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Homework 5

1. Let x, y and z be real numbers. Do the following
   - Prove that if $x \cdot z = y \cdot z$ and $z \neq 0$, then $x = y$.
   - Prove that if $x \neq 0$ then $x^2 > 0$.

1.**Proof:**

$$
\begin{aligned}
x \cdot y &= y \cdot z \\
(x \cdot z) \cdot \frac{1}{z} &= (y \cdot z) \cdot \frac{1}{z} && \text{by M1 since } x \cdot z = y \cdot z \text{ , } \frac{1}{z} = \frac{1}{z} \text{ , and } \frac{1}{z} \text{ exists since } z \neq 0 \\
x \cdot (z \cdot \frac{1}{z}) &= y \cdot (z \cdot \frac{1}{z}) && \text{by M3} \\
x \cdot 1 &= y \cdot 1 && \text{by M5} \\
x &= y && \text{by M4}
\end{aligned}
$$

2.**Proof:** By the trichotomy law, we consider three exhaustive cases. (i). $x > 0$ (ii) $x = 0$ (iii) $x < 0$. Since in this case $x \neq 0$, we only need to consider two exhaustive cases where $x > 0$ or $x < 0$.

(i) $x > 0$

$$
\begin{aligned}
0 \cdot x &< x \cdot x && \text{by } O4 \text{ since } 0 < x \text{ and } x > 0 \\
0 &< x \cdot x && \text{by Theorem } 3.22(b) \\
0 &< x^2 && \text{by the definition of } x^2
\end{aligned}
$$

Therefore we have shown that $x^2 > 0$ when $x > 0$.
(ii) $x < 0$

$$
\begin{aligned}
x \cdot x &> 0 \cdot x && \text{by Theorem } 3.22(b) \text{ since } x < 0 \text{ and } x < 0 \\
x \cdot x &> 0 && \text{by Theorem3.22}(b) \\
x^2 &> 0 && \text{by the definition of } x^2
\end{aligned}
$$

Therefore we have shown that $x^2 > 0$ when $x < 0$.
Therefore we have proved if $x \neq 0$ then $x^2 > 0$.

2. Let S and T be nonempty bounded subsets of $\mathbb{R}$ with $S \subseteq T$. Prove that $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.

Proof: In order to prove $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$, we need to show (i). $\inf(T) \leq \inf(S)$ (ii). $\inf(S) \leq \sup(S)$ (iii).$\sup(S) \leq \sup(T)$.

First we prove $\inf(T) \leq \inf(S)$. In oder to show $\inf(T) \leq \inf(S)$, we assume $\inf(T) > \inf(S)$. Since $\inf(T) > \inf(S)$, there exists $s' \in S$ such that $s' < \inf(T)$ by the definition of infimum. Since $s \subseteq T$ and $s' \in S$, $s' \in T$. Because $\inf(T) \leq t$ for all $t \in T$ and $s' \in T$, s' cannot be smaller than $\inf(T)$. Therefore $\inf(T) > \inf(S)$ leads to contradiction. Hence $\inf(T) \leq \inf(S)$.

Then we need to prove $\inf(S) \leq \sup(S)$. Let $s \in S$. By defintion of infimum, $\inf(S) \leq s$. By definition of supremem, $s \leq \sup(S)$. Therfore $\inf(S) \leq s \leq \sup(S)$. Hence we have shown $\inf(S) \leq \sup(S)$.

Finally we need to prove $\sup(S) \leq \sup(T)$. In order to show $\sup(S) \leq \sup(T)$, we assume $\sup(S) > \sup(T)$ which means $\sup(T) < \sup(S)$. By definition of supremem if $\sup(T) < \sup(S)$ then there exists $s' \in S$ such that $s' > \sup(T)$. Since $s \subseteq T$ and $s' \in S$, $s' \in T$. Because $\sup(T) \geq t$ for all $t \in T$ and $s' \in T$, s' cannot be greater than $\inf(T)$. Therefore $\sup(S) > \sup(T)$ leads to contradiction. Hence $\sup(S) \leq \sup(T)$.

Therefore we have proved $\inf(T) \leq \inf(S) \leq \sup(S) \leq \sup(T)$.

The function is clearly differentiable and thus continious on domain $[1, 9]$ Thus the conditions of MVT is satisfied.

$$
\begin{aligned}
f'(c) &= \frac{f(b) - f(a)}{b - a} \\
\frac{1}{2} \times \frac{1}{\sqrt{x}} &= \frac{3 - 1}{8} \\
\frac{1}{\sqrt{x}} &= \frac{1}{2} \\
x &= 2
\end{aligned}
$$

(b). NO Assume there exists a function f such that $f'(x) \geq 10$, $\forall x \in \mathbb{R}$. we know $\exists c \in \mathbb{R}$ such that $f'(c) = \dfrac{f(4) - f(0)}{4} = 1 \leq 10$. Therefore we found a contradiction thus there does not exist that function f such that $f'(x) \geq 10$, $\forall x \in \mathbb{R}$.