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Matrix Models: Foundations and the BFSS matrix model

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Abstract: 这个笔记第一部分简要介绍了BFSS(Banks-Fischler-Shenker-Susskind)矩阵模型以及整理了相关文献，第二部分是对一般的矩阵模型的介绍。仍在不断补充中。

目录

1 BFSS Matrix Model

1.1 计算

想要真正理解一个物理概念，计算是必不可少的。因此我们将首先从“如何计算BFSS矩阵模型的哈密顿量”出发，介绍这个物理学概念应当如何计算。

BFSS矩阵模型是一种矩阵量子力学，可以看成1+9维$\mathcal{N}=1$的super Yang-Mills的降维，因此在计算上可以借鉴Yang-Mills场的技巧。它的作用量为：

$$S=\frac{1}{2g^2}\int dtTr((D_tX_i)^2+\frac{1}{2}[X_i,X_j]^2-i\Psi^T{}_\alpha C_{10}\gamma^0{}_{10}D_t\Psi_\beta+\Psi^T{}_\alpha C_{10}\gamma^i{}_{\alpha\beta}[X_i,\Psi_\beta])$$

其中$D_t=\partial_t-i[A(t),\cdot]$扮演协变导数的角色，$A(t)$是一维的规范场。$X^i$可以看成是矢量场，$\Psi_\alpha$看成旋量场，他们两个都是$N\times N$的无迹厄米矩阵（对应$SU(N)$场论的降维）。由于在十维，$i,j=1,2,...,9$，$\alpha,\beta=1,2,...,32$。$C_{10}$是十维的电荷共轭矩阵。

由于在$D=10$时，旋量场有Majorana-Weyl表示（可以见我另外一篇笔记 [1]），我们通过选取合适的分解，有 [2]:

$$\Psi=\theta\otimes\begin{pmatrix}1\\0\end{pmatrix}.$$

$$\Gamma^i=\gamma^i\otimes\sigma_1,$$
$$\Gamma^0=1_{16}\otimes i\sigma_2,$$
$$C_{10}=C_9\otimes i\sigma_2$$

且我们可以通过选取γ^i为实的对称矩阵（也就是C_9为单位矩阵）的基，并把含有Γ_0的项和含Γ_i的项合并在一起。最终我们可以给出理论的哈密顿量：

$$H = \frac{1}{2}Tr(g^2P^2 - \frac{1}{2g^2}[X^i, X^j]^2 - \theta^T\Gamma_i[X^i, \theta]) \tag{1.1}$$

这里的$\alpha, \beta = 1, 2, ..., 16$。要计算这个哈密顿量，首先我们需要选取一个$N \times N$的无迹厄米矩阵的基。我们选取$su(N)$李代数的生成元$T^a$。它满足：

$$TrT_A = 0, \tag{1.2}$$

$$Tr(T_AT_B) = \delta_{AB}, \tag{1.3}$$

$$[T_A, T_B] = if_{ABC}T_C. \tag{1.4}$$

此时我们就可以分解这些矩阵。根据正则量子化，有对易关系：

$$X_i = X_i^AT_A. \tag{1.5}$$

$$\theta_\alpha = \theta_\alpha^AT_A. \tag{1.6}$$

此时γ矩阵的关系是$\{\gamma^i, \gamma^j\} = 2\delta^{ij}$。

下面以费米子项来逐步介绍如何计算。费米子项是

$$Tr\theta_\alpha\gamma^i_{\alpha\beta}[X_i, \theta_\beta]$$

我们将X，θ这两个矩阵分解，并把系数提出来：

$$\begin{aligned}
&[X_i, \theta_\beta] \\
&= [X_i^AT_A, \theta_\beta^BT_B] \\
&= X_i^A\theta_\beta^B[T_A, T_B] \\
&= if_{ABC}X_i^C\theta_\beta^BT^C
\end{aligned}$$

最后一步用到了(1.4)。注意到迹是对T^a取的，γ矩阵与此无关。因此我们可以把所有只含有矢量和旋量指标的项提出来并计算：

$$\begin{aligned}
&Tr\theta_\alpha\gamma^i_{\alpha\beta}[X_i, \theta_\beta] \\
&= Tr\theta_\alpha^AT^A\gamma^i_{\alpha\beta}[X_i, \theta_\beta] \\
&= iTr\theta_\alpha^AT^A\gamma^i_{\alpha\beta}f_{CBD}X_i^C\theta_\beta^BT^D \\
&= if_{CBD}\theta_\alpha^A\gamma^i_{\alpha\beta}X_i^C\theta_\beta^BTrT^AT^D \\
&= if_{CBA}\theta_\alpha^A\gamma^i_{\alpha\beta}X_i^C\theta_\beta^B
\end{aligned}$$

最后一行用到了(1.3)。

1.2 背景简介

1997年，BFSS几人提出了一个猜想：

猜想：无限动量参考系下的M理论与大N极限下N个D0膜动力学之间等价。

而BFSS矩阵模型作为一个$0+1$维大N矩阵量子力学模型描述了这个D0膜动力学。由于BFSS矩阵模型对偶到高维的时空，所以研究怎么从这个矩阵模型对偶到高维黑洞的形成也是一个研究的热点。

1.3 要点梳理

这一部分我将简要整理提出该猜想的原始论文[3]和一篇Review[4]中的一些要点。

[5]是一个关于“无限动量参考系（Infinite Momentum Frame, IMF）是什么”的参考视频。

值得一提的是，IMF是唯一被证明可以将弦论写成哈密顿量形式的参考系。他们猜想的是**IMF下的M理论唯一的动力学自由度是D0膜**，为什么？

这是因为：在IMF中，所有的系统都是由partons组成的，这些partons只能带正的纵向动量，因为在无限动量极限下，这些partons的频率会由于time-dilated而趋于无穷，负和零的纵向动量会被integrated out。而在type-II A弦理论中，微扰弦都在纵向上携带零动量，因此不能作为一个独立的动力学自由度；anti-D0膜有负的动量；只有D0膜才具有纵向动量。所以，D0膜将是唯一的动力学自由度。

那么这个猜想所指的精确的动力学表达式是什么？在讨论这个之前我们应该注意到，除了D0膜的坐标之外，还有一些自由度：连接D0膜之间的开弦。他们对于D0膜之间的作用是有用的，因此我们不能丢掉他们。于是我们选择了矩阵：以9个N乘N的矩阵来描述坐标空间。而由于这是一个超对称理论，则也会出现superpartners。出于简单以及Lorentz不变性方面的考虑，这个量子力学的拉氏量最终可以写成：

$$L = \frac{1}{2gl_s} Tr\{\dot{X}^a \dot{X}^a + \frac{1}{2}[X^a, X^b]^2 + \theta^T(i\dot{\theta} - \Gamma_a[X^a, \theta])\} \tag{1.7}$$

其中，X是9个$N\times N$的玻色矩阵，θ是有16个分量的旋量，他们都是$SO(9)$的表示。可以大致看成，第一项是动能，第二项是势能，第三项则包含了超对称的费米子。这个理论有规范不变性，矩阵本身是属于$U(N)$群的表示。而其哈密顿量则可以写作：

$$H = \frac{1}{2} Tr\{g^2 P^2 - \frac{1}{2g^2}[X^i, X^j]^2 - \theta^T \Gamma_i [X^i, \theta]\} \tag{1.8}$$

BFSS猜想最有力的一个证据是他们可以解释超膜理论一直被人诟病的**“不稳定性”**。

在BFSS之前，Nicolai等人就使用了矩阵来regularize经典超膜理论，这种方法用$N\times N$矩阵中包含的有限自由度来代表膜在时空中嵌入的无限自由度。然而这种理论遇到了难以解释的困难：在弦论中，弦在Hilbert space中的specturm of

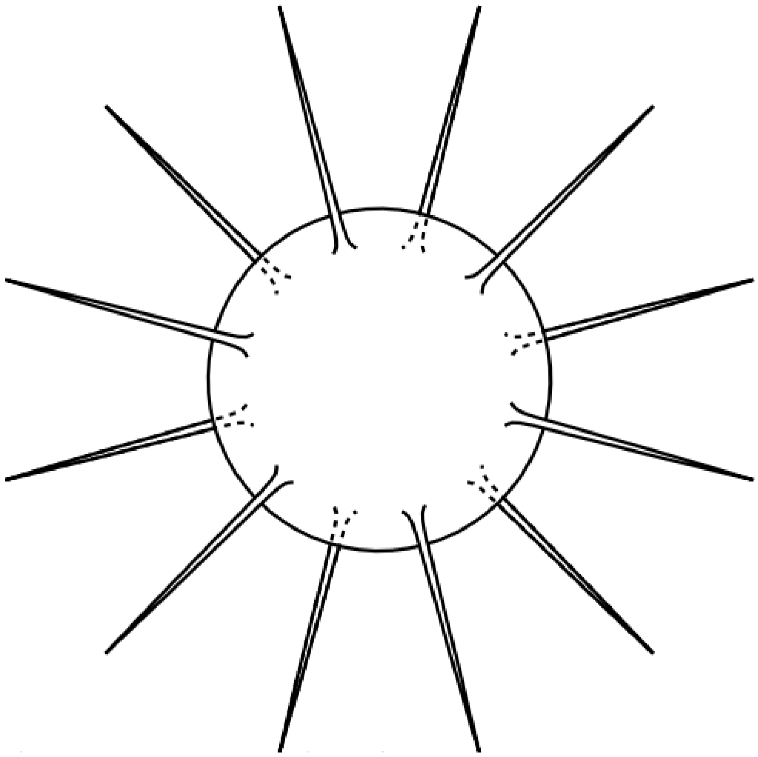

图 **1**. Fig.1 in [4]

state**应该与target space里的基本粒子状态一一对应**；所以类比弦论，学者们普遍希望massless particle specturm包含graviton，以及massless state和massive excitations之间有mass gap，也就是这个理论有离散的spectrum of states；然而在超膜理论中膜是不稳定的，膜上面会有很多窄的尖刺出来，这个解释不成立。我们引用[4]中的Fig.1来说明这个问题：

在弦论中，弦的长度和能量是成正比的，所以能量越大，弦的长度越大。弦论中没有“不稳定性”的问题。而在膜理论中，如果膜上有一个接近圆柱形的尖刺（spike）,如图1所示，那么能量为 $E = 2\pi r L T$ ，其中r为尖刺的半径，L为尖刺的长度，T为膜的张力。当L非常大而$r \ll 1/TL$ 时，能量依然很小，但此时尖刺非常长。由于量子化的膜是有涨落的（E是有不确定度的），所以对应着这些小的能量涨落，膜上就会不断有很多尖刺出现。这种“不稳定性”在矩阵化后的膜理论中表现为一些moduli space中的“平坦方向”（flat directions）。

例如在经典的理论中，假设我们有两个玻色矩阵：

$$X^1 = \begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}, X^2 = \begin{pmatrix} 0 & y \\ y & 0 \end{pmatrix}$$

那么势能项$Tr[X^1, X^2]^2$就正比于 $x^2 y^2$ 。当x或者y等于0时，另一个参数就没有限制，在moduli space中形成了一个flat direction。在量子玻色膜理论中，通过非对角元的零点能量，我们可以发现这种表面上的不稳定性被移除掉了，这个系统实际

上有着分立的能谱；然而在超对称膜理论中依然存在这种不稳定性。学者们证明了这种超对称矩阵量子力学的能谱是连续的：

> **定理：** 对于任意的$\epsilon > 0$和任意的$E \in [0, \infty)$，存在一个可归一化的态ψ，使得
> $$||(H-E)\psi||^2 < \epsilon \tag{1.9}$$

这也就很难解释说明如何在target space里有一个离散的particle spectrum。

那么BFSS是如何解释这个问题的呢？

原因在于：**矩阵量子力学的Hilbert space是包括多个粒子的状态的。它是一个二次量子化的理论**。让我们进一步说明这一点。

type II-A理论中的D0膜对应着M理论中有单位纵向动量的graviton。先看哈密顿量对应单粒子的谱。

1. 当矩阵的N=1时，纵向动量为0时的一些态对应着的是静止的D0膜。这些态构成了16个旋量的带有 2^8个分量的表示，而这正好对应着supergraviton的256个状态的量子数。

2. 在$N > 1$时，$U(N)$的对称性可以看成$U(1) \times SU(N)$的对称性。 $U(1)$的部分描述了理论的中心自由度。

   具体来说，我们可以把X 矩阵拆为：
   $$X = \frac{X_{\text{质心运动}}}{N} I + X_{\text{相对运动}}$$
   其中 I是单位矩阵。那么后一部分代表相对运动的矩阵就属于$SU(N)$的伴随表示。进一步也可以把哈密顿量拆成：
   $$H = H_{\text{质心运动}} + H_{\text{相对运动}}$$
   其中相对运动的哈密顿量可以看成10维的super-Yang Mills 理论简化到0维的空间上。对于这个相对运动的部分，我们相信这个对应着的单粒子态有唯一一个E=0的可归一化的束缚态，正确地描述了无限动量参考系下的11维的无质量粒子。我们下面就讨论这个相对运动的矩阵。

3. 然后我们再看对应着距离很远的多粒子态的谱。当距离很远时，矩阵X的元素相应变得很大，则哈密顿量中对易子项也变得很大，对应着很大的能量。所以在粒子距离很远时，只有当矩阵X的构型在“平坦方向”上（在这，矩阵互相对易）时，我们才能得到有限的能量。具体而言，考虑相互对易的块对角矩阵X：
   $$X = \begin{pmatrix} {X_1}^i & 0 & 0 & \dots \\ 0 & {X_2}^i & 0 & \\ 0 & 0 & {X_3}^i & \\ \dots & & & \dots \end{pmatrix}$$

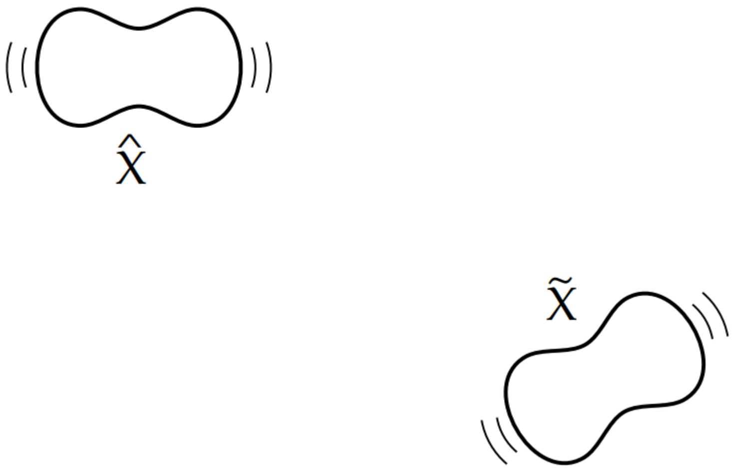

图 2. Fig.2 in[4]

其中X_a^i是$N_a \times N_a$的矩阵，$N_1+N_2+...+N_n=N$。这个矩阵就可以看成描述了是n个相互解耦合的系统，每个系统包含N_a个D0膜。如上所述，当两个系统的相对距离

$$R=\left|\frac{trX_a}{N_a}-\frac{trX_b}{N_b}\right|$$

非常大时，有限能量状态对应flat direction。

对应着的物理直觉是，例如我们考虑有两个分块矩阵的情况，如图2所示，根据上述（3）的讨论，我们让每个分块矩阵对应着一个$H=0$ 的graviton，两个graviton分得很开又相对速度很小时，自然可以构造出任意小的总能量E 。对应（式3），我们稍作推广，就能理解BFSS矩阵模型为什么会有连续的谱、它是如何解释“平坦方向”的。

1.4 文献整理

[6]：这篇文章讨论了将BFSS矩阵模型中$SU(N)$对称性视作一个global symmerty时的情况。

[7]：之前我们只认为M theory和矩阵模型的联系都只是在大N层面下成立的，而这一篇文章则猜想有限N下矩阵模型与M theory的discrete light-front quantized sector有关。为什么BFSS 矩阵理论要求$N\to\infty$，而10维N个$D0$膜给出的退耦矩阵理论对任何有限N都成立？Susskind在这篇文章中提出后者描述了一个类光紧致的11维M理论.DLCQ矩阵理论给出了11维M理论紧致在一个有限类光空间半径R上的统一描述，对任何N都成立，等价于10维DKPS极限，当$N\to\infty$时给出11维非紧致的BFSS矩阵理论.

[8]：BFSS矩阵模型与D0膜近视界几何中存在全息对偶。这篇文章主要讨论了在’t Hooft coupling下D0膜动力学的引力对偶。

2 Matrix Models

2.1 Introduction

Matrix models are physical models in which the dynamical quantities are square matrices, and are usually taken at the limit when the size of matrix tends to infinity. Matrix models are in a sense the simplest examples of quantum gauge theories, namely, they are qauantum gauge theories in zero dimensions in which the spacetime dependence has been removed [9].

The simplest instance is one Hermitian matrix model. Let's see how it works.

First we review the Gaussian integrals which are generally used in quantum field theory (QFT). We have the formula

$$Z_0 = \int_{-\infty}^{\infty} dM e^{\frac{a}{2}M^2} = \sqrt{\frac{2\pi}{a}}. \tag{2.1}$$

Define the (normalized) moments

$$< M^k >= \frac{1}{Z_0}\int_{-\infty}^{\infty} dM M^k e^{\frac{a}{2}M^2}.$$

By taking derivatives of (2.1) with respect to a , we could get the formula

$$< M^{2k} >= (2k-1)!!a^{-k}$$

The above Gaussian integral could be viewed as a 1×1 matrix model. The partition fuction of one Hermitian matrix models always takes the form [9]

$$Z = \int [dM] e^{-NTrV(M)},$$

where M is an $N \times N$Hermitian matrix, and $V(M)$ is a matrix valued function, generally

$$V(M) = \sum_{k=0}^{\infty} g_k M^k.$$

The integral is over Haar measure

$$[dM] = \prod_J dM_{JJ} \prod_{I<J} d\mathrm{Re}M_I J d\mathrm{Im}M_{IJ}.$$

Every matrix element M_{ij} is a variable. Clearly the integral is invariant under the symmetry

$$M \to UMU^\dagger, U \in U(N). \tag{2.2}$$

Since the matrix is Hermitian, it could be diagonalised using a unitary matrix U as $M = U^\dagger D_M U$ where $D_M = \mathrm{diag}(\lambda_1, \lambda_2, ..., \lambda_\mathrm{N})$ is a real diagonal matrix. Therefore we could change N^2 element variables to N eigenvalue variables, and the group factor have been integrated out. We finally get the eigenvalue version of partition function

$$Z = \mathrm{C} \int \prod_{\mathrm{I}=1}^{\mathrm{N}} \mathrm{d}\lambda_\mathrm{I} \mathrm{e}^{-\mathrm{N}\sum_{\mathrm{i}=1}^{\mathrm{N}} \mathrm{V}(\lambda_\mathrm{i})} \Delta^2(\lambda_1, \lambda_2, ..., \lambda_\mathrm{N}),$$

where $\Delta^2(\lambda_1, \lambda_2, ..., \lambda_N)$ is the Vandermonde determinant with $\Delta(\lambda_1, \lambda_2, ..., \lambda_N) = \prod_{I<J}(\lambda_I - \lambda_J)$ and C is the volume of group $U(N)$.

The common way to study the large N limit is to introduce the normalized spectral density of eigenvalues

$$\rho(\lambda) = \frac{1}{N} \sum_{i=1}^{N} \delta(\lambda - \lambda_i),$$

which satisfiying

$$\int_{-a}^{a} d\lambda \rho(\lambda) = 1.$$

As in QFT, we define the normalized expectation values in matrix model

$$< Q(M) > \equiv \frac{1}{Z} \int [dM] e^{-NTrV(M)} Q(M).$$

We could view this as amplitudes in QFT. Because we have the symmetry (2.2), we need to define observables those are invariant under the symmetry. So we take trace. A more general defination of amplitudes is

$$\mathcal{M}_N^{(k_1,k_2,...,k_n)} \equiv \frac{1}{\mathcal{M}_N} \int [dM] e^{-N\mathrm{Tr}V(M)} \prod_{i=1}^{n} \frac{1}{N} \mathrm{Tr} M^{k_1} = \frac{1}{N^n} < \prod_{i=1}^{n} \mathrm{Tr} M^{k_i} > .$$

2.2 Saddlepoint Approximation

There are various methods to solve the Hermitian one matrix model. Two of the most prominent ones are the saddle point approximation and the method of orthogonal polynomials. Other methods contain direct recursion relations for planar graphs, and loop equations. We will first introduce the saddle point approximation.

We could start from the integral over reals [9].

$$I = \int dx e^{-Nf(x)}, \tag{2.3}$$

where N is a positive integer, and $f(x)$ is a real valued function. We want to study the behavior of I when N eventually be taken to be large. In the large N limit the exponential causes the integrand to peak sharply at minima of function $f(x)$,while all other values are suppressed. So we could use Taylor series to expand $f(x)$ around the minima x_e

$$f(x) = f(x_e) + \frac{f''(x_e)}{2!}\delta x^2 + ..., \delta x \equiv x - x_e.$$

Then we could redefine$\delta x = \delta\xi/\sqrt{N}$, and see we could approximate (2.3) in the large N expansion:

$$I \approx e^{-Nf(x_e)} \int d\delta x e^{-\frac{N}{2}f''(x_e)\delta x^2}(1-\frac{N}{3!}f^{(3)}(x_e)\delta x^3+...) \approx \sqrt{\frac{2\pi}{Nf''(x_e)}}e^{-Nf(x_e)}(1+O(1/N)).$$

The same method is applied to matrix models. Recall the eigenvalue version partition function of one Hermitian matrix model takes the form

$$Z = \mathrm{C}\int \prod_{\mathrm{I=1}}^{\mathrm{N}} \mathrm{d}\lambda_{\mathrm{I}} \mathrm{e}^{-\mathrm{N}\sum_{\mathrm{i=1}}^{\mathrm{N}}\mathrm{V}(\lambda_{\mathrm{i}})}\Delta^2(\lambda_1, \lambda_2, ..., \lambda_{\mathrm{N}}).$$

To have the same form with (2.3), we rewrite the partition function as[1]

$$Z = \mathrm{C}\int \prod_{\mathrm{I=1}}^{\mathrm{N}} \mathrm{d}\lambda_{\mathrm{I}} \mathrm{e}^{-\mathrm{N}^2\mathrm{S_{eff}}[\lambda]},$$

where

$$S_{eff}[\lambda] \equiv \frac{1}{N}\sum_{I=1}^{N} V(\lambda_I) - \frac{1}{N^2}\sum_{J\neq I}\log|\lambda_{\mathrm{I}} - \lambda_{\mathrm{J}}|.$$

The minima is where $S'_{eff} = 0$. So the saddle point equations for $S[\lambda]$ are given by

$$V'(\lambda_I) = \frac{2}{N}\sum_{J\neq I}(\lambda_I - \lambda_J)^{-1}. \tag{2.4}$$

In the large N limit, we use the normalized spectral density of eigenvalues to rewrite the RHS:

$$V'(\lambda) = 2Principle[\int_{-a}^{a} d\mu \frac{\rho(\mu)}{\lambda - \mu}],$$

while the integral is taken the principal value through the real axis.

To solve the saddle point equations, an elegant way is to introduce resolvent

$$R(z) \equiv \frac{1}{N}\mathrm{Tr}(z\mathcal{I} - M)^{-1} = \frac{1}{N}\sum_{i=1}^{N}\frac{1}{z-\lambda_i} \xrightarrow{\text{large } N} \int_{-a}^{a} d\mu \frac{\rho(\mu)}{z-\mu},$$
$$z \in C/\{\lambda_i\}.$$

[1]As we focus on the solving method in the context, we will put any useless constant in C from now on.

where $\mathcal{I}$ is the $N \times N$ identity matrix.

By multiplying $\frac{1}{\lambda_i - z}$ on the both side of (2.4) and summing over i , we have

$$R^2(z) - V'(z)R(z) = -P(z),$$

with

$$P(z) = \frac{1}{N}\sum_{i=1}^{N}\frac{V'(z) - V'(\lambda_i)}{z - \lambda_i}.$$

By observing (2.2) we know that the resolvent should have these qualities:

$$\begin{aligned}
\mathrm{res_a} &: \lim_{\mathrm{z}\to\infty} \mathrm{R(z)} = \frac{1}{\mathrm{z}}, \\
\mathrm{res_b} &: \rho(\mathrm{x}) = \frac{1}{2\pi \mathrm{i}}(\mathrm{R(x - i\epsilon) - R(x + i\epsilon)}), \\
\mathrm{res_c} &: \mathrm{V'(x) = R(x - i\epsilon) + R(x + i\epsilon)}.
\end{aligned}$$

where I use the variable x to denote on the real axis $[-a, a]$.

By using the condition $\mathrm{res_a}$, we find the root

$$R(z) = \frac{1}{2}(V'(z) - \sqrt{V'(z)^2 - 4P(z)}).$$

And with condition $\mathrm{res_b}$, we could find the eigenvalue spetrum $\rho(z)$.

2.3 Loop Equation Approach

2.3.1 Schwinger-Dyson Equations

Another different way to solve matrix models are loop equations. The Loop equations are nothing but the matrix model versions of the Schwinger-Dyson equations. So let's first briefly introduce the Schwinger-Dyson Equations.[2] The Schwinger-Dyson equations can be understood as the quantum equations of motion for Green's functions with proper contact terms included. In deriving classical motion equations, we use the stationary of action: under any infinitesimal change of fields $\phi^\alpha \to \phi^\alpha + \delta\phi^\alpha$. That's the same for generation functions. Let's take the example in quantum mechanics from [12] as an instance.

The generating functional

$$Z[J] = \int Dx \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)]$$

is given by

$$L = \frac{1}{2}(\frac{d}{dt}x(t))^2 - \frac{1}{2}x(t)^2 - gx(t)^4$$

[2]This part is copied from another note of mine [11]

Give an infinitesimal change of integration

$$x(t) \to x(t) + \delta x(t),$$

(for convenience I will omit the integral limits and some variable (t)) The stationary condition yields:

$$\begin{aligned}
&0{=}\delta Z\\
&{=}\delta(\int Dx \exp[i \int dt(L + Jx)]\\
&= \int Dx\delta(\exp[i \int dt(L + Jx))\\
&= \int Dx \exp[i \int dt(L + Jx)] \int dt(\delta L + J\delta x)
\end{aligned}$$

$$\begin{aligned}
&\int dt(\delta L + J\delta x)\\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + \tfrac{\partial L}{\partial(\partial_t x)}\delta\partial_t x + J\delta x)\\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + \tfrac{\partial L}{\partial(\partial_t x)}\partial_t\delta x + J\delta x)\\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) + \int dt\tfrac{\partial L}{\partial(\partial_t x)}\partial_t\delta x\\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) + \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)}\delta x) - \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)})\delta x\\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) - \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)})\delta x
\end{aligned}$$

The last line use the proper boundary condition. Insert it to the equation above we have:

$$\begin{aligned}
&\tfrac{\partial L}{\partial x} = -x - 4gx^3\\
&\tfrac{\partial L}{\partial(\partial_t x)} = \partial_t x\\
&\int dt(\delta L + J\delta x)\\
&= \int dt(-x - 4gx^3 + J - \partial^2{}_t x)\delta x
\end{aligned}$$

$$\begin{aligned}
0 &= \int Dx \exp[i \int dt_1(L + Jx)] \int dt_2(-x - 4gx^3 + J - \partial^2{}_t x)\delta x\\
&= \int dt_2 \int Dx\delta x(\exp[i \int dt_1(L + Jx)](-x - 4gx^3 + J - \partial^2{}_t x))
\end{aligned}$$

As this should be satisfied for any possible choice of ε, we change the integrate of Dx and dt, and get

$$\int Dxe^{i\int dt(L+Jx)}(-x - 4gx^3 + J - \partial^2{}_t x) = 0$$

Now we use the property of generating function

$$\frac{\delta^n Z[J]}{i^n\delta^n J} = \int Dx \cdot x^n \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)],$$

and get the equation of motion:

$$(\partial^2{}_t + 1)\frac{\delta}{i\delta J(t)}Z[J] + 4g\frac{\delta^3}{i^3\delta^3 J(t)}Z[J] = J(t)Z[J].$$

We can further take the function derivatives $n-1$ times with respect to $J(t_n)$ and set source to zero:

$$\frac{\delta}{i\delta J(t)\delta J(t_1)...\delta J(t_{n-1})}Z[J] = i^{n-1}\int Dx\cdot x(t)x(t_1)...x(t_{n-1})\cdot \exp[i\int\limits_{-\infty}^{\infty} dt(L+J(t)x(t)],$$
$$\frac{\delta}{i^3\delta^3 J(t)}Z[J] = i^{n-1}\int Dx\cdot x^3(t)x(t_1)...x(t_{n-1})\cdot \exp[i\int\limits_{-\infty}^{\infty} dt(L+J(t)x(t)],$$
$$\frac{\delta}{\delta J(t_1)...\delta J(t_{n-1})}(J(t)Z[J]) = \sum_{k=1}^{n-1}\frac{\delta J(t)}{\delta J(t_k)}\frac{\delta Z[J]}{\delta J(t_1)...\delta J(t_{k-1})\delta J(t_{k+1})...\delta J(t_{n-1})} + J(t)\frac{\delta Z[J]}{\delta J(t_1)...\delta J(t_{n-1})}$$
$$= \sum_{k=1}^{n-1}\delta(t-t_k)i^{n-2}\int Dx\cdot x(t_1)...x(t_{k-1})x(t_{k+1})...x(t_{n-1})\cdot \exp[i\int\limits_{-\infty}^{\infty} dt(L+J(t)x(t)]$$

So:

$$i^{n-1}(\partial^2{}_t+1)G_n(t,t_1,t_2,...,t_{n-1}) + 4gi^{n-1}G_{n+2}(t,t_1,t_2,...,t_{n-1})$$
$$= i^{n-2}\sum_{k=1}^{n-1}\delta(t-t_k)G_{n-2}(t_1,t_2,...,t_{k-1},t_{k+1},...,t_{n-1})$$

There is an equivalent way to derive Schwinger-Dyson equations:

$$0 = \int Dx\frac{\delta}{\delta x(t)}\{F(x)\exp[i\int\limits_{-\infty}^{\infty} dt(L+J(t)x(t)]\},$$

where $F(x)$ is polynomially functional of x .

2.3.2 Loop Equations

We have introduced the loop equations (or Schwinger-Dyson equations). Let's how this works in matrix models.

From now on we define the large trace $TrI = \frac{1}{N}trI = 1$. The Schwinger-Dyson equation for one matrix model is:

$$0 = \int dM\frac{\partial}{\partial M_{ij}}((M^k)_{ij}e^{-S(M)}) \tag{2.5}$$

And it gives:

$$< TrM^kV'(M) >= \sum_{l=0}^{k-1} < TrM^l >< TrM^{k-l-1} >. \tag{2.6}$$

Derivation

Here is the detailed derivation of the formula:

$$< TrM^kV'(M) >= \sum_{l=0}^{k-1} < TrM^l >< TrM^{k-l-1} >.$$

Look at (2.5)

$$0 = \int dM \frac{\partial}{\partial M_{ij}}((M^k)_{ij} e^{-NtrV(M)}),$$

The derivative can act on both terms, giving

$$\begin{aligned}
&0 = \textstyle\int dM \frac{\partial}{\partial M_{ij}}((M^k)_{ij} e^{-NtrV(M)}) \\
&= \textstyle\int dM e^{-NtrV(M)} \frac{\partial}{\partial M_{ij}}(M^k)_{ij} + \int dM (M^k)_{ij} \frac{\partial}{\partial M_{ij}}(e^{-NtrV(M)}) \\
&= \textstyle\int dM e^{-NtrV(M)} \frac{\partial}{\partial M_{ij}}(M^k)_{ij} - \int dM (M^k)_{ij} N e^{-NtrV} \frac{\partial}{\partial M_{ij}}(-trV),
\end{aligned}$$

and that is

$$\int dM e^{-S} \frac{\partial}{\partial M_{ij}}(M^k)_{ij} = \int dM e^{-S} (M^k)_{ij} N \frac{\partial}{\partial M_{ij}}(-trV).$$

Use the formula below [13]:

$$\frac{\partial (M^n)_{kl}}{\partial M_{ij}} = \sum_{r=0}^{n-1} (M^r)_{ki} (M^{n-1-r})_{jl}$$

$$\frac{\partial tr(V(M)}{\partial M} = V'(M)^T,$$

where V' is the scalar derivative of V . Summing over i, j , the left hand side is:

$$\begin{aligned}
&\textstyle\sum_{i,j} \int dM e^{-S} \frac{\partial}{\partial M_{ij}}(M^k)_{ij} \\
&= \textstyle\sum_{i,j} \int dM e^{-S} \sum_{r=0}^{n-1} (M^r)_{ii} (M^{n-1-r})_{jj} \\
&= \textstyle\int dM e^{-S} \sum_{r=0}^{n-1} tr(M^r) tr(M^{n-1-r}) \\
&= \textstyle\sum_{r=0}^{n-1} < tr(M^r) tr(M^{n-1-r}) >.
\end{aligned}$$

and the right hand side is:

$$\begin{aligned}
&\textstyle\sum_{i,j} \int dM e^{-S} (M^k)_{ij} N \frac{\partial}{\partial M_{ij}}(-trV) \\
&= \textstyle\sum_{i,j} \int dM N e^{-S} (M^k)_{ij} V'(M)_{ji} \\
&= \textstyle\int dM N e^{-S} tr(M^k V'(M)) \\
&= N < tr(M^k V'(M)) > .
\end{aligned}$$

Combining the two equation we have:

$$\sum_{r=0}^{n-1} tr(M^r)tr(M^{n-1-r}) = N < tr(M^k V'(M)) > .$$

And that's (2.6) using the large N factorization.

Then we introduce resolvent

$$R(z) =< Tr\frac{1}{z-M} >= \sum_{n=0}^{\infty} G_n z^{-n-1}.$$

By multiply z^{-n} and sum them from $n = 1$ to ∞ , we can get:

$$R(z)^2 + P(z) = V'(z)R(z), \tag{2.7}$$

where $P(z) =< Tr\frac{V'(z)-V'(M)}{z-M} >$.

Derivation

Let's briefly show the derivation of (2.7):

$$R(z)^2 + P(z) = V'(z)R(z)$$

in the case $V(z) = \frac{1}{2}z^2 + \frac{g}{4}z^4$. The Schwinger-Dyson equation (2.7) here is

$$G_{n+1} + gG_{n+3} = \sum_{p=0}^{n-1} G_p G_{n-p-1}.$$

By multiply z^{-n} and sum them from $n = 1$ to ∞ , we can get:

$$\sum_{n=1}^{\infty} G_{n+1}z^{-n} + gG_{n+3}z^{-n} = \sum_{p=0}^{n-1} G_p z^{-n} G_{n-p-1}$$

The first term is related to $z^2R(z) - k_1$, and the second term is related to $gz^4R(z) - k_2$. The right hand side is $zR^2(z)$. Reorganize the equation and collect $k_1 + k_2$ by $P(z)$ reproduce the (2.7).

Solving (2.7) leads to

$$R(z) = \frac{1}{2}V'(z) - \frac{1}{2}\sqrt{V'(z)^2 - 4P(z)},$$

where we choose the negative root to satisfy $\lim_{z\to\infty}(zR(z)) = 1$.

The moments then can be compute via the rosolvent by simple contour integration formula:

$$G_n = \frac{1}{2\pi i} \oint_\Gamma z^n R(z) dz,$$

where the contour Γ must encircle all branch points of the rosolvent.

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