CSCI 2200 HW3
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Problem 4.27(c) $\forall n : (P(n) \implies Q(n)$

Prove: You can prove it use direct proof. Show that Q(n) cannot be false when P(n) is true.

Disprove: You can disprove it by showing that Q(n) is false when P(n) is true.

Problem 4.27(d) $\forall x : ((\forall n : (P(n))) \implies Q(x))$

Prove:Firstly prove $P(n)$ is correct because for all n, $P(n)$ must be true. Since the first part of the if then statement is correct, $Q(x)$ must also be true

Disprove: Show either $P(n)$ is false, or $Q(n)$ is false.

Problem 5.12(h) For $n \geq -1$, prove by induction:

$$10^0 + 10^1 + 10^2 + ... + 10^n < 10^{n+1}$$

Base Case: n = 1

$10^0 + 10^1 = 11$

$10^2 = 100$

$\because 11 \leq 100$

$\therefore$ the base case holds.

Inductive step:

Assume: $10^0 + 10^1 + ... + 10^n \leq 10^{n+1}$ holds

Prove: $10^0 + 10^1 + ... + 10^{n+1} \leq 10^{n+2}$

$10^0 + 10^1 + ... + 10^n + 1 \leq 10^{n+1} + 10^{n+1}$

$10^0 + 10^1 + ... + 10^n + 1 \leq 2 \cdot 10^{n+1}$

$10^{n+2} = 10^{n+1} \cdot 10 = 10 \cdot 10^{10^n + 1}$

$10^{n+2} = 10^{n+1} \cdot 10 = 10 \cdot 10^{n+1}$

$\because 10 \cdot 10^{n+1} \geq 2 \cdot 10^{n+1} \therefore 10^0 + 10^1 + ... + 10^n + 1 \leq 10 \cdot 10^{n+1}$

$\leq 10^{n+2}$

$\therefore 10^0 + 10^1 + ... + 10^n + 1 \leq 10^{n+2}$

$\square$

Problem 5.12(J) is on the next page

Problem 5.12(j) For $n \geq -1$, prove by induction:

$$(1+x)^n \geq 1+nx \text{ when } x \geq -1$$

Base Case: n = 1

1+x = 1+x $\therefore$ the base case holds.

Inductive step:

Assume: $(1+x)^n \geq 1+nx$ holds

Prove: $(1+x)^{n+1} \geq 1+(n+1)x$

$(1+x)^n \cdot (1+x) \geq (1+nx) \cdot (1+x)$

$(1+x)^{n+1} \geq (1+x+nx+nx^2)$

$\because (1+x+nx+nx^2) - (1+nx+x) = nx^2$ and nx^2 because $x^2 \geq 0$ and $n \geq 1$

$\therefore 1+x+nx+nx^2 \geq (1+nx+x)$

$\therefore (1+n)^{n+1} \geq 1+nx+x$

$\square$

Problem 5.26(f). Find a formula for the quantity of interest:

$$S(n) = \sum_{i=1}^{n} \frac{n}{(n+1)!}$$

$S(1) = \frac{1}{2!} = \frac{1}{2}$

$S(2) = \frac{1}{2!} + \frac{2}{3!} = \frac{1}{2} + \frac{2}{6} = \frac{5}{6}$

$S(3) = \frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} = \frac{5}{6} + \frac{3}{24} = \frac{20}{24} + \frac{3}{24} = \frac{23}{24}$

The Assumption here is

$$\sum_{i=1}^{n} \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

Base case:

$$S(1) = \frac{(1+1)! - 1}{(1+1)!} = \frac{1}{2}$$

We have verified the base case is valid

Assume:

$$\sum_{i=1}^{n} \frac{n}{(n+1)!} = \frac{(n+1)! - 1}{(n+1)!}$$

Prove:

$$\sum_{i=1}^{N+1} \frac{n+1}{(n+2)!} = \frac{(n+2)! - 1}{(n+2)!}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n+1}{(n+2)!} = \frac{(n+1)! - 1}{(n+1)!} + \frac{n+1}{(n+2)!}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n+1}{(n+2)!} = \frac{((n+1)! - 1) \cdot (n+2)}{(n+2)!} + \frac{n+1}{(n+2)!}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n+1}{(n+2)!} = \frac{((n+2)! - (n+2)) + (n+1)}{(n+2)!}$$

$$\frac{1}{2!} + \frac{2}{3!} + \frac{3}{4!} + ... + \frac{n+1}{(n+2)!} = \frac{((n+2)! - 1}{(n+2)!}$$

$$\therefore \sum_{i=1}^{N+1} \frac{n+1}{(n+2)!} = \frac{(n+2)! - 1}{(n+2)!}$$

$\square$

Problem 5.71. Distinct numbers $X_1, ..., X_n$ must be placed in n boxes separated by $<$ and $>$ signs so that all inequalities are obeyed between consecutive numbers. For example, $5, 7, 3, 8$ can be placed in $\square < \square > \square < \square$ as follows, $5 < 8 > 3 < 7$. Prove this can always be done no matter what the numbers and inequalities are.

$P(n)$: n distinct numbers can be placed in boxes separeted by $n-1$ inequality signs for all $n \in \mathbb{N}$.

Base case: $P(1)$: 1 distinct number can be placed in boxes separeted by 0 inequality sign.

Let us pick an arbitray number X. We can prove that X can be placed on its own with 1 box and no inequality signs.

$$X$$

Therefore we have verified the base case.

Inductive step:

Assume: $P(n)$ holds

Prove: $P(n+1)$: $n+1$ distinct numbers can be placed in boxes separeted by n inequality signs for all $n+1 \in \mathbb{N}$.

The inequality connecting Box_{n-1} and box_n can either be $>$ or $<$. Consider two exhuastive cases.

Case 1: The inequality sign is $<$. Find the largest number X_{max} in $X_1, X_2, ...X_n$. Then, the other n numbers can be placed in the first n boxes(assumption in inductive steps). Since X_{max} is larger than any number in $X_1, X_2, ...X_n$, we can always place it in the last box since the last inequality sign is $<$.

Case 2: The inequality sign is $>$. Find the largest number X_{min} in $X_1, X_2, ...X_n$. Then, the other n numbers can be placed in the first n boxes(assumption in inductive steps). Since X_{min} is smaller than any number in $X_1, X_2, ...X_n$, we can always place it in the last box since the last inequality sign is $>$.

Hence, we have shown that p(n+1) is true when p(n) is true.

$\square$

Problem 6.6. Let $H_n = 1/1 + 1/2 + \ + 1/n$, the nth Harmonic number, and $S_n = H_1/1 + H_2/2 + \ + H_n/n$. (a) Prove $S_n \leq H_n^2/2 + 1$ by induction. What goes wrong? (b) Prove the stronger claim $S_n \leq H_n^2/2 + (1/12 + 1/22 + \ + 1/n^2)/2$. Why is this stronger?

(a). Base case: $n = 1$, $H_1 = 1$, $S_1 = H_1/1 = 1$, $\therefore S_1 \leq H_n^2/2 + 1$

Inductive Step:

Assume: $S_n \leq H_n^2/2 + 1$

Prove: $S_{n+1} \leq H_{n+1}^2/2 + 1$

$$H_{n+1}^2 = \frac{H_n^2}{2} + \frac{H_n}{n+1} + \frac{1/2}{(n+1)^2}$$

$$S_{n+1} \leq \frac{H_n^1}{2} + 1 + \frac{H_{n+1}}{n+1}$$

$$S_{n+1} \leq \frac{H_n^1}{2} + 1 + \frac{H_n + \frac{1}{n+1}}{n+1} + 1$$

$$S_{n+1} \leq \frac{H_n^1}{2} + 1 + \frac{H_n + \frac{1}{n+1}}{n+1} + 1$$

$$S_{n+1} \leq \frac{H_n^2}{2} + 1 + \frac{H_n}{n+1} + \frac{1}{(n+1)^2} + 1$$

$$\because \frac{H_n^2}{2} + 1 + \frac{H_n}{n+1} + \frac{1}{(n+1)^2} \geq \frac{H_n^2}{2} + \frac{H_n}{n+1} + \frac{1/2}{(n+1)^2}$$

$$\therefore S_{n+1} \nleq H_{n+1}^2/2 + 1$$

(b). Base case: $n = 1$, $H_1 = 1$, $S_1 = H_1/1 = 1$, $\therefore S_1 \leq H_n^2/2 + \frac{\frac{1}{1^2}}{2}$

Inductive Step:

Assume: $S_n \leq \frac{H_n^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})}{2}$

Prove: $S_{n+1} \leq \frac{H_{n+1}^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{(n+1)^2})}{2}$

$$S_{n+1} \leq \frac{H_n^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})}{2} + \frac{H_{n+1}}{n+1}$$

$$S_{n+1} \leq \frac{H_n^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})}{2} + \frac{H_n + \frac{1}{n+1}}{n+1}$$

$$S_{n+1} \leq \frac{H_n^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})}{2} + \frac{H_n}{n+1} + \frac{1}{(n+1)^2}$$

$$S_{n+1} \leq \frac{H_n^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{n^2})}{2} + \frac{H_n}{n+1} + \frac{1/2}{(n+1)^2} + + \frac{1/2}{(n+1)^2}$$

$$S_{n+1} \leq \frac{H_{n+1}^2}{2} + \frac{(\frac{1}{1^2} + \frac{1}{2^2} + ... + \frac{1}{(n+1)^2})}{2}$$

It is stronger because it is more precise. It replace a sequence of number greater than one.

$\square$