Equalization

Combating ISI/frequency selectivity

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The communications system chain

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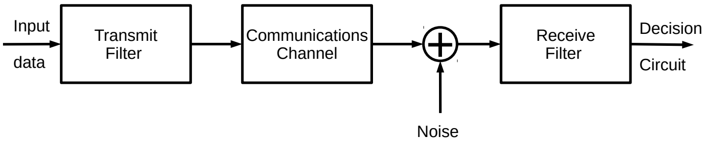


The communications system chain

- The transmit and receive filters should be selected such as to eliminate Intersymbol Interference (ISI) and limiting the bandwidth of the communications signal.
- Nyquist criterion, if $x(t)$ is the cascaded impulse response of the transmit filter, the channel and the receive filter, it must hold that:

$$x(nT) = \begin{cases} 1, & n = 0 \\ 0, & n \neq 0. \end{cases} \tag{1}$$

or equivalently that:

$$\sum_{m=-\infty}^{\infty} X\left(f + \frac{m}{T}\right) = T \tag{2}$$

Raised cosine filtering

- By designing the transmit and receive filters such that:

$$X(f) = \begin{cases} T, & 0 \leq |f| \leq \frac{(1-\alpha)}{2T} \\ \frac{T}{2}\left[1 + \cos\frac{\pi T}{a}\left(|f| - \frac{1-a}{2T}\right)\right], & \frac{1-a}{2T} \leq f \leq \frac{1+a}{2T} \\ 0, & |f| > \frac{1+a}{2T} \end{cases} \tag{3}$$

  we can satisfy this condition.
- System design with an ideal channel:
  - The channel impulse response $H_C(f) = 1$
  - We can select the transmit filter such that: $H_T(f) = \sqrt{X(f)}e^{-j2\pi f t_0}$
  - and the receive filter such that:

$$H_R(f) = H^{\star}(f). \tag{4}$$

Channel induced distortions

Amplitude and phase distortions

- The channel frequency response can be expressed as:

$$H_C(f) = |H_C(f)| \, e^{j\Theta(f)}. \tag{5}$$

- Two types of distortion can be recognized:
  - Amplitude distortion: If $H_C(f)$ is not-constant over the signal spectrum, different frequency components are treated differently by the channel.
  - Phase distortion: If $\Theta(f)$ is not a linear function, the channel introduces different deals for the different channel frequency components.
  - Both effects result in introducing Intersymbol Interference.

Combating channel effects with known channel

- Assuming knowledge of $H_C(f)$ at both the transmitter and the receiver, we can construct the transmit $h_T(t)$ and receive filters $h_R(t)$ such that:

$$H_T(f)\, H_C(f)\, H_R(f) = X_{rc}(f) \tag{6}$$

- Drawbacks:
  - If the channel response changes with time (e.g., wireless channel), we need to readjust our transmit and/or receive filters such as to adjust them to these changes.
  - Perfect channel knowledge is required in order to totally avoid Intersymbol Interference.

Practical transmit and receive filtering

- Since in most cases the channel response is unknown and/or time varying, we select fixed transmitter and receiver filters:

$$H_T(f) = \begin{cases} \sqrt{X_{rc}(f)}e^{-j2\pi f t_0}, & |f| \leq W, \\ 0, & |f| > W \end{cases} \tag{7}$$

  and

$$|H_R(f)| = |H_T^\star(f)|. \tag{8}$$

- The cascaded communication chain results in pulse per symbol having a shape: $x(t) = h_T(t) \star h_C(t) \star h_R(t) = x_{rc}(t) \star h_c(t)$, which does not satisfy the Nyquist criterion

The received signal

- The received signal is expressed as:

$$y(t) = \sum_{n=-\infty}^{\infty} a_n x(t - nT) + n(t), \tag{9}$$

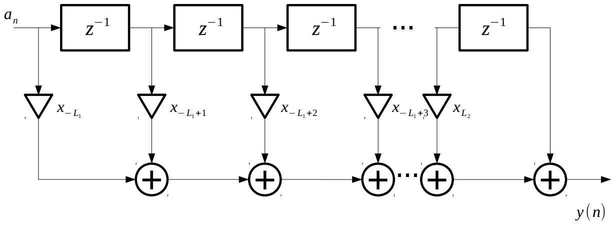
where $n(t)$ the AWGN.

- After sampling the received signal at time instances that are multiples of the symbol duration, we obtain that:

$$\begin{aligned} y_m = y(mT) &= \sum_{n=-\infty}^{\infty} a_n x_{m-n} + n_m \\ &= \underbrace{x_0 a_m}_{\text{Desired signal}} + \underbrace{\sum_{n=-\infty, n\neq m}^{+\infty} a_n x_{m-n}}_{ISI} + \underbrace{n_m}_{\text{Noise}}. \end{aligned} \tag{10}$$

Discrete time equivalent channel filter

- In practice, the length of pulse $x(t)$ is small enough so that only a limited number of symbols are influenced.
- We therefore have that $x_n = 0$ for $n < -L_1$ and $n > L_2$.



Optimal (Maximum Likelihood) detection

Optimal detection for frequency selective channels

- Given a vector of observations $\{y_m\}$, we need to determine the vector of input symbols $\{a_m\}$.
- The channel can then be represented as a trellis diagram and a state diagram with M^L, $(L = L_1 + L_2)$ states.
- Assuming knowledge of the channel $\{x_n\}$, detection can then be done using Viterbi algorithm,
- As channel order L increases, the complexity increases exponentially.
- Practical only for small values of M and L.
- Suboptimal schemes:
    - Linear equalizers
    - Decision feedback equalizer

Trellis and State diagram representation for frequency selective channels

Assuming a causal FIR channel:

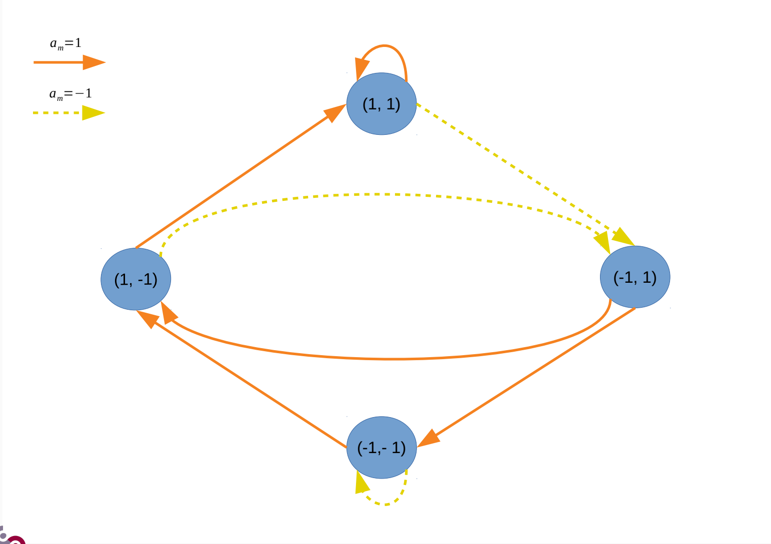
- Accounting for the fact that $x_n = 0$ for $n < 0$ and $n > L$ we can write y_m as:

$$y_m = y(mT) = \underbrace{x_0 a_m}_{\text{Desired signal}} + \underbrace{\sum_{n=1,L_2}^{+\infty} a_{m-n} x_n}_{ISI} + \underbrace{n_m}_{\text{Noise}} . \tag{11}$$

- We can describe the channel as a state diagram where each state corresponds to the vector $(a_{m-1}, \ldots, a_{m-L})$

State diagram representation of a channel: Example

Transmission of BPSK over a channel with 3 taps:



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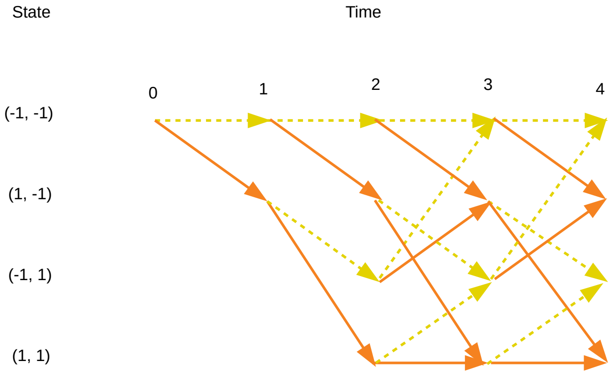
Trellis diagram representation of a channel: Example

Transmission of 4 BPSK symbols over a channel with 3 taps (assuming a starting state (-1, -1)):



Optimal detection for frequency selective channels

- Given the vector of observations corresponding to the transmission of M symbols:

$$\mathbf{y} = [y_M, y_{M-1}, \ldots, y_0] \tag{12}$$

we decide in favor of the transmit sequence:

$$\mathbf{y} = [a_M, a_{M-1}, \ldots, a_0] \tag{13}$$

that maximizes the likelihood function:

$$p\left(y_M, \ldots, y_0 | a_M, \ldots, a_0\right) \tag{14}$$

- Equivalently we select the sequences that maximizes the log-likelihood function

$$\begin{aligned} \log p\left(y_M, \ldots, y_0 | a_M, \ldots, a_0\right) &= \log p\left(y_M | a_M, \ldots, a_{M-L}\right) \\ &\quad + \log p\left(y_{M-1}, \ldots, y_0 | a_{M-1}, \ldots, a_0\right) \end{aligned} \tag{15}$$

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Viterbi algorithm

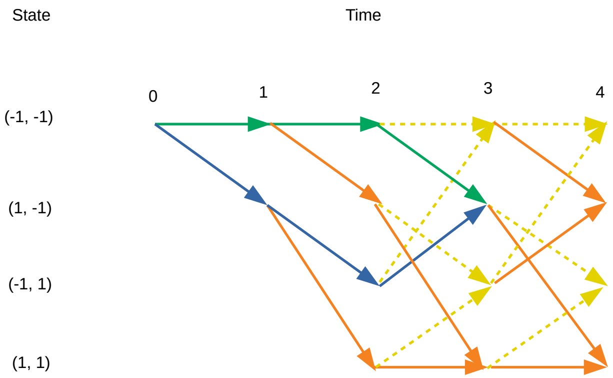
- When observing sample value y_M at the end of the M-th transmit period, associate to the transmition from state $(a_{M-1}, \ldots, a_{M-L})$ to state $(a_M, \ldots, a_{M-L+1})$ a weigh equal to :

$$-\log p\left(y_M | a_M, \ldots, a_{M-L}\right) \tag{16}$$

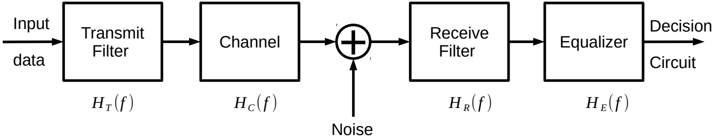
- The sequence detection problem then becomes a shortest path routing problem.
- If there are more than one paths that arrive at a particular state after observing y_M, we only need to consider the one that results in the shortest cost.

Viterbi algorithm

Out of the two paths that arrive at state $(1,-1)$ at the end of the third transmission period, select the one that results in the shortet path and eliminate the other one.



(Linear) Channel equalizers



Linear channel equalizers

(Linear) Channel equalizers

- Preset Equalizers
  - The filter gains are selected in the beginning of the transmission and remain constant for the whole system duration.
  - Suitable for systems where the channel impulse response remains fixed during transmission.
- Adaptive equalizers: The tap weights of the equalizer are updated during system operation.

Zero forcing equalizer

- If $H_C\left(f\right) = \left|H_C\left(f\right)\right| e^{j\theta_c(f)}$, the equalizer frequency response is selected to be equal to:

$$H_e\left(f\right) = \frac{1}{|H_c(f)|} e^{-j\theta_c(f)}, \;\; |f| \leq W. \tag{17}$$

- The ISI effect is totally eliminated.
- If $H_T\left(f\right)$ and $H_R\left(f\right)$ satisfy the Nyquist criterion, the out of the equalizer can be expressed as:

$$y_m = y\left(mT\right) = a_m + \nu_m, \tag{18}$$

  where ν_m is Gaussian noize.
- If the channel impulse response causes deep fades in some frequency components, then the equalizer "amplifies" noise components in the particular frequencies.
- Generally, the noise variance at the output of the equalizer is higher that noise variance of the channel.

FIR implementation of equalizers

- For practical purposes,equalizers are implemented as FIR filters
- Symbol spaced equalizer: The time delay τ between adjacent taps is equal to the symbol period T
- Fractionally spaced equalizer: $\tau < T$.
- A common choice for τ is $\tau = T/2$.
- The impulse response of the equalizer:

$$h_e(t) = \sum_{n=-N}^{N} c_n \delta(t - n\tau), \tag{19}$$

- Frequency response of the equalizer:

$$H_e(f) = \sum_{n=-N}^{n=N} c_n e^{-j2\pi f n\tau} \tag{20}$$

- N should be chosen large enough (i.e., $2N + 1 > L$) such as the length of the filter is at least as large as the ISI span.

FIR implementation of the zero-forcing equalizer

- Ignoring noise, if $h_T(t) \star h_R(t) = x(t)$, the equalizer output is equal to:

$$z(t) = \sum_{n=-N}^{N} c_n x(t - n\tau) \tag{21}$$

- Samples obtained at mT:

$$z(mT) = \sum_{n=-N}^{N} c_n x(mT - n\tau), m \in \mathbb{Z}. \tag{22}$$

- How can we determine the coefficients c_k such as to eliminate ISI?

FIR implementation of the zero-forcing equalizer

- The $2N+1$ tap weight gains need to be determined.
- Assuming knowledge of $x(n)$ we can construct $2N+1$ equations such as to determine c_n as follows:

$$z(mT) = \sum_{n=-N}^{N} c_n x(mT - n\tau) = \begin{cases} 1, & m = 0 \\ 0, & m = \pm 1, \pm 2, \dots, \pm N \end{cases} \tag{23}$$

- Equivalent formulation:

$$\mathbf{Xc} = \mathbf{z} \tag{24}$$

where:

$$\mathbf{c} = \left[c_{-N}, c_{-N+1}, \dots, c_{-1}, c_0, c_1, \dots, c_N\right]^T \tag{25}$$

$$\mathbf{z} = \left[\underbrace{0, \dots, \dots, 0}_{\text{N elements}}, 1, \underbrace{0, \dots, 0}_{\text{N elements}}\right]^T \tag{26}$$

FIR implementation of the zero-forcing equalizer

and

$$\mathbf{X} = \begin{bmatrix} x(-NT+N\tau) & x(-NT+(N-1)\tau) & \cdots & x(-NT) & \cdots & x(-NT+N\tau) \\ \vdots & \vdots & \ldots & \vdots & \vdots & \vdots \\ x(NT+N\tau) & x(NT+(N-1)\tau) & \cdots & x(NT) & \cdots & x(NT+N\tau) \end{bmatrix} \tag{27}$$

Remark: Inverting an FIR channel requires an IIR equalizer. However by letting $N \to \infty$, the presented FIR equalizer converges to a channel inverting equalizer.

Example

Consider a communication channel where the channel impulse response is such that $x(t) = h_T(t) \star h_C(t) \star h_R(t)$, with

$$x(t) = \frac{1}{1 + \left(\frac{2t}{T}\right)^2} \tag{28}$$

where $1/T$ is the symbol rate. Determine the coefficients of a 5-tap half-symbol spaced zero forcing equalizer.

Solution: Since we have a five tap filter, half-symbol spaced equalizer, the equalizer filter will have $N = 2$ and it will be defined by the vector of coefficients:

$$\mathbf{c} = [c_{-2}, c_{-1}, c_0, c_1, c_2]^T . \tag{29}$$

As explained earlier, the vector of coefficients $\mathbf{c}$ can be obtained by solving the linear system:

$$\mathbf{Xc} = \mathbf{z} \tag{30}$$

where $\mathbf{z} = [z(-2T), z(-T), z(0), z(T, z(2T)]^T = [0, 0, 1, 0, 0]^T$

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Example (continued)

and

$$\mathbf{X} = \begin{bmatrix} x(-T) & x(-3T/2) & x(-2T) & x(-5T/2) & x(-3T) \\ x(0) & x(-T/2) & x(-T) & x(-3T/2) & x(-3T) \\ x(T) & x(T/2) & x(0) & x(-T/2) & x(-T) \\ x(2T) & x(3T/2) & x(T) & x(T/2) & x(0) \\ x(3T) & x(5T/2) & x(2T) & x(3T/2) & x(T) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{5} & \frac{1}{10} & \frac{1}{17} & \frac{1}{26} & \frac{1}{37} \\ 1 & \frac{1}{2} & \frac{1}{5} & \frac{1}{10} & \frac{1}{17} \\ \frac{1}{5} & \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{5} \\ \frac{1}{17} & \frac{1}{10} & \frac{1}{5} & \frac{1}{2} & 1 \\ \frac{1}{37} & \frac{1}{26} & \frac{1}{17} & \frac{1}{10} & \frac{1}{5} \end{bmatrix} \tag{31}$$

As a result, we obtain that $\mathbf{c} = \mathbf{X}^{-1}\mathbf{z}$.

MMSE equalizer

- Key idea: Instead of selecting c_n such as to eliminate ISI, select c_n such as to minimize the mean squares error between the received quantity and the transmitted symbol.
- Mathematical formulation: Expressing again the sampled equalizer output as:

$$z\left(mT\right) = \sum_{n=-N}^{N} c_n y\left(mT - n\tau\right), \tag{32}$$

the objective is to minimize:

$$MSE = \mathbb{E}\left\{\left|z\left(mT\right) - a_m\right|^2\right\} \tag{33}$$

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MMSE equalizer

Assuming real symbols a_m (e.g., PAM symbols) we have that:

$$MSE = \mathbb{E}\left\{a_m^2\right\} + \sum_{n=-N}^{N}\sum_{k=-N}^{N} c_n c_k Ry\left(n-k\right) - 2\sum_{k=-N}^{N} c_k R_{ay}\left(k\right), \quad (34)$$

where:

$$R_y\left(n-k\right) = \mathbb{E}\left\{y\left(mT-n\tau\right)y\left(mT-k\tau\right)\right\} \quad (35)$$

and

$$R_{ay}(k) = \mathbb{E}\left\{y\left(mT-k\tau\right)a_m\right\} \quad (36)$$

The optimal (Minimum Mean Square Error - MMSE) equalizer is obtained as the solution to the following linear system:

$$\mathbf{Rc} = \mathbf{r} \quad (37)$$

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MMSE equalizer

where:

$$\mathbf{R} = \begin{bmatrix} R_y(0) & R_y(1) & \cdots & R_y(2N) \\ R_y(-1) & R_y(0) & \cdots & R_y(2N-1) \\ \vdots & \vdots & \ddots & \vdots \\ R_(y)(-2N) & R_y(-2N+1) & \cdots & R_y(0) \end{bmatrix} \tag{38}$$

and

$$\mathbf{r} = \left[R_{ay}\left(-N\right), \ldots R_{ay}(N)\right]^T. \tag{39}$$

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MMSE equalizer

- Implementing an MMSE equalizer in practice, requires knowledge of the autocorrelation and cross correlation functions R_y and $R_{a,y}$.
- In practice, these functions can be estimated using ensemble averages obtained by transmitting a pilot signal know both to Tx and Rx.
- In such a case, R_y and R_{ay} are estimated as:

$$\hat{R}_y(n) = \frac{1}{K}\sum_{k=1}^{K} y(kT - n\tau)\, y(kT) \tag{40}$$

and

$$\hat{R}_{ay}(n) = \frac{1}{K}\sum_{k=1}^{K} y(kT - n\tau)\, y(kT) \tag{41}$$

Adaptive equalizers

- For both ZF and MMSE equalizer, we determine the coefficient vector $\mathbf{c}$ by solving a linear system of the form:

$$\mathbf{c} = \mathbf{B}^{-1}\mathbf{d}. \tag{42}$$

- For practical purposes we wish to find ways for reaching the solution avoiding the use of matrix inversion.
- Iterative, gradient descent methods can be used.
  - Input: Initial estimate $\mathbf{c}_0$, matrix $\mathbf{B}$, vector $\mathbf{d}$
  - Step k, $k = 1, \ldots$: Set:

$$\mathbf{c}_k = \mathbf{c}_{k-1} - \Delta \mathbf{g}_{k-1} \tag{43}$$

  where Δ a step size parameter, and $\mathbf{g}_k$ the gradient vector corresponding to our optimization criterion.

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Adaptive implementation of MMSE equalizer

- For the MMSE equalizer, it can be proven that:

$$\mathbf{g}_k = -\mathbb{E}\left\{(a_k - z_k)\,\mathbf{y}_k\right\} \tag{44}$$

- By replacing $\mathbf{g}_k$ by the estimate

$$\hat{\mathbf{g}}_k = -\left(a_k - z_k\right)\mathbf{y}_k \tag{45}$$

  the update rule:

$$\mathbf{c}_k = \mathbf{c}_{k-1} + \Delta\left(a_k - z_k\right)\mathbf{y}_k \tag{46}$$

  is obtained.

- Applying this iterative process we obtain the so-called stochastic gradient of Least Mean Squares (LMS) algorithm.
- One iteration corresponds to one symbol transmission.

Exploiting prior decisions

In practice adaptive equalizers operate in two modes

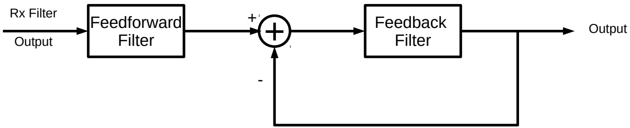
- Training mode: An initial mode where a known sequence is transmitted such as to obtain a relatively reliable estimate of the channel,
- Decision directed mode: The equalizer uses the outputs of the detector device $\tilde{a}_k$ instead of the unknown a_k and uses $\tilde{\mathbf{g}}_k = -\left(\tilde{a}_k - z_k\right)\mathbf{y}_k$ instead of the unknown $\mathbf{g}_k$.
- Infrequent detection errors during decision directed operation do not strongly influence the performance.
- The use of decision directed mode limits the need of frequent training data transmission.

Decision feedback equalizer

Drawbacks of linear equalizers

- In practice, linear equalizers exhibit poor performance in case of severe ISI effects.
- ISI severity is not necessarily connected to the length N of the FIR channel.
- It is rather connected to whether or not the channel exhibits frequency nulls.
- Channels with deep fades in the frequency domain can not be efficiently equalized by a linear equalizer.
- This is due to the fact that linear equalizers tend to enhance noise components in deeply faded components.

Decision feedback equalizer



Decision feedback equalizer

- Feedforward filter: A linear equalizer of the form presented earlier (e.g., equalizer)
- Feedback filter
  - A symbol spaced FIR filter aiming at further suppressing ISI.
  - It subtracts from the current output of the feedforward filter, a linear combination of previous symbol decisions.
  - The weights for this linear combination is normally determined using the LMS algorithm.