Prepared for submission to JHEP

Condensed Notes on General Relativity and Holographic Principles

Xinran Su

E-mail: suxr5@mail2.sysu.edu.cn

Abstract: This informal note provides a brief summary of key concepts in General Relativity and Holographic Principles. It is a condensed compilation of important points extracted from related materials, intended to offer a quick overview and serve as a reference for my future study.

Contents

A General reletivity

A.1 Introduction

A diffeomorphism $\phi : M \to M$ is called an isometry if $\phi^* g_{ab} = g_{ab}$.

A vector field ξ^a on (M, g_{ab}) is called a **Killing vector field** (Section 4.3 in [1], the same below) if the one-parameter group of diffeomorphisms it generates is a one-parameter group of isometries.

A vector field ξ^a on (M, g_{ab}) is called a Killing vector field if $\mathcal{L}_\xi g_{ab} = 0$.

Killing Equation:

$$\nabla_a \xi_b + \nabla_b \xi_a = 0 \tag{A.1}$$

Theorem: If there exists a coordinate system $\{x^\mu\}$ such that all components of g_{ab} satisfy $\partial g_{\mu\nu}/\partial x^1 = 0$, then $(\partial/\partial x^1)^a$ is a Killing vector field in the coordinate region.

From this theorem, it is easy to see that the two-dimensional Euclidean space has three independent Killing fields, which are $(\partial/\partial x)^a$, $(\partial/\partial y)^a$, and $(\partial/\partial \phi)^a$.

Definition: A curve $\gamma(t)$ on (M, ∇_a) is called a **geodesic** (Section 3.3) if its tangent vector $T^a \equiv \partial/\partial t$ satisfies the geodesic equation $T^b \nabla_b T^a = 0$. Thus, a necessary and sufficient condition for a curve to be a geodesic is that its tangent vector is parallel transported along the curve. The parameter that makes the curve a geodesic is called the **affine parameter** of the curve.

Mathematically, geodesics can be seen as the generalization of straight lines in curved space; when a metric definition exists, geodesics can be defined as the locally shortest path between two points in the space.

The worldline of a free particle (subject to no forces other than gravity) is a geodesic in the curved spacetime (M, g_{ab}) (i.e., the four-force is zero).

Theorem: Let ξ^a be a Killing vector field and T^a be the tangent vector of a geodesic. Then $T^a\nabla_a(T^b\xi_b) = 0$, i.e., $T^b\xi_b$ is constant along the geodesic.

Energy-Momentum Tensor T_{ab} satisfies:
For any observer $(p, (e_\mu))$, with $(e_0)^a = Z^a$,
(a) $\mu \equiv T_{ab}Z^aZ^b = T_{00}$ is the energy density measured by this observer;
(b) $w_i = -T_{ab}Z^a(e_i)^b = -T_{0i}$ is the i-th component of the 3-momentum density (energy flux density) measured by this observer;
(c) $T_{ab}(e_i)^a(e_j)^b = T_{ij}$ is the ij-th component of the 3-stress tensor measured by this observer.
The conservation equation $\nabla^a T_{ab} = 0$ implies conservation of energy, 3-momentum, and angular momentum.

The **Riemann curvature tensor** field $R_{abc}^{\ \ \ d}$ of the derivative operator ∇_a is defined as:

$$(\nabla_a\nabla_b - \nabla_b\nabla_a)\omega_c = R_{abc}^{\ \ \ d}\omega_d, \quad \forall \omega_c \in \mathscr{F}(0,1) \tag{A.2}$$

A metric whose Riemann tensor is zero is called a flat metric. Both Euclidean and Minkowski metrics are flat metrics.

Given a (0,4) tensor R_{abcd}, there is only one independent contraction of its six contractions, denoted by $R_{ac} \equiv R_{abc}^{\ \ \ b}$, and called the **Ricci tensor**. The quantity $g^{ac}R_{ac}$ is denoted by R and called the **scalar curvature**.

Einstein tensor:

$$G_{ab} \equiv R_{ab} - \frac{1}{2}Rg_{ab} \tag{A.3}$$

which satisfies $\nabla^a G_{ab} = 0$.

Setting $G_{ab} = 8\pi T_{ab}$, we obtain the **Einstein field equations**:

$$R_{ab} - \frac{1}{2}Rg_{ab} = 8\pi T_{ab} \tag{A.4}$$

When $T_{ab} = 0$, (A.4) reduces to

$$R_{ab} - \frac{1}{2}Rg_{ab} = 0 \tag{A.5}$$

called the **vacuum Einstein field equations**. It can be shown that when $T_{ab} = 0$, R equals zero, thus (A.5) is equivalent to $R_{ab} = 0$.

A static, spherically symmetric metric satisfying the vacuum Einstein field equations is called the **Schwarzschild vacuum solution**, or Schwarzschild solution for short.

When a star has a charge, the exterior is no longer a vacuum but is filled with an electromagnetic field. A spacetime with only an electromagnetic field and no other matter fields is called an electrovac spacetime.

In this case, T_{ab} is the energy-momentum tensor of some electromagnetic field F_{ab}, given by

$$T_{ab} = \frac{1}{4\pi}\left(F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \tag{A.6}$$

By substituting into (A.4), we obtain the Einstein-Maxwell equations:

$$G_{ab} = 2\left(F_{ac}F_b^c - \frac{1}{4}g_{ab}F_{cd}F^{cd}\right) \tag{A.7}$$

$$\nabla^a F_{ab} = 0 \tag{A.8}$$

$$\nabla_{[a}F_{bc]} = 0 \tag{A.9}$$

The solution to these equations is known as the **Reissner-Nordström solution**.

A point where the metric becomes undefined is called a **singularity** (Section 9.4). If the metric appears poorly behaved at a point due to a poor choice of coordinates and this can be resolved by choosing a different coordinate system, it is called a coordinate singularity. If the metric tensor itself is genuinely poorly behaved at a point, it is called a true singularity or a spacetime singularity.

For example, in the Schwarzschild metric, $r = 2M$ is a coordinate singularity, while $r = 0$ is a true singularity.

A spacetime is called singular if there exist incomplete timelike or null (but not spacelike) geodesics in an inextendible spacetime, meaning the affine parameter range is not $(-\infty, \infty)$. This indicates the presence of spacetime singularities.

The Kruskal extension of Schwarzschild spacetime is shown in Figure (1).

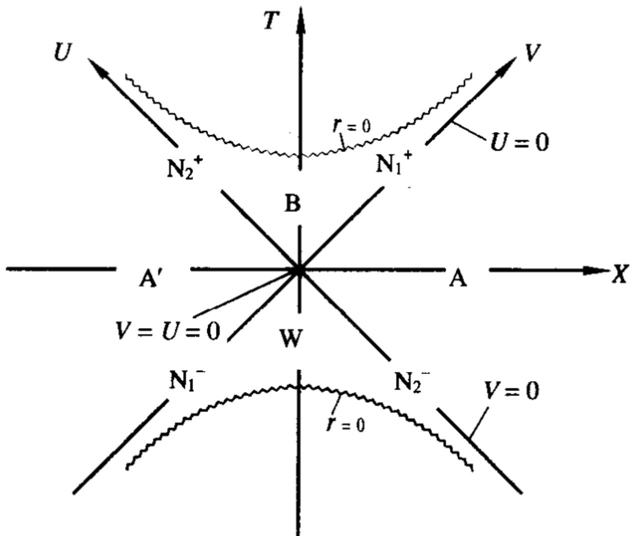
In the figure, the line $r = 0$ corresponds to $X^2 - T^2 = -1$, indicating a spacetime singularity. The Kruskal extension is valid for $r > 0$, so the region above $r = 0$ is not part of the extended spacetime. In the diagram with T and X as coordinates, the (radial) null curves are straight lines at $\pm 45°$.

The pre-extension spacetime is in region A, which corresponds to the original $\{t, r\}$ coordinate system. Since $r = 2M$ corresponds to $X^2 - T^2 = 0$, i.e., $T = \pm X$, region A represents $r > 2M$. Regions B, W, and A' are products of the extension.

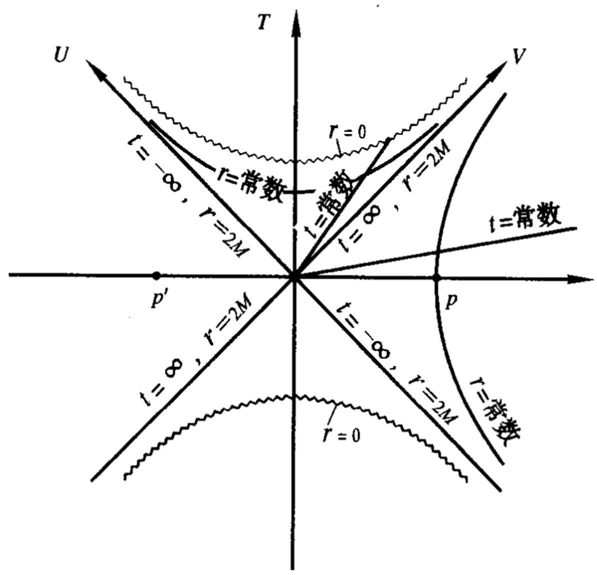
Consider the connected manifold $A \cup N_1^+ \cup B$. Any future-directed null curve starting from a point in region A with decreasing r will inevitably cross N_1^+ into region B. Conversely, any future-directed timelike or null curve starting from a point in region B cannot cross N_1^+ into region A. This indicates that N_1^+ is a one-way membrane: once an object (including photons) crosses it into region B, it can never return to region A (only falling into the singularity). Therefore, region B is called a **black hole**, and N_1^+ is called the **event horizon**.

B Holographic Duality

This part refers to [6].



(a) A 和 A′代表两个无因果联系的渐近平直区. B 为黑洞区, W 为白洞区.
N_1^+ 和 N_2^+ 是黑洞的事件视界. 锯齿状曲线代表奇性所在处(不属于时空)



(b) 双曲线上 $r=$ 常数, 过原点直线上 $t=$ 常数, 两条过原点的 45°斜直线上 $r=2M, t=\pm\infty$.

图 9-13 施瓦西时空的 Kruskal 最大延拓

Figure 1. Kruskal extension of Schwarzschild spacetime

Can massless spin-2 particles (gravitons) arise as bound states in a theory of massless spin-1 (photons, gluons) and spin- 1 /2 particles (protons, electrons)?

Weinberg-Witten Theorem forbids the existence of massless spin-2 particles, which is a hallmark of gravity, in the same spacetime a QFT lives.

> Theorem 1 : A theory that allows the construction of a Lorentz-covariant conserved 4-vector current J^μ cannot contain massless particles of spin $> \frac{1}{2}$ with non-vanishing values of the conceived charge $\int J^0 d^3x$.
> Theorem 2 : A theory that allows a conserved Lorentz-covariant stress tensor $T^{\mu\nu}$ cannot contain massless particles of spin > 1.

But there is a loophole: emergent gravity can live in a different spacetime, as in a holographic duality.

B.1 Black Hole

A null hypersurface $\mathcal{K}$ is called a **Killing horizon** if there is a Killing vector field K^a in the spacetime such that $K^a\mid_{\mathcal{K}}$ is orthogonal to $\mathcal{K}$.

The **surface gravity** κ is defined by:

$$\nabla^a(K^b K_b) = -2\kappa K^a \quad (\text{on } \mathcal{K}) \tag{B.1}$$

In Hawking radiation, the **temperature** $T_{BH} = \frac{\kappa}{2\pi}$ and the black hole entropy $S_{BH} = \frac{A}{4}$.

$$\delta E = T\delta S - P\delta V \tag{B.2}$$

$$\delta M = \frac{1}{8\pi}\kappa\delta A + \Omega\delta J \tag{B.3}$$

Planck scale represents the energy scale at which the quantum effects of gravity become strong.

Schwarzshild radius : the classical gravity becomes strong Two important scale:

$$r_C = \frac{\hbar}{mc} : \ \textit{Reduced Compton wavelength} \tag{B.4}$$

$$r_S = \frac{2G_N m}{c^2} : \ \textit{Schwarzschild radius.} \tag{B.5}$$

However, black holes can make quantum gravity effects manifest at macroscopic level, at length scales of $O(r_s)$, we will discuss this later.

Black hole geometry is the solution of Einstein equation with zero cosmology constant.

(A.5)'s solution is what we called **Schwarzschild metric solution**:

$$ds^2 = -fdt^2 + \frac{1}{f}dr^2 + r^2(d\theta^2 + sin\theta^2 d\phi^2) \tag{B.6}$$
$$f = 1 - \frac{2G_N M}{r} = 1 - \frac{r_s}{r}$$

The event horizon is defined at

$$r = r_s = 2G_N M \tag{B.7}$$

It is only a coordinate singularity (not an intrinsic singularity), where t (Schwarzschild time) and r coordinates become singular at the horizon. It takes a free-fall traveler a finite proper time to reach the horizon, but infinite Schwarzschild time. Once inside the horizon, a traveler cannot send signals to outside, nor can she/he escape.

Area of a spatial section:

$$A = 4\pi r_s^2 = 16\pi G_N^2 M^2 \tag{B.8}$$

We will show that a black hole has a temperature, as viewed by a stationary observer outside the black hole. First we need to define the temperature. Recall that in QFT, to describe a system at finite temperature (T), we analytically continue to Euclidean signature, i.e. $t \to -i\tau$. And let τ to be periodic: $\tau \sim \tau + \hbar\beta$, with $\beta = \frac{1}{T}$. Notice that here we have also adapted the convention $k_B = 1$. Conversely if Euclidean continuation of a QFT is periodic in time direction, we can conclude that **the QFT is at finite temperature**. This is exactly what we are going to do to interpret the temperature of a black hole.

Through the periodicity for θ in 2π, we could know the $\tau \sim \tau + \frac{2\pi}{\kappa}$ (Lecture 3 in [6]). And an observer at $r = \infty$ will feel a temperature $T = \frac{1}{\beta} = \frac{\hbar\kappa}{2\pi} = \frac{\hbar}{8\pi G_N m}$.

Then recall thermodynamic relations:

$$\frac{dS}{dE} = \frac{1}{T(E)} = \frac{8\pi G_N m}{\hbar} \tag{B.9}$$

since for a black hole $E = m$:

$$S(E) = \int \frac{dE}{T(E)} = \frac{4\pi G_N E^2}{\hbar} + const = \frac{4\pi r_s^2}{4\hbar G_N} = \frac{A_H}{4\hbar G_N} \tag{B.10}$$

So:

$$T_{BH} = \frac{\hbar K}{2\pi}, S_{BH} = \frac{A_H}{4\hbar G_N} \tag{B.11}$$

The equation (B.11) is in fact universal, applying to all black holes in Einstein gravity with matter fields (including those from string theory).

In this section, we list the mostly quoted results about general back holes.

No hair theorem: a stationary, asymptotically flat black hole is characterized by its:
1. mass M
2. angular momentum J
3. conserved gauged charges (e.g. electric charge Q).

This means after the process: star $\rightarrow$ black hole, all features of the stars have lost (classically).

Now we summarize four laws of black hole mechanics:

• 0th law: surface gravity κ is constant over the horizon.

• 1st law:

$$dM = \frac{\kappa}{8\pi G_N} dA + \Omega dJ + \Phi dQ \tag{B.12}$$

where Ω is the angular frequency at the horizon, $\Phi$is the electric potential at the horizon (assume that at ∞ the potential is 0).

• 2nd law: horizon area never decreases classically.

• 3rd law: surface gravity of a black hole cannot be reduced to 0 in a finite number of steps.

B.2 Holographic duality from field theory side

Holographic principle: In quantum gravity, a regime of boundary area A can be fully described by no more than $\frac{A}{4\hbar G_N} = \frac{A}{4l_p^2}$ degrees of freedom, i.e. degree of freedom per Planck area. Black hole brings quantum gravity to a macroscopic level.

Let h be genus. Sphere is genus-0, torus is genus-1. An topological invasriant of the manifold is the Euler character:

$$\xi = 2 - 2h \tag{B.13}$$

1.For any non-planar diagram, there exists an integer h, such that the diagram can be straightened out (i.e. non-crossing) on a genus-h surface, but not on a surface with a smaller genus.

2. For any non-planar diagram, the power of N that comes from contracting propagators is given by the number of faces on such a genus-h surface, i.e. the number of disconnected regions separated by the diagram.

Given a surface composed of polygons with F faces, E edges and V vertices, the Euler character satisfy:

$$\xi = F + V - E = 2 - 2h \tag{B.14}$$

In general, a vacuum diagram has the following dependence on g^2 and N:

$$A \sim (g^2)^E (g^2)^{-V} N^F \tag{B.15}$$

where E is the number of propagators, V is the number of vertices, F is the number of faces. This does not give a sensible $N \to \infty$ limit or 1/N expansion, since there is no upper limit on F. However, 't Hooft suggests that we can take the limit $N \to \infty$ and $g^2 \to 0$ but keep $\lambda = g^2 N$ fixed. Then

$$A \sim (g^2 N)^{E-V} N^{F+V-E} = \lambda^{L-1} N^{\xi} = \lambda^{L-1} N^{2-2h} \tag{B.16}$$

where L is the number of loops.

Consider the two belowed theories:

$$\mathcal{L} = -\frac{1}{g^2} Tr[\frac{1}{2}(\partial\Phi)^2 + \frac{1}{4}\Phi^4] \tag{B.17}$$

$$\mathcal{L} = \frac{1}{g_{YM}^2}[-\frac{1}{4} Tr F_{\mu\nu} F^{\mu\nu} - i\bar{\Psi}(\not{D} - m)\Psi] \tag{B.18}$$

(B.17) is invariant under $U(N)$ transformation. (B.18) is invariant under local $U(N)$ transformation. In (B.18), allowed operators must be gauge invariant.

So if we consider gauge theories: $\mathcal{L} = \mathcal{L}(A_\mu, \Phi)$, the allowed operators will be like :$\mathrm{Tr}(F_{\mu\nu}F^{\mu\nu}), \mathrm{Tr}(\Phi^n)$)...(Single-trace operators)

We denote single-trace operators as $\mathcal{O}_k$.So general observables will be correlation functions of gauge invariant operators, here we focus on local operators:

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n) >_c \tag{B.19}$$

For any Lagrangian of matrix valued fields of the form

$$\mathcal{L} = \frac{N}{\lambda} Tr(...) \tag{B.20}$$

because log$Z$ evaluates the sum of all vacuum diagrams, we would have

$$logZ = \sum_{h=0}^{\infty} N^{2-2h} f_h(\lambda) \tag{B.21}$$

Consider

$$Z[J_1, ..., J_n] = \int DA_\mu D\Phi ... exp(iS_{eff}) \tag{B.22}$$

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n) >_c = \frac{\delta^n \log Z}{\delta J_1(x_1)...\delta J_n(x_n)}|_{J_1=...=J_n=0} \frac{1}{(iN)^n} \tag{B.23}$$

Applying (B.21), we have

$$< \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)...\mathcal{O}_n(x_n) >_c \sim N^{2-n}[1 + O(\frac{1}{N^2} + ...)] \tag{B.24}$$

In (C.1), for vacuum processes:

$$Z_{string} = \sum_{\text{all closed surfaces}} e^{-S_{NG}} = \sum_{h=0}^{\infty} e^{-\lambda\xi} \sum_{\text{surface with given topology}} e^{-S_{NG}} \tag{B.25}$$

here $\xi = 2 - 2h$ denotes the weight for different topologies. We define $g_s = e^{\lambda}$.There is a remarkable fact about string theory: summing over topology of all surfaces automatically includes interactions of strings.

$$\begin{aligned} \text{a large N gauge theory} &= \text{a string theory} \\ \frac{1}{N}\text{expansion} &= \text{pertubative expansion in } g_s \\ \text{large N limit (classical theory of glue balls)} &= \text{classical string theory} \\ \text{single-trace operators(glue balls)} &= \text{string states} \end{aligned} \tag{B.26}$$

B.3 D brane

A Dp-brane in general spacetime dimension D breaks translational T^D and Lorentz symmetries $SO^+(1, D)$ (Poincare symmetry in D dimensions) into:

$$(T^p \otimes SO^+(1, p)) \otimes SO(D - 1 - p) \tag{B.27}$$

The Poincare symmetry along the Dp-brane is preserved, while in the directions perpendicular to the brane (the traverse directions) only the spatial rotation symmetry is unbroken.

From the spacetime symmetries preserved by a D3-brane, given in equation (B.27) , one can guess the form of the spacetime metric deformed by a stack of N D3-branes (sitting at position r = 0):

$$ds^2 = f(r)(-dt^2 + \sum_{i=1}^{3} dx_i^2) + h(r)(dr^2 + r^2 d\Omega_e^2) \tag{B.28}$$

The SUGRA solution gives:

$$f(r) = \frac{1}{h(r)} = H^{-1/2}(r), H(r) = 1 + \frac{(R^4)^4}{r}, R^4 = N\frac{4}{\pi^2}G_N T_3 = N4\pi g_s \alpha' \tag{B.29}$$

Now, we has 2 descriptions of N D-branes:

1. Description A: D-branes in flat spacetime where open strings can end.

2. Description B: Deformed spacetime metric given in this equation with F_5 fluxes on S^5 where only closed strings can propagate.

Here we have AdS/CFT duality

From the general relativity Einstein's field equation point of view, AdS is a spacetime of constant curvature with negative cosmological constant:

$$\mathcal{R}_{MN} - \frac{1}{2}g_{MN}(\mathcal{R} - 2\Lambda) = 0; \Lambda < 0 \tag{B.30}$$

C Some Foundations of String theory

The simplest form of S_{string} is the Nambu-Goto action

$$S_{NG} = T \int_{\Sigma} dA \tag{C.1}$$

$dA = \sqrt{-\mathrm{det} h_{ab}} d\sigma d\tau$ is the infinitesimal area of the world sheet with the induced matrix $h_{ab} = g_{\mu\nu} \partial_a X^\mu \partial_b X^\nu$.

Nambu-Goto Action and The Polyakov Action (eliminate the square root by introducing another field).

$$potential\, energy = T \times (spatial\, length\, of\, string) \tag{C.2}$$

T is the energy per unit length as claimed. We learn that the string energy increases linearly with length:

$$T = \frac{1}{2\pi\alpha'} \tag{C.3}$$

where α' is referred to as the "universal Regge slope".

Length scale l_s:

$$\alpha' = l_s^2 \tag{C.4}$$

Lightcone coordinates:

$$\sigma^{\pm} = \tau \pm \sigma \tag{C.5}$$

When strings interact with each other, all the information about interacting strings is already contained in the free theory described by the Polyakov action.

To see the difference made by topologies, we argument Polyskov action by

$$S_{string} = S_{Poly} + \lambda\xi \tag{C.6}$$

$$\xi = \frac{1}{4\pi} \int d^2\sigma \sqrt{g} R \tag{C.7}$$

Define string coupling constant as $g_s = e^{\lambda}$.

Schematically, the path integral takes the form

$$Z = \int D\phi e^{-S[\phi]} \tag{C.8}$$

where ϕ collectively denote all the fields.

References

[1] Canbin Liang, *Introduction to Differential Geometry and General Relativity*,

[2] Michio Kaku, *Introduction to superstrings and M theory.*

[3] Polchinski, *String Theory.*

[4] Wald,*General Relativity.*

[5] David Tong,*String theory*

[6] Hong Liu,MIT 8.821 String Theory and Holographic Duality