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2013 Mathematical Contest in Modeling (MCM) Summary Sheet
(Attach a copy of this page to your solution paper.)

Type a summary of your results on this page. Do not include
the name of your school, advisor, or team members on this page.

We seek the perfect brownie pan, one which mitigates the age-old problem of uneven cooking while fitting smoothly into modern ovens. In our quest for the ideal pan geometry we simulate a range of shapes known as **superellipses**, from circles to squares, ellipses to rectangles, squircles to rhombi.

We measure the utility of a pan based on metrics of packing efficiency and cooking evenness. Standard-sized square and circular pans serve as baselines for these metrics. Packing theory helps us determine near-optimal arrangements of superellipses as we employ both rectangular and hexagonal pan tessellations. We then develop a quasi-three-dimensional model of heat diffusivity. We efficiently solve the model's differential equation numerically while **accurately simulating** the brownie cooking process for arbitrary pan shapes and sizes. Using this flexible heat model we define cooking evenness as a measure of deviation from an ideal "cookedness" value which was determined from empirical baking studies.

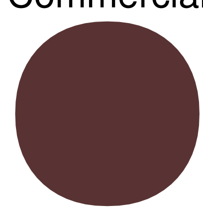
We examine two main oven sizes: a typical **residential oven** and a standard **commercial oven**. For each we investigate the effects of changing pan area, pan roundness, and the ratio of superelliptical pan dimensions on both packing efficiency and cooking evenness to understand variations in our ultimate metric: pan utility. The optimal pan shapes determined are shown below and depend on the different oven dimensions. The optimal pan shape for residential ovens is almost square, while the optimal commercial shape is almost circular. Conducting sensitivity analysis on the oven dimensions ensures that **our results are valid even for non-standard oven sizes.**

We verify our cooking model by:

- Comparing our approach and assumptions to scientific literature on cake baking.
- Carefully estimating unknown brownie batter properties, such as thermal diffusivity, using known values for ingredients and similar batters.
- Cross-checking our model's predicated cook times and internal batter temperatures with values reported by **expert chefs**.

Residential

Commercial



Throughout the process we focus on the needs of today's busy brownie bakers: we select pan sizes suited for standard batch sizes, we choose pan shapes that are easy to handle and clean, and we use a quantitative measure of cooking completion that both **protects against foodborne illness** (according to the U.S. Department of Health and Human Services) and is easy to measure.

Finally, we use our model to test the popular Baker's Edge pan, which is designed to maximize edges. The pan's manufacturer claims that its unique geometry minimizes cooking time and maximizes evenness. Our simulation agrees that the pan cooks brownies slightly more quickly than standard pan shapes of the same size, but we find that the evenness of brownies cooked in a Baker's Edge pan is very low, so overall these pans are **much less desirable** than the vast range of superelliptical pans we test.

Batter, Batter, Everywhere

Contents

1 Introduction

Food engineering is a multidisciplinary field which uses applied physical sciences to model the processes which occur in food. Food engineers strive to understand the process of creating complex foodstuffs and translate that understanding to efficient and safe manufacturing processes. The problem of designing the most effective brownie pan is an example of a food engineering problem, requiring knowledge of both geometric packing theory and the physical processes involved in baking.

We seek a brownie pan shape that optimizes the amount of brownies we can cook at one time in an rectangular oven (packing efficiency) while also cooking all areas of the brownie batter consistently (evenness). Optimal packing is especially important for commercial applications where high yield is crucial. Evenness is also critical to ensure quality, both in taste and food safety. Overcooked brownies tend to be dehydrated, possessing an undesirable tough and crunchy texture. On the other hand, undercooked brownies have the potential to transmit the salmonella virus if the egg in the batter is not fully cooked [26].

We tackle the problem of modeling the baking process of brownies to ensure evenness using a 3-D thermal conduction simulation. We seek maximum yield (space efficiency) using a geometric pan tessellation algorithm. To balance these two criteria we seek a pan shape whose edges are round enough to provide cooking evenness but regular enough to fill a rectangular oven rack well.

First we will describe our interpretation of the problem in Section 2 then provide an overview of our terminology in Section 3. From there we will discuss the pan shapes we explored in Section 4 and how we fit them onto oven racks in Section 5. Section 6 describes our model of brownie heating, including justifications of our heating assumptions, derivations of parameter values, and comparisons to other scientific baking models. We carefully define our metrics of packing efficiency and cooking evenness in Section 7 before presenting our results and sensitivity analysis in Section 8. Finally, we summarize the strengths and weakness of our model in Section 9.1.

Previous Work

Before we go on to describe our own work, we will first give a brief overview of other's work in this area as several of the mathematical and scientific issues that arise from this problem have been studied by others.

Pan Design

Circular, square, and rectangular baking pans are all commonly used to cook brownies. Hexagonal, oval and elliptical pans in various dimensions are available for purchase online, but are probably much less common than the standard circular, square and rectangular shapes. Baking pans are also produced in novelty shapes such as hearts, flowers, stars, and Christmas trees, however, these shapes are rife with corners so they are difficult to pack onto a rectangular oven rack and they result in unevenly baked final products. While these pans make for a beautiful presentation of a cake, brownies are typically served cut into small pieces, ruining the effect of an exciting pan shape.

In an effort to provide more consistent cooking results, some manufacturers have turned to other shapes. One popular option is called *Baker's Edge*, a rectangular brownie pan with extra metal intrusions into the center, creating a longer batter perimeter (see Figure 1) [6]. Finally, we use our model to test the manufacturer's claims of the popular Baker's The company claims that this pan maximizes edges, minimizes cooking time, and increase evenness over standard pan shapes. We predict that the long perimeter and large number of corners will lead to low evenness, and test this prediction with our model in section 8.4 below.

To our knowledge, there are no scientific studies that model or experiment brownie cooking with a range of pan shapes. The American magazine *Consumer Reports* have tested at least brownie pan made up of a series of small rectangles, but they do not provide details on the evenness of bake [10]. We were only able to locate scientific papers studying the standard circular and rectangular baking pans.

Figure 1: *Baker's Edge* Pan. The pan's manufacturer claims that this pan cooks brownies quickly and evenly, but we predict otherwise [6].

Baking

We could not locate any scientific papers empirically studying or theoretically modeling the process of baking brownies. Therefore we turned to several studies on cake baking for comparison purposes. We found the empirical results in [3, 24, 5, 4, 18] and especially the review of even more empirical results in [22] to be helpful for deriving parameter values for our model, as will be discussed in Section 6.6 below.

We also drew inspiration from theoretical heating models in [20] and [23]. Heat transmission occurs through three processes: conduction, convection, and radiation. Conduction is the process of energy transmission by direct contact, such as touching a cold surface. Convection is the process of energy transmission by bulk movement of a fluid, such as when hot air rises. Transfer via electromagnetic energy is called conduction, and is the process by which the sun heats the earth. While all three of these processes are important, conduction contributes the most to thermal conduction in an oven.

The authors of [20] modeled the effects of conductive heating only, finding this adequate to fit experimental data on cake baking. By contrast, the authors of [23] modeled conductive, radiative, and convective heat transfer, and therefore their model was more complex. We decided to model conductive heat transfer only. In 9 we will compare our modeling approach to both of these papers and justify our simpler approach.

2 Problem Interpretation

We set out to solve the problem of finding the optimum pan shape to ensure both maximum yield per batch and baking consistency. We focus on two goals: learning how brownies cook as influenced by pan shape, and how pans can be arranged efficiently in an oven. To that end we initially divide the problem into two parts, individually treating the packing efficiency of each pan shape and simulating the temperature throughout the brownie over time.

Our goal is to find a single optimal pan shape and size for commercial bakers and another, possibly the same, shape and size for residential bakers using home ovens. We wish to provide each type of baker with pans of a single shape with which to make their brownies, as it is unreasonable to ask bakers to purchase multiple pan shapes, which will be difficult to store when not in use and which require a specific arrangement in the oven to achieve optimal cooking. We also restrict our attention to pans that have uniform horizontal cross-sections, that is, pans with vertical walls that are perpendicular to their base. Finally, we do not consider interlocking pan shapes because, although they may achieve higher packing efficiency, we consider them to be difficult to place in and remove from the oven, and cumbersome to store between bakes.

We will consider two metrics when selecting pan shape: packing efficiency (E) and cooking evenness (H), both arranged on a scale of 0 to 1. We use these values and a gain, p, to create a total utility function to describe how well a specific pan produces brownies. These metrics are discussed in more detail in section 7.

Packing efficiency can be non-dimensionalized and determined relative to W/L and will be reported in terms of percentage covered. However, our model of cooking relies on heat transfer equations involving thermal diffusivity which cannot be non-dimensionalized as such. To make our results concrete we will focus on two cases:

- **Residential Ovens** Interior dimensions of typical residential ovens are approximately 61cm W x 53 cm H x 41cm D (24" wide x 21" high x 16" deep) [17].
- **Commercial Ovens** Commercial ovens typically have interior dimensions of 51cm W x 74cm H x 71cm D (20" wide x 29" high x 28" deep) [11].

In addition, we consider several discrete pan sizes, 200 cm^2, 400 cm^2, and 600 cm^2, which correspond to 1/2, 1, and 3/2 time a traditional brownie recipe which fills a 20cm $\times$ 20cm (8 inch $\times$ 8 inch) pan to a depth of 2.5 cm (1 inch). This will allow bakers to easily convert their existing recipes and creates sufficiently small pans to allow various arrangements within the oven.

To model the heating of our brownies we use the three-dimension heat equation, which we solve numerically by discretizing both space and time. As the brownies cook, they absorb thermal energy from the oven. We define a measure "cookedness" (C) to represent the thermal energy absorption by each unit of volume over time. Finally, we define the evenness (H) in the cookedness values of different sections of the pan, comparing cookedness values at every section to an ideal cookedness values.

We make a number of assumptions when modeling the cooking of our brownies, which are discussed in detail in Section 6.1. It should be noted here that we assume that all ovens have a uniform temperature over space and time, so that while we consider both residential and commercial ovens to have two racks, we treat both racks identically. We also assume that pans of brownie batter cook independently of one another.

We then perform sensitivity analysis on the gain assigned to each term in the utility function to determine what effect that has on our "optimal" pan choice.

3 Terminology

All values of variables and parameters are reported with the following units:

Distance - Centimeters - cm

Temperature - Degrees Celsius - °C

Mass - Grams - g

Time - Seconds or Minutes - s or min

Energy - Joules - J

3.1 Shape and Size Terms

Variables and terms related to the shapes and sizes of pans and ovens include:

A - Area of each pan

W - Width of an oven rack (horizontal distance between side walls of the oven)

L - Length of an oven rack (depth, distance between the back wall of the oven and the door of oven)

n - A shape parameter of our pans

E - Packing efficiency

N - Number of pans that can fit into the oven, $N/2$ on each rack.

3.2 Heat Terms

Variables related to the heat model include:

T - Temperature, measured in ° Celsius (°C). Temperature is expressed as a function of spatial dimensions x, y, and z and time t.

t - Time, measured in seconds (s) or minutes (min)

t_{done} - Time done, time when brownies are removed from the oven, measured in minutes because on the order of 30 minutes [min]

(x,y,z) - Position in the pan, measured in centimeters (cm).

C - Cookedness, a measure of thermal energy absorbed by brownie batter over time, in Joule minutes [J min]

C* - Ideal Cookedness, absorbed energy required to cook brownies to ideal texture, in Joule minutes [J min]

H - Evenness, a comparison of cookedness over all locations, in Joule minutes [J min]

We also define a number of parameters used in the heat model, whose values we will discuss below in Section 6.6.

α - Thermal diffusivity of brownie batter, $[\mathrm{m}^2/\mathrm{s}]$

ρ - Density of brownie batter, $[\mathrm{g}/\mathrm{cm}^3]$

c_p - Specific heat of brownie batter, $[\mathrm{J}/\mathrm{g\ K}]$

k - Thermal conductivity, in Watts per meter Kelvin, $[\mathrm{W}/\mathrm{m\ K}]$

T_{done} - The minimum temperature the batter must reach to be considered cooked, [°C]

t_{done} - When the minimum batter temperature in the pan is T_{done}, the brownies are considered done [°C]

T_{oven} - Temperature of the oven [°C]

T_0 - Initial batter temperature [°C]

3.3 Optimization

Variables related to our optimization include:

p - A weight on the importance of packing efficiency and evenness in the overall score.

S_p - An overall scale defining the total ability of a pan to cook brownies at a specific p weight.

4 Shapes: Superellipses

Most people who are familiar with baking brownies can attest that the corners are always more well done than the center. Thus, rectangular pans are less desirable in terms of cooking consistent brownies than round pans, which do not have corners. On the other hand, rectangular pans can be packed in a rectangular oven more tightly and easily than circular pans. Thus, we wish to test pan shapes which are somewhat rounded but somewhat rectangular.

To create pans with shapes intermediate between circles and rectangles we used the class of shapes known as superellipses or Lamé curves. Although first discovered by Lamé in the early 1800s, such shapes were popularized by the Danish poet and designer Piet Hein [13]. Hein found superellipses to be the “simplest and most pleasing closed curve that mediates fairly between these two clashing tendencies” of rectangles and ellipses [13]. Superellipses are represented by the Cartesian equation

$$\left|\frac{x}{a}\right|^n + \left|\frac{y}{b}\right|^n = 1$$

for positive numbers a, b and n, and can by written parametrically as

$$x = \pm a \cos^{\frac{2}{n}}(\theta)$$
$$y = \pm b \cos^{\frac{2}{n}}(\theta)$$

for $\theta \in [0, \pi/2)$. The parameters a and b represent the “stretch” in the x and y directions, like the semiaxes of an ellipse. The shape parameter n represents the “roundness” of the shape.

Figure 2 shows superellipses for $a = b = 1$ for values of n between 1 and 5 in increments of 0.5. When $n = 1$ the superellipse is a perfect square with sides at 45° degrees to the axes (the innermost shape in Figure 2), $n = 2$ produces a circle (shown by a thicker line in Figure 2), and as n increases beyond $n = 2$ the shape become more “square-like” again (gray). If we take $a \neq b$ the superellipses are *unevenly* stretched in the x and y direction, resulting in elongated shapes. In this situation $n = 1$ creates a rhombus, $n = 2$ an ellipse, and large n approach a rectangle, as shown in Figure 3. For more information on the calculation of the area of superellipses and boxes that bound them, see Appendix A.

While the idea of superellipses can be extended to round out other polygons using “superformulas” [14], here we only consider superellipses.

5 Packing Model

We model the packing of superellipses by first bounding each superellipse by a polygon (rectangle or hexagon) and then tessellating that polygon over the area of the oven rack.

5.1 Bounding

Figure 4 shows the bounding of superellipses for the even stretch case $a = b$ and for various values of the shape parameter n from 1.5 to 5. For each value of n, there are two choices for bounding boxes: the bounding box parallel to the x and y axes (shown in dark blue) and a rotated bounding box (shown in red). The non-rotated bounding box (dark blue) has a fixed width of 2 units (for $a = b = 1$) and thus fixed area of $4ab = 4$ units. For $n \geq 2$ this box touches the superellipses at four points along the axes.

The other option is the rotated, red boxes. The red boxes are rotated 45° (for $a = b = 1$) and touch the ellipses off the axes (in the first quadrant where $\theta = \pi/4$). The area calculation for the rotated red boxes is presented in Appendix A. For $n < 2$ the rotated red boxes have smaller areas than the outer dark blue box, by symmetry for $n = 2$ the boxes have equal size, and for $n > 2$ the rotated red boxes are larger than the blue box. These areas are summarized in Figure 5, in colors corresponding to Figure 4.

For the case of $a \neq b$, superellipses can be bound in a similar manner, with the bounding polygon being a rhombus instead of a square. However, we predict that stretched rhombi will not tile the rectangular oven racks well, so for the case of $a \neq b$ we only use the non-rotated bounding box for all values of n.

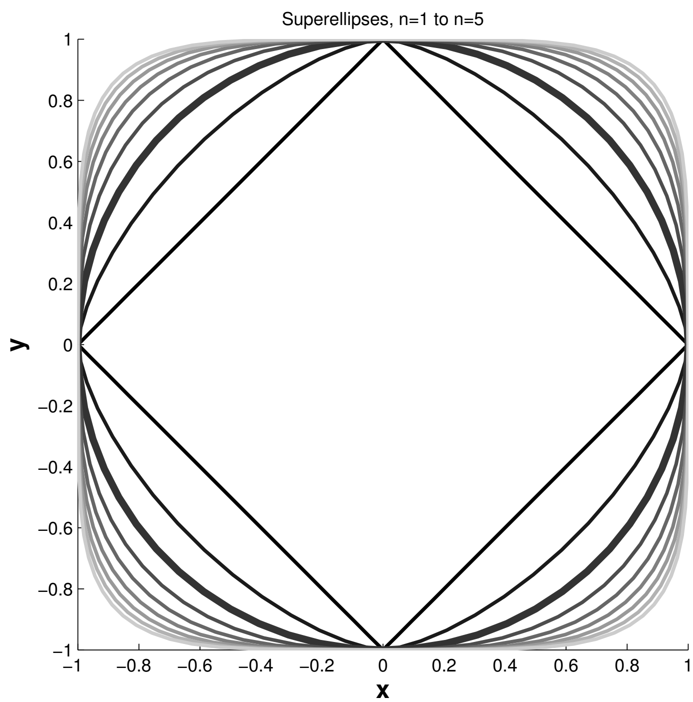


Figure 2: Superellipses with equal stretch $a = b = 1$ and shape parameter n from $n = 1$ (innermost, darkest) to 5 (outermost, lightest). Thickest curve is a circle ($n = 2$).

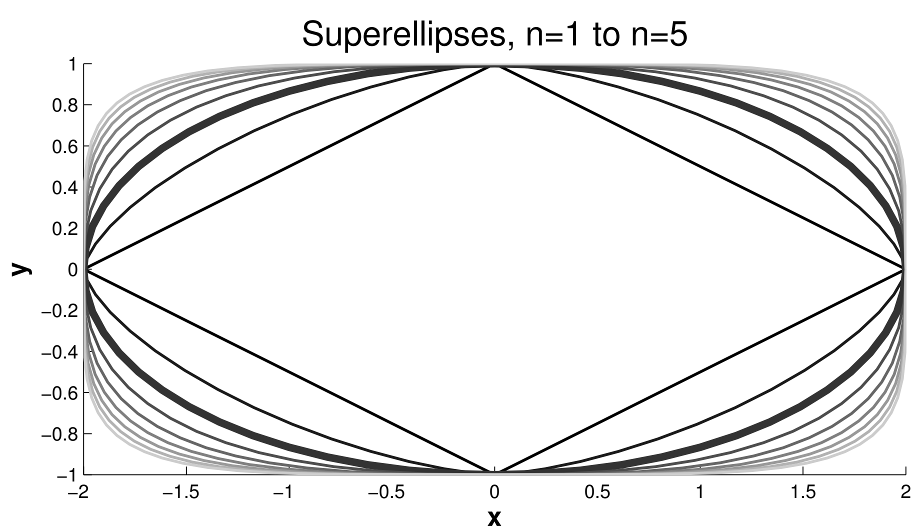


Figure 3: Superellipses, with double stretch in x-axis, $a = 2$, $b = 1$ and shape parameter n from $n = 1$ (innermost, darkest) to 5 (outermost, lightest). Thickest curve is an ellipse ($n = 2$).

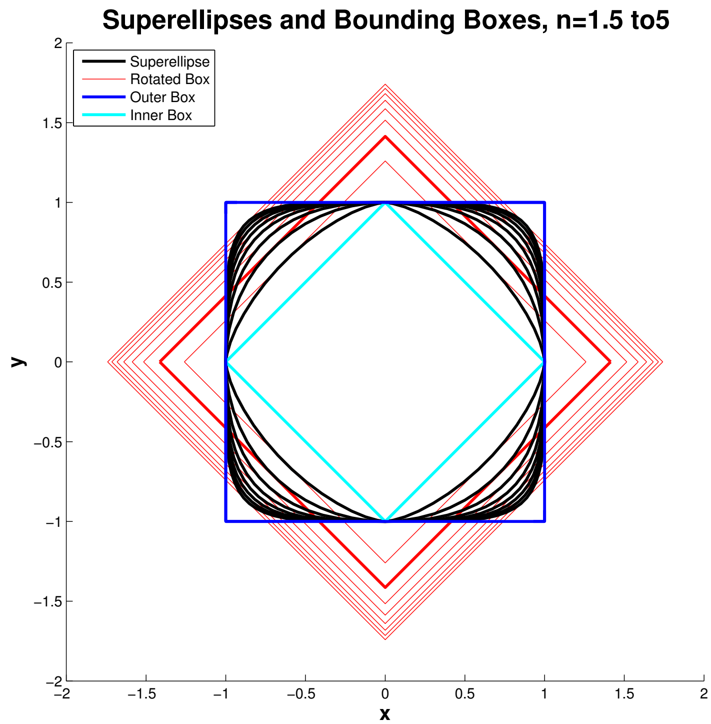


Figure 4: Superellipses and Bounding Boxes, $a = b = 1$. Superellipses are shown in black from $n = 1.5$ (innermost) to 5 (outermost). These superellipses are bounded inside by a cyan box corresponding to $n = 1$ and outside by an dark blue box corresponding to $n = \infty$. Each superellipse can also be bound outside by an rotated red box, concentric to the inner box.

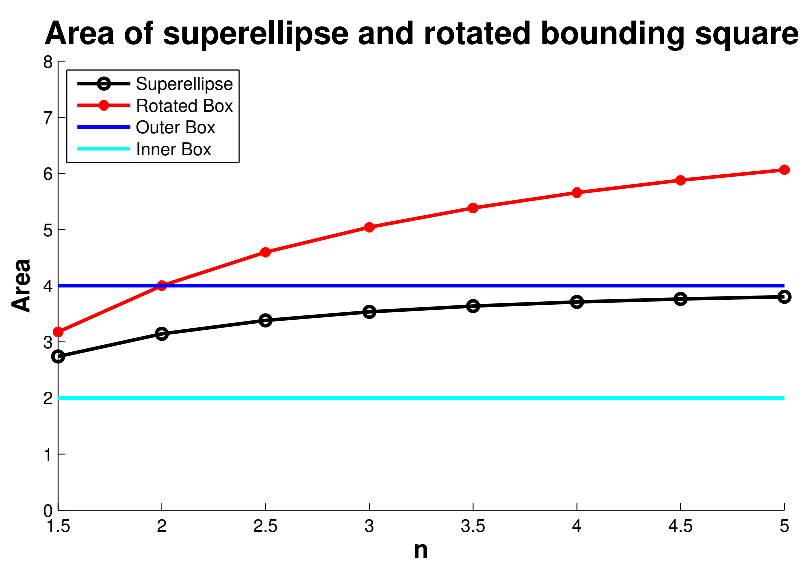


Figure 5: Area of Superellipses and Bounding Boxes, $a = b = 1$. The area of superellipses from $n = 1.5$ to $n = 5$ are shown in black, while the limit for $n = \infty$ (the outer blue box in the figure above) is shown in blue, and is 4. The inner cyan box in Figure 4 has area 2, shown here in cyan. The rotated bounding boxes (shown in red above) have area here shown in red. When $n = 2$, the superellipse is a circle and both the red and dark blue boxes have the same area.

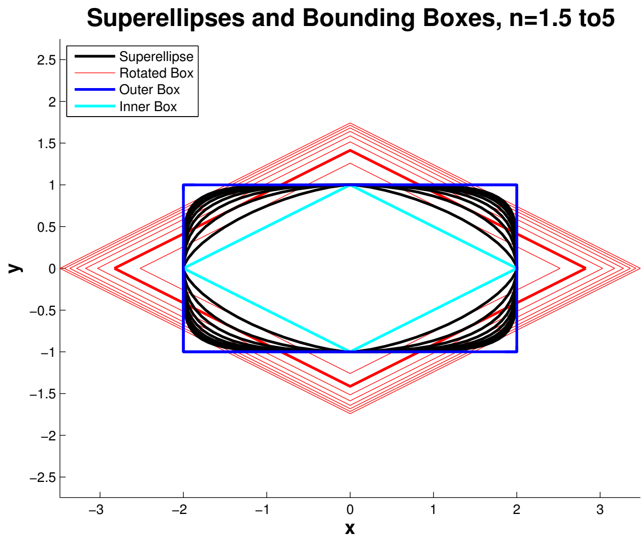


Figure 6: Superellipses and Bounding Boxes, $a \neq b$. Superellipses are shown in black from $n = 1.5$ (innermost) to 5 (outermost). These superellipses are bounded inside by a cyan diamond corresponding to $n = 1$ and outside by an dark blue box corresponding to $n = \infty$. Each superellipse can also be bound outside by an rotated red diamond, concentric to the inner diamond.

5.2 Tiling

Though higher packing efficiencies can be achieved for irregular packing schemes, we consider only regular packing methods which could be easily used in a residential setting. Figure 7 shows the optimal packing scheme for 10 unit squares and a certain W and L. Though this arrangement is optimal, it will not be considered in our solution, as exact placement of pans in this configuration would be difficult.

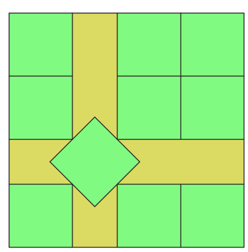


Figure 7: Optimal packing of a square inside a square.

5.2.1 Equal Stretch

For a general superellipse with $a = b$ we consider two possibilities: (1) tiling rectangularly in their original (x, y) orientation as shown in Figure 2 and (2) tiling rectangularly with a 45° rotation. An exception is made for the case of circles ($a = b$, $n = 2$), for which hexagonal tiles are employed (see section 5.2.3).

For $n > 2$ we can place the superellipse in the blue box and tile those boxes in the oven in their original orientation. When $n < 2$ we can rotate the superellipse, enclose it in a red box, and tile the oven rack using the rotated boxes in the same manner as above.

In both cases tiling is achieved by simply placing the squares side by side until the edge of the rack is reached. The number of pans can be analytically determined for fixed W, L, and A by the relations:

$$\begin{aligned} a &= \sqrt{A} \\ N &= \text{floor}\left(\frac{W}{2 \cdot a}\right) \cdot \text{floor}\left(\frac{L}{2 \cdot a}\right); \end{aligned}$$

Figure 8 illustrates these two schemes for superellipses with $n = 1.5$, $n = 2$, and $n = 3$. The original x axis is indicated in blue, so that we see for $n = 1.5$ the superellipses are rotated, and for $n = 3$ they maintain the original orientation.

At $n = 2$, by symmetry, both bounding boxes are equivalent. However, for some oven shapes and pan sizes we can do even better, using a hexagonal tiling pattern.

5.2.2 Unequal Stretch

When $a \neq b$ the superellipse will resemble an ellipse or rectangle with rounded corners, depending on the value of n chosen. When $n = 1$ the superellipse resembles a rhombus with rounded corners, when $n = 2$ the superellipse is a true ellipse, and as n goes to ∞ the shape approaches a rectangle with sharper corners (see Figure 3).

For the case of $a \neq b$ we consider tiling with the outer bounding rectangle only. Using the rotated bounding boxes would give stretched rhombi, which we predict would not tile a rectangular oven rack well in most cases.

Note that the relationship between a and b does not change the percentage of the outer bounding rectangle that the superellipses take up, since the area of the super ellipse is $4ab\frac{\Gamma\left(1+\frac{1}{n}\right)^2}{\Gamma\left(1+\frac{2}{n}\right)}$ while the area of the bounding box is $4ab$ (see Appendix A, here Γ is the continuous analog of the factorial function). Therefore the area ratio of the superellipse to the limiting rectangle is dependent on n only.

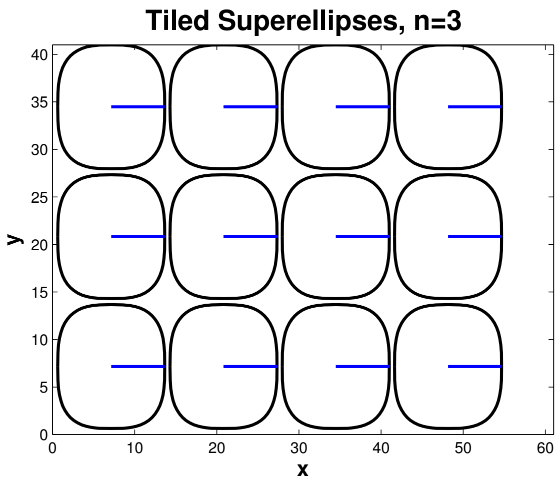
5.2.3 Circles

We treat the case of $a = b$ and $n = 2$, that is, circles, separately. Optimal packing for circles is achieved using a hexagonal tessellation, first proved by László Fejes Tóth in 1940 [27]. Figure 9 shows circles tessellated in both a hexagonal and rectangular pattern. Hexagonal tessellation a density of 0.906 (shown in Figure 9 (a)), while rectangular tessellation achieves a density of 0.785 (shown in Figure 9 (b)).

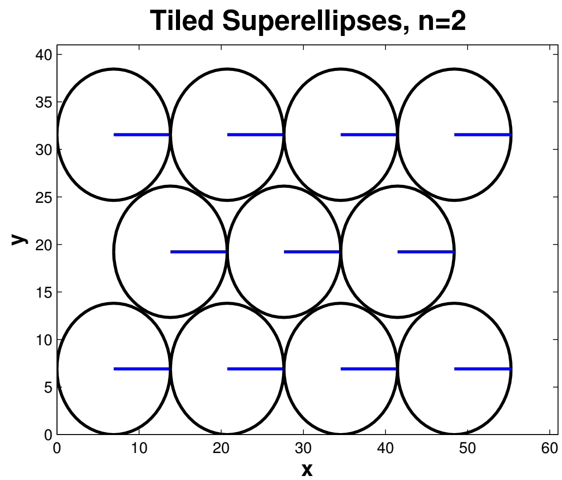
For the case of $n = 2$, the number of pans is given by hexagon tessellation and is:

$$\begin{aligned} a &= \sqrt{\frac{A}{\pi}} \\ s &= \frac{2 \cdot a}{\sqrt{3}} \\ N_{L1} &= floor\left(\frac{L - 2 \cdot s}{3 \cdot s}\right) + 1 \\ N_{W1} &= floor\left(\frac{W}{\sqrt{3} \cdot s}\right) \\ N_{L2} &= floor\left(\frac{L - 1.5 \cdot s}{3 \cdot s}\right) \\ N_{W2} &= floor\left(\frac{L - \sqrt{3} \cdot s}{\sqrt{3} \cdot s}\right) \\ N &= 2 \cdot (N_{W1} \cdot N_{L1} + \cdot N_{W2} \cdot N_{L2}) \end{aligned}$$

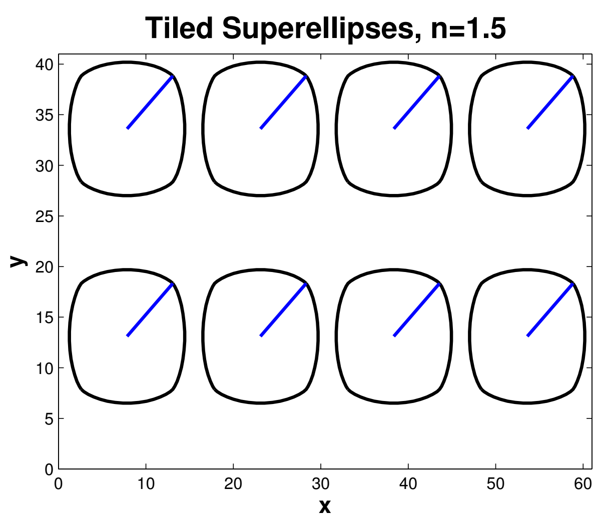
for a hexagon, where the subscripts 1 and 2 represent the two offset rows of a hexagon tessellation.



(a) Rectangular Tiling of $n = 3$ Superelliptic Pans



(b) Hexagonal Tiling of $n = 2$ Superelliptic Pans (Circles)



(c) Rectangular Tiling of $n = 1.5$ Superelliptic Pans

Figure 8: Tiling of superellipses over the space in the oven. The blue lines show the original x axis orientation. For n$\leq$2 optimal packing is achieved by rotating the superellipses by $\theta = \pi/4$.

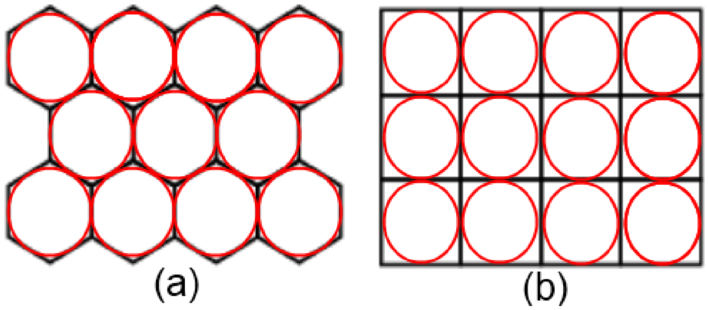


Figure 9: Tessellation of a circle in a hexagon and square pattern.

6 Heat Model

6.1 Assumptions

We divide assumptions into two categories, weak and strong, based on their validity. We believe strong assumptions are reasonable even though we may lack specific evidence or data to support them. Weak assumptions were are made to simplify the modeling problem.

6.1.1 Strong Assumptions

We assume:

1. **Pans have uniform horizontal cross-section**, i.e. the pan's vertical walls are perpendicular to the base. We do not expect this to have an effect on optimal pan selection.

2. **Pan shapes are superellipses to the extent allowed by a discretized model**. Variations in shape due to discretization are no more than 0.0625 cm and also cause small variations in pan area.

3. **The brownie batter is thin such that heating in the vertical (z) axis is uniform.** Since the brownie batter is only 2.5 cm thick (1") and the top and bottom area is significantly greater than the area of the sides, heat is transmitted through the vertical axis is evenly and quickly. This will not affect the determined optimal 2-D shape because pans are uniform vertically .

4. **The pan is a perfect thermal conductor.** Aluminum, a common pan material, has a thermal diffusivity over 800 times higher than batters of common baked goods [9, 22], thus transmitting heat almost 3 orders of magnitude faster than these batters and faster than our simulation time-step of two seconds. Therefore, we assume the pan itself instantly transmits thermal energy.

5. **The oven rack does not affect heating**. Oven racks are typically made of wire, resulting in low contact area and low thermal conduction, and low impact on air flow.

6. **The pan and the oven transfer heat identically**. The oven air and the pan are at the same temperature by assumption 4, but we further assume that they transfer heat to the batter identically.

7. **A pan of brownies is done when the coolest part of the batter reaches a set temperature which we call T_{done}.** Several sources recommend removing brownies from the oven when the internal temperature reaches a certain value (the specific temperature is discussed in Section 6.6 below). Other bakers look for a "set" center, or a clean toothpick inserted into the center of the batter[16, 7]. All of these metrics correspond to a certain minimum temperature in the pan to set the batter or leave

the toothpick clean[1]. Therefore, we use a single minimum temperature threshold, T_{done}, to determine doneness.

Note that it may be possible to achieve a better evenness value by cooking the batter for more or less time. However, a baker cannot determine this optimal evenness easily when deciding when to remove the brownies. Using a fixed internal temperature is measured by a thermometer, or by a texture measure (center set, toothpick clean), and is therefore a realistic way and easy way for bakers to determine doneness.

8. **The batter's initial temperature is a uniform $T_0 = 25°$C at time zero.** Batter is mixed in its creation, homogenizing any temperature fluctuations. The use of melted butter and chocolate will result in a batter slightly above room temperature.

6.1.2 Weak Assumptions

We assume:

10. **All pans are heated independently**. We assume that even if the pans are close together, the metal pan is a sufficiently good conductor or sufficient air flows between the pans such that each pan is heated evenly. Therefore pans do not affect their neighbors and can be modeled in isolation.

11. **The oven temperature is uniform in time.** Thermal PID control loops in ovens are capable of maintaining set temperature to within 2 °C, though they are not typically found in consumer ovens [8]. The use of a convection setting in a residential oven can produce temperature variations of 11 °C (6% temperature variation at 167 °C). Typical residential ovens typically have temperature variations as large as 31 °C at the center of the oven (19% variation at 167 °C), though the addition of a cookie sheet will decrease this variation to 17 °C. Thus, it is at least possible to well control an oven air temperature in time.

12. **The oven temperature is uniform in space.** Typical temperature variations in ovens from top to bottom are 3 °C, as hot air tends to rise to the top [8]. This assumption is made to simplify modeling and may produce slight variations in the top and bottom rack, but are likely negligible.

13. **Batter density, specific heat, thermal diffusivity, and volume do not change throughout cooking.** This is not completely realistic since some moisture is lost during baking. For some baked goods that rise and become very light, such as bread, muffins, and light cakes, these values may change significantly over time. Brownie batter consistency, however, is more stable over time because it has much less liquid (milk, water) than typical cake batter. In fact, culinary historian Nancy Baggett suggests that brownies may be "the fortunate result of someone forgetting the milk in a chocolate cake!"[2]. Even for cakes, the specific heat has been estimated to vary by only 5%, see discussion in Section 6.6.

14. **Conduction is the primary mode of heat transfer.** Previous studies only model conduction and achieve good agreement with empirical results. See [20] and Section 6.7.

6.2 Set-up

We use a modified version of the **three-dimensional heat equation** to model the batter's heat diffusion. We denote temperature (in °C) by T, which is a function of Cartesian spatial dimensions x, y, and z

[1]Leaving the toothpick clean could also be argued to be a function of the moisture content of the batter. However since moisture content presumably decreases monotonically over time when temperature monotonically increases, we can also view toothpick cleanliness as a function of temperature.

(measured in cm), and time t (measured in seconds). In three-dimensions the heat equation stipulates that the temperature T of the batter at position (x, y, z) at time t satisfies

$$\frac{\partial T}{\partial t} = \alpha \cdot \left(\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} \right) \tag{1}$$

where α is the thermal diffusivity of the material (see section 6.6.4)

In addition to Equation (1) we have **boundary conditions** at time $t = 0$:

- The temperature of the batter is uniformly T_0.
- The temperature at all points outside the batter is equal to T_{oven}, that is, the oven has uniform temperature and the pan conducts thermal energy perfectly, as discussed in Assumptions 4 and 11 (Section 6.1).

6.3 Discretization

Since we are interested in a large variety of pan shapes, we do not solve the partial differential heat equation (1) analytically for each geometry. Rather we discretize both space and time and solve using a numerical simulation.

We use a uniform grid ($\Delta x = \Delta y = 0.125$cm) in the non-vertical spatial dimensions . Due to the computational intensive nature of solving equation 1 in 3 dimensions, we compact the vertical dimension, z, into two layers, reducing the computation to a quasi-3D equation. This is justified by the geometry of the problem, specifically the ratio of W and L to z, which is always large ($\approx$ 10:1 for a typical 20cm x 20cm pan (8" x 8")). Since we're interested in determining the optimum 2D pan shape, modeling variations in temperature in x and y is important, with z being less so. While our model of temperature diffusion may be less accurate in the vertical axis, that inaccuracy will be consistent across the entire pan and will not affect the 2D pan selection.

We thus bin the vertical dimension into two equal bins, essentially slicing the brownies in half. The top slice of batter is exposed to the oven's hot air above and more batter below. The bottom slide is exposed to the pan below and more batter above. As the pan is a near perfect thermal conductor (Assumption 4) the pan and oven air are at the same temperature, resulting in vertical symmetry. We exploit this symmetry by modeling only one layer in z.

Figure 10 illustrates how we discretize space. The shaded box represents a single simulation point, with width Δx, depth Δy, and height Δz. The height Δz is shown to scale, half the brownie thickness, however, the width Δx and the depth Δy are not shown to scale.

We use a uniform time step of 2 seconds. The time step is determined through von Neumann stability analysis, which gives an upper bound on the time step to ensure a stable simulation [15]. The analysis sets the maximum time step at:

$$\Delta t \leq \frac{\Delta x^2}{2 \cdot \alpha} \tag{2}$$

We use choose a time step four times smaller than the maximum calculated time step in order to be confident in the time stability, but still have reasonably computation times.

6.4 Numerical Approach

Let $T_{i,j}(k)$ represent the temperature of grid point $(x = i, y = j)$ at time t (so position $(x, y) = (i\Delta x, j\Delta y)$ and $t = u\Delta t$. Recall that we model all regions identically in the vertical axis, so we do not need a z subscript.

We estimate the temperature change between time steps linearly as

$$\begin{aligned} T_{i,j}(u+1) &\approx T_{i,j}(u) + \Delta t \cdot \alpha \frac{\partial T_{i,j}(u)}{\partial t} \\ &= T_{i,j}(u) + \Delta t \cdot \alpha \cdot \left(\frac{\partial^2 T}{\partial x^2}\Big|_{i,j,u} + \frac{\partial^2 T}{\partial y^2}\Big|_{i,j,u} + \frac{\partial^2 T}{\partial z^2}\Big|_{i,j,u} \right) \end{aligned}$$

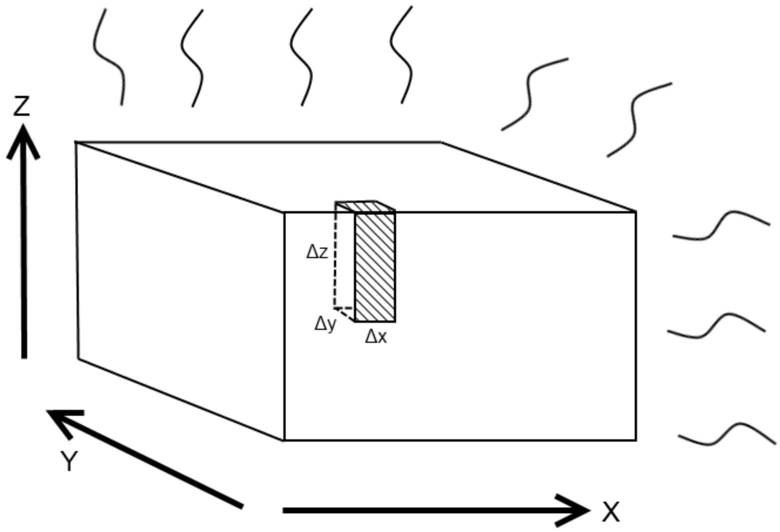


Figure 10: Schematic of model discretization. Δz is to scale at half the thickness, Δx and Δy are each 0.125cm and have been greatly enlarged for illustrative purposes.

Next, we approximate each second spatial derivative by a secant line of the first spatial derivatives, which is also approximated by a secant line. So in the x dimension

$$\frac{\partial^2 T}{\partial x^2}\Big|_{i,j,u} \approx \frac{\frac{\partial T}{\partial x}\Big|_{\text{right}} - \frac{\partial T}{\partial x}\Big|_{\text{left}}}{\Delta x} \approx \frac{\frac{T_{i+1,j}(u)-T_{i,j}(u)}{\Delta x} - \frac{T_{i,j}(u)-T_{i-1,j}(u)}{\Delta x}}{\Delta x}$$

$$= \frac{T_{i-1,j}(u) - 2T_{i,j}(u) + T_{i+1,j}(u)}{(\Delta x)^2}$$

and similar for y in index j:

$$\frac{\partial^2 T}{\partial y^2}\Big|_{i,j,u} \approx \frac{T_{i,j-1}(u) - 2T_{i,j}(u) + T_{i,j+1}(u)}{(\Delta y)^2}$$

In the vertical dimension the process is similar, but slightly modified. Recall that we consider the batter into two identical slices, an upper slice and a lower slice. The upper slice is heated by the oven above it, but tempered by the batter below it. The lower slice is heated by the oven below it, but tempered by the batter above it. Since we do not distinguish between the pan, the oven rack, and the hot oven air (see Assumption 6) we treat these slices identically.

Let us consider the upper slice (arbitrarily). There we can take the analog of *right* in the x axis as *above* in the z axis, and so the analog of $T_{i+1,j}(u)$ is the temperature above the slice at time u, which is just the constant oven temperature T_{oven}. The analog of *left* in the x axis, $T_{i-1,j}(u)$, is the temperature *below* the slice at time u, just the temperature of the other slice at time step k. Due to the vertical symmetry, this is equal to the temperature of the current slice at time u. Therefore

$$\frac{\partial^2 T}{\partial z^2}\Big|_{i,j,u} \approx \frac{\frac{\partial T}{\partial z}\Big|_{\text{above}} - \frac{\partial T}{\partial z}\Big|_{\text{below}}}{\Delta z} \approx \frac{\frac{T_{\text{oven}}-T_{i,j}(u)}{\Delta z} - \frac{T_{i,j}(u)-T_{i,j}(u)}{\Delta z}}{\Delta x}$$

$$= \frac{T_{\text{oven}} - T_{i,j}(u)}{(\Delta z)^2}$$

However, because Δz is so much greater than Δx and Δy we introduce a corrective factor for the heat diffusivity across the vertical axis, β. β is given by

$$\beta = \frac{\Delta z^2}{3} = \frac{1.25^2}{3} = 0.52083\bar{3}$$

Pulling each of these spatial second derivatives together we obtain our heat change estimate for the temperature of the grid point (i, j) at time step $u + 1$ at either z location:

$$T_{i,j}(k) \approx T_{i,j}(u) + \alpha \Delta t \left(\frac{T_{i-1,j}(u) - 2T_{i,j}(u) + T_{i+1,j}(u)}{(\Delta x)^2} + \frac{T_{i,j-1}(u) - 2T_{i,j}(u) + T_{i,j+1}(u)}{(\Delta y)^2} + \beta \cdot \frac{T_{\text{oven}} - T_{i,j}(u)}{(\Delta z)^2} \right)$$

6.5 Simulation Approach

In the computing environment MATLAB we represent $T(u)$ as a matrix with indices i and j and update the entire matrix at once, using submatrices of the temperature matrix from the previous time step. Thus we are able to simulate arbitrary pan sizes and geometries quite quickly (order of seconds or a few minutes), even using small steps in x, y, t (0.125 cm, 0.125 cm, and 2 seconds, respectively).

6.6 Parameter Values

The model requires a number of parameters specific to the baked good. Values used in the model are presented here with their justification and validation. Some parameters are recipe dependent, for which we verified our values with a range of different recipes. For brevity, here we report numbers only for a specific brownie recipe [25] that we found to give roughly average parameter values.

First, we use the following parameters:

- **Oven temperature,** T_{oven} is a constant 167 °C (350 °F). This is a common baking temperature for a wide range of baked goods, including all brownie recipes the authors have personal experience with and our selected recipe.
- The **mass** of brownie batter in the pan is 1.2kg, according to our selected recipe and masses of individual ingredients taken from WolframAlpha.
- Brownie batter **initial temperature** is 25 °C, slightly above room temperature[7, 20] (see Assumption 8).

We also use the following additional parameters, whose values we justify in more depth in the subsections below:

- T_{done}, the temperature of the coolest part of the batter when we declare the brownies done and remove them from the oven. We use T_{done}= 90°C .
- ρ Density of the batter, 1.2 $^{\text{g}}/_{\text{cm}^3}$
- c_p, Specific heat of the batter, set as 1.8 $\frac{\text{J}}{\text{gK}}$.
- α, Thermal diffusivity of brownie batter, estimated at $1.0 \cdot 10^{-7}$ $\frac{\text{m}^2}{\text{s}}$.
- C*, Ideal Cookedness, converted to 140 Joule minutes.

6.6.1 T_{done}

We consider the brownies to be done cooking when the minimum batter temperature reaches T_{done}= 90°C . According to the U.S. Department of Health and Human Services, this value should be at least 70° C to avoid salmonella infection from the eggs in the batter [26], but we do not want it to exceed 105 70° Celsius[19]. At least one baker recommends a value between 88° Celsius and 93° Celsius [1] for brownies. Furthermore, studies of cake-baking cite a similar temperature, including 93° C [20] and 90° C [24] after similar cook times (roughly 30 minutes).

6.6.2 Density

Before calculating other quantities specific to brownie batter, we first selected a specific brownie recipe [25]. Since our brownie recipe, like all other we are aware of, does not involve beating or whipping, we assume no significant portion of air was added to the batter during mixing. Therefore we calculate the batter density using an average of the constituent ingredient's densities, weighted by the molar volume fraction of those constituents. Ingredient densities were taken from Wolfram Alpha, and resulted in a brownie batter density of 1.2 $\mathrm{g/cm^3}$.

This is in comparison to cake densities, which start from 0.8 $\mathrm{g/cm^3}$[22]. This is qualitatively consistent since cake batter typically includes beaten butter, which incorporates air and thus decreases the cake batter density. In fact, many brownie recipes recommend stirring as little as possible. Also, from the author's personal experience, brownie batter does not float in water, indicating that its density should be greater than 1 $\mathrm{g/cm^3}$, in opposition to cake batter which floats.

We further check our brownie batter density value by using it, along with the recipe mass, to calculate the volume of brownie batter.

$$\text{Volume} = (\text{density}) \cdot (\text{mass}) = \rho \cdot \text{mass} = 1.2\mathrm{g/cm^3} \cdot 1211\mathrm{g} = 1.04\text{liter}$$

This is almost the exact amount needed to fill a standard 20cm by 20cm square pan to a height of 2.5 centimeters (1 liter).

6.6.3 Specific Heat

The specific heat of a mixture of substances is found by taking an average weighted by the mass fraction of the constituents. We calculate a brownie batter specific heat as 1.8 $\frac{\mathrm{J}}{\mathrm{gK}}$ and consider it constant throughout the baking process.

This is in comparison to specific heat values for cake, which range from 2.516-2.658$\frac{\mathrm{J}}{\mathrm{g\,K}}$ and 2.950 - 2.800$\frac{\mathrm{J}}{\mathrm{g\,K}}$, over the baking process and depend on the cake recipe. While specific heat has been shown to vary while baking, the variation is on the order of 5% [18, 22], and is not treated in our model.

6.6.4 Thermal Diffusivity

Substances with higher thermal diffusivity change temperature more quickly, while substances with lower thermal diffusivity change temperature more slowly. The thermal diffusivity of a substance can be calculated as

$$\alpha = \frac{k}{\rho \cdot c_p} \tag{3}$$

in units of $\mathrm{cm^2/s}$ where k is the thermal conductivity, ρ is the density, and c_p the specific heat.

Due to a lack of empirical baking data for brownies, the value for thermal diffusivity was estimates from studies of dense yellow cake. Cooking both brownies and cakes with an oven temperature of 167°C for 30 minutes results in roughly the same internal batter temperature (see 6.6.1 above), which implies that the oven's heat is diffusing into both batters at similar rates. The thermal diffusivity for brownie batter is taken to be $1.0 \cdot 10^{-7}$ $\mathrm{m^2/s}$.

Also, according to [2], brownie and cake batter differ mainly in their milk content. But since milk has a thermal diffusivity of approximately $1.25 \cdot 10^{-7}$ $\mathrm{m^2/s}$[12], changing the milk content should not produce a large variation in cake's thermal diffusivity value.

Using $1.0 \cdot 10^{-7}$ $\mathrm{m^2/s}$ as the thermal diffusivity value for brownie batter, together with our earlier estimates of brownie batter density and specific heat, we can calculate brownie batter thermal conductivity:

$$k = \alpha \cdot \rho \cdot c_p = 10^{-7}\frac{m^2}{s} \cdot 1.19\frac{g}{cm^3} \cdot 1.8\frac{J}{gK} = 0.216\frac{W}{m \cdot K}$$

This is very close to cake batter's thermal conductivity of approximately 0.223 $\mathrm{W/m\,K}$ [22].

6.6.5 Ideal Cookedness

We estimate the ideal cookedness, C^* as 140 Joule minutes for an entire vertical section (volume $\Delta x \Delta y 2\Delta z$). This represents the amount of thermal energy that each vertical section of brownie batter should absorb throughout cooking to have the ideal taste. (see more detail in section 7.2 below).

As is derived in section 7.2 below, total cookedness from start time $t = 0$ to end time $t = t_{\text{done}}$ is measured as

$$C = c_p \cdot (\text{mass}) \cdot \int_{t=0}^{t_{\text{done}}} T(t) \cdot \Delta t$$

where c_p is the specific heat, mass is the mass of the batter (in grams), T(t) is the temperature of the batter at time t and Δt is the time step. Note that c_p is measured in $^{\text{J}}/_{\text{gK}}$, mass in grams, temperature in °C (equivalently K), and time in seconds, we need to convert between seconds and minutes to get C in Joule minutes.

We have already discussed estimates of brownie specific heat c_p and mass above. Let $I = \int_{t=0}^{t_{\text{done}}} T(t)$ and I^* be the idea value of this integral, which has units of $°K$ minutes or °C minutes. To calculate I^* we approximated the integrals of temperature-time profiles for cakes from a variety of empirical studies [3, 24, 5, 4, 18] using trapezoidal sections. We found a range of ideal I values centered around approximately 1650°C minutes.

Therefore, estimating the mass of a vertical section using its volume and density, we have

$$\begin{aligned} C^* &= c_p \cdot (\text{mass}) \cdot I^* \\ &= c_p \cdot (\Delta x \cdot \Delta y \cdot 2\Delta z \cdot \rho) \cdot I^* \\ &= 1.8 \frac{J}{gK} \cdot (0.125\ cm \cdot 0.125\ cm \cdot 2.5\ cm \cdot 1.2 \frac{g}{cm^3}) \cdot 1650°C\text{min} \\ &\approx 140 J \cdot \text{min} \end{aligned}$$

6.7 Comparison to Other Baking Models

Although we were not able to preform rigorous experimental tests on brownie baking to validate our model, we can compare our modeling approach to baking models in the scientific literature. Unfortunately, the closest that food engineers have come to modeling the baking of brownies is baking cake, so we cannot expect our model to match these papers exactly.

We focused on two papers that modeled cake baking using a differential equation approach, [20] and [23]. The authors of [20] modeled the baking of a Genoise cake in a cylindrical shape pan. Like us, they used the three-dimensional heat equation to model heat transfer from the oven to the batter. They began by assuming, as we did, that the batter's thermal diffusivity was constant over the baking process. Since they assumed a cylindrical pan shape they were able to solve the heat equation analytically, whereas we used numerical solution methods. When the authors verified their results with empirical tests, they "found that conduction is the primary mechanism of heat transfer and that the diffusion equation provides a theoretical framework for describing the baking process." This provides support to our modeling decision of focusing solely on conductive heat transfer.

However, the authors of [20] were able to better fit their empirical data even better by creating a second, more complicated model where thermal diffusivity changed over time. They explained the need to change thermal diffusivity over time by noting the changes in the cake batter make-up over time: loss of moisture content, increased volume, etc. We do not feel that it is necessary to change our thermal diffusivity value over time in our simulation because brownie batter changes its content less over time. Brownie batter has less moisture than cake batter to begin with and the volume is roughly constant over time[2].

The authors of [23] present no empirical results, only a theoretical model of cupcake baking that is even more complex then the models in [20]. Like the authors of [20], they consider cylindrical pans only, however [23]'s authors attempted to model all forms of heat transfer: conduction, radiation and convection. Their equations are therefore more complicated, for example their heat diffusion equation includes a term for moisture evaporation (convection). To solve their more involved equations they used a numerical scheme of finite difference approximation similar to our approach. They did not validate their model with experimental data. Their primary motivation for modeling convective and radiative heat transfer in addition to conductive was to more accurately the effects of the batter's moisture loss over time. However, once again we feel that moisture changes are much less significant in brownie batter, and therefore our simplified approach is reasonable.

7 Metrics

To judge the design of our pans, we use two metrics: packing efficiency (E) and evenness of heat (H), and combine them into an overall score S.

7.1 Packing

To measure the quality of the packing of a particular shape of pan onto an oven rack, we define the metric packing efficiency, denoted by E. Recall that we pack circular pans into hexagonal tiles and all other superellipses ($n \neq 2$) into rectangular tiles and these hexagons and rectangles are tessellated over the area of the oven rack to determine N, the maximum number of pans. We can then define packing efficiency as:

$$E = \frac{A_{\text{Used}}}{A_{\text{Total}}}$$

where the used area A_{Used} is the area covered by the pans, and the total area A_{Total} is simply the oven rack size. While the oven has two racks, we assume these racks are identical so the number of racks does not change the packing efficiency ratio.

Note that this measure of packing efficiency is directly proportional to the number of pans (N) of a fixed area (A) of any given shape for fixed oven width to length ratio W/L. Note also that the packing efficiency does not depend on the actual dimensions of the oven, only the relationship between W, L, and A. If we double the oven width W, double the oven length L, and quadruple the pan area A, the packing efficiency will not change.

Also note that the packing efficiency is highly dependent on the number of rows and columns of pans we can fit onto an oven rack. Very small changes in the pan area, the pan shape, or the oven dimensions can make an additional row or column of pans, and thus several extra pans (N), possible. Due to the typical sizes of pans and oven racks (200cm^2 and 4000cm^2, respectively), an additional row or column of pans can cause a significant jump in the packing efficiency E.

7.2 Evenness

To measure the evenness of the heat throughout the pan, we first define a measure of "cookedness" (for lack of a better work), C. As the batter sits in the oven, it cooks by absorbing heat energy over time. We call this absorbed heat energy over time the cookedness, and measure it in Joule minutes. At location $x = i$ and $y = j$ (since we assume temperature is uniform in z) the cookedness of the small prism of batter at time $t = \tau$ is:

$$C_{i,j}(\tau) = c_p \cdot (\text{mass}) \cdot \sum_{t=0}^{\tau} T_{i,j}(t)] \cdot \Delta t$$

where c_p is the specific heat of the batter, mass is the mass of the batter section (density ρ times $\Delta x \Delta y \Delta z$), $T_{i,j}(t)$ is the temperature of that batter section at time t, and Δt is the time step (2 seconds). Note that c_p is measured in J/gK, mass in grams, temperature in °Celsius (which is equivalent to degrees K in terms of temperature difference), and time in seconds so cookedness $C_{i,j}$ is measured in J-m, Joule minutes. If we were able to model temperature over time continuously the sum would be replaced by an integral with respect to t. While cookedness varies over time, we only care about the cookedness values when the brownies are done. So, unless otherwise noted, the time τ is presumed to be the time of removal from the oven ($\tau = t_{\text{done}}$), and we often leave out τ to represent cookedness as $C_{i,j}$, or as an entire matrix of cookedness values C.

Evenness, H, is a measure of the overall closeness of the cookedness values of all the batter in a pan to an ideal cooknedness, C^*. The more similar the cookedness values throughout the pan, the more even we consider that pan, and the higher that pan's value of evenness, H. Since we wish to compare evenness, H, with packing efficiency, E, we wish to normalize the H value to fall within a similar range to E values (roughly 0 to 1). To do so, we first define a non-normalized measure of cooking evenness, G, which we normalize to calculate H.

G is defined as the Frobenius norm of the deviation between the batter cookedness at all points, C_ij, and the ideal cookedness, C^*.

$$G = \sqrt{\sum_i \sum_j (C^* - C_{i,j})^2}$$

Note that the Frobenius norm is equivalent to treating the matrix of cookednesses, C, as a vector and taking its Euclidean norm (2-norm). Note that since G is a sum of squared deviations from the ideal cookedness, high values of G correspond to high cookedness variability and thus low evenness.

We wish to alter G to produce a normalized measure of evenness H where a square pan has an evenness of zero (low) and a circular pan has an evenness of one (high). However, the evenness values of a square and circular pan depend on the area of the pan A, due to the thermal diffusivity. Since we are only interested in comparing evenness values for pans of the same area, we will normalize G differently depending on the value of A. Therefore we define H as

$$H = \frac{G_A^{\text{square}} - G_A}{G_A^{\text{square}} - G_A^{\text{circle}}}$$

where the subscripts A are added to make explicit the dependence of the normalization on the pan area. Note that G appears in the negative so a low value of G (low variability) creates a high value of H (high evenness). Also note that it is theoretically possible for a pan shape to have worse evenness than the square, and thus have a negative H value, or have a better evenness than a circle, resulting in an H value above 1.

7.3 Combined Score

Combining these metrics to determine the best pan, we define an overall score S_p as a weighted average of the efficiency (E) and evenness (H) metrics:

$$S_p = p \cdot E + (1 - p) \cdot H$$

where the weight p ranges from zero to one. Note that this score is dependent upon the shape of both packing efficiency (E) and cooking evenness (H). We only compare scores for pans of different shapes with the same fixed area since the evenness is not normalized with respect to pan area.

8 Results

8.1 Standard Square and Circle Pans

First we present our model's results for a standard square (20cm × 20cm, 8" × 8") pan and a standard circular pan of the same area (22.6cm, 9" diameter). These pans both have an area of 400 cm^2, which holds the standard brownie batch of 1 L at a height of 2.5 cm.

8.1.1 Square Heating

We begin with a typical square brownie pan, 20cm by 20cm. We fill the pan with 1 liter of batter at 25°C , filling it to a height of 2.5cm, then place it in the 167°C oven. Recall that this batter amount is approximately one batch in a variety of brownie recipes (see section 6.6).

Figure 11 shows the temperature of the brownie batter as cooks. Throughout the simulation we assume the oven temperature stays constant (dark red). Initially the batter is a constant temperature (dark blue in upper left hand figure), and heats up from the edges inward (orange to light blue). We stop the simulation at approximately 30 minutes, when the coolest point (the center), reaches $T_{\text{done}} = 90°\text{C}$, and call this time t_{done}. As expected, while the brownies bake the corners are hotter than the edges which are hotter than the center.

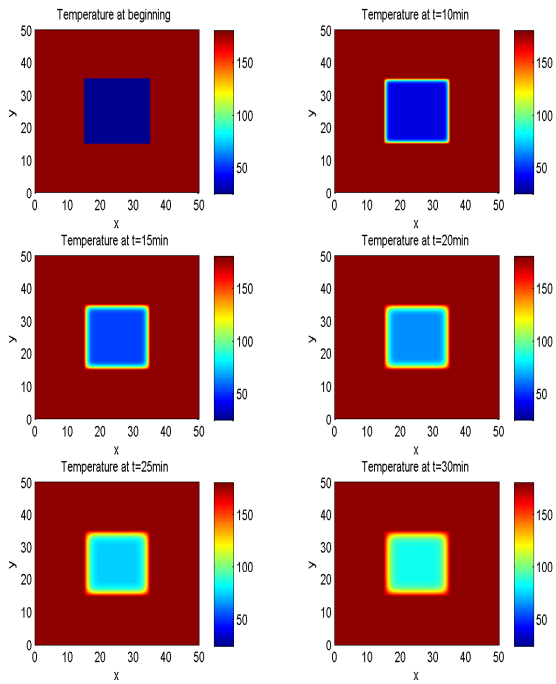


Figure 11: Temperature change over time of square 20cm × 20cm pan. The x and y axes are measured in cm, while temperature is indicated by the color, in °C.

After cooking for approximately half an hour, each part of the batter has absorbed a certain amount of thermal energy, which we term its "cookedness," C, measured in Joule-minutes. The more thermal energy absorbed, the more fully cooked the batter. From empirical studies on cake batter, we estimated an ideal cookedness, C^* as 140 Joule-minutes (Section 6.6.5). Figure 12 shows the difference between the cookedness value at each tiny section of the pan and the ideal cookedness value (x and y axes measured in cm). Dark blue indicates no deviation, while red indicates highly overcooked areas. Not surprisingly, most of the center area is cooked just right, while the edges are overcooked and the corners are the most overdone. Note that some of the center is cooked slightly less than ideal, and the very corners are cooked at about 275 Joule minutes above the ideal 140 Joule minutes. This means the very corners receive approximately 415 Joule minutes of cookedness. This value is reasonable since if we assume that the very corner of the batter is heated at the oven temperature of 167°C for 30 minutes then the cookedness would be approximately 422 Joule minutes.

We have defined a metric, normalized evenness (H), that measures the similarity of the cookedness values across the entire pan. This relies on a non-normalized measure (G) of total deviation from ideal cookedness for a square and circle pan. Note that for G a lower value represents less deviation and thus more evenness. For the square 20cm × 20cm pan this value is approximately $G = 13500$ (measured in Joule minutes), which

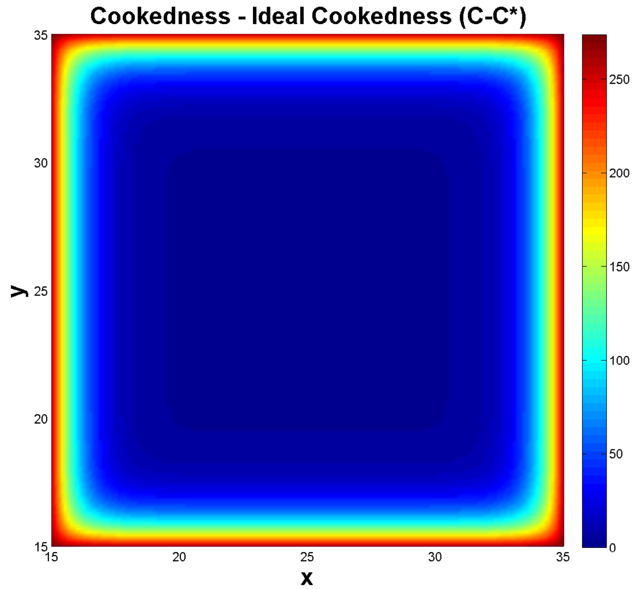


Figure 12: Deviation from Ideal Cooknedness at t_{done} for square 20cm $\times$ 20cm pan (400cm^2, in Joule minutes.

becomes normalized to a H value of zero evenness.

Validation of our model is accomplished by comparing predicted cook times to the range of cook times round for common brownie recipes. This model predicts that the brownies will be done (as defined by the time when the temperature at all points is greater than 90°C) after thirty 30 minutes, consistent with the range of 25-40 minutes found for most recipes. Qualitatively our results are also as predicted. The corners are overcooked, the edges somewhat overcooked, and the center region is cooked almost ideally.

8.1.2 Circle Heating

The other standard brownie pan choice is a circular pan with 22.6cm (9 inch) diameter (400 cm^2 area). We repeat the simulation as with the square pan: 1 liter of batter, starting at 25°C cooks in an oven at 167°C until its minimum temperature is 90°C . Figure 13 shows the resulting cookedness for the circular pan. The area outside the pan is set to zero (dark blue) for reference. The edges of the batter absorb the most heat and therefore have the highest cookedness (dark red). Note that the maximum cookedness is comparable to the square's maximum cookedness. This is likely due to our model's discretization of space, such that many points along the circle's boundary look like "corners" because we cannot model a smooth circle over a square grid.

We had predicted that the overall evenness value would be higher for the circle than for the square due to its decreased perimeter and lack of corners. To find the value of H for the square we first find G, a measure of total deviation from ideal cookedness. For this circular pan, the G value is approximately 12700 (measured in Joule minutes). This is less deviation than with a square pan (approximately 13500). When normalized to an evenness value, we get an H of 1 for the circular pan.

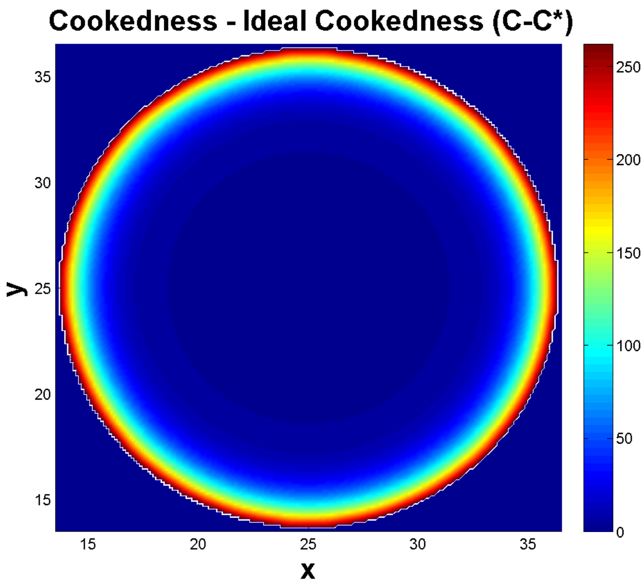


Figure 13: Deviation from Ideal Cookedness at t_{done} for circle pan 23cm in diameter (400cm^2), in Joule minutes.

8.2 Shape Variations

8.2.1 Equal Stretch

Now that we have modeled and examined the conventional square and circular pan shapes, we turn our attention to intermediate pan shapes. Recall that we use superellipses as our intermediate shapes and that superellipses which are controlled by a shape parameter n. First we will consider shapes that are equally stretched in the x and y direction, that is, stretch factors $a = b$. Thus when $n = 1$ the superellipse is a square, $n = 2$ is a circle, and larger n's are squares with increasingly less rounded corners (see Figure 2).

Heating of Superellipses with Equal Stretch While our ultimate goal is to compare these different shapes for evenness (H) and packing efficiency (E), we first examine several properties of their heat simulation.

We begin by examining how badly overcooked the corners are in superellipses over different values of n. We define the corner to be the location on the outer surface of the pan that is furthest from the center. For $1 \leq n < 2$ this occurs at $\theta = 0$ in the parametric representation, for $n = 2$ the shape is a circle and all points are equidistant, and for $n > 2$ the corner is at $\theta = \pi/4$.

Figure 14 shows the cookedness values for superelliptical pans with standard area $A = 400$ cm^2 over a range of shape parameters n. As expected, the perfectly square pan ($n = 1$) has the worst corner cookedness (approximately 413 Joule minutes) and the circular pan shape ($n = 2$) has the least corner cookedness, approximately 402 Joule minutes. As n increases beyond $n = 2$, the shape gets more square-like, the "corner" moves further away from the center, and the corner cookedness in general increases. There are some jumps in the plot that are most likely due to inaccuracies in the discretization of the space. We measure distance down 0.125cm by 0.125cm grid points, so we do not actually model a superellipse as a true rounded superellipse. But note that all of these variations in corner cookedness across n are quite small when we consider that the ideal cookedness is 140 Joule minutes. **No matter which superellipse we choose, the "corners" will be overcooked.**

But while some areas around the edge will always be overcooked, we could also examine the center of each pan. As it turns out, the cookedness values at the center of each pan at the end of baking match up to

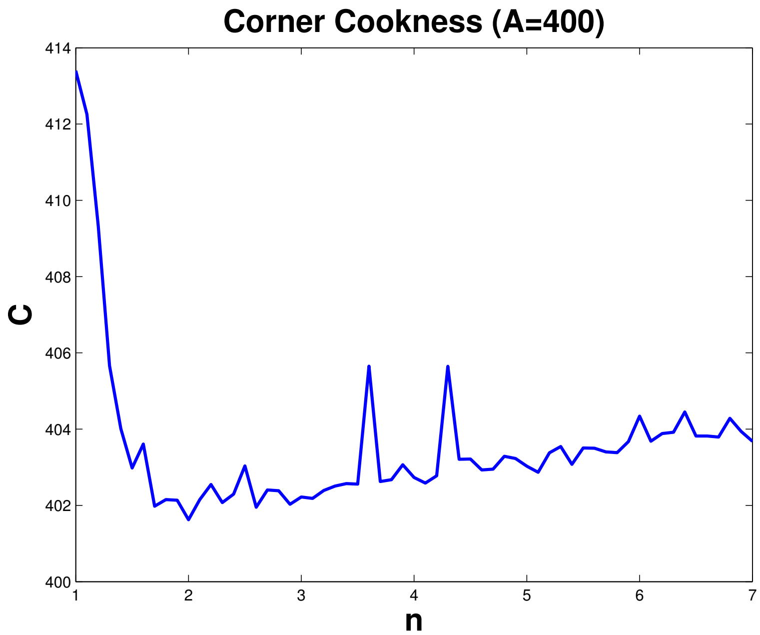


Figure 14: Cookedness (C, in Joule minutes) of Evenly Stretched ($a = b$) Superelliptical Pans with Area $A = 400$ cm^2 over Shape Parameter Range $1 \leq n \leq 7$. The corner is defined as the furthest point from the center.

4 digits after the decimal place for all values of n between 1 and 7 so we omit such a plot.

More than any particular point in our pans, be it corner or center, we are more interested in how evenly distributed cookedness is across the entire pan. To determine that we can use our evenness metric H. Figure 15 shows the evenness (H) for superelliptical pans with area 400 cm^2 over a range of shape parameter n. The square pan ($n = 1$) has zero evenness, while the circular pan ($n = 2$) has evenness of one, as these values were used to normalize the range of H. Values of n between $n = 1$ and $n = 2$ and above $n = 2.2$ all correspond to shapes that are intermediate between squares and circles, and they have intermediate evenness values. Again some jaggedness in this plot is expected due to the discretization of space. Interestingly, $n = 2$ does not display the optimal evenness, $n \approx 2.2$ seems to preform even better. This may be due to inaccuracies in our discretization.

Packing of Superellipses with Equal Stretch Next we turn our attention to the packing efficiency of these shapes. We expect that, in general, shapes that are more rectangular ($n = 1$ or large n) will have a greater packing efficiency than rounder shapes ($n = 2$). This intuition comes from the idea that, fixing area A, the rounder shapes have a larger average radius, and they do not fill up their rectangular bounding boxes as well. The circle $n = 2$ is a special case where we use hexagonal packing instead of rectangular packing (see Section 5 above).

Packing efficiency for superellipses with n between 1 and 7 are shown in Figure 16. The dimensions of a bounding box surrounding a "high-n" pan will approach the dimensions of an $n = 1$ pan, and thus will have equal packing efficiency, except in rare cases where the width of the bounding box is a near integer divisor of the over width. For a fixed A, as n approaches 2 the bounding box necessary to encompass the superellipse becomes bigger, reducing packing efficiency, except for the case where $n = 2$, when hexagon packing is used. Hexagon packing is still not as efficient as square packing in this case, and so in general the best packing will be achieved for low ($1 < n < 1.5$ and high $3 < n < 7$) n.

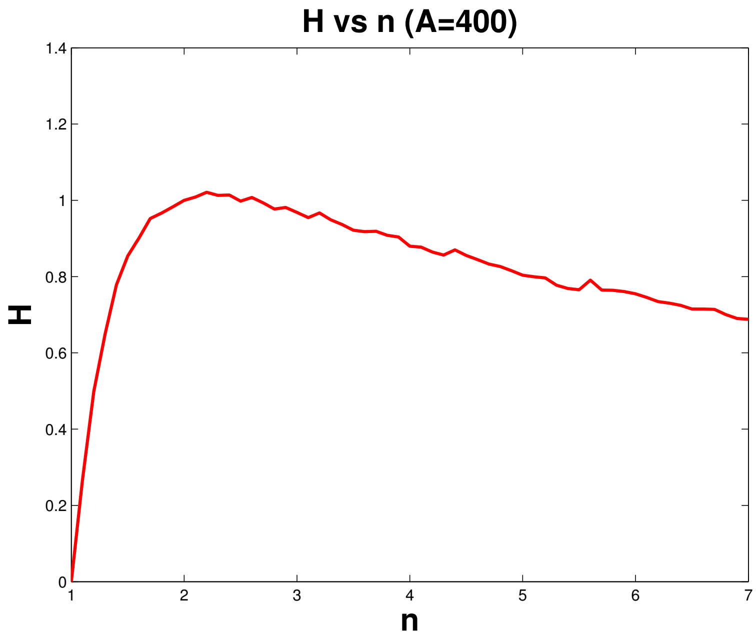


Figure 15: Evenness (H) of Evenly Stretched ($a = b$) Superelliptical Pans with Area $A = 400$ cm^2 over Shape Parameter Range $1 \leq n \leq 7$.

As in our discussion of heat dispersion, we will set the area of all pans in question to be a constant 400 cm^2. But packing efficiency is also dependent on the ratio of W and L, so we consider two cases: residential ovens and commercial ovens.

Figure 16 shows the packing efficiency of superellipses with even stretching ($a = b = 1$) with area 400 cm^2 over various values of shape parameter n. The blue line shows the packing efficiency for a residential oven. Using perfectly square pans ($n = 1$) the packing efficiency corresponds to fitting 6 pans on each rack (three across, two back) for a total of $N = 12$ and almost perfect packing efficiency. For $n = 1.1$ to 5 we can only fit 2 pans per rack (two across, one back) for a total of $N = 4$, and $n = 5.1$ to 6.3 allows 4 pans per rack (two across, two back, total $N = 8$ over two racks) before the pans because close enough to squares again and we can fit 6 per rack (three across, two back, $N = 12$ over both racks). As expected, the more rectangular the pan shape, the better the packing efficiency. In a commercial oven, the number of pans we can pack, and thus the packing efficiency, vary less at this value of $A = 400$cm^2. In fact, for almost all values of n we can fit exactly 6 pans on each rack (two by three), for a total of $N = 12$ pans. The minimum is 3 and occurs at $n = 2$ due to their increased dimensions, even with hexagonal packing.

Of course, these packing efficiency results are quite sensitive to the relationship between pan area A, oven rack width W and oven rack length L. Small variations in any one of these values could mean one extra or one fewer row or column of pans, and a drastic shift in packing efficiency at any value of n. However, large variations are also possible in the right combination without adding any extra rows or columns of pans. Note also that once we have a range of n for which the number of rows and columns of pans, N does not change, and the packing efficiency is also constant because we hold the area of each pan constant.

8.2.2 Overall Score

Now that we have seen the performance of superelliptical pans with even stretch ($a = b$) in terms of both evenness of cooking, H, and packing efficiency, E, we can consider these two factors together in an overall score S. First, we will consider the standard pan area of $A = 400$cm^2, then we briefly present results for

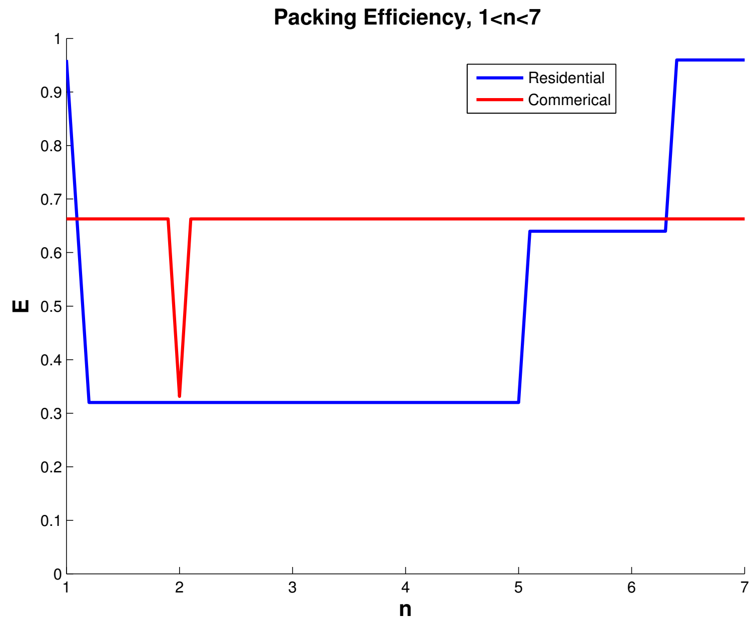


Figure 16: Packing efficiency (E) of Evenly Stretched ($a = b$) Superelliptical Pans with Area $A = 400$ cm^2 over Shape Parameter Range $1 \leq n \leq 7$. Blue line represents typical residential oven dimensions, red line represents commercial oven.

different sizes.

To combine the packing efficiency and evenness scores into an overall score S, we weight them by factors p (for E) and $1-p$ (for H). Thus a weight of $p = 1$ corresponds to packing efficiency only, a weight of $p = 1/2$ to an even average of packing efficiency and evenness, and a weight of $p = 0$ to evenness only.

First we consider residential ovens (61 cm by 41 cm) Figure 17 presents overall scores for a range of superelliptical pans ($a = b$, $1 \leq n \leq 5$) with a few different values for the weight p. The magenta curve corresponds to $p = 1$, packing efficiency, the blue curve to $p = 0$, evenness only, and purple curves to intermediate weights. In the perfectly square ($n = 1$) or nearly perfectly square cases, the cookedness is extremely uneven, but they pans tile the oven rack almost perfectly. For this shape, the weight of p makes a big difference in the overall score. For approximately $1.3 \leq n \leq 6.2$ the packing efficiency drops significantly so the weight p does not matter in this range. Thus the value of n that maximizes evenness, approximately $n = 2.2$, maximizes all the overall scores over the range $n = 1.3$ to $n = 6.2$. For $n = 6.2 \leq n \leq 7$ the packing efficiency is high while the evenness is low. When considering the entire range $1 \leq n \leq 7$, a weight of $p = 0$ would select $n = 2.2$ as optimal, while all other weights would select $n = 6.5$.

In Figure 18 we see similar results but for commercial ovens (51 cm by 71 cm). For the values of n shown, $1 \leq n \leq 5$ the packing efficiency is constant and so the evenness controls the overall score. Once again the optimal value for evenness falls around $n = 2.2$, as the evenness is not effected by the size of the oven.

Thus, it is clear the overall score depends on the weights you place on both evenness and packing efficiency.

8.2.3 Unequal Stretch

Now that we have examined the case of even stretch in the x and y directions ($a = b$), we turn to shapes with uneven stretch. Recall that the ratio a/b represents the aspect ratio of our superellipse, and for $a > b$ the superellipses will be wider in x than y. Thus the superellipse at $n = 1$ is no longer a square but a rhombus with non-right angles, the superellipse with $n = 2$ is a standard ellipse, and when $n > 2$ the superellipses

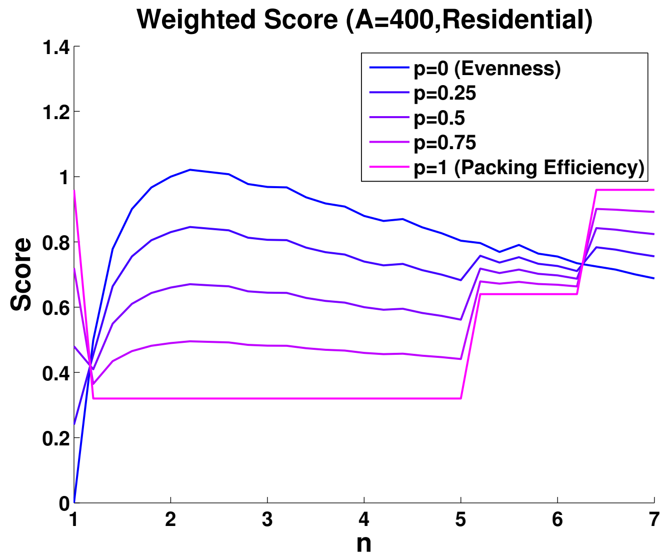


Figure 17: Overall Scores for Superelliptical Pans with Even Stretch ($a = b$), Pan Area 400 cm^2, Residential Oven

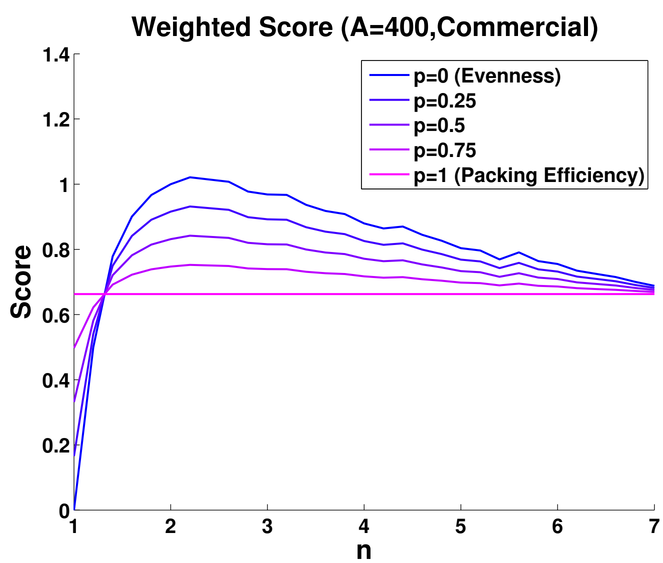


Figure 18: Overall Scores for Superelliptical Pans with Even Stretch ($a = b$), Pan Area 400 cm^2, Commercial Oven

approach a rectangle with side lengths $2a$ and $2b$ (see figure 3).

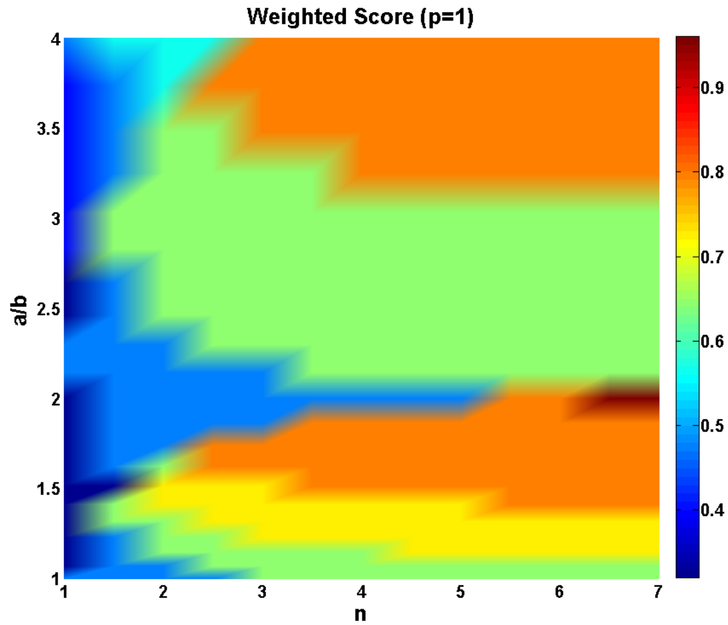
Figure 19 shows variations with the change in the ratio of the semimajor axes, a/b. As might be expected, high a/b at low n (long rectangle with relatively sharp corners) produces very uneven cooking while higher n values (more rounded shapes) produce more even results (see Figure 19b). The range of evenness values for long shapes can extend farther than the typical range of 0 for a square and 1 for a circle, indicating more and less even shapes than our baselines. Since packing efficiency is still measured on the scale 0 to 1, Score values have limited use due to the different scales and are not calculated.

Pan shapes with high a/b ratio result in more even heating due to the closeness of each point in the pan to an edge. In contrast to low a/b shapes, in which cooking of the center is only determined by thermal transfer from the top and bottom, heat from the side of the pan is able to permeate the batter at all points, increasing evenness and decreasing cooking time. This configuration produces an extremely high evenness rating compared to other shapes tested. It is expected that further increasing a/b would further improve evenness, though at some point the brownies will become too small for effective distribution/consumption. Thus, the limit of the length of the pan and thus the evenness is set by practicality, rather than heating.

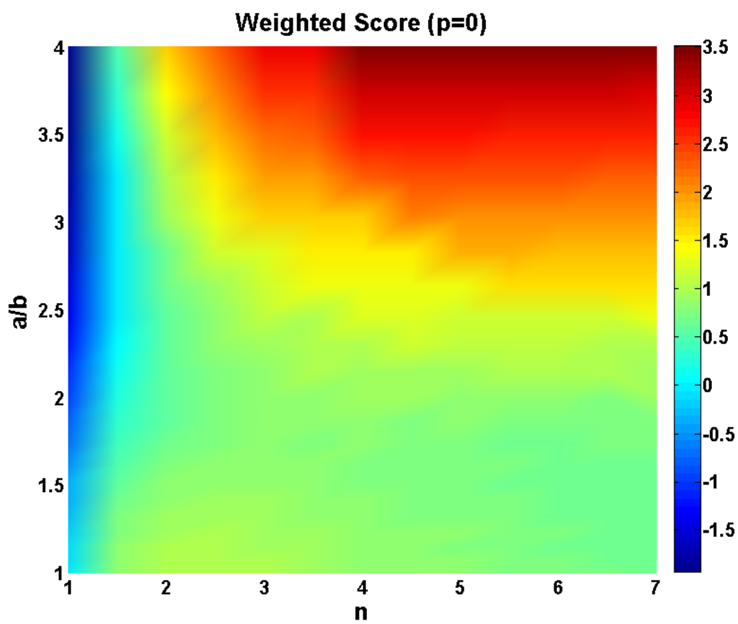
8.3 Sensitivity Analysis of A

Residential ovens typically have dimensions of 61 cm W x 41 cm D. Figures 20 show evenness, packing efficiency, and the value of the utility function for various values of n for various values of the gain, p. All cases are for equal stretching, that is

The value of A's investigated in this report, 200 cm^2, 400 cm^2, and 600 cm^2 were selected because they represent 1/2, 1, and 3/2 batches of standard brownie recipes, such that cooks can easily adjust their recipes. In addition, they were chose such that the area of one pan was not so large a fraction of the total oven area that they did not allow for interesting packing.



(a) Packing Efficiency



(b) Evenness

Figure 19: Scores for Superelliptical Pans with Unequal Stretch over Shape Parameter n. Upper plot shows packing efficiency E (overall score weight $p = 1$) and bottom plot shows evenness H (overall score weight $p = 0$). Scores are indicated by the color, red meaning high packing efficiency or evenness, blue meaning low packing efficiency or evenness.

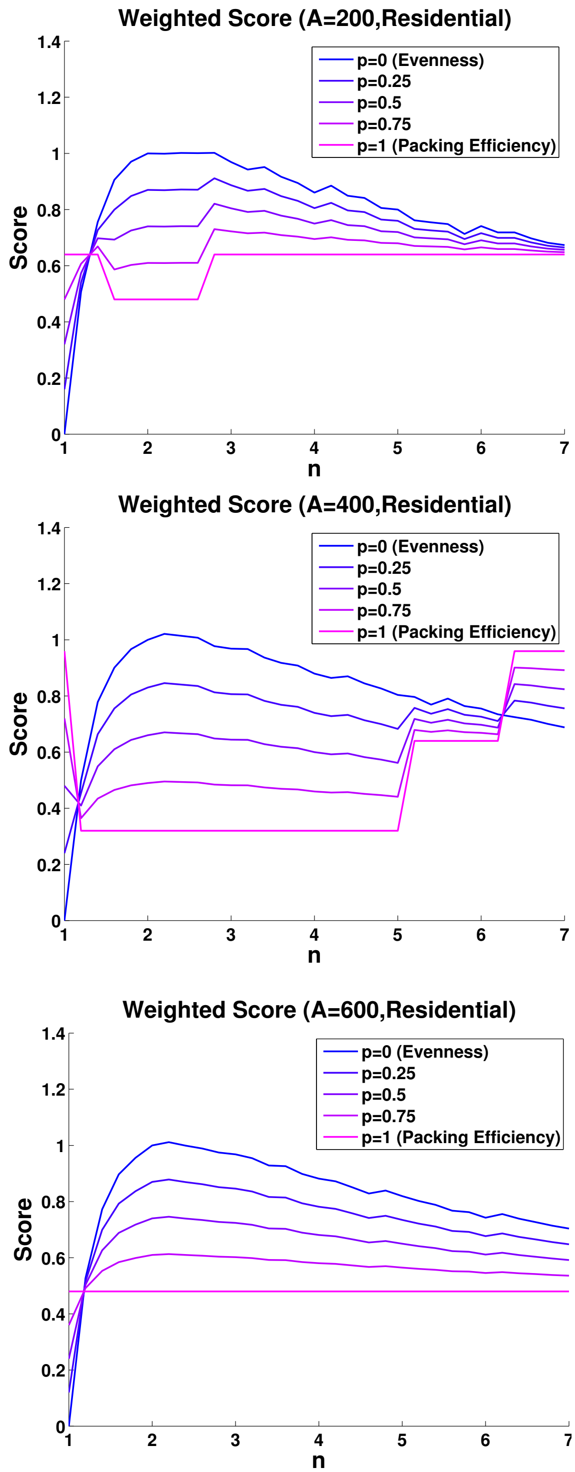
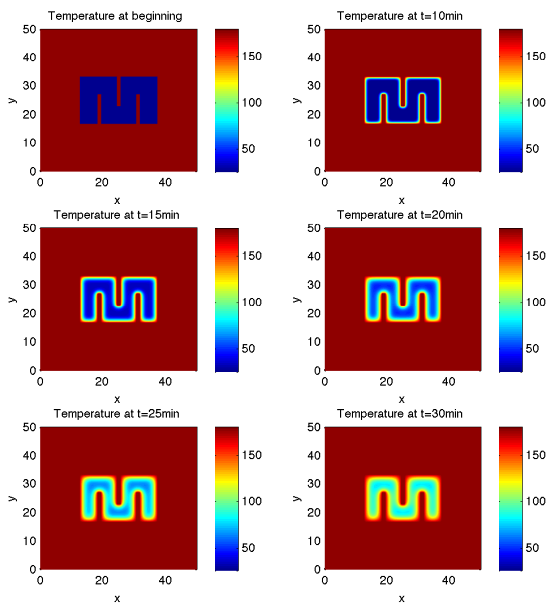


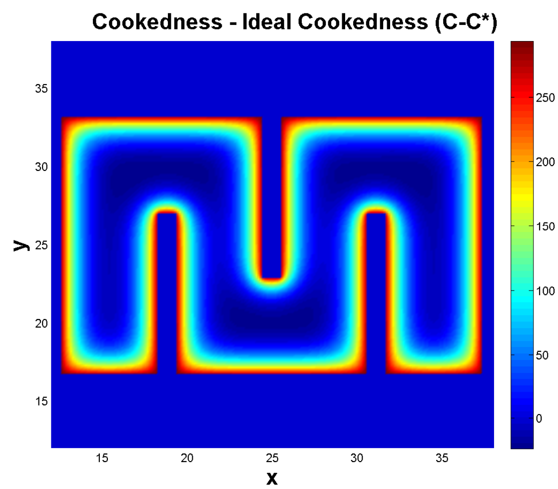
Figure 20: Overall scores for superelliptical pans over a range of shape parameters (n) and score weights p. Residential ovens are shown in the left subfigures, commercial ovens on the right. From top to bottom pan areas of 200cm^2 (half batch), 400cm^2(standard batch) and 600cm^2(one and half batch). Blue lines correspond to cooking evenness, pink lines correspond to packing efficiency.

8.4 *Baker's Edge*

As a final model validation we use our model to analyze the *Baker's Edge* pan (see Figure 1), testing the manufacturer's claims that the pan is more even and required less cooking time compared to a rectangular pan [6]. We tested the pan's shape by scaling it slightly to have a batter area of 400 cm^2 so that we can directly compare to a standard batch of brownies cooked in our superelliptical pans. The results are shown in Figure 21, the temperature of the batter over time on the left, and the deviation from ideal cookedness at the time of oven removal on the right.



(a) Temperature at time intervals during baking for the Baker's Edge.



(b) Cookedness minus Ideal Cookedness for the Baker's Edge.

Figure 21: Baker's Edge Pan, $A = 400$

Our model finds that the Baker's Edge pan is far *worse* than any of the superelliptical pans we tested, producing an evenness of -5.3 J-min. As predicted, the unevenness is due to this pans long perimeter and large number of corners. However we find that it does require slightly less time to cook than a traditional rectangular pan, as claimed by its creators.

9 Conclusion

We have successfully created a quasi-3d heat transfer model and validated it and its input parameters by through literature review, independent calculation of thermal conductivity and consistency with published cake values, and prediction of brownie batter cooking time. In addition we tested a number of superellipses ranging from circle to square and ellipse to rounded rectangle and rhombi, determining utility values for each pan as a function of E, H, and p. In addition we did sensitivity analysis on the value of area for each pan we set.

9.1 Strengths and Weaknesses

We identify a number of strengths and weaknesses with our model, which we list below.

9.1.1 Strengths

- Accurately predicts brownie cook time.
- Metrics produce results consistent with intuition and problem statement. Pans with corners cook less evenly than pans without corners.
- Internal temperature is a good value with which to measure the state of the brownies because cooks are able to measure directly measure temperature and determine when their brownies are done.
- Agrees with Baker's Edge faster cooking time claim.

9.1.2 Weaknesses

- We use a quasi-3D model which exploits thermal symmetry in the z direction based on the high thermal conductivity of the pan. The model doesn't fully model 3D heat transfer and may not capture variations in cookedness in the z direction which may contribute to overcookedness.
- Assumes oven temperature is constant. In reality, the best thermal control loops are capable of maintaining oven temperature to within 2 °C [8], however significantly higher variations exist in ovens without these.
- The discretization means that our different pans do not have exactly the same area. We saw variations on the order of 5% for the worst case: very long, thin pans for which the addition or subtraction of a row along the long edge represents a lot of area. This will also affect any calculation in which area is used.
- Several parameter values were estimated from other baked goods, due to lack of data for brownies. Attempts were made to validate uncertain values by calculating known brownie values, and produced good agreement. In addition the model itself was validated by predicting cooking time based on typical cooking temperatures and produced results consistent with typical baking times. Results may be subject to variation in brownie recipe, etc.
- We assumed the parameter values for the brownie batter such as specific heat, density, and thermal diffusivity remained constant over time. A more realistic model would vary these to better model the change in batter composition over the cooking process. Thermal diffusivity tends to increase while baking which would cause better heat penetration from the sides over time, expanding the colored borders see in temperature and evenness plots. This would likely decrease evenness, as a wider border indicates causes additional higher cookedness values.

In general, we find that the evenness of a brownie is not strongly influenced by the shape of the pan for equal and low stretch ($a/b = 1$ or small). Corner and edge values have similar cookedness values on the order of 400 J-m. Since the corners were defined as overdone in the problem statement, it seems that all edges end up being overdone and so an optimal shape to maximum evenness is a circle simply due to its lower perimeter. It is clear the baking process is dominated by heat transfer in the z direction instead of transfer from the sides of the pan, as heat only penetrated 0.5cm inward from the edge and only affected 7% area of the circular pan, as measured by significantly elevated temperatures compared to the center. Excepting pans such as Baker's Edge [6], whose purpose is to increase the perimeter of the brownie to produce additional crispy brownies for those who enjoy them, the majority of the brownie is not affected by heat transfer from the edge.

For pans with extremely high a/b ($a/b > 10$), evenness is significantly increased because the brownies cook faster. At this scale, heat from the sides of the pan is transmitted to the center and increases the temperature, reducing the time required to reach 90°C . These pans, however, are impractical in that they are thin and long, often longer than a conventional oven and thinner than a typical brownie slice.

For a typical brownie batch ($A = 400$ cm^2) and equal stretch we find the optimal residential pan shape to be a superellipse with $n = 6.5$ and $n = 2.2$ for commercial ovens. Figure 20 can be used to determine the optimal pan selection for other values of A and Figure 19 can be used to determine optimal pan for variable a/b.

Additional work might include a full 3D treatment of heat flow, empirical testing of brownie batter to confirm parameter values, and testing of additional classes of shapes known as supershapes [14] or interlocking shapes such as offset sine curves.

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A Calculations

Superellipse Area

To find the area of superellipses, we exploit symmetry in both the x and y directions and restrict our attention to the first quadrant. Thus the area is four times the area of the shape in the first quadrant [21]

$$A = 4\int_0^a b\left(1-\left(\frac{x}{a}\right)^n\right)^{\frac{1}{n}} dx = 4ab \cdot \frac{\Gamma\left(1+\frac{1}{n}\right)^2}{\left(1+\frac{2}{n}\right)}$$

where Γ is the gamma function, a continuous extension of the factorial function.[2]

[2] $\Gamma(x) = \int_0^\infty t^{x-1} \cdot e^{-t} dt$, and for positive integer x, $\Gamma(x) = (x-1)!$. For all positive x, Γ satisfies $\Gamma(x+1) = x \cdot \Gamma(x)$

Superellipse Bounding Rectangles

For superellipses with even stretching ($a = b = 1$), we considered two choices of bounding rectangles: one choice with sides parallel to the x and y axes (as defined by our parameterization above) and one with sides forming 45° angles with the axes (see figure4), Here we calculate the areas of these rectangles.

The rotated rectangle makes a 45° angle with the axes and intersects the superellipse at four points. In the first quadrant this intersection occurs at $\theta = \pi/4$, which for $a = b = 1$ is at

$$x = y = \cos^{2/n}\left(\frac{\pi}{4}\right) = \sin^{2/n}\left(\frac{\pi}{4}\right) = \left(\frac{1}{2}\right)^{1/n}$$

Since this rectangle's sides form a 45° degree angle with the axes, the line segment connecting this point and the origin is perpendicular to the rectangle's side in the first quadrant and therefore parallel to the rectangle's sides in the second and fourth quadrants. Therefore, the length of this rectangle's side is twice the distance between this point and the origin, that is $A = (2d)^2$ where

$$d = \sqrt{\left(\left(\frac{1}{2}\right)^{1/n}\right)^2 + \left(\left(\frac{1}{2}\right)^{1/n}\right)^2} = \sqrt{2 \cdot (2^{-1/n})^2} = \sqrt{2^{1-\frac{2}{n}}}$$

Thus the area of the rotated rectangle is $A = (2d)^2 = 4 \cdot 2^{1-\frac{2}{n}} = 2^{3-\frac{2}{n}}$.

Since the area of the outer, non-rotated rectangle is $(2a) \cdot (2b) = 4ab = 4a^2 = 4$ square units (where we assume $a = b$ is one unit), we can use our formula to verify which bounding rectangle is better. Whenever $A = 2^{3-\frac{2}{n}} < 4 = 2^2$ we should use the rotated rectangle. This occurs when $3 - \frac{2}{n} < 2$, which corresponds to $n < 2$ as we expected.