Portfolio Optimisation with Market Data

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1 Executive Summary

Nowadays, portfolio optimisation is frequently used in financial investment. Portfolio optimisation methods have several theoretical foundations, including Modern Portfolio Theory, Utility Theory and Stochastic Processes Theory. This report firstly introduces these theory and deduces formulas for static and dynamic optimisation with different objectives using Lagrange methods. Then, real-market data of asset returns are used to realise four portfolio optimisation strategy and compare their effectiveness.

2 Introduction

2.1 Background of Portfolio Optimisation

Portfolio optimisation is to choose the best asset allocation with the purpose of maximising factors like expected return and minimising costs such as financial risk[2]. Here, we introduce three basic theory of portfolio optimisation.

2.1.1 Modern Portfolio Theory

Proposed by Harry Markowitz in 1952, Modern Portfolio Theory demonstrated a trade-off between risk and expected return[4]. Portfolios meeting the criterion that maximizing expected return given the amount of risk are called efficient portfolios and they can be graphically represented by a curve called efficient frontier. All the points on the efficient frontier are well-diversified.

2.1.2 Utility Theory

Utility Theory serves as a reference for individual decision considering their risk appetite[3]. For a risk-averse investor, the utility function U(X) has two characters: $U'(X) > 0$, $U''(X) < 0$. One investment is considered to be worthwhile if $E(U(X - S)) > U(0)$. In this report, we assume that the investor is risk-averse, $U(X) = -e^{-\gamma x}, \gamma > 0$.

2.1.3 Stochastic Processes Theory

Stochastic processes is a descriptive mathematical theory which demonstrates random phenomenon evolution over time, for example, Brownian motion process[1]. It is commonly used in financial industry, such as analysing portfolio behavior, modelling stock price movement, and estimating a rational option price.

2.2 Summary of the following work

This report aims to perform portfolio optimisation among 8 assets for two years. Our strategy consists of static and dynamic portfolio optimisation with maximum utility and minimum variance respectively. Real-word market data of 8 assets during 2 time intervals are used, where the 8 assets are Australian Equities (AEQ), Developed Market Equities (DEQ), Emerging Market Equities (EMEQ), Australian Listed Property (ALP), Hedge Funds (HF), Australian Fixed Interest (AFI), Global Fixed Interest (GOV), Cash (CASH), and the 2 time intervals are (A) from January 2007 to December 2010 inclusive, and (B) from January 2016 to December 2019 inclusive.

In the following report, we firstly conduct a theoretical analysis of these optimisation problems. Then, using market data, we estimate parameters and demonstrate real-world results of 4 strategy. Finally, we compare the results with each other and realised data during the period from January 2020 to December 2021.

3 Theoretical Analysis

3.1 Static Portfolio Optimisation with Maximum Utility

Firstly, we consider the investor statically investing their wealth for two years. We try to solve the utility maximisation problem:

$$\begin{aligned} &\max\ \mathbb{E}\Big[U(\boldsymbol{w}^T\boldsymbol{R})\Big], \\ &\text{s.t.}\ \sum_{i=1}^{8} w_i = 1, \end{aligned} \tag{1}$$

where $\boldsymbol{w} = (w_1, \cdots, w_8)^T$ is the vector of portfolio weights, $U(x) = -e^{-\gamma x}$ with $\gamma > 0$, $\boldsymbol{R}$ denote joint two-year returns for the eight assets.

$$\begin{aligned} \max\ \mathbb{E}\Big[U(\boldsymbol{w}^T\boldsymbol{R})\Big] &= \max\ \mathbb{E}[-e^{-\gamma \boldsymbol{w}^T R}] \\ &\Leftrightarrow \min\ \mathbb{E}[e^{-\gamma \sum_{i=1}^{8} w_i^T R_i}] \end{aligned} \tag{2}$$

As $\boldsymbol{R} = (R_1, \cdots, R_8)$ is multivariate normally distributed with mean $\boldsymbol{\mu}$ and covariance matrix C, we can get that $\gamma \sum_{i=1}^{8} w_i^T R_i$ is normally distributed with mean $\gamma \boldsymbol{w}^T \boldsymbol{\mu}$ and variance $\gamma^2 \boldsymbol{w}^T C \boldsymbol{w}$.

According to the associated characteristic function,

$$\mathbb{E}[e^{-\gamma \sum_{i=1}^{8} w_i^T R_i}] = e^{-\gamma \boldsymbol{w}^T \boldsymbol{\mu} + \gamma^2 \boldsymbol{w}^T C} \tag{3}$$

Then we can get the simplified objective function

$$\begin{aligned} \max\ \mathbb{E}\Big[U(\boldsymbol{w}^T\boldsymbol{R})\Big] &= \min\ [-e^{\gamma \boldsymbol{w}^T \boldsymbol{\mu} + \frac{1}{2}\gamma^2 \boldsymbol{w}^T C \boldsymbol{w}}] \\ &\Leftrightarrow \min\ \Big[-\frac{1}{\gamma}\boldsymbol{w}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{w}^T C \boldsymbol{w}\Big] \end{aligned} \tag{4}$$

Let t denote $\frac{1}{\gamma}$, and let $\boldsymbol{e}$ denote a vector of 1s with length 8, then the original problem is equivalent of

$$\begin{aligned} &\min\ -t\boldsymbol{w}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{w}^T C \boldsymbol{w}, \\ &\text{s.t.}\ \boldsymbol{w}^T\boldsymbol{e} = 1 \end{aligned} \tag{5}$$

We can get the optimal weight $\boldsymbol{w}^*$ using the method of Lagrange multipliers:

$$L(\boldsymbol{w}, \lambda) = -t\boldsymbol{w}^T\boldsymbol{\mu} + \frac{1}{2}\boldsymbol{w}^T C \boldsymbol{w} - \lambda(\boldsymbol{w}^T\boldsymbol{e} - 1), \quad \lambda \in \mathbb{R} \tag{6}$$

$$\nabla_{\boldsymbol{w}} L(\boldsymbol{w}, \lambda) = 0 \quad \text{and} \quad \frac{\partial L(\boldsymbol{w}, \lambda)}{\partial \lambda} = \boldsymbol{w}^T\boldsymbol{e} - 1 = 0 \tag{7}$$

Solving this equation set, we can get the optimal weight

$$\begin{aligned}\lambda &= \frac{1 - t\boldsymbol{e}^T C^{-1}\boldsymbol{\mu}}{\boldsymbol{e}^T C^{-1}\boldsymbol{e}} \\ \boldsymbol{w}^* &= C^{-1}(t\boldsymbol{\mu} + \lambda\boldsymbol{e})\end{aligned} \tag{8}$$

3.2 Static Portfolio Optimisation with Minimum Variance

We try to conduct portfolio optimisation with minimum variance by solving the following problem:

$$\begin{aligned}&\min\ \boldsymbol{w}^T C\boldsymbol{w}, \\ &\text{s.t. } \boldsymbol{w}^T\boldsymbol{\mu} = \mu_V, \\ &\quad\ \ \boldsymbol{w}^T\boldsymbol{e} = 1,\end{aligned} \tag{9}$$

where μ_V denotes the expected return of the portfolio.

$$L(\boldsymbol{w}, \lambda, \eta) = \boldsymbol{w}^T C\boldsymbol{w} - \lambda(\boldsymbol{w}^T\boldsymbol{e} - 1) - \eta(\boldsymbol{w}^T\boldsymbol{\mu} - \mu_V), \quad \lambda, \eta \in \mathbb{R} \tag{10}$$

$$\nabla_{\boldsymbol{w}} L = 2C\boldsymbol{w} - \lambda\boldsymbol{e} - \eta\boldsymbol{\mu} = 0 \tag{11}$$

$$\boldsymbol{w} = \frac{1}{2}C^{-1}(\lambda\boldsymbol{e} + \eta\boldsymbol{\mu}) \tag{12}$$

Substituting $\boldsymbol{w}$ into $\boldsymbol{w}^T\boldsymbol{\mu} = \mu_V$ and $\boldsymbol{w}^T\boldsymbol{e} = 1$, we can get the solution using Cramer's Rule,

$$\boldsymbol{w} = \frac{\begin{vmatrix} 1 & \boldsymbol{e}^T C^{-1}\boldsymbol{\mu} \\ \mu_V & \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\end{vmatrix} C^{-1}\boldsymbol{e} + \begin{vmatrix}\boldsymbol{e}^T C^{-1}\boldsymbol{e} & 1 \\ \boldsymbol{\mu}^T C^{-1}\boldsymbol{e} & \mu_V\end{vmatrix} C^{-1}\boldsymbol{\mu}}{\begin{vmatrix}\boldsymbol{e}^T C^{-1}\boldsymbol{e} & \boldsymbol{e}^T C^{-1}\boldsymbol{\mu} \\ \boldsymbol{\mu}^T C^{-1}\boldsymbol{e} & \boldsymbol{\mu}^T C^{-1}\boldsymbol{\mu}\end{vmatrix}} \tag{13}$$

3.3 Dynamic Portfolio Optimisation with Maximum Utility

Then we consider the situation that the investor will adjust his portfolio weights at the beginning of the second year.

For $k = 1, 2$, let $\boldsymbol{V^k} := (V_1^k, \cdots, V_8^k)$ denote returns of the 8 asset classes for the k-th year. $\boldsymbol{V^1}$ and $\boldsymbol{V^2}$ are i.i.d.copies of $\boldsymbol{R}$, where $\boldsymbol{R}$ denote joint annual returns for the eight assets. $\boldsymbol{w} = (w_1, \cdots, w_8)^T$ is the vector of portfolio weights at the beginning of the first year, and $\boldsymbol{u} = (u_1, \cdots, u_8)^T$ is the vector of portfolio weights at the beginning of the second year.

Since the return is yearly compounded, the return of the portfolio over the two-year investment period is

$$G(\boldsymbol{w}, \boldsymbol{u}) = (1 + \boldsymbol{w}^T\boldsymbol{V}^1)(1 + \boldsymbol{u}^T\boldsymbol{V}^2) - 1 \tag{14}$$

The problem can be formulated as

$$\begin{aligned}&\max\ \mathbb{E}[U(G(\boldsymbol{w}, \boldsymbol{u}))], \\ &\text{s.t. } \sum_{i=1}^{8}\boldsymbol{w}_i = \sum_{i=1}^{8}\boldsymbol{\mu}_i = 1\end{aligned} \tag{15}$$

Note $\boldsymbol{\mu} = \boldsymbol{\mu}(\boldsymbol{V}^1)$ may depend on the realisation of $\boldsymbol{V}^1$.

$$\begin{aligned}\max_{\boldsymbol{w},\boldsymbol{u}} \mathbb{E}[U(G(\boldsymbol{w}, \boldsymbol{u}))] &= \max_{\boldsymbol{w},\boldsymbol{u}} \mathbb{E}[-e^{-\gamma[(1+\boldsymbol{w}^T\boldsymbol{V}^1)(1+\boldsymbol{u}^T\boldsymbol{V}^2)-1]}] \\ &\Leftrightarrow \max_{\boldsymbol{w},\boldsymbol{u}}\ -\mathbb{E}[e^{-\gamma[(\boldsymbol{u}^T\boldsymbol{V}^2+\boldsymbol{w}^T\boldsymbol{V}^1+\boldsymbol{w}^T\boldsymbol{V}^1\boldsymbol{u}^T\boldsymbol{V}^2)}] \\ &\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}} \mathbb{E}[e^{-\gamma[(1+\boldsymbol{u}^T\boldsymbol{V}^2+\boldsymbol{w}^T\boldsymbol{V}^1+\boldsymbol{w}^T\boldsymbol{V}^1\boldsymbol{u}^T\boldsymbol{V}^2)}] \\ &\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}} \mathbb{E}[e^{-\gamma\boldsymbol{w}^T\boldsymbol{V}^1}e^{-\gamma(\boldsymbol{w}^T\boldsymbol{V}^1+1)\boldsymbol{u}^T\boldsymbol{V}^2}]\end{aligned} \tag{16}$$

Then we use Dynamic Programming Principle.

By the end of year 2 (t=2): $\boldsymbol{V}^2$ is known and there is no control left to optimise and receive the final optimal utility.

By the end of year 1 (t=1): suppose $\boldsymbol{V}^1$ and $\boldsymbol{w}$ are known, we need to pick $\boldsymbol{u}$ to maximise the utility in second year.

$$
\begin{aligned}
V(\boldsymbol{u}, \boldsymbol{V}^1 = \boldsymbol{v}^1) &= \min_{\boldsymbol{u}} \ e^{-\gamma \boldsymbol{w}^T \boldsymbol{V}^1} \mathbb{E}[e^{-\gamma(\boldsymbol{w}^T \boldsymbol{V}^1+1)\boldsymbol{u}^T \boldsymbol{V}^2} | \boldsymbol{V}^1 = \boldsymbol{v}^1] \\
&\Leftrightarrow \min_{\boldsymbol{u}} \ \mathbb{E}[e^{-\gamma(\boldsymbol{w}^T \boldsymbol{v}_1+1)\boldsymbol{u}^T \boldsymbol{V}^2}] \\
&\Leftrightarrow \min_{\boldsymbol{u}} \ e^{-\gamma(\boldsymbol{w}^T \boldsymbol{v}_1+1)\boldsymbol{u}^T \boldsymbol{\mu}^2 + \frac{1}{2}\gamma^2(\boldsymbol{w}^T \boldsymbol{v}^1+1)^2 \boldsymbol{u}^T C_2 \boldsymbol{u}} \\
&\Leftrightarrow \min_{\boldsymbol{u}} \ -\gamma(\boldsymbol{w}^T \boldsymbol{v}_1 + 1)\boldsymbol{u}^T \boldsymbol{\mu}^2 + \frac{1}{2}\gamma^2(\boldsymbol{w}^T \boldsymbol{v}^1 + 1)^2 \boldsymbol{u}^T C_2 \boldsymbol{u}
\end{aligned}
\tag{17}
$$

Then we plug in characteristic function, and let $m = \frac{1}{\gamma(\boldsymbol{w}^T \boldsymbol{v}_1+1)}$ to simplify. We substitute it back to equation (16).

$$
\begin{aligned}
V(\boldsymbol{u}) &= \min_{\boldsymbol{u}} \ -m\boldsymbol{u}^T \boldsymbol{\mu}_2 + \frac{1}{2}m^2 \boldsymbol{u}^T C_2 \boldsymbol{u} \\
&= \min_{\boldsymbol{u}} \ m^{-2}(-m\boldsymbol{u}^T \boldsymbol{\mu}_2 + \frac{1}{2}m^2 \boldsymbol{u}^T C_2 \boldsymbol{u})
\end{aligned}
\tag{18}
$$

Hence, the original optimisation problem is equivalent of

$$
\begin{aligned}
&\min_{\boldsymbol{u}} \ V(\boldsymbol{u}) = -m\boldsymbol{u}^T \boldsymbol{\mu}_2 + \frac{1}{2}m^2 \boldsymbol{u}^T C_2 \boldsymbol{u}, \\
&\text{s.t. } \ \boldsymbol{u}^T \boldsymbol{e} = 1
\end{aligned}
\tag{19}
$$

Similar to 3.1, we can use the method of Lagrange multipliers and get the optimal weight $\boldsymbol{u}^*$,

$$
L(\boldsymbol{u}, \lambda) = -m\boldsymbol{u}^T \boldsymbol{\mu}_2 + \frac{1}{2}\boldsymbol{u}^T C_2 \boldsymbol{u} - \lambda(\boldsymbol{w}^T \boldsymbol{e} - 1)
\tag{20}
$$

$$
\begin{aligned}
\lambda &= \frac{1}{\boldsymbol{e}^T C^{-1} \boldsymbol{e}} - \frac{\boldsymbol{e}^T C^{-1} \boldsymbol{\mu}_2}{\gamma(\boldsymbol{w}^T \boldsymbol{v}^1 + 1)\boldsymbol{e}^T C^{-1} \boldsymbol{e}} \\
\boldsymbol{u}^* &= C^{-1}(\frac{\boldsymbol{\mu}_2}{\gamma(\boldsymbol{w}^T \boldsymbol{v}^1 + 1)} + \lambda \boldsymbol{e})
\end{aligned}
\tag{21}
$$

Therefore,

$$
\boldsymbol{u}(\boldsymbol{V}^1)^* = \frac{1}{\gamma(\boldsymbol{w}^T \boldsymbol{V}_1 + 1)}(C^{-1}\boldsymbol{\mu}_2 - \frac{b}{a}C^{-1}\boldsymbol{e}) + C^{-1}\frac{\boldsymbol{e}}{a},
\tag{22}
$$

where $a = \boldsymbol{e}^T C^{-1} \boldsymbol{e}$ and $b = \boldsymbol{\mu}_2^T C^{-1} \boldsymbol{e}$.

$$
\boldsymbol{u}^* \boldsymbol{V}^2 = \frac{\eta}{\gamma(\boldsymbol{w}^T \boldsymbol{V}_1 + 1)} + \xi,
\tag{23}
$$

where $\eta = \boldsymbol{\mu}_2^T(C^{-1}\boldsymbol{\mu}_2 - \frac{b}{a}C^{-1}\boldsymbol{e})$, $\xi = C^{-1}\frac{\boldsymbol{e}}{a}\boldsymbol{\mu}_2$,

Consider the beginning of the first year (t=0), we can simplify it by substituting $\boldsymbol{u}^* \boldsymbol{V}^2$ into (16)

$$
\begin{aligned}
V_0(\boldsymbol{w}) &= \min_{\boldsymbol{w},\boldsymbol{u}} \ \mathbb{E}[e^{\gamma \boldsymbol{w}^T \boldsymbol{V}^1} e^{-\eta - \gamma(\boldsymbol{w}^T \boldsymbol{V}^1+1)\xi}] \\
&\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}} \ \mathbb{E}[e^{-\gamma(1+\xi)\boldsymbol{w}^T \boldsymbol{V}^1}] \\
&\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}} \ -\gamma(1+\xi)\boldsymbol{w}^T \boldsymbol{\mu}_0 + \frac{1}{2}\gamma^2(1+\xi)^2 \boldsymbol{w}^T C \boldsymbol{w} \\
&\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}} \ k^{-2}(-k\boldsymbol{w}^T \boldsymbol{\mu}_0 - \frac{1}{2}\boldsymbol{w}^T C \boldsymbol{w}), \quad k = \frac{1}{\gamma^2(1+\xi)^2}
\end{aligned}
\tag{24}
$$

Therefore, the original optimisation problem is equivalent of

$$\begin{aligned} &\min\ -k\boldsymbol{w}^T\boldsymbol{\mu}_0 - \frac{1}{2}\boldsymbol{w}^T C\boldsymbol{w}, \\ &\text{s.t. } \boldsymbol{w}^T\boldsymbol{e} = 1 \end{aligned} \tag{25}$$

Then we use the method of Lagrange multipliers and get the optimal weight $\boldsymbol{w}^*$,

$$L(\boldsymbol{w},\lambda) = -k\boldsymbol{w}^T\boldsymbol{\mu}_0 + \frac{1}{2}\boldsymbol{w}^T C\boldsymbol{w} - \lambda(\boldsymbol{w}^T\boldsymbol{e} - 1) \tag{26}$$

$$\begin{aligned} \lambda &= \frac{1 - k\boldsymbol{e}^T C^{-1}\boldsymbol{\mu}_0}{\boldsymbol{e}^T C^{-1}\boldsymbol{e}} \\ \boldsymbol{w}^* &= C^{-1}(k\mu_{\boldsymbol{0}} + \lambda\boldsymbol{e}) \end{aligned} \tag{27}$$

Here, $\boldsymbol{w}^*$ depends on k, which is correlated to $\boldsymbol{\mu}_2$.

3.4 Dynamic Portfolio Maximisation with Minimum Variance

We try to conduct dynamic portfolio optimisation with minimum variance. Consider the following optimisation problem:

$$\begin{aligned} &\min \operatorname{Var}(G(\boldsymbol{w},\boldsymbol{u})), \\ &\text{s.t. } \mathbb{E}[G(\boldsymbol{w},\boldsymbol{u})] \geq z \\ &\sum_{i=1}^{8} w_i = \sum_{i=1}^{8} u_i = 1, \end{aligned} \tag{28}$$

Because of the constraint $\mathbb{E}[G(\boldsymbol{w},\boldsymbol{u})] = z$, we cannot use Dynamic Programming Principle directly. So, we add a Lagrange multiplier 2λ.

$$\begin{aligned} \tilde{V}(\lambda) &:= \min_{\boldsymbol{w},\boldsymbol{u}}\{\mathbb{E}[G(\boldsymbol{w},\boldsymbol{u})] - z^2 - 2\lambda(\mathbb{E}[G(\boldsymbol{w},\boldsymbol{u}) - z)\} \\ &= \min_{\boldsymbol{w},\boldsymbol{u}}\{\mathbb{E}[(G(\boldsymbol{w},\boldsymbol{u}) - \lambda)^2]\} - (\lambda - z)^2 \\ &\Leftrightarrow \min_{\boldsymbol{w},\boldsymbol{u}}\{\mathbb{E}[(G(\boldsymbol{w},\boldsymbol{u}) - \lambda)^2]\} \end{aligned} \tag{29}$$

Then we can use Dynamic Programming Principle, supposing that we get optimal control $\boldsymbol{W}^*(\lambda^*) = (\boldsymbol{w}^*(\lambda^*), \boldsymbol{u}^*(\lambda^*))$.

If we can find λ^* such that $\mathbb{E}(G) = z$ under $\boldsymbol{W}^*(\lambda^*)$, then $\tilde{\lambda^*}$ is the value for original problem and $\boldsymbol{W}^*(\lambda^*)$ is the optimal control.

When $t = 1$,

$$\begin{aligned} V_1(r) &= \min_{\boldsymbol{u}}\ \mathbb{E}[(G-\lambda)^2|\boldsymbol{V}^1 = r] \\ &= \min_{\boldsymbol{u}}\ \mathbb{E}[((1+r)(1+\boldsymbol{V}^2) - 1 - \lambda)^2] \\ &= \min_{\boldsymbol{u}}\ (1+r)^2\mathbb{E}[(\boldsymbol{V}^2)^2] + 2(r+1)(r-\lambda)\mathbb{E}[\boldsymbol{V}^2] + (r-\lambda)^2 \end{aligned} \tag{30}$$

As $\mathbb{E}[\boldsymbol{V}^2] = \boldsymbol{u}^T\boldsymbol{\mu}_1$ and $\operatorname{Var}[\boldsymbol{V}^2] = \boldsymbol{u}^T C\boldsymbol{u}$, we can get $\mathbb{E}[(\boldsymbol{V}^2)^2] = \boldsymbol{u}^T C\boldsymbol{u} + (\boldsymbol{u}^T\boldsymbol{\mu}_1)^2$, and we get the one period minimisation problem:

$$\begin{aligned} V_1(r) = &\min_{\boldsymbol{u}}\ \{(1+r)^2\boldsymbol{u}^T C\boldsymbol{u} + (1+r)^2(\boldsymbol{u}^T\boldsymbol{\mu}_1)^2 + 2(r+1)(r-\lambda)\boldsymbol{u}^T\boldsymbol{\mu}_1 + (r-\lambda)^2\} \\ &\text{s.t. } \boldsymbol{e}^T\boldsymbol{u} = 1 \end{aligned} \tag{31}$$

We can solve it using Lagrange multipliers ζ ($\zeta \in \mathbb{R}$) and KKT conditions.

$$L(\boldsymbol{u}, \zeta) = (1+r)^2 \boldsymbol{u}^T C \boldsymbol{u} + (1+r)^2 (\boldsymbol{u}^T \boldsymbol{\mu}_1)^2 + 2(r+1)(r-\lambda)\boldsymbol{u}^T \boldsymbol{\mu}_1 + (r-\lambda)^2 - \zeta(\boldsymbol{e}^T \boldsymbol{u} - 1) \tag{32}$$

$$\nabla_{\boldsymbol{u}} L(\boldsymbol{u}, \zeta) = 2(1+r)^2 C\boldsymbol{u} + 2(1+r)^2 \boldsymbol{\mu}_1^T \boldsymbol{u} \boldsymbol{\mu} + 2(1+r)(r-\lambda)\boldsymbol{\mu}_1 - \zeta \boldsymbol{e} = 0 \tag{33}$$

$$\frac{\partial L(\boldsymbol{u}, \zeta)}{\partial \zeta} = \boldsymbol{e}^T \boldsymbol{u} - 1 = 0\boldsymbol{e} = 0 \tag{34}$$

Solving the equation set, we can get

$$\zeta^* = \frac{2(1+r)^2 + 2(1+r)(r+\lambda)\boldsymbol{e}^T D^{-1} \boldsymbol{\mu}_1}{\boldsymbol{e}^T D^{-1} \boldsymbol{\mu}_1} \tag{35}$$

$$\boldsymbol{u}^* = \frac{r\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}}{(1+r)b}, \tag{36}$$

where $\boldsymbol{x} = \frac{D^{-1}\boldsymbol{e} + D^{-1}(a\boldsymbol{e} - b\boldsymbol{\mu}_1)}{b}$, $\quad \boldsymbol{y} = \frac{b\boldsymbol{\mu}_1 - a\boldsymbol{e}}{b}$, $\quad \boldsymbol{z} = \frac{D^{-1}\boldsymbol{e}}{b}$, and $a = \boldsymbol{e}^T D^{-1} \boldsymbol{\mu}_1, \quad b = \boldsymbol{e}^T D^{-1} \boldsymbol{e}$.
Therefore,

$$V_1^*(r) = (r\boldsymbol{x}^T + \lambda \boldsymbol{y}^T + \boldsymbol{z}^T) C (r\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}) + (\boldsymbol{\mu}_1^T (r\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}))^2 + 2(r-\lambda)\boldsymbol{\mu}_1^T (r\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}) + (r-\lambda)^2 \tag{37}$$

When $t = 0$, plug random return $\boldsymbol{V}^1$ into r,

$$\begin{aligned} \min_{\boldsymbol{w}} \mathbb{E}\{&(\boldsymbol{V^1}\boldsymbol{x}^T + \lambda \boldsymbol{y}^T + \boldsymbol{z}^T) C (\boldsymbol{V^1}\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}) + (\boldsymbol{\mu}_1^T (\boldsymbol{V^1}\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}))^2 + \\ &2(\boldsymbol{V^1} - \lambda)\boldsymbol{\mu}_1^T (\boldsymbol{V^1}\boldsymbol{x} + \lambda \boldsymbol{y} + \boldsymbol{z}) + (\boldsymbol{V^1} - \lambda)^2\} \end{aligned} \tag{38}$$

This is a standard format of objective function, which can be solve by Lagrange method.

4 Computational Results

4.1 Parameter Estimation

Firstly, as each asset follows a lognormal returns distribution, we estimate the parameters $\boldsymbol{a}$ and B for period A and period B.

For period A,

$$\boldsymbol{a} = \begin{pmatrix} -0.00010 \\ -0.00248 \\ 0.00600 \\ -0.01763 \\ 0.00377 \\ 0.00519 \\ 0.00663 \\ 0.00454 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.00259 & 0.00260 & 0.00297 & 0.00277 & 0.00088 & -0.00022 & 0.00002 & -0.00003 \\ 0.00260 & 0.00305 & 0.00338 & 0.00290 & 0.00098 & -0.00023 & 0.00002 & -0.00003 \\ 0.00297 & 0.00338 & 0.00485 & 0.00290 & 0.00130 & -0.00035 & 0.00004 & -0.00004 \\ 0.00277 & 0.00290 & 0.00290 & 0.00542 & 0.00079 & -0.00004 & 0.00013 & -0.00003 \\ 0.00088 & 0.00098 & 0.00130 & 0.00079 & 0.00051 & -0.00012 & -0.00001 & -0.00001 \\ -0.00022 & -0.00023 & -0.00035 & -0.00004 & -0.00012 & 0.00008 & 0.00004 & 0.00000 \\ 0.00002 & 0.00002 & 0.00004 & 0.00013 & -0.00001 & 0.00004 & 0.00007 & -0.00000 \\ -0.00003 & -0.00003 & -0.00004 & -0.00003 & -0.00001 & 0.00000 & -0.00000 & 0.00000 \end{pmatrix}$$

For period B,

$$\boldsymbol{a} = \begin{pmatrix} 0.00847 \\ 0.00877 \\ 0.00873 \\ 0.00851 \\ 0.00287 \\ 0.00373 \\ 0.00360 \\ 0.00150 \end{pmatrix}$$

$$B = \begin{pmatrix} 0.00076 & 0.00052 & 0.00031 & 0.00044 & 0.00013 & -0.00000 & -0.00002 & 0.00000 \\ 0.00052 & 0.00092 & 0.00077 & 0.00021 & 0.00024 & -0.00007 & -0.00006 & -0.00000 \\ 0.00031 & 0.00077 & 0.00108 & -0.00001 & 0.00025 & -0.00007 & -0.00002 & -0.00000 \\ 0.00044 & 0.00021 & -0.00001 & 0.00109 & 0.00006 & 0.00013 & 0.00013 & 0.00000 \\ 0.00013 & 0.00024 & 0.00025 & 0.00006 & 0.00010 & -0.00001 & -0.00000 & -0.00000 \\ -0.00000 & -0.00007 & -0.00007 & 0.00013 & -0.00001 & 0.00006 & 0.00005 & 0.00000 \\ -0.00002 & -0.00006 & -0.00002 & 0.00013 & -0.00000 & 0.00005 & 0.00006 & 0.00000 \\ 0.00000 & -0.00000 & -0.00000 & 0.00000 & -0.00000 & 0.00000 & 0.00000 & 0.00000 \end{pmatrix}$$

Let $R_{t,n}^i$ denote the n-month return for asset i.

$$\log(1+R_{t,n}^i) = \log(S_{t+n-1}^i) - \log(S_{t-1}^i) = \sum_{k=1}^{n} \log(S_{t+k}^i - S_{t+k-1}^i)) = \sum_k X_{t+k}^i$$

$$R_{t,n}^i = e^{\sum_k X_{t+k}^i} - 1$$

Let Y^i denote $\sum_k X_{t+k}^i$, mean of $Y^i = nE(X^i) = n\boldsymbol{a}$. As X_t^i are i.i.d. multivariate normally distributed, covariance of different X_t^i must be 0. So, covariance of $Y^i = nB$.

Then, let the random vector $\boldsymbol{R}^{(1)} := (R_1^{(1)}, \cdots, R_8^{(1)})^T$ (respectively, $\boldsymbol{R}^{(2)} := (R_1^{(2)}, \cdots, R_8^{(2)})^T$) model the joint annual (respectively, two-year) returns for the eight assets. For $k = 1, 2$,

$$\mu_i^{(k)} = \mathbb{E}[R_i^{(k)}] = e^{ka_i + \frac{k}{2}B_{i,i}} - 1 \tag{39}$$

$$\begin{aligned} c_{i,j}^{(k)} &= \mathbb{E}[(e^{Y_i}-1)(e^{Y_j}-1)] - \mathbb{E}[e^{Y_i}-1]\mathbb{E}[e^{Y_j}-1] \\ &= \mathbb{E}[e^{Y_i+Y_j}] - \mathbb{E}[e^{Y_i}]\mathbb{E}[e^{Y_j}] \\ &= e^{ka_i+ka_j+\frac{1}{2}(kB_{i,i}+kB_{j,j}+2kB_{i,j})} - e^{ka_i+\frac{k}{2}B_{i,i}}e^{ka_j+\frac{k}{2}B_{j,j}} \end{aligned} \tag{40}$$

$$\rho_{i,j}^{(k)} = \frac{c_{i,j}^{(k)}}{\sqrt{c_{i,i}^{(k)} c_{j,j}^{(k)}}} \tag{41}$$

Then we can estimate the real-world parameter using Python.
In period A, $k = 1$,

$$\mu^1 = \begin{pmatrix} 0.01447 \\ -0.01141 \\ 0.10643 \\ -0.16396 \\ 0.04951 \\ 0.06479 \\ 0.08333 \\ 0.05606 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 0.03252 & 0.03182 & 0.04078 & 0.02869 & 0.01124 & -0.00280 & 0.00022 & -0.00037 \\ 0.03182 & 0.03643 & 0.04533 & 0.02928 & 0.01232 & -0.00289 & 0.00026 & -0.00037 \\ 0.04078 & 0.04533 & 0.07339 & 0.03274 & 0.01822 & -0.00490 & 0.00054 & -0.00061 \\ 0.02869 & 0.02928 & 0.03274 & 0.04700 & 0.00832 & -0.00045 & 0.00140 & -0.00034 \\ 0.01124 & 0.01232 & 0.01822 & 0.00832 & 0.00682 & -0.00167 & -0.00019 & -0.00018 \\ -0.00280 & -0.00289 & -0.00490 & -0.00045 & -0.00167 & 0.00111 & 0.00050 & 0.00006 \\ 0.00022 & 0.00026 & 0.00054 & 0.00140 & -0.00019 & 0.00050 & 0.00102 & -0.00003 \\ -0.00037 & -0.00037 & -0.00061 & -0.00034 & -0.00018 & 0.00006 & -0.00003 & 0.00003 \end{pmatrix}$$

$$\rho^1 = \begin{pmatrix} 1.00000 & 0.92429 & 0.83480 & 0.73379 & 0.75503 & -0.46525 & 0.03777 & -0.40144 \\ 0.92429 & 1.00000 & 0.87665 & 0.70746 & 0.78194 & -0.45452 & 0.04293 & -0.38406 \\ 0.83480 & 0.87665 & 1.00000 & 0.55751 & 0.81477 & -0.54264 & 0.06266 & -0.44504 \\ 0.73379 & 0.70746 & 0.55751 & 1.00000 & 0.46512 & -0.06275 & 0.20159 & -0.31115 \\ 0.75503 & 0.78194 & 0.81477 & 0.46512 & 1.00000 & -0.60722 & -0.07062 & -0.42985 \\ -0.46525 & -0.45452 & -0.54264 & -0.06275 & -0.60722 & 1.00000 & 0.47083 & 0.33040 \\ 0.03777 & 0.04293 & 0.06266 & 0.20159 & -0.07062 & 0.47083 & 1.00000 & -0.17212 \\ -0.40144 & -0.38406 & -0.44504 & -0.31115 & -0.42985 & 0.33040 & -0.17212 & 1.00000 \end{pmatrix}$$

$k = 2$,

$$\mu^2 = \begin{pmatrix} 0.02915 \\ -0.02268 \\ 0.22418 \\ -0.30104 \\ 0.10147 \\ 0.13378 \\ 0.17360 \\ 0.11527 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 0.06799 & 0.06483 & 0.09322 & 0.04949 & 0.02406 & -0.00603 & 0.00048 & -0.00079 \\ 0.06483 & 0.07254 & 0.10122 & 0.04925 & 0.02572 & -0.00608 & 0.00056 & -0.00078 \\ 0.09322 & 0.10122 & 0.18508 & 0.06165 & 0.04265 & -0.01152 & 0.00130 & -0.00143 \\ 0.04949 & 0.04925 & 0.06165 & 0.06791 & 0.01468 & -0.00081 & 0.00253 & -0.00061 \\ 0.02406 & 0.02572 & 0.04265 & 0.01468 & 0.01506 & -0.00373 & -0.00042 & -0.00040 \\ -0.00603 & -0.00608 & -0.01152 & -0.00081 & -0.00373 & 0.00252 & 0.00116 & 0.00013 \\ 0.00048 & 0.00056 & 0.00130 & 0.00253 & -0.00042 & 0.00116 & 0.00240 & -0.00006 \\ -0.00079 & -0.00078 & -0.00143 & -0.00061 & -0.00040 & 0.00013 & -0.00006 & 0.00006 \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} 1.00000 & 0.92305 & 0.83097 & 0.72823 & 0.75193 & -0.46091 & 0.03747 & -0.39823 \\ 0.92305 & 1.00000 & 0.87359 & 0.70168 & 0.77815 & -0.44961 & 0.04253 & -0.38046 \\ 0.83097 & 0.87359 & 1.00000 & 0.54989 & 0.80788 & -0.53345 & 0.06174 & -0.43840 \\ 0.72823 & 0.70168 & 0.54989 & 1.00000 & 0.45895 & -0.06169 & 0.19839 & -0.30598 \\ 0.75193 & 0.77815 & 0.80788 & 0.45895 & 1.00000 & -0.60568 & -0.07049 & -0.42915 \\ -0.46091 & -0.44961 & -0.53345 & -0.06169 & -0.60568 & 1.00000 & 0.47071 & 0.33033 \\ 0.03747 & 0.04253 & 0.06174 & 0.19839 & -0.07049 & 0.47071 & 1.00000 & -0.17208 \\ -0.39823 & -0.38046 & -0.43840 & -0.30598 & -0.42915 & 0.33033 & -0.17208 & 1.00000 \end{pmatrix}$$

In period B, $k = 1$,

$$\mu^1 = \begin{pmatrix} 0.11202 \\ 0.11706 \\ 0.11762 \\ 0.11486 \\ 0.03564 \\ 0.04620 \\ 0.04457 \\ 0.01811 \end{pmatrix}$$

$$C^1 = \begin{pmatrix} 0.01137 & 0.00785 & 0.00467 & 0.00663 & 0.00180 & -0.00003 & -0.00030 & 0.00001 \\ 0.00785 & 0.01382 & 0.01162 & 0.00321 & 0.00335 & -0.00097 & -0.00082 & -0.00000 \\ 0.00467 & 0.01162 & 0.01630 & -0.00014 & 0.00346 & -0.00104 & -0.00028 & -0.00001 \\ 0.00663 & 0.00321 & -0.00014 & 0.01639 & 0.00079 & 0.00189 & 0.00184 & 0.00003 \\ 0.00180 & 0.00335 & 0.00346 & 0.00079 & 0.00126 & -0.00013 & -0.00005 & -0.00000 \\ -0.00003 & -0.00097 & -0.00104 & 0.00189 & -0.00013 & 0.00084 & 0.00061 & 0.00001 \\ -0.00030 & -0.00082 & -0.00028 & 0.00184 & -0.00005 & 0.00061 & 0.00077 & 0.00001 \\ 0.00001 & -0.00000 & -0.00001 & 0.00003 & -0.00000 & 0.00001 & 0.00001 & 0.00000 \end{pmatrix}$$

$$\rho^1 = \begin{pmatrix} 1.00000 & 0.62615 & 0.34321 & 0.48600 & 0.47563 & -0.01085 & -0.10045 & 0.09011 \\ 0.62615 & 1.00000 & 0.77405 & 0.21299 & 0.80398 & -0.28581 & -0.25226 & -0.03280 \\ 0.34321 & 0.77405 & 1.00000 & -0.00867 & 0.76479 & -0.28024 & -0.08054 & -0.03855 \\ 0.48600 & 0.21299 & -0.00867 & 1.00000 & 0.17461 & 0.50803 & 0.52058 & 0.22979 \\ 0.47563 & 0.80398 & 0.76479 & 0.17461 & 1.00000 & -0.12564 & -0.05597 & -0.09045 \\ -0.01085 & -0.28581 & -0.28024 & 0.50803 & -0.12564 & 1.00000 & 0.76300 & 0.26280 \\ -0.10045 & -0.25226 & -0.08054 & 0.52058 & -0.05597 & 0.76300 & 1.00000 & 0.17614 \\ 0.09011 & -0.03280 & -0.03855 & 0.22979 & -0.09045 & 0.26280 & 0.17614 & 1.00000 \end{pmatrix}$$

$k = 2$,

$$\mu^2 = \begin{pmatrix} 0.23659 \\ 0.24783 \\ 0.24908 \\ 0.24291 \\ 0.07256 \\ 0.09453 \\ 0.09113 \\ 0.03654 \end{pmatrix}$$

$$C^2 = \begin{pmatrix} 0.02825 & 0.01956 & 0.01163 & 0.01649 & 0.00414 & -0.00008 & -0.00069 & 0.00002 \\ 0.01956 & 0.03468 & 0.02914 & 0.00799 & 0.00776 & -0.00228 & -0.00191 & -0.00001 \\ 0.01163 & 0.02914 & 0.04097 & -0.00035 & 0.00802 & -0.00243 & -0.00066 & -0.00001 \\ 0.01649 & 0.00799 & -0.00035 & 0.04100 & 0.00183 & 0.00440 & 0.00430 & 0.00007 \\ 0.00414 & 0.00776 & 0.00802 & 0.00183 & 0.00269 & -0.00028 & -0.00012 & -0.00001 \\ -0.00008 & -0.00228 & -0.00243 & 0.00440 & -0.00028 & 0.00184 & 0.00134 & 0.00002 \\ -0.00069 & -0.00191 & -0.00066 & 0.00430 & -0.00012 & 0.00134 & 0.00167 & 0.00001 \\ 0.00002 & -0.00001 & -0.00001 & 0.00007 & -0.00001 & 0.00002 & 0.00001 & 0.00000 \end{pmatrix}$$

$$\rho^2 = \begin{pmatrix} 1.00000 & 0.62496 & 0.34196 & 0.48458 & 0.47477 & -0.01082 & -0.10019 & 0.08990 \\ 0.62496 & 1.00000 & 0.77299 & 0.21198 & 0.80269 & -0.28485 & -0.25143 & -0.03271 \\ 0.34196 & 0.77299 & 1.00000 & -0.00861 & 0.76323 & -0.27915 & -0.08025 & -0.03842 \\ 0.48458 & 0.21198 & -0.00861 & 1.00000 & 0.17405 & 0.50668 & 0.51919 & 0.22904 \\ 0.47477 & 0.80269 & 0.76323 & 0.17405 & 1.00000 & -0.12557 & -0.05595 & -0.09043 \\ -0.01082 & -0.28485 & -0.27915 & 0.50668 & -0.12557 & 1.00000 & 0.76294 & 0.26275 \\ -0.10019 & -0.25143 & -0.08025 & 0.51919 & -0.05595 & 0.76294 & 1.00000 & 0.17611 \\ 0.08990 & -0.03271 & -0.03842 & 0.22904 & -0.09043 & 0.26275 & 0.17611 & 1.00000 \end{pmatrix}$$

4.2 Static and Dynamic Portfolio Optimisation

Firstly, we calculate the optimal weight for static portfolio optimisation with maximum utility.

$$\boldsymbol{w}_A^* = \begin{pmatrix} 10.99916 \\ -7.82542 \\ 5.14694 \\ -13.88266 \\ -0.90056 \\ 23.35996 \\ 22.58386 \\ -38.48128 \end{pmatrix}, \quad \boldsymbol{w}_B^* = \begin{pmatrix} 6.17733 \\ 13.45915 \\ 3.58395 \\ -6.09538 \\ -35.90376 \\ 36.26839 \\ 36.88278 \\ -53.37247 \end{pmatrix}$$

	Period A	Period B
Expected return	8.34951	6.44320
Variance	8.23233	6.40737

Comparing between 2 periods, We can observe that the portfolio strategy of period A is more conservative with smaller absolute value of optimal weight, while the portfolio strategy of period B is more radical, in which it shorts 53 unit of cash. This might be because during period A, the market was in recession because of the 2008 financial crisis. The expected return and variance of period A are both higher than those of period B. This strategy suits those investors with high risk appetite, and higher risk brings with it higher return.

Then, we compare the portfolio for period B with the realised return for the period from January 2020 to December 2021. The expected return and variance of the realised result is much

	Period B	Realised
Expected return	6.44320	0.11735
Variance	6.40737	0.74479

smaller than those of period B. This strategy works worse in the period from January 2020 to December 2021 than in period B. It might be because of the large market volatility and economic recession due to Covid-19 pandemic.

Secondly, we calculate the optimal weight for static portfolio optimisation with minimum variance.

$$\boldsymbol{w}_A^* = \begin{pmatrix} 0.00333 \\ -0.00703 \\ 0.00172 \\ 0.00310 \\ 0.02187 \\ -0.02474 \\ 0.05370 \\ 0.94806 \end{pmatrix}, \quad \boldsymbol{w}_B^* = \begin{pmatrix} 0.06156 \\ 0.13492 \\ 0.03359 \\ -0.06273 \\ -0.34891 \\ 0.35224 \\ 0.37704 \\ 0.45230 \end{pmatrix}$$

	Period A	Period B
Expected return	0.11722	0.10000
Variance	0.00003	0.00064

We can observe that the portfolio strategy of period A is again more conservative with smaller absolute value of optimal weight, while the portfolio strategy of period B is more radical. The expected return of period A is larger than that of period B, and the variance of period A is smaller than that of period B, which means the strategy performs apparently better in period A.

Then, we compare the portfolio for period B with the realised return for the period from January 2020 to December 2021. The expected return of the realised result is smaller than those

	Period B	Realised
Expected return	0.10000	0.00138
Variance	0.00064	0.00007

of period B. The variance of the realised result is also smaller than that of period B. This strategy does not show its advantage for the period from January 2020 to December 2021.

Comparing the above portfolios with maximum utility and minimum variance (Figure 1), the expected return and variance of portfolio with maximum utility are both higher than those with minimum variance. This shows that higher risk brings with it higher return. The portfolio with maximum utility suits investors with high risk appetite, while the portfolio with minimum variance suits investors with risk averse.

Thirdly, we calculate the optimal weight for dynamic portfolio optimisation with maximum

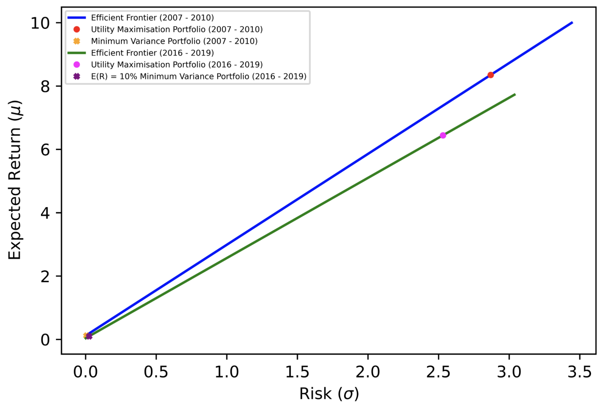

Figure 1: Efficient Frontiers and Optimal Portfolios

utility for period B.

$$\boldsymbol{w}^* = \begin{pmatrix} 6.95317 \\ 15.67296 \\ 4.48066 \\ -6.73083 \\ -40.34057 \\ 38.74593 \\ 38.28513 \\ -56.06646 \end{pmatrix}, \quad \boldsymbol{u}^* = \begin{pmatrix} 31.21465 \\ 70.35693 \\ 20.12322 \\ -30.20875 \\ -181.13145 \\ 173.97444 \\ 171.83762 \\ -255.16667 \end{pmatrix}$$

We can observe that the optimal weight of the first year is similar to the optimal weight of static strategy with maximum utility in period B. However, the optimal weight of the second year is much more radical. The annually expected return of this portfolio is 7.46612, which is larger than that of static portfolio in period B. Generally speaking, dynamic strategy works better.

Finally, we calculate the optimal weight for dynamic portfolio optimisation with minimum variance for period B. It has the same trend with the dynamic portfolio before as well as the static portfolio with minimum variance. But the portfolio now is much more conservative.

$$\boldsymbol{w}^* = \begin{pmatrix} 0.00220 \\ 0.00229 \\ 0.00223 \\ -0.00157 \\ -0.00854 \\ 0.01058 \\ 0.01012 \\ -0.96273 \end{pmatrix}, \quad \boldsymbol{u}^* = \begin{pmatrix} 0.12415 \\ 0.08702 \\ 0.07476 \\ -0.10193 \\ -0.07818 \\ 0.17152 \\ 0.17693 \\ -0.18589 \end{pmatrix}$$

4.3 Discussion of extension

Here, we change γ from 1 to 100, which represents the investor's risk tolerance, to see the different performance of portfolio optimisation. We can observe from the plot that, as γ increases, the optimal portfolio gets closer to the vertex, with both expected return and variance becoming smaller.

5 Conclusions

Higher risk brings with it higher return. Portfolio optimisation with maximum utility suits investors with high risk appetite, while portfolio optimisation with minimum variance suits investors with risk averse. If the expected return is higher than the return at vertex on Mean-Variance Frontier, it's better to use

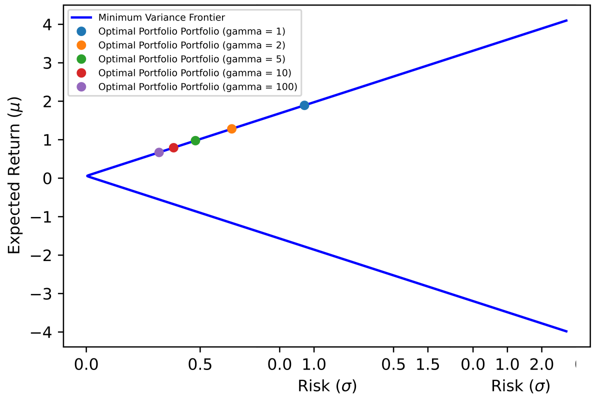

Figure 2: Utility Maximisation Portfolios with Varied γs

strategy with maximum utility. If the expected return is lower than the return at vertex, it's better to use strategy with minimum variance.

The limitation is that we assume there is no transaction cost and tax in the trading process. In real-market situation, the expected return will be influenced by these other costs.

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Appendix

```
log_returns_A = np.log(data_2007_2010+1)
a_A = (log_returns_A).mean(axis=0)
B_A = np.cov(log_returns_A,rowvar=False)
def q3_mu(num_period, mu, cov):
    var = np.diag(num_period*cov)
    return np.exp(num_period*mu+0.5*var)-1
def q3_cov(num_period,mu,cov):
    cov = num_period*cov
    mu = num_period*mu
    result = np.zeros((8,8))
    var = np.diag(cov)
    for i in range(8):
        for j in range(8):
            result[i][j] = np.exp(mu[i]+mu[j]+0.5*(var[i]+var[j]+2*cov[i][j]))\\
            -np.exp(mu[i]+0.5*var[i])*np.exp(mu[j]+0.5*var[j])
    return result
mu_A_k1 = q3_mu(12,a_A,B_A)
c_A_k1 = q3_cov(12,a_A,B_A)
```

```
def rho(num_period,mu,cov):
    cov = q3_cov(num_period,mu,cov)
    result = np.zeros((8,8))
    for i in range(8):
        for j in range(8):
            result[i][j] = cov[i][j]/(cov[i][i]*cov[j][j])
    return result
rho_A_k1 = rho(12,a_A,B_A)
gamma = 1 #coefficient of absolute risk-aversion
t = 1/gamma
n = 8
mu = np.asarray(mu_A_k2)[0:n]
S = np.asarray(c_A_k2)[0:n,0:n]
r = mu.reshape(n,1)
e = np.ones((n,1))
s_inv = np.linalg.inv(S)
a = float(np.matmul(np.matmul(e.T,s_inv),e))
b = float(np.matmul(np.matmul(r.T,s_inv),e))
c = float(np.matmul(np.matmul(r.T,s_inv),r))
d = a*c-b**2
alpha = (1/a)*np.matmul(s_inv,e)
beta = np.matmul(np.linalg.inv(S),r - (b/a)*e)
lambda_1 =
    (1-t*float(np.matmul(np.matmul(e.T,s_inv),r)))/float(np.matmul(np.matmul(e.T,s_inv),e))
w_utility_A = np.matmul(s_inv,t*r+lambda_1*e)
var_stat_A = float(np.matmul(np.matmul(w_utility_A.T,S),w_utility_A))
mu_stat_A = float(np.matmul(w_utility_A.T,r))
n = 8
mu = np.asarray(mu_A_k2)[0:n]
S = np.asarray(c_A_k2)[0:n,0:n]
r = mu.reshape(n,1)
e = np.ones((n,1))
s_inv = np.linalg.inv(S)
w_mim_var_A = np.matmul(s_inv,e)/np.matmul(e.T,np.matmul(s_inv,e))
var_mim_var_A= float(np.matmul(np.matmul(w_mim_var_A.T,S),w_mim_var_A))
mu_mim_var_A = float(np.matmul(w_mim_var_A.T,r))
mu = np.asarray(mu_B_k1)
S = np.asarray(c_B_k1)
r = mu.reshape(n,1)
e = np.ones((n,1))
s_inv = np.linalg.inv(S)
a = float(np.matmul(np.matmul(e.T,s_inv),e))
kesai = float(np.matmul(np.matmul(r.T,s_inv),e/a))
t = 1/(gamma*(1+kesai))
lambda_1 =
    (1-t*float(np.matmul(np.matmul(e.T,s_inv),r)))/float(np.matmul(np.matmul(e.T,s_inv),e))
w_utility_B = np.matmul(s_inv,t*r+lambda_1*e)
t = 1/gamma*(np.matmul(w_utility_B.T,r)+1)
lambda_2 =
    (1-t*float(np.matmul(np.matmul(e.T,s_inv),r)))/float(np.matmul(np.matmul(e.T,s_inv),e))
u = np.matmul(s_inv,t*r+lambda_2*e)
G_wu=(1+np.matmul(w_utility_B.T,r))*(1+np.matmul(u.T,r))-1
r_annual=np.sqrt(G_wu+1)-1
```