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In this exercise, the objective is to try to go further in the control of the refrigerant system. Nowadays, the temperature in a refrigerant system is governed by an hysteresis control, that turns on the cooling system if the temperature is greater than a maximal temperature, and turns it off if this temperature becomes below a minimal temperature. Generally, the temperature oscillates between 2^oC and 5^oC.

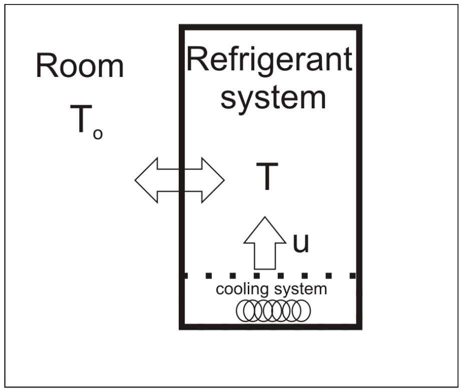

FIGURE 1 – The refrigerant system

To improve the quality of the regulation, and to reduce the energy consumption of the system, we will investigate in this exercise on the benefit of an MPC controller.

The modeling of the system is a simple first order that can be described by the following equation :

$$T(k+1) = aT(k) + b_u u(k) + b_e T_o(k) \tag{1}$$

in which T is the temperature in the refrigerant system, T_o is the temperature outside the refrigerant system, u is the cooling power injected in the system. The coefficients. For the implementation, the values of the parameters are : $a = 0.9934$, $b_u = -2\ 10^{-3}$ and $b_e = 6.6\ 10^{-3}$ and the sampling time is chosen equal to 2 minutes (this numerical value is just for your curiosity) and the time unit is the minute (maybe useful if you go into simulink).

The cooling power is limited, between $u_{min} = 0$W and $u_{max} = 100$W.

The notation N_p will be used for the length of the prediction equation. The temperature $T(k)$ is known at each time k.

From the state space model (1), the first point is to build a prediction model over the prediction horizon [1].

This model links the predicted output sequence $T(K+1|k)$ with the current state $T(k)$, the input sequence $U(K|k)$ and the predicted ambient temperature $T_o(K|k)$ of the room in which the refrigerant system is.

$$T(K+1|k) = FT(k) + H_u U(K|k) + H_e T_o(K|k)$$

Q1a. What are the dimensions of the matrices F, H_u and H_e ?

Q1b. What are the expressions of these matrices ?

环境温度

The ambient temperature is hard to predict. It will be supposed that it will be constant over the prediction horizon and equal to the current ambient temperature $\forall i = 0, \ldots, N_\mathrm{p}-1$, $T_o(k+i|k) = T_o(k)$.

1. The notation $Z(K|k)$ is used to denote the vector $\begin{pmatrix} z(k|k) \\ z(k+1|k) \\ \vdots \\ z(k+N_\mathrm{p}-1|k) \end{pmatrix}$.

Q2 Can you explain why it is important to consider it in the prediction model and not equal to 0 ?

In order to avoid oscillations, the control objective is to maintain the temperature *around* 3°C. This value will be used to build the desired temperature over the prediction horizon $W(K+1|k)$. The criterion to be minimized, in which $\lambda > 0$ is a parameter, is the following one :

$$J = (W(K+1|k) - T(K+1|k))^{\mathrm{T}}(W(K+1|k) - T(K+1|k)) + \lambda U(K|k)^{\mathrm{T}} U(K|k) \tag{2}$$

About the explicit solution

Let us suppose that the size of the prediction horizon N_{p} is equal to 1. ($N_{\mathrm{p}} = 1$).

Q3. What is the relation between a_u, b_u and λ that guarantee the stability of the closed loop system ?

Let us go back to the general case. ($N_{\mathrm{p}} \geq 1$)

Q4a. Give the explicit expression of $u^{\star}(K|k)$ that minimizes the criterion, without considering the constraints on the power.

Q4b. In order to be sure that the power constraints are fulfilled, the saturation is applied after the optimization, even if this can imply suboptimality. What is the relation between the control to be applied $u(k)$, the bounds on the power u_{min} and u_{max} and the first element of the best input sequence $u^{\star}(k|k)$?

Figures 2 and 3 present the behavior of the closed loop system, with the same value of λ, but for two different prediction horizon lentgh N_{p}. Some disturbances can be seen around 7 :00, 12 :00 and 19 :00 which coincide with time of use of the refrigerant system (door opening, ...). The energy consumption of these two experiments is given in table reftab :EnergyConsumption.

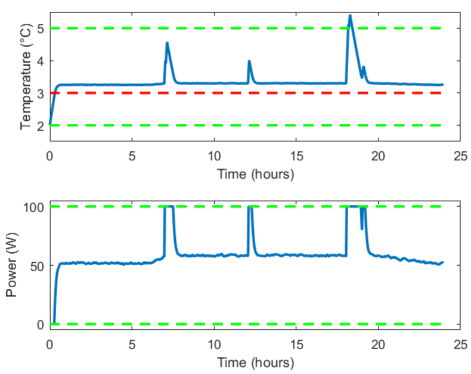

FIGURE 2 – Saturation after optimization, with $N = 1$

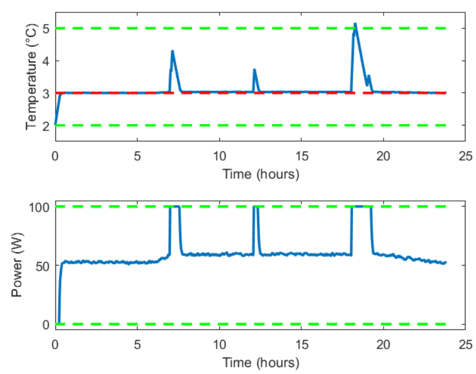

FIGURE 3 – Saturation after optimization, with $N = 5$

N	1	5
Energy consumed (kWh)	1.35	1.38

TABLE 1 – Energy consumption

Q5. What do you think of these results ? How do you think they can be improved ?

About the optimal solution

Let us now integrate directly the constraints in the optimization problem. To solve this problem, we propose to use the Matlab function `quadprog`. Here are the main lines of the documentation of `quadprog` :

Doc 1. *Help on* **quadprog**. *The* `[x,J]=quadprog[H,f,A,b,Aeq,beq,lb,ub]` *function will give the optimal solution, and the corresponding value* $J = \frac{1}{2}x^{\mathrm{T}}Hx + f^{\mathrm{T}}x$ *of the following problem :*

$$x = arg\min_{x} \frac{1}{2}x^{\mathrm{T}}Hx + f^{\mathrm{T}}x$$

subject to :

$$\begin{cases} Ax < b \\ A_{eq}x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

Q6. Give all the arguments to use `quadprog` function [2].

Q7. The value J returned by `quadprog` at one particular moment is $J = -0.0038$. Even if we do not care about the numerical value, as the orginal criterion (2) to be minimized is quadratic and positive (it is a sum of squares), how do you explain that the value returned by `quadprog` is negative ? $J = (W(K+1|k) - T(K+1|k))^{\mathrm{T}}(W(K+1|k) - T(K+1|k)) + \lambda U(K|k)^{\mathrm{T}}U(K|k)$

The result of this new control law is illustrated in figure 4. As you can see, the result is almost the same as the previous ones, and because of the disturbances, some constraints on the temperature are not fulfilled. A way to ensure a good temperature, would be to decrease the temperature reference (figure 7), but as a consequence, this will increase the energy consumption (table 2).

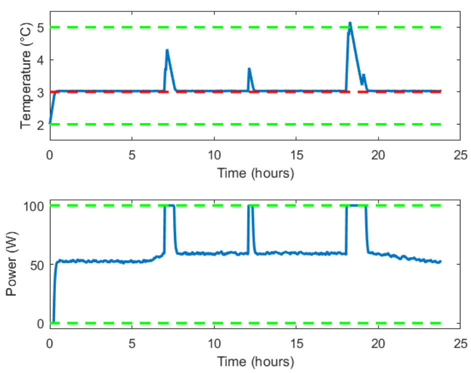

FIGURE 4 – Saturation in the optimization, with $T_{ref} = 3$

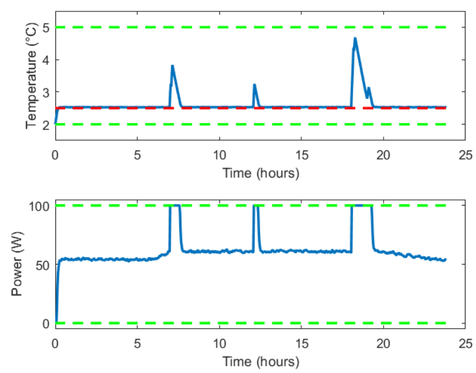

FIGURE 5 – Saturation in the optimization, with $T_{ref} = 2.5$

T_{ref}	2.5	3
Energy consumed (kWh)	1.43	1.38

TABLE 2 – Energy consumption

We propose in the last part to improve the control, based on the simple fact : to try to anticipate the periods of use of the refrigerant system and to decrease the temperature reference only for these moments.

Linear programming and energy efficiency

The main idea is first to change the optimization criterion : instead of minimizing a quadratic criterion, we will directly minimize the energy consumption over the prediction horizon while fulfilling constraint on the temperature.

The criterion to minimize is then :

2. Remember that all the matrices in this doc have no relation with the matrices defined in the previous question

$$J_2 = \sum_{i=0}^{N_\mathrm{p}-1} u(k+i|k)$$

subject to

$$T(K+1|k) \leq T_{max}(K+1|k)$$

and the power constraints, $\forall i = 0, \ldots, N-1$, $u_{min} \leq u(k+i|k) \leq u_{max}$. 约束

In this optimization problem, $T_{max}(K+1|k)$ is the maximal temperature allowed in the refrigerant system, which depends on the periods of use, and which is supposed to be known.

To solve this problem, we propose to use `linprog`. The documentation of `linprog` is given in the doc 2.

Doc 2. *Help on* `linprog`*. The* `x=linprog[c,A,b,Aeq,beq,lb,ub]` *function will give the optimal solution of the following problem :*

$$x = arg \min_x c^\mathrm{T} x$$

subject to :

$$\begin{cases} Ax \leq b \\ A_{eq}x = b_{eq} \\ lb \leq x \leq ub \end{cases}$$

Q8a. Give all the arguments required to use `linprog` to solve this problem. TO ensure the feasibility!!

Q8b. What kind of problems could occur using this control algorithm ? In few words, give ideas (and just the ideas) to solve them.

Figures 6 and 7 present the result of such a control, and their energy consumption is given in table 3.

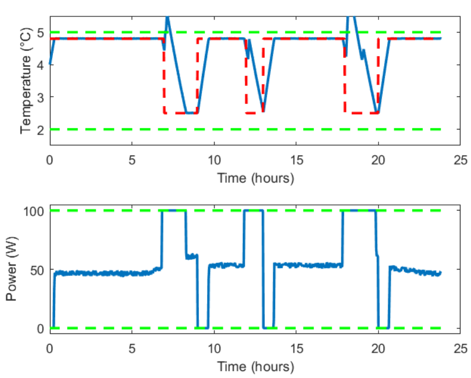

FIGURE 6 – Anticipation with $N = 5$

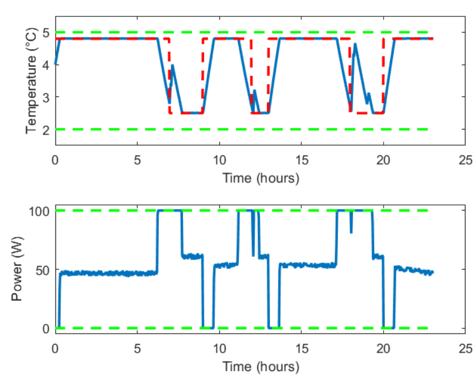

FIGURE 7 – Anticipation with $N = 30$

N	5	30
Energy consumed (kWh)	1.26	1.30

TABLE 3 – Energy consumption

Q9. Make a balance on the advantages and drawbacks of these different strategies.

Implementation in Matlab/Simulink

Q10. Implement the different strategies in Matlab/Simulink.

Q8b) Problem of unfeasability $\longrightarrow$ we need to add slack variable to relax the constraints on the temperature.

Q9) ~~Alt~~ Linear Programming : + Interesting, but need a large prediction horizon to anticipate $N \geq 30$!

+ Need a solver

QP : we could use explicite solution (easy in practice, $N = 1$ seems enough), but we need to decrease the temperature reference $\rightarrow$ this will increase the energy consumption.

MPC: Homework Part

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1 Question 1

1.1 Dimensions of Matrices

- F: $N_p \times 1$;
- H_u: $N_p \times N_p$;
- H_e: $N_p \times N_p$;

1.2 Expressions of Matrices

$$F = \begin{pmatrix} a & a^2 & a^3 & ... & a^{N_p} \end{pmatrix}^{\mathrm{T}} \tag{1.1}$$

$$H_u = \begin{pmatrix} b_u & 0 & 0 & 0 \\ ab_u & b_u & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ a^{N_p-1}b_u & \cdots & ab_u & b_u \end{pmatrix} \tag{1.2}$$

$$H_e = \begin{pmatrix} b_e & 0 & 0 & 0 \\ ab_e & b_e & 0 & 0 \\ \vdots & \ddots & \ddots & 0 \\ a^{N_p-1}b_e & \cdots & ab_e & b_e \end{pmatrix} \tag{1.3}$$

2 Question 2

The outside temperature has an obvious effect on this system because every time when you open or close the refrigerator, there will exist an heat exchange. And if it is equal to 0, the expression corresponds to a system which has no connection with the outside. Therefore, it should not be 0.

3 Question 3

3.1 $N_p = 1$

When N_p is equal to 1, the loss function is:

$$J = (T(k+1) - 3)^2 + \lambda u(k)^2 \tag{3.1.1}$$

Combined with the description of the system:

$$T(k+1) = aT(k) + b_u u(k) + b_e T_o(k) \tag{3.1.2}$$

The loss function is rewritten as;

$$J = (aT(k) + b_u u(k) + b_e T_o(k) - 3)^2 + \lambda u(k)^2 \tag{3.1.3}$$

Q2| "T_0" has to be considered

→ it influences the behavior of the (Σ) : if there is no input

$T \xrightarrow{k \to \infty} T_0$

→ In this particular case, $T_0 = 0°$! Is not coherent!

Let us consider the first order condition:

$$\frac{\partial J}{\partial u} = 2b_u(aT(k) + b_u u(k) + b_e T_o(k) - 3) + 2\lambda u(k) = 0 \tag{3.1.4}$$

Then we get:

$$u(k) = \frac{b_u}{\lambda + b_u^2} \times (3 - aT(k) - b_e T_o(k)) \tag{3.1.5}$$

Further:

$$T(k+1) = a(1 - \frac{b_u^2}{\lambda + b_u^2})T(k) + (1 - \frac{3b_u^2}{\lambda + b_u^2})b_e T_e(k) + \frac{3b_u^2}{\lambda + b_u^2} \tag{3.1.6}$$

Therefore, to satisfy the stability of this closed loop system:

$$|a(1 - \frac{b_u^2}{\lambda + b_u^2})| < 1 \tag{3.1.7}$$

3.2 General Case

Generally, the loss function could be expressed like this:

$$J = (FT(k) + H_u U(K) + H_e T_o(K) - W)^T (FT(k) + H_u U(K) + H_e T_o(K) - W) + \lambda U^T U \tag{3.2.1}$$

The first-order condition:

$$2(H_u^T H_u + \lambda I)U(K) + 2H_u^T(FT(k) + H_e T_o(K) - W) = 0 \tag{3.2.2}$$

Then, we get the relation between U and $T(k)$:

$$U^*(K) = (H_u^T H_u + \lambda I)^{-1} H_u^T (W - FT(k) - H_e T_o(K)) \tag{3.2.3}$$

Finally:

$$T(K+1) = (I - (H_u^T H_u + \lambda I)^{-1} H_u^T)FT(k) + (I - (H_u^T H_u + \lambda I)^{-1} H_u^T)H_e T_o(K) + (H_u^T H_u + \lambda I)^{-1} H_o^T W \tag{3.2.4}$$

$$Norm(EigenValue((I - (H_u^T H_u + \lambda I)^{-1} H_u^T)F)) < 1 \tag{1}$$

4 Question 4

4.1 Explicit expression of $u^*(K|k)$

The explicit expression of $U^*(K|k)$ is exactly shown in equation 3.2.3

4.2 Relations among $u(k)$, u_{min}, u_{max} and $u^*(k|k)$

- if $u^* > u_{max}$: $u(k) = u_{max}$;
- if $u_{min} \leq u^* \leq u_{max}$: $u(k) = u^*(k|k)$;
- if $u^* < u_{min}$: $u(k) = u_{min}$;

5 Question5

Ever time when the disturbance happens, it would cause a significant loss of power. Supervisory control over the disturbance could be added to decrease the energy loss.

⑥ $H_{quad} = 2 \times \left[H_u^T H_u + \lambda I_d \right]$; $I_d = \text{eye}(N)$

$f_{quad} = \left[2 (W - Fx - H_e T_o(k|k))^T H_u \right]^T$

$= 2 H_u^T (W - Fx - H_e T_o(k|k))$

$A, b, A_{eq}, B_{eq} = [\,]$

$lb = u_{min} \times \text{ones}(N,1)$

$ub = u_{max} \times \text{ones}(N,1)$

⑦ $J = \frac{1}{2} u^T H_q u + f_q^T u + \text{cste} > 0$

6 Question 6

Let's define:

$$T^{+} = FT(k) + H_u U(K) + H_e T_o(K) - W \tag{6.1}$$

Then we define the new variable:

$$Z = \begin{pmatrix} U(K) \\ T^{+} \end{pmatrix} \tag{6.2}$$

The initial problem could be rewritten as:

$$J = \frac{1}{2} Z^T H_z Z \tag{6.3}$$

where

$$H_z = \begin{pmatrix} R_U & 0 \\ 0 & Q_T \end{pmatrix} \tag{6.4}$$

with

$$R_U = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda \end{pmatrix} \quad Q_T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.5}$$

The constraint is:

$$(-H_U \quad I_{N_p \times N_p}) Z = FT(k) + H_e T_o(K) - W \tag{6.6}$$

7 Question 7

Possible reason 1: quadprog() returns negative value means that the optimisation failed.
Function help: `https://ww2.mathworks.cn/help/optim/ug/quadprog.html`
Possible reason 2:

8 Question 8

8.1 Applicability of LP

We define: $c^T = (1, 1, ..., 1)_{1 \times N_p}$ and the objective function is:

$$J_2 = \sum_{i=0}^{N_p - 1} u(k+i|k) = c^T U(K) \tag{8.1}$$

with the constraint:

$$H_u U(K) \leq T_{max}(K+1|k) - H_e T_o(K) - FT(k) \tag{8.2}$$

$$lb = u_{min} \times c, ub = u_{max} \times c \tag{8.3}$$

8.2 Potential Problems

This control law might lead to the loss of stability of temperature.
Also,the optimiser tempeture is alway 0 whatever the initial value To is.
Possible solution: add an extra penalty part in the loss function J_2 to achieve the stability.

9 Question 9

Compare these three strategies above and make a balance.

10 Question 10