# Reading notes for Hausdorff measure and dimensions

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**Abstract**: This reading notes covers the main parts of the first four chapters of FRACTAL GEOMETRY by Kenneth Falconer with the order slightly changed. Also, some exercises of Chapter four are selected and solved by virtue of Mass Distribution Principle.

## Part I

# Hausdorff measure

## 1 bacic information

**1.1 Def($\delta$-cover)** If $\{U_i\}$ is a countable/finite collection of sets of diameter at most $\delta$ that cover $F$, i.e. $F \subseteq \bigcup_{i=1}^{+\infty}$ with $0 \leq |U_i| \leq \delta$ for each i, then we say that $\{U_i\}$ is a $\delta$-cover of $F$.

**1.2 Def(s-dimensional Hausdorff measure)** $F \subseteq \mathbb{R}^n, s > 0, \delta > 0$. Then $\mathcal{H}^s_\delta(F)$:= inf $\{\sum_{i=1}^{+\infty} |U_i|^s : \{U_i\}$ is a $\delta$-cover of $F\}$. Note that $\mathcal{H}^s_\delta(F)$ increases as $\delta \to 0$. Hence we can define $\mathcal{H}^s(F) := \lim_{\delta \to 0} \mathcal{H}^s_\delta(F)$.

**1.3 Scaling property**

(a) $\lambda > 0, F \subseteq \mathbb{R}^n, \mathcal{H}^s(\lambda F) = \lambda^s \mathcal{H}^s(F)$.

(b) $F \subseteq \mathbb{R}^n, f : F \to \mathbb{R}^n$, s.t. $|f(x) - f(y)| \leq C|x - y|^\alpha$ for $x, y \in F$. $C > 0, \alpha > 0$. Then $\mathcal{H}^{s/\alpha}(f(F)) \leq C^{s/\alpha}\mathcal{H}^s(F)$.

A special case is that $f$ is an isometry, i.e. $|f(x)-f(y)| = |x-y|$. Replacing $f$ by

$f^{-1}$ in 1.3 we can obtain $\mathcal{H}^s(f(F)) = \mathcal{H}^s(F)$. Since the translation and rotation are isometry, then Hausdorff measure are translation and rotation invariant.

## Part II

# Hausdorff dimension

If $t > s$, then $\sum_i |U_i|^t = \sum_i |U_i|^{t-s}|U_i|^s \leq \delta^{t-s} \sum_i |U_i|^s$. Taking infima we obtain $\mathcal{H}^t_\delta(F) \leq \delta^{t-s}\mathcal{H}^s_\delta(F)$. Hence, if $\mathcal{H}^s(F) < +\infty$, then let $\delta \to 0$ we have $\mathcal{H}^t(F) = 0$. If $\mathcal{H}^t(F) = +\infty$, let $\delta \to 0$ we have $\mathcal{H}^s(F) = +\infty$. That is to say, there is a critical value of s at which $\mathcal{H}^s(F)$ jumps from $\infty$ to 0 as s increases. This critical value is called the Hausdorff dimension of $F$, written as $dim_H F$. Formally, $dim_H F := \inf\{s \geq 0 : \mathcal{H}^s(F) = 0\} = \sup\{s \geq 0 : \mathcal{H}^s(F) = +\infty\}$. Then

$$\mathcal{H}^s(F) = \begin{cases} +\infty, & 0 \leq s < dim_H F \\ 0, & s > dim_H F \end{cases}$$

If $s = dim_H F, \mathcal{H}^s(F)$ maybe 0, infinite of satisfy $0 < \mathcal{H}^s(F) < +\infty$.

## 2 Properties of Hausdorff dimension

**2.1 Prop(Monotonicity)**. If $E \subseteq F$, then $dim_H F \leq dim_H F$.

**2.2 Prop(countable stability)**. Suppose $\{F_i\}$ is a countable sequence of sets, then $dimH \bigcap_{i=1}^{+\infty} = \sup_i dim_H F$.

**2.3 Prop**. If $F$ is a countable set, then $dim_H F = 0$.

(Note if $F_i$ is a singleton, then $\mathcal{H}^0(F_i) = 1$ and hence $dim_H F_i = 0$. Then this prop follows from prop 3.2).

**2.4 Prop**. If n is an integer then $\mathcal{H}^n(F) = C_n^{-1}\mathcal{L}^n(F)$. Where $C_n^{-1}$ denotes $\mathcal{L}^n(B(0,1))$.

**2.5 Prop**. If $F$ is an open set in $\mathbb{R}^n$, then $dim_H F = n$.

proof: since $F$ is open, then $F$ contains a ball,say, $B(x,r)$. Note that $\mathcal{H}^n(B(x,r)) = C_n^{-1}\mathcal{L}^n(B(x,r)) \in ]0,+\infty[$. $\Rightarrow dim_H B(x,r) = n$. By monotonicity, $dim_H F \geq dim_H B(x,r) = n$. On the other hand, $F$ can be contained in countably many balls, by countable stability, $dim_H F \leq n$. $\square$

**2.6 Prop**. Let $F \subseteq \mathbb{R}^n$ and suppose $f : F \to \mathbb{R}^n$ satisfies *Hölder* condition:

$|f(x)-f(y)| \le C|x-y|^{\alpha}$. Then $dim_H f(F) \le \frac{1}{\alpha} dim_H F$.

**2.7 Corollary**

(a)If $f: F \to \mathbb{R}^n$ is Lipschitz then $dim_H f(F) \le dim_H F$. (In the case, $C = 1$).

(b)If $f: F \to \mathbb{R}^n$ is bi-Lipschitz, then $dim_H f(F) = dim_H F$.

Hausdorff dimension of a set alone tells us little about its topological Properties. However, we can know a set with Hausdorff dim less than 1 is very sparse from the following Prop.

**2.8 Prop**. A set $F \subseteq \mathbb{R}^n$ with $dim_H F < 1$ is totally disconnected(any subset of $F$ with more than one element is disconnected).

# 3 Calculation of Hausdorff dimension

**3.1 Example**. Let $F$ be the Cantor dust. Then $1 \le \mathcal{H}^1(F) \le \sqrt{2}$ and hence $dim_H F = 1$.

Sol: Let $E_k$ denote the k-th stage of the construction. Then $E_k$ consists of $4^k$ squares of side $4^{-1}$ and diameter $\sqrt{2} \cdot 4^{-k}$. Taking $E_k{}' s$ as a $\delta$-cover of $F$ where $\delta = \sqrt{2} \cdot 4^{-k}$. Then $\mathcal{H}^1_\delta(F) \le 4^k \cdot 4^{-k}\sqrt{2} = \sqrt{2}$. Let $k \to +\infty$ then giving $\mathcal{H}^1(F) \le \sqrt{2}$. This give s the upper estimate. For the lower estimate, first let $P$ denote the orthogonal projection onto the x-axis. Note $P$ is Lipschitz: $|P(x) - P(y)| \le |x-y|$ and $P(F) = [0,1]$. By 1.3 we have $\mathcal{H}^1([0,1]) = \mathcal{H}^1(P(F)) \le \mathcal{H}^1(F)$. On the other hand, $1 = \mathcal{L}^n([0,1]) = \mathcal{H}^1([0,1])$. Hence, $1 \le \mathcal{H}^1(F) \le \sqrt{2}$, which implies $dim_H F = 1$. □

**3.2 Example**. Let F denote the Cantor set. Set $s = log2/log3$. Then $dim_H F = s$ and $\frac{1}{2} \le \mathcal{H}^s(F) \le 1$.

Sol: Intuitively, $F$ can be split into a left part and a right part: $F_L := F \cap [0, \frac{1}{3}], F_R := F \cap [\frac{2}{3}, 1]$. Clearly, $F = F_L \cup F_R$-disjoint. Thus, $\mathcal{H}^s(F) = \mathcal{H}^s(F_L) + \mathcal{H}^s(F_R) = (\frac{1}{3})^s \mathcal{H}^s(F) + (\frac{1}{3})^s \mathcal{H}^s(F)$. If we have $0 < \mathcal{H}^s(F) < +\infty$ then we can get $1 = 2 \cdot (\frac{1}{3})^s$ or $s = log2/log3$.

Rigorous calculation: Again, let $E_k$ denote the k-th stage of the construction. Then $E_k$ consists of $2^k$ intervals, each of length $3^{-k}$. Taking the intervals of $E_k$ as a $3^{-k}$-cover of $F$. Then $\mathcal{H}^s_{3^{-k}}(F) \le 2^k 3^{-ks} = 1$. Let $k \to +\infty$ giving $\mathcal{H}^s(F) \le 1$. For the lower estimate, we will show $\sum\limits_i |U_i|^s \ge \frac{1}{2} = 3^{-k}$ for any cover $\{U_i\}$ of $F$. By the compactness of $F$, we can assume each $U_i$ is closed subintervals of $[0,1]$ and they are finite many W.L.O.G. For each $U_i$, we can choose interger $k$ s.t. $3^{-(k+1)} \le |U_i| < 3^{-k}$. Then $U_i$ only intersect at most one k-level interval since the distance of two different k-level invervals is at least $3^{-k}$.

If $j \geq k$,then by the construction, $U_i$ intersects at most $2^{j-k}$ j-level intervals of $E_j$. Note $2^{j-k} = 2^j 3^{-sk} \leq 2^j 3^s |U_i|^s$. We can choose j large enough so taht $3^{-(j+1)} \leq |U_i|$ for all i, then, since $\{U_i\}$ intersect all $2^j$ bascin intervals of length $3^j$, counting intervals gives $2^j \leq \sum\limits_i 2^j 3^s |U_i|^s \Rightarrow \sum\limits_i |U_i|^s \geq \frac{1}{2} = 3^{-s}$. Hence, $\mathcal{H}^s(F) \geq \frac{1}{2}$. This show $0 < \mathcal{H}^s(F) < +\infty$ and we obtain $dim_H F = s$.

## 4 Equivalent definitions of Hausdorff dimensions

• $\mathcal{B}^s_\delta(F) := \inf \big\{ \sum\limits_i |B_i|^s : \{B_i\}$ is a $\delta$-cover of $F$ by balls$\big\}$. Run a similar process we can also define Hausdorff measure and dimension by $\mathcal{B}^s_\delta(F)$. Furthermore,the dimensions defined by the two measures are the same.
• We can also get the same values for Hausdorff measure and dimension if we simply use open sets or closed sets as $\delta$-covers. Moreover, if $F$ is compact, we merely need to consider $\delta$-covers by finite collections of sets to get same results.
• Sometimes we will introduce guage function $h : \mathbb{R}^+ \Rightarrow \mathbb{R}^+$, which is $\uparrow$ and continous to get a sharper indication of dimension. $\mathcal{H}^s_\delta(F) = \inf \big\{ \sum h(|U_i|) : \{U_i\}$ is a $\delta$-cover of $F\big\}$. Note if $h(t) = t^s$ then the def is exactly the same as Hausdorff measure.

## Part III

# Alternative definitions of dimension

## 5 Box-counting dimensions

**5.1 Def(lower and upper box-counting dimensions)**.
Let $F \subseteq \mathbb{R}^n, F \neq \varnothing$, bounded and let $N_\delta(F)$ be the smallest number of sets of diameter at most $\delta$ which can cover $F$. The lower and upper box-counting dimensions of $F$ respectively are defined as $\underline{dim}_B F = \varliminf\limits_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$ $\quad \overline{dim}_B F = \varlimsup\limits_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$. If $\underline{dim}_B F = \overline{dim}_B F$, then $dim_B(F) := \lim\limits_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$.
• Equivalent def of box-counting dimensions. Consider cubes: $[m_1\delta, (m_1+1)\delta] \times \cdots \times [m_n\delta, (m_n+1)\delta]$, where $m_i$ are integers. Let $N^{'}_\delta(F)$ be the number of $\delta$-mesh cubes that intersect $F$. Then we have $N^{'}_\delta(F)$ sets of diameter $\delta\sqrt{n}$ that

cover $F$. Thus, $N_{\delta\sqrt{n}}(F) \le N'_\delta(F)$ by the def of $N_{\delta\sqrt{n}}(F)$. If $\delta\sqrt{n} < 1$,then

$$\frac{\log N_{\delta\sqrt{n}}(F)}{-\log(\delta\sqrt{n})} \le \frac{\log N'_\delta(F)}{-\log\sqrt{n} - \log\delta} \le \frac{\log N'_\delta(F)}{-\log\delta} \quad \Longrightarrow$$
$$\underline{dim}_B F \le \underline{\lim}_{\delta\to 0} \frac{\log N_{\delta\sqrt{n}}(F)}{-\log\delta} \quad \overline{dim}_B F \le \overline{\lim}_{\delta\to 0} \frac{\log N_{\delta\sqrt{n}}(F)}{-\log\delta}$$

On the other hand, any set of diameter at most $\delta$ can be contained in $3^n$ mesh cubes of side $\delta$ Thus, $N'_\delta(F) \le 3^n N_\delta(F)$. Then. if $\delta < 1$ we have $\frac{\log N'_\delta(F)}{-\log\delta} \le \frac{n\log 3}{-\log\delta} + \frac{\log N_\delta(F)}{-\log\delta}$. Let $\delta \to 0$ we get

$$\underline{\lim}_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \le \underline{dim}_B F \quad \overline{\lim}_{\delta\to 0} \frac{\log N'_\delta(F)}{-\log\delta} \le \overline{dim}_B F$$

Hence, to find box dimensions we can replace $N_\delta(F)$ by $N'_\delta(F)$, which denotes the number of mesh cubes of side $\delta$ that intersect $F$.

• Another def: let $N'_\delta(F)$ denote the largest number of disjoint balls of radius of $\delta$ with centers in $F$. Let $B_1 \cdots B_{N'_\delta(F)}$ be these balls. Observe that any point of $F$ must be whtnin distance of $\delta$ of certain $B_i$. Hence, the $N'_\delta(F)$ balss concentric with $B_i$ but of radius $2\delta$ can cover $F$, giving $N_{4\delta}(F) \le N'_\delta(F)$. On the other hand: Suppose $U_1 \cdots U_k$ be any collection of sets of diam at most *delta* that covers $F$. Then, each $B_i$ must contain at least one $U_j$. Also, $\{B_i\}$ are disjoint, giving $N'_\delta(F) \le N_\delta(F)$. Again we can show this gives an euqivalent definition.
Rmk: Note that the box-counting dimension is determined by the power law. In other words, if two definitions have the "measurements" that have a power relation with each other, then one may show the two definitions are euqivalent.

**5.2 Equivalent definitions for Box-counting dimensions**.

$$\underline{dim}_B F = \underline{\lim}_{\delta\to 0} \frac{N_\delta(F)}{-\log\delta} \quad \overline{dim}_B F = \overline{\lim}_{\delta\to 0} \frac{N_\delta(F)}{-\log\delta}.$$

If they coincide, then the box-counting dim of $F$ is defined by $dim_B F := \lim_{\delta\to 0} \frac{N_\delta(F)}{-\log\delta}$, where $N_\delta(F)$ is any of the following:

(1)the smallest number of closed balls of radius $\delta$ that cover $F$.

(2)the smallest number of cubes of side $\delta$ that cover $F$.

(3)the number of $\delta$-mesh cubes that intersect $F$.

(4)the smallest number of sets of diameter at most $\delta$ that cover $F$.

(5)the largest number of disjoint balls of radius $\delta$ with centers in $F$.

**5.3 Prop**. If $F \subseteq \mathbb{R}^n$, then

$$\underline{dim}_B F = n - \overline{\lim_{\delta \to 0}} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta} \quad \overline{dim}_B F = n - \underline{\lim_{\delta \to 0}} \frac{\log \mathcal{L}^n(F_\delta)}{\log \delta}$$

where $F_\delta = \left\{ x \in \mathbb{R}^n \middle| d(x, F) < \delta \right\}$.

Proof: suppose $\delta < 1$ and $F$ is covered by $N_\delta(F)$ balls of radius $\delta$, then it's clear that $F$ can be covered by the concentric balls of radius $2\delta$. Hence, $\mathcal{L}^n(F_\delta) \leq N)\delta(F) \cdot C_n \cdot (2\delta)^n$. $\Longrightarrow$

$$\frac{\log \mathcal{L}^n(F_\delta)}{-\log \delta} \leq \frac{\log 2^n C_n + n \log \delta + \log N_\delta(F)}{-\log \delta} \Longrightarrow$$

$$\begin{cases} \underline{\lim_{\delta \to 0}} \frac{\log \mathcal{L}^n(F_\delta)}{-\log \delta} \leq -n + \underline{dim}_B F \\ \overline{\lim_{\delta \to 0}} \frac{\log \mathcal{L}^n(F_\delta)}{-\log \delta} \leq -n + \overline{dim}_B F \end{cases} \tag{1}$$

On the other hand, it's clear that any ball centered in $F$ of radius $\delta$ is contained in $F_\delta$. Plus, these $N_\delta(F)$ balls are disjoint and hence $N_\delta C_n \delta^n \leq \mathcal{L}^n(F_\delta)$. Similarly, we can obtain the opposite inequality of (1).

# 6 The relationship between box-counting dimension and Hausdorff dimension

Let $N_\delta(F)$ denote the smallest number of sets of diam at most $\delta$ that cover $F$. $\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_i |U_i|^s : \{U_i\} \text{ cover } F \right\} \leq N_\delta(F)\delta^s$. If $\mathcal{H}^s(F) > 1$, then $\log N_\delta(F) + s \log \delta$. Hence, when $\delta$ is sufficiently small, $s \leq \underline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log \delta} = \underline{dim}_B F$. Let $s \uparrow dim_H F$ we obtain $dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F$. If we assume $dim_B F = \lim_{\delta \to 0} \frac{\log N_\delta(F)}{-\log \delta}$, then if $s < dim_B F, \exists \alpha > 1$ s.t. $\log N_\delta(F) / -log\delta > \alpha s$ provided $\delta$ is sufficiently small. Then $\log N_\delta(F) / -log\delta > \alpha s = \log \delta^{-\alpha s} \Rightarrow N_\delta(F) \cdot \delta^s > \delta^{-\alpha}$. Let $\delta \to 0$ we obtain $N_\delta(F)\delta^s \to \infty$. Similarly, $N_\delta(F)\delta^s \to 0$ if $s > dim_B F$. But note that $N_\delta(F)\delta^s = \inf \left\{ \sum_i \delta^s : \{U_i\} \text{ is a (finite) cover of } F \right\}$ while $\mathcal{H}^s_\delta(F) = \inf \left\{ \sum_i \{U_i\}^s : \{U_i\} \text{ is a } \delta\text{-cover of } F \right\}$. That is to say, box-counting dimension involves small sets but perhaps widely varying size.

**6.1 Example**. Let $F$ be a Contor set. Then $\underline{dim}_B F = \overline{dim}_B F = \log 2 / \log 3$.

Sol: All k-level intervals of $E_k$ of length $3^{-k}$ gives a cover of $F$. Then $N_\delta(F) \leq 2^k$. If $3^{-k} < \delta < 3^{-k+1} \Longrightarrow \overline{dim}_B F = \overline{\lim_{\delta \to 0}} \frac{\log N_\delta(F)}{-\log} \leq \overline{\lim_{k \to \infty}} \frac{\log 2^k}{\log 3^{k-1}} =$

$\log 2/\log 3$. On the other hand, any interval of length $\delta$ with $3^{-k-1} \leq \delta < 3^{-k}$ intersects at most one k-levle interval of $E_k$. But $|E_k| = 2^k$ so at least $2^k$ intervals of length $\delta$ are required to cover $F$. Hence $N_\delta(F) \geq 2^k$, giving $\underline{dim}_B F \geq \log 2/\log 3$. $\square$

# 7 Properties and problems of box-counting dimension

**7.1 Prop**

(a)A smooth m-dimensional submanifold of $\mathbb{R}^n$ has $dim_B F = m$

(b)$\underline{dim}_B F and \overline{dim}_B F$ are monotonic.

(c)$\overline{dim}_B F$ is finitely stable, i.e. $\overline{dim}_B(E \cap F) = \max\{\overline{dim}_B E, \overline{dim}_B F\}$

(This does not hold for $dim_B F$)

(d)$\underline{dim}_B$ and $\overline{dim}_B$ are bi-Lipschitz invariant.

Proof of (d): Suppose $|f(x) - f(y)| \leq C|x - y|$ and $F$ is covered by $N_\delta(F)$ sets of diam at most $\delta$. Then $f(F)$ can be covered by $N_\delta(F)$ sets that are image of the sets above of diam at most $c\delta$. Thus, $N_{c\delta}(f(F)) \leq N_\delta(F) \Rightarrow \overline{dim}_B f(F) \leq \overline{dim}_B F$. Since $f$ is bi-Lipschitz, replacing $f$ by $f^{-1}$ and we get $\overline{dim}_B F \leq \overline{dim}_B f(F)$. Thus, $\overline{dim}_B F = \overline{dim}_B f(F)$. Similarly, $\underline{dim}_B F = \underline{dim}_B f(F)$. $\square$

**7.2 Prop**. $\underline{dim}_B \overline{F} = \underline{dim}_B F \quad \overline{dim}_B \overline{F} = \overline{dim}_B F$

Proof: Let $N_\delta(F)$ denote the smallest number of closed balls of radius $\delta$ that cover $F$. Then $N_\delta(F) = N_\delta(\overline{F})$. $\square$

**7.3 Corollary** If $F$ is a dense subset of an open region of $\mathbb{R}^n$, then $dim_B F = n$.

Proof: It follows from $dim_B U = n$ provided $U$ is open in $\mathbb{R}^n$.

# 8 Modifid box-counting dimensions

Now we decompose $F$ into a countable pieces in such a way that the largest one has a dimension as small as possible.

**8.1 Def**

$$\underline{dim}_{MB} F := \inf\Big\{\sup_i \underline{dim}_B F_i : F \subseteq \bigcup_{i=1}^{+\infty} F_i\Big\}$$

$$\overline{dim}_{MB} F := \inf\Big\{\sup_i \overline{dim}_B F_i : F \subseteq \bigcup_{i=1}^{+\infty} F_i\Big\}$$

In particular, if we set $F_1 = F, F_j = \varnothing, j \geq 2$, then it gives that $\underline{dim}_{MB}F \leq \underline{dim}_B F$ and $\overline{dim}_{MB}F \leq \overline{dim}_B F$.

• if $F$ is countable then $\overline{dim}_{MB}F \leq \overline{dim}MBF = 0$.

• $0 \leq dim_H F \leq \underline{dim}_{MB}F \leq \overline{dim}_{MB}F \leq \overline{dim}_B F \leq n$. The importance of modified def is that it's useful to test for compact sets.

**8.2 Prop**. Let $F \subseteq \mathbb{R}^n$ be compact. If for any open sets $V$ that intersect $F$ then we have $\overline{dim}_B(F \cap V) = \overline{dim}_B F$, then $\overline{dim}_B F = \overline{dim}_{MB}F$. A similar result holds for lower box-counting dimensions.

Proof: Let $F \subseteq \bigcup\limits_{i=1}^{+\infty} F_i, F_i$ closed. By Baire's Category Theorem, $\exists F_i, \exists V$ open s.t. $F \cap V \subseteq F_i$.Then $\overline{dim}_B(F \cap V) \leq \overline{dim}_B(F_i)$. But $\overline{dim}_B(F \cap V) = \overline{dim}_B(F) \implies \overline{dim}_B(F_i) = \overline{dim}_B(F)$. By prop 7.2, $\overline{dim}_{MB}F = \inf\big\{\sup \overline{dim}_B F_i : F \subseteq \bigcup\limits_{i=1}^{+\infty} F_i, F_i \text{ is closed}\big\} \geq \overline{dim}_B F$. The opposie direction is clear. $\square$

# 9 Packing measures and dimensions

Note that the box dimensions are not defined in terms of measure. In order to solve this problem we need to introduce another def of dimensions which canbe defined through measure but in the meantime closely related to box-counting dimensions.

**9.1 Def**. For $s \geq 0, \delta > 0, \mathcal{P}^s_\delta(F) := \sum\big\{\sum\limits_i |B_i|^s : \{B_i\}$ is a collection of disjoint balls of radius at most $\delta$ with centers in $F\big\}$.

Observe that $\mathcal{P}^s_\delta \downarrow$ with $\delta$, then $\mathcal{P}^s_0(F) = \lim\limits_{\delta\to 0}\mathcal{P}^s_\delta(F)$ exsits. However, if we consider $F = \mathbb{Q} = \{F_j\}$, where $F_j$ is singleton, then $\mathcal{P}^s_0(F) = +\infty$ while $\sum\limits_{j=1}^{+\infty}\mathcal{P}^s_0(F_j) = 0$ since $F_j$ is singleton. Hence $\mathcal{P}^s_0(F)$ is not a measure. Therefore we modify this def to $\mathcal{P}^s(F) := \inf\big\{\sum\limits_i \mathcal{P}^s_0(F_i) : F \subseteq \bigcup_{i=1}^{+\infty} F_i\big\}$, which is called s-dimensional packing measure. Then $dim_P F := \sup\big\{s : \mathcal{P}^s(F) = +\infty\big\} = \inf\big\{s : \mathcal{P}^s(F) = 0\big\}$. Then we have:

(1) $dim_P E \leq dim_P F$ if $E \subseteq F$.

(2) $dim_P(\bigcup_{i=1}^{+\infty} F_i) = \sup\limits_i dim_P F_i$.

It's clear that $dim_P(\bigcup\limits_{i=1}^{+\infty} F_i) \geq \sup\limits_i dim_P F_i$. Other way around, if $s > dim_P F_i, \forall i$, then $\mathcal{P}^s \bigcup\limits_i F_i \leq \sum\limits_i \mathcal{P}^s(F_i) = 0 \implies dim_P(\bigcup\limits_i F_i) \leq s$. Let $s \to \sup\limits_i dim_P F_i$ giving the desired inequality.

Next we will show packing dimension is just the same as the modified upper box dimension.

**9.2 Lemma**. $dim_P F \leq \overline{dim}_B F$

**9.3 Prop**. $dim_P F = \overline{dim}_{MB} F$

Proof: If $F \subseteq \bigcup_{i=1}^{+\infty}$, then $dim_p F \leq dim_P(\bigcup_{i=1}^{+\infty} F_i) = \sup_i dim_P F_i \leq \sup_i \overline{dim}_B F_i$. Then $dimPF \leq \overline{MB}_F$. Conversely, if $s > dim_P F$ then $\mathcal{P}^s(F) = 0$, so that by the def of $\mathcal{P}^s(F)$ there exsits $\{U_i\}$ s.t. $\forall i, \mathcal{P}_0^s(F_i) < +\infty$. Hence, $\forall i$, if $\delta$ is small enough, then $\mathcal{P}_\delta^s(F_i) < +\infty$. Let $N_\delta(F_i)$ denote the largest number of disjoints balls of radius $\delta$ with centers in $F_i$. Then, since $\mathcal{P}_\delta^s(F_i) < +\infty, N_\delta(F_i)\delta^s \leq \mathcal{P}_\delta^s < +\infty$. Then $\overline{dim}_B F_i \leq s, \forall i$. Hence $\overline{dim}_{MB} F \leq s$. □

To summarize, we have: $dim_H F \leq \underline{dim}_{MB} F \leq \overline{dim}_{MB} F = dim_P F \leq \overline{dim}_B F$.

**9.4 Corollary**. Let $F$ be compact and $\overline{dim}_B(F \cap V) = \overline{dim}_B F$ for all open sets $V$ that intersect $F$. Then $dim_P F = \overline{dim}_B F$.

# Part IV

# Techniques for calculating dimensions

## 10 Basic methods

**10.1 Prop** Suppose $F$ can be converd by $n_k$ sets of diameter at most $\delta_K$ with $\delta_k \to 0$ as $k \to +\infty$.Then:

(1) $dim_H F \leq \underline{dim}_B F \leq \underline{\lim}_{k\to+\infty} \frac{\log n_k}{-\log \delta_k}$.

(2) If $n_k \delta_k^s$ remains bounded as $k \to +\infty$, then $\mathcal{H}^s(F) < +\infty$.

(3) If $\delta_k \to 0$ but $\delta_{k+1} \geq c\delta_k$ for some $0 < c < 1$, then $\overline{dim}_B F \leq \overline{\lim}_{k\to+\infty} \frac{\log n_k}{-\log \delta_k}$.

Proof: (1) is clear. (2) $n_k \delta_k^s$ is bdd and $\mathcal{H}_{sk}^s(F) \leq n_k \delta_k^s$ gives $\mathcal{H}^s(F) < +\infty$. (3) If $\delta_{k+1} < \delta < \delta_k$, then, with $N_k(F)$ the least number of sets in a $\delta$-cover of $F$, we have:

$$\frac{\log N_\delta(F)}{-\log \delta} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_k} = \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log \frac{\delta_{k+1}}{\delta_k}} \leq \frac{\log N_{\delta_{k+1}}(F)}{-\log \delta_{k+1} + \log C}$$

Provided $k$ is sufficiently large. Hence $\overline{dim}_B(F) \leq \overline{\lim\limits_{k\to+\infty}} \frac{\log N_\delta(F)}{-\log\delta}$. $\square$

# 11 Mass distribution principle

**11.1 Theorem**(Mass distribution principle)

Let $\mu$ be a mass distribution on $F$ and for some $s$ there are numbers $C > 0$ s.t. $\forall\epsilon > 0$ we have $\mu(U) < C|U|^s$ for all sets $U$ with $|U|^s \leq \epsilon$. Then $\mathcal{H}^s(F) \geq \mu(F)/C$ and $s \leq dim_H F \leq \underline{dim}_B F \leq \overline{dim}_B F$.

Proof: If $\{U_i\}$ is a $\epsilon$-cover of $F$, then $0 < \mu(F) \leq \mu(\bigcup\limits_i U_i) \leq \sum\limits_i \mu(U_i) \leq C\sum\limits_i |U_i|^s$. Taking infima, $\mathcal{H}^s_\epsilon \geq \mu(F)/C$. Let $\epsilon \to 0$ we obtain $\mathcal{H}^s(F) \geq \mu(F)/C$. Since $\mu(F) > 0$ we get $dim_H F \geq s$. $\square$

**11.2 Example** Let $F_1 = F \times [0,1] \subseteq \mathbb{R}^n$, where $F$ is the Cantor set. Then $dim_B F_1 = dim_H F_1 = 1 + \frac{\log 2}{\log 3}$, with $0 < \mathcal{H}^s(F) < +\infty$.

Sol: $\forall k, E_k$ cover $F$. A column of $3^k$ squares of side $3^{-k}$ covers the part of $F_1$ above each interval of $E_k$. Taking them together, $F_1$ can be coverd by $2^k \cdot 3^k$ squares of side $3^{-k}$. Hence $\mathcal{H}^s_{\sqrt{2}\cdot 3^{-k}}(F_1) \leq 3^k \cdot 2^k(\sqrt{2}\cdot 3^{-k})^s = (3\cdot 2\cdot 3^{-1-\frac{\log 2}{\log 3}})^k 2^{s/2} = 2^{s/2}$, so $\mathcal{H}^s F_1 \leq 2^{s/2}$ and $dim_H F_1 \leq \underline{dim}_B F_1 \leq \overline{dim}_B F_1 \leq s$. We define a mass distribution $\mu$ on $F$, by taking the naturual mass distribution on $F$ and spreading it uniformly along the intervals above $F$. Any set $U$ is contained in a square of side $|U|$. If $3^{-(k+1)} \leq |U| < 3^{-k}$ then $U$ lies above at most one k-level interval of $F$ of side $3^{-k}$, so $\mu(U) \leq |U|2^{-k} = |U|3^{-k\log 2/\log 3} \leq |U|(3|U|)^{\log 2/\log 3} = 3^{\log 2/\log 3}|U|^s = 2|U|^s$. By Thm 11.1, $\mathcal{H}^s(F_1) \geq s$. $\square$

**11.2 A frequently occurred construction and the method to construct a mass distribution associated with this construction**

Suppose $E$ is a bdd Borel set in $\mathbb{R}^n$. Let $\mathcal{E}_0$ consist of the single set $E$. For $k = 1, 2\cdots$ we let $\mathcal{E}_k$ be a collection of disjoint Borel subsets of $E$ s.t. each set $U$ in $\mathcal{E}_k$ is contained in one of the sets of $\mathcal{E}_{k-1}$ and contains a finite number of sets in $\mathcal{E}_{k+1}$. Furthermore, we require the maximum diameter of the sets in $\mathcal{E}_k$ tends to 0 as $k \to \infty$.

Now we define a mass distribution on $E$ by repeated subdivision. We let $\mu(E)$ satisfy $0 < \mu(E) < +\infty$, and we split this mass between the sets $U_1 \cdots U_m$ in $\mathcal{E}_1$ by defining $\mu(U_i)$ in such a way that $\sum\limits_{i=1}^m \mu(U_i) = \mu(E)$. Similarly, we assign massed to the sets of $\mathcal{E}_2$ s.t. if $U_1 \cdots U_m$ are sets of $\mathcal{E}_2$ contained in a set $U$ of $\mathcal{E}_1$, then $\sum\limits_{i=1}^m \mu(U_i) = \mu(U) \cdots\cdots$ And we define $\mu(A) = 0$ provided $A \cap E_k = \varnothing$

for some $k$, where $E_k$ is the union of all the sets in $\mathcal{E}_k$.
Let $\mathcal{E}$ denote the collection of sets that belong to $\mathcal{E}_k$ with the subsets of $\mathbb{R}^n E_k$. and then we can construct a mass distribution now.

**11.3 Theorem** Let $\mu$ be a defined on a collection of sets $\mathcal{E}$ as above, then $\mu$ can be extended to a measure on$\mathbb{R}^n$. The value of $\mu(A)$ is unique if $A \in \mathcal{B}(\mathbb{R}^n)$ and supp$\mu \subseteq \bigcap_{i=1}^{+\infty} \overline{E_k}$.

The following are some exercises solved by virtue of the mass distribution priciple method.

**11.4 Exercise** Use the mass distribution methond to calculate the dimension of Cantor dust.

Sol: Supper estimate: we just take the $\delta$-cover to be the squares of $E_k$ where $\delta = \sqrt{2} \cdot 4^{-k}$. Then $\mathcal{H}^1_{\sqrt{2}\cdot 4^{-k}}(F) \leq 4^k(\sqrt{2} \cdot 4^{-k})^1 = \sqrt{2}$

Lower estimate: For each square $A_k$ with side $4^{-k}$, define $\mu(A_k) = 4^{-k}$. Then $\mu$ satisfies the prerequisite condition of a mass distribution. By Thm 11.3 we $\mu$ can be extended to a mass distribution on $\mathbb{R}^n$. Then we restrict $\mu$ on $F$ by $\mu(A) = \mu(A \cap F)$ for any subset $A$ of $\mathbb{R}^n$. Claim: $\forall U \subseteq \mathbb{R}^n$, if $4^{-(k+1)} \leq |U| < 4^{-k}$, then $U$ insetsects at most one k-level basic square of side $4^{-k}$. This is because for those k-levle basic squares of $E_k$, the distance between each other is at least $4^{-k}$. Hence, $\mu(U) \leq \mu(A_j) = 4^{-k} < 4|U|$, where $A_j$ is the k-level square that $U$ intersetcts. Thus, by the mass distribution principle, $\mathcal{H}^1(F) \geq \frac{\mu(F)}{4}$.

Combine the upper and lower estimates together we can get $dim_H F = 1$. □

**11.5 Exercise** Let $F$ be the Contor set. Calculate the $dim_H$ of $F \times F$.

Sol: In order to construct a mass distribution, we need: $(\frac{1}{q})^s \cdot 4 = 1 \Rightarrow s = \log 2/\log 3$. Note that the distance of any two basic squares of $E_k$ is at least $2^{-k}$, then run a similar proof of Exercise 11.4 we can obtain $0 < \mathcal{H}^s(F \times F) < +\infty$.
□

**11.6 Exercise** Directly calculate $dim_H$ of the uniform canot set with $m = 3$ and $r = \frac{4}{15}$.

Sol: set $s = \log 3/(-\log \frac{4}{15})$. For any k-level interal $I$ of $E_k$, define $\mu(I) = |I|^s$. Suppose $I_1, I_2, I_3$ are the three (k+1)-level subintervals distributed by $I$. Then by taking some algebra we can find $\mu(I_i) = \frac{1}{3}|I|^s, i = 1, 2, 3$. Then, $\mu$ can be extended to a distribution measure on $F$ with $\mu(I) = |I|^s$.

Lower estimate: For any set $U$, since supp$\mu \subseteq F$, W.L.O.G. we can assume the endpoints of $U$ can be covered by one basic interval, say, $I \in E_k$. We let $I$ to be the smallest one. Let $I_1, I_2, I_3$ be the (K+1)-level intevals contained in $I$. Then

$U$ intersects at least two of them, otherwise $U$ can be covered by a smaller basic interval, contradiction. W.L.O.G. we can assume $U$ intersects $I_1, I_2, I_3$. The space between $I_i$ is $(|I| - 3|I_3|)/2 = |I|(1 - \frac{3|I_i|}{|I|})/2 = |I|(1 - 3^{1-\frac{1}{s}})/3 \geq C|I|/3$ where $C := 1 - 3^{1-\frac{1}{s}}$. Then, $|U| \geq \frac{1}{3}C|I|$. On the other hand, $\mu(U) \leq 3\mu(I_i) = 3|I_i|^s = |I|^s \leq \left(\frac{3|U|}{C}\right)^s \frac{3^s}{C^s}|U|^s$. By the mass distribution principle, $\mathcal{H}^s(F) > 0$. $\square$

# 12 The relationship of Hausdorff measure and mass distribution

**12.1 Theorem** Let $\mu$ be a mass distribution on $\mathbb{R}^n$, $F \in \mathcal{B}(\mathbb{R}^n)$ and $C \in ]0, +\infty[$. Then:

(a) If $\overline{\lim\limits_{r\to 0}} \frac{\mu(B(x,r))}{r^s} < C$ for all $x \in F$ ,then $\mathcal{H}^s \geq \frac{\mu(F)}{C}$.

(b) If $\overline{\lim\limits_{r\to 0}} \frac{\mu(B(x,r))}{r^s} > C$ for all $x \in F$ ,then $\mathcal{H}^s(F) \leq \frac{2^s \mu(\mathbb{R}^n)}{C}$.

Proof: (a) $\forall \delta > 0, F_\delta := \left\{x \in F : \mu(B(x,r)) < Cr^s \text{ for all } 0 < r \leq \delta\right\}$. Let $\{U_i\}$ be a $\delta$-cover of $F$ and thus of $F_\delta$. $\forall i, \forall x \in U_i, U_i \subseteq B := B(x, |U_i|)$. Then by def of $F_\delta, \mu(U_i) \leq \mu(B) < C|U_i|^s$. Hence, $\mu(F_\delta) \leq \sum\limits_i \mu(U_i) \leq C \sum\limits_i |U_i|^s \Rightarrow \mu(F_\delta) \leq C\mathcal{H}^s_\delta(F) \leq C\mathcal{H}^s(F)$. Simultaneously, $\overline{\lim\limits_{r\to 0}} \frac{\mu(B(x,r))}{r^s} < C$ ensures that $F_\delta \uparrow F$ as $\delta \downarrow 0$. Hence, $\mu(F) \leq C\mathcal{H}^s(F)$. i.e. $\mathcal{H}^s(F) \geq \frac{\mu(F)}{C}$.

(b) Consider the collection $\left\{B(x,r) : x \in F, 0 < r \leq \delta \text{ and } \mu(B(x,r)) > Cr^s\right\}$ and use the cover lemma. $\square$

**12.2 Corollary** If $\lim\limits_{r\to 0} \frac{\mu(B(x,r))}{r^s}$ converges for all $x \in F$, then $dim_H F = s$.

# 13 Subsets of finite measure

**13.1 Theorem** Let $F$ be a Borel subset of $\mathbb{R}^n$ with $0 < \mathcal{H}^s(F) \leq +\infty$. Then there is a compact set $E \subseteq F$ s. t. $0 < \mathcal{H}^s(F) < +\infty$.

**13.2 Prop** Let $F$ be a Borel set satisfying $0 < \mathcal{H}^s(F) < +\infty$. Then, there is a constant b and a compact set $E \subseteq F$ with $\mathcal{H}^s(E) > 0$ s.t. $\mathcal{H}^s(E \cap B(x,r)) \leq br^s$ for all $x \in \mathbb{R}^n$ and $r > 0$.

Proof: In Thm 12.1(b), take $\mu$ as the restriction of $\mathcal{H}^s$ to $F$, i.e. $\mu(A) = \mathcal{H}^s(F \cap A)$. Then, if $F_1 := \left\{x \in \mathbb{R}^n : \overline{\lim\limits_{r\to 0}} \mathcal{H}^s(F \cap B(x,r))/r^s > 2^{1+s}\right\}$, it follows that $\mathcal{H}^s(F - 1) \leq 2^s \cdot 2^{-(1+s)} \mu(F) = \frac{1}{2}\mathcal{H}^s(F)$. Thus $\mathcal{H}^s(F \setminus F_1) \geq \frac{1}{2}\mathcal{H}^s(F) > 0$. Set $E_1 = \frac{F}{\setminus} F_1$, then $\mathcal{H}^s(E_1) > 0$ and $\overline{\lim\limits_{r\to 0}} \mathcal{H}^s F \cap B(x,r)/r^s \leq 2^{1+s}$ for $x \in E_1$.

By Egoroff, $\exists$ compact set $E \subseteq E_1$ with $\mathcal{H}^s(E) > 0$ and $r_0 > 0$ s.t. $\forall x \in E, \forall 0 < r < r_0, \mathcal{H}^s(F \cap B(x,r))/r^s \leq 2^{2+s}$. However, we have $\mathcal{H}^s(F \cap B(x,r))/r^s \leq \mathcal{H}^s(F)/r_0^s$ if $r \geq r_0$. Hence, $\forall x\ in \mathbb{R}^n, \forall r > 0, \mathcal{H}^(F \cap B(x,r)) \leq br^s$ where $b = \max 2^{2+s, \mathcal{H}^s(F)/r_0^s}$. $\square$

Proof: By Thm 13.1, $\exists F_1 \in \mathcal{B}(\mathbb{R}^n)$ with $0 < \mathcal{H}^s(F) \leq +\infty$. Then $\exists$ compact set $F_1 \subseteq F$ s.t. $0 < \mathcal{H}^s(F_1) < +\infty$. Then apply Prop 13.2 to $F_1$ and we can get the result. $\square$

# 14 Potential theoretic methods

**14.1 Def** For $s \geq 0$, the s-potential at a point x of $\mathbb{R}^n$ du to the mass distribution $\mu$ on $\mathbb{R}^n$ is defined as $\phi_s(x) := \int \frac{d\mu(y)}{|x-y|^s}$. The s-energy of $\mu$ is $I_s(\mu) := \int \phi_s(x) d\mu(x) = \iint \frac{d\mu(x)d\mu(y)}{|x-y|^s}$.

**14.2 Theorem** Let $F \subseteq \mathbb{R}^n$ ,then

(a) If there is a mass distribution $\mu$ on $F$ with $I_s(\mu) < +\infty$ then $\mathcal{H}^s(F) = +\infty$ and hene $dim_H F \geq s$.

(b) If $F$ is a Borel set with $\mathcal{H}^s(F) > 0$ then there exsits a mass distribution $\mu$ on $F$ with $I_t(\mu) < +\infty$ for all $0 < t < s$.

Proof: (a)Define $F_1 = \{x \in F : \overline{\lim_{r \to 0}} \mu(B(x,r))/r^s > 0\}$. If $x \in F_1$, we may find $\epsilon > 0$ and sequence of numbers $r_i \downarrow 0$ s.t. $\mu(B(x,r)) \geq \epsilon r_i^s$. Since $\mu(x) = 0$ (otherwise $I_s(\mu) = +\infty$), it follows from the continuity of $\mu$ that, by taking $q_i (0 < q_i < r_i)$ small enough, we can get $\mu(A_i) \geq \frac{1}{4}\epsilon r_i^s$, Where $A_i = B(x, r_i) \setminus B(x, q_i)$. Taking subsequences if necessary, we may assume that $r_{i+1} < q_i$ for all $i$, so that the $A_i$ are disjoint annuli centered at x. Hence , for all $x \in F_1, \phi_s(x) = \int \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{+\infty} \int_{A_i} \frac{d\mu(y)}{|x-y|^s} \geq \sum_{i=1}^{+\infty} \frac{1}{4}\epsilon r_i^s r_i^s = +\infty$. Note $|x-y|^{-s} \geq r_i^{-s}$ on $A_i$. But $I_s(\mu) = \int \phi_s(x d\mu(x)) < +\infty <$ so $\phi_s(x) < +\infty$ for $\mu$-a.e. x. Hence, $\mu(F_1) = 0$. Since $\overline{\lim_P} r \to 0 \mu(B(x,r))/r^s = 0$ for $x \in F \setminus F_1$. By Thm 14.2(a), $\mathcal{H}^s(F) \geq \mathcal{H}^s(F \setminus F_1) \geq \frac{\mu(F \setminus F_1)}{C} \geq \frac{\mu(F) - \mu(F_1)}{C} = \frac{\mu(F)}{C}$ for all $C > 0$. Hence, $\mathcal{H}^s(F) = +\infty$.

(b) By Corollary 13.3, there exists a compact set $E \subseteq F$ with $0 < \mathcal{H}^s(E) < +\infty$ and $b \in \mathbb{R}$ s.t. $\mathcal{H}^s(E \cap B(x,r)) \leq br^s$ for all $x \in \mathbb{R}^n$ and $r > 0$. Let $\mu$ be the restriction of $\mathcal{H}^s$ to $E$, so that $\mu(A) = \mathcal{H}^s(E \cap A)$. Then $\mu$ becomes a mass distribution on $F$. Fix $x \in \mathbb{R}^n$ and write $m(r) := \mu(B(x,r))\mathcal{H}^s(E \cap B(x,r)) \leq br^s$. Then, $\forall 0 < t < s, \phi_t(x) = \int_{|x-y| \leq 1} \frac{d\mu(y)}{|x-y|^t} + \int_{|x-y|>1} \frac{d\mu(y)}{|x-y|^t} \leq \int_0^1 r^{-t} dm(r) +$

$\mu(\mathbb{R}^n) = [r^{-t}m(r)]_0^1 + t\int_0^1 r^{-(t+1)}m(r)dr + \mu(\mathbb{R}^n) \leq b + bt\int_0^1 r^{s-t-1}dr + \mu(\mathbb{R}^n) = b(1+\frac{t}{s-t}) + \mathcal{H}^s(F) := C$. Thus, $\forall x \in \mathbb{R}^n, \phi_t(x) \leq C$. Then $I_t(\mu) = \inf\left\{\phi_t(x)d\mu(x) \leq C\mu\mathbb{R}^n < +\infty\right\}$. $\square$