

# Introduction to Schwinger-Dyson Equation

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ABSTRACT: In this note I will introduce the Schwinger-Dyson equation, and give a derivation to an example.

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## 1 Schwinger-Dyson equations

The Schwinger-Dyson equations can be understood as the quantum equations of motion for Green's functions with proper contact terms included. In deriving classical motion equations, we use the stationary of action: under any infinitesimal change of fields  $\phi^\alpha \rightarrow \phi^\alpha + \delta\phi^\alpha$ . That's the same for generation functions. Let's take the example in quantum mechanics from [3] as an instance.

The generating functional

$$Z[J] = \int Dx \exp[i \int_{-\infty}^{\infty} dt (L + J(t)x(t))]$$

is given by

$$L = \frac{1}{2} \left( \frac{d}{dt} x(t) \right)^2 - \frac{1}{2} x(t)^2 - g x(t)^4$$

Give an infinitesimal change of integration

$$x(t) \rightarrow x(t) + \delta x(t),$$

(for convenience I will omit the integral limits and some variable ( $t$ )) The stationary condition yields:

$$\begin{aligned} 0 &= \delta Z \\ &= \delta \left( \int Dx \exp[i \int dt (L + Jx)] \right) \\ &= \int Dx \delta(\exp[i \int dt (L + Jx)]) \\ &= \int Dx \exp[i \int dt (L + Jx)] \int dt (\delta L + J\delta x) \\ &= \int dt (\delta L + J\delta x) \\ &= \int dt \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial(\partial_t x)} \delta \partial_t x + J\delta x \right) \\ &= \int dt \left( \frac{\partial L}{\partial x} \delta x + \frac{\partial L}{\partial(\partial_t x)} \partial_t \delta x + J\delta x \right) \\ &= \int dt \left( \frac{\partial L}{\partial x} \delta x + J\delta x \right) + \int dt \frac{\partial L}{\partial(\partial_t x)} \partial_t \delta x \\ &= \int dt \left( \frac{\partial L}{\partial x} \delta x + J\delta x \right) + \int dt \partial_t \left( \frac{\partial L}{\partial(\partial_t x)} \delta x \right) - \int dt \partial_t \left( \frac{\partial L}{\partial(\partial_t x)} \right) \delta x \\ &= \int dt \left( \frac{\partial L}{\partial x} \delta x + J\delta x \right) - \int dt \partial_t \left( \frac{\partial L}{\partial(\partial_t x)} \right) \delta x \end{aligned}$$

The last line use the proper boundary condition. Insert it to the equation above we have:

$$\begin{aligned}
\frac{\partial L}{\partial x} &= -x - 4gx^3 \\
\frac{\partial L}{\partial(\partial_t x)} &= \partial_t x \\
\int dt(\delta L + J\delta x) \\
&= \int dt(-x - 4gx^3 + J - \partial_t^2 x)\delta x
\end{aligned}$$

$$\begin{aligned}
0 &= \int Dx \exp[i \int dt_1(L + Jx)] \int dt_2(-x - 4gx^3 + J - \partial_t^2 x)\delta x \\
&= \int dt_2 \int Dx \delta x (\exp[i \int dt_1(L + Jx)](-x - 4gx^3 + J - \partial_t^2 x))
\end{aligned}$$

As this should be satisfied for any possible choice of  $\varepsilon$ , we change the integrate of  $Dx$  and  $dt$ , and get

$$\int Dx e^{i \int dt(L+Jx)} (-x - 4gx^3 + J - \partial_t^2 x) = 0$$

Now we use the property of generating function

$$\frac{\delta^n Z[J]}{i^n \delta^n J} = \int Dx \cdot x^n \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t))],$$

and get the equation of motion:

$$(\partial_t^2 + 1) \frac{\delta}{i \delta J(t)} Z[J] + 4g \frac{\delta^3}{i^3 \delta^3 J(t)} Z[J] = J(t) Z[J].$$

We can further take the function derivatives  $n - 1$  times with respect to  $J(t_n)$  and set source to zero:

$$\begin{aligned}
\frac{\delta}{i \delta J(t) \delta J(t_1) \dots \delta J(t_{n-1})} Z[J] &= i^{n-1} \int Dx \cdot x(t)x(t_1) \dots x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t))], \\
\frac{\delta}{i^3 \delta^3 J(t)} Z[J] &= i^{n-1} \int Dx \cdot x^3(t)x(t_1) \dots x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t))], \\
\frac{\delta}{\delta J(t_1) \dots \delta J(t_{n-1})} (J(t) Z[J]) &= \sum_{k=1}^{n-1} \frac{\delta J(t)}{\delta J(t_k)} \frac{\delta Z[J]}{\delta J(t_1) \dots \delta J(t_{k-1}) \delta J(t_{k+1}) \dots \delta J(t_{n-1})} + J(t) \frac{\delta Z[J]}{\delta J(t_1) \dots \delta J(t_{n-1})} \\
&= \sum_{k=1}^{n-1} \delta(t - t_k) i^{n-2} \int Dx \cdot x(t_1) \dots x(t_{k-1}) x(t_{k+1}) \dots x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t))]
\end{aligned}$$

So:

$$\begin{aligned}
&i^{n-1} (\partial_t^2 + 1) G_n(t, t_1, t_2, \dots, t_{n-1}) + 4g i^{n-1} G_{n+2}(t, t_1, t_2, \dots, t_{n-1}) \\
&= i^{n-2} \sum_{k=1}^{n-1} \delta(t - t_k) G_{n-2}(t_1, t_2, \dots, t_{k-1}, t_{k+1}, \dots, t_{n-1})
\end{aligned}$$

That reproduce the result from [3].

## References

- [1] Mark Srednicki, *Quantum Field Theory*
- [2] August Geelmuyden, [The Schwinger-Dyson equations](#)
- [3] Yongwei Guo and Wenliang Li, [Solving anharmonic oscillator with null states: Hamiltonian bootstrap and Dyson-Schwinger equations](#)