种群演化模型中的渐近结构与保结构算法

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Outline

1. Long time adaptive dynamics of an age-structured population model
2. Kinetic models leading to volume-exclusion PKS in the diffusive limit
3. Other asymptotic structures and conclusions

Motivation

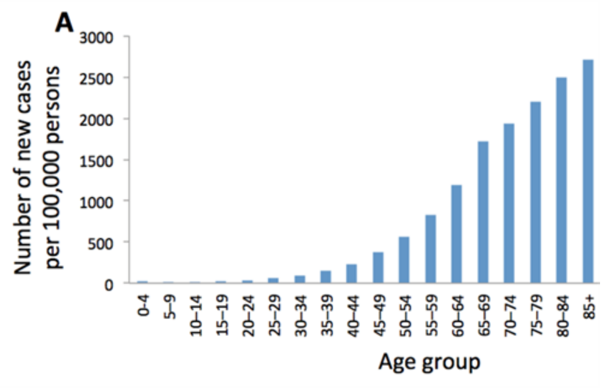


Figure: Estimated 2016 age-specific incidence rates for all cancers combined. Cancer mainly affects individuals beyond reproductive age. [1]

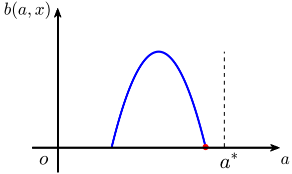
[1] AIHW analysis of the Australian Cancer Database

Age-structured Population

Introduce

- x: inherited trait.
- $b(a,x)$: birth rate satisfying

$$0 < a^* := \inf\{\, a \,|\, b(a,x) = 0, \quad \forall x \geq 0\} < \infty.$$



- $d(a,x)$: death rate satisfying

$$\lim_{a\to\infty} d(a,x) = \infty.$$

- $n(t,a,x)$: population density.
- $\rho(t) = \iint n(t,a,x)\, dadx$

Renewal Equation

Assume no death and consider only ageing, then

$$n(t+s, a+s) = n(t, a), \quad \forall s \geq 0.$$

[2] Also called **McKendrick-Von Foerster Equation**: Introduced by **McKendrick** for epidemiology, and then re-discovered by **von Foerster** for the cell division cycle

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The **renewal equation** [2]

$$\frac{\partial}{\partial t} n(t, a) + \frac{\partial}{\partial a} n(t, a) = 0.$$

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Boundary condition – new borns

$$n(t, 0) = \int_0^{a^*} b(a) n(t, a) \, da.$$

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The model without mutation

Including the death term and the trait x,

$$\begin{cases} \varepsilon \partial_t n_\varepsilon + \partial_a n_\varepsilon + d(a,x) n_\varepsilon = -\lambda_\varepsilon(t) n_\varepsilon, \\ n_\varepsilon(t, a=0, x) = \int_0^{a^*} b(a,x) n_\varepsilon \, da, \end{cases}$$

where

- ε – long time evolution,
- $\lambda_\varepsilon(t)$ – artificial term s.t. $\rho(t) \equiv 1$.

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- $\lambda_\varepsilon(t)$ – artificial term s.t. $\rho(t) \equiv 1$.

Completely decoupled system in x-direction.

Dirac Concentration

Under proper assumptions,

$$n(t,a,x) \rightharpoonup \delta(x-\bar{x}(t))N(a,\bar{x}(t)) \text{ as } \varepsilon \to 0 \quad (t \to \infty),$$

where $\bar{x}(t)$ is the fittest trait. [a]

[a] Samuel Nordmann, Benoit Perthame and Cecile Taing (2018)

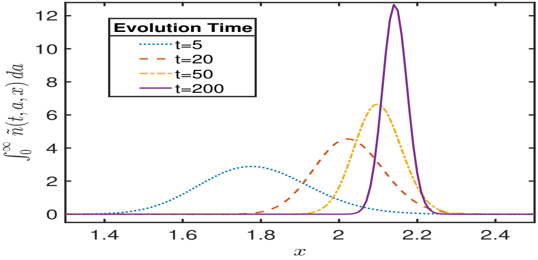

Figure: ε-dependent mesh needed to capture the Dirac concentration?

Idea to deal with concentration

WKB ansatz

Taking

$$n_\varepsilon(t,a,x) = e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} q_\varepsilon(t,a,x),$$

we expect $u_\varepsilon(t,x)$ is easy to solve with

$$e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} \rightharpoonup \delta(x-\bar{x}(t)),$$

while $q_\varepsilon(t,a,x)$ is fully regular.

Equations of q_ε and u_ε

Substituting $n_\varepsilon(t,a,x) = e^{\frac{u_\varepsilon(t,x)}{\varepsilon}} q_\varepsilon(t,a,x)$,

$$\begin{cases} q_\varepsilon \partial_t u_\varepsilon + \varepsilon \partial_t q_\varepsilon + \partial_a q_\varepsilon + d(a,x) q_\varepsilon = -\Lambda(x) q_\varepsilon + (\Lambda(x) - \lambda_\varepsilon(t)) q_\varepsilon, \\ q_\varepsilon(t, a=0, x) = \int_0^{a^*} b(a,x) q_\varepsilon(t,a,x)\, da. \end{cases}$$

[3] Existence is from the Krein-Rutman theorem.

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As $\varepsilon \to 0^+$, we expect

$$q_\varepsilon(t,a,x) \to N(a,x),$$

where $(\Lambda(x), N(a,x))$ is the eigenpair of

$$\begin{cases} \partial_a N(a,x) + d(a,x) N(a,x) = -\Lambda(x) N(a,x), \\ N(a=0,x) = \displaystyle\int_0^{a^*} b(a,x) N(a,x)\, da, \end{cases}$$

with $N > 0$, $\int_0^{a^*} N(a,x)\, da = 1$ and $-\Lambda(x)$ being the leading eigenvalue.[3]

[3] Existence is from the Krein-Rutman theorem.

Properties

Theorem (Maximum principle of q_ε)

$$0 < \underline{\gamma}(x)N(a,x) \le q_\varepsilon(t,a,x) \le \overline{\gamma}(x)N(a,x),$$

if it is true initially.

Theorem (Conservation law)

$$\int_0^{+\infty} q_\varepsilon(t,a,x)\Phi(a,x)\,da \equiv \int_0^{+\infty} q_\varepsilon^0(a,x)\Phi(a,x)\,da, \quad \forall x.^a$$

[a] Here $\Phi(a,x)$ solves the dual eigenproblem satisfying $\int_0^\infty N(a,x)\Phi(a,x)\,da = 1$.

Theorem (Limiting equations)

$$\max_x u(t,x) = 0, \text{ and } q(t,a,x) = \rho^0(x)N(a,x),$$

where $\rho^0(x) := \int_0^{+\infty} q^0(a,x)\Phi(a,x)\,da$.

Properties (discrete version)

Theorem (Maximum principle)

$$0 < \underline{\gamma_k} N_{j,k} \le q_{j,k}^n \le \overline{\gamma_k} N_{j,k},$$

if it is true initially.

Theorem (Conservation law)

$$\Delta a \sum_{j=1}^{K_a} q_{j,k}^n \phi_{j-1,k} \equiv \Delta a \sum_{j=1}^{K_a} q_{j,k}^0 \phi_{j-1,k}.$$

Theorem (Asymptotic preserving)

$$\max_k \; u_k^{n+1} = 0, \text{ and } q_{j,k}^n = \rho_k^0 N_{j,k},$$

where $\rho_k^0 = \Delta a \sum_{j=1}^{K_a} q_{j,k}^0 \phi_{j-1,k}$.

Proof of Maximal Principle

Theorem (Maximum principle)

$$0 < \underline{\gamma_k} N_{j,k} \le q_{j,k}^n \le \overline{\gamma_k} N_{j,k},$$

if it is true initially.

Proof:

- The discrete **general relative entropy**

$$\sum_{j=1}^{K_a} \phi_{j-1,k} N_{j,k} H\left(\frac{q_{j,k}^{n+1}}{N_{j,k}}\right) \le \sum_{j=1}^{K_a} \phi_{j-1,k} N_{j,k} H\left(\frac{q_{j,k}^{n}}{N_{j,k}}\right),$$

  where $H(\cdot)$ is an arbitrary convex function.
- Upper bound: Taking $H(u) = (u - \overline{\gamma_k})_+^2$.

Capture Concentration with a Coarse Mesh

Reconstruction:

$$N_{\mathrm{f}} = Q_{\mathrm{f}} e^{\frac{u_{\mathrm{f}}}{\varepsilon}},$$

where Q_{f} and u_{f} are the interpolated numerical solutions on a fine mesh.

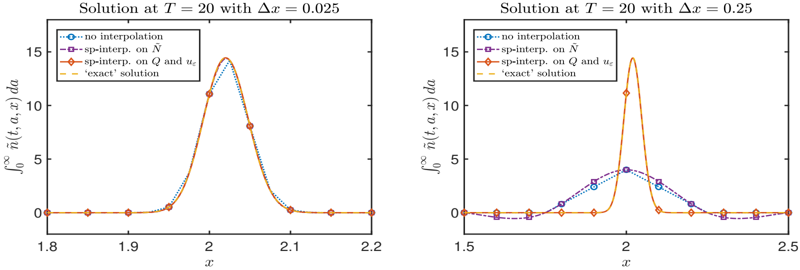

Figure: Comparison of $\int_0^\infty n(t,a,x)\,\mathrm{d}a$ with the 'exact' one. Here $\varepsilon = 0.01$ and $\Delta x = 0.025$ (left) or 0.25 (right).

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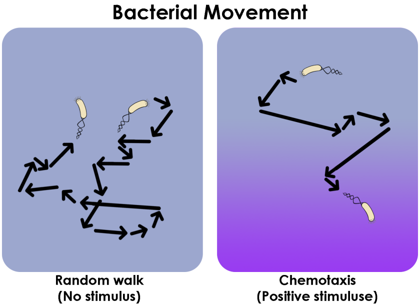

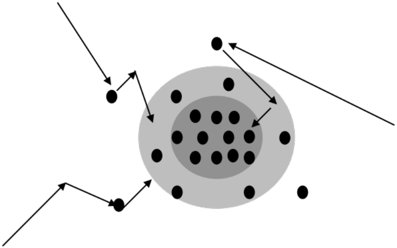

Figure: Cell movement without/with chemotaxis [4] (left) and illustration of pattern forming (right).

- run and tumble: cells move along a straight line and then reorient
- chemotaxis: cells adapt movement in response to a chemical stimulus

[4] http://2016.igem.org/Team:Technion_Israel/Chemotaxis

Patlak-Keller-Segel (PKS) model

Classical PKS

$$\partial_t \rho = D_\rho \Delta \rho - \beta \nabla \cdot \{\rho \nabla c\} + r_0 \rho (1 - \rho/\rho_{\max})_+,$$
$$- D_c \Delta c = \rho - c.$$

- ρ – cell density, c – chemical concentration
- D_ρ, D_c – diffusion constant
- Tendency of finite-time blow-up

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Volume-exclusion PKS

$$\partial_t \rho = \nabla \left(D_\rho (q(\rho) - q'(\rho)\rho) \nabla \rho - \beta q(\rho) \rho \nabla c \right) + r_0 \rho (1 - \rho/\rho_{\max})_+,$$

- Density-dependent motility and chemotactic sensitivity
- Well defined: $q(\rho) - q'(\rho)\rho > 0$ if $q'(\rho) \leq 0$
- For example, $q(\rho) = (1 - \rho/\bar{\rho})_+$

Framework of kinetic model

$$\partial_t f + \mathbf{v}\cdot\nabla(F[\rho](t,\mathbf{x},\mathbf{v})f) = \lambda q(\rho)(-f+\rho T(\mathbf{v},\rho,\nabla c)) + r_0 f(1-\rho/\rho_{\max})_+$$

- $f(t,\mathbf{x},\mathbf{v})$ – density, $\rho(t,\mathbf{x}) = \int_V f(t,\mathbf{x},\mathbf{v})\mathrm{d}\mathbf{v}$ with V unit sphere
- λ – constant turning rate
- $T(\mathbf{v},\rho,\nabla c)$ – prob. of jumping to velocity $\mathbf{v} \Rightarrow \int_V T(\mathbf{v},\rho,\nabla c)\,\mathrm{d}\mathbf{v} = 1$

Remark

We assume cells will only make a turn

- *if they are not already trapped in a high density region*
- *in directions where cell density is not too large*

$$F[\rho](t,\mathbf{x},\mathbf{v}) = q\big(\rho(t,\mathbf{x}+\mathbf{v})\big),\ T(\mathbf{v},\rho,\nabla c) = \tilde{c}(t,\mathbf{x})\psi(\mathbf{v},\nabla c)\,q(\rho(t,\mathbf{x}+\mathbf{v}))$$

Diffusion scaling

- Assume

$$\tau_{run} \ll \tau_{drift} \ll \tau_{diff}$$

where

$$\tau_{run} = \frac{1}{\lambda}, \quad \tau_{drift} = \frac{L}{s}, \quad \tau_{diff} \approx \frac{L^2\lambda}{s^2}.$$ [5]

- Introduce $\varepsilon \ll 1$ such that

$$\tau_{run} = \mathcal{O}(1), \quad \tau_{drift} = \mathcal{O}(\frac{1}{\varepsilon}), \quad \tau_{diff} = \mathcal{O}(\frac{1}{\varepsilon^2}).$$

[5] Here s is characteristic speed and L is characteristic length scale

Scaled linearized kinetic model

The scaling leads to the dimensionless equation

$$\varepsilon^2\partial_t f + \varepsilon \mathbf{v}\cdot\nabla(\tilde{F}_\varepsilon(\rho)f) = q(\rho)\left(-f + \rho\tilde{T}_\varepsilon(\mathbf{v},\rho,\nabla c)\right) + \varepsilon^2 r_0 f(1-\rho/\rho_{\max})_+\,,$$

where

$$\begin{aligned}
\tilde{F}_\varepsilon &= q(\rho(t,\mathbf{x})) + \varepsilon q'(\rho(t,\mathbf{x}))\mathbf{v}\cdot\nabla\rho(t,\mathbf{x}),\\
\tilde{T}_\varepsilon &= \psi_0(\mathbf{v}) + \varepsilon\Big(\psi_1(\mathbf{v},\nabla c) + \frac{q'(\rho)}{q(\rho)}(\mathbf{v}\cdot\nabla\rho)\psi_0(\mathbf{v})\Big),\\
\psi_\varepsilon(\mathbf{v},\nabla c) &= \psi_0(\mathbf{v}) + \varepsilon\psi_1(\mathbf{v},\nabla c),
\end{aligned}$$

and

$$<\psi_0>=1,\ <\psi_1>=0,\ <\mathbf{v}\psi_0>=\mathbf{0},\ \psi_1(\mathbf{v},\nabla c)=\phi(\mathbf{v})\cdot\nabla c.$$ [6]

[6] $<f>:=\int_V f\mathrm{d}\mathbf{v}$

Macroscopic limit

Theorem (formal)

The limit $\varepsilon \to 0$ of f^ε is $f^0 = \rho(t, \mathbf{x})\psi_0(\mathbf{v})$, where ρ solves

$$\partial_t \rho - \nabla \cdot \Big(D_0\big(q(\rho) - \rho q'(\rho)\big)\nabla \rho - \beta \rho q(\rho) \nabla c\Big) = 0 \ ,$$

with D_0 and β given by

$$D_0 = \langle (\mathbf{v} \otimes \mathbf{v}) \psi_0(\mathbf{v}) \rangle \qquad \text{and} \qquad \beta = \langle \mathbf{v} \otimes \phi(\mathbf{v}) \rangle \ .$$

where $\langle f(\mathbf{v}) \rangle = \int_V f(\mathbf{v})\,\mathrm{d}\mathbf{v}$.

Micro-macro decomposition [7]

We decompose

$$f(t,\mathbf{x},\mathbf{v}) = \rho(t,\mathbf{x})\psi_0(\mathbf{v}) + \varepsilon g(t,\mathbf{x},\mathbf{v}),$$

where g is the perturbation satisfying $\langle g\rangle = 0$.

Equation for ρ

Integrating over $\mathbf{v}$*, we have*

$$\partial_t\rho + \langle \mathbf{v}\cdot\nabla(q(\rho)g)\rangle + \nabla\cdot(q'(\rho)\rho D_0\nabla\rho) = r_0\rho\,(1-\rho/\rho_{\max})_+ \,.$$

Equation for g

By defining $\Pi f(t,\mathbf{v},\mathbf{x}) = \langle f(t,\mathbf{v},\mathbf{x})\rangle\psi_0(\mathbf{v})$ *and taking* $I-\Pi$ *to the kinetic equation, we get the equation for* g*.*

[7] M. Lemou and L. Mieussens (2008)

1D finite difference discretization

Equation for ρ

$$\delta_t^+ \rho_j^n + \langle v_k \delta_x (q(\bar\rho_*^n) g_{*,k}^{n+1})_j \rangle_h + D_h \delta_x (\bar\rho_*^n q'(\bar\rho_*^n) \delta_x \rho_*^{n+1})_j = \cdots . \ ^a$$

[a] Here $\langle \eta_{j,k}^n \rangle_h := \Delta v \sum_k \eta_{j,k}^n$.

Equation for g

$$\frac{g_{j+\frac12,k}^{n+1} - g_{j+\frac12,k}^n}{\Delta t} + \cdots = \frac{1}{\varepsilon^2} S_{j+\frac12,k}^{n,n+1} + \cdots ,$$

where

$$S_{j+\frac12,k}^{n,n+1} = -v_k \psi_0(v_k) q(\bar\rho_{j+\frac12}^n) \delta_x \rho_{j+\frac12}^{n+1} + \psi_1(v_k, \delta_x c_{j+\frac12}^n) \Phi_{j+\frac12}^{n+1,n} - q(\bar\rho_{j+\frac12}^n) g_{j+\frac12,k}^{n+1}$$

and $\Phi_{j+\frac12}^{n+1,n}$ *is a semi-implicit upwind approx. of* $q(\rho)\rho$ *at* $x = x_{j+\frac12}$.

Decoupling of ρ and g

Define

$$\frac{\tilde{g}^{n+1}_{j+\frac{1}{2},k} - g^{n}_{j+\frac{1}{2},k}}{\Delta t} + \cdots = \frac{1}{\varepsilon^2}\tilde{S}^{n,n+1}_{j+\frac{1}{2},k} + \cdots,$$

where

$$\tilde{S}^{n,n+1}_{j+\frac{1}{2},k} = -v_k\psi_0(v_k)q(\bar{\rho}^n_{j+\frac{1}{2}})\delta_x\rho^{n}_{j+\frac{1}{2}} + \psi_1(v_k,\delta_x c^n_{j+\frac{1}{2}})\Phi^{n,n}_{j+\frac{1}{2}} - q(\bar{\rho}^n_{j+\frac{1}{2}})\tilde{g}^{n+1}_{j+\frac{1}{2},k}.$$

Remark

- $\tilde{g}^{n+1}_{j+\frac{1}{2},k}$ *can be computed explicitly.*
- *The equation for* ρ^{n+1} *now depends on* $\tilde{g}^{n+1}$ *instead of* g^{n+1}*.*

2D Patterns for different initial data

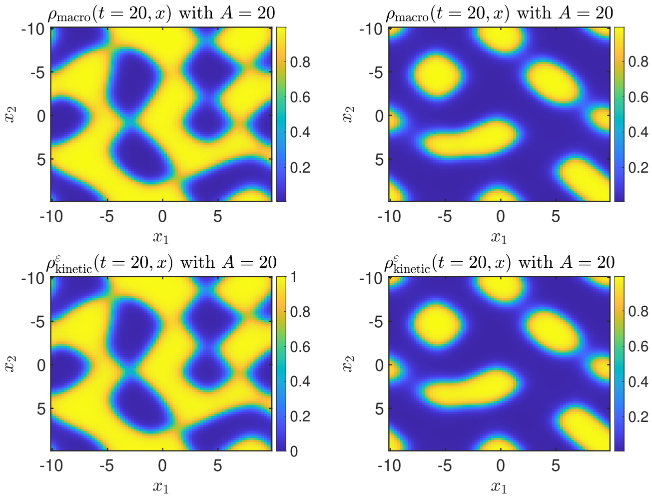

Figure: Comparison between $\rho_{\text{macro}}(t,\mathbf{x})$ (first row) and $\rho^{\varepsilon}_{\text{kinetic}}(t,\mathbf{x})$ with $\varepsilon = 10^{-2}$ (second row) for different initial conditions.

Convergence in ε

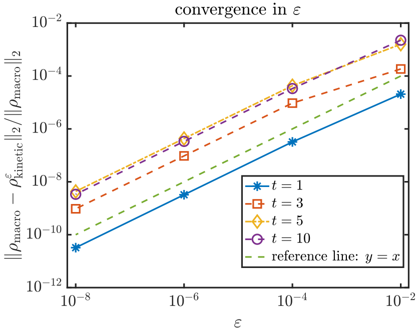

Figure: Convergence of the relative L_2-error in ε at $t = 1,\ 3,\ 5,\ 10.$

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Hele-Shaw limit of a porous medium type model

A model of tumor growth

$$\partial_t n - \nabla(n \nabla p) = nG(p),$$

where $p = n^\gamma$ is the internal pressure.
Its incompressible limit as $\gamma \to \infty$ is the Hele-Shaw problem:

$$\begin{cases} -\Delta p_\infty = G(p_\infty), & \text{in } \Omega(t), \\ V = -\partial_\nu p_\infty, & \text{on } \partial\Omega(t). \end{cases}$$

We analysed the stability and AP property of the simple upwind scheme (1D here for brevity)

$$\frac{\mathrm{d}}{\mathrm{d}t} n_i = \frac{n_{i+\frac{1}{2}} q_{i+\frac{1}{2}} - n_{i-\frac{1}{2}} q_{i-\frac{1}{2}}}{\Delta x} + n_i G(p_i), \text{ with } q_{i+\frac{1}{2}} = \frac{p_{i+1} - p_i}{\Delta x}.$$

Conclusions

- Several asymptotic structures appearing in population evolution are presented.
- We introduced how to design the asymptotic preserving (AP) schemes for the problems introduced.

References

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Thank you all for your attention!