# Semidefinite Programming Relaxations of the Traveling Salesman Problem

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Joint work with Sam Gutekunst

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# Outline

1. Introduction: The Traveling Salesman Problem and Linear Programming

2. Semidefinite Relaxations of the Traveling Salesman Problem

3. Proof Sketch: An SDP with Unbounded Integrality Gap

4. Corollaries and Open Questions

# The Traveling Salesman Problem (TSP)

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Images from
www.math.uwaterloo.ca/tsp

## The (Symmetric, Metric) TSP

- Complete undirected graph $K_n$
- Edge costs $c_{ij}$ for distinct $i, j \in [n] = \{1, 2, ..., n\}$ with $c_{ij} = c_{ji}$ and $c_{ij} \leq c_{ik} + c_{kj}$ for all distinct $i, j, k$

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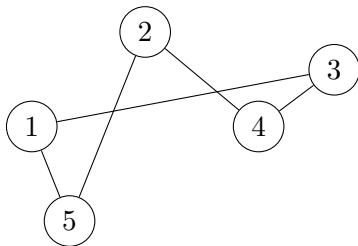


### Goal

Find a minimum-cost Hamiltonian cycle: the cheapest cycle visiting every city exactly once.

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...but that doesn't mean that finding the solution to any particular instance is hard.

# TSP in the Media

The Washington Post
*Democracy Dies in Darkness*

Quantum computers are straight out of science fiction. Take the "traveling salesman problem," where a salesperson has to visit a specific set of cities, each only once, and return to the first city by the most efficient route possible. As the number of cities increases, the problem becomes exponentially complex. It would take a laptop computer 1,000 years to compute the most efficient route between 22 cities, for example. A quantum computer could do this within minutes, possibly seconds.

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"Like reporting the US National Debt is $4" – Bill Cook

# Bill Cook
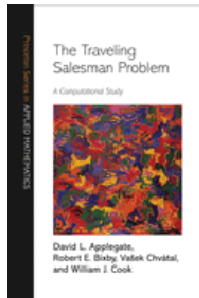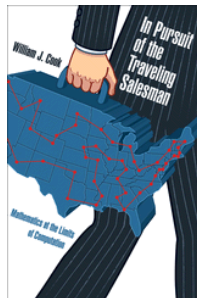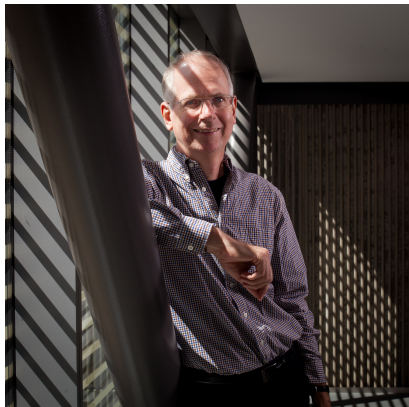
# The TSP: by Picture
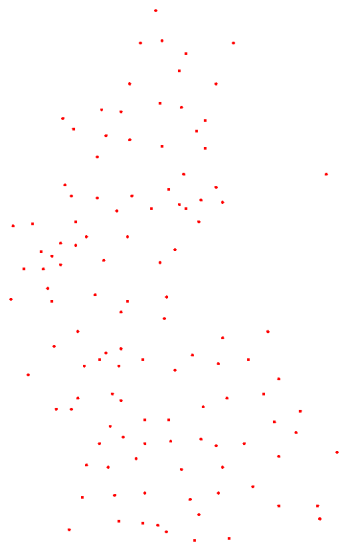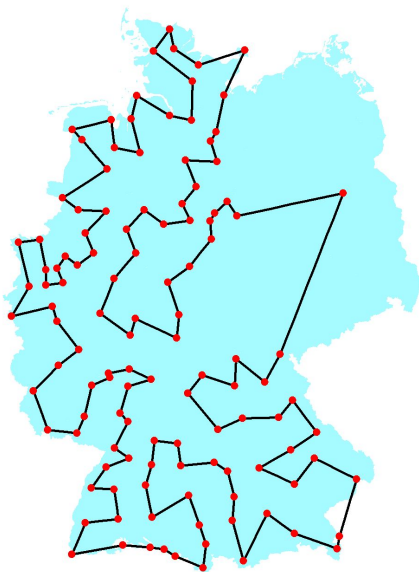
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M. Grötschel (1977)

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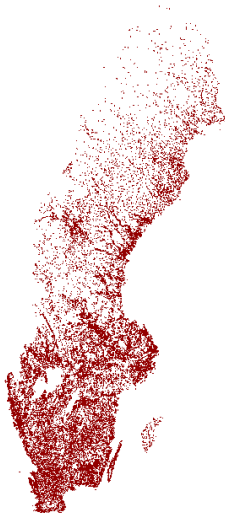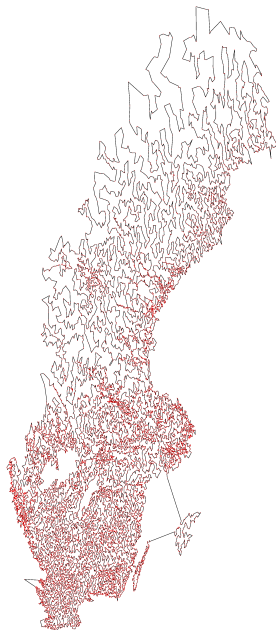
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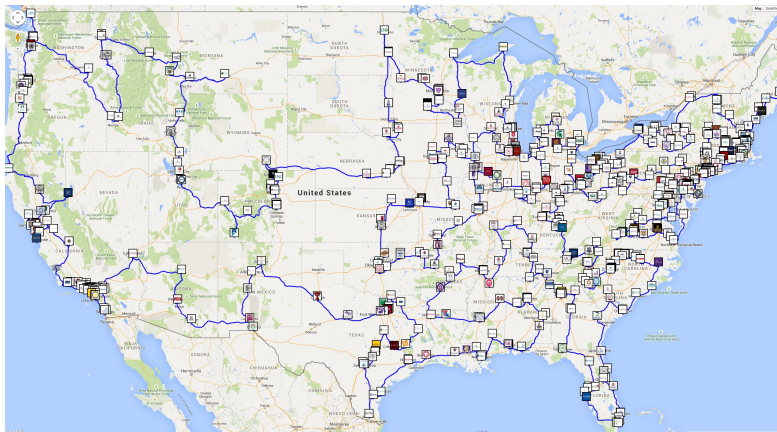
# The TSP: by Picture



Tour of 647 college campuses from Forbes' list of America's Top Colleges

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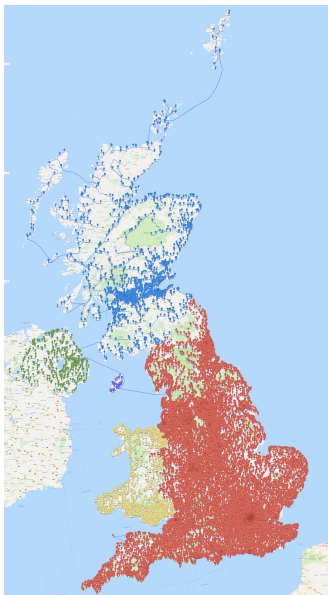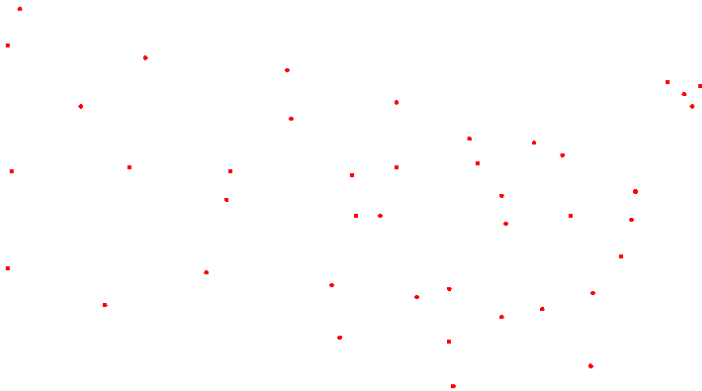
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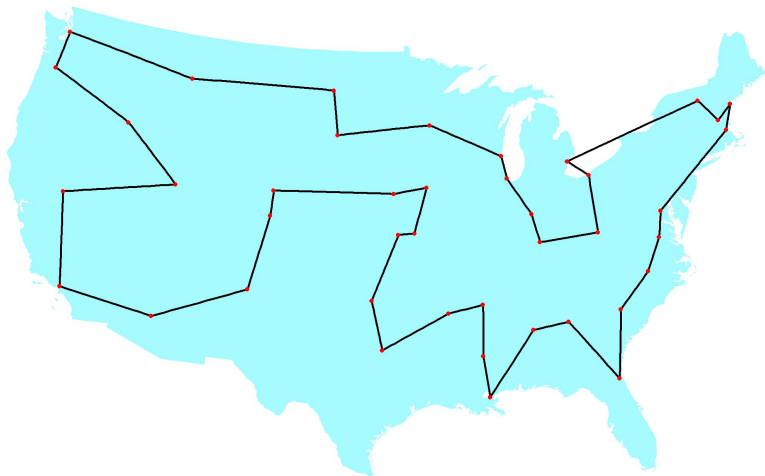


Solved by Dantzig, Fulkerson, and Johnson (1954)

## Dantzig, Fulkerson, Johnson Method

- Write a linear program (LP) using variables $x_e$
- Idea: if $x_e = 1$ then edge $e$ is in tour, else if $x_e = 0$ edge $e$ is not in tour.
- Since a linear program, can only restrict $0 \leq x_e \leq 1$
- Start with linear constraints that are satisfied by any integer tour
- If solution to LP is not integer, add more constraints (*cutting planes*) satisfied by any integer tour, but not by the current LP solution.

## The Subtour Elimination LP Relaxation (1954)

Let $\delta(S) := \{e = \{i,j\} : |\{i,j\} \cap S| = 1\}$ be the set of edges with exactly one endpoint in $S$, and let $\delta(v) := \delta(\{v\})$.

$$
\begin{array}{lll}
\min & \sum_{e \in E} c_e x_e & \\
\text{subject to} & \sum_{e \in \delta(v)} x_e = 2, & v = 1, \ldots, n \\
& \sum_{e \in \delta(S)} x_e \geq 2, & S \subset V : S \neq \emptyset, S \neq V \\
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### Remarks

- If we required that $x_e \in \{0, 1\}$ be integral, this is an integer program that exactly solves the TSP.
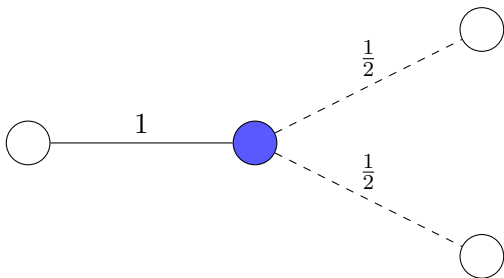
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### Remarks

- If we required that $x_e \in \{0, 1\}$ be integral, this is an integer program that exactly solves the TSP.
- With $0 \leq x_e \leq 1$, it is a **relaxation** of the TSP and can only find cheaper solutions.

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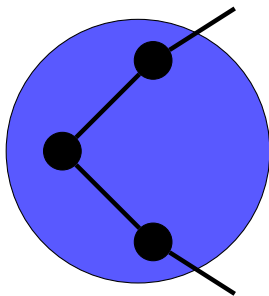
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### Remarks

The closer the value of the linear program to the value of the optimal integral solution, the easier it is to find using cutting planes or other standard techniques of integer programming (such as *branch-and-bound*).

## The Subtour Elimination LP Relaxation (1954)

| Random Uniform Euclidean | | | | TSPLIB | | | |
| --- | --- | --- | --- | --- | --- | --- | --- |
| Name | %Gap | Opttime | HKtime | Name | %Gap | Opttime | HKtime |
| E1k.0 | 0.77 | 1406 | 2.13 | dsj1000 | 0.61 | 410 | 3.68 |
| E1k.1 | 0.64 | 3855 | 2.15 | pr1002 | 0.89 | 34 | 2.40 |
| E1k.2 | 0.72 | 1211 | 2.02 | si1032 | 0.08 | 25 | 11.32 |
| E1k.3 | 0.62 | 956 | 1.92 | u1060 | 0.65 | 571 | 3.62 |
| E1k.4 | 0.69 | 330 | 1.69 | vm1084 | 1.33 | 605 | 2.40 |
| E1k.5 | 0.59 | 233 | 2.42 | pcb1173 | 0.96 | 468 | 1.70 |
| E1k.6 | 0.79 | 2940 | 1.67 | d1291 | 1.18 | 27394 | 4.54 |
| E1k.7 | 0.94 | 8003 | 1.95 | rl1304 | 1.55 | 189 | 4.08 |
| E1k.8 | 1.01 | 4347 | 1.65 | rl1323 | 1.65 | 3742 | 4.49 |
| E1k.9 | 0.61 | 189 | 2.14 | nrw1379 | 0.43 | 578 | 2.40 |
| E3k.0 | 0.71 | 533368 | 9.57 | fl1400 | 1.74 | 1549 | 9.83 |
| E3k.1 | 0.67 | 425631 | 10.54 | u1432 | 0.29 | 224 | 2.42 |
| E3k.2 | 0.74 | 342370 | 9.41 | fl1577 | 1.66 | 6705 | 38.19 |
| E3k.3 | 0.67 | 147135 | 10.30 | d1655 | 0.94 | 263 | 6.51 |
| E3k.4 | 0.73 | | 8.07 | vm1748 | 1.35 | 2224 | 4.43 |
| Random Clustered Euclidean | | | | u1817 | 0.90 | 449231 | 5.01 |
| C1k.0 | 0.54 | 337 | 9.83 | rl1889 | 1.55 | 10023 | 11.45 |
| C1k.1 | 0.41 | 534 | 10.84 | d2103 | 1.44 | – | 8.19 |
| C1k.2 | 0.42 | 320 | 8.79 | u2152 | 0.62 | 45205 | 8.10 |
| C1k.3 | 0.53 | 214 | 7.63 | u2319 | 0.02 | 7068 | 3.16 |
| C1k.4 | 0.58 | 768 | 9.36 | pr2392 | 1.22 | 117 | 5.75 |
| C1k.5 | 0.58 | 139 | 9.29 | pcb3038 | 0.81 | 80829 | 7.26 |
| C1k.6 | 0.73 | 1247 | 7.07 | fl3795 | 1.04 | 69886 | 123.66 |
| C1k.7 | 0.58 | 449 | 13.24 | fnl4461 | 0.55 | – | 12.47 |
| C1k.8 | 0.34 | 140 | 10.40 | rl5915 | 1.56 | – | 42.00 |
| C1k.9 | 0.66 | 703 | 9.61 | rl5934 | 1.38 | | 56.15 |
| C3k.0 | 0.62 | 16009 | 53.03 | pla7397 | 0.58 | – | 55.42 |
| C3k.1 | 0.61 | 17754 | 126.49 | rl11849 | 1.02 | | 102.41 |
| C3k.2 | 0.70 | 18237 | 80.39 | usa13509 | 0.66 | – | 120.20 |
| C3k.3 | 0.57 | 6349 | 71.57 | d15112 | 0.52 | | 90.13 |
| C3k.4 | 0.57 | 4845 | 44.02 | | | | |
| Random Matrices | | | | | | | |
| M1k.0 | 0.01 | 60 | 5.47 | M3k.0 | 0.00 | 612 | 40.35 |
| M1k.1 | 0.03 | 137 | 5.51 | M3k.1 | 0.01 | 546 | 39.52 |
| M1k.2 | 0.01 | 151 | 5.63 | M10k.0 | 0.00 | 1377 | 367.84 |
| M1k.3 | 0.01 | 169 | 5.26 | | | | |

From Johnson, McGeoch 2002

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### Integrality Gap

The *integrality gap* of an LP relaxation is the worst-case ratio (for any set of metric and symmetric edge costs) of

$$\frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}}.$$

## The Subtour Elimination LP Relaxation (1954)

### Theorem (Wolsey 1980, Shmoys and W 1990)

Christofides' 1976 algorithm produces a Hamiltonian cycle whose cost is within a factor of $\frac{3}{2}$ of the subtour LP:

$$\text{Optimal TSP Solution} \leq \text{Christofides' Cycle}$$
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$$\leq \frac{3}{2}\text{Optimal LP Solution}$$
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### Corollary

The integrality gap of this relaxation is at most $\frac{3}{2}$. That is, for any set of metric and symmetric edge costs,

$$\frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \leq \frac{3}{2}.$$
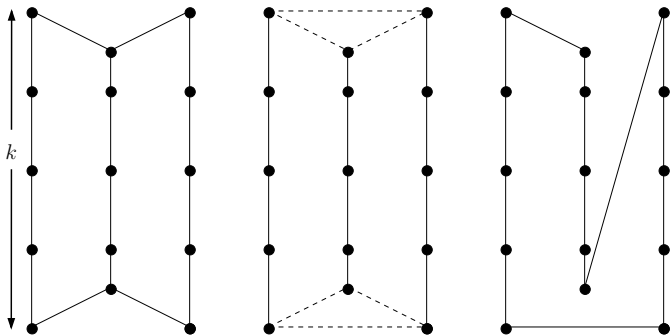
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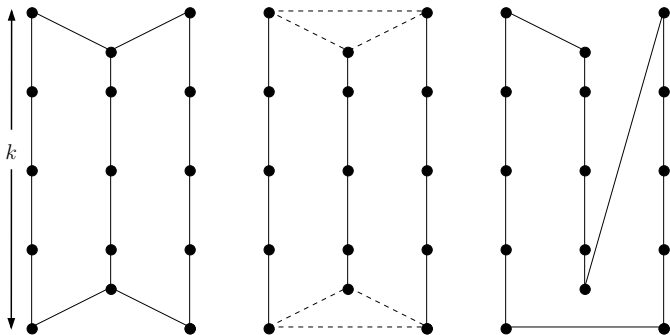


The example shows the integrality gap of this relaxation is at least $4/3$. Thus, for any set of metric and symmetric edge costs,

$$\frac{4}{3} \le \frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \le \frac{3}{2}.$$

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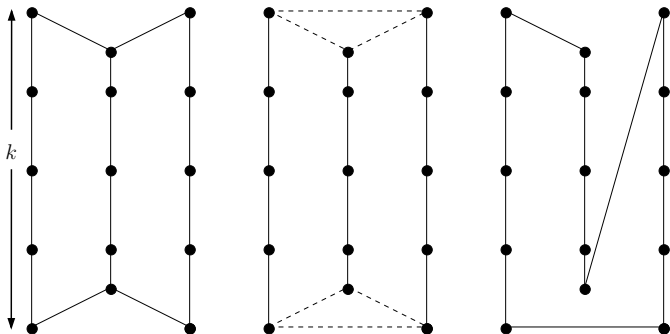


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$$\frac{4}{3} \leq \frac{\text{Optimal TSP Solution}}{\text{Optimal LP Solution}} \leq \frac{3}{2}.$$

Open problem: Prove tight bound on integrality gap.

# Looking Under Rocks



## Idea

Instead of LP relaxations, try SDP relaxations.

## Semidefinite Programs (SDPs)

A semidefinite program is similar to a linear program, except that we can take a matrix of variables and enforce that the matrix is positive semidefinite. Let $X \succeq 0$ denote that $X$ is positive semidefinite.

Recall that for real symmetric $X$, $X \succeq 0$ if and only if

- $y^T X y \geq 0$ for all $n$-vectors $y$;
- $X$ has all nonnegative eigenvalues.

$$
\begin{array}{ll}
\min & \sum_{i,j=1}^n C_{ij} X_{ij} \\
\text{subject to} & \sum_{i,j} a_{ijk} X_{ij} = b_k \quad k = 1, \ldots, m \\
& X \succeq 0 \\
& X = (X_{ij}) \qquad \text{real, symmetric}
\end{array}
$$

We can solve SDPs efficiently.

# A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs.
Let $J$ denote the all-ones matrix, and $e$ denote the all-ones vector.

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}\text{trace}\,(CX) = \tfrac{1}{2}\sum_{i,j=1}^n C_{ij}X_{ij} \\
\text{subject to} \quad & Xe = 2e \\
& X_{ii} = 0, && i = 1, ..., n \\
& 0 \leq X_{ij} \leq 1, && i, j = 1, ..., n \\
& 2I - X + \left(2 - 2\cos\left(\tfrac{2\pi}{n}\right)\right)(J - I) \succeq 0 \\
& X \text{ a real, symmetric } n \times n \text{ matrix.}
\end{aligned}
$$

## Theorem (Cvetković, Čangalović, and Kovačević-Vujčić 1999)

This semidefinite program is a relaxation of the TSP: the adjacency matrix of any Hamiltonian cycle is feasible and has the appropriate objective value.

## A First SDP Relaxation (1999)

Let $C = (c_{ij})_{i,j=1}^{n}$ be the matrix of edge costs.

| min | $\frac{1}{2} \text{trace} (CX)$ | |
|---|---|---|
| subject to | $Xe = 2e$ | |
| | $X_{ii} = 0,$ | $i = 1, ..., n$ |
| | $0 \leq X_{ij} \leq 1,$ | $i, j = 1, ..., n$ |
| | $2I - X + J - \left(2 - 2\cos\left(\frac{2\pi}{n}\right)\right) I \succeq 0$ | |
| | $X$ a real, symmetric $n \times n$ matrix. | |

$X$ is a fractional adjacency matrix of $K_n$ :

for $e = \{i, j\}, X_{ij} = X_{ji}$ is the proportion of edge $e$ used.

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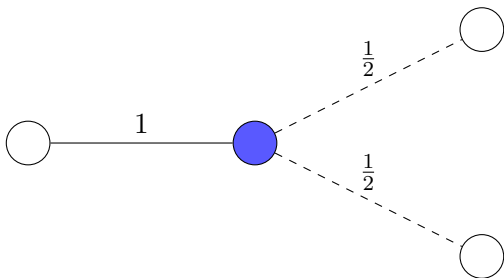\end{array}
$$

The weighted graph corresponding to $X$ (as a weighted adjacency matrix) is at least as connected as a cycle graph, with respect to algebraic connectivity

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### Theorem (Goemans and Rendl, 2000)

This SDP is weaker than the Subtour Elimination LP: any feasible solution for the Subtour LP is also feasible for this SDP.

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$$

### Theorem (Gutekunst and W, 2018)

This SDP has an unbounded integrality gap.

## A Second SDP Relaxation (2008)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs and $S^n$ be the set of real, symmetric $n \times n$ matrices. Also let $d = \lfloor \frac{n}{2} \rfloor$.

$$
\begin{aligned}
\min \quad & \tfrac{1}{2}\text{trace}\left(C X^{(1)}\right) \\
\text{subject to} \quad & X^{(k)} \geq 0, && k = 1, \ldots, d \\
& \sum_{j=1}^d X^{(j)} = J - I, \\
& I + \sum_{j=1}^d \cos\left(\tfrac{2\pi j k}{n}\right) X^{(j)} \succeq 0, && k = 1, \ldots, d \\
& X^{(k)} \in S^n, && k = 1, \ldots, d.
\end{aligned}
$$

Theorem (de Klerk, Pasechnik, and Sotirov 2008)

This semidefinite program is a relaxation of the TSP. Moreover, it is incomparable with the Subtour Elimination LP.

## A Second SDP Relaxation (2008)

### Idea

Let $\mathcal{C}$ be a Hamiltonian cycle. For $i = 1, ..., d = \lfloor \frac{n}{2} \rfloor$, let $X^{(i)}$ be the $i$th distance matrix of $\mathcal{C}$:

$$X^{(i)}_{jk} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$



$$X^{(1)} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

# A Second SDP Relaxation (2008)

## Idea

Let $\mathcal{C}$ be a Hamiltonian cycle. For $i = 1, ..., d = \lfloor \frac{n}{2} \rfloor$, let $X^{(i)}$ be the $i$th distance matrix of $\mathcal{C}$:

$$X_{jk}^{(i)} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$



$$X^{(2)} = \begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \end{pmatrix}$$

# A Second SDP Relaxation (2008)

$$
\begin{array}{lll}
\min & \frac{1}{2}\mathrm{trace}\left(CX^{(1)}\right) & \\
\text{subject to} & X^{(k)} \geq 0, & k = 1, \dots, d \\
& \sum_{j=1}^{d} X^{(j)} = J - I, & \\
& I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \dots, d \\
& X^{(k)} \in S^n, & k = 1, \dots, d.
\end{array}
$$

For $i = 1, \dots, d = \lfloor \frac{n}{2} \rfloor$, these quickly follow from

$$
X_{jk}^{(i)} = \begin{cases} 1, & j \text{ and } k \text{ are distance } i \text{ apart in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}
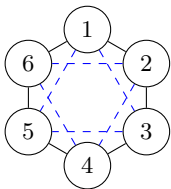$$

# A Second SDP Relaxation (2008)

$$\begin{array}{lll} \min & \frac{1}{2}\text{trace}\left(C X^{(1)}\right) \\ \text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\ & \sum_{j=1}^{d} X^{(j)} = J - I, \\ & I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\ & X^{(k)} \in S^n, & k = 1, \ldots, d. \end{array}$$

- The distance matrices of a cycle form an *association scheme*.

- This is an application of a more general statement about association schemes.

(See de Klerk, Filho, Pasechnik 2012)

- The distance matrices of a cycle are *circulant matrices*.

- Linear combinations of circulant matrices are circulant.

- Circulant matrices have well-understood eigenvalues.

(see Gutekunst and W. 2018)

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& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}$$

$$\begin{pmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\
m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_1 & m_2 & m_3 & \cdots & m_0
\end{pmatrix}$$

- The distance matrices of a cycle are *circulant matrices*.

- Linear combinations of circulant matrices are circulant.

- Circulant matrices have well-understood eigenvalues.

(see Gutekunst and W. 2018)

## A Second SDP Relaxation (2008)

### Goal

For $X_{st}^{(j)} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \quad k = 1, \ldots, d.$$

$$\begin{pmatrix}
m_0 & m_1 & m_2 & \cdots & m_{n-1} \\
m_{n-1} & m_0 & m_1 & \cdots & m_{n-2} \\
m_{n-2} & m_{n-1} & m_0 & \ddots & m_{n-3} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
m_1 & m_2 & m_3 & \cdots & m_0
\end{pmatrix}$$

For $\omega_n = e^{-\frac{2\pi i}{n}}$,

$$\lambda_t(M) = \sum_{s=0}^{n-1} m_s \omega_n^{st}, \quad t = 1, \ldots, n-1, \quad \lambda_n(M) = \sum_{s=0}^{n-1} m_s.$$

## A Second SDP Relaxation (2008)

### Goal

For $X_{st}^{(j)} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \quad k = 1, \ldots, d.$$

$$\begin{pmatrix} 1 & \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cdots & \cos(2\pi 2k/n) & \cos(2\pi k/n) \\ \cos(2\pi k/n) & 1 & \cos(2\pi k/n) & \cdots & \cos(2\pi 3k/n) & \cos(2\pi 2k/n) \\ \cos(2\pi 2k/n) & \cos(2\pi k/n) & 1 & \ddots & \cos(2\pi 4k/n) & \cos(2\pi 3k/n) \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ \cos(2\pi k/n) & \cos(2\pi 2k/n) & \cos(2\pi 3k/n) & \cdots & \cos(2\pi k/n) & 1 \end{pmatrix}$$

$$\lambda_t(M) = \sum_{s=0}^{n-1} m_s \omega_n^{st}, \quad t = 1, \ldots, n-1, \quad \lambda_n(M) = \sum_{s=0}^{n-1} m_s.$$

## A Second SDP Relaxation (2008)

### Goal

For $X_{st}^{(j)} = \mathbb{1}_{\{s \text{ and } t \text{ are distance } j \text{ apart in } \mathcal{C}\}}$,

$$I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, \quad k = 1, \ldots, d.$$

For $t \leq n$,

$$\begin{aligned}
\lambda_t(M) &= \sum_{s=0}^{n-1} m_s \omega_n^{st} \\
&= 1 + \cos\left(\frac{2\pi kd}{n}\right) \omega_n^{dt} + \sum_{s=1}^{d-1} \cos\left(\frac{2\pi sk}{n}\right) \left(\omega_n^{st} + \omega_n^{(n-s)t}\right) \\
&= \cdots \\
&= \begin{cases} 2d, & \text{if } k = t = d \\ d, & \text{if } k \neq d, t \in \{k, n-k\} \\ 0, & \text{else.} \end{cases}
\end{aligned}$$

## A Second SDP Relaxation (2008)

Let $C = (c_{ij})_{i,j=1}^n$ be the matrix of edge costs and $S^n$ be the set of real, symmetric $n \times n$ matrices. Also let $d = \lfloor \frac{n}{2} \rfloor$.

$$
\begin{array}{lll}
\min & \frac{1}{2}\text{trace}\left(C X^{(1)}\right) \\
\text{subject to} & X^{(k)} \geq 0, & k = 1, \ldots, d \\
& \sum_{j=1}^d X^{(j)} = J - I, \\
& I + \sum_{j=1}^d \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, & k = 1, \ldots, d \\
& X^{(k)} \in S^n, & k = 1, \ldots, d.
\end{array}
$$

### Theorem (Gutekunst and W, 2018)

This SDP has an unbounded integrality gap. That is, there exists no constant $\alpha > 0$ such that

$$
\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP}}(C)} \leq \alpha
$$

for all cost matrices $C$ with metric, symmetric edge costs.

## Our Main Theorem: Proof Sketch

Let $n$ be even and consider the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$
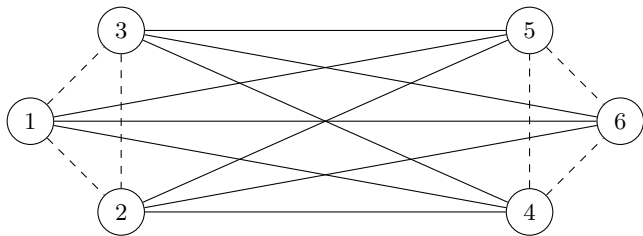


$$\underline{\phantom{xxxxx}} \quad c_e = 1$$

$$\text{- - - - -} \quad c_e = 0$$

## Our Main Theorem: Proof Sketch

Let $n$ be even and consider the cost matrix

$$\hat{C} := \begin{pmatrix} 0 & \cdots & 0 & 1 & \cdots & 1 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & \cdots & 1 \\ 1 & \cdots & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & \cdots & 1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d.$$

$\hat{C}$ corresponds to:

- a *cut semimetric*: costs where, for some $S \subset V$, $c_{ij} = 1$ if $\{i, j\} \in \delta(S)$ and $c_{ij} = 0$ otherwise.
- an instance of *Euclidean TSP*: vertices $1, ..., \frac{n}{2}$ are at $0 \in \mathbb{R}^1$ and vertices $\frac{n}{2} + 1, ..., n$ are at $1 \in \mathbb{R}^1$. Costs are given by the Euclidean distance between corresponding vertices.

## Our Main Theorem: Proof Sketch

**Theorem (Gutekunst and W, 2018)**

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\text{OPT}_{\text{TSP}}(\hat{C})$.

**Corollary**

There exists no constant $\alpha > 0$ such that

$$\frac{\text{OPT}_{\text{TSP}}(C)}{\text{OPT}_{\text{SDP}}(C)} \leq \alpha$$

for all cost matrices $C$ with metric, symmetric edge costs.

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For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Strategy:

1. Look within a class of feasible solutions that respect the symmetry of $\hat{C}$.

2. Exploit the structure of such solutions by reducing the SDP to an LP *for solutions in that class*.

3. Find a feasible solution to the LP achieving the desired cost.

## Our Main Theorem: Proof Sketch

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For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\text{OPT}_{\text{TSP}}(\hat{C})$.

Candidate solutions:

$$X^{(j)} = \left( \begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d - 1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$



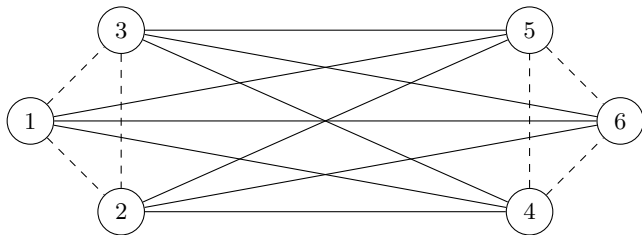$$X_e^{(j)} = b_j$$

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# Our Main Theorem: Proof Sketch

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For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.



$X_e^{(j)} = b_j$, cost 1

$X_e^{(j)} = a_j$, cost 0

**TSP Solutions**

$$\mathrm{OPT}_{\mathrm{TSP}}(\hat{C}) = 2$$

**SDP Solutions**

$$\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) = \frac{1}{2}\mathrm{trace}\left(CX^{(1)}\right)$$

$$= 0 \times 2\binom{n/2}{2}a_1 + 1 \times \left(\frac{n}{2}\right)^2 b_1$$

## Our Main Theorem: Proof Sketch

Theorem (Gutekunst and W, 2018)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Let

$$X^{(j)} = \left( \begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$
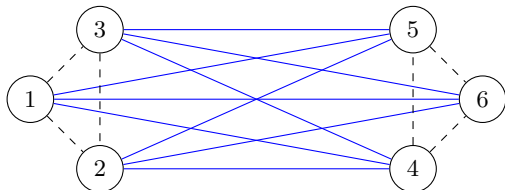
Want to verify that it satisfies

$$\begin{aligned}
& X^{(k)} \geq 0, && k = 1, \ldots, d \\
& \textstyle\sum_{j=1}^d X^{(j)} = J - I, \\
& I + \textstyle\sum_{j=1}^d \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0, && k = 1, \ldots, d \\
& X^{(k)} \in S^n, && k = 1, \ldots, d,
\end{aligned}$$

so need $a_j \geq 0$, $b_j \geq 0$, $\sum_{j=1}^d a_j = 1$.

## Our Main Theorem: Proof Sketch

### Theorem (Gutekunst and W, 2018)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

Let

$$X^{(j)} = \left( \begin{pmatrix} a_j & b_j \\ b_j & a_j \end{pmatrix} \otimes J_d \right) - a_j I_n, \quad b_j = \begin{cases} \frac{4}{n} - \left(1 - \frac{2}{n}\right) a_j, & j \leq d-1 \\ \frac{2}{n} - \left(1 - \frac{2}{n}\right) a_j, & j = d. \end{cases}$$

The SDP constraint $I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0$ becomes

$$\left( \begin{pmatrix} a^{(k)} & b^{(k)} \\ b^{(k)} & a^{(k)} \end{pmatrix} \otimes J_d \right) + (1 - a^{(k)})I_n \succeq 0,$$

where $a^{(k)}$ and $b^{(k)}$ are linear combinations of $a_1, ..., a_d$.

## Our Main Theorem: Proof Sketch

**Theorem (Gutekunst and W, 2018)**

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT_{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT_{TSP}}(\hat{C})$.

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for

$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \quad b^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) b_i.$$

## Our Main Theorem: Proof Sketch

### Theorem (Gutekunst and W, 2018)

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- The eigenvalues of $A \otimes B$ are $\lambda_i(A)\lambda_j(B)$.
- $J_d$ has one eigenvalue $d$, all other eigenvalues are zero.
- The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are $a + b$ and $a - b$.

## Our Main Theorem: Proof Sketch

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- The eigenvalues of $\begin{pmatrix} a & b \\ b & a \end{pmatrix}$ are $a + b$ and $a - b$.

So eigenvalues are

$$1 - a^{(k)}, \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} + b^{(k)}\right), \quad 1 - a^{(k)} + \frac{n}{2}\left(a^{(k)} - b^{(k)}\right).$$

## Our Main Theorem: Proof Sketch

**Theorem (Gutekunst and W, 2018)**

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\text{OPT}_{\text{TSP}}(\hat{C})$.

The SDP constraint $I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0$ becomes

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For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

The SDP constraint $I + \sum_{j=1}^{d} \cos\left(\frac{2\pi jk}{n}\right) X^{(j)} \succeq 0$ becomes

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So eigenvalues are

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for

$$a^{(k)} = \sum_{i=1}^{d} \cos\left(\frac{2\pi ik}{n}\right) a_i, \quad b^{(k)} = -\left(1 - \frac{2}{n}\right)a^{(k)} - \frac{2}{n}.$$

## Our Main Theorem: Proof Sketch

### Theorem (Gutekunst and W, 2018)

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

**Intermediate step:** Rewriting $b^{(k)}$ in terms of $a^{(k)}$, and imposing that the eigenvalues, $a_j$, and $b_j$, are nonnegative, and finding minimum-cost solution becomes linear program:

$$
\begin{aligned}
\max \quad & a_1 \\
\text{subject to} \quad & \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i \geq -\frac{2}{n-2}, \quad k = 1, ..., d \\
& \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i \leq 1, \qquad k = 1, ..., d \\
& \sum_{i=1}^{d} a_i = 1 \\
& a_i \leq \frac{4}{n-2}, \qquad i = 1, ..., d-1 \\
& a_d \leq \frac{2}{n-2} \\
& a_i \geq 0, \qquad i = 1, ..., d.
\end{aligned}
$$

## Our Main Theorem: Proof Sketch

**Theorem (Gutekunst and W, 2018)**

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\mathrm{OPT}_{\mathrm{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n} \mathrm{OPT}_{\mathrm{TSP}}(\hat{C})$.

$$
\begin{array}{lll}
\max & a_1 & \\
\text{subject to} & \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i & \geq -\frac{2}{n-2}, \quad k = 1, ..., d \\
& \sum_{i=1}^{d} \cos\left(\frac{2\pi i k}{n}\right) a_i & \leq 1, \qquad k = 1, ..., d \\
& \sum_{i=1}^{d} a_i & = 1 \\
& a_i & \leq \frac{4}{n-2}, \quad i = 1, ..., d-1 \\
& a_d & \leq \frac{2}{n-2} \\
& a_i & \geq 0, \qquad i = 1, ..., d.
\end{array}
$$
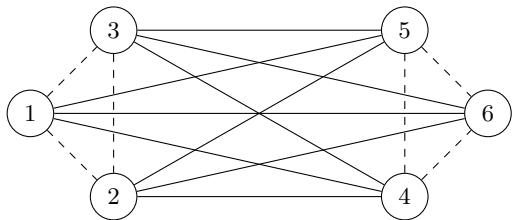
Guess and verify that the following solution is feasible.

$$
a_j = \frac{2}{n-2}\left(\cos\left(\frac{\pi j}{d}\right) + 1\right), \quad j = 1, ..., d.
$$

## Our Main Theorem: Proof Sketch

**Theorem (Gutekunst and W, 2018)**

For $\hat{C} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes J_d$, we have $\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{\pi^2}{n} = \frac{\pi^2}{2n}\text{OPT}_{\text{TSP}}(\hat{C})$.

Thus we find solutions where



$$\text{—} \quad b_1 = \frac{1 - \cos\left(\frac{\pi}{d}\right)}{n} \sim \frac{1}{n^3}$$

$$\text{---} \quad a_1 = \frac{2\cos\left(\frac{\pi}{d}\right) + 2}{n-2}$$

$$\text{OPT}_{\text{SDP}}(\hat{C}) \leq \frac{n^2}{4}b_1 \sim \frac{1}{n}.$$

## Summary

- The 2008 SDP relaxation has an unbounded integrality gap
- To show that it produces arbitrarily bad solutions, we:
  1. Looked within a class of feasible solutions that respect the symmetry of $\hat{C}$.
  2. Exploited the structure of such solutions by reducing the SDP to an LP *for solutions in that class*.
  3. Found a feasible solution to the LP achieving the whose cost decreases like $\frac{1}{n^3}$.
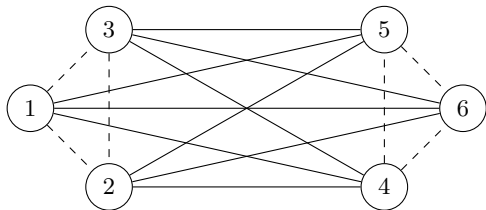
## Corollaries of Our Theorem

### Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

### Corollary

The SDP is non-monotonic, unlike the TSP and subtour elimination LP.

We've found SDP solutions costing $\frac{n^2}{4}b_1 \approx \frac{1}{n}$, which become arbitrarily small with $n$



$$— \quad b_1 = \frac{1-\cos\left(\frac{\pi}{d}\right)}{n} \sim \frac{1}{n^3}$$

$$-- \quad a_1 = \frac{2\cos\left(\frac{\pi}{d}\right)+2}{n-2}$$

## Corollaries of Our Theorem

### Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

### Corollary

The earlier SDP of Cvetković, Čangalović, and Kovačević-Vujčić has an unbounded integrality gap: the same $X^{(1)}$ we found is feasible (and has exactly the same algebraic connectivity as a cycle).
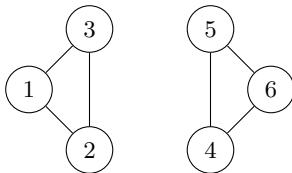
## Corollaries of Our Theorem

### Theorem (Gutekunst and W, 2018)

The SDP has an unbounded integrality gap.

### Corollary

A related SDP from de Klerk, de Oliveira Filho, and Pasechnik 2012 for the $k$-cycle cover problem also has an unbounded integrality gap.

# Open Questions

1. How does this SDP perform on special cases of the TSP?

- We've shown that the integrality gap is unbounded on the general metric and symmetric TSP, as well as on Euclidean TSP.

- On *graphic* TSP (where edge costs correspond to shortest paths in a connected input graph), the integrality gap is at most 2. Is it strictly better?

## Open Questions

1. How does this SDP perform on special cases of the TSP?
2. If you combine both this SDP and the subtour LP, can you guarantee an integrality gap of $1.5 - \epsilon$ for any $\epsilon > 0$?

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1. How does this SDP perform on special cases of the TSP?
2. If you combine both this SDP and the subtour LP, can you guarantee an integrality gap of $1.5 - \epsilon$ for any $\epsilon > 0$?
3. De Klerk and Sotirov introduced a stronger SDP in 2012. Does this SDP have a bounded integrality gap?

# Big Open Questions

**Open Problem**

Prove tight bound on integrality gap of Subtour LP.

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Is there some other way of understanding the surprising practical effectiveness of the Subtour LP?

Thanks for your attention.