$$\frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x+(n/2-i)h\right)$$

approximates the n-th derivative. The method is of order 2.

Remark VII.2.18

For odd n, the function is not evaluated on the grid (as for $n = 1$). The problem may be remedied taking the average on the two nearest points of the grid.

Similarly,

Proposition VII.2.19

The nth-order forward difference quotient

$$\frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x+(n-i)h\right) \quad = \quad \frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x_{j+n-i}\right)$$

approximates the n-th derivative. The method is of order 1.

Proposition VII.2.20

The nth-order backward difference quotient

$$\frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x-ih\right) \quad = \quad \frac{1}{h^n}\sum_{i=0}^{n}(-1)^i\binom{n}{i}f\left(x_{j-i}\right)$$

approximates the n-th derivative. The method is of order 1.

$x = x_j$ needs to be chosen so the x_{j+n-i} / x_{j-i} are defined.

VII.2.3. Approximating the solution of a PDE

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) \quad x \in \,]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

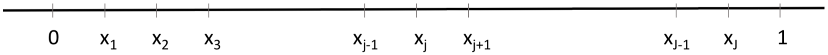
Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

We proved the existence and uniqueness of $u \in C^2([0,1])$.
(Exercise E.V.2 Lab 5)

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0,1[\\ u(0) = u(1) = 0 \end{cases}$$



Consider a uniform mesh on $[0,1]$:
$(x_j)_{j\in\{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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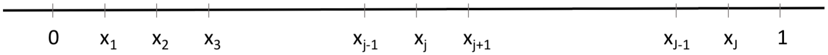


Consider a uniform mesh on $[0,1]$:
$(x_j)_{j \in \{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

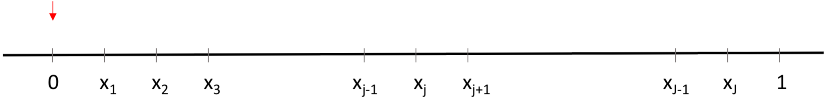
Define

- $c_j = c(x_j)$,
- $f_j = f(x_j)$.

And let u_j be the unknowns that will approximate $u(x_j)$.

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0,1[\\ u(0) = u(1) = 0 \end{cases}$$



Consider a uniform mesh on $[0,1]$:
$(x_j)_{j \in \{0,\ldots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_0$

$$u_0 = u(0) = 0$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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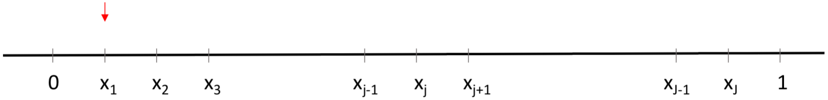

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$u_0 \qquad = \quad 0$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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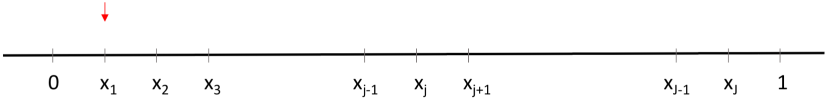

Consider a uniform mesh on $[0,1]$: $\quad u_0 \quad = \quad 0$

$(x_j)_{j\in\{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_1$

$$-u''(x_1) + c(x_1)u(x_1) = f(x_1)$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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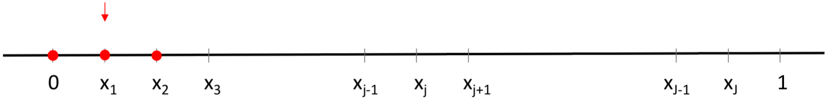


Consider a uniform mesh on $[0,1]$: $(x_j)_{j \in \{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

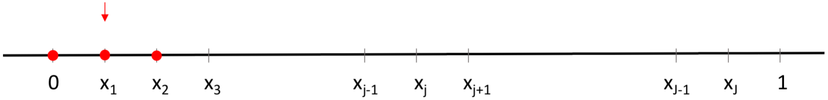
$u_0 \qquad = \quad 0$

For $x = x_1$

$$-\frac{u_2 - 2u_1 + u_0}{h^2} + c_1 u_1 = f_1$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0,1[\\ u(0) = u(1) = 0 \end{cases}$$



Consider a uniform mesh on $[0,1]$: u_0 $=$ 0
$(x_j)_{j \in \{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_1$

$$-\frac{u_2 - 2u_1}{h^2} + c_1 u_1 = f_1$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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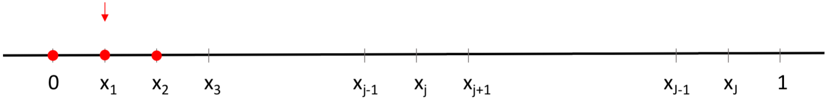

Consider a uniform mesh on $[0,1]$: $\qquad u_0 \qquad = \quad 0$

$(x_j)_{j\in\{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_1$

$$\frac{2u_1 - u_2}{h^2} + c_1 u_1 = f_1$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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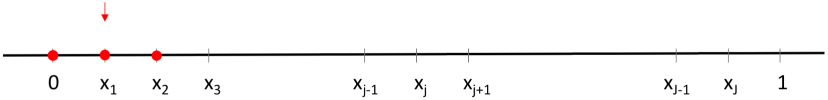

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$(x_j)_{j\in\{0,\ldots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_1$

$$\left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 = f_1$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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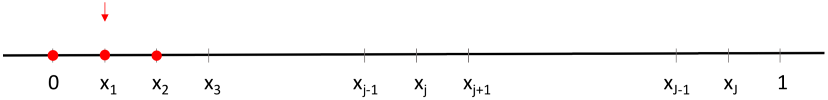

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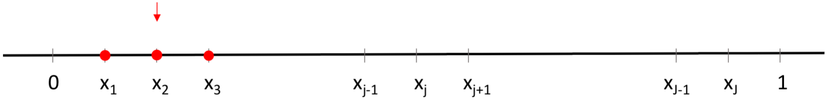

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For $x = x_2$

$$-u''(x_2) + c(x_2)u(x_2) = f(x_2)$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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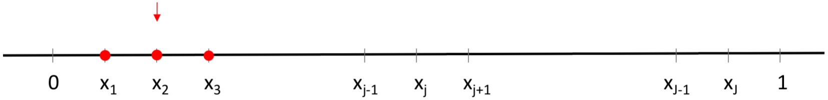

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For $x = x_2$

$$-\frac{u_3 - 2u_2 + u_1}{h^2} + c_2 u_2 = f_2$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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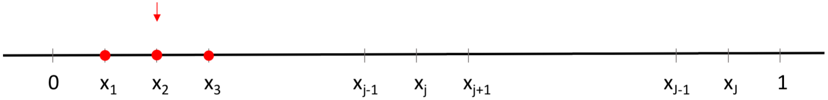

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For $x = x_2$

$$-\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 = f_2$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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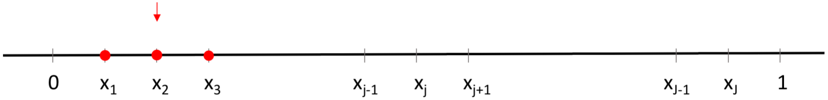

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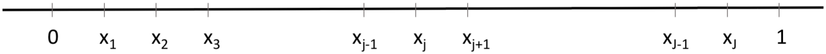

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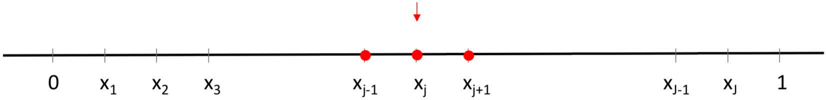

Consider a uniform mesh on $[0,1]$: $(x_j)_{j \in \{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_j$ with $j \in \{2, \dots, J-1\}$

$$-u''(x_j) + c(x_j)u(x_j) = f(x_j)$$

$$\begin{array}{lcc} u_0 & = & 0 \\ \left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 & = & f_1 \\ -\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 & = & f_2 \\ \vdots & \vdots & \vdots \end{array}$$

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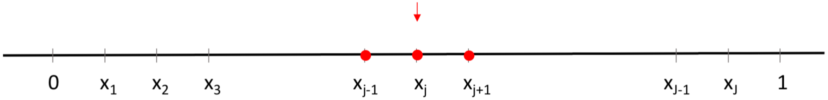

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$$-\frac{u_{j+1} - 2u_j + u_{j-1}}{h^2} + c_j u_j = f_j$$

$$\begin{array}{lcl} u_0 & = & 0 \\ \left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 & = & f_1 \\ -\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 & = & f_2 \\ \vdots & \vdots & \vdots \end{array}$$

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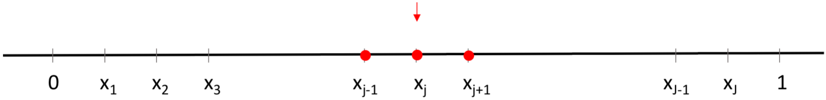

Consider a uniform mesh on $[0,1]$: $(x_j)_{j\in\{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_j$ with $j \in \{2, \dots, J-1\}$

$$-\frac{1}{h^2}u_{j-1} + \left(\frac{2}{h^2} + c_j\right)u_j - \frac{1}{h^2}u_{j+1} = f_j$$

$$\begin{array}{lcc} u_0 & = & 0 \\ \left(\frac{2}{h^2} + c_1\right)u_1 - \frac{1}{h^2}u_2 & = & f_1 \\ -\frac{1}{h^2}u_1 + \left(\frac{2}{h^2} + c_2\right)u_2 - \frac{1}{h^2}u_3 & = & f_2 \\ \vdots & \vdots & \vdots \end{array}$$

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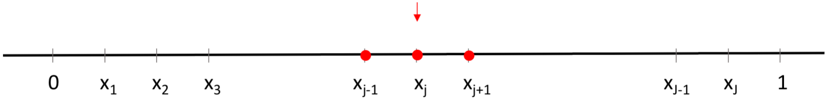

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$$\begin{aligned}
u_0 &= 0 \\
\left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 &= f_1 \\
-\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 &= f_2 \\
\vdots \qquad &\vdots \quad \vdots \\
-\frac{1}{h^2} u_{j-1} + \left(\frac{2}{h^2} + c_j\right) u_j - \frac{1}{h^2} u_{j+1} &= f_j
\end{aligned}$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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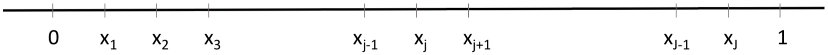

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Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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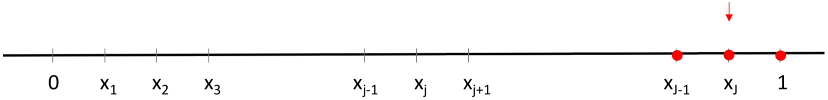

Consider a uniform mesh on $[0,1]$: $(x_j)_{j\in\{0,\dots,J+1\}}$, $x_0 = 0$, $x_{J+1} = 0$.

For $x = x_J$

$$\begin{array}{lcl} u_0 & = & 0 \\ \left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 & = & f_1 \\ -\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 & = & f_2 \\ \vdots & \vdots & \vdots \\ -\frac{1}{h^2} u_{j-1} + \left(\frac{2}{h^2} + c_j\right) u_j - \frac{1}{h^2} u_{j+1} & = & f_j \\ \vdots & \vdots & \vdots \end{array}$$

Let $f \in C^0([0,1])$ and $c \in C^0([0,1], \mathbb{R}^+)$. Consider

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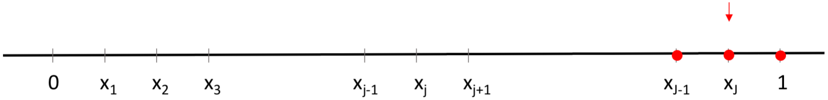

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For $x = x_J$

$$-u''(x_J) + c(x_J)u(x_J) = f(x_J)$$

$$\begin{array}{lcl} u_0 & = & 0 \\ \left(\frac{2}{h^2} + c_1\right) u_1 - \frac{1}{h^2} u_2 & = & f_1 \\ -\frac{1}{h^2} u_1 + \left(\frac{2}{h^2} + c_2\right) u_2 - \frac{1}{h^2} u_3 & = & f_2 \\ \vdots & \vdots & \vdots \\ -\frac{1}{h^2} u_{j-1} + \left(\frac{2}{h^2} + c_j\right) u_j - \frac{1}{h^2} u_{j+1} & = & f_j \\ \vdots & \vdots & \vdots \end{array}$$

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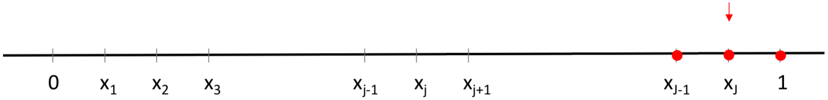

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For $x = x_J$

$$-\frac{u_{J+1} - 2u_J + u_{J-1}}{h^2} + c_J u_J = f_J$$

$$\begin{aligned}
&u_0 &&= 0 \\
&\left(\tfrac{2}{h^2} + c_1\right) u_1 - \tfrac{1}{h^2} u_2 &&= f_1 \\
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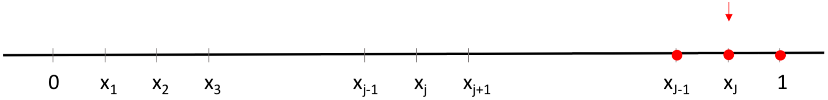

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$$-\frac{-2u_J + u_{J-1}}{h^2} + c_J u_J = f_J$$

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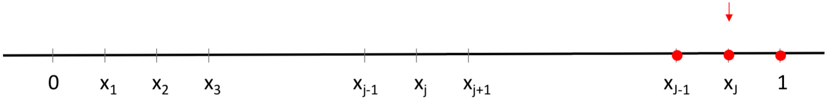

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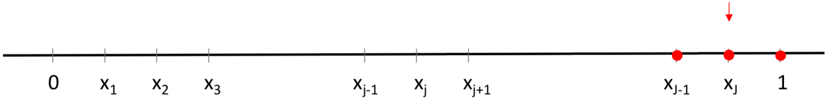

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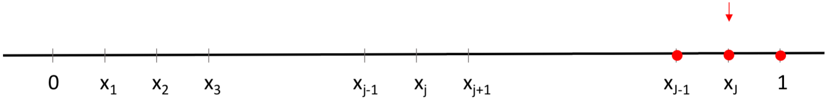

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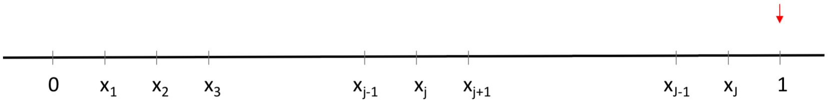

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$(A_h + C_h)U_h = F_h$ and $u_0 = u_{J+1} = 0$

$$\Updownarrow$$

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Theorem VII.2.21

For all $J \geq 1$, the linear system

$$(A_h + C_h)U_h = F_h$$

has a unique solution U_h.

The proof was given in the lab session an hour ago (Exercice E.V.2)

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Definition VII.2.22

The matrix A_h is the **FDM-Matrix of the Laplacian** *in dimension 1.*

Maximum Principle

Definition VII.2.23

We say that a vector v (resp. a matrix M) is non-negative if all of its components are non-negative. We note $v \geq 0$ (resp. $M \geq 0$).

Definition VII.2.24

A matrix $M \in \mathcal{M}_q(\mathbb{R})$ is **monotone** *if*

$$\forall v \in \mathbb{R}^q, \; Mv \geq 0 \Rightarrow v \geq 0$$

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A monotone matrix is non-singular.

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A monotone matrix is non-singular.

Lemma VII.2.26

A matrix M is monotone iff $M^{-1} \geq 0$.

Maximum Principle

Lemma VII.2.27

$A_h + C_h$ is monotone.

Proof: Lab session to come.

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If $F_h \geq 0$

Maximum Principle

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Proof: Lab session to come.

If $F_h \geq 0$ then $(A_h + C_h)U_h \geq 0$

Maximum Principle

Lemma VII.2.27

$A_h + C_h$ is monotone.

Proof: Lab session to come.

If $F_h \geq 0$ then $(A_h + C_h)U_h \geq 0$ then $U_h \geq 0$.

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If $F_h \geq 0$ then $(A_h + C_h)U_h \geq 0$ then $U_h \geq 0$.

Theorem VII.2.28 (Maximum Principle)

If $f \geq 0$ then U_h has only non-negative components.

Maximum Principle

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Theorem VII.2.28 (Maximum Principle)

If $f \geq 0$ then U_h has only non-negative components.

Compare with Theorem V.2.6.

VII.2.4. Convergence

Let's get rid of a possible misconception

U_h is **not** a discretization of the solution.

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We have discretized the problem, not the solution.

Now, we need to compare u_j and $u(x_j)$
and verify if the former approximates the latter.

Projection onto the mesh

Define the projection Π_h of a function on the mesh by its pointwise evaluation on the nodes (x_j):

$$\begin{array}{rcl} \Pi_h \ : \ C([0,1]) & to & \mathbb{R}^J \\ u & \mapsto & [u(x_j)]_{j\in\{1,\ldots,J\}} \end{array}$$

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We have $F_h = \Pi_h(f)$

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We have $F_h = \Pi_h(f)$ and $C_h = \Pi_h(c)$.
We don't have $U_h = \Pi_h(u)$ in general, let us measure the gap:

$$E_h = U_h - \Pi_h u = \begin{pmatrix} u_1 - u(x_1) \\ \vdots \\ u_J - u(x_J) \end{pmatrix}$$

Convergence

Definition VII.2.29

The numerical method is **convergent** *if*

$$\lim_{h \to 0} \|E_h\|_h = 0$$

where $\| \cdot \|_h$ *is a norm on* $\mathbb{R}^J$.

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where $\|\cdot\|_h$ *is a norm on* $\mathbb{R}^J$.

Furthermore, if $\|E_h\|_h = O(h^p)$ *the method is said to be of* **order** p

Consistency

Definition VII.2.30

Define

$$\mathcal{E}_h = A_h \Pi_h u - F_h \in \mathbb{R}^J$$

We say the Finite Difference Method is **consistent** *with the PDE if*

$$\lim_{h\to 0} \max_{j\in\{1,\ldots,J\}} |(\mathcal{E}_h)_j| = \lim_{h\to 0} \|\mathcal{E}_h\|_\infty = 0$$

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What follows makes no mathematical sense but tries to describe the meaning of this definition:

$$\textit{consistency} \iff \lim_{\text{mesh size}\to 0} (PDE - FDM) = 0$$

Consistency

Example

In our earlier problem

$$\begin{cases} -u''(x) + c(x)u(x) = f(x) & x \in]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

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$$\begin{cases} -u''(x)+c(x)u(x) = f(x) & x \in \,]0,1[\\ u(0) = u(1) = 0 \end{cases}$$

take $f \in C^2(0,1)$ *and* $c = 0$ *(Poisson's equation in dimension 1).*

Consistency

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take $f \in C^2(0,1)$ *and* $c = 0$ *(Poisson's equation in dimension 1). Then* $u \in C^4(0,1)$*.*

Consistency

Example

Consider the Poission equation in dimension 1 with a right hand side $f \in C^2(0,1)$. The solution $u \in C^4(0,1)$.

Consistency

Example

Consider the Poisson equation in dimension 1 with a right hand side $f \in C^2(0,1)$. The solution $u \in C^4(0,1)$.

The second-order central difference quotient is a finite difference approximation of the second derivative of order 2:

$$\frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - u''(x_j) = O(h^2)$$

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Using the Taylor-Lagrange inequality, we could be more precise:

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Using the Taylor-Lagrange inequality, we could be more precise:

$$\left| \frac{u(x_{j+1}) - 2u(x_j) + u(x_{j-1})}{h^2} - u''(x_j) \right| \leq \|u^{(4)}\|_\infty \frac{h^2}{12}$$

Consistency

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Consider the Poisson equation in dimension 1 with a right hand side $f \in C^2(0,1)$. The solution $u \in C^4(0,1)$.

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The FDM is consistent with the PDE in the order 2.
Note: The remainder vanishes if $u \in \mathbb{R}_3[X]$.

Stability

Definition VII.2.31

Consider a numerical method $A_h U_h = F_h$ where

- *U_h is the the unknown,*
- *F_h is the input data and,*
- *A_h is a non-singular matrix.*

Consider $\|\cdot\|$ the matrix norm induced by the norm $\|\cdot\|$ on $\mathbb{R}^J$.

The numerical method is **stable** *if there exists a constant C independent of J such that*

$$\|A_h^{-1}\| \leq C.$$

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$$|||A_h^{-1}||| \leq C.$$

Stability means small errors in the input data F_h don't get out of hand.

Stability

The convergence in $\mathbb{R}^d$ does not depend on the norm we choose, since $\mathbb{R}^d$ has a finite dimension.

However, since C must **not** depend on h (i.e. must not depend on $d = J$), we can have stability for a norm and not for another norm. Therefore, it is necessary to say which norm $\|\cdot\|$ we take for stability (and later for convergence).

Convergence

$$E_h = U_h - \Pi_h u$$

Convergence

$$A_h E_h = A_h U_h - A_h \Pi_h u$$

Convergence

$$A_h E_h = F_h - A_h \Pi_h u$$

Convergence

$$A_h E_h = -\mathcal{E}_h$$

Convergence

$$E_h = -A_h^{-1}\mathcal{E}_h$$

Convergence

$$\|E_h\| = \| - A_h^{-1} \mathcal{E}_h\|$$

Convergence

$$\|E_h\| = \|A_h^{-1}\mathcal{E}_h\|$$

Convergence

$$\|E_h\| \leq \|A_h^{-1}\| \, \|\mathcal{E}_h\|$$

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(Stability) $\quad \|A_h^{-1}\| \leq C$

Convergence

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(Stability) $\|A_h^{-1}\| \leq C$

(Consistency) $\lim_{h\to 0} \|\mathcal{E}_h\| = 0$

Convergence

$$\|E_h\| \leq \|A_h^{-1}\| \, \|\mathcal{E}_h\|$$

$$\left.\begin{array}{ll} \text{(Stability)} & \|A_h^{-1}\| \leq C \\ \text{(Consistency)} & \lim_{h\to 0} \|\mathcal{E}_h\| = 0 \end{array}\right\} \Rightarrow$$

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Theorem VII.2.32

A finite difference method that is both **consistent** *and* **stable** *is* **convergent**.

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Theorem VII.2.33

Let A_h be the matrix associated to the Poisson's equation in dimension 1.

$$\forall J \geq 1, \qquad \|A_h^{-1}\|_\infty \leq \frac{1}{8}.$$

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Let $[A_h^{-1}]_{ij}$ be the component of A_h^{-1} on line i and column j.

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Proof:

Let $[A_h^{-1}]_{ij}$ be the component of A_h^{-1} on line i and column j.
We have established: (last lecture)

$$\|A_h\|_\infty = \max_{1 \leq i \leq J} \sum_{j=1}^{J} |[A_h^{-1}]_{ij}|$$

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Proof:

Let $[A_h^{-1}]_{ij}$ be the component of A_h^{-1} on line i and column j.
We know that A_h is monotone so $A_h^{-1} \geq 0$ therefore

$$\|A_h\|_\infty = \max_{1\leq i\leq J} \sum_{j=1}^{J} [A_h^{-1}]_{ij}$$

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Proof:

Let i be an integer in $[1, J]$
and $g \in \mathbb{R}^J$ be a vector for which each component $g_j = 1$.

$$\|A_h\|_\infty = \max_{1 \leq i \leq J} \sum_{j=1}^{J} [A_h^{-1}]_{ij} g_j$$

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Consider $-u'' = 1$ with the initial condition $u(0) = u(1) = 0$.

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We can look for a solution in $\mathbb{R}_2[x]$

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Let $u(x) = \frac{1}{2}\left(-x^2 + x\right)$. It is a solution.

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Since $u \in \mathbb{R}_2[x]$, we have $u''(x_i) = \frac{u(x_{i+1}) - 2u(x_i) + u(x_{i-1})}{h^2}$

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It follows that $w_i = u(x_i)$.

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QED

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Corollary VII.2.34

The finite difference method for the Poisson's problem in dimension 1 converges in $\|\cdot\|_\infty$. The order is 2.

Programming the FDM for $f = 1$

```
import numpy as np
from scipy import sparse
from scipy.sparse.linalg import dsolve
import matplotlib
import matplotlib.pyplot as plt

J = 10
h = 1.0/(J+1)

# Building Ah
diagonal = np.ones(J)*2.0
side_diagonal = np.ones(J-1)*(-1.0)
h2_Ah = sparse.diags([side_diagonal,diagonal,side_diagonal],
                     [-1,0,1], format="csr")
Ah = h2_Ah*(1/(h**2))

print("h2_Ah =\n",h2_Ah.A)
print("Ah =\n",Ah.A)
```

```
h2_Ah =
 [[ 2. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  2. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  2. -1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  2. -1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  2. -1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0. -1.  2. -1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0. -1.  2. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0. -1.  2. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0. -1.  2. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0. -1.  2.]]
Ah =
 [[ 242. -121.    0.    0.    0.    0.    0.    0.    0.    0.]
 [-121.  242. -121.    0.    0.    0.    0.    0.    0.    0.]
 [   0. -121.  242. -121.    0.    0.    0.    0.    0.    0.]
 [   0.    0. -121.  242. -121.    0.    0.    0.    0.    0.]
 [   0.    0.    0. -121.  242. -121.    0.    0.    0.    0.]
 [   0.    0.    0.    0. -121.  242. -121.    0.    0.    0.]
 [   0.    0.    0.    0.    0. -121.  242. -121.    0.    0.]
 [   0.    0.    0.    0.    0.    0. -121.  242. -121.    0.]
 [   0.    0.    0.    0.    0.    0.    0. -121.  242. -121.]
 [   0.    0.    0.    0.    0.    0.    0.    0. -121.  242.]]
```

Programming the FDM for $f = 1$

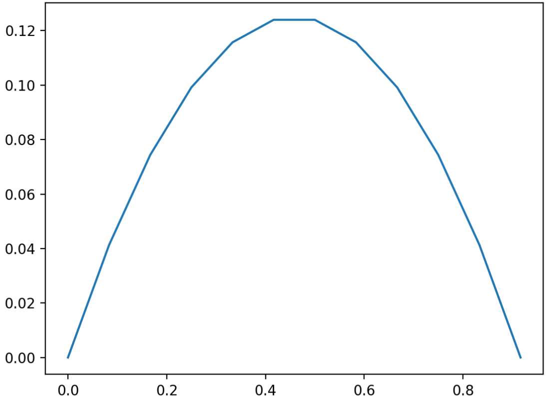
```
# Building b
b = np.ones(J)

# Solving for u
u = dsolve.spsolve(Ah, b)

# Plotting the solution
x = np.linspace(0.0, 1.0, num=J+2)
u = np.concatenate(([0], u, [0]))
fig = plt.figure()
ax = fig.gca()
ax.plot(x, u)
plt.show()
```

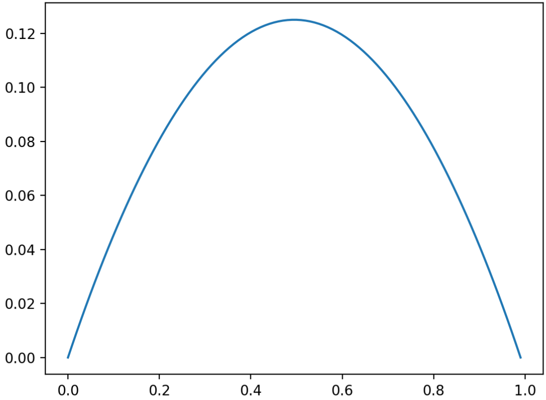
Approximation of the solution

$$J = 10$$



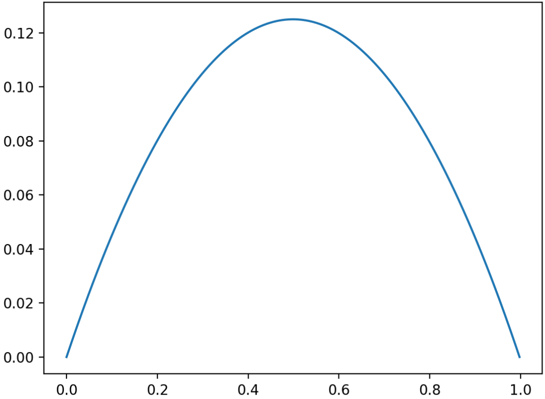
Approximation of the solution

$$J = 100$$



Approximation of the solution

$$J = 500$$

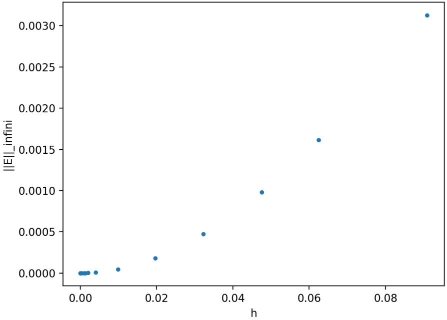


Approximation of the solution

Program available for download at

https://cagnol.link/fdmp1

Estimating the error



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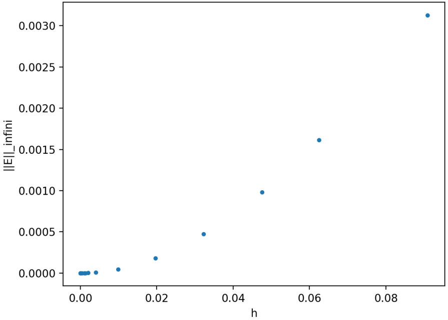

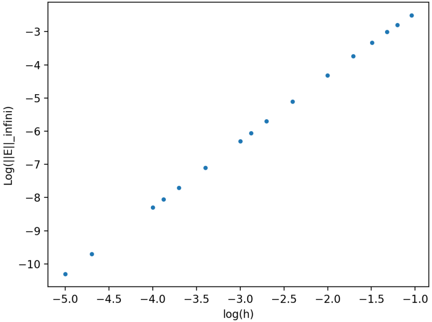

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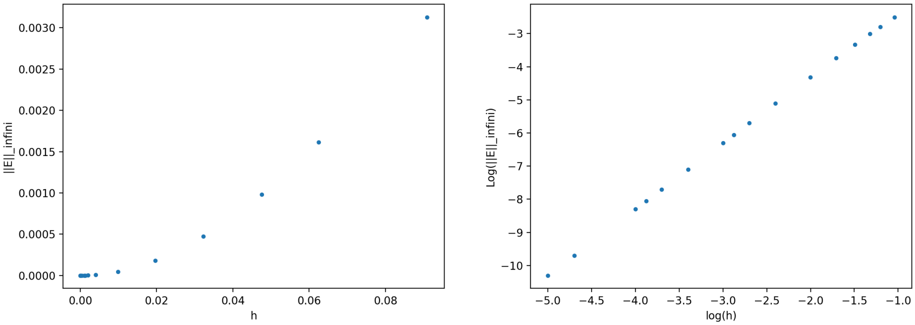
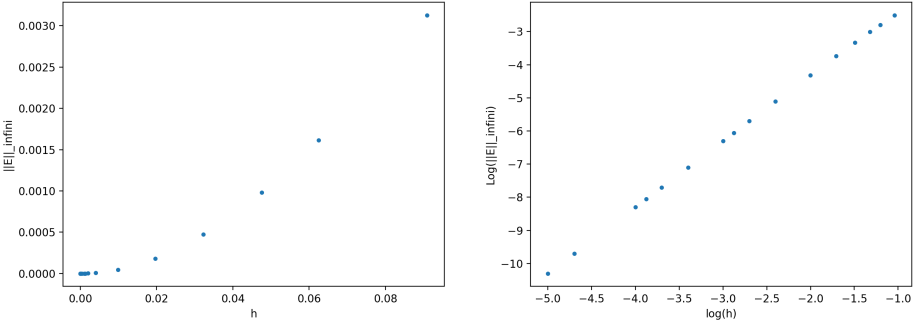


Slope of the line on the log-scale graph: 1.97

Estimating the error



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Program available for download at https://cagnol.link/fdmp1cr

VII.3. FDM in dimension $d > 1$

VII.3.1. Meshing

Structured vs. Unstructured meshes

For **Finite Element Method** we often mesh using simplices (triangles in 2D).

The mesh does not follow a pattern structure.

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The mesh follows a pattern structure: we tessellate Ω.

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The mesh follows a pattern structure: we tessellate Ω.

Definition VII.3.1

A **structured mesh** *is a mesh that can be produced by replicating an elementary cell.*
A mesh that is not structured is called **unstructured**.

With a structured mesh, every vertex can be easily numbered by a d-tuple: (i, j) if $d = 2$ and (i, j, k) if $d = 3$.

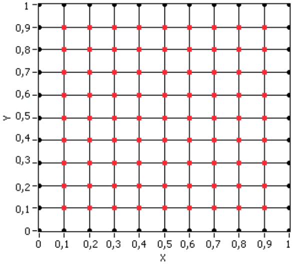
Example of a structured mesh

Consider a square $\Omega \subset \mathbb{R}^2$:

We tessellate Ω with $(J+1)^2$ elementary squares.

$J+2$ points subdivize $[0,1]$ on the x-axis.
$J+2$ points subdivize $[0,1]$ on the y-axis.

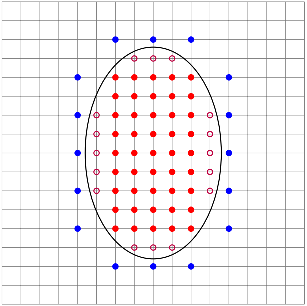
There are J^2 points (in red) that are not on $\partial\Omega$. Points $P_n = (x_j, y_j)$ are numbered from 1 to J^2 with $n = J(i-1)+j$



$J = 9$

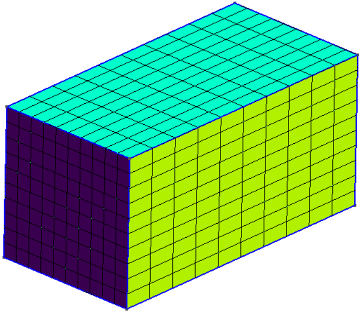
Example of a structured mesh

For a more complicated $\Omega \subset \mathbb{R}^2$



Example of a structured mesh

An example for $\Omega \subset \mathbb{R}^3$



Credit: CFDyna

VII.3.2. Finite Difference Approximation of the Partial Derivatives

First order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions.

First order Partial Derivatives for $d = 2$

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$$
\begin{aligned}
\frac{\partial f}{\partial x}(x,y) &= \frac{f(x+h,y)-f(x,y)}{h} + O(h) && \text{(forward)}\\
\frac{\partial f}{\partial x}(x,y) &= \frac{f(x,y)-f(x-h,y)}{h} + O(h) && \text{(backward)}\\
\frac{\partial f}{\partial x}(x,y) &= \frac{f(x+\frac{h}{2},y)-f(x-\frac{h}{2},y)}{h} + O(h^2) && \text{(central)}
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial f}{\partial y}(x,y) &= \frac{f(x,y+k)-f(x,y)}{k} + O(k) && \text{(forward)}\\
\frac{\partial f}{\partial y}(x,y) &= \frac{f(x,y)-f(x,y-k)}{k} + O(k) && \text{(backward)}\\
\frac{\partial f}{\partial y}(x,y) &= \frac{f(x,y+\frac{k}{2})-f(x,y-\frac{k}{2})}{k} + O(k^2) && \text{(central))}
\end{aligned}
$$

First order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions.
For $x = x_i$ and $y = y_j$

$$\frac{\partial f}{\partial x}(x_j, y_i) = \frac{f(x_{j+1},y_i) - f(x_j,y_i)}{x_{j+1} - x_j} + O(h) \qquad \text{(forward)}$$

$$\frac{\partial f}{\partial x}(x_j, y_i) = \frac{f(x_j,y_i) - f(x_{j-1},y_i)}{x_i - x_{j-1}} + O(h) \qquad \text{(backward)}$$

$$\frac{\partial f}{\partial x}(x_j, y_i) = \frac{f(x_{j+1},y_i) - f(x_{j-1},y_i)}{x_{j+1} - x_{j-1}} + O(h^2) \qquad \text{(central)}$$

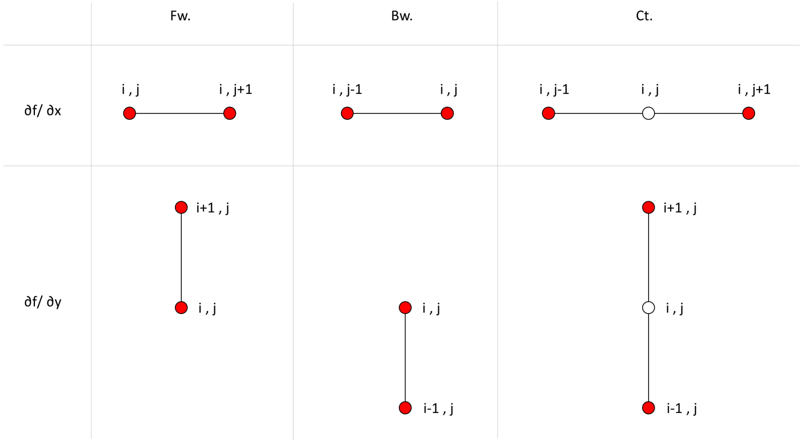
$$\frac{\partial f}{\partial y}(x_j, y_i) = \frac{f(x_j,y_{j+1}) - f(x_j,y_i)}{y_{i+1} - y_i} + O(k) \qquad \text{(forward)}$$

$$\frac{\partial f}{\partial y}(x_j, y_i) = \frac{f(x_j,y_i) - f(x_j,y_{j-1})}{y_i - y_{i-1}} + O(k) \qquad \text{(backward)}$$

$$\frac{\partial f}{\partial y}(x_j, y_i) = \frac{f(x_j,y_{i+1}) - f(x_j,y_{i-1})}{y_{i+1} - y_{i-1}} + O(k^2) \qquad \text{(central)}$$

First order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions.
For $x = x_i$ and $y = y_j$, the stencils are:



Second order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions. The central difference quotient for the second derivative are given by:

Second order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions. The central difference quotient for the second derivative are given by:

$$
\begin{aligned}
\frac{\partial^2 f}{\partial x^2}(x,y) &= \frac{f(x+h,y)-2f(x,y)+f(x-h,y)}{h^2} + O(h^2)\\
\frac{\partial^2 f}{\partial y^2}(x,y) &= \frac{f(x,y+k)-2f(x,y)+f(x,y-k)}{k^2} + O(k^2)\\
\frac{\partial^2 f}{\partial x \partial y}(x,y) &= \frac{f(x+\frac{h}{2},y+\frac{k}{2})+f(x-\frac{h}{2},y-\frac{k}{2})}{hk}\\
&\quad - \frac{f(x+\frac{h}{2},y-\frac{k}{2})+f(x-\frac{h}{2},y+\frac{k}{2})}{hk} + O(h^2,k^2).
\end{aligned}
$$

You can adapt these formulas to forward and backward difference quotients.

Second order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions. For $x = x_i$ and $y = y_j$. The central difference quotient for the second derivative are given by:

$$\frac{\partial^2 f}{\partial x^2}(x_j, y_i) = \frac{f(x_{j+1},y_i) - 2f(x_j,y_i) + f(x_{j-1},y_i)}{(x_{j+1} - x_{j-1})^2} + O(h^2)$$

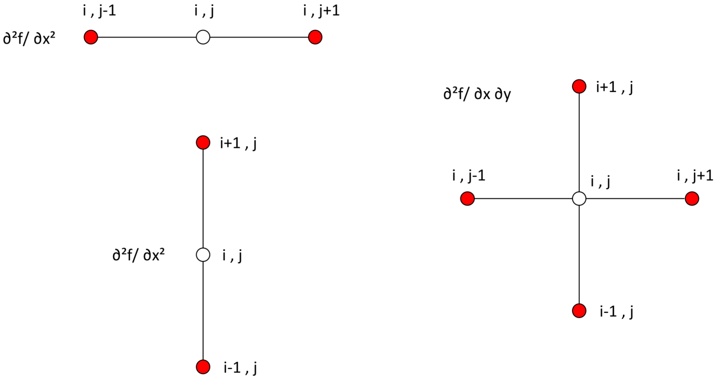
$$\frac{\partial^2 f}{\partial y^2}(x_j, y_i) = \frac{f(x_j,y_{i+1}) - 2f(x_j,y_i) + f(x_j,y_{i-1})}{(y_{i+1} - y_{i-1})^2} + O(k^2)$$

$$\begin{aligned}\frac{\partial^2 f}{\partial x \partial y}(x_j, y_i) &= \frac{f(x_{j+1},y_{i+1}) + f(x_{j-1},y_{i-1})}{(x_{j+1} - x_{j-1})(y_{i+1} - y_{i-1})} \\ &\quad - \frac{f(x_{j+1},y_{i-1}) + f(x_{j-1},y_{i+1})}{(x_{j+1} - x_{j-1})(y_{i+1} - y_{i-1})} + O(h^2, k^2).\end{aligned}$$

You can adapt these formulas to forward and backward difference quotients.

Second order Partial Derivatives for $d = 2$

What we did in dimension one generalizes to higher dimensions.
For $x = x_i$ and $y = y_j$. The stencils are:



Laplace Operator for $d = 2$

Since $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ we have

$$\begin{aligned}\Delta f(x,y) = & \frac{f(x+h,y) - 2f(x,y) + f(x-h,y)}{h^2} \\ & + \frac{f(x,y+k) - 2f(x,y) + f(x,y-k)}{k^2} + O(h^2, k^2)\end{aligned}$$

Laplace Operator for $d = 2$

Since $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ we have

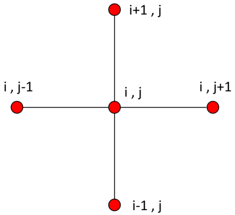
$$\Delta f(x_j, y_i) = \frac{f(x_{j+1}, y_i) - 2f(x_j, y_i) + f(x_{j-1}, y_i)}{(x_{j+1} - x_{j-1})^2} + \frac{f(x_j, y_{i+1}) - 2f(x_j, y_i) + f(x_j, y_{i-1})}{(y_{i+1} - y_{i-1})^2} + O(h^2, k^2)$$

Laplace Operator for $d = 2$

Since $\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2}$ we have

$$\Delta f(x_j, y_i) = \frac{f(x_{j+1}, y_i) - 2f(x_j, y_i) + f(x_{j-1}, y_i)}{(x_{j+1} - x_{j-1})^2} + \frac{f(x_j, y_{i+1}) - 2f(x_j, y_i) + f(x_j, y_{i-1})}{(y_{i+1} - y_{i-1})^2} + O(h^2, k^2)$$

The (five-point) stencil is



VII.3.3. Approximating the PDE

Let us discretize Δu

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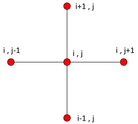
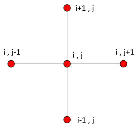


$$\frac{1}{h^2}((u(x_{j+1}, y_i) - 2u(x_j, y_i) + u(x_{j-1}, y_i)) + (u(x_j, y_{i+1}) - 2u(x_j, y_i) + u(x_j, y_{i-1})))$$

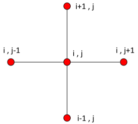
Let us discretize Δu



$$\frac{1}{h^2}((u(x_{j+1}, y_i) - 2u(x_j, y_i) + u(x_{j-1}, y_i)) + (u(x_j, y_{i+1}) - 2u(x_j, y_i) + u(x_j, y_{i-1})))$$

$$\frac{1}{h^2}(u(x_j, y_{i-1}) + u(x_{j-1}, y_i) - 4u(x_j, y_i) + u(x_{j+1}, y_i) + (u(x_j, y_{i+1})))$$

Let us discretize Δu



$$\frac{1}{h^2}(u(x_j, y_{i-1}) + u(x_{j-1}, y_i) - 4u(x_j, y_i) + u(x_{j+1}, y_i) + (u(x_j, y_{i+1})))$$

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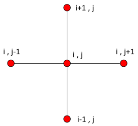
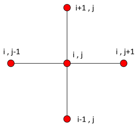


$$\frac{1}{h^2}(u(x_j, y_{i-1}) + u(x_{j-1}, y_i) - 4u(x_j, y_i) + u(x_{j+1}, y_i) + (u(x_j, y_{i+1})))$$

$$\frac{1}{h^2}(u_{j,i-1} + u_{j-1,i} - 4u_{j,i} + u_{j+1,i} + u_{j,i+1})$$

Let us discretize Δu



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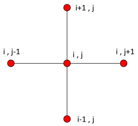


$$\frac{1}{h^2}(u_{j,i-1} + u_{j-1,i} - 4u_{j,i} + u_{j+1,i} + u_{j,i+1})$$

Going from dual-index to single-index with $(i,j) \mapsto n = J(i-1) + j$

$$\frac{1}{h^2}(u_{j+J(i-2)} + u_{j-1+J(i-1)} - 4u_{j+J(i-1)} + u_{j+1+J(i-1)} + u_{j+Ji})$$

where u_n is replaced by zero for all out-of-range n.

The discretization of $-\Delta u = f$ is given by:

$$\frac{1}{h^2}(-u_{j+J(i-2)} - u_{j-1+J(i-1)} + 4u_{j+J(i-1)} - u_{j+1+J(i-1)} - u_{j+Ji}) = f_{j-1+J(i-1)}$$

The discretization of $-\Delta u = f$ is given by:

$$\frac{1}{h^2}\left(-u_{j+J(i-2)} - u_{j-1+J(i-1)} + 4u_{j+J(i-1)} - u_{j+1+J(i-1)} - u_{j+Ji}\right) = f_{j-1+J(i-1)}$$

$$A_h = \frac{1}{h^2}\begin{pmatrix} \mathbf{T} & -\mathbf{I} & 0 & \dots & 0 \\ -\mathbf{I} & \mathbf{T} & -\mathbf{I} & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -\mathbf{I} & \mathbf{T} & -\mathbf{I} \\ 0 & \dots & 0 & -\mathbf{I} & \mathbf{T} \end{pmatrix} \quad \text{with}$$

$$\mathbf{T} = \begin{pmatrix} 4 & -1 & 0 & \dots & 0 \\ -1 & 4 & -1 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & -1 & 4 & -1 \\ 0 & \dots & 0 & -1 & 4 \end{pmatrix} \qquad \mathbf{I} = \begin{pmatrix} 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}$$

```
import numpy as np
from scipy import sparse

J = 4

diagonal_T = np.ones(J**2)*4.0

side_diagonal_T = np.ones(J**2-1)*(-1.0)
side_diagonal_T[np.arange(1,J**2)%J==0] = 0

diagonal_I = np.ones(J**2-J)

h2_Ah = sparse.diags([-diagonal_I,
                      side_diagonal_T,
                      diagonal_T,
                      side_diagonal_T,
                      -diagonal_I],
                     [-J,-1,0, 1,J], format="csr")

print(h2_Ah.A)
```

```
[[ 4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  4.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  0.  0.  0.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  0.  0. -1.  4.  0.  0.  0. -1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0. -1.  0.  0.  0.  4. -1.  0.  0. -1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4.  0.  0.  0. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0.  0.  4. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4.]]
```

```
[[ 4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  4.  0.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.  0.]
 [-1.  0.  0.  0.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.  0.]
 [ 0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.  0.]
 [ 0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.  0.  0.  0.]
 [ 0.  0.  0. -1.  0.  0. -1.  4.  0.  0.  0. -1.  0.  0.  0.  0.]
 [ 0.  0.  0.  0. -1.  0.  0.  0.  4. -1.  0.  0. -1.  0.  0.  0.]
 [ 0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.  0. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4.  0.  0.  0. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0.  0.  4. -1.  0.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.  0.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4. -1.]
 [ 0.  0.  0.  0.  0.  0.  0.  0.  0.  0.  0. -1.  0.  0. -1.  4.]]
```

$$\begin{bmatrix} T & -I & & \\ -I & T & -I & \\ & -I & T & -I \\ & & -I & T \end{bmatrix}$$

Approximating the solution to the PDE for $f = 1$

```
Ah = h2_Ah*(1/(h**2))
```

Approximating the solution to the PDE for $f = 1$

```
Ah = h2_Ah*(1/(h**2))
```

```
b = np.ones(J**2)
```

Approximating the solution to the PDE for $f = 1$

```
Ah = h2_Ah*(1/(h**2))

b = np.ones(J**2)

# We need to add this at the beginning of the file
# from scipy.sparse.linalg import dsolve:

u = dsolve.spsolve(Ah, b)
```

Approximating the solution to the PDE for $f = 1$

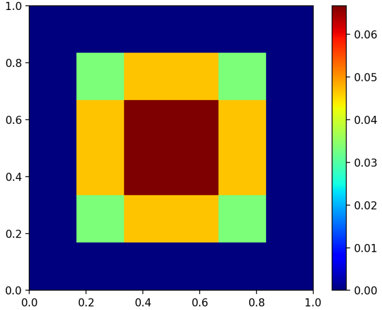
```
z = np.empty([J+2, J+2])

for i in range(0,J+2):
  for j in range(0,J+2):

    n = j+J*(i-1)   # Going from two indices to one

    if i==0 or i==J+1:
        z[i,j] = 0.0
    if j==0 or j==J+1:
        z[i,j] = 0.0
    if i>0 and j>0 and i<J+1 and j<J+1:
        z[i, j] = u[n-1]
        # elements of u are numbered starting at 0
```

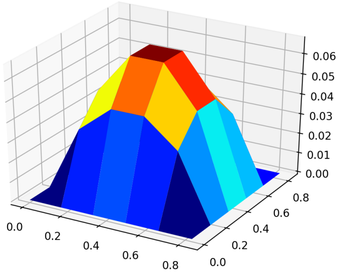
Approximating the solution to the PDE for $f = 1$

```
# We need to add this at the beginning of the file
# import matplotlib.cm as cm
# from matplotlib import pyplot as plt
# from mpl_toolkits.mplot3d import Axes3D

# First way to plot the data: showing values using colors
plt.imshow(z, cmap=cm.jet, extent=[0.0, 1.0, 0.0, 1.0])
plt.colorbar()
plt.show()

# Second way to plot the data: we create a 3D-plot
fig = plt.figure()
ax = fig.gca(projection='3d')
x = np.linspace(0.0, 1.0, num=J+2)
y = np.linspace(0.0, 1.0, num=J+2)
X, Y = np.meshgrid(x, y)
surf = ax.plot_surface(X, Y, z, linewidth=0, cmap=cm.jet)
plt.show()
```

Approximating the solution to the PDE for $f = 1$

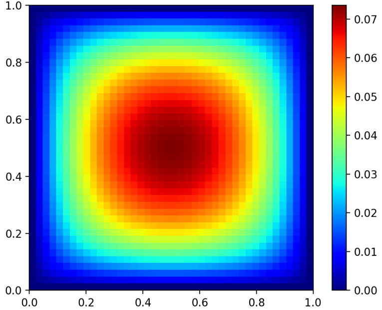
$$J = 4$$

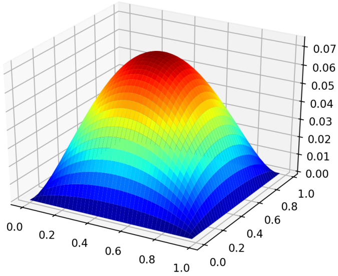




Approximating the solution to the PDE for $f = 1$

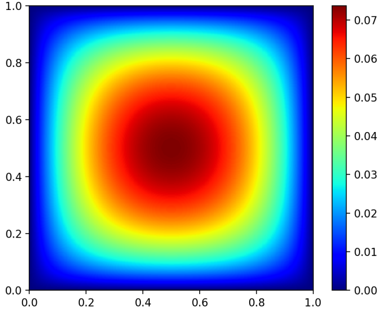
$$J = 40$$

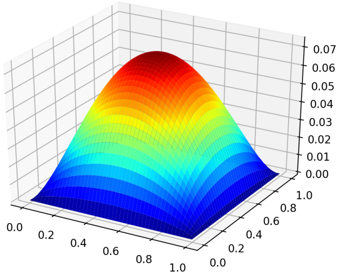




Approximating the solution to the PDE for $f = 1$

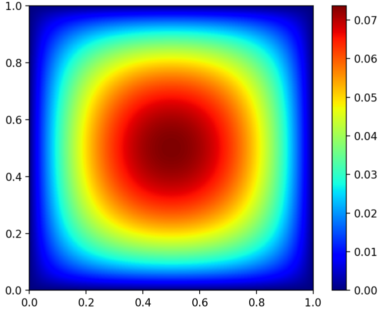
$$J = 400$$

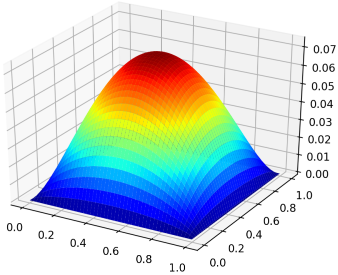




Approximating the solution to the PDE for $f = 1$

$$J = 800$$





Approximating the solution to the PDE for $f = 1$

Program available for download at

https://cagnol.link/fdmp2

VII.4. FEM-FDM comparison

FDM	FEM
Consistency error	Céa Lemma
Consistency	interpolation
L^∞-stability	L^2-coercivity