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MATH-2400 INTRODUCTION TO DIFFERENTIAL EQUATIONS SPRING 2021

Homework-5

Assigned Friday March 5,2021

Due 5 PM, Friday March 12, 2021

NOTES

1. Practice problems listed below and taken from the textbook are for your own practice, and are not to be turned in.
2. Legible, handwritten solutions will be acceptable, but the use of a typesetting system such as LaTeX is strongly recommended. **Do not turn in your rough attempt at solving a problem; once you have worked out the solution, copy it neatly or typeset it before submission, after removing all false starts.**
3. Please write your solutions clearly and coherently, with the work displayed in a sequential manner and sufficient explanation provided so that your strategy and approach are transparent to the reader.
4. Figures, if any, should be neatly drawn by hand, properly labelled and captioned.
5. The assignment is to be submitted electronically to LMS as a single pdf file. Be sure that the pages are properly oriented and well lighted. **Please do not e-mail your homework submission to the TAs or the instructors.**

Practice Problems from the textbook (not to be turned in)

Exercises from Chapter 3, pages 58-59: 1(d,f,h,p), 2(c,d,g), 3(c,j).
Exercises from Chapter 3, page 63: 1(c,d,e), 3(b).

Subjective part: problems to be turned in

1. (20 points) Solve the initial-value problem

$$y'' - 4y' = 4te^{4t} + 4t, \quad y(0) = 1,\ y'(0) = 1.$$

solution: Let us first consider the homogeneous DE first. Let $y = e^{rt}$, the characteristic equation is

$$r^2 - 4r = 0$$

which is equivalent to $r(r-4) = 0$. This gives us the result $r = 0$ or $r = 4$.

Thus $y_h(t) = C_1 + C_2e^{4t}$

Next we need to find a particular solution of the DE. We observed that the forcing function of DE consists of 2 parts. Let us denote y_{p1} as a particular solution to the DE $y'' - 4y' = 4te^{4t}$ and y_{p2} as a particular solution to the DE $y'' - 4y' = 4t$. Then $y_p = y_{p1} + y_{p2}$.

The guess for the solution to y_{p1} is $y_{p1} = (a + bt)t$ (since there is no y) and the guess for y_{p2} is $y_{p2} = (cte^{4t} + de^{4t})t$. Then the solution is $y_p = (a + bt)t + (cte^{4t} + de^{4t})t$.

Then we have

$$y'(t) = a + 2bt + de^{4t} + 4de^{4t}t + 4ce^{4t}t^2 + 2e^{4t}t$$
$$y''(t) = 2b + c(2e^{4t} + 16e^{4t}t^2 + 16e^{4t}t) + d(8e^{4t} + 16e^{4t}t))$$

Then let us substitute $y'(t)$ and $y''(t)$ into the DE,

$$y'' - 4y' = 4te^{4t} + 4t$$

$$2b + c(2e^{4t} + 16e^{4t}t^2 + 16e^{4t}t) + d(8e^{4t} + 16e^{4t}t)) - 4(a + 2bt + de^{4t} + 4de^{4t}t + 4ce^{4t}t^2 + 2e^{4t}t) = 4te^{4t} + 4t$$

$$-4a + 2b + (4d + 2c)e^{4t} - 8bt + 8ce^{4t}t = 4t + 4e^{4t}t$$

Then we have the following system of equations

$$\begin{aligned} -4a + 2b &= 0 \\ 4d + 2c &= 0 \\ -8b &= 4 \\ 8c &= 4 \end{aligned}$$

Then we have the following solutions $a = -\frac{1}{4}, b = -\frac{1}{2}, c = \frac{1}{2}$, and $d = -\frac{1}{4}$

Thus $y_p = -\frac{t^2}{2} + \frac{1}{2}e^{4t}t^2 - \frac{t}{4} - \frac{1}{4}e^{4t}t$

Then we have $y(t) = y_p(t) + y_h(t) = -\frac{t^2}{2} + \frac{1}{2}e^{4t}t^2 - \frac{t}{4} - \frac{1}{4}e^{4t}t + C_1 + C_2e^{4t}$.

Substituting $y(0) = 1$ gives us $c_1 + c_2 = 1$

$$y'(t) = -\frac{e^{4t}}{4} + 2e^{4t}t^2 - t + +4c_2e^{4t} - \frac{1}{4} \tag{1}$$

Substituting $y'(0)$ gives us $4c_2 - \frac{1}{2} = 1$

Thus we have $c_1 = \frac{5}{8}$ and $c_2 = \frac{3}{8}$

Then we have $y(t) = -\frac{t^2}{2} + \frac{1}{2}e^{4t}t^2 - \frac{t}{4} - \frac{1}{4}e^{4t}t + \frac{5}{8} + \frac{3}{8}e^{4t}$

After simplification we have $y(t) = \frac{1}{8}(-4t^2 - 2t + e^{4t}(4t^2 - 2t + 3) + 5)$

2. (20 points) Find the solution of the IVP

$$y'' + 4y' + 5y = 4e^{-2t}\cos t, \quad y(0) = 2, \;\; y'(0) = 0.$$

solution: Let us first consider the homogeneous DE first. Let $y = e^{rt}$, the characteristic equation is

$$r^2 + 4r + 5 = 0$$

Let us obtain the roots using the formula $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$. Thus the roots are $-2 \pm i$.
For $r = -2 + i$ the complex solution is

$$Y_1(t) = e^{(-2+i)t} = e^{-2t}cost + ie^{-2t}sint$$

Thus the general solution is $y_h(t) = C_1e^{-2t}cost + C_2e^{-2t}sint$.

Next we need to find a particular solution of the DE. We make an educated guess. Let $y_p = t(ae^{-2t}cost + be^{-2t}sint)$. Then we calculate y_p'

$$y_p' = ae^{-2t}cos(t) - 2ae^{-2t}tcos(t) - ae^{-2t}tsin(t) + be^{-2t}tcos(t) + be^{-2t}sin(t) - 2be^{-2t}tsin(t)$$

$y_p'' = -ae^{-2t}tcos(t) + acos(t)(-4e^{-2t} + 4e^{-2t}t) - 2a(e^{-2t} - 2e^{-2t}t)sin(t) + 2bcos(t)(e^{-2t} - 2e^{-2t}t)$
$-be^{-2t}tsin(t) + b(-4e^{-2t} + 4e^{-2t}t)sin(t)$

Substitute into the DE gives us:

$$y_p'' + 4y_p' + 5y_p = 4e^{-2t}cos(t)$$

$-ae^{-2t}tcos(t) + acos(t)(-4e^{-2t} + 4e^{-2t}t) - 2a(e^{-2t} - 2e^{-2t}t)sin(t) + 2bcos(t)(e^{-2t} - 2e^{-2t}t)$
$-be^{-2t}tsin(t) + b(-4e^{-2t} + 4e^{-2t}t)sin(t) + 4(ae^{-2t}cos(t) - 2ae^{-2t}tcos(t) - ae^{-2t}tsin(t) + be^{-2t}tcos(t) +$
$be^{-2t}sin(t) - 2be^{-2t}tsin(t)) + 5(t(ae^{-2t}cost + be^{-2t}sint)) = 4e^{-2t}cos(t)$

After simplication we have: $2be^{-2t}cos(t) - 2ae^{-2t}sin(t) = 4e^{-2t}cos(t)$

Then we know $2b = 4 \wedge -2a = 0$. Thus $a = 0 \wedge b = 2$. Therefore the particular solution is $y_p(t) = 2e^{-2t}tsin(t)$

Then the solution is $y(t) = C_1e^{-2t}cost + C_2e^{-2t}sint + 2e^{-2t}tsin(t)$.

Substitute $y(0) = 2$ in gives us $c_1 = 2$

We take the derivative of y, which gives us $y'(t) = 2e^{-2t}tcos(t) + 2e^{-2t}sin(t) - 4e^{-2t}tsin(t) - 2c_1e^{-2t}cos(t) - c_1e^{-2t}sin(t) + c_2e^{-2t}cos(t) - 2c_2e^{-2t}sin(t)$ Substitute $y'(0) = 0$ in gives us $-2c_1 + c_2 = 0$

Thus $c_1 = 2$ and $c_2 = 4$

Thus the DE is $y(t) = 2e^{-2t}(cos(t) + (t + 2)sin(t))$

3. (20 points) Consider the DE

$$ty'' - (1+t)y' + y = t^2e^{2t}, \quad t > 0.$$

(a) Check that $y_1(t) = 1 + t$ and $y_2(t) = e^t$ are solutions of the homogeneous DE. $y_1' = 1 \wedge y_1'' = 0 \wedge y_2' = e^t \wedge y_2'' = e^t$

Substitution: $t \times 0 - (1+t) + (1+t) == 0$

$te^t - (1+t)e^t + e^t == 0$ Thus they are the solution for homogenous DE.

(b) Use variation of parameters to find the general solution of the DE.

We seek the solution in the form $y_p = u(t)(1+t) + v(t)e^t$.

The DE can be written as $y'' - (\frac{1+t}{t})y' + (\frac{1}{t}y) = te^{2t}$

Then the wrosnskian gives us $e^t - e^t(t+1) = -te^t = y_1y_1' - y_2y_1'$

Then $u' = \frac{t^2e^{2t}(e^t)}{-te^t}$, $v' = \frac{t^2e^{2t}(t+1)}{-te^t}$

Therefore $u = \int \frac{t^2e^{2t}(e^t)}{-te^t} = \frac{1}{4}(1-2t)$ (perform an integrate by parts, where f = t,df = dt, g = $\frac{e^{2t}}{2}$, dg= e^{2t} dt) $v = \int \frac{t^2e^{2t}(t+1)}{-te^t} = e^t(t^2 - t + 1)$

Thus $y_p = \frac{1}{4}(1-2t) + e^t(t^2 - t + 1)$

Thus $y = C_1(1+t) + C_2e^t + \frac{1}{4}(1-2t) + e^t(t^2 - t + 1)$