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Note on Majorana-Weyl Spinor

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Abstract: In this note, I have restructured the introduction to Majorana-Weyl spinors from [4] and provided fundamental insights into the concepts of spinor fields.

Contents

1 Definition

The definitions are as follows:

Weyl spinor: spinors with definite γ_{D+1} eigenvalue(± 1).

Majorana spinor: fermions equals to its charge conjugation, or spinors who have real representation and make Dirac equation real(The two situations are the same, because it is how we define the charge conjugation matrix[1]).

Majorana-Weyl spinor: the Majorana condition can be satisfied for a spinor with a definite chirality. Or to apply both the Weyl and Majorana conditions. It's possible in $D = 8k$ (Euclidean spacetime) and $D = 8k + 2$ (Minkowski spacetime).

2 Weyl Spinor

We start with 4 dimension spinor field.

Weyl equations:

$$\partial_0 \psi = \pm \sigma^i \partial_i \psi$$

describe massless particles with light velocity. ψ have two components. To describe the massive particles , we couple the two Weyl equations and get **Dirac equation**:

$$(i\gamma^\mu \partial_\mu - m)\psi = 0$$

where

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}$$

Here ψ_R and ψ_L are the two component Weyl spinors, and ψ is a 4 component Dirac spinor.

[1]Look for equation (42) in [4]

In D dimension, if γ is in the Clifford Algebra

$$\{\gamma^i, \gamma^j\} = 2g^{ij}$$

And we define an extra gamma matrix (if D=2k is even. In fact the 2k+1 representation is also given by it)

$$\gamma_{D+1} = (-1)^{\frac{1}{4}(D+2)}\gamma_0\gamma_1...\gamma_{D-1}$$

That play a role as γ_5 in 4d. Then

$$\gamma^2{}_{D+1} = 1$$

The eigenvalue of γ_{D+1} is $\lambda^2 = 1$.Since gamma matrices are traceless, eg

$$tr\gamma^5 = tr(\gamma^0\gamma^0\gamma^5) = -tr(\gamma^0\gamma^5\gamma^0) = -tr(\gamma^0\gamma^0\gamma^5) = -tr\gamma^5$$

The eigenvalue are half 1 and half -1. We can therefore write any spinor uniquely as

$$\psi = \psi_+ + \psi_-$$

which $\psi_\pm$ has eigenvalue ± 1. A spinor with a definite γ_{D+1} eigenvalue is called a **Weyl spinor**.

More specificly, we could define projection operators as

$$P_L{=}\frac{1}{2}(1-\gamma^5), P_R = \frac{1}{2}(1+\gamma^5)$$

They act on Dirac spinors, giving left-handed and right-handed spinors. Left-handed and right-handed spinors correspond to eigenvalues of gamma matrices being negative one and positive one respectively.

$$\gamma^5 P_L\psi = -P_L\psi, \gamma^5 P_R\psi = P_R\psi$$

3 Majorana Spinor

Let's come back to the 4d example. A transformation to Weyl equation gives

$$\partial_0\psi_C = \mp\sigma^i\partial_i\psi_C$$

where $\psi_C = C\psi = \eta_C\sigma^2\psi^*$. This equation represents a particle with opposite helicity. The charge conjugate operation C means change a particle to its antiparticle.

Majorana means a neutral particle equals to its antiparticle, that is,

$$\psi = \psi^C = C\bar{\psi}^T = C(\psi^\dagger\gamma^0)^T$$

(there are different defination of charge conjugation matrix, first is $\psi' = C\psi$) . Another description is the spinor is real (real gamma matrices and fermions belongs to real representation). They are equivlent.

A real representation is if you can find a unitary matrix U that

$$\rho(T^a)^T = -U\rho(T^a)U^{-1}$$

Through calculation we know U is either symmetric or antisymmetric. But if we'd like to find a basis where all representation matrices of group elements are real, we need U to be symmetric.

In Euclidean spacetime, there are two possibilities for the representation to qualify as "real",

$$\rho(T^a)^T = -C\rho(T^a)C^{-1}$$
$$\rho(T^a)^T = +T\rho(T^a)T^{-1}$$

As calculation in [4], the real in strict sense is only the case when $D = 8k, 8k+1, 8k+7$.If we want to make Dirac equations real, we need gamma matrices could be taken real, and fermions belongs to real representations of gauge group. Majorana situation are when the Dirac equations is real and a real representation of spinors. Through calculate we found that the Majorana situation need the T is symmetric. It is in $D = 8k, 8k+1, 8k+2$.

4 Majorana-Weyl Spinor

Weyl spinors leaves in even dimensions. And if we need Majorana-Weyl spinors, we need the Majorana condition can be satisfied for a spinor with a definite chirality, i.e. Majorana and Weyl conditions could be imposed simultaneously. If so, we need C and T to commute with γ_{D+1} then the irreducible representations of Spin(N) with definite chirality under γ_{D+1} are each self-conjugate. Therefore, we cannot have Majorana-Weyl condition in $8k+2$ but in $8k$.

Above we talk about Euclidean spacetime. Through a construction we could found in Minkowski spacetime we only need to plus 2 dimensions. $C\gamma^0$ play the role of T above.

References

[1] Lewis H. Ryder,*Quantum Field Theory*

[2] Steven Weinberg,*The Quantum Theory of Fields, Vol. 3 Supersymmetry. (2000)*

[3] Neil Lambert,Supersymmetry and Gauge Theory

[4] Hitoshi Murayama,Notes on Clifford Algebra and Spin(N) Representations