Partial Differential Equations

Chapter II - Introduction to PDE and Modeling

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The Engineering Program of CentraleSupélec

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II.1. Definitions

II.1.1. Definition of a PDE

Partial Differential Equations

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The **order** *or the PDE is determined by the term with the highest derivatives.*

Definition II.1.3

A PDE is **linear** *if it is linear in the unknown function and its derivatives.*

II.1.2. The different types of PDEs

Classification of PDEs

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where A, B, C, D, E, F, and G are functions defined on $\mathbb{R}^2$.

The PDE is said to be

- **elliptic** if $B^2 - 4AC < 0$
- **parabolic** if $B^2 - 4AC = 0$ and $(A, B, C) \neq (0, 0, 0)$
- **hyperbolic** if $B^2 - 4AC > 0$ or $(A, B, C) = (0, 0, 0)$

Classification of PDEs

Example

$\Delta u = f$

Classification of PDEs

Example

$$\frac{\partial^2 u}{\partial x_1^2} + \frac{\partial^2 u}{\partial x_2^2} = f$$

Classification of PDEs

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$\Delta u = f$ *is an elliptic PDE.*

The classification elliptic/parabolic/hyperbolic can be extended to higher order linear PDEs and non-linear PDEs.

$\Delta u = f$ in dimension 3 is an elliptic PDE.

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is an hyperbolic PDE. It is an evolution equation.

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$\partial_{tt} u - c^2 \partial_{xx} u = f$	waves (1D)	No	H

II.1.3. Wellposeness

Wellposedness

PDE: infinite number of solutions
Problem: PDE + conditions

- **boundary conditions:** value on $\partial\Omega$ (e.g. Dirichlet)
- **initial condition (Cauchy problem) for evolution equations:** value at $t = 0$.

Well-Posedness

Definition II.1.4

Let E and F be two spaces. Let $f \in F$ be a data and $\mathcal{A} : E \to F$. We are looking for solutions $u \in E$ to $\mathcal{A}(u) = f$ (the PDE).

The PDE is **well-posed in the sense of Hadamard** *if:*

- *a solution u exists,*
- *the solution u is unique,*
- *u changes "continuously" with respect to$f$.*

A PDE that is not well-posed is said to be **ill-posed**

II.1.4. Boundary Conditions

Boundary

Let $d \in \mathbb{N}^*$ and $\Omega \subset \mathbb{R}^d$.

Definition II.1.5 (Boundary)

We defined (CIP) the **boundary** *of* Ω *by* $\overline{\Omega} \setminus \mathring{\Omega}$.
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Example

$d = 1$.
The boundary of $]a, b[$ *is* $\{a, b\}$.
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The boundary of $]-\infty, b]$ *is* $\{b\}$.
The boundary of $\mathbb{R}$ *is* $\emptyset$.

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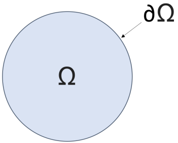
Example

$d = 2$.
The boundary of the disk is the circle.



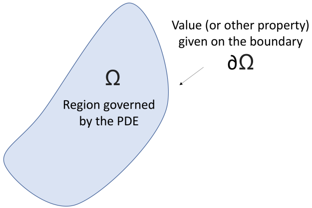
Boundary Conditions

Definition II.1.6

Let $\Omega \subset \mathbb{R}^d$ be a regular open set of class C^1.

Consider a PDE $\mathcal{A}u = f$ on Ω where $\mathcal{A}$ is the differential operator, u is the unknown and f is the data.

The **Boundary Condition** *(B.C.) is an equation that u must satisfy on $\partial\Omega$ (or part of $\partial\Omega$).*



Boundary Conditions

Definition II.1.7

A **Dirichlet boundary condition** *is* $u = g$ *on* $\partial\Omega$ *for a given* g*.*

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Example

$$\begin{cases} -\Delta u(x) = f(x) & x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases}$$

is an elliptic PDE with homogeneous Dirichlet boundary condition.

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Example

$$\begin{cases} \frac{\partial^2 u}{\partial t^2}(t,x) - \Delta u(t,x) = f(t,x) & (t,x) \in]0,T[\times\Omega \\ u|_{\partial\Omega}(t) = 0 & t \in]0,T[\\ u(0,x) = \phi(x) \text{ and } \frac{\partial u}{\partial t}(0,x) = \psi(x) & x \in \Omega \end{cases}$$

is an hyperbolic PDE with homogeneous Dirichlet boundary condition and with a given initial condition.

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Definition II.1.8

A **Neumann boundary condition** *is* $\frac{\partial u}{\partial n} = g$ *on* $\partial\Omega$ *for a given* g.

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II.2. Differential Calculus

II.2.1. Outward Normal Vector Field

Definition

An open set $\Omega \subset \mathbb{R}^d$ *is a* **regular open set of class** C^1 *if for each point* $x_0 \in \partial\Omega$ *there exist:*

- *a radius* $r > 0$ *and*
- *a* C^1 *function* $\gamma : \mathbb{R}^{d-1} \to \mathbb{R}$

such that

$$\Omega \cap B(x_0, r) = \{x \in B(x_0, r) : x_d > \gamma(x_1, \dots, x_{d-1})\}$$

(after reorienting the coordinates axes if necessary)

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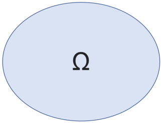

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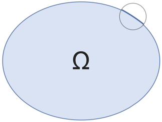

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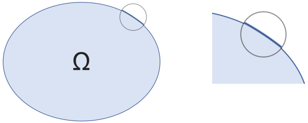

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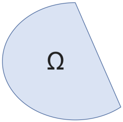

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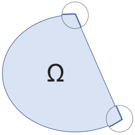

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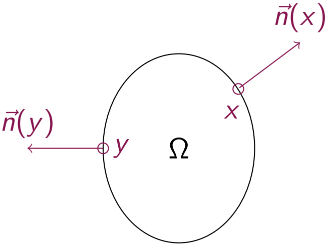
In other words, the boundary of Ω is locally the graph of a C^1 function.

Definition II.2.1

Let Ω will be a regular open set of class C^1 bounded in one or more direction.

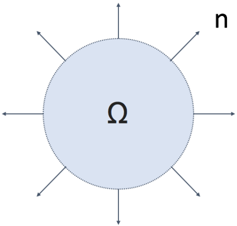
The **outward unit normal vector field** *to Ω is defined for each point of $x \in \partial\Omega$ as the unit vector normal to the tangent plane to Ω in x pointing towards the exterior.*



It is usually denoted $n(x) = [n_1(x), \dots, n_d(x)]^T$.

Example

Let $d = 2$ and Ω be the open disk or radius 1 centered in $(0, 0)$.



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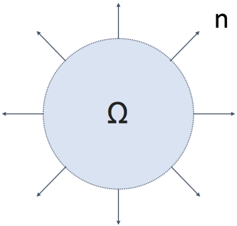
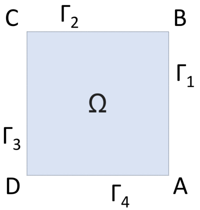


$$\forall x = (x_1, x_2) \in \partial\Omega, \;\; n(x) = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Example

Let $d = 2$ and Ω be an open square.

Denote A, B, C, D its vertices and Γ_1, Γ_2, Γ_3 and Γ_4 its edges. $\partial\Omega = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$.



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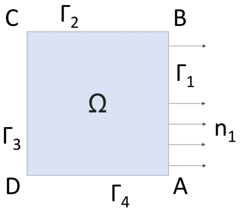

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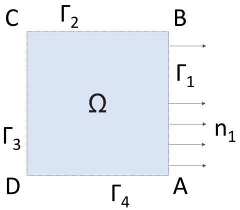

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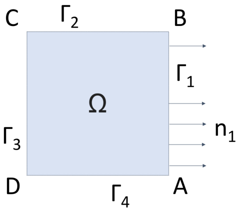

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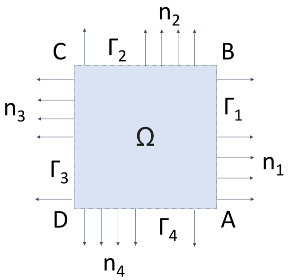

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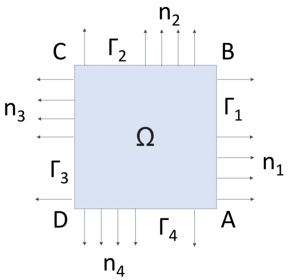
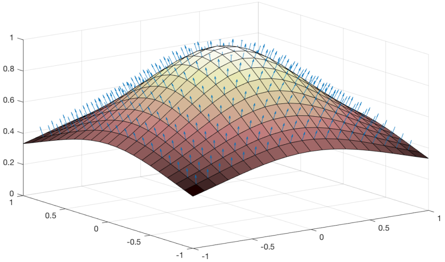


Note: Ω is not a regular open set and we cannot compute the unit outward normal vector field on the entire boundary (vertices)

Example

Let $d = 3$ *and* $\Omega = \left\{(x, y, z) \in \mathbb{R}^3,\ z \leq \frac{1}{1+x^2+y^2}\right\}.$



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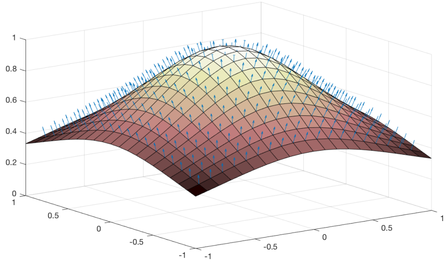


Let $f(x, y) = \frac{1}{1+x^2+y^2}$.

Can we characterize the normal field at (x, y)*?*

II.2.2. Partial Derivatives

Differentiability

Let $d, N \in \mathbb{N}^*$ and $U \subset \mathbb{R}^d$ be a non-empty open set.

Definition

*The function $f : U \to \mathbb{R}^N$ is **differentiable** at $x_0 \in U$ if there exists*

- *a linear application $L : \mathbb{R}^d \to \mathbb{R}^N$ and*
- *a neighborhood $V \in \mathcal{V}(x_0)$*

such that

$$\forall x \in V, \quad f(x) - f(x_0) = L(x - x_0) + o(\|x - x_0\|).$$

Partial Derivatives and Differential

Definition (Partial Derivatives)

Let $\mathbf{B} = (e_1, \ldots, e_d)$ *be a basis of* $\mathbb{R}^d$*. For* $i \in \{1, \ldots, d\}$*, we note*

$$\frac{\partial f}{\partial x_i}(x_0) = \lim_{s \in \mathbb{R},\, s \to 0} \frac{f(x_0 + s e_i) - f(x_0)}{h}.$$

It is the **partial derivative** *of* f *with respect to* x_i *in* x_0*.*
It is also denoted $\partial_{x_i} f(x_0)$*.*

Definition (Differential)

Differential of $f = \begin{bmatrix} f_1 \\ \vdots \\ f_N \end{bmatrix}$ *in* x_0*:* $D\, f(x_0) = \begin{bmatrix} \partial_{x_1} f_1(x_0) & \ldots & \partial_{x_d} f_1(x_0) \\ \vdots & \ddots & \vdots \\ \partial_{x_1} f_N(x_0) & \ldots & \partial_{x_d} f_N(x_0) \end{bmatrix}$

Case $N = 1$

If $N = 1$ then

- f is a scalar valued function
- L is a linear form

Definition

The **gradient** *of f at x_0, denoted* $\operatorname{grad} f(x_0)$ *or* $\nabla f(x_0)$, *is the vector such that* $\forall h \in \mathbb{R}^d,\ Lh = \langle \nabla f(x_0), h \rangle_{\mathbb{R}^d}$

$$\nabla f(x_0) = \begin{bmatrix} \partial_{x_1} f(x_0) \\ \vdots \\ \partial_{x_d} f(x_0) \end{bmatrix} \qquad \textit{in the basis } \mathbf{B}$$

The components of the gradient are the coefficients appearing in the equation of the tangent space to the graph, which, in turn gives the components of the normal vector field.

Case $N = d$

If $N = d$, then L is an endomorphism of $\mathbb{R}^d$.

Definition

We call **divergence** *of f at x_0, denoted $\operatorname{div} f(x_0)$ or $\nabla \cdot f(x_0)$ the trace of the matrix $Df(x_0)$.*

$$\nabla \cdot f(x_0) = \sum_{1 \le i \le d} \frac{\partial f_i}{\partial x_i}(x_0) \qquad \textit{in the basis } \mathbf{B}$$

The divergence of a vector field is a scalar function that measures the tendency of a vector field's to close in toward a point or to repel from a point.

Laplace Operator

Definition

The **Laplace operator** *or* **Laplacian** *is a differential operator defined by the divergence of the gradient. It is denoted* Δ.

Let $f : U \subset \mathbb{R}^d \to \mathbb{R}$. *The Laplacian of* f *at* x_0 *is*

$$\Delta f(x_0) = div(\nabla f)(x_0) = \frac{\partial^2 f}{\partial x_1^2}(x_0) + \ldots + \frac{\partial^2 f}{\partial x_d^2}(x_0).$$

The notation ∇^2 *can also be found to represent the Laplacian.*

The multi-index notation

Definition

A d-dimensional **multi-index** *is an element of* $\mathbb{N}^d$*.*

It is an d-tuple $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ *of non-negative integers.*

Example

$(1, 0, 4)$ *is a 3-dimensional multi-index.*

The multi-index notation

Definition

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ *be a* d*-dimensional* **multi-index**.
The sum of components is denoted $|\alpha| = \alpha_1 + \alpha_2 + \cdots + \alpha_d$.

Definition (Higher-order partial derivative)

Let $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_d)$ *be a* d*-dimensional* **multi-index**.

$$D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1} \partial x_2^{\alpha_2} \ldots \partial x_d^{\alpha_d}}$$

Example

Let $\alpha = (1, 0, 4)$ *and* $f : \mathbb{R}^3 \to \mathbb{R}$ *be a* C^5*-function.*

$$D^\alpha f = \frac{\partial^5 f}{\partial x_1 \partial x_3^4}$$

II.2.3. Integration by Parts in Higher dimension

Support

Definition II.2.2

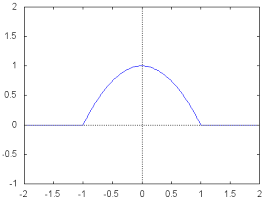
Let f be a function defined from an open interval $I \subset \mathbb{R}$ to $\mathbb{R}$. The **support** *of f, denoted $\mathrm{supp}(f)$, is the closure of the set of points where f does not vanish:*

$$\mathrm{supp}(f) = \overline{\{x \in I \mid f(x) \neq 0\}}$$

Example

$$\begin{array}{rcl} f : \mathbb{R} & \to & \mathbb{R} \\ x & \mapsto & \begin{cases} 1 - x^2 & \text{if} \quad |x| < 1 \\ 0 & \text{if} \quad |x| \geq 1 \end{cases} \end{array}$$

$$\mathrm{supp}(f) = [-1, 1]$$



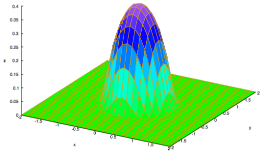
Support

Definition II.2.3

Let f be a function defined on $\Omega \subset \mathbb{R}^d$.
The **support** *of f, denoted $\mathrm{supp}(f)$, is the closure of the set of points where f does not vanish:*

$$\mathrm{supp}(f) = \overline{\{x \in \Omega \,|\, f(x) \neq 0\}}$$

Example



The support is the closed disk centered in $(0,0)$ with radius 1.

Integration

Theorem II.2.4 (Green's Formula)

Let $u \in C^1(\overline{\Omega})$ with compact support in $\overline{\Omega}$. Then

$$\forall i \in \{1, \dots, d\}, \quad \int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds.$$

Integration

Theorem II.2.4 (Green's Formula)

Let $u \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then*

$$\forall i \in \{1, \ldots, d\}, \quad \int_\Omega \frac{\partial u}{\partial x_i}(x)\ dx = \int_{\partial\Omega} u(s) n_i(s)\ ds.$$

Remark

If $d = 1$ *and* $\Omega =]a, b[$ *with* $a < b$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\ dx = \int_{\partial\Omega} u(s) n_i(s)\ ds$$

can be written

$$\int_\Omega \frac{du}{dx}(x)\ dx = \int_{\{a,b\}} u(s) n(s)$$

Integration

Theorem II.2.4 (Green's Formula)

Let $u \in C^1(\overline{\Omega})$ with compact support in $\overline{\Omega}$. Then

$$\forall i \in \{1, \ldots, d\}, \quad \int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds.$$

Remark

If $d = 1$ and $\Omega =]a, b[$ with $a < b$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds$$

can be written

$$\int_\Omega \frac{du}{dx}(x)\, dx = \sum_{s \in \{a,b\}} u(s) n(s)$$

Integration

Theorem II.2.4 (Green's Formula)

Let $u \in C^1(\overline{\Omega})$ with compact support in $\overline{\Omega}$. Then

$$\forall i \in \{1, \ldots, d\}, \quad \int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds.$$

Remark

If $d = 1$ and $\Omega =]a, b[$ with $a < b$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds$$

can be written

$$\int_\Omega \frac{du}{dx}(x)\, dx = u(a)n(a) + u(b)n(b)$$

Integration

Theorem II.2.4 (Green's Formula)

Let $u \in C^1(\overline{\Omega})$ with compact support in $\overline{\Omega}$. Then

$$\forall i \in \{1, \ldots, d\}, \quad \int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds.$$

Remark

If $d = 1$ and $\Omega =]a, b[$ with $a < b$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, dx = \int_{\partial\Omega} u(s) n_i(s)\, ds$$

can be written

$$\int_\Omega \frac{du}{dx}(x)\, dx = u(b) - u(a)$$

Integration by parts

Corollary II.2.5 (IP 1)

Let $u, v \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then* $\forall i \in \{1, \dots, d\}$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, v(x)\, d\lambda^{(d)} = -\int_\Omega u(x)\, \frac{\partial v}{\partial x_i}(x)\, d\lambda^{(d)} + \int_{\partial\Omega} uv\, n_i\, d\sigma.$$

Integration by parts

Corollary II.2.5 (IP 1)

Let $u, v \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then* $\forall i \in \{1, \ldots, d\}$,

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Corollary II.2.6 (IP 2)

Let $u \in C^2(\overline{\Omega})$ *and* $v \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then*

$$\int_\Omega (\Delta u)\, v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} v\, \nabla u \cdot n$$

Integration by parts

Corollary II.2.5 (IP 1)

Let $u, v \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then* $\forall i \in \{1, \ldots, d\}$,

$$\int_\Omega \frac{\partial u}{\partial x_i}(x)\, v(x)\, d\lambda^{(d)} = -\int_\Omega u(x)\, \frac{\partial v}{\partial x_i}(x)\, d\lambda^{(d)} + \int_{\partial\Omega} uv\ n_i\ d\sigma.$$

Corollary II.2.6 (IP 2)

Let $u \in C^2(\overline{\Omega})$ *and* $v \in C^1(\overline{\Omega})$ *with compact support in* $\overline{\Omega}$. *Then*

$$\int_\Omega (\Delta u)\, v = -\int_\Omega \nabla u \cdot \nabla v + \int_{\partial\Omega} v\, \frac{\partial u}{\partial n}$$

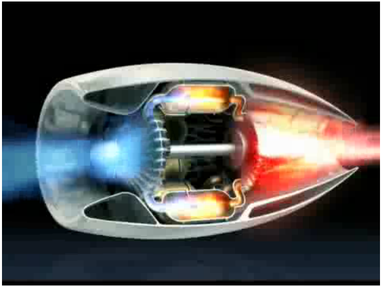
where $\frac{\partial u}{\partial n} = \nabla u \cdot n$ *is called the* **normal derivative**.

II.3. The importance of PDEs in modeling

II.3.1. Temperature in a Reactor

The Problem

We are interested in the temperature in this reactor.



Credit: Rendermedia Ltd.

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We are interested in the temperature in this reactor.

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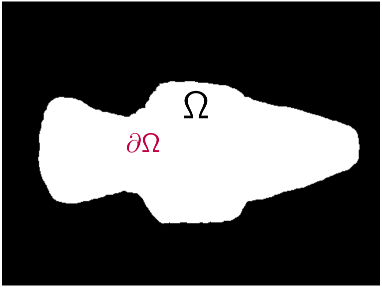
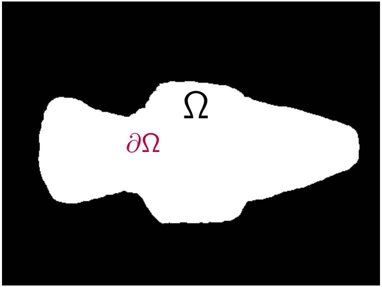


The Problem

We are interested in the temperature in this reactor.



Variables: $t \in [0, +\infty[$,
$x \in \Omega$

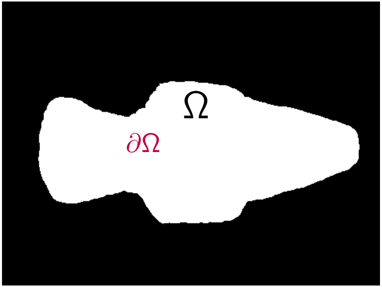
Unknown: $(t, x) \mapsto \theta(t, x)$

Data: ρ (density),
u (velocity),
f (source)

Parameters: $x \mapsto c_v(x)$,
$x \mapsto \kappa(x) > 0$

The Problem

We are interested in the temperature in this reactor.



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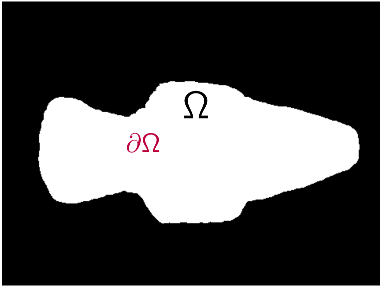
Parameters: $x \mapsto c_v(x)$,
$x \mapsto \kappa(x) > 0$

Heat Equation :

$$\rho c_v \partial_t \theta + \mathrm{div}_x(\rho c_v\, \theta\, u) - \mathrm{div}_x(\kappa \nabla_x(\theta)) = f \text{ in }]0, +\infty[\times\Omega.$$

The Problem

We are interested in the temperature in this reactor.



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Parameters: $x \mapsto c_v(x)$,
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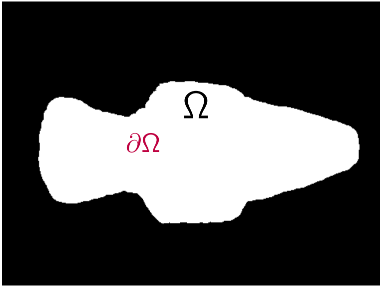
Heat Equation when the fluid does not move:

$$\rho c_v \partial_t \theta + \mathrm{div}_x(\rho c_v\, \theta\, u) - \mathrm{div}_x(\kappa \nabla_x(\theta)) = f \text{ in }]0, +\infty[\times\Omega.$$

The Problem

We are interested in the temperature in this reactor.



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Unknown: $(t, x) \mapsto \theta(t, x)$

Data: ρ (density),
u (velocity),
f (source)

Parameters: $x \mapsto c_v(x)$,
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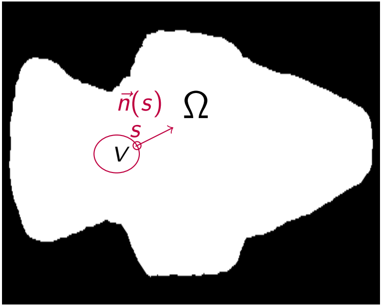
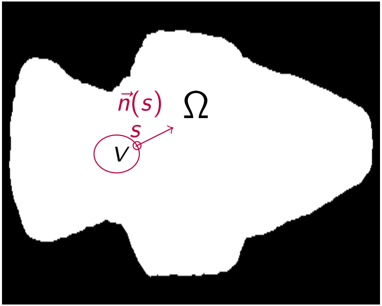
Let us establish the heat equation when the fluid does not move:

$$\rho c_v \partial_t \theta + \mathrm{div}_x(\rho c_v\, \theta\, u) - \mathrm{div}_x(\kappa \nabla_x(\theta)) = f \text{ in }]0, +\infty[\times \Omega.$$

Energy Conservation

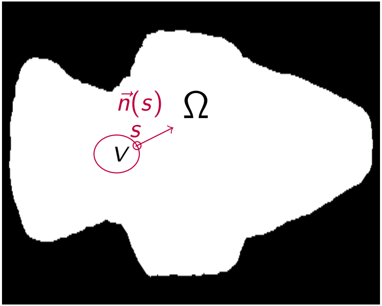


Energy Conservation



Density of energy in V:
$(t, x) \mapsto \rho c_v(x)\theta(t, x)$

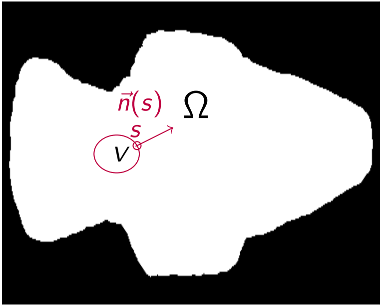
Energy Conservation



Density of energy in V:
$(t,x) \mapsto \rho c_v(x)\theta(t,x)$

Energy in V:
$$t \mapsto \iiint_V \rho c_v(x)\theta(t,x)\,dx$$

Energy Conservation



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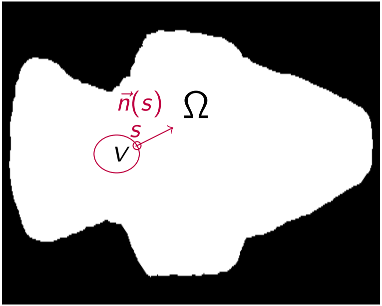
Energy flux through ∂V:
$(t,s) \mapsto q(t,s)$

Energy Conservation



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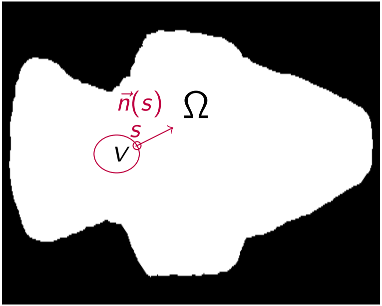
Energy sources/sinks in V:
$(t,x) \mapsto f(t,x)$

Energy Conservation



Density of energy in V:
$(t,x) \mapsto \rho c_v(x)\theta(t,x)$

Energy in V:
$t \mapsto \iiint_V \rho c_v(x)\theta(t,x)\,dx$

Energy flux through ∂V:
$(t,s) \mapsto q(t,s)$

Energy sources/sinks in V:
$(t,x) \mapsto f(t,x)$

$$\frac{d}{dt}\iiint_V \rho c_v(x)\theta(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

Deriving the Heat Equation

$$\frac{d}{dt}\iiint_V \rho c_v(x)\theta(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

Deriving the Heat Equation

$$\underbrace{\frac{d}{dt}\iiint_V \rho c_v(x)\theta(t,x)\,dx}_{=\iiint_V \partial_t(\rho c_v\theta)(t,x)\,dx} = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

Deriving the Heat Equation

$$\iiint_V \partial_t(\rho c_v \theta)(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

Deriving the Heat Equation

$$\iiint_V \partial_t(\rho c_v \theta)(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

Green's formula (Theorem II.2.4) for $w \in (C^1(\overline{\Omega}))^3$:

$$\iint_{\partial V} w(s)\cdot n(s)\mathrm{d}s = \iiint_V \mathrm{div}_x(w)(x)\,dx$$

Deriving the Heat Equation

$$\iiint_V \partial_t(\rho c_v \theta)(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

$$\iiint_V (\rho c_v)(x)\partial_t \theta(t,x)\,dx + \iiint_V \mathrm{div}_x(q)(t,x)\,dx = \iiint_V f\,(t,x)\mathrm{d}x$$

Deriving the Heat Equation

$$\iiint_V \partial_t(\rho c_v \theta)(t,x)\,dx = -\iint_{\partial V} q(t,s)\cdot n(s)\,ds + \iiint_V f(t,x)\,dx$$

$$\iiint_V (\rho c_v)(x)\partial_t\theta(t,x)\,dx + \iiint_V \mathrm{div}_x(q)(t,x)\,dx = \iiint_V f(t,x)\mathrm{d}x$$

This is true for all V thus

$$\forall t > 0,\ \forall x \in \Omega,\ (\rho c_v)(x)\,\partial_t\theta(t,x) + \mathrm{div}_x(q)(t,x) = f(t,x)$$

Deriving the Heat Equation

$$\forall t > 0,\ \forall x \in \Omega, \quad \rho c_v(x)\partial_t\theta(t,x) + \operatorname{div}_x(q)(t,x) = f\,(t,x)$$

This is not a PDE yet. We need to link q to θ.

Deriving the Heat Equation

$$\forall t > 0,\ \forall x \in \Omega, \quad \rho c_v(x)\partial_t\theta(t,x) + \mathrm{div}_x(q)(t,x) = f\,(t,x)$$

This is not a PDE yet. We need to link q to θ.

The Fourier law gives: $q = -\kappa\nabla_x(\theta)$

Deriving the Heat Equation

$$\forall t > 0,\ \forall x \in \Omega, \quad \rho c_v(x)\partial_t\theta(t,x) + \mathrm{div}_x(q)(t,x) = f(t,x)$$

This is not a PDE yet. We need to link q to θ.

The Fourier law gives: $q = -\kappa\nabla_x(\theta)$
We derive:

$$\rho c_v \partial_t\theta - \mathrm{div}_x(\kappa\nabla_x(\theta)) = f \quad \text{in} \quad \mathbb{R}^{+*}\times\Omega$$

Initial conditions / Boundary conditions

- Initial Condition – When $t = 0$: θ^0 is given. (Cauchy)
- Boundary Condition – On $\partial\Omega$:
  - A set value: $\theta|_{\partial\Omega} = g$ given. **Dirichlet**
  - A set flow: $\nabla_x(\theta)|_{\partial\Omega} \cdot n = h$ given. **Neumann**
  - Both: $\partial\Omega = \partial\Omega_D \sqcup \partial\Omega_N$: $\theta|_{\partial\Omega_D} = g$ and $\nabla_x(\theta)|_{\partial\Omega_N} \cdot n = h$

From Physics to Mathematics

In mathematics, the quantities are dimensionless.
We need to go through a nondimensionalization.

Name	Variable	Unit	Nondimensionalization
Time (s)	t	T	$t = Tt^*$
Length (m)	x	L	$x = Lx^*$
Temperature (K)	θ	Θ	$\theta = \Theta\theta^*$
Velocity ($m.s^{-1}$)	u	U	$u = Uu^*$
Th. Cond. ($W.m^{-1}.K^{-1}$)	κ	$\mathcal{K}$	$\kappa = \mathcal{K}\kappa^*$
Source ($kg.m^2.s^{-3}$)	f	F	$f = Ff^*$

Nondimensionalized equation

$$\begin{cases} \partial_t \theta = \dfrac{1}{T} \partial_{t^*} \theta \\ \partial_{x_i} \theta = \dfrac{1}{L} \partial_{x_i^*} \theta \\ \operatorname{div}_x \theta = \dfrac{1}{L} \operatorname{div}_{x^*} \theta \end{cases}$$

Nondimensionalized equation

$$\begin{cases} \partial_t \theta = \dfrac{1}{T} \partial_{t^*} \theta \\ \partial_{x_i} \theta = \dfrac{1}{L} \partial_{x_i^*} \theta \\ \mathrm{div}_x \theta = \dfrac{1}{L} \mathrm{div}_{x^*} \theta \end{cases}$$

In the equation

$$\frac{\rho c_v \Theta}{T} \partial_{t^*} \theta^* + \frac{\rho c_v \Theta U}{L} \mathrm{div}_{x^*} (\theta^* \, u^*) - \frac{\Theta \mathcal{K}}{L^2} \Delta_{x^*} \theta^* = F \, f^*$$

Nondimensionalized equation

$$\begin{cases} \partial_t \theta = \dfrac{1}{T} \partial_{t^*} \theta \\ \partial_{x_i} \theta = \dfrac{1}{L} \partial_{x_i^*} \theta \\ \mathrm{div}_x \theta = \dfrac{1}{L} \mathrm{div}_{x^*} \theta \end{cases}$$

Finally:

$$1\, \partial_{t^*} \theta^* + \frac{T}{T_C} \mathrm{div}_{x^*} (\theta^*\, u^*) - \frac{T}{T_D} \Delta_{x^*} \theta^* = \frac{T}{T_S} f^*$$

With characteristic times

$$T_C = \frac{L}{U}, \qquad T_D = \frac{\rho c_v L^2}{\mathcal{K}} \qquad \text{et} \qquad T_S = \frac{\rho c_v \Theta}{F}.$$

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C} \mathrm{div}_x(\theta\, u) - \frac{T}{T_D} \Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C} \mathrm{div}_x(\theta\, u) - \frac{T}{T_D} \Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type
$T << T_D, T_C$	$\partial_t \theta = f$	**ODE**

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C} \mathrm{div}_x(\theta\, u) - \frac{T}{T_D} \Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type
$T << T_D, T_C$	$\partial_t \theta = f$	**ODE**
$T >> T_C >> T_D$	$-\Delta_x \theta = f$	**diffusion (steady)**

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C}\mathrm{div}_x(\theta\, u) - \frac{T}{T_D}\Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type
$T << T_D, T_C$	$\partial_t \theta = f$	**ODE**
$T >> T_C >> T_D$	$-\Delta_x \theta = f$	**diffusion (steady)**
$T \sim T_D << T_C$	$\partial_t \theta - \Delta_x \theta = f$	**diffusion time-dep.**

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C} \mathrm{div}_x(\theta\, u) - \frac{T}{T_D} \Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type
$T << T_D, T_C$	$\partial_t \theta = f$	**ODE**
$T >> T_C >> T_D$	$-\Delta_x \theta = f$	**diffusion (steady)**
$T \sim T_D << T_C$	$\partial_t \theta - \Delta_x \theta = f$	**diffusion time-dep.**
$T \sim T_C << T_D$	$\partial_t \theta + \mathrm{div}_x(\theta u) = f$	**transport**

Several Behaviors

$$\partial_t \theta + \frac{T}{T_C} \mathrm{div}_x(\theta\, u) - \frac{T}{T_D} \Delta_x \theta = \frac{T}{T_S} f$$

Behavior	Equation	Type
$T << T_D, T_C$	$\partial_t \theta = f$	**ODE**
$T >> T_C >> T_D$	$-\Delta_x \theta = f$	**diffusion (steady)**
$T \sim T_D << T_C$	$\partial_t \theta - \Delta_x \theta = f$	**diffusion time-dep.**
$T \sim T_C << T_D$	$\partial_t \theta + \mathrm{div}_x(\theta u) = f$	**transport**
$T_D \sim T \sim T_C$	$\partial_t \theta + \mathrm{div}_x(\theta u) - \Delta_x \theta = f$	**transport-diffusion time-dep.**

II.3.2. Air Pollution / Traffic Flow

The Transport equation can be used to model air pollution

Consider the transport equation.

$$\begin{cases} \partial_t u + c \cdot \nabla_x u = f & \text{in }]0, +\infty[\times\mathbb{R} \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R} \end{cases}$$

where $c > 0$ is a constant and f is a given function.

It can be used to model the air pollution.

u represents the density of pollutant.

The Transport equation can be used to model traffic flow

Consider the transport equation in one (space) dimension (note: $d = 2$).

$$\begin{cases} \partial_t u + c\,\partial_x u = f & \text{in }]0, +\infty[\times\mathbb{R} \\ u(0,\cdot) = u_0 & \text{on } \mathbb{R} \end{cases}$$

where $c > 0$ is a constant and f is a given function.

It can be used to model the traffic flow.

u represents the density of vehicles.

Characteristics

Consider the transport equation in one dimension (note: $d = 2$).

$$\begin{cases} \partial_t u + c\,\partial_x u = 0 & \text{in }]0, +\infty[\times \mathbb{R} \\ u(0, \cdot) = u_0 & \text{on } \mathbb{R} \end{cases}$$

where $c > 0$ is a constant.

Characteristics:
Find $t \mapsto X(t)$ s.t. $w : t \mapsto u(t, X(t))$ is constant.

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If $\frac{dX}{dt} = c$ then $\frac{dw}{dt} = 0$ then w is constant. Thus $X(s) = cs + X(0)$ therefore $u(t, X(t)) = u^0(X(0))$

II.3.3. Black and Scholes

Another example of a parabolic PDE

Consider a contract giving the right to buy or sell a stock at a date T at a strike price K.

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source: wikipedia images

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Another example of a parabolic PDE

Consider a contract giving the right to buy or sell a stock at a date T at a strike price K. With some hypotheses, we derive

$$\begin{cases} \frac{\partial u}{\partial t} - \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 u}{\partial x^2} + rx\frac{\partial u}{\partial x} - ru = 0 \\ u(x, T) = \max(x - K, 0) \\ u(0, t) = 0 \text{ and } \lim_{x\to\infty} \frac{u(x,t)}{x} = 1 \end{cases}$$

where

- $t \in [0, T]$ is the time
- $x(t)$ the price of the stock at time t,
- K the strike price
- $u(x(t), t)$ is the price of the option
- r is the annualized risk-free interest rate
- σ is the standard deviation of the stock's returns.

II.4. Outline of the work to be done

What's next?

(From the general introduction)

We would like wellposedness theorems for elliptic PDEs.

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- The theory of distributions:
  Chapter III. On Friday
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Then, we will approximate the solutions of the elliptic PDEs.
After which, we will discuss parabolic PDEs.