Participation3

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February 2023

1 Argue that nonnegative multiples of convex functions are convex

Let $h(x) = \beta f(x)$
We have

$$\begin{aligned}
&h(\alpha x + (1-\alpha)y) \\
&\leq \beta[\alpha f(x) + (1-\alpha)f(y)] \\
&= \alpha\beta f(x) + (1-\alpha)\beta f(y) \\
&= \alpha h(x) + (1-\alpha)h(y)
\end{aligned}$$

Therefore $h(x)$, nonnegative multiples of a convex function, is convex.

2 Argue that affine functions (functions of the form $f(x) =< a, x > +b$) are convex.

$$\begin{aligned}
&f(\alpha x + (1-\alpha)y) \\
&=< a, \alpha x + (1-\alpha)y > +b \\
&= \alpha < a, x > +(1-\alpha) < a, y > +b \\
&= \alpha(< a, x > +b) + (1-\alpha)(< a, y > +b) + b - \alpha b - b + \alpha b \\
&= \alpha(< a, x > +b) + (1-\alpha)(< a.y > +b) \\
&\leq \alpha f(x) + (1-\alpha)f(y)
\end{aligned}$$

Therefore affine functions (functions of the form $f(x) =< a, x > +b$) are convex.

3 Argue that if g is convex and f is affine, then the composition $g(f(\cdot))$ is a convex function.

Given f is an affine function, we have $f(\alpha x + (1-\alpha)y) = \alpha f(x) + (1-\alpha)f(y)$

$$
\begin{aligned}
&g(f(\alpha x + (1-\alpha)y)) \\
&= g(\alpha f(x) + (1-\alpha)f(y)) \\
&\leq \alpha g(f(x)) + (1-\alpha)g(f(y))
\end{aligned}
$$

Therefore $g(f(\cdot))$ is a convex function.

4 Argue that the sum of convex functions is convex

let $h(x) = \sum_{i=1}^{\infty} f(\alpha x + (1-\alpha)y), \forall x, y \in dom(f) \wedge \forall \alpha \in [0,1]$ We proof h is convext through induction.
Base Case: i = 1, f is convex as a given condition.
Induction step: Assume $h_k(x) = \sum_{i=1}^{k} f(\alpha x + (1-\alpha)y)$ is convex, prove $h_{k+1}(x) = h_k(x) + f(x)$ is convex.

$$
\begin{aligned}
&(h_k + f)(\alpha x + (1-\alpha)y) \\
&= \alpha h_k(x) + (1-\alpha)h_k(y) + \alpha f(x) + (1-\alpha)f(y) \\
&= \alpha((h_k + f)(x)) + (1-\alpha)((h_k + f)(y)) \\
&= \alpha(h_{k+1}(x)) + (1-\alpha)(h_{k+1}(y))
\end{aligned}
$$

Thus h_{k+1} is convex.
Therefore we can conclude the sum of convex functions h is convex.

5 Argue that the OLS problem is a convex optimization problem given x^2 is convex

since $X\beta - y$ is convex, $(X\beta - y)^2$ is convex given x^2 is convex. Thus $\sum(X\beta - y)^2 = (\sqrt{\sum(X\beta - y)^2})^2 = ||X\beta - y||_2^2$ is also convex since the sum of convex functions is convex. Since $\frac{1}{n}$ is a nonnegative number, $\frac{1}{n}||X\beta - y||_2^2$ is convex given nonnegative multiples of a convex function is also convex. Thus the OLS problem is a convex optimization problem.