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hw02

1.
Claim: This graph does not have a cut-edge.
We prove this by contradiction.
Proof: Assume $\exists e \in E(G)$ such that e is a cut-edge. Cut edge e will give us 2 separate components, G_1 and G_2. Since $\forall v \in V(G)$ we have $dv = 2k$(even) where $k \in \mathbb{N}$, $\exists v \in V(G)$ such that $d(v) = 2k+1$(odd) after cutting the edge e. Thus we have a vertex $v \in V(G_1)$ in one component with an odd degree, and the rest of the vertices in G_1 has even degrees. Thus the sum of degree in G_1 is an odd number. By the hand shaking theorem, $|E(G_1)|$ will not be an integer, which is impossible. Thus we have found a contradiction. Therefore such cut-edge e does not exist.

2.
The figure on the left is the graph generated using the graphic sequence, where the vertices are arranged in increasing order. The figure on the right is the tree generated using Prüfer code.
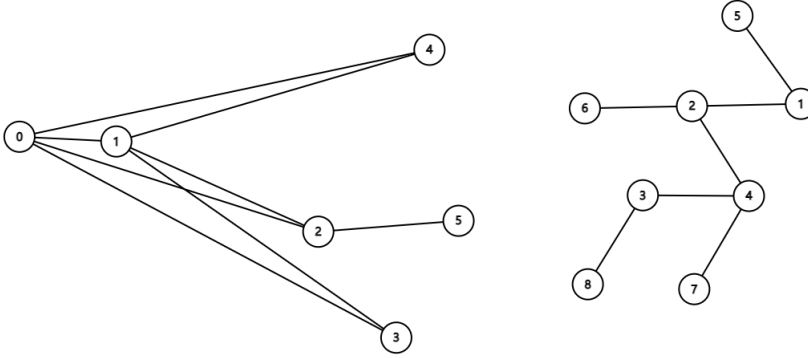
3.
(a).Let $P(n)$ denote the statment that the number of leaves l on a tree T with $|V(T)| = n$ is bounded below by the maximum degree of of the tree $k = \triangle(T)$.

   Base Case: $n = 1$. We have a graph with 1 vertex. This is trivial. $n = 2$, we have a graph with two leaf vertices and an *edge*. $k < l$. $n = 3$. we have a graph with 2 leaf vertices and 2 edges.

   Inductive Step: Assume $p(1 \leq k' < n)$ holds, and prove $p(n)$ holds.

   Consider the case $k' = n - 1$, and we have the tree T'. In order to prove $P(n)$ we need to add one vertex u to T'. We could only add a leaf since adding a non-leaf edge creats a cycle, which violates the tree's property. Assume the vertex $v \in V(T')$ has the largest degree. We consider two exhaustive cases:

   1. u is connected with v. In this case, the number of leaf vertices and the maximum degree both increased by one. Thus $k = k + 1 \wedge l = l + 1$. Since $l \geq k$ from our inductive hypothesis, $l + 1 \geq k + 1$. Thus in this case $P(n)$ holds.

   2. u is not connected with v. In this case, the number of leaf vertices is increased by one and the maximum degree stays the same. Thus $k = k \wedge l = l + 1$. Since $l \geq k$ from our inductive hypothesis, $l + 1 \geq k$. Thus in this case $P(n)$ holds.

   Therefore P(n) holds in all cases, which means the number of leaves on a tree T with $|V(T)| = n$ is bounded below by the maximum degree of of the tree.

(b). Let us denote the vertex with the largest degree $v \in V(G)$. Vertex v has k edges, which implies there are k unique paths since trees contains no cycle. The vertex that has the maximal $u - v$ path with vertex v is a leaf node, other wise the path could be extended and the path would not be maximal. Thus the k unique path will reach to at least k leaf nodes, or more if one vertex on the path has degree greater than 2. Therefore $l \geq k$ and the number of leaves on a tree T with $|V(T)| = n$ is bounded below by the maximum degree of of the tree.

(c). From the handshake theorem, we know $\sum_{v \in V(G)} deg(v) = 2|E|$. From the properties of trees, we know $|E| = |V| - 1$. For the non-leaf vertices, the degree of each vertex is at least two to stay connected. Let u denote the vertex with maximal degree in G, and let l denote the cardinality of the set of leaf vertices. Let leaf be the set of leaf vertices. Then we have the following formula.

$$\sum_{v \in V(G)} deg(v) \geq deg(u) + \sum_{v \in V(G) \setminus L \cup \{u\}} deg(v) + \sum_{v \in leaf} deg(v)$$
$$2(|E|) \geq k + 2(|V| - |leaf| - 1) + |leaf|$$
$$2(|V| - 1) \geq k + 2|V| - 2l - 2 + l$$
$$l \geq k$$

Therefore $l \geq k$ and the number of leaves on a tree T with $|V(T)| = n$ is bounded below by the maximum degree of of the tree.

4.

We prove case one by contradiction. Assume $|V(B_1)| \geq |V(B_2)|$ and there does not exist a leaf vertex in B_1, which means $\nexists v_1 \in V(B_1)$ such that $deg(v_1) = 1$. Then $\forall v_1 \in V(B_1)$, $deg(v_1) \geq 2$. Therefore $\sum_{v_1 \in V(B_1)} deg(v_1) \geq 2|V(B_1)|$. Since $\forall v_1 \in V(B_1)$, v_1 is connected to $v_2 \in B_2 \setminus B_1$, $\sum_{v_2 \in V(B_2)} deg(v_2) = \sum_{v_1 \in V(B_1)} deg(v) \geq 2|V(B_1)|$. Thus we have $\sum_{v \in V(B_1) \cup V(B_2)} deg(v) \geq 4|V(B_1)|$. By the handshake theorem, there are $|E'| = 2|V(B_1)|$ edges. According to the properties of tree, $|E| = |V(B_1)| + |V(B_2)| - 1$. $|E'|$ must equal to $|E|$ by the property of tree, which means

$$|V(B_1)| + |V(B_2)| - 1 = 2|V(B_1)|$$
$$|V(B_1)| = |V(B_2)| - 1$$

From our assumption, we know $|V(B_1)| \geq |V(B_2)|$. Therefore we have a contradiction. Then there must $\exists v_1 \in V(B_1)$ such that $deg(v_1) = 1$, which means there must exist a leaf vertex in B_1. When $|V(B_1)| = |V(B_2)|$, this holds for both components and thus each of them must contain a leaf node.

5. Since graph G is connected, $|E(G)| \geq |V(G)| - 1$. Because $|V(G)| > |E(G)|$, it could only be the case that $|E(G)| = |V(G)| - 1$. Therefore G is a tree. Assume the maximal length path $P_1, P_2 \subseteq G$ have no intersections and $|P_1| = |P_2| = k$. Let us denote the set $P_1 = \{u_1, ..., u_k\}$ and the set $P_2 = \{v_1, ..., v_k\}$. For $u_i \in P_1$ and $v_j \in P_2$, there exists a unique path $p_3 = \{q_1, ..., q_n\}$ connecting u_i and v_j. $\exists a \in p_3$ and $a \notin P_1 \cup P_2$ since by definition P_1 and P_2 are not connected. Thus we have the non-empty set $Q = \{v : v \in P_3 \setminus (P_1 \cup P_2)\}$. Let us denote the path in P_1 starting from the end vertex in P_1 and end with vertex u_i with length greater than or equal to $\dfrac{k}{2}$ as P_1'. Like wise we have P_2'. Since $|P_1'| + |P_2'| + |Q| = \dfrac{k}{2} + \dfrac{k}{2} + |Q| \geq k$, we know P_1 and P_2 are not the longest path. Thus there's a contradiction and thus P_1 and P_2 must have common vertices.

6. We have the following recurrence relation.
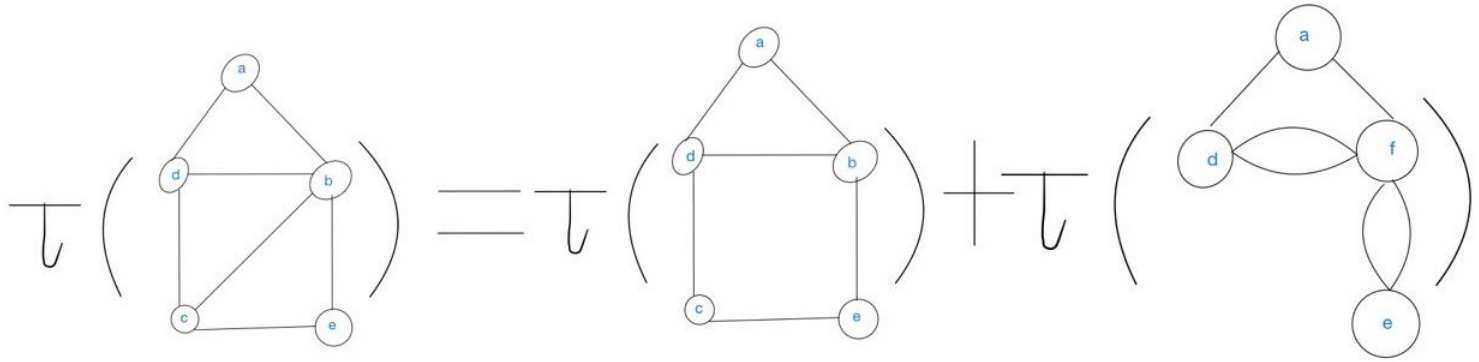
The spanning trees for the first graph on the RHS are 11 as we discussed in lecture 7. The spanning trees for the second graph on the RHS is the sum of the number of spanning trees for the following graphs.
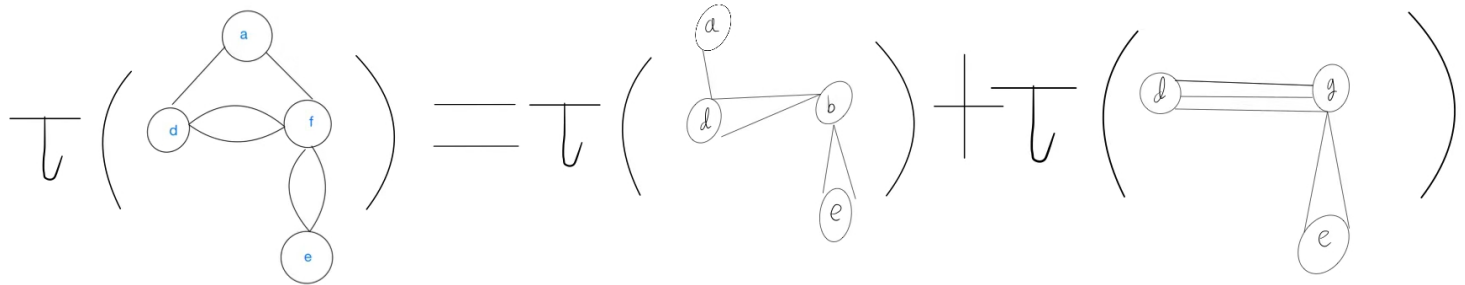
The number of spanning trees on the first graph on RHS is $2 \times 2 = 4$ because there are two edges connecting d and b and two edges connecting d and e. The number of spanning trees on the second graph on RHS is $3 \times 2 = 6$ because there are three edges connecting d and g and two edges connecting g and e.

Thus there are in total $11 + 4 + 6 = 21$ spanning trees for the graph.

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7.

(a).

$v \in G$	E_1	E_2	E_3	E_4	E_5	E_6
a	0	0	0	0	0	0
b	8	7	7	7	7	7
c	∞	∞	∞	∞	8	8
d	1	1	1	1	1	1
e	∞	∞	∞	12	10	10
f	∞	3	3	3	3	3
g	∞	∞	6	6	6	6

The order of processing order of verticies is $\{a, d, f, g, b, c, e\}$

(b). We connect the smallest edges.

iteration 1: connect ad

iteration 2: connect df

iteration 3: connect fg

iteration 4: connect gb

iteration 5: connect be

iteration 6: connect ec

Thus we have the spanning tree $T = \{ad, df, fg, gb, be, ec\}$.