Information theory

Gaussian channel capacity

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The Gaussian channel

- A discrete time channel model.
- At time i, the channel input X_i is contaminated by Gaussian noise Z_i in order to produce the channel output Y_i, i.e.,

$$Y_i = X_i + Z_i, \quad Z_i \sim \mathcal{N}(0, N).$$

- Noise Z_i is assumed to be independent of the signal X_i
- The most common channel model for wired, wireless and satellite communications links.

Capacity considerations for a Gaussian channel

- We consider that signal X can take any real value and therefor it can encode an infinite number of bits provided that noise is absent.
- It can also be proven that even in the presence of noise, if no constraint is introduced on signal X_i, the capacity is still infinite.
- To study the capacity the introduction of an average power constraint is introduced:

$$\frac{1}{n}\sum_{i=1}^{n} x_i^2 \leq P \tag{1}$$

Definition of information capacity for the Gaussian Channel

Definition: In the presence of a power constraint P we define the information capacity of the Gaussian channel as:

$$C = \max_{p_X(x):\mathbb{E}\{X^2\}\leq P} I\left(X;Y\right).$$

Calculating the mutual entropy:

$$\begin{aligned} I\left(X;Y\right) &= h\left(Y\right) - h\left(Y|Z\right) \\ &= h\left(Y\right) - h\left(X+Z|X\right) \\ &= h\left(Y\right) - h\left(Z|X\right) \\ &= h\left(Y\right) - h\left(Z\right). \end{aligned}$$

where $h\left(Z\right) = \frac{1}{2}\log 2\pi eN$.

Question: How can we maximize the mutual information?

Gaussian distribution and entropy properties

Theorem: Let $\mathbf{X} \in \mathbb{R}^n$ be a zero mean random vector, having a covariance matrix $\mathbf{K} = \mathbb{E}\left\{\mathbf{X}\mathbf{X}^T\right\}$. We then have that $h\left(\mathbf{X}\right) \leq \frac{1}{2}\log\left(2\pi e\right)^n |\mathbf{K}|$, with equality if and only if $\mathbf{X} \sim \mathcal{N}\left(\mathbf{0}, \mathbf{K}\right)$.

Proof: For simplicity let us use notation $\phi_K\left(k\right)$ for the zero mean multivariate Gaussian distribution. Based on the non-negativity of the Kullback-Leibler distance, we then have that for any multivariate distribution $g\left(\cdot\right)$ it holds that:

$$0 \leq D\left(g||\phi_K\right) = -h\left(g\right) - \int g \log \phi_K$$

However, we have that:

$$\log \phi_K\left(\mathbf{x}\right) = -\frac{1}{2}\log\left(2\pi\right)^n |K| - \mathbf{x}^T\mathbf{K}^{-1}\mathbf{x}. \tag{2}$$

As a result $\int g \log \phi_K$ depends only on $\mathbf{K}$ and it holds that $\int g \log \phi_K = \int \phi_k \log \phi_K = h\left(\phi_K\right)$. Hence, we obtain that $h\left(\phi_K\right) \geq h\left(g\right)$.

Information capacity of Gaussian channel

- Selecting $p_X(x)$ such as to maximize $I(X;Y)$, subject to the constraint $\mathbb{E}\left\{X^2\right\} \leq P$ is equivalent to maximizing $h(Y)$.
- For any value of $\mathbb{E}\left\{X^2\right\}$, this is achieved if Y is Gaussian.
- Gaussianity of Y is achieved if X is Gaussian. The variance of Y is then $N + \mathbb{E}\left\{X^2\right\}$ and $h(Y)$ becomes:

$$h(Y) = \frac{1}{2} \log 2\pi e \left(\mathbb{E}\left\{X^2\right\} + N\right).$$

- As a result, information capacity becomes:

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N}\right).$$

- It can be proven that this information capacity is actually the capacity of the channel, i.e., the maximum number of transmitted bits per transmission sample that we can achieve.

Capacity of bandlimited channels

- Let us assume that we are given a communications channel occupying the bandwidth $[-W, W]$.
- Assuming noise of power spectral density $N_0/2$ Watts/Hertz, the noise power (variance) is going to be equal to $N_0 W$.
- By sampling the received signal with a sampling period equal to $\frac{1}{2W}$. The noise variance of each of the $2WT$ noise sample is going to be equal to $N_0/2$ while the variance of the transmitted signal will be equal to $P/2W$. The number of transmitted bits per transmitted sample is then equal to:

$$C = \frac{1}{2} \log \left(1 + \frac{P}{N_0}\right) \tag{3}$$

and the toal number of bits per second will be:

$$C = W \log_2 \left(1 + \frac{P}{N_0 W}\right) \tag{4}$$