1. Elementary Algebra:

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1.1 $\frac{(z^3)^2}{z^2 \cdot z^8} = \frac{z^6}{z^{10}} = z^{-4}$

1.2 $12^2 \cdot 3^x \cdot 2^{2x} = 12^{-2}$

$\log(12^2) + \log(3^x) + \log(4^x) = -2\log(12)$

$x\log(3) + x\log(4) = -4\log(12)$

$x(\log(3) + \log(4)) = -4\log(12)$

$x\log(12) = -4\log(12)$

$x = -4.$

1.3. P.S. I'm assuming the equation is $x^{-1} \cdot y^{-1} = 5$ that's given,

so $xy = \frac{1}{5}$, $(xy)^3 = (\frac{1}{5})^3 \Rightarrow x^3 \cdot y^3 = \frac{1}{125}$

1.4 $\frac{\sqrt{3^{10}}}{\sqrt{9^3}} = \frac{3^5}{9^{\frac{3}{2}}} = \frac{3^5}{3^3} = 3^2 = 9$

1.5. a) T ; b). T; c). F; d). F.

1.6 $\frac{7x-10}{2} \geq 9$ $\quad 7x - 10 \geq 18$

$7x \geq 28$

$x \geq 4$

2. Functions of one Variable

2.1. Set the linear relationship as $y = ax+b$, where x is measure in Celsius, y is measure in Fahrenheit

$$\begin{cases} b = 32 \\ 100a + 32 = 212 \end{cases} \quad \therefore y = 1.8x + 32 \qquad \begin{cases} y = 1.8x + 32 \\ x = y \end{cases} \quad \therefore -0.8y = 32, \quad y = x = -40.$$

$\therefore$ When the temperature is -40, then both scales are the same.

2.2 $f(y) = 3y + 3$, now $f(y) = 54 \Rightarrow 3y + 3 = 54 \quad y = 17.$

2.3 $10^{4x^2-16x+3} = 1000 \Rightarrow 4x^2 - 16x + 3 = 3$

$$4x^2 - 16 = 0.$$
$$x^2 = 4$$
$$x = \pm 2$$

2.4 Let's set GDP as X, T as years.

$$X \cdot (1+3.2\%)^T = 2X$$
$$(1+3.2\%)^T = 2$$
$$T \log(1+3.2\%) = \log(2)$$
$$T = \frac{\log(2)}{\log(1.032)} \doteq 22 \text{ years.}$$

2.5 $\ln\left(\frac{1}{e^{-5}}\right) = \ln(e^5) = 5\ln(e) = 5$

3. Calculus

3.1 $\sum_{i=0}^{\infty}\left(\frac{1}{13^i} + 0.25^i\right) = \sum_{i=0}^{\infty}\left(\frac{1}{13}\right)^i + \sum_{i=0}^{\infty} 0.25^i$

$= \left(\frac{1}{13}\right)^0 + \sum_{i=1}^{\infty}\left(\frac{1}{13}\right)^i + (0.25)^0 + \sum_{i=1}^{\infty} 0.25^i$

$= 1 + \left(\frac{1}{13}\right) \times \frac{1-\cancel{\left(\frac{1}{13}\right)^{\infty}}^{=0}}{1-\left(\frac{1}{13}\right)} + 1 + 0.25 \times \frac{1-\cancel{0.25^{\infty}}^{=0}}{1-0.25}$

$= 1 + \frac{1}{13} \times \frac{13}{12} + 1 + \frac{1}{4} \times \frac{4}{3} = \frac{29}{12}$

3.2. $\lim_{x \to 4} \frac{2x-8}{2} = 0$

3.3 $f(x) = x^3 - 4$, $f'(x) = 3x^2$, when $x = -1$, $f'(x) = 3$

3.4 $\frac{d}{dx}\frac{2x^2+x}{x-32} = \frac{4x+1}{x-32} - \frac{2x^2+x}{(x-32)^2} = \frac{(4x+1)(x-32)-2x^2-x}{(x-32)^2}$

$= \frac{2x^2-128x-32}{(x-32)^2}$

3.5 $\frac{d^2}{dx^2} 4x^{-3} + 4 = \frac{d}{dx}(-12x)^{-4}$

$= 48x^{-5}$

3.6 $f(x) = \frac{1}{x+2}$

$\because \lim_{x \to -2^+} \frac{1}{x+2} \neq \lim_{x \to -2^-} \frac{1}{x+2}$

$\lim_{x \to -2^+} \frac{1}{x+2} = \infty$

$\therefore$ It's not continuous at $x = -2$.

$\lim_{x \to -2^-} \frac{1}{x+2} = -\infty$

3.7. $f(x) = \frac{\ln x}{x}$, $\frac{d}{dx}\left(f(x) = \frac{\ln x}{x}\right) = \frac{1-\ln x}{x^2} = 0$

$1 - \ln x = 0 \quad x = e.$

So there's only one stationary point at $x = e$.

$$\frac{d^2}{dx^2}\left(\frac{\ln x}{x}\right) = -2x^{-3}(1-\ln x) - x^{-2} \cdot \frac{1}{x}$$

$$= -2x^{-3}(1-\ln x) - x^{-3}$$

$$= \frac{-2(1-\ln x)-1}{x^3}$$

Set $-2(1-\ln x) - 1 = 0$

$-2(1-\ln x) = 1 \quad (1-\ln x) = -\frac{1}{2} \quad -\ln x = -\frac{3}{2} \quad x = e^{\frac{3}{2}}$

So when $x > e^{\frac{3}{2}}$, $f''(x) = \frac{-2(1-\ln x)-1}{x^3} > 0$ $\boxed{\text{convex}}$

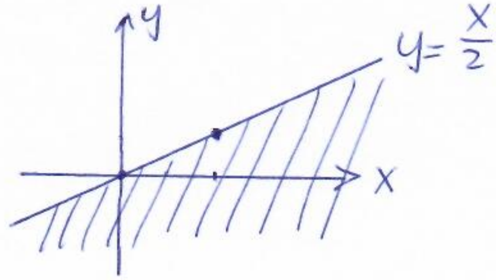
when $x < e^{\frac{3}{2}}$, $f''(x) = \frac{-2(1-\ln x)-1}{x^3} < 0$ $\boxed{\text{concave}}$

$\therefore$ The stationary point at $x = e$ is a local maxima and also a global maxima.

3.8 $f(x,y) = x^3y^2 \quad f(2,3) = 2^3 3^2 = 72$

3.9. $f(x,y) = \ln(x-2y)$

$x - 2y > 0 \Rightarrow x > 2y, \ y < \frac{x}{2}$



3.10. $\frac{\partial^2}{\partial x^2} x^5 + x^2y^3$

$= \frac{\partial}{\partial x} 5x^4 + 2y^3x$

$= 20x^3 + 2y^3.$

3.11. $f(x,y) = \sqrt{xy} - 0.25x - 0.25y.$

$\frac{\partial^2 f}{\partial x^2} = \frac{\partial f}{\partial x}\left(\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} - 0.25\right) = -\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}$

$\frac{\partial^2 f}{\partial x \partial y} = \frac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$

$\frac{\partial^2 f}{\partial y^2} = \frac{\partial f}{\partial y}\left(\frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} - 0.25\right) = -\frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{2}}$

$\frac{\partial^2 f}{\partial y \partial x} = \frac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}}$

$$\therefore H = \begin{bmatrix} -\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}} & \frac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}} \\ \frac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}} & -\frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{2}} \end{bmatrix}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} - 0.25 = 0. \qquad \frac{\partial f}{\partial y} = \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} - 0.25 = 0.$$

$$\frac{1}{2}x^{-\frac{1}{2}}y^{\frac{1}{2}} = 0.25 \quad ① \qquad \frac{1}{2}x^{\frac{1}{2}}y^{-\frac{1}{2}} = 0.25 \quad ②$$

$\frac{②}{①}$: $y = \frac{1}{4}x$

$$Det(H) = \left(-\frac{1}{4}x^{-\frac{3}{2}}y^{\frac{1}{2}}\right)\left(-\frac{1}{4}x^{\frac{1}{2}}y^{-\frac{3}{2}}\right) - \left(\frac{1}{4}x^{-\frac{1}{2}}y^{-\frac{1}{2}}\right)^2$$

$$= \frac{1}{16}x^{-1}y^{-1} - \frac{1}{16}x^{-1}y^{-1} = 0$$

$\therefore$ the second derivative test is inconclusive. Any points on the solution of F.O.C. ($y = \frac{1}{4}x$) could be any of a minimum, maximum or saddle point.

3.12 max x^3y^3 s.t. $x+y=2$

let $\mathcal{L} = x^3y^3 + \lambda(2-x-y)$

F.O.C. $\frac{\partial \mathcal{L}}{\partial x} = 3x^2y^3 - \lambda = 0$ ①

$\frac{\partial \mathcal{L}}{\partial y} = 3x^3y^2 - \lambda = 0$ ②

$\frac{\partial \mathcal{L}}{\partial \lambda} = 2 - x - y = 0$ ③

$\frac{①}{②}$: $x^{-1}y - 1 = 0$

$x = y$ ④.

Take ④ into ③:

$x = y = 1$, $\lambda = 3$

$\therefore x^3y^3 = 1$

check S.O.C.

$$|H| = \begin{vmatrix} 0 & -1 & -1 \\ -1 & 6xy^3 & 9x^2y^2 \\ -1 & 9x^2y^2 & 6x^3y \end{vmatrix}$$

$Det(H) = 9x^2y^2 + 9x^2y^2 - 6xy^3 - 6x^3y$

$= 18x^2y^2 - 6xy^3 - 6x^3y$

when $x = y = 1$

$Det(H) = 18 - 6 - 6 = 6 > 0$.

$\therefore (1,1)$ is a maximum.

4. Linear Algebra:

4.1. $\begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 11 & 8 \\ 6 & 17 & 6 \\ 5 & 6 & 5 \end{bmatrix}$

4.2 $\begin{bmatrix} 1 & 4 & 1 \\ 2 & 1 & 2 \end{bmatrix} \cdot \begin{bmatrix} 2 & 3 \\ 4 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 19 & 9 \\ 9 & 11 \end{bmatrix}$

4.3 $\begin{bmatrix} 3.3 & 5.1 & 4.7 \\ 2 & 6.1 & 1.23 \\ 4 & 5.76 & 0 \end{bmatrix}^T = \begin{bmatrix} 3.3 & 2 & 4 \\ 5.1 & 6.1 & 5.76 \\ 4.7 & 1.23 & 0 \end{bmatrix}$

4.4. $\det \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} = 2 \times 5 - 3 \times 4 = 10 - 12 = -2$

5. Probability Theory.

5.1. Sample space = {HHHH, HHHT, HHTH, HHTT,
HTHH, HTHT, HTTH, HTTT,
THHH, THHT, THTH, THTT,
TTHH, TTHT, TTTH, TTTT}

5.2 Drug   No Drug   Positive   Negative

D   ND   P   N

We know: $P(P|D) = 99\%$

$P(N|ND) = 99.5\%$

$P(D) = 1\% \Rightarrow P(ND) = 99\%$

We need to find: $P(D|P) = ?$

$$P(D|P) = \frac{P(D \& P)}{P(P)}$$

$$P(P|D) = \frac{P(P \& D)}{P(D)} = 99\% \Rightarrow P(P \& D) = 99\% \times 1\% = 0.99\%$$

$$P(P) = \underbrace{P(P \& D)}_{0.99\%} + P(P \& ND)$$

$$P(N|ND) = \frac{P(N \& ND)}{P(ND)} = 99.5\%$$

$$\Rightarrow P(N \& ND) = 99.5\% \times 99\% = 98.505\%$$

$$\therefore P(P \& ND) = P(ND) - P(N \& ND) = 0.495\%$$

$$\therefore P(P) = 0.99\% + 0.495\% = 1.485\%$$

$$\therefore P(D|P) = \frac{P(D \& P)}{P(P)} = \frac{0.99\%}{1.485\%} = 66.67\%$$

5.3.

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12.

$$E(X) = 2 \times \frac{1}{36} + 3 \times \frac{2}{36} + 4 \times \frac{3}{36} + 5 \times \frac{4}{36} + 6 \times \frac{5}{36}$$

$$+ 7 \times \frac{6}{36} + 8 \times \frac{5}{36} + 9 \times \frac{4}{36} + 10 \times \frac{3}{36} + 11 \times \frac{2}{36}$$

$$+ 12 \times \frac{1}{36}$$

$$= 7$$