Problem Presentation 3

Problem statement: Define a relation R on $\mathbb{Z}$ by $x\mathsf{R}y$ iff $x-y=4k$ for some integer k. Verify that R is an equivalence relation and describe the equivalence class E_5. How many distinct equivalence classes are there?

Proof. In order to prove R is an equivalence relation, we need to show R is reflexive, symmetric, and transitive.

In order to prove R is reflexive, we need to show $x\mathsf{R}x$. Substuting x into the relation gives us $x-x=4k$ which means $0=4k$. When $k=0$, this statements hold and $0 \in \mathbb{Z}$.

In order to prove R is symmetric, we need to show $x\mathsf{R}y \implies y\mathsf{R}x$. Substuting x and y into $x\mathsf{R}y$ gives us $x\mathsf{R}y : x-y=4k_1$. Multiplying both sides of the equation gives us $y-x=-4k_1$. Let us assume there exists a k_2 such that $k_2=-k_1$. Therefore $y-x=4k_2$ which means $y\mathsf{R}x$ holds. Hence we have proved $x\mathsf{R}y \implies y\mathsf{R}x$.

In order to prove R is transitive, we need to show if $x\mathsf{R}y$ and $y\mathsf{R}z$, then $x\mathsf{R}z$. Substituting x and y into $x\mathsf{R}y$ gives us $x\mathsf{R}y : x-y=4k_1$. Substituting y and z into $y\mathsf{R}z$ gives us $y\mathsf{R}z : y-z=4k_2$. Adding these two equations together gives us $x-y+y-z=4k_1+4k_2=4(k_1+k_2)=x-z$. Let $k_3=k_1+k_2$. Since k_3 is the sum of two integers, k_3 is also an integer. Therefore $x-z=4k_3$ which means $x\mathsf{R}y$. Hence we have proved $x\mathsf{R}y$ and $y\mathsf{R}z$, then $x\mathsf{R}z$.

Therefore we have proved R is reflexive, symmetric, and transitive. Thus R is an equivalence relation.

$\square$

$E_5 = \{y \in \mathbb{Z} : y\mathsf{R}5 : y-5=4k, k \in \mathbb{Z}\} = \{..., -3, 1, 5, 9, 13, ...\}$. For all y in real numbers such that $y-5$ is a multiple of 4 with some integer k. The increment between each numbers in the set is 4. Therefore there should be other three sets that are equivalence class of $\mathbf{R}$, $E_3 = \{..., -5, -1, 3, 7, 11, ...\}$ and $E_4 = \{..., -4, 0, 4, 8, 12, ...\}$, and $E_6 = \{..., -2, 2, 6, 10, 14, ...\}$. All the other equivalence classes are identical to any of these four sets.