Abstract

1.

The equation of $n_\varepsilon(x,\theta)$

$$\begin{cases} -D(\theta)\Delta_x n_\varepsilon - \varepsilon^2\Delta_\theta n_\varepsilon = n_\varepsilon(K(x)-\rho_\varepsilon(x)), \\ \rho_\varepsilon = \int_0^1 n_\varepsilon(x,\theta)d\theta \end{cases} \tag{1.1}$$

Set $n_\varepsilon(x,\theta) = W_\varepsilon(x,\theta)e^{u(\theta)/\varepsilon}$, then

$$\begin{aligned} &\nabla_x n_\varepsilon = \nabla_x W_\varepsilon\ e^{u/\varepsilon}, \\ &\Delta_x n_\varepsilon = \Delta_x W_\varepsilon\ e^{u/\varepsilon}, \\ &\nabla_\theta n_\varepsilon = \nabla_\theta W_\varepsilon\ e^{u/\varepsilon} + W_\varepsilon\ e^{u/\varepsilon}\frac{\nabla_\theta u}{\varepsilon}, \\ &\Delta_\theta n_\varepsilon = \Delta_\theta W_\varepsilon\ e^{u/\varepsilon} + 2\nabla_\theta W_\varepsilon\ e^{u/\varepsilon}\ \frac{\nabla_\theta u}{\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{|\nabla_\theta u|^2}{\varepsilon^2} + W_\varepsilon e^{u/\varepsilon}\frac{\Delta_\theta u}{\varepsilon} \end{aligned} \tag{1.2}$$

and (1.1) will be

$$\begin{aligned} &-D(\theta)\Delta_x W_\varepsilon\ e^{u/\varepsilon} - \varepsilon^2\left(\Delta_\theta W_\varepsilon\ e^{u/\varepsilon} + 2e^{u/\varepsilon}\ \frac{\nabla_\theta W_\varepsilon\cdot\nabla_\theta u}{\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{|\nabla_\theta u|^2}{\varepsilon^2} + W_\varepsilon e^{u/\varepsilon}\frac{\Delta_\theta u}{\varepsilon}\right) \\ &= W_\varepsilon e^{u/\varepsilon}(K(x)-\rho_\varepsilon(x)), \\ &-D(\theta)\Delta_x W_\varepsilon\ e^{u/\varepsilon} - \varepsilon^2\Delta_\theta W_\varepsilon\ e^{u/\varepsilon} - 2e^{u/\varepsilon}\ \nabla_\theta W_\varepsilon\cdot\nabla_\theta u\varepsilon - W_\varepsilon e^{u/\varepsilon}|\nabla_\theta u|^2 - W_\varepsilon e^{u/\varepsilon}\Delta_\theta u\varepsilon \\ &= W_\varepsilon e^{u/\varepsilon}(K(x)-\rho_\varepsilon(x)), \end{aligned} \tag{1.3}$$

i.e.,

$$-D(\theta)\Delta_x W_\varepsilon - \varepsilon^2\Delta_\theta W_\varepsilon - 2\varepsilon\nabla_\theta W_\varepsilon\cdot\nabla_\theta u - W_\varepsilon|\nabla_\theta u|^2 - W_\varepsilon\Delta_\theta u\varepsilon = W_\varepsilon(K(x)-\rho_\varepsilon(x)), \tag{1.4}$$

$$\begin{cases} -D(\theta)\Delta_x W_\varepsilon - \varepsilon^2\Delta_\theta W_\varepsilon - 2\varepsilon\nabla_\theta W_\varepsilon\cdot\nabla_\theta u = W_\varepsilon(K(x)-\rho_\varepsilon(x)), \\ -|\nabla_\theta u|^2 - \varepsilon\Delta_\theta u = 0 \end{cases} \tag{1.5}$$

$$\begin{cases} -D(\theta)\Delta_x W_\varepsilon - \varepsilon^2\Delta_\theta W_\varepsilon - 2\varepsilon\nabla_\theta W_\varepsilon\cdot\nabla_\theta u = W_\varepsilon(K(x)-\rho_\varepsilon(x)) + W_\varepsilon H, \\ -|\nabla_\theta u|^2 - \varepsilon\Delta_\theta u = -H \end{cases} \tag{1.6}$$

The eigenvalue problem

$$-D(\theta)\Delta_x W_\varepsilon - W_\varepsilon(K(x)-\rho_\varepsilon(x)) = W_\varepsilon H, \tag{1.7}$$

$$\rho_\varepsilon(x) = \int_0^1 W_\varepsilon(x,\theta)e^{u(\theta)/\varepsilon}d\theta \tag{1.8}$$

$$\varepsilon\to 0, \qquad \rho_\varepsilon(x)\to\int_0^1 W_\varepsilon(x,\theta)\delta(\theta-\theta_m)d\theta = \bar{W}, \tag{1.9}$$

where $\bar{W}(x)$ satisfies

$$-D(\theta_m)\Delta_x\bar{W} = \bar{W}(K(x)-\bar{W})$$

1. initial value $W_\varepsilon(x,\theta), u(\theta)$;
2. calculate ρ_ε;
3. eigenvalue H;
4. the second equation, calculate $u(\theta)$;
5. the first equation, calculate W_ε.

$$\partial_t n_\varepsilon - D(\theta)\Delta_x n_\varepsilon - \varepsilon^2\Delta_\theta n_\varepsilon = n_\varepsilon(K(x) - \rho_\varepsilon(x))$$

Set $n_\varepsilon(x,\theta,t) = W_\varepsilon(x,\theta,t)e^{u(\theta,t)/\varepsilon}$, then

$$\begin{aligned}
&\partial_t n_\varepsilon = \partial_t W_\varepsilon e^{u/\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{\partial_t u}{\varepsilon},\\
&\nabla_x n_\varepsilon = \nabla_x W_\varepsilon\ e^{u/\varepsilon},\\
&\Delta_x n_\varepsilon = \Delta_x W_\varepsilon\ e^{u/\varepsilon},\\
&\nabla_\theta n_\varepsilon = \nabla_\theta W_\varepsilon\ e^{u/\varepsilon} + W_\varepsilon\ e^{u/\varepsilon}\frac{\nabla_\theta u}{\varepsilon},\\
&\Delta_\theta n_\varepsilon = \Delta_\theta W_\varepsilon\ e^{u/\varepsilon} + 2\nabla_\theta W_\varepsilon\ e^{u/\varepsilon}\ \frac{\nabla_\theta u}{\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{|\nabla_\theta u|^2}{\varepsilon^2} + W_\varepsilon e^{u/\varepsilon}\frac{\Delta_\theta u}{\varepsilon}
\end{aligned} \tag{1.10}$$

and

$$\begin{aligned}
&\partial_t W_\varepsilon e^{u/\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{\partial_t u}{\varepsilon}\\
&- D(\theta)\Delta_x W_\varepsilon\ e^{u/\varepsilon} - \varepsilon^2\left(\Delta_\theta W_\varepsilon\ e^{u/\varepsilon} + 2e^{u/\varepsilon}\ \frac{\nabla_\theta W_\varepsilon\cdot\nabla_\theta u}{\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{|\nabla_\theta u|^2}{\varepsilon^2} + W_\varepsilon e^{u/\varepsilon}\frac{\Delta_\theta u}{\varepsilon}\right)\\
&= W_\varepsilon e^{u/\varepsilon}(K(x) - \rho_\varepsilon(x)),\\
&\partial_t W_\varepsilon e^{u/\varepsilon} + W_\varepsilon e^{u/\varepsilon}\frac{\partial_t u}{\varepsilon}\\
&- D(\theta)\Delta_x W_\varepsilon\ e^{u/\varepsilon} - \varepsilon^2\Delta_\theta W_\varepsilon\ e^{u/\varepsilon} - 2e^{u/\varepsilon}\ \nabla_\theta W_\varepsilon\cdot\nabla_\theta u\varepsilon - W_\varepsilon e^{u/\varepsilon}|\nabla_\theta u|^2 - W_\varepsilon e^{u/\varepsilon}\Delta_\theta u\varepsilon\\
&= W_\varepsilon e^{u/\varepsilon}(K(x) - \rho_\varepsilon(x)),
\end{aligned} \tag{1.11}$$

and

$$\begin{cases}
\partial_t W_\varepsilon - D(\theta)\Delta_x W_\varepsilon - \varepsilon^2\Delta_\theta W_\varepsilon - 2\varepsilon\nabla_\theta W_\varepsilon\cdot\nabla_\theta u = W_\varepsilon(K(x) - \rho_\varepsilon(x)),\\
\dfrac{\partial_t u}{\varepsilon} - |\nabla_\theta u|^2 - \varepsilon\Delta_\theta u = 0
\end{cases} \tag{1.12}$$

so

$$\begin{cases}
\partial_t W_\varepsilon - D(\theta)\Delta_x W_\varepsilon - \varepsilon^2\Delta_\theta W_\varepsilon - 2\varepsilon\nabla_\theta W_\varepsilon\cdot\nabla_\theta u = W_\varepsilon(K(x) - \rho_\varepsilon(x)) + W_\varepsilon H,\\
\dfrac{\partial_t u}{\varepsilon} - |\nabla_\theta u|^2 - \varepsilon\Delta_\theta u = -H
\end{cases} \tag{1.13}$$

where H and ρ_ε are obtained by the eigenvalue problem

$$-D(\theta)\Delta_x W_\varepsilon - W_\varepsilon(K(x) - \rho_\varepsilon(x)) = W_\varepsilon H, \tag{1.14}$$

and

$$\rho_\varepsilon(x) = \int_0^1 W_\varepsilon(x,\theta)e^{u(\theta)/\varepsilon}d\theta \tag{1.15}$$

The step to solve the problem:

- initial value $t = 0$: $W_\varepsilon(x,\theta,0), u(\theta,0)$;
- calculate $\rho_\varepsilon(x,0)$;
- solve the eigenvalue problem and obtain H;
- solve the second equation, obtain $u(\theta)$;
- solve the first equation, obtain W_ε.

Discretization

1D:

$$\rho_\varepsilon(x_j) = \int_0^1 W_\varepsilon(x_j,\theta)e^{u(\theta)/\varepsilon}d\theta = \sum_{i=1}^{M}\int_{\theta_{i-1}}^{\theta_i} W_\varepsilon(x_j,\theta)e^{u(\theta)/\varepsilon}d\theta \tag{1.16}$$

The choose of numerical integration?

The eigenvalue problem: Sturm-Liouville theory. We obtain $H_0, H_1, \cdots, H_M$.

$$-D(\theta)\frac{W_\varepsilon(x_{j+1},\theta)-2W_\varepsilon(x_j,\theta)+W_\varepsilon(x_{j-1},\theta)}{\delta x^2} - W_\varepsilon(x_j,\theta)(K(x_j)-\rho_\varepsilon(x_j)) = W_\varepsilon(x_j,\theta)H, \qquad j=1,2,\cdots,N-1 \tag{1.17}$$

with the Neumann boundary condition:

$$W_\varepsilon(x_0,\theta_j) = W_\varepsilon(x_1,\theta_j), \qquad W_\varepsilon(x_N,\theta_j) = W_\varepsilon(x_{N-1},\theta_j)$$

the matrix:

$$-D(\theta)\frac{1}{\delta x^2}(-2diag(ones(n,1),0)+diag(ones(n-1,1),1)+diag(ones(n-1,1),-1)) - diag(K_j-\rho_j)$$

i.e.

$$-D(\theta)\frac{1}{\delta x^2}\begin{bmatrix} -1 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ 0 & 1 & -2 & \cdots & 0 \\ & & \cdots & & \\ 0 & \cdots & 1 & -2 & 1 \\ 0 & \cdots & 0 & 1 & -1 \end{bmatrix}\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-2} \\ w_{N-1}\end{pmatrix} - \begin{bmatrix} K_1-\rho_1 & 0 & \cdots & 0 \\ 0 & K_2-\rho_2 & \cdots & 0 \\ & & \ddots & \\ 0 & 0 & \cdots & K_{N-1}-\rho_{N-1}\end{bmatrix}\begin{pmatrix} w_1 \\ w_2 \\ \vdots \\ w_{N-2} \\ w_{N-1}\end{pmatrix} \tag{1.18}$$

Then we solve the second equation by finite difference method:

$$\frac{\partial_t u(\theta,t)}{\varepsilon} - |\nabla_\theta u(\theta,t)|^2 - \varepsilon\Delta_\theta u(\theta,t) = -H(\theta) \tag{1.19}$$

In 1D case,

$$\frac{\partial_t u(\theta,t)}{\varepsilon} - (\partial_\theta u(\theta,t))^2 - \varepsilon\partial_{\theta\theta} u(\theta,t) = -H(\theta) \tag{1.20}$$

and the forward Euler,

$$\begin{aligned} &\frac{u(\theta_i,t_{m+1})-u(\theta_i,t_m)}{\varepsilon\tau} - \left(\frac{u(\theta_{i+1},t_m)-u(\theta_i,t_m) or u(\theta_i,t_m)-u(\theta_{i-1},t_m)}{\delta\theta}\right)^2 \\ &-\varepsilon\frac{u(\theta_{i+1},t_{m+1})-2u(\theta_i,t_{m+1})+u(\theta_{i-1},t_{m+1})}{\delta\theta^2} = -H(\theta_i) \end{aligned} \tag{1.21}$$

i.e.

$$\begin{aligned} &\frac{u(\theta_i,t_{m+1})-u(\theta_i,t_m)}{\varepsilon\tau} - \varepsilon\frac{u(\theta_{i+1},t_{m+1})-2u(\theta_i,t_{m+1})+u(\theta_{i-1},t_{m+1})}{\delta\theta^2} = -H(\theta_i) + dthetaupwind^2, \\ &u(\theta_i,t_{m+1})-u(\theta_i,t_m) - \varepsilon^2\tau\frac{u(\theta_{i+1},t_{m+1})-2u(\theta_i,t_{m+1})+u(\theta_{i-1},t_{m+1})}{\delta\theta^2} = \varepsilon\tau(-H(\theta_i)+dthetaupwind^2), \\ &u(\theta_i,t_{m+1}) + \frac{2\varepsilon^2\tau}{\delta\theta^2}u(\theta_i,t_{m+1}) - \frac{\varepsilon^2\tau}{\delta\theta^2}u(\theta_{i+1},t_{m+1}) - \frac{\varepsilon^2\tau}{\delta\theta^2}u(\theta_{i-1},t_{m+1}) = u(\theta_i,t_m) + \varepsilon\tau(-H(\theta_i)+dthetaupwind^2), \end{aligned} \tag{1.22}$$

obtain $u(\theta_i,t_{m+1})$, and we solve the first equation

$$\partial_t W_\varepsilon(x,\theta,t) - D(\theta)\partial_{xx}W_\varepsilon(x,\theta,t) - \varepsilon^2\partial_{\theta\theta}W_\varepsilon - 2\varepsilon\partial_\theta W_\varepsilon\ \partial_\theta u = W_\varepsilon(K(x)-\rho_\varepsilon(x)) + W_\varepsilon H, \tag{1.23}$$

which is discreted as

$$
\begin{aligned}
&\frac{W_\varepsilon(x_j,\theta_i,t_{m+1})-W_\varepsilon(x_j,\theta_i,t_m)}{\tau}-D(\theta)\frac{W_\varepsilon(x_{j+1},\theta_i,t_{m+1})-2W_\varepsilon(x_j,\theta_i,t_{m+1})+W_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2}\\
&-\varepsilon^2\frac{W_\varepsilon(x_j,\theta_{j+1},t_{m+1})-2W_\varepsilon(x_j,\theta_j,t_{m+1})+W_\varepsilon(x_j,\theta_{j-1},t_{m+1})}{\delta\theta^2}\\
&-2\varepsilon dwupwind\frac{u(\theta_{j+1},t_{m+1})-u(\theta_{j-1},t_{m+1})}{2\delta\theta}\\
&=W_\varepsilon(K(x)-\rho_\varepsilon(x))+W_\varepsilon H,
\end{aligned}
\tag{1.24}
$$

i.e.

$$
\begin{aligned}
&\frac{W_\varepsilon(x_j,\theta_i,t_{m+1})-W_\varepsilon(x_j,\theta_i,t_m)}{\tau}-D(\theta)\frac{W_\varepsilon(x_{j+1},\theta_i,t_{m+1})-2W_\varepsilon(x_j,\theta_i,t_{m+1})+W_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2}\\
&-\varepsilon^2\frac{W_\varepsilon(x_j,\theta_{j+1},t_{m+1})-2W_\varepsilon(x_j,\theta_j,t_{m+1})+W_\varepsilon(x_j,\theta_{j-1},t_{m+1})}{\delta\theta^2}\\
&=2\varepsilon dwupwind\frac{u(\theta_{j+1},t_{m+1})-u(\theta_{j-1},t_{m+1})}{2\delta\theta}+W_\varepsilon(K(x)-\rho_\varepsilon(x))+W_\varepsilon H,
\end{aligned}
\tag{1.25}
$$

$$
\begin{aligned}
&W_\varepsilon(x_j,\theta_i,t_{m+1})-\frac{\tau D}{\delta x^2}[W_\varepsilon(x_{j+1},\theta_i,t_{m+1})-2W_\varepsilon(x_j,\theta_i,t_{m+1})+W_\varepsilon(x_{j-1},\theta_i,t_{m+1})]\\
&-\frac{\varepsilon^2\tau}{\delta\theta^2}[W_\varepsilon(x_j,\theta_{j+1},t_{m+1})-2W_\varepsilon(x_j,\theta_j,t_{m+1})+W_\varepsilon(x_j,\theta_{j-1},t_{m+1})]\\
&=\tau(2\varepsilon dwupwind\frac{u(\theta_{j+1},t_{m+1})-u(\theta_{j-1},t_{m+1})}{2\delta\theta}+W_\varepsilon(K(x)-\rho_\varepsilon(x))+W_\varepsilon H)+W_\varepsilon(x_j,\theta_i,t_m),
\end{aligned}
\tag{1.26}
$$

$$
\begin{aligned}
&W_\varepsilon(x_j,\theta_i,t_{m+1})-\frac{\tau D}{\delta x^2}W_\varepsilon(x_{j+1},\theta_i,t_{m+1})+\frac{2\tau D}{\delta x^2}W_\varepsilon(x_j,\theta_i,t_{m+1})-\frac{\tau D}{\delta x^2}W_\varepsilon(x_{j-1},\theta_i,t_{m+1})\\
&-\frac{\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_{i+1},t_{m+1})+\frac{2\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_i,t_{m+1})-\frac{\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_{i-1},t_{m+1})\\
&=\tau(2\varepsilon dwupwind\frac{u(\theta_{i+1},t_{m+1})-u(\theta_{i-1},t_{m+1})}{2\delta\theta}+W_\varepsilon(K(x)-\rho_\varepsilon(x))+W_\varepsilon H)+W_\varepsilon(x_j,\theta_i,t_m),
\end{aligned}
\tag{1.27}
$$

$$
\begin{aligned}
&W_\varepsilon(x_j,\theta_i,t_{m+1})+\frac{2\tau D}{\delta x^2}W_\varepsilon(x_j,\theta_i,t_{m+1})+\frac{2\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_i,t_{m+1})\\
&-\frac{\tau D}{\delta x^2}W_\varepsilon(x_{j+1},\theta_i,t_{m+1})-\frac{\tau D}{\delta x^2}W_\varepsilon(x_{j-1},\theta_i,t_{m+1})\\
&-\frac{\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_{i+1},t_{m+1})-\frac{\varepsilon^2\tau}{\delta\theta^2}W_\varepsilon(x_j,\theta_{i-1},t_{m+1})\\
&=\tau(2\varepsilon dwupwind\frac{u(\theta_{i+1},t_{m+1})-u(\theta_{i-1},t_{m+1})}{2\delta\theta}+W_\varepsilon(K(x)-\rho_\varepsilon(x))+W_\varepsilon H)+W_\varepsilon(x_j,\theta_i,t_m),
\end{aligned}
\tag{1.28}
$$

$$
\begin{aligned}
&\partial_t n_\varepsilon - D(\theta)\Delta_x n_\varepsilon - \varepsilon^2 \Delta_\theta n_\varepsilon = n_\varepsilon (K(x) - \rho_\varepsilon(x)), \\
&\rho_\varepsilon = \int_0^1 n_\varepsilon(x,\theta) d\theta
\end{aligned}
\tag{1.29}
$$

$$
\begin{aligned}
&\frac{n_\varepsilon(x_j,\theta_i,t^{m+1}) - n_\varepsilon(x_j,\theta_i,t^m)}{\delta t} - D(\theta_i)\frac{n_\varepsilon(x_{j+1},\theta_i,t_{m+1}) - 2n_\varepsilon(x_j,\theta_i,t_{m+1}) + n_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2} \\
&-\varepsilon^2 \frac{n_\varepsilon(x_j,\theta_{i+1},t_{m+1}) - 2n_\varepsilon(x_j,\theta_i,t_{m+1}) + n_\varepsilon(x_j,\theta_{i-1},t_{m+1})}{\delta\theta^2} - n_\varepsilon(x_j,\theta_i,t_{m+1})\left(K(x_j) - \rho_\varepsilon(x_j)\right) = 0
\end{aligned}
\tag{1.30}
$$

$$
\begin{aligned}
&n_\varepsilon(x_j,\theta_i,t^{m+1}) - D(\theta_i)\delta t \frac{n_\varepsilon(x_{j+1},\theta_i,t_{m+1}) - 2n_\varepsilon(x_j,\theta_i,t_{m+1}) + n_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2} \\
&- \varepsilon^2 \delta t \frac{n_\varepsilon(x_j,\theta_{i+1},t_{m+1}) - 2n_\varepsilon(x_j,\theta_i,t_{m+1}) + n_\varepsilon(x_j,\theta_{i-1},t_{m+1})}{\delta\theta^2} - \delta t n_\varepsilon(x_j,\theta_i,t_{m+1})\left(K(x_j) - \rho_\varepsilon(x_j)\right) = n_\varepsilon(x_j,\theta_i,t^m)
\end{aligned}
\tag{1.31}
$$

$$
\begin{aligned}
&n_\varepsilon(x_j,\theta_i,t^{m+1}) + 2D(\theta_i)\delta t/\delta x^2 n_\varepsilon(x_j,\theta_i,t^{m+1}) - D(\theta_i)\delta t \frac{n_\varepsilon(x_{j+1},\theta_i,t_{m+1}) + n_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2} \\
&+ 2\varepsilon^2 \delta t/\delta\theta^2 n_\varepsilon(x_j,\theta_i t_{m+1}) - \varepsilon^2 \delta t \frac{n_\varepsilon(x_j,\theta_{i+1},t_{m+1}) + n_\varepsilon(x_j,\theta_{i-1},t_{m+1})}{\delta\theta^2} \\
&- \delta t n_\varepsilon(x_j,\theta_i,t_{m+1})\left(K(x_j) - \rho_\varepsilon(x_j)\right) = n_\varepsilon(x_j,\theta_i,t^m)
\end{aligned}
\tag{1.32}
$$

$$
\rho_\varepsilon = \int_0^1 n_\varepsilon(x,\theta) d\theta = \sum_{i=1}^M \int_{\theta_{i-1}}^{\theta_i} n_\varepsilon(x,\theta) d\theta
$$

$$
\begin{aligned}
&- D(\theta)\Delta_x n_\varepsilon - \varepsilon^2 \Delta_\theta n_\varepsilon = n_\varepsilon(K(x) - \rho_\varepsilon(x)),\\
&\rho_\varepsilon = \int_0^1 n_\varepsilon(x,\theta)d\theta
\end{aligned}
\tag{1.33}
$$

$$
\begin{aligned}
&2D(\theta_i)/\delta x^2 n_\varepsilon(x_j,\theta_i,t^{m+1}) - D(\theta_i)\frac{n_\varepsilon(x_{j+1},\theta_i,t_{m+1}) + n_\varepsilon(x_{j-1},\theta_i,t_{m+1})}{\delta x^2}\\
&+ 2\varepsilon^2/\delta\theta^2 n_\varepsilon(x_j,\theta_i t_{m+1}) - \varepsilon^2\frac{n_\varepsilon(x_j,\theta_{i+1},t_{m+1}) + n_\varepsilon(x_j,\theta_{i-1},t_{m+1})}{\delta\theta^2}\\
&- n_\varepsilon(x_j,\theta_i,t_{m+1})\left(K(x_j) - \rho_\varepsilon(x_j)\right) = 0
\end{aligned}
\tag{1.34}
$$