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Introduction to Schwinger-Dyson Equation

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Abstract: In this note I will introduce the Schwinger-Dyson equation, and give a derivation to an example.

Contents

1 Schwinger-Dyson equations

The Schwinger-Dyson equations can be understood as the quantum equations of motion for Green's functions with proper contact terms included[2]. In deriving classical motion equations, we use the stationary of action: under any infinitesimal change of fields $\phi^\alpha \to \phi^\alpha + \delta\phi^\alpha$. That's the same for generation functions. Let's take the example from[3] as an instance.

The generating functional

$$Z[J] = \int Dx \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)]$$

is given by

$$L = \frac{1}{2}(\frac{d}{dt}x(t))^2 - \frac{1}{2}x(t)^2 - gx(t)^4$$

Give an infinitesimal change of integration

$$x(t) \to x(t) + \delta x(t),$$

(for convenience I will omit the integral limits and some variable (t)) The stationary condition yields:

$$\begin{aligned}
0 &= \delta Z \\
&= \delta(\int Dx \exp[i \int dt(L + Jx)] \\
&= \int Dx \delta(\exp[i \int dt(L + Jx)) \\
&= \int Dx \exp[i \int dt(L + Jx)] \int dt(\delta L + J\delta x)
\end{aligned}$$

$$\begin{aligned}
&\int dt(\delta L + J\delta x) \\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + \tfrac{\partial L}{\partial(\partial_t x)}\delta\partial_t x + J\delta x) \\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + \tfrac{\partial L}{\partial(\partial_t x)}\partial_t\delta x + J\delta x) \\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) + \int dt \tfrac{\partial L}{\partial(\partial_t x)}\partial_t\delta x \\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) + \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)}\delta x) - \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)})\delta x \\
&= \int dt(\tfrac{\partial L}{\partial x}\delta x + J\delta x) - \int dt\partial_t(\tfrac{\partial L}{\partial(\partial_t x)})\delta x
\end{aligned}$$

The last line use the proper boundary condition. Insert it to the equation above we have:

$$\begin{aligned}
&\frac{\partial L}{\partial x} = -x - 4gx^3 \\
&\frac{\partial L}{\partial(\partial_t x)} = \partial_t x \\
&\int dt(\delta L + J\delta x) \\
&= \int dt(-x - 4gx^3 + J - \partial^2{}_t x)\delta x
\end{aligned}$$

$$\begin{aligned}
0 &= \int Dx \exp[i \int dt_1(L + Jx)] \int dt_2(-x - 4gx^3 + J - \partial^2{}_t x)\delta x \\
&= \int dt_2 \int Dx\delta x(\exp[i \int dt_1(L + Jx)](-x - 4gx^3 + J - \partial^2{}_t x))
\end{aligned}$$

As this should be satisfied for any possible choice of ε, we change the integrate of Dx and dt, and get

$$\int Dxe^{i\int dt(L+Jx)}(-x - 4gx^3 + J - \partial^2{}_t x) = 0$$

Now we use the property of generating function

$$\frac{\delta^n Z[J]}{i^n \delta^n J} = \int Dx \cdot x^n \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)],$$

and get the equation of motion:

$$(\partial^2{}_t + 1)\frac{\delta}{i\delta J(t)} Z[J] + 4g\frac{\delta^3}{i^3\delta^3 J(t)} Z[J] = J(t)Z[J].$$

We can further take the function derivatives $n-1$ times with respect to $J(t_n)$ and set source to zero:

$$\begin{aligned}
&\frac{\delta}{i\delta J(t)\delta J(t_1)...\delta J(t_{n-1})} Z[J] = i^{n-1} \int Dx \cdot x(t)x(t_1)...x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)], \\
&\frac{\delta}{i^3\delta^3 J(t)} Z[J] = i^{n-1} \int Dx \cdot x^3(t)x(t_1)...x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)], \\
&\frac{\delta}{\delta J(t_1)...\delta J(t_{n-1})}(J(t)Z[J]) = \sum_{k=1}^{n-1} \frac{\delta J(t)}{\delta J(t_k)} \frac{\delta Z[J]}{\delta J(t_1)...\delta J(t_{k-1})\delta J(t_{k+1})...\delta J(t_{n-1})} + J(t)\frac{\delta Z[J]}{\delta J(t_1)...\delta J(t_{n-1})} \\
&= \sum_{k=1}^{n-1} \delta(t - t_k) i^{n-2} \int Dx \cdot x(t_1)...x(t_{k-1})x(t_{k+1})...x(t_{n-1}) \cdot \exp[i \int_{-\infty}^{\infty} dt(L + J(t)x(t)]
\end{aligned}$$

So:

$$\begin{aligned}
&i^{n-1}(\partial^2{}_t + 1)G_n(t, t_1, t_2, ..., t_{n-1}) + 4gi^{n-1}G_{n+2}(t, t_1, t_2, ..., t_{n-1}) \\
&= i^{n-2} \sum_{k=1}^{n-1} \delta(t - t_k)G_{n-2}(t_1, t_2, ..., t_{k-1}, t_{k+1}, ..., t_{n-1})
\end{aligned}$$

That reproduce the result from [3].

References

[1] Mark Srednicki, *Quantum Field Theory*

[2] August Geelmuyden, The Schwinger-Dyson equations

[3] Yongwei Guo and Wenliang Li, Solving anharmonic oscillator with null states: Hamiltonian bootstrap and Dyson-Schwinger equations