2A-SG8-MPC
Model Predictive Control

CentraleSupélec

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CHAPTER 5

CONSTRAINTS:
THE MAIN ADDED VALUE OF MPC

5.1 INTRODUCTION MOTIVATION EXAMPLE

EXERCICE 5.1

Let us consider the following system:

$$\begin{cases} x(k+1) = \begin{pmatrix} 1.1 & 2 \\ 0 & 0.95 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 0.0787 \end{pmatrix} u(k) \end{cases}$$

The objective is to bring the system to the origin $x = 0$.

We suppose here that there exist contraints on the input :

$$u_{min} \leq u(k) \leq u_{max}$$

The objective criterion is the classic one:

$$J^{N_p} = \sum_{i=1}^{N_p} \left(x^\intercal(k+i|k) Q x(k+i|k) + \boldsymbol{\lambda} u^2(k+i-1|k) \right)$$

EXERCICE 5.1
PROPOSITION 1

- Proposition 1 : we do not consider the constraints in the optimization problem

  - We get : $U^{*,unc}(K|k)$
  - We pick up the first element of the sequence : $u^{*,unc}(k|k)$
  - We saturate the input :

$$u^{unc}(k) = \max(u_{min}\,; \min(u^{*,unc}(k|k)\,; u_{max})),$$

EXERCICE 5.1
PROPOSITION 2

- Proposition 2 : we use the weighting term λ to reduce the value of the control to apply (this is called *detuning*)

$$J^{N_p} = \sum_{i=1}^{N_p} \left(x^\intercal(k+i|k) Q x(k+i|k) + \boldsymbol{\lambda} u^2(k+i-1|k) \right)$$

- We increase the value of λ until $U^{*,\lambda}(K|k)$ is in the bound

- We apply the control

$$u^\lambda(k) = U^{*,\lambda}(k|k)$$

EXERCICE 5.1
PROPOSITION 3

- Proposition 3 : we integrate directly the constraints in the optimization solver

$$U^*(K|k) = \arg\min_{U(K|k)} \tilde{J}^{N_p}\big(x(k), U(K|k)\big)$$
$$subject\ to\ \forall i = 1, Np,\ \ h_u\big(u(k+i-1|k)\big) \leq 0$$

- We apply the first element

$$u^*(k) = U^*(k|k)$$

EXERCICE 5.1
THE RESULTS (1)

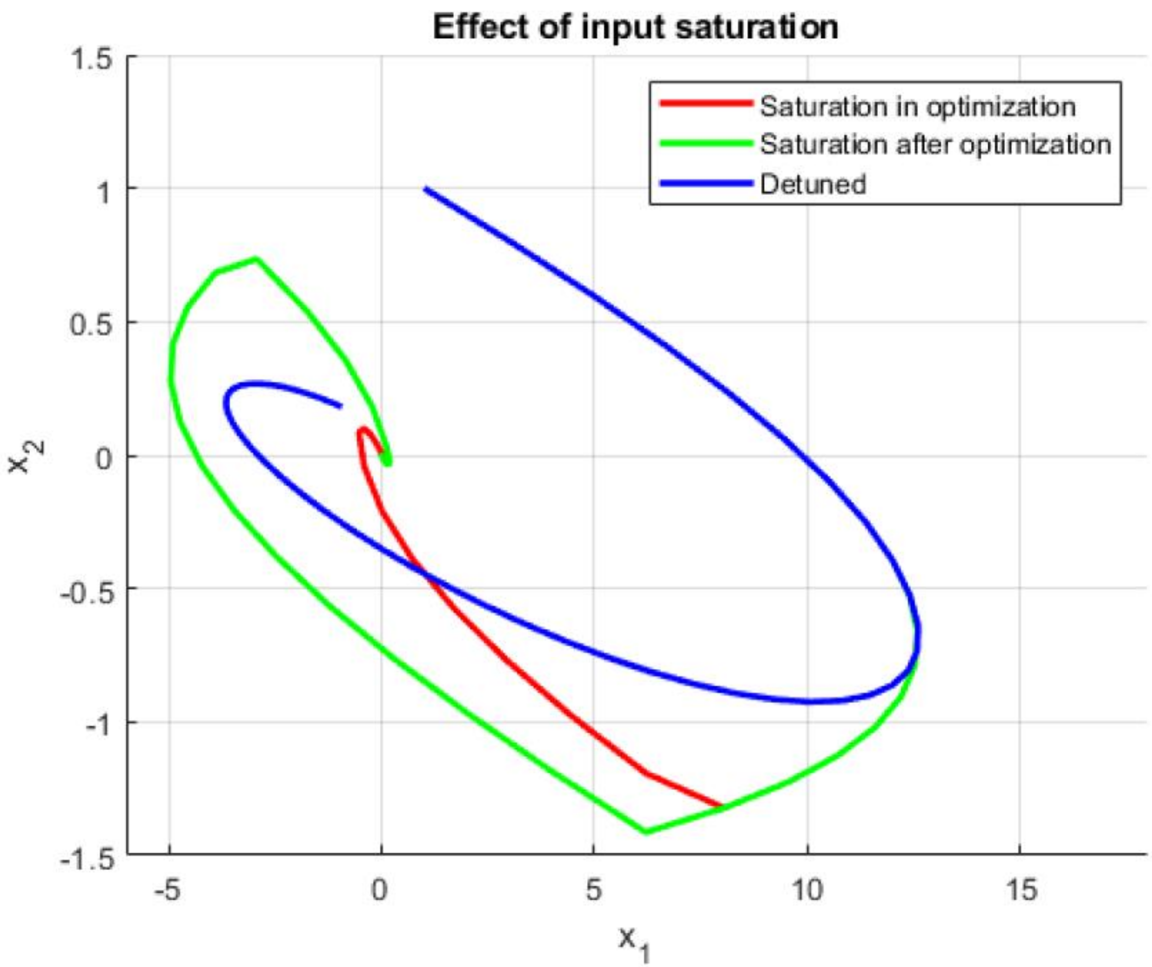

FIG: State trajectory 状态轨迹

EXERCICE 5.1
THE RESULTS (2)

Norm

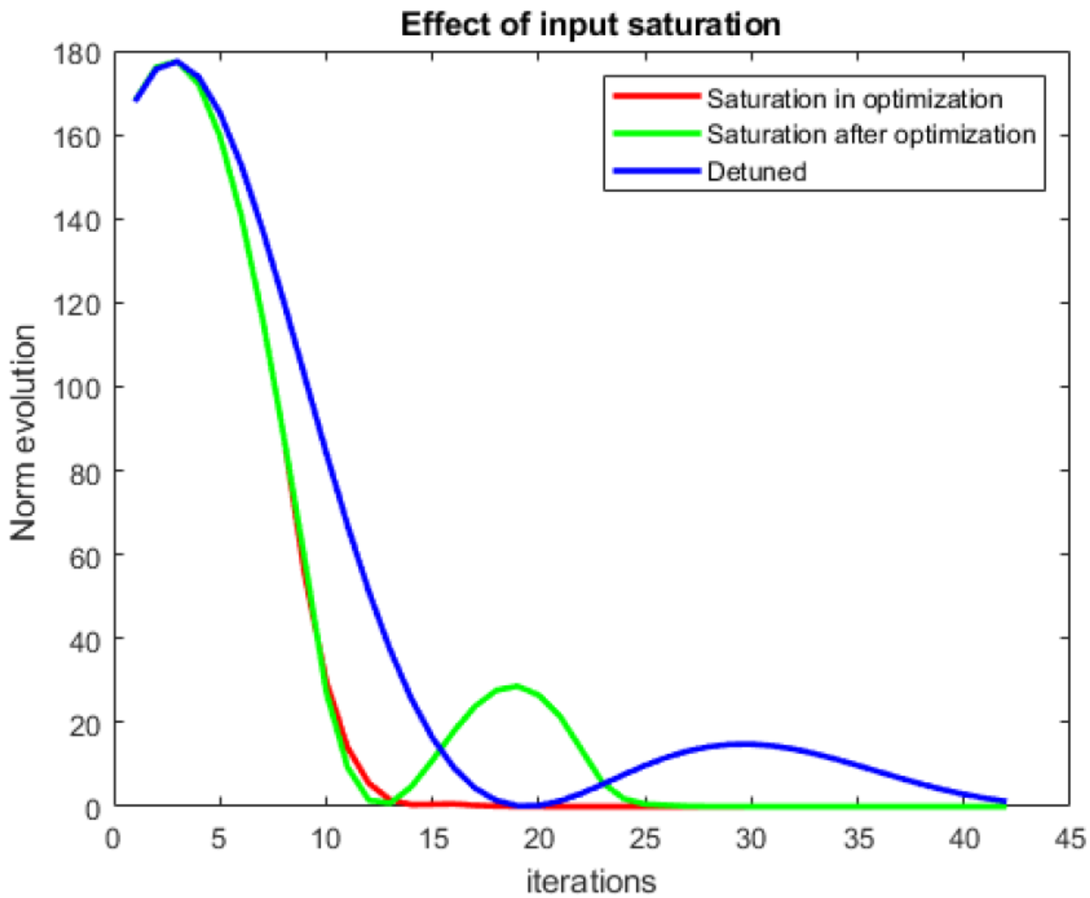

FIG: Norm of the state

5.2 DIFFERENT WAYS TO DEAL WITH CONSTRAINTS

AMONG THE 3 PROPOSITIONS THE BIG MATCH

Proposition	Advantages	Drawbacks
Saturation after optimization	Explicite solution	Suboptimal Analysis not possible if constraints
Detuning	Explicite solution Analysis possible	Suboptimal
Constraints in the optimization	Optimal	Solver is required Analysis not possible

WHICH KIND OF CONSTRAINTS?

约束的种类：输入的上下界，输入的速率/快慢，输出/状态

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Constraints on the input bounds :

$$\forall i = 1, Np, \qquad u_{min} \leq u(k+i-1|k) \leq \mathrm{u_{max}}$$

Physcial constraints, Saturation of the actuators, power limitations, …

Easy ones ! But have to be fulfilled

约束：
输入界限，输入速率，状态/输出约束

Constraints on the input rates :

$$\forall i = 1, Np, \qquad \Delta u_{min} \leq \Delta u(k+i-1|k) \leq \Delta \mathrm{u_{max}}$$

Same : easy ones ! But have to be fulfilled

Constraints on the state (output):

$$\forall i = 1, Np, \qquad x(k+i|k) \in \boldsymbol{X}$$

Comfort, quality of service, …

can be relaxed! => **soft constraints**

Safety, Security

cannot be relaxed !!! => ***Feasibility issues ???***

Guarantees ???

SOFT CONSTRAINTS RELAXATION
AN EXAMPLE

ε: slack variable

$$\min_{U,\varepsilon} \; C_u^T U + C_\varepsilon^T \varepsilon$$

$$y(k+1) \leq y_{max}$$

$$Fx + HU \leq y_{max}$$

$$-\varepsilon + HU \leq y_{max} - Fx$$

N_p constraints

$$\varepsilon \geq 0 \qquad \varepsilon \in \mathbb{R}^{N_p}$$

$$C_\varepsilon \gg C_u$$

SOFT CONSTRAINTS RELAXATION
AN EXAMPLE

+ LINPROG.

$$\min_z c^T z$$

$$Az \leq b$$

$$Z = \begin{pmatrix} U \\ \varepsilon \end{pmatrix} \text{ : variable optm.}$$

$$C = \begin{pmatrix} C_u \\ C_\varepsilon \end{pmatrix}$$

$$\begin{cases} U_{min} \leq U \leq U_{max} \\ 0 \leq \varepsilon \end{cases}$$

$$\underbrace{-\varepsilon + HU}\leq Y_{max} - Fx$$

$$\underbrace{\begin{pmatrix} H & -I \end{pmatrix}}_{A} \begin{pmatrix} U \\ \varepsilon \end{pmatrix} \leq \underbrace{Y_{max} - Fx}_{b}$$

$$lb = \begin{pmatrix} U_{min} \\ \hline 0 \end{pmatrix} I_{np}$$

$$Ub = \begin{pmatrix} U_{max} \\ \hline inf \end{pmatrix}$$

$$lb \leq$$

MPC : REMEMBER THE IMPORTANT REMARK 2 !
SEE CHAPTER 2, AROUND SLIDE 26 …

Remark 2

$$U^*(K|k) = \arg\min_{U(K|k)} \tilde{J}^{N_p}\big(x(k), U(K|k)\big)$$

Only the first element of this sequence is applied:

$$u(k) = u^*(k|k) = \boldsymbol{\pi}(\boldsymbol{x}(\boldsymbol{k}))$$

MPC = nonlinear state feedback

How can we provide stability guarantees ????

5.3 CONSTRAINTS AND STABILITY GUARANTEES

DLQR AND CONSTRAINTS
THE PICTURE TO REMEMBER (SLIDE FROM MARK CANNON)

stabilizing linear controller satisfies constraints

= Limit of DLQR !

FEASABILITY AND STABILITY 可解性和稳定性
THE MAIN IDEAS

$$\min_{U} \sum_{i=k}^{k+N_p-1} l_i(x_i, u_i) \boxed{+ \mathbf{l_F}(\mathbf{x_{N_p}})}$$

Terminal Cost

subject to :

$$U = \{u_0, u_1, \dots, u_{N_p-1}\}$$

$$x_{i+1} = f(x_i, u_i, i)$$

$$x_i \in \boxed{X_i}, \quad u_i \in \boxed{U_i}$$

$$x_0 = x(k)$$

$$\boxed{\boldsymbol{x_{k+N_p} \in X_F}}$$

Input Sequence

Prediction Model

State and Input Constraints

(assumption: polytopic sets)

Measurement = Initialization

Terminal Contraint

$\mathbf{l_F}(\cdot)$ and $\boldsymbol{X_F}$ mimic an infinite horizon

$\boldsymbol{X_F}$: feasible set

3 THEORETICAL NOTIONS ARE REQUIRED

- Invariant Set – Control Invariant Set

定集
非线性系统的稳定性
迭代可解性

- Stability tools for nonlinear systems

- Recursive feasability

5.3.1 INVARIANCE

INVARIANCE

POSITIVE INVARIANT SET

For an autonomous system $x(k+1) = f(x(k))$,
O is said to be a positive invariant set if:

$$\forall x(k) \in O \Rightarrow x(k+1) \in O$$

- *Maximum positive invariant set*: the largest one
- **Important remark: if the invariant set fulfills the constraints, the trajectory will never violate the constraints**

5.3.2 STABILITY TOOLS FOR NONLINEAR (DISCRETE-TIME) SYSTEMS

STABILITY OF AN EQUILIBRIUM POINT

If you are close, you stay close

Stability

An equilibrium point x_e is stable, if and only if

$$\forall\epsilon > 0, \ \exists\delta_\epsilon \text{ such that } |x(0) - x\_e| \leq \delta_\epsilon \Rightarrow \forall k, |x(k) - x_e| \leq \epsilon$$

稳定平衡点

STABILITY OF AN EQUILIBRIUM POINT

If you are close, you stay close and you go back to the equilibrium point

Asymptotic Stability

An equilibrium point x_e is asymptotically stable, if and only if

A.s.渐进稳定

- x_e is stable
- $\lim_{k\to\infty} |x(k) - x_e| = 0$ *(= attractivity)*

Most common usage [edit]

A *positive-definite function* of a real variable *x* is a complex-valued function $f : \mathbb{R} \to \mathbb{C}$ such that for any real numbers $x_1, \ldots, x_n$ the $n \times n$ matrix

$$A = (a_{ij})_{i,j=1}^{n}\,, \quad a_{ij} = f(x_i - x_j)$$

is positive *semi*-definite (which requires *A* to be Hermitian; therefore $f(-x)$ is the complex conjugate of $f(x)$).

In particular, it is necessary (but not sufficient) that

$$f(0) \geq 0\,, \quad |f(x)| \leq f(0)$$

(these inequalities follow from the condition for $n = 1, 2$.)

A function is *negative definite* if the inequality is reversed. A function is *semidefinite* if the strong inequality is replaced with a weak (≤, ≥ 0).

STABILITY OF AN EQUILIBRIUM POINT
LYAPUNOV FUNCTION

Lyapunov Function Lyapunov函数

Consider the origin $x = 0$ as the equilibrium point. Let $\Omega \subset \mathbb{R}^n$ be a closed and bounded set containing the origin. $\alpha: \mathbb{R}^n \to \mathbb{R}^+$ is a continuous positive function.

A continuous function $V: \mathbb{R}^n \to \mathbb{R}$, finite for all $x \in \Omega$ and such that:

- $V(0) = 0$
- $\forall x \in \Omega - \{0\},\ V(x) > 0$
- Along the trajectory of the dynamical system $x(k+1) = f(x(k))$,

$$\forall x \in \Omega\,, \qquad V\big(f(x)\big) - V(x) \leq -\alpha(x)$$

Is called a Lyapunov function

STABILITY OF AN EQUILIBRIUM POINT
LYAPUNOV – MY SUPER HERO

Theorem: Lyapunov function

1. If the system $x(k+1) = f(x(k))$, with 0 as an equilibrium point admits a Lyapunov function, then 0 is stable.

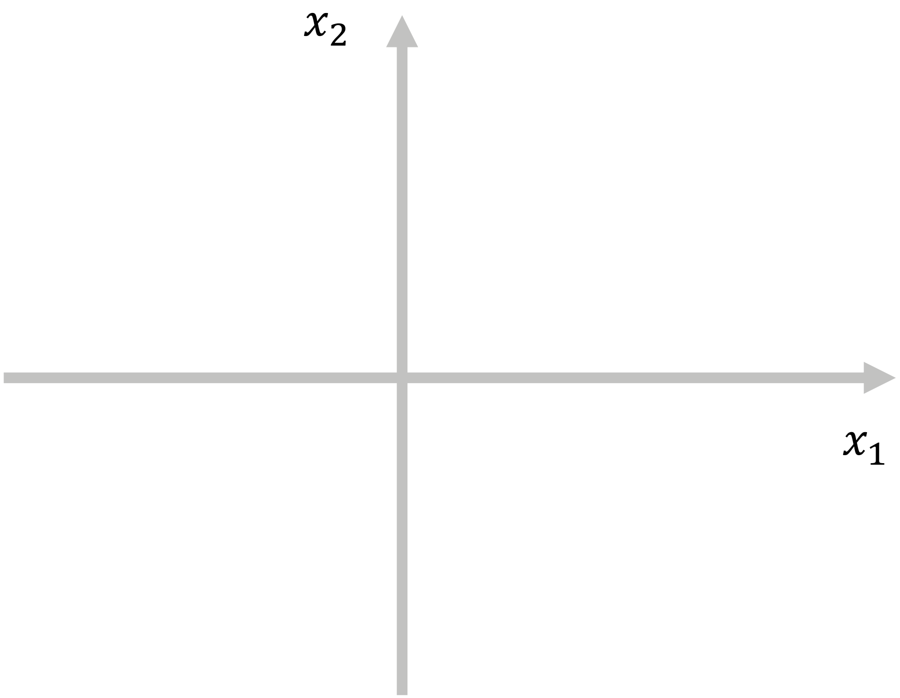
2. Moreover if $\alpha\colon \mathbb{R}^n \to \mathbb{R}^+$ is a continuous positive **definite** function, then 0 is asymptocitally stable

3. (The magic : converse theorem). If 0 is asymptotically stable, then there exists a Lyapunov function!

The challenge is of course to find the Lyapunov function!

稳定性

STABILITY OF AN EQUILIBRIUM POINT
LYAPUNOV FUNCTION – THE EASY GRAPH



LYAPUNOV FUNCTION FOR LINEAR SYSTEM? (SEE COURSE 3) 线性系统的Lyapunov方程

Let us consider a linear system: $x(k+1) = Ax(k) + Bu(k)$

Remember the $\boxed{\text{dLQR problem: } J^{\infty} = \frac{1}{2}\sum_{i=k}^{\infty}\left(x_i^{\intercal} Q x_i + u_i^{\intercal} R u_i\right)}$

dLQR Control

$$u(k) = -Kx(k)$$
$$K = (R + B^{\intercal}PB)^{-1}B^{\intercal}PA$$
$$P = A^{\intercal}(P - PB(R + B^{\intercal}PB)^{-1}B^{\intercal}P)A + Q$$

Discrete Riccati Equation

$$P = A^{\intercal}(P - PB(R + B^{\intercal}PB)^{-1}B^{\intercal}P)A + Q$$

LYAPUNOV FUNCTION FOR LINEAR SYSTEM?

Discrete Riccati Equation

$$P = A^\intercal (P - PB(R + B^\intercal PB)^{-1} B^\intercal P)A + Q$$

We have seen that *under good conditions*, there is an unique solution of the discrete Riccati Equation.*

The additional info: P is a definite positive matrix

The magic: $\boldsymbol{V(x) = x^\intercal P x}$ **is a Lyapunov function!**

V is a convex function … and $\{x \in \mathbb{R}^n : V(x) \leq \alpha\}$ is a convex set

The magic of the magic: $\{x \in \mathbb{R}^n : V(x) \leq \alpha\}$ **is positive invariant set**

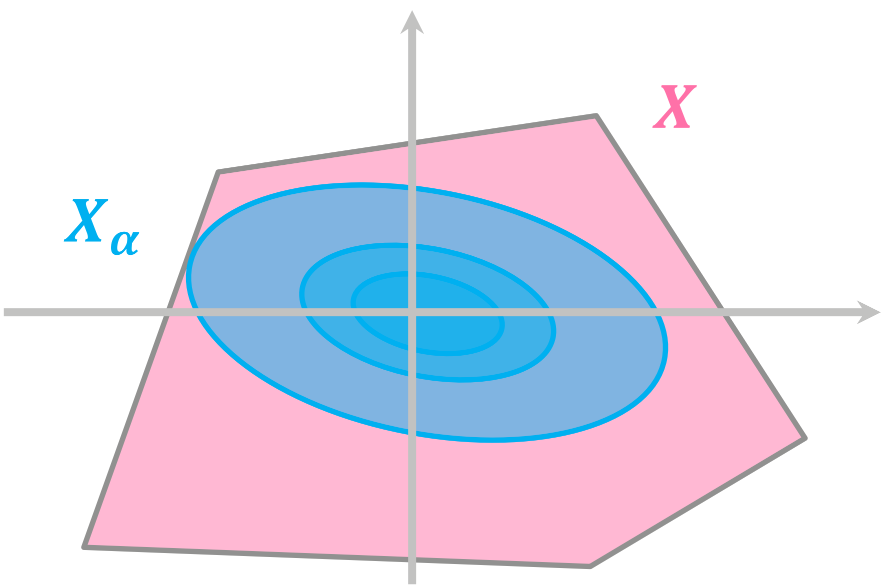
* $Q = C^\intercal C$, (A, B) stabilizable and (A, C) observable.

DLQR AND CONSTRAINTS
WHERE IS MY INVARIANT SET?

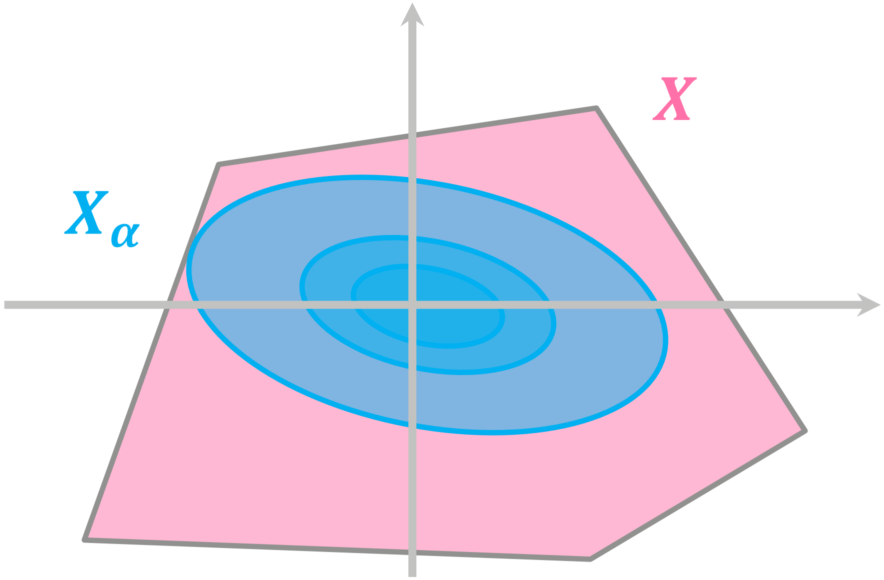
Let us consider a linear system: $x(k+1) = Ax(k) + Bu(k)$

We have constraints on state $x \in X$ and input $u \in U$

1. Choice of Q, R =>dLQR : we get K **and** P
2. $u = -Kx$, $V = \mathrm{x}^\intercal P x$
3. Looking for the best α to define $\mathrm{X}_\alpha = \{x: V(x) \leq \alpha\}$
4. **What is the best???**



DLQR AND CONSTRAINTS
WHERE IS MY INVARIANT SET? (2)



X_α: is the feasible set X_F we were looking for!

- It depends on Q and R !
- There exist technics to compute it! (out of this course)
- ***Do not forger it is a feasible positive invariant set!!!***

5.3.3 RECURSIVE FEASABILITY

RECURSIVE FEASABILITY

$$\min_U \sum_{i=k}^{k+N_p-1} l_i(x_i,u_i) + \mathbf{l_F}(\mathbf{x_{N_p}})$$

subject to :

$U=\{u_0,u_1,\ldots,u_{N_p-1}\}$	Input Sequence
$x_{i+1}=f(x_i,u_i,i)$	Prediction Model
$x_i \in X_i, \quad u_i \in U_i$	State and Input Constraints *(assumption: polytopic sets)*
$x_0=x(k)$	Measurement = Initialization
$x_{k+N_p} \in X_F$	Terminal Contraint

DEFINITION

递归可解性！！

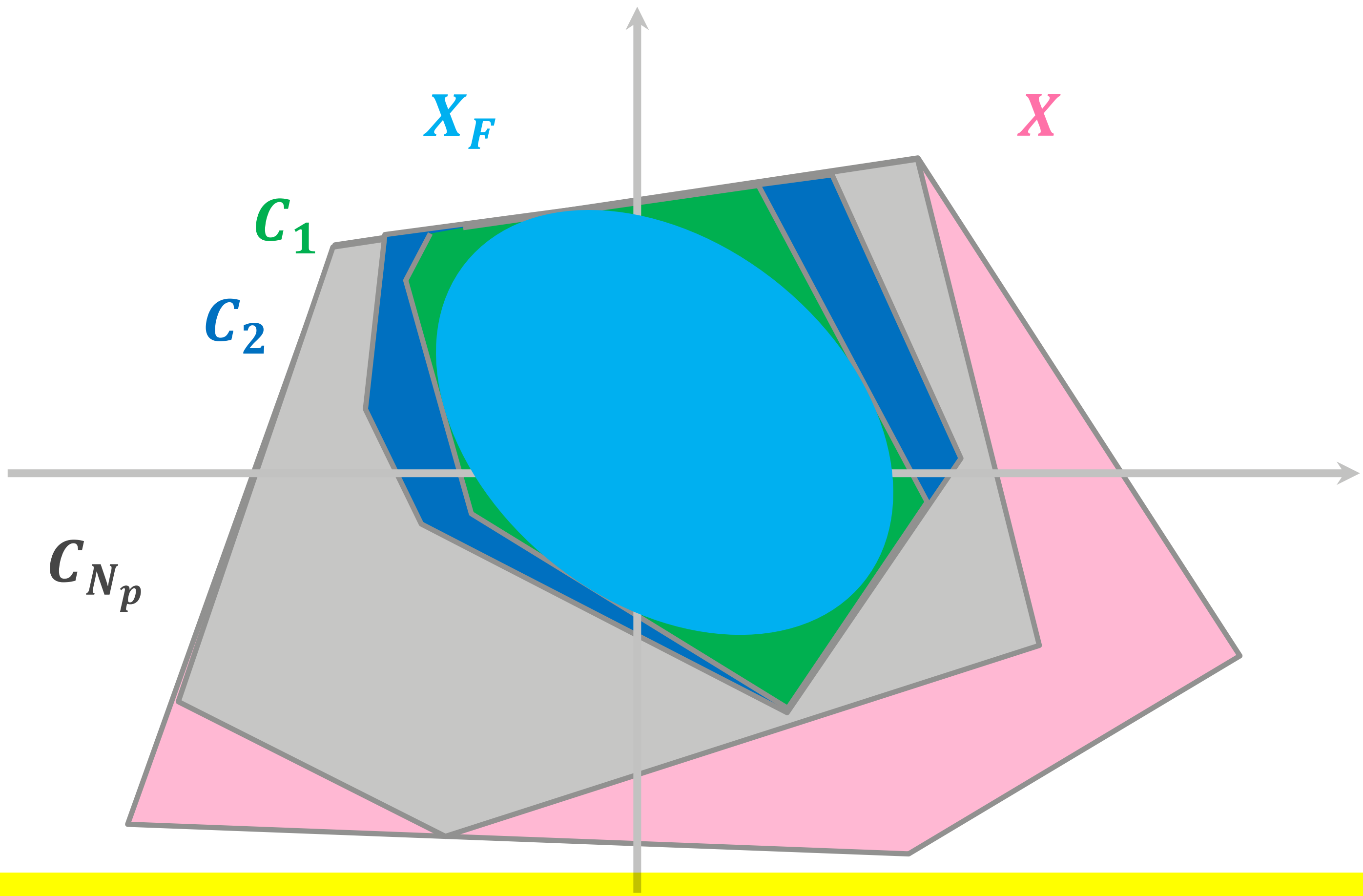
If there is a solution at time k, there will be a solution at time $k+1$

- Suppose at time k, starting from $x(k)$, the MPC has a solution. We get, the optimal input sequence $\{u(k|k), \ \ldots u(k+N_p-1|k)\}$
- In the constraint, we imposed that $x\left(k+N_p|k\right) \in X_F$
- At time k+1, what could be a feasible input sequence ?

Terminal constraint provides recursive feasability

RECURSIVE FEASABILITY
THE IMPACT OF N_p

C_i: There exists a feasible input sequence of length i



If we increase the value of N_p we increase the control set with guarantees

There exist technics to compute them (out of the course)
Be carefull with the dépendance of X_F

5.3.4 ASYMPTOTIC STABILITY OF OUR MPC

ASYMPTOTIC STABILITY OF MPC
THE 3 ASSUMPTIONS

1. The stage cost $l(x,u)$ is positive definite

   Already checked with Q and R !

2. Terminal set X_F is **positive invariant** under a local control law $\pi_{loc}(x)$

   Already checked with $\boldsymbol{u = -Kx}$, from the dLQR approach

3. Terminal cost l_F is a Lyapunov function in the terminal set X_F :

$$\forall x \in X_F, \qquad l_F(x(k+1)) - l_F\big(x(k)\big) \leq -l(x(k), \pi_{loc}(k))$$

So … just one last point to define !

Under these 3 assumptions, the closed-loop system under the MPC control law $\pi(x)$

$$x(k+1) = Ax(k) + B\pi(x)$$

is asymptotically stable

WHERE IS MY LYAPUNOV FUNCTION?

At time k

$$J^*\big(x(k)\big) = \sum_{i=k}^{k+N_p-1} l(x_i^*, u_i^*) \ + \mathbf{l_F}(\boldsymbol{x}_{\boldsymbol{N_p}}^*)$$

At time $k+1$: we have $\left\{u_1^*, u_2^*, \dots, -Kx_{N_p}^*\right\}$: **feasible** and **suboptimal**

$$\begin{aligned} J^*\big(x(k+1)\big) &\leq \sum_{i=k+\mathbf{1}}^{k+N_p-1} l(x_i^*, u_i^*) + l(\boldsymbol{x}_{\boldsymbol{N_p}}^*, -\boldsymbol{K}x_{N_p}^*) + \boldsymbol{l_F}\left((\boldsymbol{A}-\boldsymbol{BK})\,\boldsymbol{x}_{\boldsymbol{N_p}}^*\right) \\ &= J^*\big(x(k)\big) - l(x_k, u_0^*) - l_F\left(x_{N_p}^*\right) + +l(\boldsymbol{x}_{\boldsymbol{N_p}}^*, -\boldsymbol{K}x_{N_p}^*) + \boldsymbol{l_F}\left((\boldsymbol{A}-\boldsymbol{BK})\,\boldsymbol{x}_{\boldsymbol{N_p}}^*\right) \\ &= J^*\big(x(k)\big) - l(x_k, u_0^*) \underbrace{+\boldsymbol{l_F}\left((\boldsymbol{A}-\boldsymbol{BK})\,\boldsymbol{x}_{\boldsymbol{N_p}}^*\right) - l_F\left(x_{N_p}^*\right) + l(\boldsymbol{x}_{\boldsymbol{N_p}}^*, -\boldsymbol{K}x_{N_p}^*)}_{\leq\, 0,\ \text{by asumption 3}} \end{aligned}$$

Then we have:

$$J^*\big(x(k+1)\big) - J^*\big(x(k)\big) \leq -l(x_k, u_0^*)$$

J^* is a Lyapunov function! The closed-loop is AS !

WHERE IS MY LYAPUNOV FUNCTION?
A WORD ON ASSUMPTION 3 – IN PRACTICE

What is our final cost $\mathbf{l}_{\mathbf{F}}\left(\boldsymbol{x}_{N_p}\right)$?

$$\mathbf{l}_{\mathbf{F}}\left(\boldsymbol{x}_{N_p}\right) = x_{N_p}^{\intercal} P x_{N_p}$$

Discrete Riccati Equation

$$P = A^{\intercal}(P - PB(R + B^{\intercal}PB)^{-1}B^{\intercal}P)A + Q$$

It still depends on Q and R !

5.4 CONCLUSION

2A-SG8-MPC-CHAPTER 5
QUESTIONS?

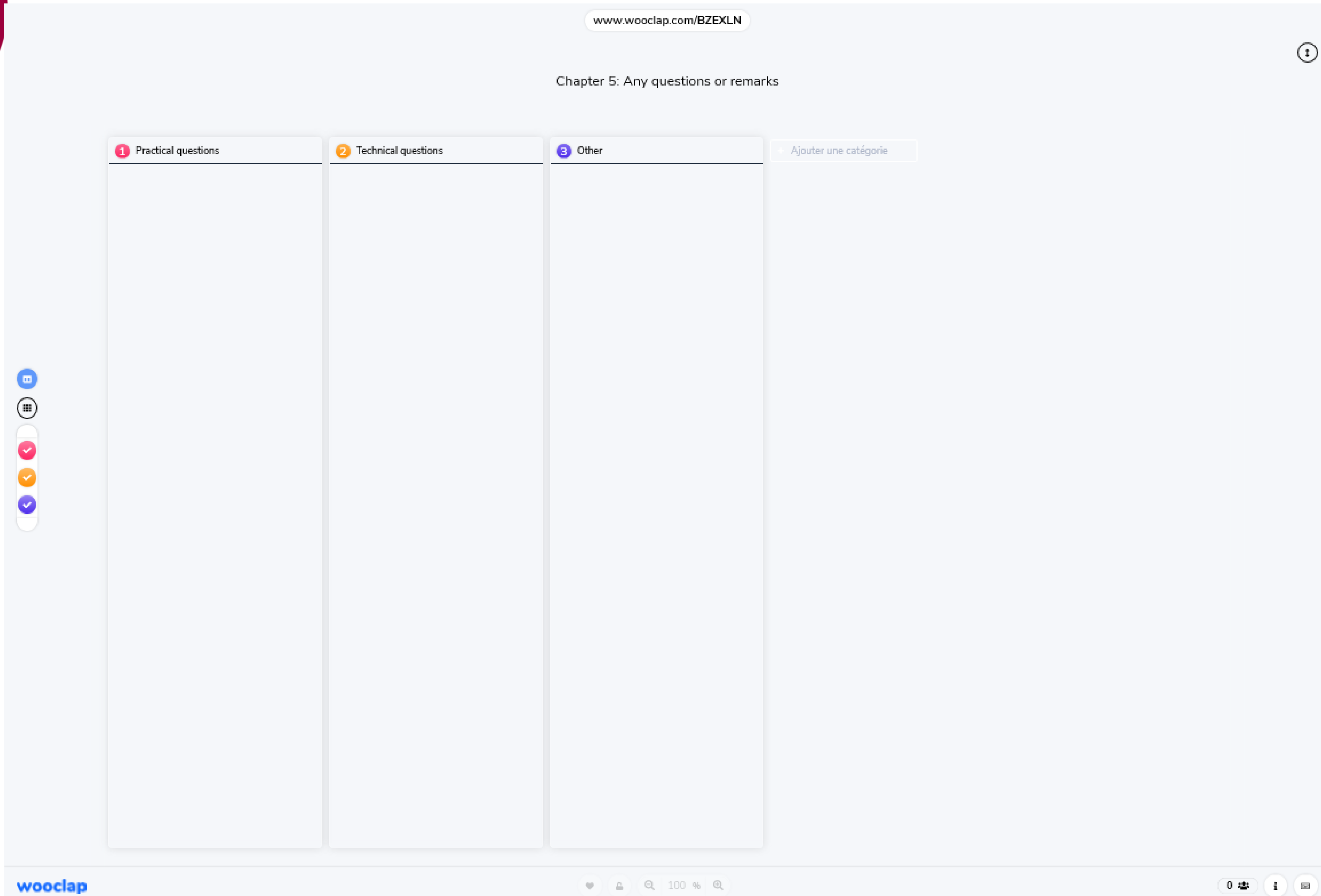
Expected skills

- How to deal with constraints in practice
  - Hard constraints
  - Soft constraints and relaxation
- The main ideas of the fundamentals to study more advanced MPC
  - The importance of dLQR
  - Recursive feasability + Lyapunov function

Check your knowledge by reading/looking at the slides of Mark Cannon, Available on Edunao (maybe teams too)

Prepare your questions for nex time!

2A-SG8-MPC-CHAPTER 5 QUESTIONS?



2A-SG8-MPC-CHAPTER 5
THE RYTHM