Monte Carlo Method in Option Pricing

RE Group13

Jie Li, Junyou Wang, Shawn Xu, Xinran Zhang, Xinying Wang
School of Mathematics and Statistics
University of Sydney

September 12, 2022

Overview

Background

Background

- Origin - "Buffon's Needle" experiment to determine π
- Reducing generation cost - Computing power developed

Applications in Option Pricing

- Price options with numerous sources of uncertainty and random features in option price
- Deal with the curse of dimensionality
  The Monte Carlo method works well against that because the computational complexity of the method no longer depends on the dimensionality

Basic Monte Carlo

Introduction

- Generation of random objects
- Repeat the experiment

Basic idea

- Law of Large Numbers
  $\int_a^b f(x)\,\mathrm{d}x = (b-a)\int_a^b f(x)\frac{1}{b-a}\,\mathrm{d}x = (b-a)Ef(x)$
  Where $X \sim U(a,b)$
  E.g. $\int_0^1 sin(x)e^{x^2}\,\mathrm{d}x = E[sin(x)e^{x^2}] \approx \frac{1}{N}\sum_{i=1}^{n} sin(x_i)e^{x_i^2}$
- Statistical inference
- Accuracy and reliability(central limit theorem)
  Let $x_1, x_2, ..., x_n$ be iid random variable with $E(X_i) = \mu$ and $var(X_i) = \sigma^2 < \infty$
  Then $\frac{(\frac{1}{n}\sum_{i=1}^{n} X_i - \mu)}{\frac{\sigma}{\sqrt{n}}} \to N(0,1)$

3.1 Basic idea of variance reduction

- To search for alternative and more accurate estimators

- Find a function g, such that

$$E\,g(x) = E\,f(x),\; Var\,g(x) \ll Var\,f(x)$$

- Typically needed in simulations of highly complex models with high computation cost per replication, and in simulations of rare events with too many replications in crude Monte Carlo

3.2 Different variance reduction techniques

Classic variance reduction techniques

- Systematic sampling
  — Antithetic variables, Stratified sampling
- Importance sampling
- Control variates
- Partial integration

Modern variance reduction techniques

- Quasi Monte Carlo
- weighted Monte Carlo
- Latin hypercube sampling
- ...

3.3 Antithetic variables and importance sampling

- **Antithetic variables**

– Assume $X \overset{d}{=} -X$,

$cov(f(x), f(-x)) \leqslant 0$

– $E\, f(x) = E\,\frac{f(x)+f(-x)}{2}$

$\approx \frac{1}{N}\sum_{i=1}^{N} \frac{f(x_i)+f(-x_i)}{2}$

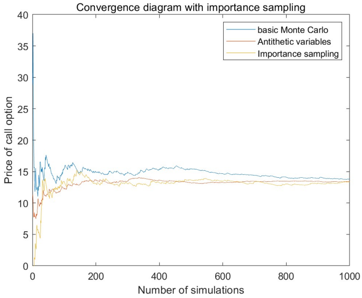
- **Importance sampling**

– Suppose $p(f(x) = 0) \approx 1$

– Find g_β, s.t.

$E\, f(x) = E\, g_\beta(y) \approx \frac{1}{N}\sum_{i=1}^{N} g_\beta(y_i)$,

$p(g_\beta(y_i) \neq 0)$ is large



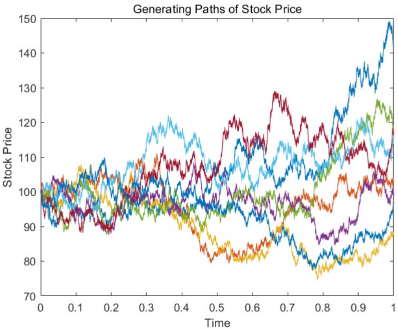
4.1 Generating Random Paths

Basic Idea

- Start from initial stock price
- Simulate paths with the same time increment
- Average each terminal point

Methods

- Euler method
- Milstein method
- Milstein method is better than Euler method **theoretically**



4.2 Euler Method

- **Basic Idea**
  - Stochastic Differential Equation(SDE)
  - Recursion with step length $\Delta t = \frac{T}{M}$
- **Discretisation of the SDE**
  $S^{m+1} = S^m + rS^m\Delta t + \sigma S^m\sqrt{\Delta t}\xi^m,\ \xi^m \sim N(0,1)\ where\ S^m = S_{m\Delta t}$
- **Error**
  - MC Error $\sim O(\frac{1}{\sqrt{N}})$
    Time Discretisation Error $\sim O(\frac{1}{\sqrt{M}})$
  - Total Error $\sim O(\frac{1}{\sqrt{N}}) + O(\frac{1}{\sqrt{M}})$
  - Given MN fixed, minimised when $M \approx N$

4.3 Milstein Method

- **From Ito's Formula**

$$Y(t+\Delta t) = Y(t) + b(Y_t,t)\Delta t + \sigma(Y_t,t)(B_{t+\Delta t} - B_t) + I + II + III + IV$$

where

$$I := \int_t^{t+\Delta t}\int_t^s Lb(Y_u,u)\,duds$$

$$II := \int_t^{t+\Delta t}\int_t^s Gb(Y_u,u)\,dBuds$$

$$III := \int_t^{t+\Delta t}\int_t^s L\sigma(Y_u,u)\,dudBs$$

$$IV := \int_t^{t+\Delta t}\int_t^s Gb(Y_u,u)\,dBudBs$$

and

$$Lf := \frac{\partial}{\partial t}f + b\frac{\partial}{\partial y}f + \frac{1}{2}\sigma^2\frac{\partial^2}{\partial y^2}f,\ Gf := \sigma\frac{\partial}{\partial y}f$$

- **Discretisation of the SDE**

$$S^{m+1} = S^m + rS^m\Delta t + \sigma S^m\sqrt{\Delta t}\xi^m + S^m\frac{\Delta t}{2}\left((\xi^m)^2 - 1\right),$$

$\xi^m \sim N(0,1)$

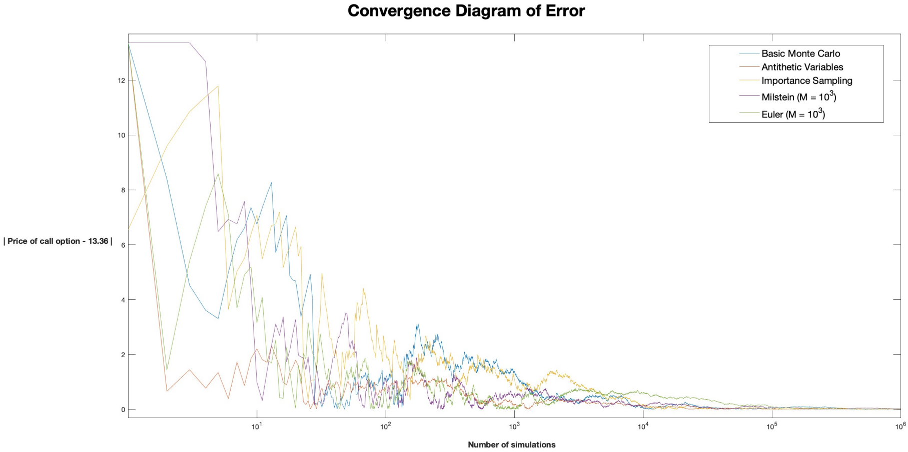
where $S^m = S_{m\Delta t}$

- **Error**

Discretisation error $\sim \Delta t \sim \frac{1}{M} \ll \frac{1}{\sqrt{M}} \sim \sqrt{\Delta t} \sim$ Euler method

5 Error's Convergence Diagram

- European call option: $S_0 = 100$ r $= 0.07$, σ $=0.25$, K $= 100$, T $= 1$.



6 Comparison Between Monte Carlo and Finite Difference

- Consider $Payoff = f(S_T^1, S_T^2, \cdots\cdots, S_T^d)$
- To achieve accuracy $\sim O(\varepsilon)$
- Monte Carlo: number of simulations N
- FD Crank-Nicolson:
  Number of time steps $M \sim \frac{1}{\Delta t} \sim \frac{1}{\Delta x}$
  Number of grids for each time step $N \sim \frac{1}{(\Delta x)^d}$

	Monte Carlo	**FD Crank-Nicolson**
Error	$\sim \frac{1}{\sqrt{N}}$	$\sim (\Delta x)^2$
Number of computations	$\sim O\left(\frac{1}{\varepsilon^2}\right)$	$\sim O\left(\frac{1}{\varepsilon^{\frac{d+1}{2}}}\right)$
Better in Dimension	$d > 3$	$d \leq 2$
Path dep. Option	Yes	No
Output(s)	$V(0, S_T^\star)$	$\big(V(t, S)\big)_{t,S}$

The End