2A - Automatique

Chapter 6

Control Science (AUT)

Discrete-Time Systems

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Preamble
About this course

Course outline

- From a continuous-time system to a discrete-time system
- Analysis of discrete-time systems
- Some numerical control actions
- Digitizing Analog Controllers
- Numerical Implementation

Some reminders
The Z-transform - Definition

Defintion

- the monolateral Z-transform of a signal discrete $x(k)$ is :

$$X(z) = \sum_{k=0}^{+\infty} x(k) z^{-k}$$

Useful example

- $x(k) = a^k$, with $a \in \mathbb{R}$. Its Z-transform is :

$$\begin{aligned} X(z) &= \sum_{k=0}^{+\infty} \left(\frac{a}{z}\right)^k \\ &= \frac{z}{z-a}, \text{ if convergence!} \end{aligned}$$

Some reminders
The Z-transform - Properties

- Linearity : $\mathcal{Z}(af + bg) = aF + bG$
- Pure delay : $\mathcal{Z}(f(k - k_0)) = z^{-k_0}F(z)$, $k_0 \in \mathbb{N}$.
- Advance : $\mathcal{Z}(f(k + n)) = z^nF(z) - z^nf(0) - z^{n-1}f(T_e) - \dots zf((n-1)T_e)$, with $n \in \mathbb{N}$ and the values of f at the past samplings $(k + n)$.
- $\mathcal{Z}(a^kf(k)) = F(\frac{z}{a})$
- BE CAREFUL : $\mathcal{Z}(f \times g) \neq FG$, but $Z(f * g) = FG$, with $*$ the convolution product

Two theorems

- Initial value theorem : $x(0) = \lim\limits_{z \to +\infty} X(z)$
- Final value theorem : $\lim\limits_{k \to +\infty} x(k) = \lim\limits_{z \to 1}(z - 1)X(z)$

Some reminders

The Z-transform - the ones to know

- $\mathcal{Z}(d(k)) = 1$,
- $\mathcal{Z}(\Gamma(k)) = \frac{z}{z-1}$,
- $\mathcal{Z}(r(k)) = \frac{z}{(z-1)^2}$,
- $\mathcal{Z}(a^k) = \frac{z}{z-a}$.

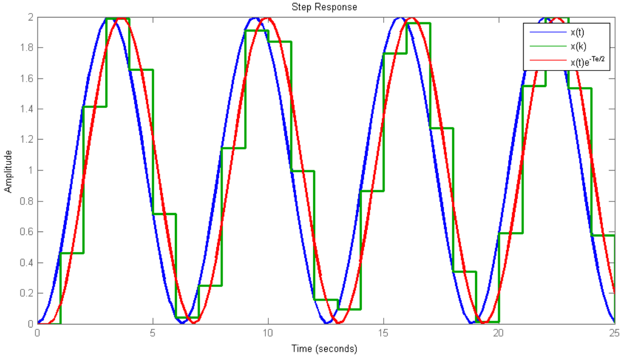
Looking for the digital sequence

- Knowing $X(z)$, what is $x(k)$?
- Example : $X(z) = \frac{z+1}{z-1}$.

Outline

1. Introduction
2. **Discrete-Time equivalents of Continuous-Time plants**
3. Analysis of a discrete-time system
4. Some digital controllers
5. Conclusion

Impact of the ADC and DAC

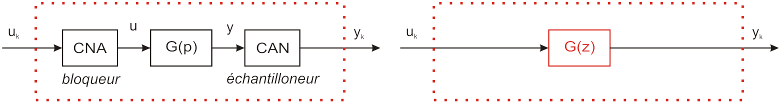


What you need to remember

- The best approximation : continuous signal delayed by half a period !
- Impact due to the delay : decrease of the phase ! (Margin, destabilization, . . .)

Towards the equivalent discrete-time system

A continuous process, governed by a digital controller



The zero-order hold (ZOH)

$$B_0(p) = \frac{1}{p}\left(1 - e^{-Tp}\right).$$

Link between $G(p)$ and $G(z)$

$$G(z) = (1 - z^{-1})\mathcal{Z}\left(\frac{G(p)}{p}\right)$$

Outline

Stability

Defintion

A system is stable (BIBO), if for any bounded input, its output is also bounded.

Theorem

A discrete system is asymptotically stable if and only if its impulse response is absolutely sommable

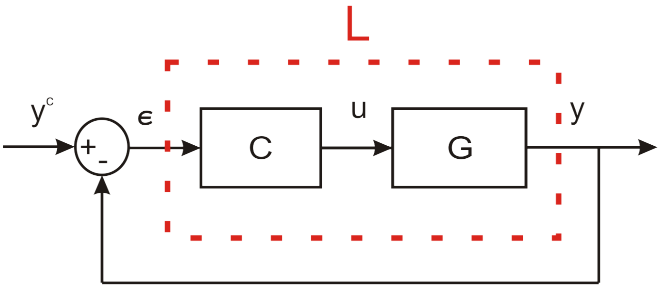
$$\sum_{k=0}^{+\infty} |g(k)| \quad < \quad +\infty$$

Characterization

A LTI discrete-time system is stable (BIBO) if and only if all the poles of its transfer function are inside the unit circle (i.e. if they are all of modulus strictly less than 1).

- Tools : Jury, Routh (homographic transform $z = \frac{1+w}{1-w}$)
- One friend : Matlab

Stability : from open-loop to closed-loop
Problem statement



Relation between L and $\frac{1}{1+L}$

Stability : from open-loop to closed-loop

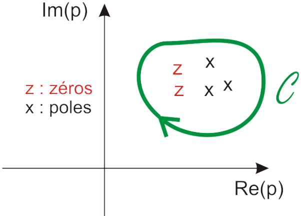
Cauchy's theorem

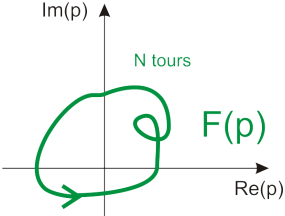
- Let us consider $F(p)$, a meromorph complex function. Let us consider $\mathcal{C}$ a closed contour.
- Z : number of zeros of F, P : number of poles of F inside the closed contour $\mathcal{C}$

Cauchy's theorem

- When p is moving on the contour $\mathcal{C}$, $F(p)$ describes a closed path
- N : number of rotations of $F(p)$ around 0, counted in the same direction of travel.

$$\boxed{N = Z - P}$$



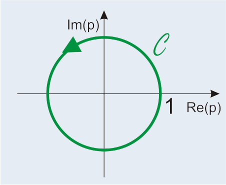


Nyquist Criterion for Discrete-Time systems

Using Cauchy for stability analysis

- The transfer to study : $\frac{1}{1+L}$
- To ensure stability : all the zeros of $(1+L)$ have to be in the unit circle.

What is Bromwitch for DT systems ?



The Nyquist plot

THe image of the unit circle by the transfer $L(z)$ is the Nyquist plot of L

- It is a close curve.

Nyquist Criterion for Discrete-Time systems

Using Cauchy for stability analysis : From $1 + L$ to L

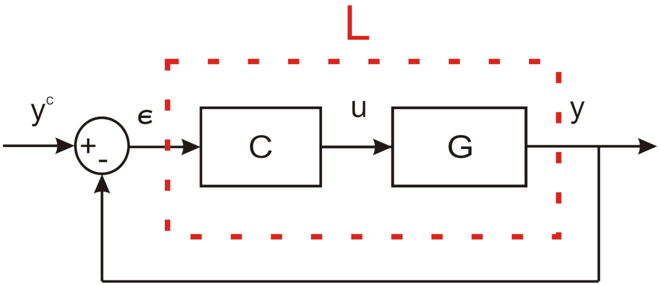
- $L = \frac{NUM}{DEN}$
- $1 + L = \frac{DEN+NUM}{DEN}$.

- 1st observation :
  - L, causal system, with n as order : the degree of DEN is greater than the degree of NUM : so $1 + L$ has exactly n !
  - L and $1 + L$ have the same poles
  - Let us denote with P, the number of unstable poles of L.
  - L has $n - P$ poles inside the unit circle

- Using Cauchy's theorem :
  - The image of $1 + L$ will do $N = n - (n - P) = P$ turns around 0
  - So the image of this circle by the transfer L must do P turns around -1 !

Nyquist Criterion for Discrete-Time systems

The criterion, finally, we can get it !



- Let us denote with P the number of poles of $L(z)$ with a modulus greater than 1.

Nyquist criterion

The transfer $S = \frac{1}{1+L}$ is asymptotically stable if and only if the Nyquist plot of L encircles P times the point -1 counter-clockwisely !

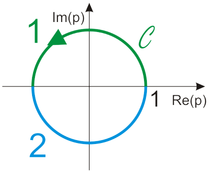
Nyquist Criterion for Discrete-Time systems
Consequences

The good news

- All the notions, studied for continuous-time systems are the same ...
- In particular phase margin, gain margin, ...

How to draw the Nyquist plot ?

- On part 1 : $z = e^{j\nu}$ with $\nu \in]0, \pi]$: everything is given by the Bode diagrams
- On part 2 : it is the symmetrical with respect to the abscissa axis.



The false bad news

- It is very difficult to draw Bode diagrams for a discrete time system
- Fortunately, Matlab is here !

Reminder 1 : Impact of the poles on behaviour
1st order systems

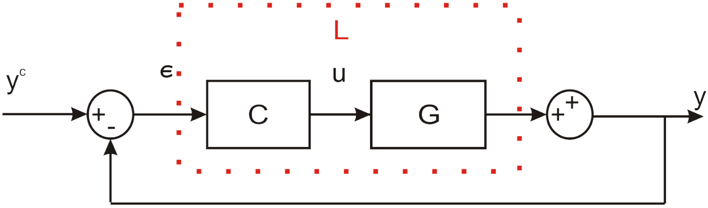
Reminder 1 : Impact of the poles on behaviour
2nd order systems

Reminder 1 : Impact of the poles on behaviour

Relation between continuous poles and discrete poles

Steady-state error

As for continuous systems, it depends on the number of integral actions

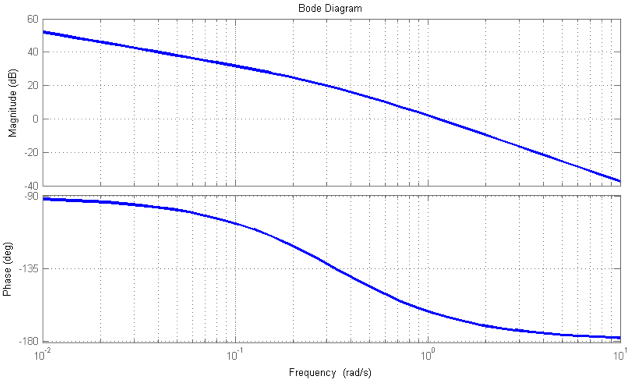


- For the study, use the notation $L(z) = \frac{N(z)}{(1-z^{-1})^m D(z)}$

Your turn to play

An example for fun (what a laugh !)

$$G(p) = \frac{K}{p(1+\tau p)},$$



- Specifications : digital control, simple gain, overshoot 10%, no steady-state error

Your turn to play

An example for fun (what a laugh !)

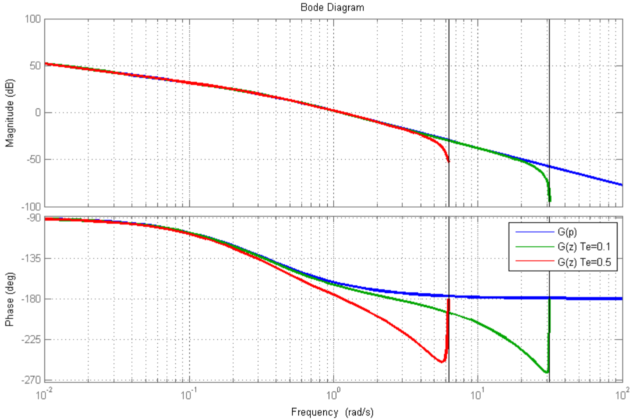
$$G(p) = \frac{K}{p(1+\tau p)},$$

- For a continuous control : what would be the value of the gain to meet these specifications ?
- What would be the bandwidth of the closed-loop system ?
- Propose a sampling period ?

Your turn to play

An example for fun : Impact of the sampling period

$$G(p) = \frac{K}{p(1 + \tau p)},$$



Your turn to play

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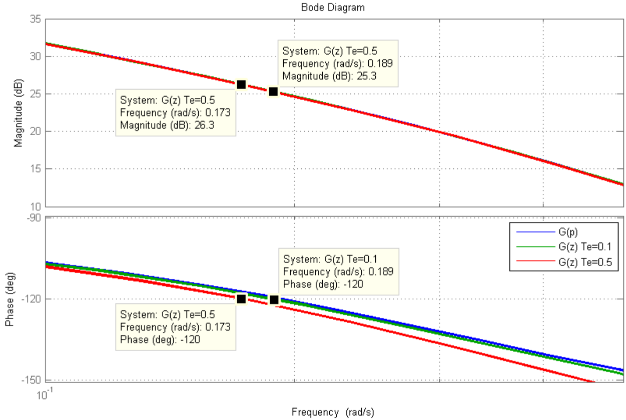

Your turn to play

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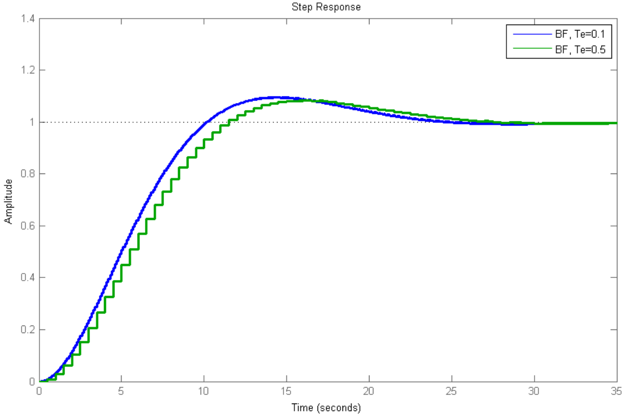


Outline

Two ways to design a controller
Among others

Digitizing Analog Controllers

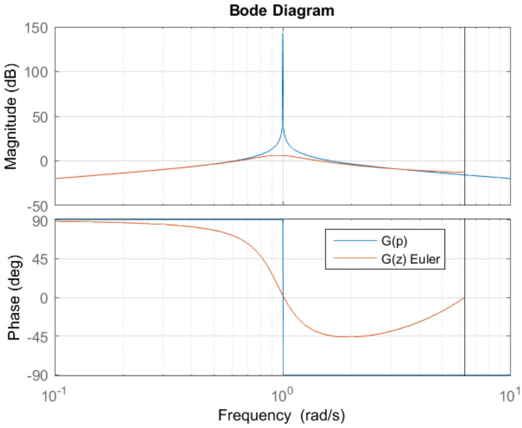
- Controller synthesis using a continuous time approach : we get $C(p)$
- Then digitizing : we get $C(z)$. (3 technics, 1 tool)

Discrete-Time Equivalents of $G(p)$, then pole placement

- Specification of the desired closed-loop behavior
- Controller synthesis directly from a discrete-time approach and a mathematical operation
- Many structures : digital PID, RST structure, ...

Digitizing technics
Method 1 : Euler's Backward Method

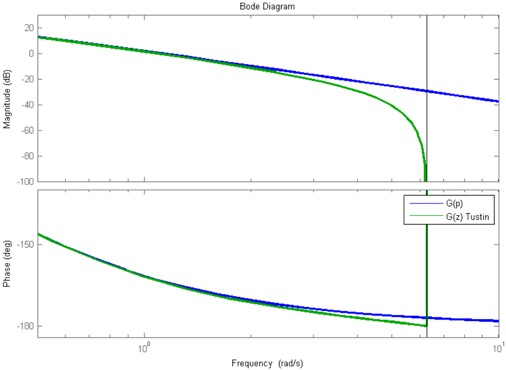
- Method : Integral calculation by the rectangle method



Digitizing technics

Method 2 : Trapezoidal Method - Tustin approximation

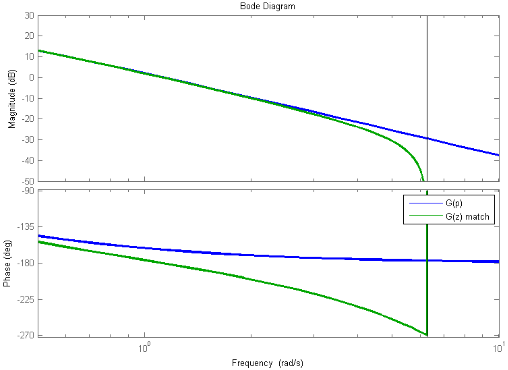
- Method : integral calculation by the trapezoidal method
- One of the more stable



Digitizing technics
Method 3 : Pole-Zero Matching

- Digitizing zero by zero, pole by pole
- Adjusting the static gain

$$G(p) = K\frac{\prod(p-r_k)}{\prod(p-p_i)} \quad \rightarrow \quad G(z) = K_d\frac{\prod\left(\frac{1-e^{r_k Te}z^{-1}}{Te}\right)}{\prod\left(\frac{1-e^{p_i Te}z^{-1}}{Te}\right)}$$



Digitizing technics

With a little help from my friend

- Using the function : `c2d`

```
SYSD = c2d(SYSC,TS,METHOD) computes a discrete-time model SYSD with
sampling time TS that approximates the continuous-time model SYSC.
The string METHOD selects the discretization method among the following:
   'zoh'       Zero-order hold on the inputs
   'foh'       Linear interpolation of inputs
   'impulse'   Impulse-invariant discretization
   'tustin'    Bilinear (Tustin) approximation.
   'matched'   Matched pole-zero method (for SISO systems only).
The default is 'zoh' when METHOD is omitted. The sampling time TS should
be specified in the time units of SYSC (see "TimeUnit" property).
```

- Euler : $p \longrightarrow \frac{1-z^{-1}}{T_s}$ - No Matlab help !
- Tustin : $p \longrightarrow \frac{2}{T_s}\frac{1-z^{-1}}{1+z^{-1}}$
- Zero-Pole Match : change term by term
- Each case requires its own technique - To be checked a posteriori

Digital PID
The whole diagram

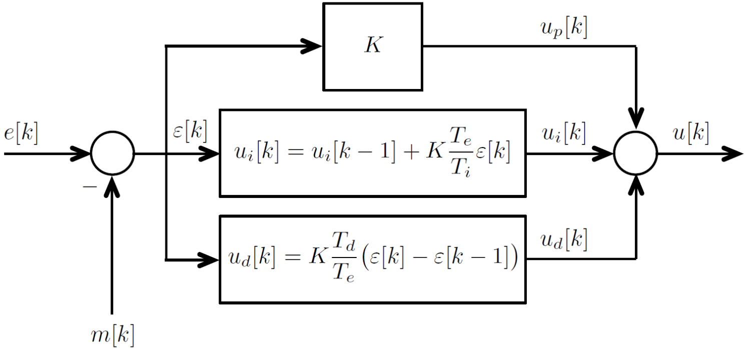
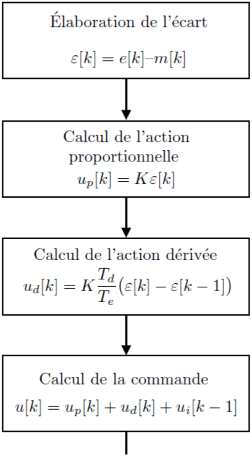
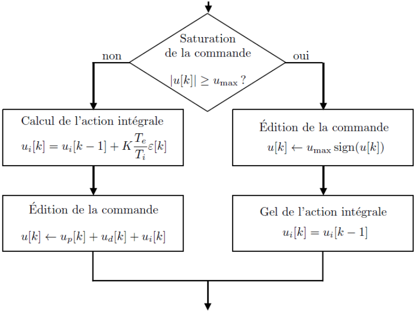


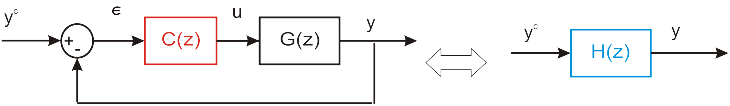
FIGURE 6.13 – Structure parallèle du correcteur PID numérique.

Digital PID
The whole diagram





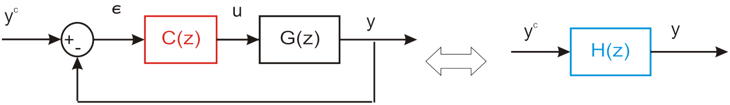
Pole placement and mathematical inversion
Methodology



- What we have : $G(z)$
- What we want : $H(z)$
- What we are looking for : $C(z)$

- What is C ?

Pole placement and mathematical inversion
Methodology



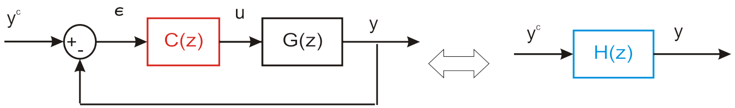
$$C = \frac{1}{G}\frac{H}{1-H}$$

So simple, but is it working ? yes . . . if 3 rules !

- C : Causality
- C : No zeros with a modulus greater than one - (Stability of the controller)
- C : No zeros with a modulus greater than one - (be careful with Nyquist, nonminimum phase system)

Pole placement and mathematical inversion

Your turn to play

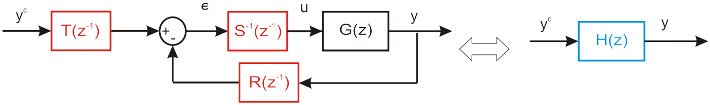


- Let us consider the continuous system, sampled with a period of 1 second, whose transfer in z is :

$$G(z) = \frac{z-2}{(z-1)(z-0.3)(z-0.5)}$$

- Determine a controller C, so that the closed-loop system behaves roughly like a first-order system, with a settling time of about 15 seconds.

The RST structure
Polynomial correction

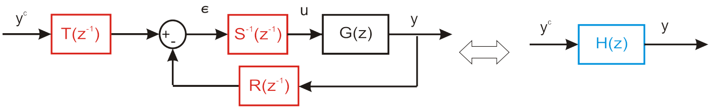


- What with have : $G(z)$, to be specified using z^{-1}
- What we want : $H(z)$, to be specified using z^{-1}
- What we are looking for : $S(z^{-1})$, $R(z^{-1})$, $T(z^{-1})$

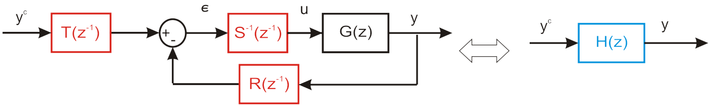
Main principles

- R,S and T are z^{-1} polynomials
- Two degrees of freedom : R and S are in the loop (disturbance), T is a precompensation

The RST structure
Polynomial correction



The RST structure
Your turn to play



- Let us consider the continuous system, sampled with a period of 1 second, whose transfer in z is :

$$G(z) = \frac{z-2}{(z-1)(z-0.3)(z-0.5)}$$

- Determine a RST controller, so that the closed-loop system behaves roughly like a first-order system, with a response time of about 15 seconds.

Outline

Conclusions
After this course

Skills

- Understand the impact of the digitizing
- Discrete-Time system analysis
- Controller synthesis using a continuous approach then digitizing
- digital PID
- Pole placement and mathematical inversion