2A-SG8-MPC
Model Predictive Control

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CHAPTER 4
WHEN STUDENTS WORK AND THE TEACHER HAS REST.

4.1 EXERCICE 1 INTRODUCTION TO REFERENCE TRACKING

EXERCICE 4.1
PART 1

Let us consider the following system:

$$\begin{cases} x(k+1) = \begin{pmatrix} 0 & 1 \\ -1 & 0.5 \end{pmatrix} x(k) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} u(k) \\ y(k) = \begin{pmatrix} 1 & 1 \end{pmatrix} x(k) \end{cases}$$

The objective is to find the control that brings the output to $w = 1$. e suppose that the state $x(k)$ is known.

To do that, we would like to try to adopt a MPC approach, considering the following criterion:

$$J_1\big(u(k|k)\big) = \big(y(k+1|k) - 1\big)^2$$

4.1.1. Give an interpretation of this criterion

4.1.2. Analyse the stability of the resulting optimal control

1) $J_1(u(k)) = (y(k+1) - 1)^2$

$$= (CA\,x(k) + CB\,u(k) - 1)^2 \qquad CB = 1$$

$$\boxed{u^* = 1 - CA\,x(k)} \qquad CA = \begin{pmatrix} -1 & 3/2 \end{pmatrix}$$

in C.L :

$$x(k+1) = (A - BCA)\,x(k) + B.1$$

$$= \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix} x(k) + B.1$$

CL-stability: 2 eigenvalues $\rightarrow \lambda_1 = 0 \; ; \; \lambda_2 = 1$

not AS !

EXERCICE 4.1
PART 2

We still would like to try to adopt a MPC approach, considering the following criterion:

$$J_2\big(u(k|k)\big) = (y(k+1|k) - 1)^2 + \lambda u(k|k)^2$$

4.1.3. Find the analytic expression of the optimal control, that depends on x and λ. u* = (1–CAx(k))/(1+lambda)

4.1.4. Analyse the stability of the resulting optimal control Lambda =< 2, a.s??

4.1.5. What is the steady-state value of the output, when applying this control. 与lambda有关

4.1.6 Implement this control in Matlab to illustrate the phenomenon

$$Ax(k) + \frac{B}{\lambda+1}(1 - CAx(k))$$

$$= \left(A - \frac{BCA}{\lambda+1}\right)x(k) + \frac{B}{\lambda+1}.$$

$$\begin{pmatrix}0&0\\1&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0.5\end{pmatrix} = \begin{pmatrix}0&0\\-1&1.5\end{pmatrix}$$

$$A - \frac{BCA}{\lambda+1} = \begin{pmatrix}0&1\\1&0.5\end{pmatrix} - \frac{\begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1&1\end{pmatrix}\begin{pmatrix}0&1\\-1&0.5\end{pmatrix}}{\lambda+1}$$

$$= \begin{pmatrix}0&1\\-1+\frac{1}{\lambda+1}&0.5-\frac{1.5}{\lambda+1}\end{pmatrix} = \begin{pmatrix}0&1\\\frac{-\lambda}{\lambda+1}&\frac{0.5\lambda-1}{\lambda+1}\end{pmatrix}$$

MPC FOR TRACKING
INTRODUCTION TO INCREMENTAL MODEL

In order to avoid any steady-state error, one idea is to add an integral action.

This can be done, considering two things :

- the incremental input as a new input : $\Delta u(k) = u(k) - u(k-1)$
- an augmented state : $\bar{x}(k) = \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix}$

- What is the expression of the augmented state space model ?

INTRODUCTION TO INCREMENTAL MODEL

In order to avoid any steady-state error, one idea is to add an integral action. This can be done, considering two things :

$u(k) = \Delta u(k) + u(k-1)$

- the incremental input as a new input : $\Delta u(k) = u(k) - u(k-1)$
- an augmented state : $\bar{x}(k) = \begin{pmatrix} x(k) \\ u(k-1) \end{pmatrix}$ $\quad x(k+1) = A\,x(k) + B\,u(k)$

$y = Cx$

- What is the expression of the augmented state space model ?

$$\bar{x}(k+1) = \left(\begin{array}{c|c} A & B \\ \hline 0 & I \end{array}\right) \bar{x}(k) + \left(\begin{array}{c} B \\ \hline I \end{array}\right) \Delta u(k)$$

$$y(k) = \left(\begin{array}{c|c} C & 0 \end{array}\right) \bar{x}(k)$$

EXERCICE 4.1
PART 3

We still would like to try to adopt a MPC approach, considering the following criterion:

$$J_3\big(u(k|k)\big) = (y(k+1|k) - 1)^2 + \lambda \Delta u(k|k)^2$$

4.1.7. Find the analytic expression of the optimal incremental control, that depends on x and λ.

4.1.8. Implement the closed-loop behavior in Matlab, and have a look on the steady state error.

4.1.9. Using Matlab, have a discussion on the stability of the closed-loop model for different values of λ.

MPC FOR TRACKING
THE OBJECTIVE FUNCTION

We now suppose that the reference to track is not constant but is time variable which is known over the prediction horizon: $W(K+1|k)$

Using the incremental model, the objective criterion to consider is:

Penalization

$$\sum_{i=1}^{N_p}\left((y(k+i|k)-w(k+i|k)\right)^{\intercal}\left((y(k+i|k)-w(k+i|k)\right)+\lambda\Delta u(k+i-1|k)^{\intercal}\Delta u(k+i-1|k)$$

What is the optimal incremental input sequence ?

Give the block diagram of the resulting control structure ?

→ Construction by recurrence:

$$y(k+1|k) = \bar{C}\,\bar{x}(k+1|k)$$

$$= \bar{C}\bar{A}\,\bar{x}(k) + \bar{C}\bar{B}\Delta u(k)$$

$$\vdots$$

$$Y(k+1|k) = \begin{pmatrix} \bar{C}\bar{A} \\ \bar{C}\bar{A}^2 \\ \vdots \\ \bar{C}\bar{A}^{N_p} \end{pmatrix} \bar{x}(k) + \begin{pmatrix} \bar{C}\bar{B} & 0 & \cdots & 0 \\ \bar{C}\bar{A}\bar{B} & \bar{C}\bar{B} & & \\ \vdots & & \ddots & \\ & \cdots & & \bar{C}\bar{B} \end{pmatrix} \Delta U(k)$$

→ criterion with vectorial form.

$$\sum_{i=1}^{N_p} \left((y(k+i|k) - w(k+i|k))\right)^{\intercal}\left((y(k+i|k) - w(k+i|k)\right) + \lambda \Delta u(k+i-1|k)^{\intercal} \Delta u(k+i-1|k)$$

$$(Y^+ - W^+)^T (Y^+ - W^+) + \lambda\, \Delta U^T \Delta U$$

$$Y^+ = \bar{F}\,\bar{x}(k) + \bar{H}\,\Delta U$$

1st-order condition

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OPTIMAL INCREMENTAL INPUT SEQUENCE

$$(\bar{F}\bar{x} + \bar{H}\Delta U - W^+)^T (*) + \lambda \Delta U^T \Delta U$$

$$\hookrightarrow 2(\bar{H}^T\bar{H} + \lambda I)\Delta U + 2\bar{H}^T(\bar{F}\bar{x} - W^+) = 0$$

$$\boxed{\Delta U^* = \underbrace{(\bar{H}^T\bar{H} + \lambda I)^{-1}\bar{H}^T}(W^+ - \bar{F}\bar{x})}$$

$$\hookrightarrow \Delta u(k) = [I \; 0 - 0]\,\Delta U^* \quad (\text{1st-element})$$

$$= H_{Np}(W^+ - \bar{F}\bar{x})$$

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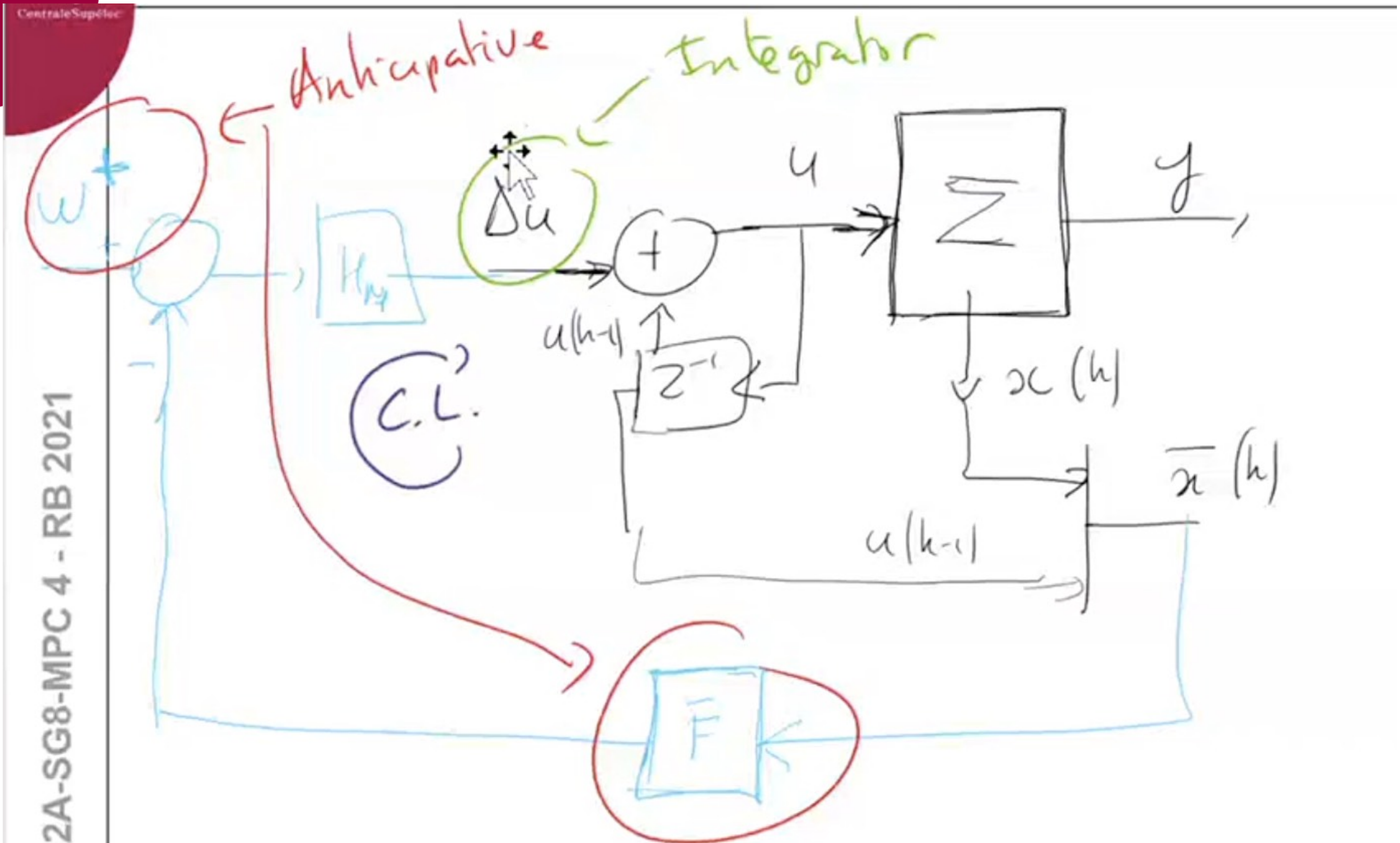
BLOCK DIAGRAM. WHERE IS THE ANTICIPATION?



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BLOCK DIAGRAM. WHERE IS THE ANTICIPATION?

MPC FOR TRACKING
SOME REMARKS

If you open a book dedicated to MPC, you may find this criterion:

$$\sum_{i=N_1}^{N_p} \big((y(k+i|k) - w(k+i|k)\big)^{\intercal} \big((y(k+i|k) - w(k+i|k)\big)$$

$$+ \lambda \sum_{i=1}^{N_u} \Delta u(k+i-1|k)^{\intercal} \Delta u(k+i-1|k)$$

Only 4 parameters:

- N_p
- λ
- N_1 : delay of system.
- N_u : control horizon (simplification of optim pb.)

4.2 EXERCICE 2 INTRODUCTION TO CONSTRAINT HANDLING

EXERCICE 4.2

See exercice available in Teams:

2A_SG8_4.2.pdf

To be finished for the 3rd of May 2021

4.3 CONCLUSION

YOUR NEW SKILLS

离散推断

IMPLEMENTING A MPC APPROACH FOR REFERENCE TRACKING

- To get the explicite solution in this case, using the incremental model
- To get the block diagram structure with the feedforward action

USING A SOLVER TO INTEGRATE CONSTRAINTS

- Analysis is still missing in this case …

To be done next course!

2A-SG8-MPC-CHAPTER 4
QUESTIONS?