CSCI 2200 HW3
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**Problem 13.10:** From 10 students, in how many ways can you choose a president and vice-president? What if two students are identical twins in every possible way? What if three students are identical triplets in every possible way?

We are counting sequence of length 2 without repetition for 10 object, so there are $\frac{10!}{(10-2)!} = 10 \times 9 = 90$ ways of choosing a president and vice-president. If two students are identical twins in every possible way, then the possibilities are $10 \times 9 - (9+8) = 73$. If there are three identical twins, then there are $10 \times 9 - (9+8+9) = 64$ ways.

**Problem 11.13:** How many functions $f : \{1, ..., 5\} \rightarrowtail \{1, ..., 10\}$ are: (a) Strictly increasing? (b) Non-decreasing?

(a). If the function is strictly increasing, then we have $f(1) < f(2) ... < f(n)$ and each value in the domain has to be mapped into a value in the range. Therefore we are taking 5 out of 10 values which gives us $\binom{10}{5} = 252$

(b). Likewise, for every $x$, we have $f(1) \leq (2) ... \leq f(n)$ and each value in the domain has to be mapped into a value in the range. Therefore the number of non-decreasing functions $= \binom{5+9}{9} = 2002$.

If random variable X has pgf $G(z) = \sum_{k=1}^{\infty} akz^k$ where a is either 0 or 1, find $p(X \leq 7)$.

**Problem 13.53(h):** The streets in a neighborhood form a rectangular grid. A child starts at home and walks to school which is 10 blocks east and 10 blocks north. How many shortest paths are there?

Ultimately it takes 10 steps north and 10 steps east to get to school, so there are 20 choices and we are choosing 10 from the 20 options. Therefore the number of possibilites is

$$\binom{20}{10} = 184756$$

Find the coefficients of $x^3$ and $x^{17}$ in: (a) $(1+\sqrt{x}+x+x^2)^{10}$ (b) $(x^{1/2}+x^{3/2}+x^{7/2})^{10}$.

The general formula is

$$\sum_{k_1+k_2+...+k_m=n} \binom{n!}{k_1, k_2, ..., k_m} \prod_{i=1}^{m} x_i^{k_i}$$

(a). At $x^3$, there are 6 possibilities. (1).$k_1 = 7, k_2 = 2, k_3 = 2, k_4 = 0$ (2).$k_1 = 8, k_2 = 1, k_3 = 0, k_4 = 1$ (3).$k_1 = 7, k_2 = 0, k_3 = 2, k_4 = 1$ (4).$k_1 = 7, k_2 = 3, k_3 = 0, k_4 = 0$ (5).$k_1 = 4, k_2 = 0, k_3 = 6, k_4 = 0$ (6).$k_1 = 5, k_2 = 1, k_3 = 4, k_4 = 0$. This gives us the coefficient of $x^3$ as $\frac{10!}{2!2!6!} + \frac{10!}{1!2!7!} + \frac{10!}{1!1!8!} + \frac{10!}{7!3!} + \frac{10!}{4!6!} + \frac{10!}{1!4!5!} = 3300$.

Likewise, there are 3 possibilities at $x^17$. (1).$k_1 = 1, k_2 = 1, k_3 = 0, k_4 = 8$ (2).$k_1 = 0, k_2 = 0, k_3 = 2, k_4 = 8$ (3).$k_1 = 0, k_2 = 3, k_3 = 0, k_4 = 7$. This gives us the coefficient of $x^{17}$ as $\frac{10!}{2!8!} + \frac{10!}{1!8!1!} + \frac{10!}{7!3!} = 255$.

(b). At $x^3$, we need to satisfy the following system of equations:

$$\begin{cases} k_1 + k_2 + k_3 & = 10 \\ \frac{1}{2}k_1 + \frac{3}{2}k_2 + \frac{7}{2}k_3 & = 3 \end{cases}$$

Which gives us the solution:

$$\begin{cases} k_2 = 16 - \frac{3k_1}{2} \\ \\ k_3 = -6 + \frac{k_1}{2} \end{cases}$$

Since $k_3 \geq 0$, $k_1 \geq 12$, which is impossible. Therefore there does not exists a term $x^3$ in the expansion. Thus coefficient $= 0$.

Likewist at $x^17$, we need to satisfy the following system of equations:

$$\begin{cases} k_1 + k_2 + k_3 & = 10 \\ \frac{1}{2}k_1 + \frac{3}{2}k_2 + \frac{7}{2}k_3 & = 17 \end{cases}$$

Which gives us the solution:

$$\begin{cases} k_2 = 9 - \frac{3k_1}{2} \\ \\ k_3 = 1 + \frac{k_1}{2} \end{cases}$$

by substuting $k_1 = 0, 2, 4, 6$ into $k_2$ and $k_3$ gives us 4 possibilities 1).$k_1 = 0, k_2 = 9, k_3 = 1$ (2).$k_1 = 2, k_2 = 2, k_3 = 6$ (3).$k_1 = 4, k_2 = 3, k_3 = 3$ (4).$k_1 = 6, k_2 = 0, k_3 = 4$. This gives us the coefficient of $x^{17}$ as $\frac{10!}{9!1!} + \frac{10!}{2!2!6!} + \frac{10!}{4!3!3!} + \frac{10!}{6!4!} = 5680$.

Problem 14.26. How many of the billion numbers $0, ..., 999999999$ contain a 1? Solve this problem in three ways:
(a) Compute how many do not contain a 1 and subtract from 1 billion ?

Since it does not contain 1, then every digit have 9 options instead of 10.The t otal number of possibilities that does not contain a one is 387420489. Substracting from $10^9$ gives us 612579511.

(b) Compute how many contain 1 one, 2 ones, . . . , 9 ones and then add them up ?

If we fix one 1 and let others vary, there are $\frac{9!}{1!8!} \times 9^8$ possibilities. It is the same for all the 1s. Therefore the total possibilities is

$$\sum_{i=0}^{8} \left( \frac{9!}{i!(9-i)!} \right) 9^i$$

which gives us the same value 612579511(credit to Mathematica).
(c) Let $A_i = \{$numbers in which the $i^{th}$ digit is one$\}$. Compute $|A_1 \cup A_2 \cup A_3 ... \cup A_9|$.

The general inclusion-exclusion principle is

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{k=1}^{n} (-1)^{k+1} \left( \sum_{1 \leqslant i_1 lt; \cdots lt; i_k \leqslant n} |A_{i_1} \cap \cdots \cap A_{i_k}| \right)$$

The formula for $|A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5 \cap A_6 \cap A_7 \cap A_8 \cap A_9|$ is therefore equal to sum of each digit is 1 minus to digits is 1 minus two digits are 1 plus 3 digits are 1 minus 4 digits are 1 plus 5 digits are 1 minus 6 digits are 1 plus 7 digits are 1 minus 8 digits are 1 plus 9 digits is 1. We can represent it as

$$\sum_{i=1}^{9} (-1)^{i+1} 9^i \binom{9}{i}$$

which is 134217729.