# Model Predictive Control Examples Sheet

Mark Cannon, Trinity Term 2020

Reading: Kouvaritakis & Cannon, Sections 2.1–2.6 and 3.1–3.3
or Maciejowski Chapters 2, 3, 6, 8

## Prediction equations

1. A system with model

$$x_{k+1} = Ax_k + Bu_k, \qquad y_k = Cx_k$$
$$A = \begin{bmatrix} 1 & 0.1 \\ 0 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

is to be controlled using an unconstrained predictive control law that minimizes the predicted performance cost

$$J_k = \sum_{i=0}^{N-1} \big(y_{i|k}^2 + \lambda u_{i|k}^2\big) + y_{N|k}^2, \qquad \lambda = 1.$$

(a). Show that the state predictions can be written in the form

$$\mathbf{x}_k = \mathcal{M}x_k + \mathcal{C}\mathbf{u}_k, \quad \mathbf{u}_k = \begin{bmatrix} u_{0|k} \\ \vdots \\ u_{N-1|k} \end{bmatrix}, \quad \mathbf{x}_k = \begin{bmatrix} x_{0|k} \\ \vdots \\ x_{N|k} \end{bmatrix}$$

and evaluate $\mathcal{C}$ and $\mathcal{M}$ for a horizon of $N = 3$.

(b). For $N = 3$, determine the matrices $H$, $F$ and $G$ in

$$J_k = \mathbf{u}_k^\top H \mathbf{u}_k + 2x_k^\top F^\top \mathbf{u}_k + x_k^\top G x_k.$$

(c). Give expressions for the derivatives $\partial J/\partial u_{i|k}$ for $i = 0, 1, 2$. Hence verify that the gradient of $J$ is $\nabla_{\mathbf{u}} J = 2H\mathbf{u} + 2Fx$.

2. (a). For the plant model and cost given in Question 1, show that the un-

constrained predictive control law for $N = 3$ is linear feedback:

$$u_k = Lx_k, \quad L = -\begin{bmatrix} 0.1948 & 0.1168 \end{bmatrix}.$$

Hence show that the closed-loop system is unstable. eig$(A + BL) = \{1.01,\ 1.93\}$.

(b). Write some Matlab code to evaluate $\mathcal{M}$ and $\mathcal{C}$ for any given $N$, and hence determine $H$ and $F$, for any horizon length $N$. Show that the predictive control law does not stabilize the system if $N < 6$.

## Infinite horizon cost and constraints

3. (a). Explain why the predictive control law of Question 1 necessarily stabilizes the system if the cost is minimized subject to $x_{N|k} = 0$. (Hint: what is the infinite horizon cost when this constraint is used?)

(b). How would you modify the cost of Question 1 in order to achieve closed loop stability without including the constraint $x_{N|k} = 0$? Why would this be preferable?

In practice the constraint $x_{N|k} = 0$ should be avoided because:

(i) it results in poor robustness to model errors and disturbances, and

(ii) it leads to very active predicted control sequences.

Closed-loop stability can be ensured instead by using the infinite horizon cost as described in Section 3.1 of the lecture notes (Lecture 3).

4. A predictive controller minimizes the predicted performance index:

$$J_k = \sum_{i=0}^{\infty} (y_{i|k}^2 + u_{i|k}^2)$$

at each time-step $k$ subject to input constraints: $-1 \leq u_{i|k} \leq 2$ for all $i \geq 0$. The system output $y$ is related to the control input $u$ via

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$
$$A = \begin{bmatrix} -2 & 1 \\ 0 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \end{bmatrix}.$$

(a). Why is the MPC optimization performed repeatedly, at $k = 0, 1, 2 \ldots$, instead of just once, at $k = 0$?

(b). If the mode 2 feedback law is $u_k = \begin{bmatrix} 2 & -1 \end{bmatrix} x_k$, show that

$$J_k = \sum_{i=0}^{N-1} \big(y_{i|k}^2 + u_{i|k}^2\big) + x_{N|k}^\top \begin{bmatrix} 13 & -1 \\ -1 & 2 \end{bmatrix} x_{N|k}$$

where $N$ is the length of the mode 1 prediction horizon.

(c). Show that the constraints

$$-1 \le u_{i|k} \le 2, \quad i = 0, 1, \ldots, N+1$$

ensure that the predictions satisfy $-1 \le u_{i|k} \le 2$ for all $i \ge 0$.

(d). Derive a bound on $J_{k+1}^* - J_k^*$, where $J_k^*$ is the optimal value of $J_k$. Hence show that $\sum_{k=0}^{\infty} \big(y_k^2 + u_k^2\big) \le J_0^*$ along trajectories of the closed loop system.

(e). Is the closed loop system stable? Explain your answer.

5. (a). Explain the function of terminal constraints in a model predictive control strategy for a system with input or state constraints. Define two principal properties that must be satisfied by a terminal constraint set.

(b). A discrete time system has the state space model

$$x_{k+1} = Ax_k + Bu_k, \quad A = \begin{bmatrix} 0.3 & -0.9 \\ -0.4 & -2.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0.5 \\ 1 \end{bmatrix}$$

and constraints

$$\big|[x]_1 + [x]_2\big| \le 1\,, \quad \big|[x]_1 - [x]_2\big| \le 1\,, \quad x = \begin{bmatrix} [x]_1 \\ [x]_2 \end{bmatrix}$$

(i). If the terminal feedback law is $u_k = Kx_k$, $K = \begin{bmatrix} 0.4 & 1.8 \end{bmatrix}$, show that the following set is a valid terminal constraint set

$$\big\{x : \big|[x]_1 + [x]_2\big| \le 1\,,\ \big|[x]_1 - [x]_2\big| \le 1\big\}.$$

(ii). Describe a procedure for determining the largest terminal constraint set for the case of a general feedback gain $K$.

(c). What are the main considerations that govern the choice of the prediction horizon $N$?

## Integral action and disturbances

6. The vertical position $y$ of a machine tool positioning platform is controlled by a motor which applies a vertical force $F$ to the platform (Figure 1). The platform has mass $M$ and carries a variable load of mass $m$; the unloaded weight of the platform is balanced by a counter-weight. The force $F$ is proportional to the voltage $V$ applied to the motor, so that $F = K_V V$ where $K_V$ is a fixed gain.

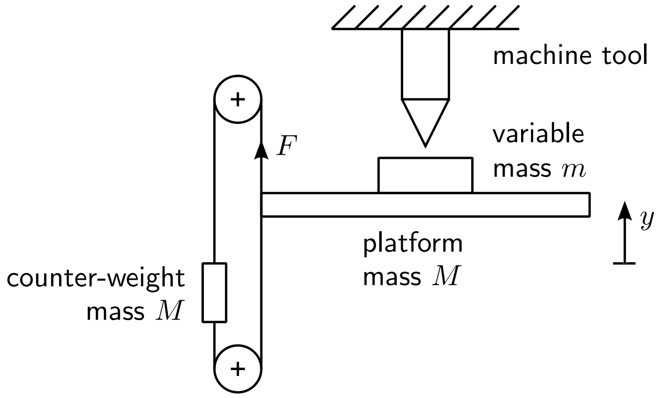


**Figure 1.** Machine tool and positioning platform

Assuming $m$ is small enough that $M + m \approx M$, the unknown load constitutes a (constant) disturbance in the discrete-time model of the system for sampling interval $T$:

$$x_{k+1} = Ax_k + Bu_k + Dw, \quad e_k = Cx_k$$

$$A = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \; B = \frac{K_V}{2M}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \; D = -\frac{g}{2M}\begin{bmatrix} T^2/2 \\ T \end{bmatrix}, \; C = \begin{bmatrix} 1 & 0 \end{bmatrix}$$

where $e$ is the error in $y$ relative to a desired steady-state height $y^0$, and

$$x_k = \begin{bmatrix} y(kT) - y^0 \\ \dot{y}(kT) \end{bmatrix}, \quad u_k = V(kT), \quad w = m.$$

(a). For the model parameters $M = 10\,\text{kg}$, $K_V = 7\,\text{N}\,\text{V}^{-1}$, $T = 0.1\,\text{s}$, the

LQ-optimal feedback law with respect to the cost

$$J_k = \sum_{i=0}^{\infty} \left(e_{k+i}^2 + \lambda u_{k+i}^2\right), \quad \lambda = 10^{-4}$$

is $u_k = Kx_k$, $K = \begin{bmatrix} -66.0 & -19.4 \end{bmatrix}$. Determine the maximum steady state error $y - y^0$ with this controller if the mass of the load is limited to the range:

$$m \leq 0.5\,\text{kg}.$$

(b). Explain how to modify the cost and model dynamics in order to obtain a stabilizing LQ-optimal controller giving zero steady-state error.

(c). The motor input voltage is subject to the constraints

$$-1 \leq V \leq 1$$

A predictive controller is to be designed based on the predicted cost:

$$J_k = \sum_{i=0}^{\infty} \left(e_{i|k}^2 + v_{i|k}^2 + \lambda u_{i|k}^2\right), \quad \lambda = 10^{-4}$$

where $v_{i+1|k} = v_{i|k} + e_{i|k}$ is the prediction of the integrated error.

(i). For a predicted input sequence with $N$ degrees of freedom, show that $J_k$ can be re-written as

$$J_k = \sum_{i=0}^{N-1} \left(e_{i|k}^2 + v_{i|k}^2 + \lambda u_{i|k}^2\right) + \|\xi_{N|k}^{\top}\|_P^2$$

and define $\xi$ and $P$. What is the implied mode 2 feedback law?

(ii). Briefly explain how the constraints on $V$ can be incorporated in a robust MPC strategy for this system (i.e. for all values of $m$ in the range $m \leq 0.5\,\text{kg}$).

7. Assume that, for the given initial condition $x(0)$, the optimization of $J$ subject to the robust constraints determined in Question 6 is initially feasible.

Will the online optimization remain feasible at all future sampling times? What can be said about the steady-state value of $y$? Will the optimal value of the cost necessarily decrease monotonically, and what can be concluded about the convergence of the state $x_k$ to zero in closed-loop operation?

8. A production planning problem involves optimizing the quantity $u$ of stock manufactured in each week. The quantity $x$ of stock that remains unsold at the start of week $k+1$ is given by

$$x_{k+1} = x_k + u_k - w_k, \quad k = 0, 1, \dots$$

where the quantity $w_k$ that is sold in each week is unknown in advance but is expected to be equal to a known constant $\hat{w}$. Limits on storage and manufacturing capacities imply that $x$ and $u$ can only take values in the intervals

$$0 \le x_k \le X, \quad 0 \le u_k \le U.$$

The desired level of stock in storage is $x^*$, and the planned values $u_{0|k}, u_{1|k}, \dots$ are to be optimized at time $k$ given a measurement of the value of $x_k$ by minimizing a cost

$$J_k = \sum_{i=0}^{\infty} e_{i|k}^2 , \quad e_{i|k} = x_{i|k} - x^*.$$

(a). What are the advantages of using a receding horizon control strategy in this application instead of an open-loop control sequence computed at $k = 0$?

(b). Assume that $w_k = \hat{w}$ for all $k = 0, 1, \dots$

(i). Show that the unconstrained optimal control law is $u_k = \hat{w} - e_k$.

(ii). Show that, for a mode 1 horizon of $N$, the infinite horizon cost can be expressed

$$J_k = \sum_{i=0}^{N} e_{i|k}^2 ,$$

and state the corresponding mode 2 feedback law.

(iii). Show that constraints are satisfied over an infinite horizon if $0 \leq x_{i|k} \leq X$ and $0 \leq u_{i|k} \leq U$ for $0 \leq i \leq N-1$, and

$$\max\{0, \hat{w} + x^* - U\} \leq x_{N|k} \leq \min\{X, \hat{w} + x^*\}.$$

What assumptions on $\hat{w}$, $x^*$, $U$ and $X$ are needed?.

(c). Assume now that the future value of $w$ is unknown and may take any value in an interval: $0 \leq w_k \leq W$. Suggest how to express the planned sequence $u_{0|k}, u_{1|k}, u_{2|k}$ in terms of the free variables in the receding horizon optimization problem, and justify your answer by determining the predictions $e_{1|k}, e_{2|k}, e_{3|k}$.

## Some answers

1. (b). $H = \begin{bmatrix} 1.025 & 0.0075 & 0 \\ 0.0075 & 1.0025 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $F = \begin{bmatrix} 0.2 & 0.12 \\ 0.05 & 0.035 \\ 0 & 0 \end{bmatrix}$, $G = \begin{bmatrix} 4 & 1.1 \\ 1.1 & 0.59 \end{bmatrix}$

2. (a). Closed loop poles for $N = 3$: eig$(A + BL) = 1.01, 1.93$

   (b).

| $N$ | 4 | 5 | 6 | 7 |
|---|---|---|---|---|
| eig$(A + BL)$ | $1.03, 1.69$ | $1.11 \pm 0.15$i | $0.86 \pm 0.10$i | 0.95,0.58 |

3. (b). In $J_k$, replace $y^2_{N|k}$ with $\|x_{N|k}\|^2_P$, $P = \begin{bmatrix} 22.46 & 4.098 \\ 4.098 & 12.79 \end{bmatrix}$

4. (d). $J^*_{k+1} - J^*_k \leq -\big(y^2_k + u^2_k\big)$

   (e). $x = 0$ is locally asymptotically stable

6. (a). $|y - y^0| \leq 0.0106\,$m in steady state

   (c). $\xi$: augmented predicted state, $\xi = \begin{bmatrix} x & v \end{bmatrix}^\top$, $P$: the solution of

   $$P - \left( \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right)^\top P \left( \begin{bmatrix} A & 0 \\ C & I \end{bmatrix} + \begin{bmatrix} B \\ 0 \end{bmatrix} K_\xi \right) = \begin{bmatrix} C^\top C & 0 \\ 0 & 1 \end{bmatrix} + \lambda K_\xi^\top K_\xi,$$

   Mode 2 feedback law: $u = K_\xi x$, e.g. LQ-optimal $K_\xi = -\begin{bmatrix} 201.4 & 29.6 & 48.2 \end{bmatrix}$

8. (c). $u_{i|k} = \hat{w} - e_{i|k} + c_{i|k}$, where $c_{i|k} =$ for $i = 0, \ldots, N - 1$ are decision variables, and $c_{i|k} = 0$ for $i \geq N$