Three parts of a numerical method:

1. Input of a problem: e.g. data (given, should not be changed in the numerical method)

   initial guess of solution (e.g. randomly generated)

2. update scheme: how to renew the guess

   key part

3. Stopping condition: e.g. max running time

   max number of updates

   based on optimality condition

Example: find a solution to the linear system

$$\begin{cases} 2x_1 + x_2 = 3 \\ -x_1 + 2x_2 = 1 \end{cases} \Rightarrow x^* = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

1. data of the problem,

$$A = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

initial guess: $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

2 update scheme:

Suppose the current guess is $x^{(k)} = \begin{bmatrix} x_1^{(k)} \\ x_2^{(k)} \end{bmatrix}$, $k \geq 0$

$$\begin{cases} x_1^{(k+1)} = (3 - x_2^{(k)})/2 \\ x_2^{(k+1)} = (1 + x_1^{(k)})/2 \end{cases}, \quad k = 0, 1, \ldots \qquad (1)$$

3. Stopping condition:

$$\| A x^{(k)} - b \| \le \underbrace{tol}_{\uparrow}$$

a small positive number

Following the update scheme in (1):

$k=0$, $x^{(0)} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \begin{cases} x_1^{(1)} = (3 - x_2^{(0)})/2 = 3/2 \\ x_2^{(1)} = (1 + x_1^{(0)})/2 = 1/2 \end{cases}$

$$A x^{(1)} - b = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 7/2 \\ -1/2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ -3/2 \end{bmatrix}$$

So $\| A x^{(1)} - b \| = \dfrac{\sqrt{10}}{2}$

$k=1$, $x^{(1)} = \begin{bmatrix} 3/2 \\ 1/2 \end{bmatrix} \Rightarrow \begin{cases} x_1^{(2)} = (3 - x_2^{(1)})/2 = 5/4 \\ x_2^{(2)} = (1 + x_1^{(1)})/2 = 5/4 \end{cases}$

$$A x^{(2)} - b = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 5/4 \\ 5/4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 15/4 \\ 5/4 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 3/4 \\ 1/4 \end{bmatrix}$$

So $\| A x^{(2)} - b \| = \dfrac{\sqrt{10}}{4}$

$k=2$, $x^{(2)} = \begin{bmatrix} 5/4 \\ 5/4 \end{bmatrix} \Rightarrow \begin{cases} x_1^{(3)} = (3 - x_2^{(2)})/2 = 7/8 \\ x_2^{(3)} = (1 + x_1^{(2)})/2 = 9/8 \end{cases}$

$$Ax^{(3)} - b = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} 7/8 \\ 9/8 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 23/8 \\ 11/8 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/8 \\ 3/8 \end{bmatrix}$$

So $\|Ax^{(3)} - b\| = \frac{\sqrt{10}}{8}$

Claim: $\|Ax^{(k)} - b\| = \frac{\sqrt{10}}{2^k}$

Rate of Convergence

Def (Q-linear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is Q-linear convergent if there is $r \in (0,1)$, such that

$$\frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} \le \underbrace{r}_{\text{called the linear rate}}, \quad \text{when } k \text{ is large enough.}$$

Example: $z^{(k)} = \|Ax^{(k)} - b\| = \frac{\sqrt{10}}{2^k}$ (from previous example)

$$z^* = \lim_{k \to \infty} z^{(k)} = 0$$

$$\frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} = \frac{\left|\frac{\sqrt{10}}{2^{k+1}} - 0\right|}{\left|\frac{\sqrt{10}}{2^k} - 0\right|} = \frac{1}{2}, \quad \forall k \ge 0$$

So $\{z^{(k)}\}$ Q-linearly converges to 0.

Def (Q-superlinear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is Q-superlinear convergent if

$$\lim_{k\to\infty} \frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} = 0$$

Example: $z^{(k)} = 1 + k^{-k}$, $k = 1, 2, \ldots$

$$z^* = \lim_{k\to\infty} z^{(k)} = 1$$

$$\frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} = \frac{(k+1)^{-(k+1)}}{k^{-k}} = \frac{(k+1)^{-1}\cdot(k+1)^{-k}}{k^{-k}} = \frac{1}{k+1}\left(\frac{k}{k+1}\right)^k \to 0$$

So $z^{(k)}$ Q-superlinearly converges to 1.

Def (Q-sublinear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is Q-sublinear convergent if

$$\lim_{k\to\infty} \frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} = 1$$

Example: $z^{(k)} = 1 + \frac{1}{k}$, $k = 1, 2, \ldots$

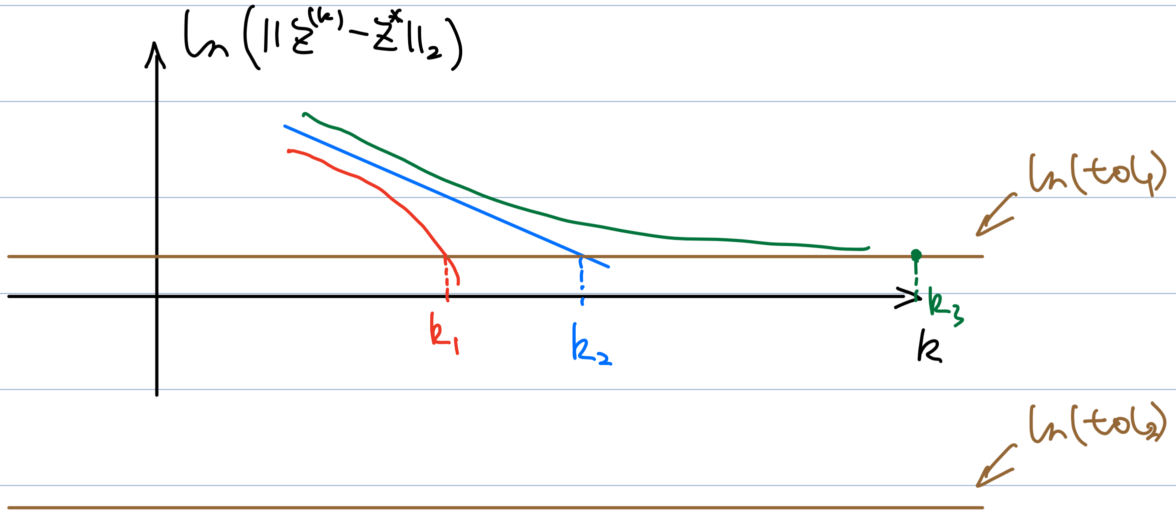
$$z^* = \lim_{k\to\infty} z^k = 1$$

$$\frac{\|z^{(k+1)}-z^*\|_2}{\|z^{(k)}-z^*\|_2} = \frac{\frac{1}{k+1}}{\frac{1}{k}} = \frac{k}{k+1} \to 1, \text{ as } k\to\infty$$

So $z^{(k)}$ Q-sublinearly converges to 1

Suppose $\frac{\|z^{(k+1)}-z^*\|_2}{\|z^{(k)}-z^*\|_2} \le r$. Then

$$\ln(\|z^{(k+1)}-z^*\|_2) - \ln(\|z^{(k)}-z^*\|_2) \le \ln r < 0$$



Def (Q-quadratic convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k\to\infty$. We say the sequence is Q-quadratic convergent if there is $M>0$, such that

$$\frac{\|z^{(k+1)}-z^*\|_2}{\|z^{(k)}-z^*\|_2^2} \le M, \quad \text{when } k \text{ is large enough.}$$

$$\Rightarrow \frac{\|z^{(k+1)}-z^*\|_2}{\|z^{(k)}-z^*\|_2} \le M\|z^{(k)}-z^*\|_2 \to 0$$

Remark: Q-quadratic $\Rightarrow$ Q-Superlinear

Q-Superlinear faster than Q-linear

Def (R-linear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is R-linear convergent if there is $\{v^{(k)}\}_{k=0}^{\infty}$ such that

$\|z^{(k)} - z^*\|_2 \le v^{(k)}$, and $\{v^{(k)}\}_{k=0}^{\infty}$ Q-linearly converges to 0

Example: $$z^{(k)} = \begin{cases} 1 + 1/2^k, & \text{if } k \text{ is even} \\ 1, & \text{if } k \text{ is odd.} \end{cases}$$

$z^{(k)} \to 1 = z^*$

$$\frac{\|z^{(k+1)} - z^*\|_2}{\|z^{(k)} - z^*\|_2} = \begin{cases} \dfrac{0}{1/2^k} = 0, & \text{if } k \text{ is even} \\ \dfrac{1/2^{k+1}}{0_+} = +\infty & \text{if } k \text{ is odd} \end{cases}$$

Let $v^{(k)} = \frac{1}{2^k}$, $\forall k \ge 0$

Then $\|z^{(k)} - z^*\|_2 \le v^{(k)}$, $\forall k \ge 0$

$v^{(k)} \to 0$

$$\frac{|v^{(k+1)} - 0|}{|v^{(k)} - 0|} = \frac{1/2^{k+1}}{1/2^k} = \frac{1}{2} \in (0,1)$$

So $\{v^{(k)}\}$ Q-linearly converges to 0, and $\{z^{(k)}\}$ R-linearly converges to 1

Def (R-superlinear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is R-superlinear convergent if there is $\{v^{(k)}\}_{k=0}^{\infty}$ such that $\|z^{(k)} - z^*\|_2 \leq v^{(k)}$, and $\{v^{(k)}\}_{k=0}^{\infty}$ Q-superlinearly converges to 0

Example:

$$z^{(k)} = \begin{cases} 1 + k^{-k} & \text{if } k = 2, 4, \ldots \\ 1, & \text{if } k = 1, 3, \ldots \end{cases}$$

R-superlinearly converges to 1

Def (R-sublinear convergence): Let $\{z^{(k)}\}_{k=0}^{\infty}$ be a sequence of vectors or scalars. Suppose $z^{(k)} \to z^*$, as $k \to \infty$. We say the sequence is R-sublinear convergent if there is $\{v^{(k)}\}_{k=0}^{\infty}$ such that $\|z^{(k)} - z^*\|_2 \leq v^{(k)}$, and $\{v^{(k)}\}_{k=0}^{\infty}$ Q-sublinearly converges to 0

Example:

$$z^{(k)} = \begin{cases} 1 + \frac{1}{k}, & \text{if } k = 2, 4, \ldots \\ 1 & \text{if } k = 1, 3, \ldots \end{cases}$$

R-sublinearly converges to 1

Remark: if $\{z^{(k)}\}$ is Q-linear convergent to z^*, then $\{z^{(k)}\}$ is also R-linear convergent to z^*