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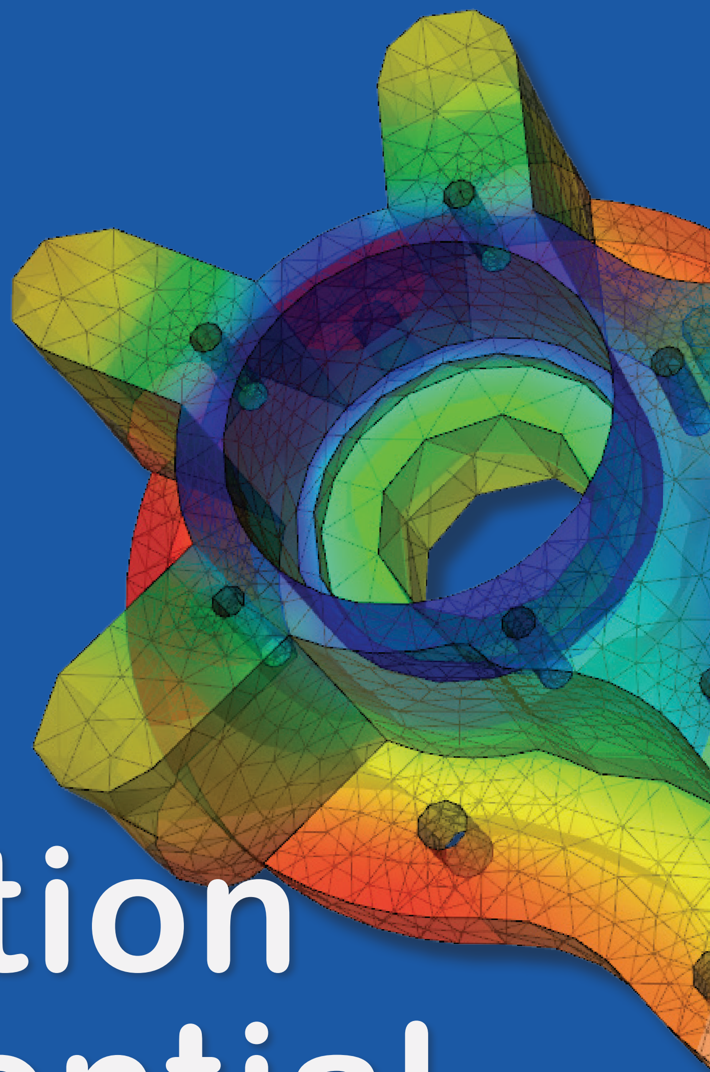
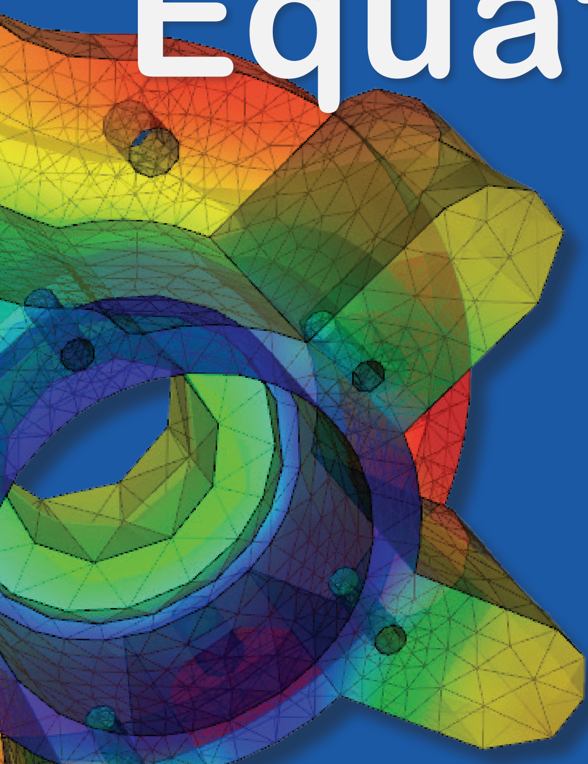
MARK H. HOLMES



Introduction to Differential Equations



Introduction to Differential Equations

Mark H. Holmes

XanEdu

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Preface

The textbook is written for MATH-2400, which is the lower division differential equations course at RPI. This is not done because there is a shortage of textbooks at this level. Quite the contrary, there are many. Most are over 600 pages in length and often with many editions (some as high as 11). After searching for a text that is suitable for RPI students, we finally decided it best to just write our own. There are several significant reasons for this, and in almost alphabetical order they are:

Cost The book we have been using costs anywhere from \$200 to \$270, depending on which version you purchase. This is typical for textbooks on this subject, and they are, in our opinion, overpriced.

Gratuitous Facts and Details So many, often esoteric, topics are covered in most textbooks that important points are hard to identify. This is likely the fault of the publishers who want their textbooks to fit into any, and every, college and university.

Emphasis As applied mathematicians we believe mathematics should be used to solve problems. This should not be interpreted to mean that the theoretical aspects of the subject are ignored. Rather, the theoretical results that are considered have constructive consequences. If you want to learn about the theoretical underpinnings of the subject, we recommend you take the upper division course on differential equations.

Evolution There is no question that having a solid foundation in how to solve basic differential equation is essential in science and engineering. However, most physically realistic problems are now solved numerically (i.e., using a computer). This has had an effect on what you need to learn about, for even the simplest differential equations. One example is something known as asymptotic stability, which plays a critical role in selecting an appropriate numerical method.

The prerequisites for this text are MATH-1020, which includes vector calculus, and some knowledge of matrices. The material requiring vectors and matrices is in Chapters 4 and 5, and at the end of Chapter 6. However, there is a fundamental connection between differential equations and linear algebra, and this connection is used throughout this textbook. The material is written so it is self-contained, so a previous course in linear algebra is not necessary. You will see comments such as "if you recall from linear algebra," which are used to indicate where the connections are, but the material required for differential equations is then written out explicitly. Occasionally there are facts, or results, from linear algebra that are needed and they are stated without proof. This is also done with other topics, such as the convergence theorem for a Fourier series. In such cases, you will either need to consult the appropriate Wikipedia page, or (even better), take the needed mathematics course to learn more about the topic.

There are solutions for many of the exercises, but they are not included in the textbook. Rather, they are available at the book's web-page, and your instructor will provide the link. Also included on the web-page is a listing of the typos, as well as plots for those exercises that reference particular plots in the text.

Chapter 1

Introduction

We begin with a question: why are most students who are majoring in engineering or science required to take an entire course dedicated to something called differential equations?

We'll start to answer this by giving a couple of examples where they arise, and this will also provide an opportunity to introduce some of the terminology used in the subject.

Example 1: Rate Laws

These describe the fluctuations, or changes, in the concentration of something. The something in this case could be the concentration of a chemical, a population of animals, or perhaps the temperature of an object. As a simple example, a radioactive isotope is unstable, and will decay by emitting a particle, transforming into another isotope. The assumption used to model such situations is that the rate of decrease in the amount of radioactive isotope is proportional to the amount currently present. To translate this into mathematical terms, let $N(t)$ designate the amount of the radioactive material present at time t. In this case we obtain the rate equation

$$\frac{dN}{dt} = -kN, \quad \text{for} \;\; 0 < t, \tag{1.1}$$

where k is a positive constant. This is a differential equation for N. Usually one knows the amount N_0 of the isotope at the beginning, which gives us the requirement that

$$N(0) = N_0. \tag{1.2}$$

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This is known as an **initial condition**. Together, (1.1) and (1.2) form what is called an **initial value problem (IVP)**. ■

Example 2: Mechanics

One of the biggest generators of differential equations is Newton's second law, which states that $F = ma$. Any situation, electrical, mechanical or otherwise, involving non-static forces will almost inevitability result in having to solve a differential equation. To illustrate, consider the simple case of dropping an object off a building. If $x(t)$ is the distance of the object from the ground, then its velocity is $v = x'(t)$, and its acceleration is $a = x''(t)$. If the forces on the object include gravity $F_g = -mg$, and air resistance $F_r = -cv$, then $F = F_g + F_r$. Together, these expressions result in the following differential equation for $x(t)$:

$$m\frac{d^2x}{dt^2} = -mg - c\frac{dx}{dt}\,. \quad \blacksquare \tag{1.3}$$

The differential equations in (1.1) and (1.3) have a few things in common, such as there is one independent variable, t, and one dependent variable, N and x. There are also differences, and an example is that (1.3) involves the second derivative and (1.1) only involves the first derivative. It is important to be able to recognize these differences as they are used in this textbook to determine how to solve the problem.

1.1 ▪ Terminology for Differential Equations

Problems involving differential equations can involve a single equation, or several equations. They can also have one independent variable, or several. There are other differences, and to help illustrate some of the possibilities we will use the following examples.

Example 1: $\dfrac{d^2y}{dt^2} - 2\dfrac{dy}{dt} + 4ty = 0$ Example 2: $\dfrac{\partial u}{\partial t} - 2\dfrac{\partial u}{\partial x} = u^3$

Example 3:
$$\frac{du}{dt} = u + v + 1$$
$$\frac{dv}{dt} = -u + v$$

Dependent variable(s): This is the variable(s) being solved for.

Example 1: y Example 2: u Example 3: u and v

Independent variable(s): These are usually time (t) and/or space (x).

Example 1: t Example 2: x and t Example 3: t

Order: The order of the highest derivative in the equation (or equations).

Example 1: second-order Example 2: first-order
Example 3: first-order

Linear or Nonlinear: A differential equation is linear if it is a linear expression of the dependent variable and its derivatives, otherwise it is nonlinear.

Example 1: linear Example 2: nonlinear (because of the u^3)
Example 3: linear

ODE or PDE: If there is one independent variable, then it is an ordinary differential equation (ODE). If there is more than one independent variable, then it is a partial differential equation (PDE).

Example 1: ODE Example 2: PDE Example 3: ODEs

Homogeneous or Inhomogeneous: A linear differential equation is homogeneous if the identically zero function is a solution. Otherwise, it is inhomogeneous.

Example 1: homogeneous since $y \equiv 0$ is a solution

Example 2: inapplicable since the equation is not linear

Example 3: inhomogeneous since $u \equiv 0$ and $v \equiv 0$ is not a solution

1.2 ▪ Solutions and Non-Solutions of Differential Equations

One of the central questions of this textbook is how to find the solution of a differential equation. The examples below are about the reverse situation, where a function is given and the question is whether it is a solution of a particular differential equation.

Example 1: Show that $y = te^{-2t}$ is a solution of $y' = -2y + e^{-2t}$.

Answer: Since $y' = e^{-2t} - 2te^{-2t} = (1-2t)e^{-2t}$, and $-2y + e^{-2t} = -2te^{-2t} + e^{-2t} = (1-2t)e^{-2t}$, it follows that $y' = -2y + e^{-2t}$. In other words, y is a solution of the differential equation. ■

Example 2: For what value(s) of r and c, if any, is $y = ce^{rt}$ a solution of the IVP: $y' + y = 0$ where $y(0) = 3$?

Answer: Since $y' = rce^{rt}$, then from the differential equation we require that $(r+1)ce^{rt} = 0$. Given that e^{rt} is never zero, we conclude that either $c = 0$ or else $r = -1$. From the initial condition $y(0) = 3$, we need $c = 3$, and so this means that $r = -1$. ■

Example 3: For what value(s) of r, if any, is $y = e^{rt}$ a solution of $y'' - y' - 6y = 0$?

Answer: Since $y' = re^{rt}$, and $y'' = r^2e^{rt}$, then from the differential equation we require that $(r^2 - r - 6)e^{rt} = 0$. Given that e^{rt} is never zero, we conclude that $r^2 - r - 6 = 0$. Solving this, we get that $r = 3$ and $r = -2$ are the only values for which $y = e^{rt}$ is a solution. ■

Example 4: For what value(s) of r and c, if any, is $y = e^{rt}$ a solution of $y' = 2y^3$?

Answer: Since $y' = re^{rt}$, then from the differential equation we require that $re^{rt} = 2e^{3rt}$. Given that e^{rt} is never zero, we need $r = 2e^{2rt}$. The left hand side is constant. The only way to have the right hand side a constant is to take $r = 0$. In this case, the differential equation becomes $0 = 2$. This is not possible, and so the answer is that no values result in a solution. ■

Exercises

1. Show that the given function $y(t)$ is a solution of the given differential equation.

   a) $y = e^{2t} - 1, \quad y' = 2y + 2$
   b) $y = te^{-t}, \quad y' + y = e^{-t}$
   c) $y = \cos(3t), \quad y'' = -9y$
   d) $y = e^{3t}, \quad y'' + y' - 12y = 0$
   e) $y = e^t + 1, \quad y'' + 2y' - 3y = -3$
   f) $y = \frac{1}{1+t}, \quad y' + y^2 = 0$
   g) $y = \tan\left(\frac{1}{3}t + 1\right), \quad 3y' = 1 + y^2$
   h) $y = \ln(1 + t^2), \quad y' = 2te^{-y}$

2. For what value(s) of r, if any, is $y = e^{rt}$ a solution of the differential equation?

   a) $y' = -2y$
   b) $3y' = y$
   c) $y' = y + 1$
   d) $y'' + 4y' = 0$
   e) $2y'' + 5y' - 3y = 0$
   f) $y'' - 4y' + 4y = 0$
   g) $y'' + y' + y = e^{-3t}$
   h) $y'' - 3y' + y = 1$
   i) $y' = -2y^3$
   j) $y' = y^2$

3. For what value of r and c is $y = ce^{rt}$ a solution of the IVP?

   a) $y' = -2y, \quad y(0) = 1$
   b) $y' + y = 0, \quad y(0) = -1$
   c) $3y' - y = 0, \quad y(0) = 3$
   d) $y' - y = 0, \quad y(0) = -1$
   e) $5y' = -2y, \quad y(0) = -7$
   f) $y' + 4y = 0, \quad y(0) = 3$

4. The following are *linear* and *homogeneous* first-order differential equations. The given function $y_1(t)$ is a solution, and you are to show that $y = cy_1$ is a solution for any value of the constant c.

   a) $y' = 2y, \quad y_1 = e^{2t}$
   b) $y' + y = 0, \quad y_1 = e^{-t}$
   c) $y' - 4y = 0, \quad y_1 = e^{4t}$
   d) $3y' = y, \quad y_1 = e^{t/3}$

5. The following are *linear* and *homogeneous* second-order differential equations. The given functions $y_1(t)$ and $y_2(t)$ are solutions, and you are to show that $y = c_1y_1 + c_1y_2$ is a solution for any value of the constants c_1 and c_2.

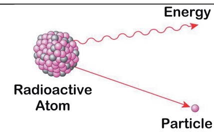
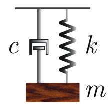
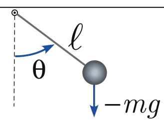
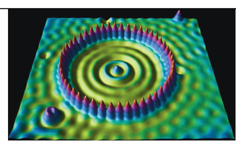
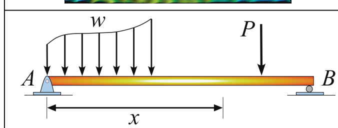
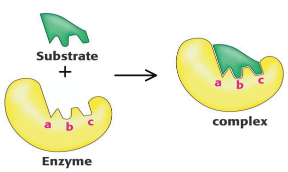
   a) $y'' - 3y' + 2y = 0,$
   $y_1 = e^{2t}, \;\; y_2 = e^t$
   b) $y'' - y' - 2y = 0,$
   $y_1 = e^{2t}, \;\; y_2 = e^{-t}$
   c) $y'' + y' = 0,$
   $y_1 = e^{-t}, \;\; y_2 = 1$
   d) $y'' + 2y' + 5y = 0,$
   $y_1 = e^{-t}\cos(2t)$
   $y_2 = e^{-t}\sin(2t)$

   Important Conclusion: From Problems 4 and 5, if $y_1(t)$ and $y_2(t)$ are solutions of a *linear* and *homogeneous* differential equation, then $c_1y_1(t) + c_2y_2(t)$ is a solution of the equation for any value of c_1 and c_2. This is known as the **principle of superposition**, and it holds for all linear homogeneous differential equations (ODES or PDEs). Moreover, as demonstrated in the following exercise, this does not (usually) hold for a nonlinear differential equation.

6. Both $y_1(t)$ and $y_2(t)$ are solutions of the given *nonlinear* differential equation. Show that (i) $y = c_1y_1(t)$ is not a solution unless $c_1 = 1$, and (ii) $y = c_1y_1 + c_2y_2$ is not a solution if c_1 and c_2 are both nonzero.

   a) $y' = t/(1 + y),$
   $y_1 = -1 + t, \;\; y_2 = -1 - t$
   b) $y' = e^{-y},$
   $y_1 = \ln(1 + t), \;\; y_2 = \ln(3 + t)$

7. Fill out the table on the next page. Assume that any constants in the equations are nonzero. Also, in the last column, the answer Inapplicable (IA) is possible. References for images: Radioactive atom [NRC, 2019], Schrödinger's equation [Eigler, 2019].

	Equation(s)	dep var(s)	indep var(s)	order	linear (L) or nonlinear (NL)	ODE or PDE	homog (H) or inhomog (IH)
	Radioactive decay $\frac{dy}{dt} = -ry$						
	Mass-Spring-Dashpot system $m\frac{d^2y}{dt^2} + c\frac{dy}{dt} + ky = \sin t$						
	Pendulum equation $\frac{d^2\theta}{dt^2} = -\frac{g}{\ell}\sin\theta$						
	Schrödinger's equation $i\hbar\frac{\partial u}{\partial t} = -\frac{\hbar^2}{2m}\frac{\partial^2 u}{\partial x^2} + Vu$						
	Beam equation $\frac{\partial^4 w}{\partial x^4} + \frac{\partial^2 w}{\partial t^2} = P$						
	Michaelis-Menten equations $\frac{dS}{dt} = -k_1ES + k_{-1}(E_0 - E)$ $\frac{dE}{dt} = -k_1ES + k_3(E_0 - E)$						

Chapter 2

First-Order Equations

This chapter concerns solving differential equations of the form

$$\frac{dy}{dt} = f(t, y).$$

There are no known analytical methods that can solve the general version of this problem. Consequently, assumptions have to be made on the function $f(t, y)$ to be able to derive a solution. The two more useful assumptions are that the function is separable or it is linear, and both are considered in this chapter. The fact is, however, that for many real world problems it is not possible to solve the differential equation(s) by hand. Consequently, the ability to determine the properties of the solution, without actually solving the problem, becomes essential. What this entails is introduced in Section 2.4.

2.1 ▪ Separable Equations

To introduce this method we begin by considering the differential equation

$$\frac{dy}{dt} = 3y^2. \tag{2.1}$$

We are going to treat the derivative as if it were a fraction, and rewrite the above equation as

$$\frac{dy}{y^2} = 3dt. \tag{2.2}$$

So, the variables have been separated in the sense that all of the y terms are on the left hand side, and the t terms are on the right. We now

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integrate both sides, which gives

$$\int \frac{dy}{y^2} = \int 3dt.$$

Carrying out the integrations, and including the usual integration constant, we have

$$-\frac{1}{y} = 3t + c. \tag{2.3}$$

Solving this for y, we obtain the solution

$$y = -\frac{1}{3t + c}\,. \tag{2.4}$$

The last step is to check on whether the separation of variable step might involve dividing by zero. This happens for (2.2) when $y = 0$. Moreover, $y \equiv 0$ is a solution of (2.1), and it is not included in (2.4). Consequently, another solution of the differential equation is

$$y \equiv 0\,. \tag{2.5}$$

The method used to solve (2.1) is rather simple, but it contains a questionable step. Namely, how is it possible to split a derivative as in (2.2)? One way to answer this is to point out that this is also done when using integration by parts, and you can review your calculus textbook on the justification. However, it is easier to reformulate the question and ask: even though the steps used to obtain (2.4) might seem to be mathematically questionable, is (2.4) a solution of the differential equation? This is easy to answer, as you simply have to substitute (2.4) into (2.1) and verify the equation is satisfied.

2.1.1 ▪ General Version

To explain how the method can be used for other problems, suppose the differential equation to solve is

$$\frac{dy}{dt} = f(t, y). \tag{2.6}$$

The method requires that it is possible to find a factorization of the form $f(t, y) = F(t)G(y)$. This means that it is possible to write the differential equation as

$$\frac{dy}{dt} = F(t)G(y). \tag{2.7}$$

Separating variables gives

$$\frac{dy}{G(y)} = F(t)dt,$$

and integrating we get

$$\int \frac{dy}{G(y)} = \int F(t)dt. \tag{2.8}$$

In theory, you carry out the above integrations, and then solve for y. How difficult this might be depends on how complicated the y integral is, and the examples that follow illustrate some of the complications that can arise. It is also important to note that the above method requires that $G(y) \neq 0$. Consequently, in addition to the solutions that come from (2.8), you must include as solutions any constant that satisfies $G(y) = 0$.

Example 1: Find all solutions of $4y' = -y^3$.

Answer: Since $f(t, y) = -\frac{1}{4}y^3$, we can take $F(t) = \frac{1}{4}$ and $G(y) = -y^3$. So, (2.8) becomes

$$-\int \frac{dy}{y^3} = \int \frac{1}{4}dt.$$

Integrating gives us

$$\frac{1}{2y^2} = \frac{1}{4}t + c,$$

which is rewritten as

$$y^2 = \frac{2}{t + 4c}.$$

From this we obtain the two solutions

$$y = \pm\sqrt{\frac{2}{t + 4c}}. \tag{2.9}$$

To check on the $G(y) = 0$ solutions, solving $G(y) = 0$ gives $y = 0$. This constant function is not included in the above expressions for y, so it is a third solution of the equation. ■

Example 2: Find the solution of the IVP: $4y' = -y^3$, where $y(0) = -3$.

Answer: The three solutions of the differential equation were derived in the previous example. Because the initial condition requires the solution to be negative, the solution we need is

$$y = -\sqrt{\frac{2}{t + 4c}}.$$

Setting $y = -3$ and $t = 0$ in this equation gives $3 = 1/\sqrt{2c}$, which means that $c = 1/18$. Therefore, the solution is

$$y = -\sqrt{\frac{18}{9t + 2}}. \quad ■$$

Example 3: Is $y' + y = t$ a separable equation?

Answer: No. For this equation, $f(t, y) = t - y$, and it is not possible to factor this as $f(t, y) = F(t)G(y)$. How to solve this equation is explained in the next section. ■

Example 4: Solve $y' = \dfrac{t}{1+y}$, where $y(0) = -2$.

Answer: Separating variables, so $(1 + y)dy = -tdt$, and then integrating gives

$$y + \frac{1}{2}y^2 = \frac{1}{2}t^2 + c.$$

To satisfy the initial condition, substitute $y = -2$ and $t = 0$ into the above equation, from which we get that $c = 0$. This leaves $y + \frac{1}{2}y^2 = \frac{1}{2}t^2$, or equivalently, $y^2 + 2y - t^2 = 0$. This is a quadratic equation in y, and solving it we get the two solutions

$$y = -1 \pm \sqrt{1 + t^2}.$$

The initial condition is needed to determine which sign to use, and since $y(0) = -2$ then we need the minus sign. Therefore, the solution of the IVP is $y = -1 - \sqrt{1 + t^2}$. ■

Example 5: Solve $y' = -\dfrac{y}{1+y}$, where $y(0) = 1$.

Answer: Separating variable yields

$$\frac{1+y}{y}dy = -dt.$$

Since $(1 + y)/y = 1/y + 1$, and $y(0) > 0$, then integrating we get that

$$y + \ln y = -t + c.$$

It is not possible to solve this for y as in the previous examples, without resorting to more advanced mathematical methods. For this reason, this is an example of what is called an *implicit solution*, and they are very common when solving nonlinear differential equations. Even so, it is still possible to find c from the initial condition. Substituting $y = 1$ and $t = 0$ into the above equation we get that $c = 1$. Therefore, the solution of the IVP is defined implicitly through the equation

$$y + \ln y = -t + 1. \quad \blacksquare \tag{2.10}$$

A few comments need to be made about separation of variables before ending this section.

Integration Constant: The integration constant plays an essential role in the solution of a differential equation. It is useful to be aware that there are different ways you can write it. As an example, instead of (2.9), you can write the solution as

$$y = \pm\sqrt{\frac{2}{t+\bar{c}}}\,,$$

where $\bar{c} = 4c$. Similarly, if the solution is found to be

$$y = \frac{3t - 2c + 4}{t + 2c - 4}\,,$$

you can write it as

$$y = \frac{3t - \bar{c}}{t + \bar{c}}\,, \tag{2.11}$$

where $\bar{c} = 2c - 4$. For both of these examples, the solution contains one undetermined constant, just as in the original version of each solution. It should also be mentioned that this simplification is often used when giving the answers to the exercises. Moreover, instead of (2.11), the answer will likely be written as

$$y = \frac{3t - c}{t + c}\,,$$

or as

$$y = \frac{3t + c}{t - c}\,.$$

Linear or Nonlinear: The method works on linear and nonlinear first-order differential equations. However, it does not work on every linear or nonlinear equation.

Non-uniqueness of Factorization: The factorization $f(t, y) = F(t)G(y)$ is not unique. For example, for $f(t, y) = y + ty$ you can take $F(t) = 1 + t$ and $G(y) = y$. You can also take $F(t) = \frac{1}{2}(1 + t)$ and $G(y) = 2y$. It makes no difference which one you use, it is just required that $f(t, y) = F(t)G(y)$. Any such factorization will lead, eventually, to the same solution for the differential equation.

Finally, when trying to solve a nonlinear differential equations there is always the question as to whether there is a solution, or if there is more than one solution. Some of the possible complications, and how to resolve them, if possible, are considered in Exercise 6.

Exercises

1. Find all of the solutions of the given differential equation.

a) $y' = -3y^4$
b) $y' = y^3 e^{-t}$
c) $y' + y^2 \sin t = 0$
d) $2y' = t/(y-3)$
e) $y' = -(2+t)e^y$
f) $(1+t)y' = -e^{3y}$
g) $y' = -e^{2t+4y}$
h) $y' = -2^y$
i) $y' + (1+3y)^3 = 0$
j) $y' = y^2 + 4y + 4$
k) $2y' = y^2 - 6y + 9$
l) $3y' = y^2 + 1$
m) $y' - te^y = te^{-y}$
n) $y' - e^{-y} = 1$
o) $y' = t(y + 1/y)$

2. Find the solution of the IVP.

a) $y' = -y^3, \quad y(0) = 5$
b) $y' = -2y^3, \quad y(0) = 0$
c) $(1+t)y' = 3 + y, \quad y(0) = 4$
d) $(4+e^t)y' + e^t y^2 = 0, \; y(0) = 1$
e) $y' = te^{-y}, \quad y(0) = -1$
f) $y' = \dfrac{1}{2+y}, \quad y(0) = 0$
g) $y' = 1 + \cos(y), \quad y(0) = \pi/2$
h) $y' = y^2 - 5y, \quad y(0) = 1$
i) $y' + e^{-2y} = 1, \quad y(0) = 1$
j) $y' = 1/(e^{-y} + e^y), \quad y(0) = 0$
k) $y' = \sqrt{1-y^2}, \quad y(0) = 0$
Hint: $y' \geq 0$

3. Find the solution of the IVP. In these problems, the independent variable is not t and the dependent variable is not y.

a) $\dfrac{dq}{dr} = -7q^3, \quad q(0) = -1$
b) $\dfrac{dp}{dr} = -4p^3, \quad p(0) = 0$
c) $3\dfrac{dh}{d\tau} = 2 + h, \quad h(0) = 2$
d) $\dfrac{dh}{dx} = h^2 - 3h, \quad h(0) = 2$
e) $(1+e^{-r})\dfrac{dz}{dr} + z^2 = 0, \quad z(0) = 6$
f) $4\dfrac{dw}{d\tau} = \tau^3 e^{-2w}\,, \quad w(0) = 0$
g) $(\theta+1)^3 \dfrac{dr}{d\theta} = r^2, \quad r(0) = 2$
h) $\dfrac{dr}{d\theta} = \dfrac{2\theta}{1+r}, \quad r(0) = 0$

4. Find the solution of the IVP in implicit form.

a) $y' = 1 + \dfrac{1}{y}\,, \quad y(0) = 1$
b) $y' = \dfrac{3}{1+y^4}\,, \quad y(0) = -1$
c) $y' = \dfrac{1+y}{2+y}\,, \quad y(0) = 5$
d) $y' = \dfrac{e^y}{1+e^y}, \quad y(0) = 2$

5. An elastic cable is hung between two poles as illustrated in Figure 2.1. It is assumed that the poles are located at $x = -L$ and $x = L$, and

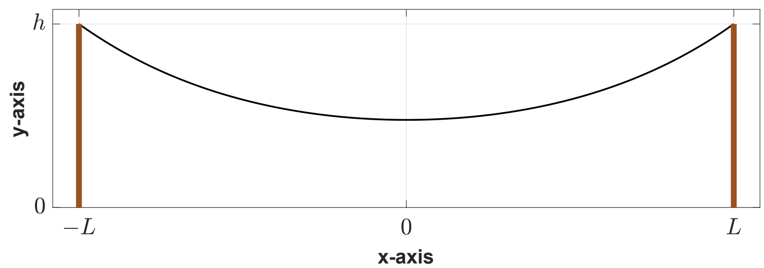

Figure 2.1. *Cable hanging between two poles, as described in Exercise 5.*

have height h. Letting $y(x)$ be the curve determined by the hanging cable, it is found that

$$a\frac{d^2y}{dx^2} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}, \quad \text{for } -L < x < L,$$

where a is a positive constant. Because of the symmetry in the problem, $y'(0) = 0$.

a) Letting $w(x) = y'(x)$, rewrite the differential equation as a first-order equation involving w and w'. Also, what is $w(0)$?

b) Solve the problem in part (a) for w.

c) Integrate $y'(x) = w(x)$, and use the condition $y(L) = h$, to determine $y(x)$. The solution you are finding is an example of what is called a catenary.

6. The following illustrate some of the complications that can arise when solving nonlinear differential equations.

a) The question is whether the implicit solution (2.10) actually has a solution. To show this, rewrite (2.10) as $\ln y = -y - t + 1$. Setting $g(y) = \ln y$, and $h(y) = -y - t + 1$, let $t = 0$ and then sketch $g(y)$ and $h(y)$ on the same axes for $0 < y < \infty$. Explain why this shows that there is exactly one solution of (2.10). Do the same thing for $t = 1$ and $t = 2$. Use this sketching procedure to determine what value y approaches as $t \to \infty$.

b) Redo Example 4, but use the initial condition $y(0) = -1$. Explain why there is no real-valued solution. Also, explain why a complication should be anticipated for this differential equation and initial condition before even trying to find the solution.

c) Solve $y' = \frac{1}{2}y^3$, where $y(0) = 1$. Explain why there is no solution for $t \geq 1$.

d) One solution of $y' = 6ty^{2/3}$ is $y_1 = 0$. Use separation of variables to find a second solution y_2. Using the initial condition $y(0) = 0$, show that there are, at least, two different solutions of the IVP.

2.2 ▪ Integrating Factor

The equation to be solved is

$$y' + p(t)y = g(t). \tag{2.12}$$

What is important here is that this equation is linear, as well as first-order.

The solution will be derived using two formulas from calculus. The first is the product rule, which states that

$$\frac{d}{dt}(\mu y) = \mu(t)y'(t) + \mu'(t)y(t). \tag{2.13}$$

The second is the Fundamental Theorem of Calculus, which states that if

$$\frac{d}{dt}(\mu y) = q(t), \tag{2.14}$$

then

$$\mu y = \int_0^t q(s)ds + c. \tag{2.15}$$

The first step is the observation that the left hand side of (2.12) resembles the right hand side of (2.13). To make it so they are exactly the same we need to multiply the differential equation by μ, which gives us

$$\mu y' + \mu p y = \mu g. \tag{2.16}$$

What we need, to get this to work, is that μ must be such that

$$\mu' = p\mu. \tag{2.17}$$

It will make the formula for the solution a bit simpler if we require $\mu(0) = 1$. The differential equation (2.17) is separable, and one finds that the solution of the IVP is

$$\mu(t) = e^{\int_0^t p(r)dr}. \tag{2.18}$$

With this choice for μ, the differential equation for y in (2.16) can be written as

$$\frac{d}{dt}(\mu y) = \mu g. \tag{2.19}$$

Integrating this we get that

$$\mu y = \int_0^t \mu(s)g(s)ds + c,$$

where c is the usual integration constant. The solution of (2.12) is therefore

$$y(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s)g(s)ds + c\right]. \tag{2.20}$$

The function $\mu(t)$, which is given in (2.18), is said to be an **integrating factor** for the original differential equation.

There are two important special cases to mention. First, suppose that the problem also has an initial condition, say $y(0) = y_0$. Since $\mu(0) = 1$, then from (2.20) the solution of the resulting IVP is

$$y(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s)g(s)ds + y_0\right]. \tag{2.21}$$

The second special case arises for the homogeneous equation $y' + p(t)y = 0$. Setting $g = 0$ in (2.20), gives us the solution

$$y(t) = ce^{-\int_0^t p(r)dr}. \tag{2.22}$$

If $y(0) = y_0$, then the resulting solution is

$$y(t) = y_0 e^{-\int_0^t p(r)dr}. \tag{2.23}$$

Example 1: Solve $y' + 3y = e^{2t}$.

Answer: Since $p = 3$, then

$$\int_0^t p(r)dr = \int_0^t 3dr = 3t.$$

From (2.18), the integrating factor is $\mu = e^{3t}$. So, since $g(t) = e^{2t}$, then from (2.20),

$$\begin{aligned} y(t) &= e^{-3t}\left[\int_0^t e^{3s}e^{2s}ds + c\right] = e^{-3t}\left[\int_0^t e^{5s}ds + c\right] \\ &= e^{-3t}\left[\frac{1}{5}e^{5s}\Big|_{s=0}^{t} + c\right] = e^{-3t}\left[\frac{1}{5}e^{5t} - \frac{1}{5} + c\right] \\ &= \frac{1}{5}e^{2t} + \bar{c}e^{-3t}, \end{aligned}$$

where $\bar{c} = c - 1/5$ is an arbitrary constant. ■

Example 2: Solve $2y' - ty = 6$, where $y(0) = 5$.

Answer: Since $p = -t/2$, then from (2.18), $\mu = e^{-t^2/4}$. Given that $g = 3$, then from (2.21) we have

$$y(t) = e^{t^2/4}\Big(\int_0^t 3e^{-s^2/4}ds + 5\Big).$$

The integral in the above expression can not be written in terms of elementary functions, and so that is the final answer. ■

Recommendation: The method requires you to memorize something. It is easier if you memorize how its derived, which means (2.16)-(2.19), rather than (2.18) and (2.20).

2.2.1 ▪ General and Particular Solutions

We have shown that the solution of the linear differential equation

$$y' + p(t)y = g(t), \tag{2.24}$$

is

$$y(t) = \frac{1}{\mu(t)}\left[\int_0^t \mu(s)g(s)ds + c\right]. \tag{2.25}$$

Any, and all, solutions of (2.24) are included in this formula, and for this reason (2.25) is said to be the **general solution**.

A useful observation about (2.25) is that it can be written as

$$y(t) = y_p(t) + y_h(t), \tag{2.26}$$

where

$$y_p(t) = \frac{1}{\mu(t)}\int_0^t \mu(s)g(s)ds, \tag{2.27}$$

and

$$y_h(t) = \frac{c}{\mu(t)} = ce^{-\int_0^t p(r)dr}. \tag{2.28}$$

The formulas for y_p and y_h are not important here. What is important is that y_p is a solution of the differential equation (2.24). It does not contain the arbitrary constant, and for this reason it is said to be a **particular solution**. In contrast, the function $y_h(t)$, which contains an arbitrary constant, is a solution of the differential equation

$$y' + p(t)y = 0. \tag{2.29}$$

This is the homogeneous equation coming from (2.24). Consequently, $y_h(t)$ is said to be the general solution of the **associated homogeneous equation**.

Example: In Example 1 above we found that the general solution is

$$y(t) = \frac{1}{5}e^{2t} + \bar{c}e^{-3t},$$

where $\bar{c}$ is an arbitrary constant. In this case, a particular solution is $y_p = \frac{1}{5}e^{2t}$, and the general solution of the associated homogeneous equation is $y_h = \bar{c}e^{-3t}$. ■

The observation in the previous paragraph that the general solution can be written as the sum of a particular solution and the general solution of the associated homogeneous equation holds for all linear differential equations (not just those that are first order). Because we are able to derive a formula for the solution, which is given in (2.25), this observation is not really needed to solve first-order linear differential equations. However, for second-order equations, which will be studied in the next chapter, this observation serves a fundamental role in finding the solution.

2.2.2 ▪ Interesting But Tangentially Useful Topics

The following topics are worth knowing about. However, you can skip this material, if you wish, as it is not required to solve any of the problems in this chapter.

Method of Undetermined Coefficients

Most first-order linear differential equations that arise in applications have constant coefficients, which means that they can be written as

$$y' + ay = g(t), \tag{2.30}$$

where a is a constant. This can be solved using an integrating factor, but there is often an easier way to find the solution. This involves making an educated guess for the particular solution. The guess depends on the specific form of the function $g(t)$, and it is the basis of what is called the method of undetermined coefficients. This is explained in Section 3.7 for second-order equations, but it works on first-order equations as well. The reason it is easier is that it avoids having to integrate anything, and you therefore do not need to remember integration rules to find the solution. In fact, for a problem such as the one in Example 1, you can solve it in your head and simply write the answer down. On the other hand, the method will not work for Example 2, and it will not work for Example 1 if $g(t) = e^{2t}$ is replaced, say, with $g(t) = \sqrt{t}$. If you want to pursue this idea a bit more, after reading Section 3.7, you should look at Exercise 4 on page 59.

Connections with Linear Algebra

For those who have taken a course in linear algebra, there is a connection between that subject and linear differential equations that is worth knowing about. To explain, a central problem in linear algebra is to solve $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A}$ is a $m \times n$ matrix. It's possible to prove that if there is a solution of this equation, then it has the form $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is the general solution of the associated homogeneous equation $\mathbf{Ax} = \mathbf{0}$. This is basically the same statement we made for the solution of the linear differential equation (2.24).

To pursue the connection a bit more, the set of vectors that satisfy $\mathbf{Ax} = \mathbf{0}$ is called the *null space* for $\mathbf{A}$. If the null space contains more than just the zero vector, then you can find a set of basis vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_k$ for the null space and write $\mathbf{x}_h = c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_k\mathbf{x}_k$, where c_1, c_2, ..., c_k are arbitrary constants. In this case the dimension of the null space is k. Now, going back to the solution of the differential equation, the general solution of the associated homogeneous equation is given in (2.28). There is one arbitrary constant, so the implication is that null space for the first-order linear homogeneous differential equation is one. As we will see in the next chapter, the null space for a second-order linear homogeneous differential equation is two, and this means the general solution of the associated homogeneous equation contains two arbitrary constants.

The previous two paragraphs illustrate the beauty, and profundity, of mathematical abstraction. Namely, it is possible to make rather significant conclusions about the solution of an equation, irrespective it is algebraic or differential, simply from the basic properties these equations have in common.

Exercises

1. Find the general solution of the given differential equation.

   a) $y' + 3y = 0$
   b) $y' - 2y = t$
   c) $4y' - y = 6 + 2t$
   d) $y' = -y + 2e^t - 1$
   e) $(3t + 2)y' + 3y = \sin(4t) + 5$
   f) $(2 + t)y' + y = 1$
   g) $y' - 3y = 1 + \sqrt{t}$
   h) $2y' + y = \frac{t}{1+t}$

2. Find the solution of the IVP.

   a) $y' - y = 4\ ,\quad y(0) = -1$
   b) $y' + 4y = 3t, \quad y(0) = 0$
   c) $5y' + y = 0, \quad y(0) = 2$
   d) $2y' = y + e^{-t} - 2, \quad y(0) = 1$
   e) $(5 + t)y' + y = -1, \quad y(0) = 2$
   f) $3y' + ty = -2, \quad y(0) = 0$

3. Find the solution of the IVP. In these problems, the independent variable is not t and the dependent variable is not y.

   a) $\dfrac{dq}{dr} + 2q = 4\ ,\quad q(0) = -1$
   b) $\dfrac{dp}{dr} + 4p = -r, \quad p(0) = 0$
   c) $2\dfrac{dw}{d\tau} - w = e^{2\tau}, \quad w(0) = 0$
   d) $\dfrac{dz}{d\tau} = 4z + 1 + \tau, \quad z(0) = 0$
   e) $(x + 7)\dfrac{dh}{dx} + h = -1,\ h(0) = 2$
   f) $(5r+1)\dfrac{dh}{dr} + 5h = 3,\ h(0) = -1$

4. Find a particular solution, and the general solution to the associated homogeneous equation, of the following differential equations.

   a) $y' - 2y = 6$
   b) $y' + y = 3e^{-t}$
   c) $7y' - y = e^{2t} + 3$
   d) $y' + 2ty = 1$

5. A Maxwell viscoelastic material is one for which the stress $T(t)$ and the strain-rate $r(t)$ satisfy

$$T + \tau \frac{dT}{dt} = \kappa r,$$

where τ and κ are positive constants. By solving this equation for T, and assuming $T_0 = T(0)$, show that

$$T = T_0 e^{-t/\tau} + \int_0^t \kappa e^{(s-t)/\tau} r(s) ds.$$

2.3 ▪ Modeling

2.3.1 ▪ Mixing

Typical mixing problems involve a continuously stirred tank, as illustrated in Figure 2.2. As an example, suppose that water, containing salt, is flowing into a well-stirred tank. At the same time, the mixture in the tank is flowing out. The goal is to determine how much salt is in the tank as a function of t.

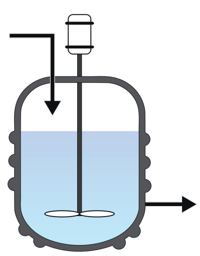

Figure 2.2. *Schematic of a continuous stirred tank.*

The quantities of interest in this problem are:

Q**:** This is the amount of salt in the tank. If the volume of water in the tank is V, and c is the concentration of salt in the water, then $Q = cV$.

R_i**:** This is the rate that salt is flowing into the tank. If the incoming volumetric flow rate is F_i, and c_i is the concentration of salt in the incoming water, then $R_i = c_i F_i$.

R_o**:** This is the rate that salt is flowing out of the tank. If the outgoing volumetric flow rate is F_o, then $R_o = cF_o$.

If the initial amount of salt in the tank is Q_0, then the resulting IVP for Q is:

$$\begin{aligned} \frac{dQ}{dt} &= R_i - R_o, \\ Q(0) &= Q_0. \end{aligned}$$

Example 1

Suppose that salt water, containing 1/2 lbs of salt per gal, is poured into a tank at 2 gal/min. Also, the water flows out of the tank at the same rate. If the tank starts out with 100 gal of water, with 10 lbs of salt per gal, find a formula for the total amount of salt in the tank.

Setup

inflow: Since $F_i = 2$, and $c_i = 1/2$, then $R_i = 1$.

outflow: Since the mixture flows out at 2 gal/min, then the volume of water in the tank stays at 100 gal. Since $F_o = 2$, and $c_o = Q/100$, then $R_o = \frac{1}{50}Q$.

$t = 0$*:* Given that at the start there are 10 lbs of salt per gal, $Q(0) = 1000$.

The resulting IVP for Q is:

$$\frac{dQ}{dt} = 1 - \frac{1}{50}Q, \tag{2.31}$$

$$Q(0) = 1000. \tag{2.32}$$

Note that because of the way the variables have been defined, Q is measured in pounds and t is measured in minutes.

Solution

Using separation of variables, or an integrating factor, one finds that $Q(t) = 50 + 950e^{-t/50}$.

Question: What is the eventual concentration of salt in the tank?

Answer using solution: Since $\lim_{t\to\infty} Q(t) = 50$, then the eventual concentration is $50/V = \frac{1}{2}$ lbs/gal.

Answer using physical reasoning: The concentration in the tank will eventually be the same as the concentration for the incoming flow, and so the answer is 1/2 lbs/gal.

Answer using math reasoning: If the eventual salt content in the tank is a constant, then that constant must be a solution of the differential equation (2.31). The constant solution is $Q = 50$, and so the eventual concentration is $50/V = \frac{1}{2}$ lbs/gal. Note that this reasoning requires that the $Q = 50$ solution is asymptotically stable, and how to determine is explained in Section 2.4. ■

Example 2

Salt water, containing 3 lbs of salt per gal, flows into a 50 gal drum at 2 gal/sec. If the drum initially contains 10 gal of pure water, find a formula for Q as a function of t.

Comments about this problem: There is no outflow, so the volume of water will increase. However, it's a 50 gal drum, so eventually it will fill and start running over. When this occurs there is outflow, at a rate equal to the incoming rate. To account for this, the problem needs to be split into two phases, one where the volume is increasing, and the second when it is a constant.

Solution

Phase 1: In this case, $R_i = 6$, $R_o = 0$, and $Q(0) = 0$. The resulting IVP is

$$\begin{aligned}\frac{dQ}{dt} &= 6\\ Q(0) &= 0\end{aligned}$$

The solution is $Q(t) = 6t$. Also, the volume of water in the tank is $V = 10 + 2t$. So, this solution for Q holds for $V \leq 50$, which means that $t \leq 20$.

Phase 2: As before, $R_i = 6$. For the outflow, the rate is 2 gal/sec and the concentration in the outflow is $Q/50$. This means that $R_o = Q/25$. Now, this phase starts at $t = 20$, and the amount of salt in the tank at this time is 120 (this comes from the solution for Phase 1). This means that the problem to solve is

$$\begin{aligned}\frac{dQ}{dt} &= 6 - \frac{1}{25}Q, \quad \text{for } \ 20 < t,\\ Q(20) &= 120.\end{aligned}$$

What is different about this problem is the time interval, which is not the usual $0 \leq t$. However, this does not interfere with our solution methods, and the solution can be found using an integrating factor or separation of variables. One finds that the general solution of the differential equation is

$$Q(t) = 150 + Ae^{-t/25}.$$

From the requirement that $Q(20) = 120$ it follows that $A = -30e^{4/5}$.

The Solution: Combining the Phase 1 and Phase 2 solutions, we have that

$$Q(t) = \begin{cases} 6t & \text{if} \quad 0 \leq t \leq 20, \\ 150 - 30e^{(20-t)/25} & \text{if} \quad 20 < t. \quad \blacksquare \end{cases}$$

2.3.2 ▪ Newton's Second Law

Suppose an object with mass m is moving along the x-axis. Letting $x(t)$ be its position, then its velocity is $v = \frac{dx}{dt}$, and its acceleration is $a = \frac{d^2x}{dt^2} = \frac{dv}{dt}$. If the object is acted on by a force F, then from Newton's second law, which states that $F = ma$, we have that

$$m\frac{dv}{dt} = F. \tag{2.33}$$

What sort of differential equation this might be depends on how F depends on v. Once (2.33) is solved for v, then the position is determined by integrating the equation

$$\frac{dx}{dt} = v. \tag{2.34}$$

To complete the problem, given the initial position $x(0) = x_0$, the integration constant can be determined.

Vertical Motion

The object is assumed to be moving vertically, either up or down. In this case, $x(t)$ is the distance of the object from the ground. It is also assumed that it is acted on by gravity, F_g, and a drag force, F_r. Consequently, $F = F_g + F_r$. As for what these forces are:

Gravitational force: Assuming the gravitational field is uniform, then $F_g = -mg$, where g is the gravitational acceleration constant. The minus sign is because the force is in the downward direction.

Drag force: As long as the object is not moving very fast, the drag is proportional to the velocity (see Exercise 10). In this case, $F_r = -cv$, where c is a positive constant (the minus sign is because the force is in the opposite direction to the direction of motion).

Units and Values: In the exercises, the value to use for g is usually stated. If it is not given, then you should leave g unevaluated. Whatever value is used, it is only approximate. If a more physically realistic value is needed, then you should probably use the Somigliana equation. Finally, weight is a force, so for an object that weighs w lbs, its mass can be determined from the equation $w = mg$.

Example: Suppose a ball with a mass of 2 kg is dropped, from rest, from a height of 1000 m. Assume that the forces acting on the object are gravity, and the drag force is due to air resistance, with $c = \frac{1}{2}$ kg/s. Assume that $g = 10\,\mathrm{m/s}^2$.

Question 1: What is the resulting IVP for v, and what problem must be solved to find x?
Answer: Since $F = -mg - cv$, where $m = 2$ and $c = 1/2$, from (2.33), the differential equation is

$$\frac{dv}{dt} = -10 - \frac{1}{4}v.$$

Since the object is dropped from rest, then the initial condition is $v(0) = 0$. Once v is known, then x is found by integrating (2.34), and using the fact that $x(0) = 1000$.

Question 2: What is the solution of the IVP, and the resulting solution for x?
Answer: Using an integrating factor, it is found that the general solution is $v = -40 + ce^{-t/4}$. Applying the initial condition we get that

$$v = 40(-1 + e^{-t/4}).$$

Integrating $x' = 40(-1 + e^{-t/4})$, yields $x = 40(-t - 4e^{-t/4}) + c$. Since $x(0) = 1000$, then $c = 1160$. So, $x = 40(-t - 4e^{-t/4}) + 1160$.

Question 3: What is the terminal velocity v_T of the object?
Answer: There are two ways to determine v_T. The easy way, assuming there is a terminal velocity, is note that $v = v_T$ is constant and it must satisfy the differential equation $v' = -10 - v/4$. From this we get that $v_T = -40\,\mathrm{m/s}$. The second way to find v_T is to take the limit $t \to \infty$ in the above solution. This yields $v_T = -40\,\mathrm{m/s}$.

Question 4: When does the object hit the ground?
Answer: It hits the ground when $x = 0$, which means that it is the value of t that satisfies $t + 4e^{-t/4} = 29$. To determine the actual value for t, it is necessary to use a numerical method to find it. However, it is possible to estimate the value by assuming it takes several seconds to hit the ground, which means that $e^{-t/4} \approx 0$. The conclusion is that $t \approx 29$ s. In comparison, the numerically computed value is about 28.997 s. ■

2.3.3 ▪ Logistic Growth or Decay

An assumption often made for the growth of the population of a species is that the population grows at a rate proportional to the current population.

If $P(t)$ is the population at time t, then this assumption results in the equation $P' = kP$. The solution is $P(t) = P(0)e^{kt}$, which means that there is exponential growth in the population. This is not sustainable in the real world, and it is more realistic to assume that the rate of growth slows down as the population increases. In fact, if the population is very large, the population should decrease instead of increase. A simple model for this is to assume that $k = r\big(1 - \frac{P}{N}\big)$, where r and N are positive constants. The resulting differential equation is

$$P' = r\Big(1 - \frac{P}{N}\Big)P, \tag{2.35}$$

which is known as the *logistic equation.* This nonlinear equation can be solved using separation of variables, and partial fractions. Doing this, in the case of when $0 < P < N$,

$$\begin{aligned} \frac{NdP}{(N-P)P} &= rdt \qquad (2.36)\\ \Rightarrow \quad \int \Big(\frac{1}{P} + \frac{1}{N-P}\Big)dP &= rt + c \\ \Rightarrow \quad \ln\frac{P}{N-P} &= rt + c \\ \Rightarrow \quad \frac{P}{N-P} &= e^{rt+c}. \end{aligned}$$

From this, we get

$$P = (N-P)\bar{c}e^{rt}, \tag{2.37}$$

where $\bar{c} = e^c$ is a positive constant. Doing the same thing for the case of when $N < P$, one again gets (2.37) except that $\bar{c}$ is a negative constant. Moreover, for the divide by zero case of when $P = 0$, you get (2.37) but $\bar{c} = 0$. In other words, except for when $P = N$, (2.37) holds with the understanding that $\bar{c}$ is an arbitrary constant. Solving (2.37) for P yields

$$P = \frac{N\bar{c}e^{rt}}{1 + \bar{c}e^{rt}}, \tag{2.38}$$

where $\bar{c}$ is an arbitrary constant. If $P(0) = P_0$, and if $P_0 \neq N$, then one finds that $\bar{c} = P_0/(N - P_0)$. When $P(0) = N$, this is a divide by zero situation in (2.36), and the resulting solution is just the constant $P(t) = N$.

The solution we have derived in (2.38) is known, not surprisingly given the name of the model, as the *logistic function* or the *logistic curve.* When

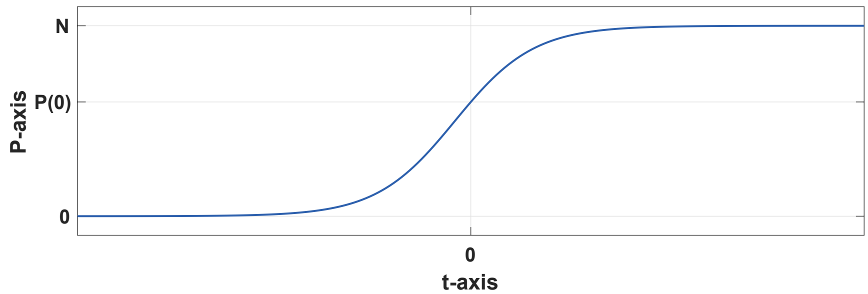

Figure 2.3. *Solution (2.38) of the logistic equation, for $-\infty < t < \infty$, in the case of when $P(0) < N$.*

plotted for $-\infty < t < \infty$ it has a S, or sigmoidal, shape as shown in Figure 2.3. It is one of those functions that appears in so many applications that it deserves its own graph in this textbook (hence Figure 2.3). ■

2.3.4 ▪ Newton's Law of Cooling

The assumption is that the rate of change of the temperature of an object is proportional to the difference between its temperature and the ambient temperature (i.e., the temperature of its surroundings). This is often referred to as Newton's law of cooling, but it also applies to heating an object.

To write down the mathematical form of this statement, we introduce the following:

T**:** This is the temperature of the object at time t.

T_a**:** This is the ambient temperature.

k**:** This is the proportionality coefficient.

If the initial temperature of the object is T_0, then the resulting IVP for T is:

$$\frac{dT}{dt} = -k(T - T_a). \tag{2.39}$$

$$T(0) = T_0. \tag{2.40}$$

This problem can be solved using an integrating factor, or by using separation of variables. It is found that $T = T_a + (T_0 - T_a)e^{-kt}$.

Example 1: Cooling a Cup of Coffee

According to the National Coffee Association, the ideal temperature for brewing coffee is 200° F, and to get the most flavor out of it, you should drink it when the coffee is between 120 and 140° F.

Question 1: If the room temperature is 70° F, what is the solution of the resulting IVP for T?

Answer: Since $T_0 = 200$ and $T_a = 70$, then

$$T = 70 + 130e^{-kt}. \tag{2.41}$$

Question 2: If the temperature is 180° F after 2 minutes, determine k.

Answer: Using the formula from part (a), $180 = 70+130e^{-2k}$. From this one finds that $k = \frac{1}{2}\ln(13/11)\,\frac{1}{\text{min}}$.

Question 3: When should you start drinking the coffee (according to the National Coffee Association)?

Answer: The time when $T = 140$ occurs when $140 = 70 + 130e^{-kt}$, from which one finds that

$$t = 2\frac{\ln(13/7)}{\ln(13/11)}\ \text{min}. \tag{2.42}$$

Question 4: What is the computed value for the answer for Question 3?

Answer: It is $t \approx 7.4$ minutes. ■

Example 2: Nonlinear Cooling

Experimentally it has been observed that for certain fluids the k in (2.39) is not constant. To account for this, according to what is known as the Dulong-Petit law of cooling, the k in (2.39) is replaced with $k(T-T_a)^{1/4}$. The resulting differential equation is

$$\frac{dT}{dt} = -k(T-T_a)^{5/4}.$$

This requires cooling, and so it requires $T \geq T_a$.

Question: As in Example 1, suppose that the room temperature is 70° F and $T(0) = 200°$ F. What is the solution of the resulting IVP?

Answer: Separating variables,

$$-\frac{dT}{(T-70)^{5/4}} = kdt$$

$$\Rightarrow$$

$$\frac{4}{(T-70)^{1/4}} = kt + c$$

$$\Rightarrow$$

$$(T-70)^{1/4} = \frac{4}{kt+c}.$$

Solving this for T, we get that

$$T = 70 + \left(\frac{4}{kt + c}\right)^4.$$

Since $T(0) = 200$, then the above equation gives us that $130 = (4/c)^4$. Solving this we obtain $c = 4/130^{1/4}$. ■

Reality Check: The models that are considered here are used to illustrate how, and where, differential equations arise. As with all models, there are numerous simplifying assumptions that are made to obtain the resulting mathematical problem. Many of these assumptions are not considered or accounted for in our examples, and the same is true for the exercises. As a case in point, Newton's Law of Cooling is usually limited to cases of when $|T - T_a| \ll |T_a|$, and its applicability depends on whether the heat flow is due to conduction, convection, or radiation. Said another way, if you want to impress your family at Thanksgiving by using the solution of the cooking a turkey exercise (see below), just make sure to check on the turkey temperature regularly to make sure your predictions are correct.

Exercises

In answering the following questions, do not numerically evaluate numbers such as $\sqrt{2}$, $\pi/3$, e^2, $\ln(4/3)$, etc. The exception to this is when the question explicitly asks you to *compute* the answer.

1. The IVP for radioactive decay was derived in Example 1, on page 1.
   a) What is the solution of the IVP for a radioactive material?
   b) If 12 mg of a radioactive material decays to 9 mg in one day, find k.
   c) The half-life of a radioactive material is the time required for it to reach one-half of the original amount. What is the half-like of the material in part (b)?
2. Radiocarbon dating uses the decay of carbon-14 to estimate how long ago something died. The assumption is that carbon-14 satisfies the radioactive decay problem derived in Example 1, on page 1.
   a) What is the solution of the IVP for a radioactive material?
   b) The half-life of a radioactive material is the time required for it to reach one-half of the original amount. The half-life of carbon-14 is 5,730 years. Use this to determine k.
   c) The amount of carbon-14, relative to carbon, is the same in all living organisms. When an organism dies the amount starts to undergo

radioactive decay. So, for radioactive dating you know N_0, as well as the current value of N. Explain how knowing N_0, N, and k can be used to determine t (which is the time that has passed since the organism died).

d) Measurements in 1991 determined that the amount of carbon-14 in the Temple Scroll, which is one of the Dead Sea scrolls found at Qumran, to be 186.18. The amount in living organisms is 238. Determine (i.e., compute) what two years the scroll could have been written in. Note that in the BC/AD system there is no year zero, so it goes from 1 BC to 1 AD.
Comment: The organism here is the parchment from the scroll, and the testing is described in Bonani et al. [1992].

Mixing

3. A tank contains 100 L of salt water with a concentration of 2 g/L. To flush the salt out, pure water is poured in at 4 L/min, and the mixture in the tank flows out at the same rate.
   a) What is the resulting IVP for the total amount $Q(t)$ of salt in the tank?
   b) Solve the IVP determined in part (a).
   c) How long does it take until the amount of salt in the tank is 1% of its original amount?
4. A tank contains 20 L of fresh water. Suppose water, containing $\frac{1}{4}$ g/L of salt, starts to flow into the tank at 2 L/min, and the well-stirred mixture flows out at the same rate.
   a) What is the resulting IVP for the amount $Q(t)$ of salt in the tank?
   b) Solve the IVP determined in part (a).
   c) How much salt is in the tank after one hour?
5. Ten years ago, a factory started operation in a pristine valley. The valley's volume is $10^6\,\mathrm{m}^3$. Each year the factory releases $10^5\,\mathrm{m}^3$ of exhaust through its smoke stacks, and this exhaust contains 1000 kg of pollutants. Assume that the well-mixed polluted air leaves the valley at $10^5\,\mathrm{m}^3/\mathrm{yr}$.
   a) What is the IVP for the amount of pollutant in the valley?
   b) How much pollutant is in the valley now?
6. A small lake contains 60,000 gal of pure water. There is an inlet stream of pure water into the lake, as well as an outlet stream, both flowing at a rate of 100 gal/min. Suppose someone starts pouring water into the lake at the rate of 10 gal/min that contains 5 lbs/gal of a chemical, and they do this for 8 hours. While this happens the inlet stream of pure water is unchanged, and the outflow rate from the lake remains at 100 gal/min.

a) What is the formula for the volume of the lake while the person is pouring?
b) What IVP must be solved to determine the amount of the chemical in the lake?
c) How much of the chemical is in the lake when the person stops pouring?
d) Once the person stops pouring, what IVP must be solved to determine how much of the chemical is in the lake?

Newton's Second Law

7. A mass of 10 kg is shot upward from the surface of the Earth with a velocity of 100 m/s. In addition to gravity, assume that there is a drag force $F_r = -cv$, where $c = 5\,\mathrm{kg/s}$. Assume that $g = 10\,\mathrm{m/s}^2$.

   a) Write down the IVP for v, and then find its solution.
   b) Find x.
   c) How high does the object get?

8. A skydiver weighing 176 lbs drops from a plane that is at an altitude of 5000 ft. Assume that $g = 32\,\mathrm{ft/s}^2$.

   a) Before the parachute opens, the forces on the skydiver are gravity and a drag force $F_r = -cv$. Assuming $v(0) = 0$, write down the IVP for v, and then find the solution.
   b) It is claimed that the terminal velocity of a person falling is 120 mph. Use this to determine c.
   c) If the parachute is opened after 10 s of free fall, what is the speed of the skydiver when it opens?
   d) Find the distance the skydiver falls before the parachute opens.
   e) When the parachute is open, the drag force increases by a factor of 8 from the free fall drag force. What is the resulting terminal velocity of the skydiver?

9. A spherical object sinking to the bottom of a lake is acted on by three forces: a drag force $F_r = -cv$, a buoyant force F_b, and gravity F_g. According to Archimedes' principle, the buoyant force is equal to the weight of the water that is displaced by the sphere.

   a) What is the formula for F_b in terms of the radius a, water density ρ, and g?
   b) The differential equation for the velocity of the sphere has the form $mv' = A - cv$. What is A?
   c) Assuming there is a terminal velocity, find a formula for it in terms of c, a, ρ, and g. What condition must be satisfied if the sphere is sinking?

d) Assuming the sphere is released from rest, solve the resulting IVP for v.

e) Assume the object is released a distance L from the bottom of the lake. Also assume that it takes a while for it to hit the bottom. Use an approximation similar to the one used in Question 4 on page 23 to derive an approximate formula for the time it takes it to hit the bottom.

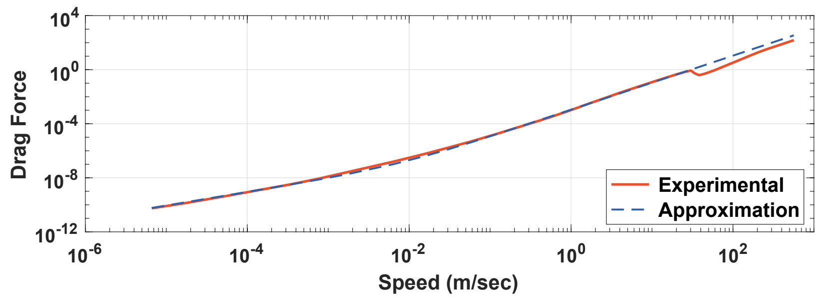

Figure 2.4. *Drag force on a smooth sphere as a function of the speed [Roos and Willmarth, 1971, NASA, 2019]. The approximation is used in Exercise 10.*

10. A spherical object falling in the atmosphere is acted on by gravity, F_g, and a drag force F_r. It is assumed that $F_r = -cv(1-\beta v)$, where v is the velocity. Both c and β are positive constants.

a) Assuming the sphere is dropped from rest, what is the resulting IVP for v?

b) Assuming there is a terminal velocity, find the formula for it.

c) The constants in F_r are $c = 24R\mu$ and $\beta = R\rho/(36\mu)$, where R is the radius of the sphere, ρ is the air density, and μ is the air viscosity. For a baseball falling in the atmosphere, $R = 0.037$, $\mu = 1.8 \times 10^{-5}$, and $\rho = 1.2$ (using kg, m, s units). Also, the mass of a baseball is 0.14. Compute the terminal velocity. How does this compare to what is the speed of a typical fastball in professional baseball?
Comment: The drag force used in this problem is close to what is observed experimentally. To demonstrate this, the experimentally determined values of the drag, and the values determined using F_r, are shown in Figure 2.4 as a function of the speed $|v|$.

Logistic Growth or Decay

11. Species can help each other, the benefits of which are called mutualism. One model for this results in the equation

$$P' = r\frac{1-\frac{P}{N}}{1+\frac{P}{N}}P,$$

where r and N are positive constants.

a) Assuming that $P(0) = \frac{1}{2}N$, solve the resulting IVP for P.

b) What is the limiting population $P(\infty) = \lim_{t\to\infty} P(t)$? Does mutualism result in a different limiting population than obtained using the logistic equation model?

12. The population of fish in a large lake can be modeled using the logistic equation. However, assuming that the fish are caught at a constant rate h, the equation for the population becomes

$$P' = r\left(1 - \frac{P}{N}\right)P - h,$$

where r and N are positive constants. In this problem take $r = 4$, $h = 750$, and $N = 1000$. Also, $P(0) = 1000$.

a) Solve the IVP for P.

b) What is the limiting population? In other words, what is $P(\infty) = \lim_{t\to\infty} P(t)$?

Cooling or Heating

13. Suppose coffee has a temperature of 200° F when freshly poured, and the room temperature is 72° F. In this exercise use Newton's law of cooling.

a) What IVP does the temperature of the coffee satisfy?

b) What is the solution of the IVP?

c) If the coffee cools to 136° F in five minutes, what is k?

d) When does the coffee reach a temperature of 150° F?

14. Redo the previous exercise but use the Dulong-Petit law of cooling.

15. To cook a turkey you are to put it into a 350°F oven, and cook it until it reaches 165°F. In answering the following questions, assume Newton's law of cooling is used.

a) Suppose the turkey starts out at room temperature, which is 70°F. What IVP does the temperature satisfy?

b) Suppose that after two hours in the oven, the temperature of the turkey is 140°F. How much longer before it is done?

c) Suppose the turkey is taken from the refrigerator, which is set to 40°F, and put directly into the oven. How much longer does it take to cook than when the turkey starts out at room temperature? The value for k is the same as in part (b).

16. A homicide victim was discovered at 1 p.m. in a room that is kept at 70°F. When discovered, the temperature of the body was 90°F, and one hour later it had dropped to 85°F.

a) Assuming Newton's Law of Cooling, and normal body temperature is 98.6°F, how long had the person been dead when the body was discovered?

b) Compute the time of death. Round your answer so it just gives the hour and minute (e.g., 7:13 a.m. or 5:32 p.m.).

17. Suppose that in Newton's Law of Cooling that k is found to depend on temperature. A common assumption is that $k = k_0 + k_1(T - T_a)$, where k_0 and k_1 are positive constants.

a) What is the resulting differential equation for T?

b) To find T it makes things easier to introduce the variable $S(t) = T(t) - T_a$. Rewrite the differential equation in part (a) in terms of S. Also, if $T(0) = T_0$, what is $S(0)$?

c) Solve the resulting IVP in part (b) for S, and then use this to show that

$$T = T_a + \frac{k_0 c e^{-k_0 t}}{1 - k_1 c e^{-k_0 t}},$$

where $c = S_0/(k_0 + k_1 S_0)$ and $S_0 = T_0 - T_a$.

d) Using (2.41), it was found you have to wait about 7.4 minutes to drink the coffee. Taking $k_0 = \frac{1}{2}\ln(13/11)$ and $k_1 = 0.01$, compute how long you need to wait using the solution for T from part (c).

2.4 ▪ Steady States and Stability

All of the applications considered in the previous section have one thing in common: the solution eventually approaches a constant value, or steady state. This is not unusual, as this is what often happens. What is of interest here is whether it is possible to determine the eventual steady state without actually having to solve the problem.

To illustrate, as explained in the previous section, the population $P(t)$ of a species is determined by solving

$$P' = f(P), \tag{2.43}$$

where, for this example, we will take

$$f(P) = 2(3 - P)P. \tag{2.44}$$

The solution of this equation is given in (2.38), and it is plotted in Figure 2.5 for the case of when $P(0) = 0.1$, and when $P(0) = 4.5$. It shows that for both initial values, the population approaches, asymptotically, $P = 3$. In both cases the approach is monotonic, either increasing or decreasing.

What is important for this discussion is that it is possible to determine the general behavior of the solution seen in Figure 2.5 without actually solving the problem. This requires the following three observations:

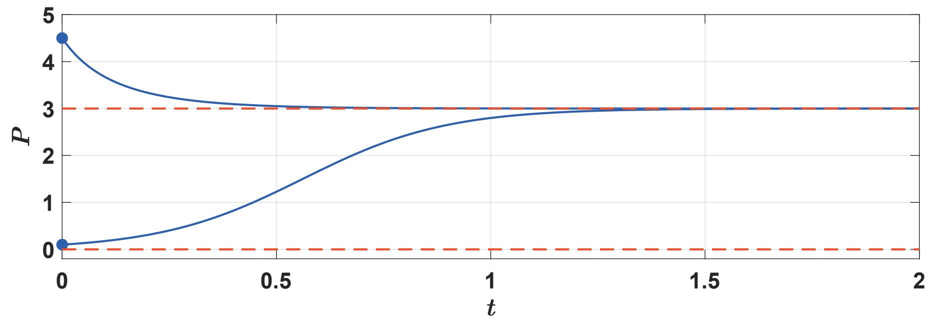

Figure 2.5. *Solution of (2.43) and (2.44) in the case of when $P(0) = 0.1$, and when $P(0) = 4.5$. The dashed red lines are the steady state values.*

Steady States: If the solution does asymptotically approach a constant value $\overline{P}$, then $P = \overline{P}$ must be a solution of the differential equation. This means that it is required that $f(\overline{P}) = 0$, and from this we get the two values $\overline{P} = 0$ and $\overline{P} = 3$. These are called **steady states** for this equation.

Unstable: Even though the initial value $P(0) = 0.1$ is close to the steady state $\overline{P} = 0$, the solution moves away from $\overline{P} = 0$. This happens because of $f(P)$. To explain, the function $f(P)$ is plotted in Figure 2.6. It shows that $f(P)$ is positive for $0 < P < 3$. So, in this interval $P'(t) > 0$, and this means that P is increasing. Similarly, since $f(P)$ is negative for $3 < P$, then P is decreasing in this interval. The arrows in Figure 2.6 indicate the corresponding movement of P. The conclusion we derive from the arrows is that if $P(0)$ is anywhere in $0 < P < 3$, then the solution will move away from the steady state at $\overline{P} = 0$. Because of this, the steady state is said to be **unstable**.

Stable: The second conclusion we make from the arrows in Figure 2.6 is

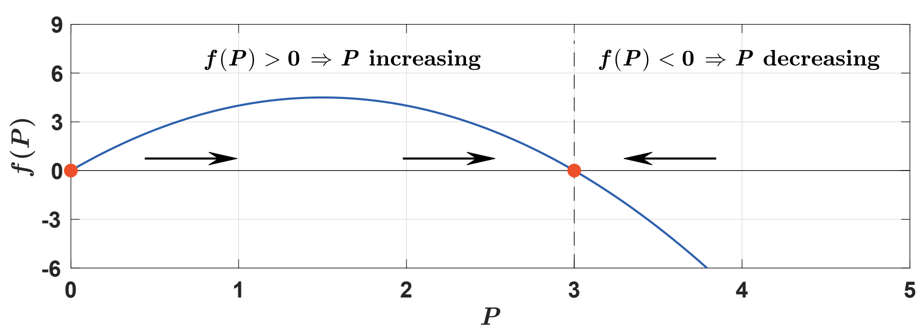

Figure 2.6. *The function $f(P)$ in (2.44). The two steady states are shown by the reds dots. The arrows indicate the direction P moves in the respective interval.*

that if $P(0)$ is anywhere in $0 < P < 3$, then the solution increases towards the steady state $\overline{P} = 3$. Moreover, if $P(0)$ is anywhere in $3 < P$, then the solution decreases towards the steady state $\overline{P} = 3$. A consequence of this is that, no matter what initial condition we pick near $P = 3$,

$$\lim_{t\to\infty} P(t) = 3.$$

For this reason, $\overline{P} = 3$ is said to be an **asymptotically stable** steady state.

The key to what guarantees that $\overline{P} = 3$ is asymptotically stable is that $f(P)$ is positive to the left of the steady state, and negative to the right of it. In other words, $f(P)$ is a deceasing function at the steady state, and this means that $f'(3) < 0$. At the unstable steady state $f(P)$ is increasing, and this means that $f'(0) > 0$.

2.4.1 ▪ General Version

The reasoning used in the above example is easily extended to more general differential equations. To do this, assume the equation is

$$y' = f(y), \tag{2.45}$$

where $f'(y)$ is a continuous function of y. Because $f(y)$ is assumed to not depend explicitly on t, the equation is said to be *autonomous*. So, $y' = 1 + y^3$ is autonomous, but $y' = t + y^3$ is not.

Steady State. *$y = Y$ is a steady state for (2.45) if it is constant and $f(Y) = 0$.*

Stability Theorem. *A steady state $y = Y$ is asymptotically stable if $f'(Y) < 0$ and it is unstable if $f'(Y) > 0$.*

The condition for asymptotic stability means that if $y(0)$ is any point close to Y, then

$$\lim_{t\to\infty} y(t) = Y.$$

In contrast, when $f'(Y) > 0$, there is a small interval around $y = Y$ so that any $y(0)$ located in the interval, with $y(0) \neq Y$, results in the solution moving away from Y (and it never returns). Consequently, in this case, Y is unstable. Where the solution goes in this case depends on the specific problem.

The condition $y(0) \neq Y$, which was made when discussing an unstable steady state, merits a comment. No matter if the steady state is stable or unstable, if $y(0) = Y$, then $y(t) = Y$ is a solution of the resulting IVP.

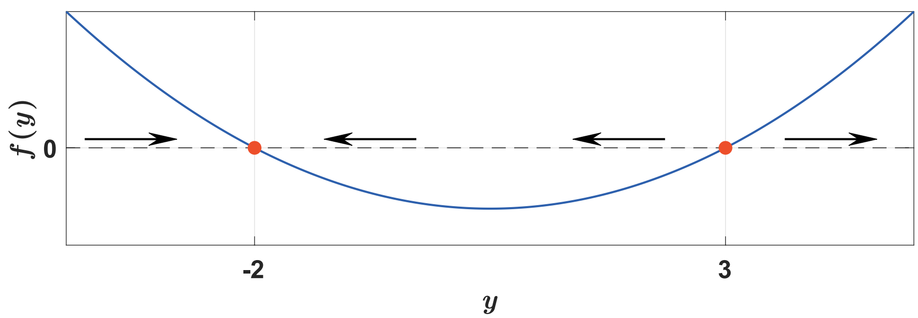

Figure 2.7. *The function $f(y)$ for Example 1. The two steady states are shown by the reds dots. The arrows indicate the direction y moves in the respective interval.*

Consequently, what is of interest is what the solution does if you start close, but not exactly at, a steady state.

Using the above stability theorem it is relatively simply to determine if a steady state is stable or unstable. It is also relatively easy to sketch the solution. The following two examples illustrate what is involved.

Example 1: Find the steady states, and determine their stability for

$$y' = y^2 - y - 6.$$

Answer: The steady states are found by solving $y^2 - y - 6 = 0$, and from this we get $Y = 3$ and $Y = -2$. To determine their stability, since $f(y) = y^2 - y - 6$, then $f' = 2y - 1$. Since $f'(3) = 5 > 0$, then $Y = 3$ is unstable, and since $f'(-2) = -5 < 0$, then $Y = -2$ is asymptotically stable. ■

2.4.2 ▪ Sketching the Solution

It is possible to expand on the above discussion and sketch the solution. As usual, it is easiest to explain how this is done by using an example.

Example 2: Sketch the solution of $y' = f(y)$, where $f(y)$ is given in Figure 2.8.

Answer: To do this it is necessary to know $y(0)$. Before picking this value, we first see what can be determined about the solution.

Steady States: The steady states are the points where $f(y) = 0$. From Figure 2.8, this happens when $y = -2$, $y = 1$, and $y = 3$. These are identified in Figure 2.9 using red dots. From the graph it is evident

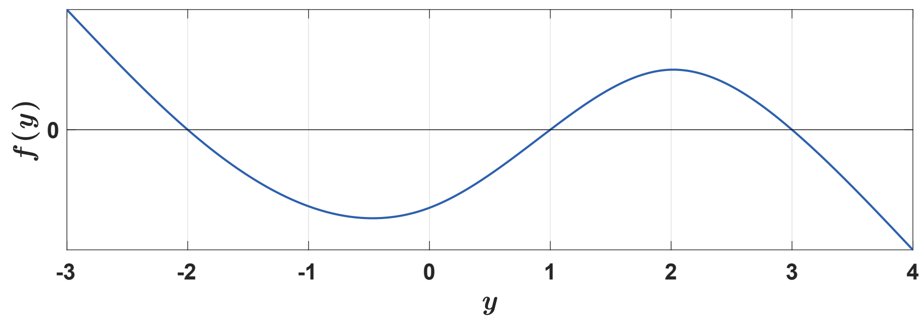

Figure 2.8. *The function $f(y)$ for Example 2.*

that $f'(-2) < 0$ and $f'(3) < 0$, and this means that $y = -2$ and $y = 3$ are asymptotically stable. Similarly, since $f'(1) > 0$, then $y = 1$ is unstable.

Increasing or Decreasing: If $f(y) > 0$, then the solution is increasing, and if $f(y) < 0$, then the solution is deceasing. The respective y intervals where this happens are shown in Figure 2.9 using arrows.

Concave Up or Concave Down: The solution curve is concave up if $y'' > 0$. Since $y' = f(y)$, then $y'' = f'(y)y'(t) = f'(y)f(y)$. So, a point of inflection, which is where $y''(t) = 0$, occurs when $f'(y) = 0$ or when $f(y) = 0$. This means that steady states, and local maximum and minimum points for $f(y)$, are inflection points. These are indicated in Figure 2.9 using vertical dotted lines. Between each of these dotted lines $y(t)$ is either concave up or concave down. Which one is determined by whether $f'(y)f(y)$ is positive or negative. For example, when $1 < y < 2$, $f(y)$ is increasing, so $f'(y) > 0$, and $f(y)$ is positive. This means that $y''(t) > 0$, and so the solution is concave up. This is indicated in Figure 2.9. What happens with the other subintervals can be determined in a similar manner, and

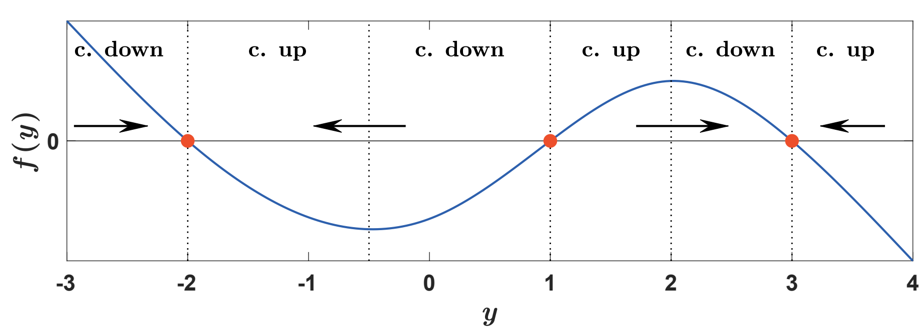

Figure 2.9. *The annotated version of the function $f(y)$ shown in Figure 2.8. Concave up (c. up) and concave down (c. down) refer to the solution $y(t)$.*

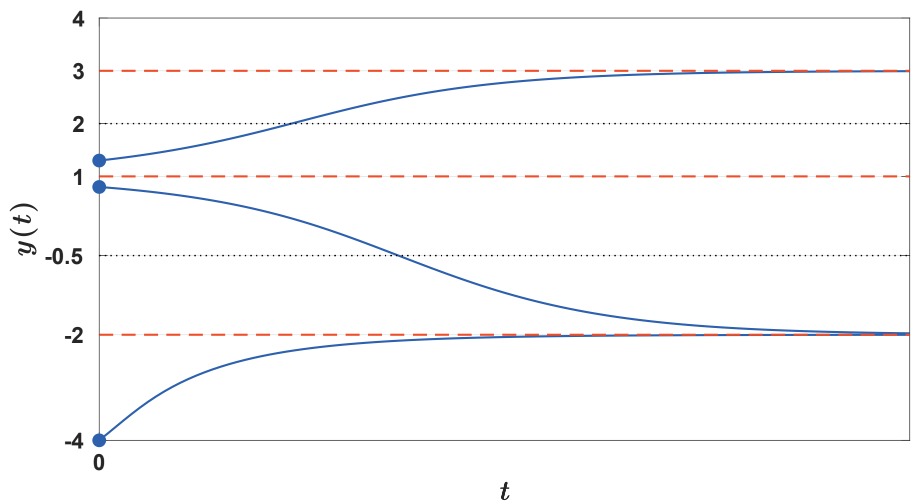

Figure 2.10. *Solution curves obtained using the information in Figure 2.9. The dashed red lines are the steady state values.*

the results are shown in the figure.

We will now use the above conclusions to sketch the solution.

$y(0) = 1.3$*:* This point is located between two steady states, specifically, $1 < y < 3$. According to Figure 2.9, $y(t)$ increases monotonically in this interval, and asymptotically approaches $y = 3$. Also, the curve is concave up until y crosses $y = 2$, after which it is concave down. A curve with these properties is shown in Figure 2.10.

$y(0) = 0.8$*:* In this case, the point is located between two steady states, namely, $-2 < y < 1$. From Figure 2.9, $y(t)$ decreases monotonically in this interval, and asymptotically approaches $y = -2$. Also, the curve is concave down until y crosses $y = -1/2$, after which it is concave up. A curve with these properties is shown in Figure 2.10.

$y(0) = -4$*:* For this initial condition, according to the information in Figure 2.9, $y(t)$ increases monotonically, and asymptotically approaches $y = -2$. Also, the curve is concave down. A curve with these properties is shown in Figure 2.10. ■

The sketching procedure outlined above leaves some things undetermined. For example, nothing was said about how steep the curves are. It could be that the actual solution rises up very quickly, or that it rises very slowly. It is possible to determine this, but this level of analysis is not considered in this text.

2.4.3 ▪ Parting Comments

A few closing comments about the material in this section are in order.

1. The test we have for stability and instability does not cover the case of when $f'(Y) = 0$. If this happens, you can determine the stability by checking on whether $f(y)$ is positive or negative on either side of the steady state. For example, $f(y) > 0$ just to the left of Y, and $f(y) < 0$ just to the right of Y, then Y is asymptotically stable.

2. When a solution moves away from an unstable steady state, it does not necessarily approach the closest stable steady state. An example of this is shown with the solution curve in Figure 2.8. Although $Y = 3$ is closer to the initial point, $f(y)$ is negative for $-2 < y < 1$, and this means the solution must decrease.

3. What are defined as steady states here are sometimes called critical points, or equilibrium points. Referring to them as a steady state is consistent with what is used for time independent solutions of partial differential equations.

4. Finally, besides unstable and asymptotically stable, there is also the possibility of neutrally stable. This is not really needed for first-order equations, so it will be defined later, in Section 4.7.

Exercises

1. For each equation, verify that $Y = 0$ is a steady state. Determine if is unstable or asymptotically stable.

   a) $y' = \sin(1 - e^y)$
   b) $y' = y^5 - 3y^2 + y$
   c) $y' = -e^y \sin(y)$
   d) $y' = (1 + y^9)\ln(1 + y)$

2. For each differential equation, find the steady states and determine if they are asymptotically stable or unstable.

   a) $y' = y^2 + y - 2$
   b) $y' = y^3 - y$
   c) $y' = 4y - y^3$
   d) $y' = e^{-y} - 2$
   e) $y' = y^4 - 3y^2 - 4$
   f) $y' = e^{2y} - 4e^y + 3$

3. Sketch the solution curve for each of the given initial conditions.

   a) $y' = y^2 + y - 2$
      $y(0) = -3$; $y(0) = 0$
   b) $y' = y^3 - y$
      $y(0) = 3/4$; $y(0) = -1/4$

c) $y' = 4y - y^3$
$y(0) = 1/2$; $y(0) = 3$

d) $y' = e^{-y} - 2$
$y(0) = 1$; $y(0) = -1$

e) $y' = y^4 - 3y^2 - 4$
$y(0) = 1$; $y(0) = -3$

f) $y' = e^{2y} - 4e^y + 3$
$y(0) = -1$; $y(0) = \frac{1}{2}\ln 6$

4. Suppose the function $f(y)$ is given Figure 2.8. For the given initial condition, sketch the corresponding solution curve, and provide an explanation why the curve has the form you have sketched. a) $y(0) = -0.6$, b) $y(0) = 2.1$, c) $y(0) = 4$, d) $y(0) = 1$.
5. For the salt water problem given in (2.31), (2.32), sketch the solution without using the formula for the solution. Make sure to explain how you do this.
6. For the logistic equation (2.35), assume that $r = 1$ and $N = 5$. Without using the formula for the solution, sketch the solution in the case of when: a) $P(0) = 1$, and (b) $P(0) = 8$. Make sure to explain how you do this.
7. The population of fish in a lake can be modeled using the logistic equation. However, assuming that the fish are caught at a constant rate h, the equation for the population becomes

$$P' = r\Big(1 - \frac{P}{N}\Big)P - h,$$

where r and N are positive constants.

a) Assuming that the loss due to fishing is small enough that $0 < h < rN/4$, find the two steady states for the equation. Label these values as P_1 and P_2, where $P_1 < P_2$.
b) Determine whether P_1 and P_2 are unstable or asymptotically stable.
c) Letting $f(P)$ be the right hand side of the differential equation, sketch $f(P)$ for $0 \leq P < \infty$. With this, answer the question in Exercise 12(b) on page 31.
d) Assuming that $P_1 < P(0) < P_2$, sketch the solution. Do the same for the case of when $P_2 < P(0)$.
e) Sketch the solution if $0 < P(0) < P_1$. In doing this remember that $P(t)$ can not be negative. Note that you will find that there is a time t_e where extinction occurs, and the differential equation does not apply to the fish population for $t_e < t$.

8. The solution of a differential equation is shown in Figure 2.11. Explain why it can not be the plot of the solution of the following differential equations. You only need to provide one reason (even though there might be several).

a) $y' = 1 + y^2$

b) $y' = y - 4$

c) $y' = (y-4)(y-3)(y-1)$

d) $y' = (y-4)(3-y)(y-1)$

e) $y' = (y-2)(4-y)$

f) $y' = (y-2)^2$

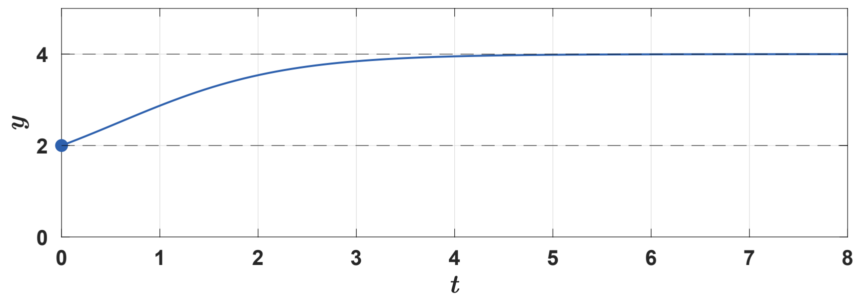

Figure 2.11. *Plot used in Exercise 8. The starting point is $y(0) = 2$.*

9. The following refer to the solution of (2.45), where $f(y)$ is continuous. Sketch a function $f(y)$ so the stated conditions hold. Make sure to provide a short explanation of why your function satisfies the conditions stated. If it is not possible to find such a function, explain why.

   a) The solution is strictly monotone decreasing, and there are no steady states.

   b) The solution is strictly monotone increasing for $y < 0$, is strictly monotone decreasing for $y > 0$, and there are no steady states.

   c) The only asymptotically stable steady states are $Y = 0$ and $Y = 2$, and the only unstable steady state is $Y = 1$.

   d) The only asymptotically stable steady state is $Y = 0$, and the only unstable steady states are $Y = -1$ and $Y = 1$.

   e) The only asymptotically stable steady state is $Y = 0$, and the only unstable steady states are $Y = 1$ and $Y = 2$.

10. This problem concerns what is known as one-sided stability, or semi-stability. The differential equation considered is

$$y' = 2(3-y)^2.$$

   a) Show that there is one steady state Y, and $f'(Y) = 0$.

   b) Sketch $f(y)$ for $-\infty < y < \infty$. Use this to explain why, except when $y = Y$, y is an increasing function of t.

   c) Using the same reasoning as for the population example, explain why, if $y(0) < Y$, then $\lim_{t\to\infty} y(t) = Y$. However, if $y(0) > Y$, then $\lim_{t\to\infty} y(t) = \infty$.

   d) Use the results from part (c) to explain why this is an example of one-sided stability.

Chapter 3

Second-Order Linear Equations

The general version of the differential equations considered in this chapter can be written as

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = f(t), \tag{3.1}$$

where $p(t)$, $q(t)$, and $f(t)$ are given. One of the reasons this equation gets its own chapter is Newton's second law, which, if you recall, is $F = ma$. To explain, if $y(t)$ is the displacement, then the acceleration is $a = y''$, and this gives us the differential equation $my'' = F$. In this chapter we are considering problems when F is a linear function of velocity y' and displacement y. Later, in Chapter 5, we will consider equations where the dependence is nonlinear. It is because of the connections with the second law that $f(t)$ in (3.1) is often referred to as the **forcing function**.

In the previous chapter, for first-order linear differential equations, we very elegantly derived a formula for the general solution. This will not happen for second-order equations. *All* of the methods derived in this chapter are, in fact, just good guesses on what the answer is. There are non-guessing methods, and one example involves using a Taylor series expansion of the solution. An illustration of how this is done can be found in Exercise 7 on page 51.

To use a guessing approach, it becomes essential to know the mathematical requirements for what can be called a general solution. This is where we begin.

Introduction to Differential Equations, M. H. Holmes, 2019

3.1 ▪ Initial Value Problem

A typical initial value problem (IVP) consists of solving (3.1), for $t > 0$, with the initial conditions

$$y(0) = y_0, \quad \text{and} \quad y'(0) = y_0', \tag{3.2}$$

where y_0 and y_0' are given numbers. Given that our solution methods involve guessing, it is important that we know when to stop guessing and conclude we have found the solution. This is why the next result is useful.

Existence and Uniqueness Theorem. *If $p(t)$, $q(t)$, and $f(t)$ are continuous for $t \geq 0$, then there is exactly one smooth function $y(t)$ that satisfies (3.1) and (3.2).*

In stating that $y(t)$ is a smooth function, it is meant that $y''(t)$ is defined and continuous for $t \geq 0$. Those interested in the proof of this theorem, or the theoretical foundations of the subject, should consult Coddington and Carlson [1997].

So, according to the above theorem, if we find a smooth function that satisfies the differential equation and initial conditions, then that is the solution, and the only solution, of the IVP.

3.2 ▪ General Solution of a Homogeneous Equation

The **associated homogeneous equation** for (3.1) is

$$\frac{d^2y}{dt^2} + p(t)\frac{dy}{dt} + q(t)y = 0. \tag{3.3}$$

We need to spend some time discussing what it means to be the general solution of this equation. So, consider Exercise 5(a), in Section 1.2. Assuming you did this exercise, you found that given solutions $y_1 = e^{2t}$ and $y_2 = e^t$ of $y'' - 3y' + 2y = 0$, then

$$y(t) = c_1 y_1(t) + c_2 y_2(t) \tag{3.4}$$

is a solution for any value of c_1 and c_2. What is important here is that this is a **general solution** of the differential equation. As in the last chapter, this means that any, and all, solutions of the differential equation are included in this formula.

This gives rise to the question: what is required so a solution like the one in (3.4) can be claimed to be a general solution? The key to answering this is the uniqueness guaranteed by the above theorem. The specifics of the analysis are not needed here. What is needed is the conclusion, which is stated next.

General Solution. *The function $y = c_1y_1(t) + c_2y_2(t)$, where c_1 and c_2 are arbitrary constants, is a general solution of (3.3) if the following are true:*

1. *y_1 and y_2 are solutions of (3.3), and*
2. *y_1 and y_2 are linearly independent.*

Stating that y_1 and y_2 are **linearly independent** means that the only constants c_1 and c_2 that satisfy

$$c_1y_1(t) + c_2y_2(t) = 0, \quad \forall t \geq 0, \tag{3.5}$$

are $c_1 = 0$ and $c_2 = 0$. This is, effectively, the same definition of linear independence used in linear algebra. The difference is that we have functions rather than vectors. Also, if it is possible to find either $c_1 \neq 0$ or $c_2 \neq 0$ so (3.5) holds, then y_1 and y_2 are said to be **linearly dependent**.

Given two solutions y_1 and y_2 of (3.3), the easiest way to determine if they are independent is to use what is called the Wronskian. To explain, the **Wronskian** of y_1 and y_2 is defined as

$$W(y_1, y_2) \equiv \det\begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix}. \tag{3.6}$$

For those unfamiliar with determinants, this can be written as

$$W(y_1, y_2) \equiv y_1y_2' - y_2y_1'.$$

The usefulness of this function is due, in part, to the next result.

Independence Test. *If y_1 and y_2 are solutions of (3.3), then y_1 and y_2 are independent if, and only if, $W(y_1, y_2)$ is nonzero.*

To explain how the Wronskian comes into this problem, (3.5) must hold on the interval $0 \leq t < \infty$. So, (3.5) can be differentiated, which gives us the equation $c_1y_1' + c_2y_2' = 0$. This, along with (3.5), provides two equations for c_1 and c_2. It is not hard to show that if $W(y_1, y_2) \neq 0$, then the only solution to these two equations is $c_1 = c_2 = 0$. Consequently, y_1 and y_2 are independent. The rest of the proof, along with some additional information, can be found in Exercises 5 and 6.

In the remainder of the chapter, except in Section 3.11 and in some of the exercises for Section 3.9, we will only consider differential equations of the form

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \tag{3.7}$$

where b and c are given constants. The reasons are that these are easier to solve, and, more importantly, they are also the most common second-order differential equations that arise in applications.

Exercises

1. Assuming $b \neq 0$, show that $y_1 = 1$ and $y_2 = e^{-bt}$ are independent solutions of $y'' + by' = 0$.
2. Assuming $\omega \neq 0$, show that $y_1 = \cos(\omega t)$ and $y_2 = \sin(\omega t)$ are independent solutions of $y'' + \omega^2 y = 0$.
3. Assuming $\omega \neq 0$, show that $y_1 = e^{\omega t}$ and $y_2 = e^{-\omega t}$ are independent solutions of $y'' - \omega^2 y = 0$.
4. Show that $y_1 = e^{-\alpha t}$ and $y_2 = te^{-\alpha t}$ are independent solutions of $y'' + 2\alpha y' + \alpha^2 y = 0$.
5. If y_1 and y_2 are solutions of (3.3), show that $\frac{d}{dt}W + p(t)W = 0$. Use this to derive Abel's formula, which is that
$$W(y_1, y_2) = ce^{-\int_0^t p(r)dr}.$$
6. Let $y_1 = (t-1)^2$ and $y_2 = -(t-1)|t-1|$.
   a) On the same axes, sketch y_1 and y_2 for $0 \leq t < \infty$.
   b) Use (3.5) to show that y_1 and y_2 are linearly independent for $0 \leq t < \infty$. Hint: Consider $t = 0$ and $t = 2$.
   c) Show that $W(y_1, y_2) = 0$. Explain why this, along with the result in part (b), do not contradict the Independence Test.

3.3 ▪ Solving a Homogeneous Equation

The solution of the homogeneous equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0 \tag{3.8}$$

can be found by assuming that $y = e^{rt}$. With this, $y' = re^{rt}$, and $y'' = r^2e^{rt}$, and so (3.8) becomes $(r^2 + br + c)e^{rt} = 0$. Since e^{rt} is never zero, we conclude that

$$r^2 + br + c = 0. \tag{3.9}$$

This is called the **characteristic equation** for (3.8). It is easily solved using the quadratic formula, which gives us that

$$r = \frac{1}{2}\big(-b \pm \sqrt{b^2 - 4c}\big). \tag{3.10}$$

There are three possibilities here:

1. there are two real-valued r's: this happens when $b^2 - 4c > 0$,
2. there is one r: this happens when $b^2 - 4c = 0$, and
3. there are two complex-valued r's: this happens when $b^2 - 4c < 0$,

The case of when the roots are complex-valued requires a short introduction to complex variables, and so it is done last.

3.3.1 ▪ Two Real Roots

When there are two real-valued roots, say, r_1 and r_2, then the two corresponding solutions of (3.8) are $y_1 = e^{r_1 t}$ and $y_2 = e^{r_2 t}$. It is left as an exercise to show they are independent. Therefore, the resulting general solution of (3.8) is

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t}.$$

3.3.2 ▪ One Real Root and Reduction of Order

When there is only one root, the second solution can be found using what is called the *reduction of order method*. To explain, if you know a solution $y_1(t)$, it is possible to find a second solution by assuming that $y_2(t) = w(t)y_1(t)$. In our case, we know that $y_1(t) = e^{rt}$, where $r = -b/2$, is a solution. So, to find a second solution it is assumed that $y(t) = w(t)e^{rt}$. Substituting this into (3.8), and simplifying, yields the differential equation

$$w'' + (2r + b)w' + r^2 + br + c = 0.$$

Since $r = -b/2$, then the above differential equation reduces to just $w'' = 0$. Integrating this once gives $w' = d_1$ and then integrating again yields $w = d_1 t + d_2$, where d_1 and d_2 are arbitrary constants. With this our second solution is $y = (d_1 t + d_2)e^{rt}$. A solution that is independent of $y_1 = e^{rt}$ is obtained by taking $d_1 = 1$ and $d_2 = 0$, which means that $y_2 = te^{rt}$. Therefore, the resulting general solution of (3.8) is

$$y = c_1 e^{rt} + c_2 te^{rt}.$$

3.4 ▪ Complex Roots

An example of a differential equation that generates complex-valued roots is

$$y'' + 4y' + 13y = 0. \tag{3.11}$$

Assuming $y = e^{rt}$, we obtain the characteristic equation $r^2 + 4r + 13 = 0$. The two solutions of this are $r_1 = -2 + 3i$ and $r_2 = -2 - 3i$. Proceeding as in the case of two real-valued roots, the conclusion is that the resulting general solution of (3.11) is

$$\begin{aligned} y &= c_1 e^{r_1 t} + c_2 e^{r_2 t} \\ &= c_1 e^{(-2+3i)t} + c_2 e^{(-2-3i)t}. \end{aligned} \tag{3.12}$$

Because complex numbers are used in the exponents, if this expression is used as the general solution, then c_1 and c_2 must be allowed to also be complex-valued.

Although solutions as in (3.12) are used, particularly in physics, there are other ways to write the solution that do not involve complex numbers. Even if (3.12) is used, there is still the question of how to evaluate an expression such as e^{3i}. For this reason, a short introduction to complex variables is needed.

3.4.1 ▪ Euler's Formula and its Consequences

The key for working with complex exponents is the following formula.

Euler's Formula. *If θ is real-valued then*

$$e^{i\theta} \equiv \cos\theta + i\sin\theta. \tag{3.13}$$

It is not possible to overemphasize the importance of this formula. It is one of those fundamental mathematical facts that you must memorize. It is also worth knowing that this is the definition of $e^{i\theta}$. Moreover, as it must, this formula is consistent with the usual rules involving arithmetic, algebra, and calculus. The examples below will provide illustrations of this fact.

Example 1: Since, by definition, $i = \sqrt{-1}$, then $i^2 = -1$, $i^3 = -i$, and $i^4 = 1$. Also,

$$\begin{aligned}(a+ib)^2 &= (a+ib)(a+ib)\\ &= a^2 - b^2 + 2iab.\end{aligned}$$

It is useful to be able to identify the real and imaginary part of a complex number. So, if $r = a + ib$, and a and b are real, then

$$\mathrm{Re}(r) = a, \quad \text{and} \quad \mathrm{Im}(r) = b.$$

Also, two complex numbers are equal only when their respective real and imaginary parts are equal. So, for example, to state that $e^{i\theta} = \frac{1}{2}\sqrt{2}(1-i)$ requires that, using Euler's formula, $\cos\theta = \frac{1}{2}\sqrt{2}$ and $\sin\theta = -\frac{1}{2}\sqrt{2}$. ■

Example 2: $e^{i\pi} = \cos\pi + i\sin\pi = -1$.

This shows that the exponential function can be negative. Moreover, since $e^{i\pi} = -1$ then, presumably, $\ln(-1) = i\pi$ (i.e., you can take the logarithm of a negative number). This is true, but there are complications related to the periodicity of the trigonometric functions, and to learn more about this you should take a course in complex variables. ■

Example 3: $e^{i\pi/2} = \cos\pi/2 + i\sin\pi/2 = i$, and $e^{-i\pi/2} = -i$. ■

Example 4: Assuming θ and φ are real-valued, then

$$\begin{aligned} e^{i\theta}e^{i\varphi} &= (\cos\theta + i\sin\theta)(\cos\varphi + i\sin\varphi) \\ &= \cos\theta\cos\varphi - \sin\theta\sin\varphi + i(\cos\theta\sin\varphi + \sin\theta\cos\varphi) \\ &= \cos(\theta+\varphi) + i\sin(\theta+\varphi) \\ &= e^{i(\theta+\varphi)}. \quad \blacksquare \end{aligned}$$

Example 5: Assuming r is real-valued, then

$$\begin{aligned} \frac{d}{dt}e^{irt} &= \frac{d}{dt}(\cos rt + i\sin rt) \\ &= -r\sin rt + ir\cos rt \\ &= ir(\cos rt + i\sin rt) \\ &= ire^{irt}. \quad \blacksquare \end{aligned}$$

The next step is to extend Euler's formula to a general complex number. With this in mind, let $z = x + iy$, where x and y are real-valued. Using the usual law of exponents,

$$\begin{aligned} e^z = e^{x+iy} &= e^x e^{iy} \\ &= e^x\big(\cos y + i\sin y\big). \end{aligned} \tag{3.14}$$

The above expression is what we need for solving our differential equations.

3.4.2 ▪ Second Representation

We return to the general solution given in (3.12). With (3.14), we get the following

$$\begin{aligned} y &= c_1e^{(-2+3i)t} + c_2e^{(-2-3i)t} \\ &= c_1e^{-2t}\big(\cos 3t + i\sin 3t\big) + c_2e^{-2t}\big(\cos 3t - i\sin 3t\big) \\ &= (c_1+c_2)e^{-2t}\cos 3t + i(c_1-c_2)e^{-2t}\sin 3t. \end{aligned}$$

We have therefore shown that the general solution can be written as

$$y(t) = d_1\bar{y}_1(t) + d_2\bar{y}_2(t), \tag{3.15}$$

where $\bar{y}_1 = e^{-2t}\cos 3t$ and $\bar{y}_2 = e^{-2t}\sin 3t$. It is not difficult to check that these two functions are solutions of (3.11), and they have a nonzero Wronskian. Moreover, since $\bar{y}_1$ and $\bar{y}_2$ do not involve complex numbers, then d_1 and d_2 in the above formula are arbitrary real-valued constants.

3.4.3 ▪ Third Representation

There is a third way to write the general solution that can be useful when studying vibration, or oscillation, problems. This comes from making the observation that given the values of d_1 and d_2 in (3.15), we can write them as a point in the plane (d_1, d_2). Using polar coordinates, it is possible to find R and φ so that $d_1 = R\cos\varphi$ and $d_2 = R\sin\varphi$. In this case,

$$\begin{aligned} y &= d_1 e^{-2t}\cos 3t + d_2 e^{-2t}\sin 3t \\ &= Re^{-2t}\big(\cos\varphi\cos 3t + \sin\varphi\sin 3t\big) \\ &= Re^{-2t}\cos(3t-\varphi). \end{aligned} \tag{3.16}$$

This last expression is the formula we are looking for. In this representation of the general solution, R and φ are arbitrary constants that satisfy $0 \le R$, and $0 \le \varphi < 2\pi$. The advantage of this form of the general solution is that it is much easier to sketch the solution, and to determine its basic properties. Its downside is that it can be a bit harder to find R and φ from the initial conditions than the other two representations.

3.5 ▪ Summary for Solving a Homogeneous Equation

To solve

$$y'' + by' + cy = 0, \tag{3.17}$$

where b and c are constants, assume $y = e^{rt}$. This leads to solving the characteristic equation $r^2 + br + c = 0$, and from this the resulting general solution is given below.

Two Real Roots: $r = r_1, r_2$ (with $r_1 \neq r_2$).

$$y = c_1 e^{r_1 t} + c_2 e^{r_2 t} \tag{3.18}$$

One Real Root: $r = \lambda$.

$$y = c_1 e^{\lambda t} + c_2 t e^{\lambda t} \tag{3.19}$$

Complex Roots: $r = \lambda \pm i\mu$ (with $\mu \neq 0$). Any of the following can be used:

$$y = c_1 e^{(\lambda+i\mu)t} + c_2 e^{(\lambda-i\mu)t}, \quad \text{where } c_1, c_2 \text{ are complex-valued} \tag{3.20}$$

$$y = d_1 e^{\lambda t}\cos(\mu t) + d_2 e^{\lambda t}\sin(\mu t), \quad \text{where } d_1, d_2 \text{ are real-valued} \tag{3.21}$$

$$y = Re^{\lambda t}\cos(\mu t - \varphi), \quad \text{where } R \ge 0, \text{ and } 0 \le \varphi < 2\pi \tag{3.22}$$

In what follows, (3.21) is used. The exception is in Section 3.10, where (3.22) is preferred because it is easier to sketch.

Example 1: Find a general solution of $y'' + 2y' - 3y = 0$.

Answer: The assumption that $y = e^{rt}$ leads to the characteristic equation $r^2 + 2r - 3 = 0$. The solutions of this are $r = -3$ and $r = 1$. Therefore, a general solution is $y = c_1 e^{-3t} + c_2 e^t$. ■

Example 2: Find the solution of the IVP: $y'' + 2y' = 0$ where $y(0) = 3$ and $y'(0) = -4$.

Answer: The assumption that $y = e^{rt}$ leads to the characteristic equation $r^2 + 2r = 0$. The solutions of this are $r = -2$ and $r = 0$. Therefore, a general solution is $y = c_1 e^{-2t} + c_2$. To satisfy $y(0) = 3$ we need $c_1 + c_2 = 3$, and for $y'(0) = -4$ we need $-2c_1 = -4$. This gives us that $c_1 = 2$, and $c_2 = 1$. Therefore, the solution is $y = 2e^{-2t} + 1$. ■

Example 3: Find the solution of the IVP: $y'' - 2y' + 26y = 0$ where $y(0) = 1$ and $y'(0) = -4$.

Answer: The characteristic equation is $r^2 - 2r + 26 = 0$, and the solutions of this are $r = 1 + 5i$ and $r = 1 - 5i$. Using (3.21), since $\lambda = 1$ and $\mu = 5$, the general solution has the form

$$y = d_1 e^t \cos(5t) + d_2 e^t \sin(5t).$$

To satisfy the initial conditions we need to find y', which for our solution is

$$y' = (d_1 + 5d_2)e^t \cos(5t) + (-5d_1 + d_2)e^t \sin(5t).$$

So, to satisfy $y(0) = 1$ we need $d_1 = 1$, and for $y'(0) = -4$ we need $d_1 + 5d_2 = -4$. This means that $d_2 = -1$, and therefore the solution of the IVP is $y = e^t \cos(5t) - e^t \sin(5t)$. ■

Example 4: Find the solution of the IVP: $y'' - 9y = 0$ where $y(0) = -2$ and $y(t)$ is bounded for $0 \leq t < \infty$.

Answer: The assumption that $y = e^{rt}$ leads to the quadratic equation $r^2 = 9$. The solutions of this are $r = -3$ and $r = 3$. Therefore, a general solution is $y = c_1 e^{-3t} + c_2 e^{3t}$. To satisfy $y(0) = 1$ we need $c_1 + c_2 = -2$. As for boundedness, e^{-3t} is a bounded function $0 \leq t < \infty$ but e^{3t} is not. This means we must take $c_2 = 0$. The resulting solution is $y = -2e^{-3t}$. ■

As you might have noticed, in the above examples the formula for the roots in (3.10) was not used. The reason is that it is much easier to remember the way the characteristic equation is derived (by assuming $y = e^{rt}$, etc) than by trying to remember the exact formula for the roots.

Exercises

1. Find Re(z) and Im(z) for the following:

   a) $z = e^{3i\pi}$
   b) $z = e^{-i\pi/3}$
   c) $z = ie^{i\pi/4}$
   d) $z = e^{2+i\pi/6}$
   e) $z = e^{-1+i\pi/2}$
   f) $z = e^{2-3i}$

2. Show that the following hold:

   a) $\dfrac{1}{i} = -i$
   b) $\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$
   c) $e^{i\theta} \neq 0, \forall\theta$
   d) $e^{-i\theta} = \dfrac{1}{e^{i\theta}}$
   e) $(e^{i\theta})^2 = e^{2i\theta}$
   f) $e^{i(\theta+2\pi)} = e^{i\theta}$
   g) $e^{i(\theta-\varphi)} = \dfrac{e^{i\theta}}{e^{i\varphi}}$
   h) $\int e^{i\theta} d\theta = -ie^{i\theta} + c$
   i) $\cos\theta = \frac{1}{2}\left(e^{i\theta} + e^{-i\theta}\right)$
   j) $\sin\theta = \frac{1}{2i}\left(e^{i\theta} - e^{-i\theta}\right)$

3. Find the general solution of the given differential equation.

   a) $y'' + y' - 2y = 0$
   b) $2y'' + 3y' - 2y = 0$
   c) $y'' + 3y' = 0$
   d) $4y'' - y = 0$
   e) $y'' = 0$
   f) $y'' - 6y' + 9y = 0$
   g) $4y'' + 4y' + y = 0$
   h) $4y'' + y = 0$
   i) $y'' - 2y' + 2y = 0$
   j) $y'' + 2y' + 5y = 0$

4. Find the solution of the IVP.

   a) $y'' - y' - 2y = 0, \quad y(0) = 0,\ y'(0) = -1$
   b) $2y'' + 3y' - 2y = 0, \quad y(0) = -1,\ y'(0) = 0$
   c) $y'' + 3y' = 0, \quad y(0) = -1,\ y'(0) = -1$
   d) $5y'' - y' = 0, \quad y(0) = -1,\ y'(0) = -1$
   e) $3y'' - y = 0, \quad y(0) = 3$, $y(t)$ is bounded for $0 \le t < \infty$
   f) $y'' - \frac{3}{2}y' - y = 0, \quad y(0) = 5$, $y(t)$ is bounded for $0 \le t < \infty$
   g) $y'' + 2y' + y = 0, \quad y(0) = -1,\ y'(0) = 0$
   h) $y'' + 9y = 0, \quad y(0) = -1,\ y'(0) = -1$
   i) $y'' + 2y' + 5y = 0, \quad y(0) = -1,\ y'(0) = -1$
   j) $y'' - y' + \frac{13}{36}y = 0, \quad y(0) = -1,\ y'(0) = 0$

5. The roots of the characteristic equation are given. You are to find the original differential equation (of the form given in (3.17)). If only one value is given, that is the only root.

a) $r = -1, 1$	d) $r = 0, 2$	g) $r = 2 \pm 5i$
b) $r = 3, 5$	e) $r = 1$	h) $r = \pm 2i$
c) $r = \pm 2$	f) $r = 0$	

6. Answer the following questions by either providing one example showing it is true, or explaining why it is not possible.
   a) Is it possible to find values for b and c so that the solution of (3.17) is such that $\lim_{t\to\infty} y = 0$, no matter what the initial conditions?
   b) Is it possible to find values for b and c so that the solution of (3.17) is a bounded function of t, no matter what the initial conditions?
   c) Is it possible to find values for b and c so that the solution of (3.17) is a periodic function of t, no matter what the initial conditions?
7. Suppose $y(t)$ satisfies the IVP: $y'' - 2y' + 2y = 0$, where $y(0) = -1$ and $y'(0) = 0$.
   a) Without solving the IVP, determine $y''(0)$.
   b) Without solving the IVP, determine $y'''(0)$, $y''''(0)$, and $y'''''(0)$.
   c) Explain how it is possible to determine the Maclaurin series expansion of $y(t)$ directly from the differential equation and initial conditions.

3.6 ▪ Solution of an Inhomogeneous Equation

We now turn to the problem of solving the inhomogeneous second-order differential equation

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t). \tag{3.23}$$

As with the homogeneous equation, the first task is to explain what form a general solution will have.

Equation (3.23) shares a property with all linear inhomogeneous differential equations. Namely, the general solution can be written as

$$y(t) = y_p(t) + y_h(t), \tag{3.24}$$

where y_p is a **particular solution** of the differential equation, and $y_h(t)$ is the **general solution of the associated homogeneous equation**. That the solution can be written in this way was discussed for linear first-order equations in Section 2.2.1. As you recall, we had solved the problem and then made the observation that the solution can be written as in (3.24). For the second-order problems we are now considering, the situation is reversed, and we will use (3.24) to construct the general solution.

The associated homogeneous equation for (3.23) is just

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0. \tag{3.25}$$

How to find the general solution of this has been discussed in some detail, and formulas for the solution are given in Section 3.5.

So, what remains is to determine how to find a particular solution of (3.23). As you should recall, a particular solution is *any* function that satisfies the differential equation. Since any function will do, we are not really picky on how this function is determined. In fact, our go-to method is nothing more than guessing what a particular solution might be. For those who prefer a more systematic approach, an alternative method is derived in Section 3.9. The guessing method, what is called the method of undetermined coefficients, is considered first.

3.6.1 ▪ Non-Uniqueness of a Particular Solution

A particular solution is only required to be a solution of the differential equation. It is possible, for any given differential equation, to have two rather different looking functions both be particular solutions. As an example, both $y = 1 - t$ and $y = 1 - t + 3e^t - 5e^{-2t}$ are particular solutions of $y'' + y' - 2y = 4t$. To explain what's going on here, the general solution of the differential equation is

$$y = 1 - t + c_1 e^t + c_2 e^{-2t},$$

where c_1 and c_2 are arbitrary constants. A particular solution of this equation is a solution with particular choices for c_1 and c_2. For the two particular solutions given earlier, the first has $c_1 = c_2 = 0$ and the second has $c_1 = 3$ and $c_2 = -5$.

For the most part, when trying to find a particular solution we will be looking for the case of when $c_1 = c_2 = 0$.

3.7 ▪ The Method of Undetermined Coefficients

The objective is to be able to find a solution, any solution, that satisfies

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t). \tag{3.26}$$

Depending on $f(t)$, it is often possible to simply guess a solution. To illustrate, suppose the equation to solve is

$$y'' + y' + 2y = 5e^{3t}. \tag{3.27}$$

This equation is asking for a function y, which if you differentiate it as indicated, and add the results together you get $5e^{3t}$. A function that will generate e^{3t} in this way is e^{3t}. In other words, it is reasonable to expect that a particular solution will have the form $y = Ae^{3t}$. Since $y' = 3Ae^{3t}$ and $y'' = 9Ae^{3t}$, then from the differential equation we require that $14Ae^{3t} = 5e^{3t}$. This will hold by taking $A = 5/14$, and therefore a particular solution is $y_p = \frac{5}{14}e^{3t}$.

Example 1: Find a particular solution of

$$y'' - 2y' + y = 2\cos 4t. \tag{3.28}$$

Answer: The functions which will, if you differentiate them once or twice, generate $\cos(4t)$ are $\cos(4t)$ and $\sin(4t)$. So, the assumption is that a particular solution can be found of the form

$$y = A\cos 4t + B\sin 4t. \tag{3.29}$$

Since $y' = -4A\sin 4t + 4B\cos 4t$, and $y'' = -16A\cos 4t - 16B\sin 4t$, then (3.28) requires that

$$(-15A - 8B)\cos 4t + (-15B + 8A)\sin 4t = 2\cos 4t. \tag{3.30}$$

Equating the coefficients of the $\cos 4t$ terms, and the coefficients of the $\sin 4t$ terms, we get that $15A + 8B = -2$ and $-15B + 8A = 0$. Solving these two equations gives us that $A = -30/289$, and $B = -16/289$. Therefore, a particular solution of (3.28) is

$$y_p = -\frac{30}{289}\cos 4t - \frac{16}{289}\sin 4t. \quad \blacksquare \tag{3.31}$$

The key observation coming from the last example is that if you believe a function needs to be included in the guess for y_p, then all of its derivatives must be included. So, looking at (3.28) you would expect that $\cos(4t)$ needs to be part of the guess, which means you must also include $\sin(4t)$. You do not need to include $4\sin(4t)$, or $-4\sin(4t)$, because $\sin(4t)$ is multiplied by an arbitrary constant in the guess (3.29), and this can account for any constant factors that might be generated by taking a derivative.

There are two situations when this guessing approach runs into trouble. One is easily fixable and this is demonstrated in the next example. The other situation is not fixable, and the cause of the difficulty is illustrated in Example 7 below.

Example 2: Find a particular solution of

$$y'' + 4y = 3\cos 2t.$$

Answer: Given what happened in the last example, you would expect that to find a particular solution you would assume that

$$y = A\cos 2t + B\sin 2t.$$

However, both $\cos 2t$ and $\sin 2t$ are solutions of the associated homogeneous equation. Because of this, the guess would give us that

$y'' + 4y = 0$, no matter what the values are for A and B. The fix is to take the guess, and for the terms that are solutions of the associated homogeneous equation, multiply them by t. So, the modified guess for this example would be

$$y = t(A\cos 2t + B\sin 2t).$$

To check that this works, since

$$y' = A\cos 2t + B\sin 2t + t(-2A\sin 2t + 2B\cos 2t),$$

and

$$y'' = 2(-2A\sin 2t + 2B\cos 2t) + t(-4A\cos 2t - 4B\sin 2t),$$

then from the differential equation we get

$$2(-2A\sin 2t + 2B\cos 2t) = 3\cos 2t.$$

Equating the coefficients of the $\cos 2t$ and $\sin 2t$ terms we get that $-4A = 0$ and $4B = 3$. Therefore, $A = 0$, $B = \frac{3}{4}$, and a particular solution is $y_p = \frac{3}{4}t\sin 2t$. ■

When using the method of undetermined coefficients, the step that requires the most thought is getting the guess correct. After that, it is relatively straightforward to find the coefficients. Consequently, in the examples below, only the appropriate guess is determined. In these examples, $y_h(t)$ is the general solution of the associated homogeneous equation, and $f(t)$ is the forcing function.

Example 3: What guess should be made for $y'' - y' - 6y = t^3 + 2$?

Answer: Since $f(t) = t^3 + 2$, then $f' = 3t^2$, $f'' = 6t$, and $f''' = 6$. So, a complete guess is $y = At^3 + Bt^2 + Ct + D$. It remains to make sure that none of the functions in this guess is a solution of the associated homogeneous equation. Since $y_h = c_1e^{3t} + c_2e^{-2t}$, and the guess does not include e^{3t} or e^{-2t}, then our guess is, indeed, complete. ■

Example 4: What guess should be made for $y'' - y' - 6y = te^{-5t}$?

Answer: The initial guess is $y = Ate^{-5t}$. However, $y' = A(e^{-5t} - 5te^{-5t})$, and this includes a new function e^{-5t}. This must be included in the guess, and so a complete guess is $y = Ate^{-5t} + Be^{-5t}$. Finally, since $y_h = c_1e^{3t} + c_2e^{-2t}$, and the guess does not include e^{3t} or e^{-2t}, then our guess is, indeed, complete. ■

Example 5: What guess should be made for $y'' - y' - 6y = 4t^2 + 1 - \sin(\pi t)$?

The guess for $f(t) = 4t^2 + 1$ is $y = A_0 t^2 + A_1 t + A_2$, and the guess for $f(t) = \sin(\pi t)$ is $y = B_0 \sin \pi t + B_1 \cos \pi t$. So, for the equation as given, a guess is

$$y = A_0 t^2 + A_1 t + A_2 + B_0 \sin \pi t + B_1 \cos \pi t.$$

Finally, since $y_h = c_1 e^{3t} + c_2 e^{-2t}$, and the guess does not include e^{3t} or e^{-2t}, then our guess is, indeed, complete. ■

Example 6: What guess should be made for $y'' + 4y' + 4y = 5e^{-2t}$?

Answer: The initial guess is $y = Ae^{-2t}$. However, for this equation, $y_h = c_1 e^{-2t} + c_2 t e^{-2t}$ and one of these functions appears in the guess. The first modification $y = Ate^{-2t}$ also appears in y_h, and this means we need to multiply by t again. Therefore, the complete guess is $y = At^2 e^{-2t}$. ■

Example 7: What guess should you make if $f(t) = \ln(1 + t)$?

Answer: The initial guess is $y = A \ln(1 + t)$. Its derivatives are $y' = A/(1 + y)$, $y'' = -A/(1 + y)^2$, $y''' = 2A/(1 + t)^3$, etc. Unlike the other examples, the list of different derivative functions does not stop. In such cases, the method of undetermined coefficients should not be used. So, the answer to the question is, there is no guess and the method described in Section 3.9 should be used. ■

3.7.1 ▪ Finding the Coefficients

In Example 1, we ended up with the equation

$$(-15A - 8B) \cos 4t + (-15B + 8A) \sin 4t = 2 \cos 4t, \ \forall t. \tag{3.32}$$

To find A and B we equated the coefficients of the $\cos 4t$ and $\sin 4t$ terms in this equation. This can be done because these functions are linearly independent, and this will be explained below. This approach does *not* require that you prove the functions are independent. Rather, if you think they might be, and you then determine values for A and B so (3.32) is satisfied based on this assumption, then you have found a particular solution.

For those uncomfortable with the "try it and see if it works" approach, another method is to evaluate (3.32) at well-chosen t values. For example, taking $t = 0$ we get that $-15A - 8B = 2$ and at $t = \pi/8$ we get that $-15B + 8A = 0$. The drawback with this is that the equations you derive for the coefficients can be complicated. It can also fail. For example, it fails if you take $t = 0$ and $t = \pi/4$. However, like the method described

if $f(t)$ contains	then $y_p(t)$ contains all of the following
e^{at}	e^{at}
$\cos(\omega t)$ or $\sin(\omega t)$	$\cos(\omega t)$, $\sin(\omega t)$
t^n	t^n, t^{n-1}, $\cdots$, 1
$t^n e^{at}$	$t^n e^{at}$, $t^{n-1}e^{at}$, $\cdots$, e^{at}
$e^{at}\cos(\omega t)$ or $e^{at}\sin(\omega t)$	$e^{at}\cos(\omega t)$, $e^{at}\sin(\omega t)$

Table 3.1. *Guesses when using the method of undetermined coefficients. Note that the exponent n must be a non-negative integer. Also, as explained in the text, adjustments are need if $y_p(t)$ contains a solution of the associated homogeneous equation.*

in the previous paragraph, if you do find values for A and B so (3.32) is satisfied, then you have found a particular solution.

Finally, the explanation of why linear independence can be used to determine A and B. In using the method of undetermined coefficients, we end up needing to satisfy equations such as

$$\alpha y_1(t) + \beta y_2(t) = -2y_1(t) + 3y_2(t), \ \forall t,$$

where α and β depend on A and B. This can be rewritten as $c_1y_1(t) + c_2y_2(t) = 0$, where $c_1 = \alpha + 2$ and $c_2 = \beta - 3$. According to the definition of linear independence, as given in (3.5), if y_1 and y_2 are independent then it must be that $c_1 = 0$ and $c_2 = 0$. In other words, $\alpha = -2$ and $\beta = 3$.

3.7.2 ▪ Odds and Ends

Most textbooks on differential equations have tables for various guesses that you should make for the method of undetermined coefficients. The fact is that they are mostly unreadable. It is much easier to just remember the rules used in formulate the guess, and the earlier examples should be reviewed for the particulars.

However, some do find a table useful, and one is provided in Table 3.1. A few comments need to be made about what is listed. First, if $f(t)$ contains t^n, as well as t^{n-1}, or t^{n-2}, or t^{n-2}, etc, then the guess for t^n is all that you need (see Example 4 above). Second, when solving (3.8), if one the functions in the left column is a solution of the associated homogeneous differential equation the guess must be modified. The needed modification was explained earlier (see Examples 2 and 6).

3.8 ▪ Solving an Inhomogeneous Equation

As stated earlier, the general solution of

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \tag{3.33}$$

can be written as

$$y(t) = y_p(t) + y_h(t), \tag{3.34}$$

where y_p is a particular solution, and y_h is the general solution of the associated homogeneous equation. We now know how to find y_p and y_h, and so we consider a few examples.

Example 1: Find a general solution of $y'' - 3y' + 2y = 5t^2 + 1$.

Step 1: Find y_h. The associated homogeneous equation is $y'' - 3y' + 2y = 0$. Assuming $y = e^{rt}$, one gets the characteristic equation $r^2 - 3r + 2 = 0$. The roots are $r = 1$ and $r = 2$, and so $y_h = c_1e^t + c_2e^{2t}$.

Step 2: Find y_p. The guess is $y = At^2 + Bt + C$, which means that $y' = 2At + B$ and $y'' = 2A$. Inserting these into the differential equation we get that

$$2At^2 + (-6A + 2B)t + 2A - 3B + 2C = 5t^2 + 1.$$

The coefficients of the t^2 terms on the left and right hand sides must be equal, as must the coefficients of the t and constant terms. This means that $2A = 5$, $-6A+2B = 0$, and $2A-3B+2C = 1$. Solving, we get that $A = 5/2$, $B = 15/2$, and $C = 37/4$.

Step 3: The general solution is

$$y = \frac{5}{2}t^2 + \frac{15}{2}t + \frac{37}{4} + c_1e^t + c_2e^{2t}. \quad \blacksquare$$

Example 2: Find the solution of the IVP: $y'' - 2y' = 4\cos(2\pi t)$ where $y(0) = 0$ and $y'(0) = 0$.

Step 1: Find y_h. The associated homogeneous equation is $y'' - 2y' = 0$. Assuming $y = e^{rt}$, one gets the characteristic equation $r^2 - 2r = 0$. The roots are $r = 0$ and $r = 2$, and so $y_h = c_1 + c_2e^{2t}$.

Step 2: Find y_p. The guess is $y = A\cos(2\pi t) + B\sin(2\pi t)$, which means that $y' = -2\pi A\sin(2\pi t) + 2\pi B\cos(2\pi t)$ and $y'' = -4\pi^2A\cos(2\pi t) - 4\pi^2B\sin(2\pi t)$. Inserting these into the differential equation we get that

$$-(4\pi^2A + 4\pi B)\cos(2\pi t) + (-4\pi^2B + 4\pi A)\sin(2\pi t) = 4\cos(2\pi t).$$

This means that $-(4\pi^2 A + 4\pi B) = 4$ and $-4\pi^2 B + 4\pi A = 0$. Solving, we get that $A = -1/(1+\pi^2)$ and $B = -1/\big(\pi(1+\pi^2)\big)$.

Step 3: The general solution is $y = A\cos(2\pi t) + B\sin(2\pi t) + c_1 + c_2 e^{5t}$, where A and B are given above.

Step 4: To satisfy $y(0) = 0$ we need $A + c_1 + c_2 = 0$, and for $y'(0) = 0$ we need $2\pi B + 2c_2 = 0$. Solving these for c_1 and c_2, the conclusion is that the solution of the IVP is

$$y = \frac{1}{1+\pi^2}\Big(-\cos(2\pi t) - \frac{1}{\pi}\sin(2\pi t) + e^{2t}\Big). \quad \blacksquare$$

Example 3: Find a general solution of $y'' + 9y = \cos(3t)$.

Step 1: Find y_h. The associated homogeneous equation is $y''+9y = 0$. Assuming $y = e^{rt}$, one gets the characteristic equation $r^2+9 = 0$. The roots are $r = 3i$ and $r = -3i$, and so $y_h = d_1\cos(3t)+d_2\sin(3t)$.

Step 2: Find y_p. The initial guess is $y = A\cos(3t) + B\sin(3t)$. However, these functions are solutions of the homogeneous equation, and so the guess must be modified to $y = t(A\cos(3t) + B\sin(3t))$. Inserting this into the differential equation we get that

$$-6A\sin(3t) + 6B\cos(3t) = \cos(3t).$$

Equating the coefficients of the $\cos 3t$ and $\sin 3t$ terms we get that $A = 0$ and $B = 1/6$ (alternatively, you can obtain these values by taking $t = 0$ and $t = \pi/6$ in the above equation).

Step 3: The general solution is therefore

$$y = \frac{1}{6}t\sin(3t) + d_1\cos(3t) + d_2\sin(3t). \quad \blacksquare$$

Exercises

1. Find the general solution of the given differential equation.

a) $y'' - y' - 6y = 6e^t$
b) $y'' + 3y' + 2y = \sin \pi t$
c) $y'' + 4y' - 5y = 2t^2$
d) $5y'' - y' = e^{-t} + 3\cos 2t$
e) $3y'' - 5y' - 2y = t^3 - 2t$
f) $8y'' - 2y' - y = 4 + 5\sin 2t$
g) $y'' + 4y = te^t$
h) $y'' - 5y' - 6y = 10t\sin(3t)$
i) $y'' - 2y' + 5y = 5t^2 + 4$
j) $y'' + 2y' + 10y = 3e^t + 1$
k) $y'' - 3y' = t^3 - 6$
l) $3y'' + y' - 2y = 3e^{-2t} - e^t$
m) $y'' - 8y' + 17y = e^{4t}\sin t$
n) $y'' - 5y' - 6y = -3\sin(t+7)$
o) $y'' + 3y' + 2y = \sin^2 t$
p) $4y'' + y' = \cos(2t)\cos(t)$

2. Find the solution of the given IVP.

   a) $y'' + y' - 2y = 3t$, $y(0) = 0$, $y'(0) = 0$
   b) $y'' + 4y = t^2$, $y(0) = 1$, $y'(0) = 0$
   c) $y'' - y' = \sin t$, $y(0) = 1$, $y'(0) = -1$
   d) $y'' + 3y' = 2t$, $y(0) = 1$, $y'(0) = 0$
   e) $y'' + 4y' + 4y = -3e^{2t}$, $y(0) = 1$, $y'(0) = 0$
   f) $4y'' - y = e^{-t/2} + 1$, $y(0) = 0$, $y'(0) = 0$
   g) $y'' + 9y = -2\sin(3t)$, $y(0) = 0$, $y'(0) = 0$
   h) $y'' + 2y' + 5y = e^{-t}$, $y(0) = -1$, $y'(0) = 0$
   i) $y'' - y' + \frac{1}{2}y = 25\cos 3t$, $y(0) = -1$, $y'(0) = 0$

3. For the following, determine a complete guess that can be used to find a particular solution (you do not need to find the coefficients).

   a) $y'' + y' - 2y = t^5 - t^2$
   b) $y'' + 4y = t\cos t$
   c) $y'' + 4y = t + \sin 2t$
   d) $y'' - y' = 1 + \sin t$
   e) $y'' + 3y' = 1 + e^{-3t}$
   f) $y'' + y' + 2y = t^3e^{-2t}$
   g) $y'' - y' + 6y = e^{-t}\cos 3t$
   h) $4y'' - y = -2(t-1)^7$
   i) $y'' - 2y' + 2y = e^{t-5}\cos t$
   j) $y'' + 4y = \cos(2t + 3)$
   k) $y'' + 25y = -3\sin(5t + 7)$
   l) $y'' + y' + y = \int_0^1 \sqrt{s}\cos(2t - s)ds$

4. The idea underlying undetermined coefficients has nothing to do with the differential equation being second-order. What is required is a linear differential equation with constant coefficients. As an example, for the first-order equation $y' + y = e^{2t}$ you assume a particular solution of the form $y = Ae^{2t}$. The associated homogeneous equation is $y' + y = 0$, which means that $y_h = c_1e^{-t}$. Finding the solution in this way is easier than using an integrating factor (which is the way it is done in Example 1 of Section 2.2). Find the general solution of the following first-order equations using the method of undetermined coefficients.

   a) $y' - 6y = 2e^t$
   b) $3y' + 2y = \sin \pi t$
   c) $y' + 3y = 2t$
   d) $5y' - y = e^{-t} + 3t$
   e) $y' - 4y = t\sin 2t$
   f) $y' - 6y = te^{-t} + 2$
   g) $3y' + 2y = e^{-t}\cos(t)$
   h) $y' - 2y = \cos(2t + 5)$

5. This problem concerns **boundary value problems** (BVPs) for a function $u(x)$. Because they involve linear differential equations with constant coefficients, the methods developed in this chapter can be used to solve them. The difference is that x, rather than t, is the independent variable. For example, to find the general solution of $u'' - 4u = 0$, for $0 < x < 2$, you assume that $u = e^{rx}$. Also, instead of two initial

conditions, there are two boundary conditions, one at each end of the spatial interval. The one complication is that it is possible that the boundary conditions are incompatible with the differential equation, in which case there is no solution. In this exercise this does not happen, and you are to find the solution $u(x)$ of the given BVP. In the next exercise a problem with an incompatibility is examined.

a) $u'' - 4u = 0$, for $0 < x < 2$; $u(0) = 0$ and $u(2) = 1$.
b) $u'' + u = 0$, for $0 < x < 1$; $u(0) = 0$ and $u(1) = -1$.
c) $u'' + u' + u = 0$, for $0 < x < 1$; $u(0) = 0$ and $u(1) = 1$.
d) $u'' - u = 5$, for $0 < x < 2$; $u(0) = 0$ and $u(2) = 0$.
e) $u'' + u' = x$, for $0 < x < 1$; $u(0) = 0$ and $u(1) = 0$.

6. The problem considers the BVP of finding the function $u(x)$ that satisfies $u'' + u = 0$, for $0 < x < L$.

a) Show that the general solution of the differential equation is $u(x) = A\cos(x) + B\sin(x)$.
b) What is the solution if $L = 1$, and the boundary conditions are $u(0) = 0$ and $u(L) = 1$?
c) Explain why there is no solution in the case of when $L = \pi$, and the boundary conditions are $u(0) = 0$ and $u(L) = 1$. What are the values of $L > 0$ for which there is no solution of this BVP?

3.9 ▪ Variation of Parameters

When the method of undetermined coefficients works, it is relatively easy to use it to find a particular solution. However, as illustrated in Example 7 of Section 3.7, it does not always work. In such cases, the method of variation of parameters can be used. Interestingly, this method is also based on a guess. Namely, to find a particular solution of

$$\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = f(t), \tag{3.35}$$

it is assumed that

$$y = u_1(t)y_1(t) + u_2(t)y_2(t), \tag{3.36}$$

where y_1 and y_2 are independent solutions of the associated homogeneous equation. As you should notice, the guess (3.36) resembles the general solution of the associated homogeneous equation. However, instead of arbitrary constants, there are now unknown functions u_1 and u_2. Our job is to find these functions. Although it might not appear to be significant right now, we are looking for a single function, y_p, yet our guess contains two unknown functions. This means that we have the option to pick one of these two functions anyway we wish. We will use this option to advantage to find y_p.

Our task is simple in that (3.36) must be a solution of (3.35). So, in preparation for substituting (3.36) into (3.35) note that

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2'.$$

We now use the option of picking u_1 or u_2 anyway we want. The specific choice is that

$$u_1' y_1 + u_2' y_2 = 0. \tag{3.37}$$

So $y' = u_1 y_1' + u_2 y_2'$, and $y'' = u_1 y_1'' + u_1' y_1' + u_2 y_2'' + u_2' y_2'$. Substituting these into (3.35), we get that

$$u_1' y_1' + u_2' y_2' + u_1(y_1'' + by_1' + cy_1) + u_2(y_2'' + by_2' + cy_2) = f. \tag{3.38}$$

Using the fact that y_1 and y_2 are solutions of the associated homogeneous equation, then the above equation reduces to

$$u_1' y_1' + u_2' y_2' = f. \tag{3.39}$$

Therefore, to find u_1 and u_2, we must solve (3.37) and (3.39). This is fairly easy. First, from (3.37), $u_2' = -u_1' y_1 / y_2$. Inserting this into (3.39) we get that

$$(y_2 y_1' - y_1 y_2') u_1' = y_2 f.$$

This can be written as

$$-W(y_1, y_2) u_1' = y_2 f,$$

where $W(y_1, y_2)$ is the Wronskian as defined in (3.6). Solving this gives us that

$$u_1(t) = -\int_0^t \frac{y_2(s) f(s)}{W(y_1(s), y_2(s))} ds. \tag{3.40}$$

Inserting into (3.37), and integrating, we obtain

$$u_2(t) = \int_0^t \frac{y_1(s) f(s)}{W(y_1(s), y_2(s))} ds. \tag{3.41}$$

Therefore, the particular solution we have found is

$$y_p(t) = -y_1(t) \int_0^t \frac{y_2(s) f(s)}{W(y_1(s), y_2(s))} ds + y_2(t) \int_0^t \frac{y_1(s) f(s)}{W(y_1(s), y_2(s))} ds. \tag{3.42}$$

Example 1: Find a particular solution of $y'' - 3y' + 2y = t$ using variation of parameters.

Step 1: Find y_1 and y_2. The associated homogeneous equation is $y'' - 3y' + 2y = 0$. Assuming $y = e^{rt}$, one gets the characteristic equation $r^2 - 3r + 2 = 0$. The roots are $r = 1$ and $r = 2$, and so $y_1 = e^t$ and $y_2 = e^{2t}$.

Step 2: Find u_1. Since $W = y_1 y_2' - y_2 y_1' = e^{3t}$, and $f = t$, then from (3.40),

$$u_1(t) = -\int_0^t \frac{e^{2s}s}{e^{3s}} ds = -\int_0^t se^{-s} ds.$$

Using integration by parts yields $u_1 = (1+t)e^{-t} - 1$.

Step 3: Find u_2. From (3.41), and using integration by parts,

$$u_2(t) = \int_0^t \frac{e^s s}{e^{3s}} ds = \int_0^t se^{-2s} ds = \frac{1}{4}\big[1 - (2t+1)e^{-2t}\big].$$

Step 4: Collecting our results,

$$\begin{aligned} y_p &= \big[(1+t)e^{-t} - 1\big]e^t + \frac{1}{4}\big[1 - (2t+1)e^{-2t}\big]e^{2t} \\ &= \frac{1}{2}t + \frac{3}{4} - e^t + \frac{1}{4}e^{2t}. \quad \blacksquare \end{aligned}$$

Example 2: Find the solution of the IVP: $y'' + 4y = \sqrt{t}$, where $y(0) = 1$, and $y'(0) = 0$.

Step 1: Find y_1 and y_2. The associated homogeneous equation is $y'' + 4y = 0$. Assuming $y = e^{rt}$, one gets the characteristic equation $r^2 = -4$. The roots are $r = \pm 2i$, and so $y_1 = \cos 2t$ and $y_2 = \sin 2t$.

Step 2: Find u_1. Since $W = y_1 y_2' - y_2 y_1' = 2$, and $f = \sqrt{t}$, then from (3.40),

$$u_1(t) = -\int_0^t \frac{1}{2}\sqrt{s}\sin(2s) ds.$$

The answer is left as a definite integral because it is not possible to express it in terms of elementary functions.

Step 3: Find u_2. From (3.41),

$$u_2(t) = \int_0^t \frac{1}{2}\sqrt{s}\cos(2s) ds.$$

Step 4: Collecting our results, the general solution is

$$y = \frac{1}{2}\int_0^t \sqrt{s}\big[\sin(2t)\cos(2s) - \cos(2t)\sin(2s)\big] ds + c_1\cos(2t) + c_2\sin(2t).$$

Step 4: To satisfy $y(0) = 1$ we need $c_1 = 1$, and for $y'(0) = 0$ we need $c_2 = 0$. Therefore, the solution is

$$\begin{aligned} y &= \frac{1}{2}\int_0^t \sqrt{s}\big[\sin(2t)\cos(2s) - \cos(2t)\sin(2s)\big] ds + \cos(2t) \\ &= \frac{1}{2}\int_0^t \sqrt{s}\sin\big(2(t-s)\big) ds + \cos(2t). \quad \blacksquare \end{aligned}$$

A couple of comments need to be made about the above discussion. First, (3.42) can always be used to find a particular solution. The price paid for always working is more work, and in some cases a lot more work. As a case in point, it is much easier to do Example 1 using undetermined coefficients. In contrast, for Example 2, undetermined coefficients does not work, and this means that variation of parameters is the method of choice. A second comment is that (3.42) is a formula that is not easily remembered, whereas the procedure for undetermined coefficients is fairly simple and therefore easily remembered. The conclusion is that, if at all possible, use undetermined coefficients. If it can not be used, for whatever reason, then use (3.42).

Exercises

1. Using variation of parameters, find a particular solution of the given differential equation.

   a) $2y'' + 3y' - 2y = 25e^{-2t}$
   b) $y'' - 2y' + 2y = 6$
   c) $y'' + y' - 2y = 3\ln(1+t)$
   d) $y'' + 3y' = t^{3/2} + 1$
   e) $5y'' - y' = \frac{2}{1+t}$
   f) $4y'' - y = \sin(1+t^2)$

2. Find the solution of the IVP where the differential equation comes from the previous problem, and the initial conditions are $y(0) = 1$ and $y'(0) = 0$.

3. The formula for a particular solution given in (3.42) applies to the more general problem of solving $y''+p(t)y'+q(t)y = f(t)$. In this case, y_1 and y_2 are independent solutions of the associated homogeneous equation $y'' + p(t)y' + q(t)y = 0$. In the following, show that y_1 and y_2 satisfy the associated homogeneous equation, and then determine a particular solution of the inhomogeneous equation.

   a) $t^2y'' - t(t+2)y' + (t+2)y = 2t^3; \quad y_1(t) = t,\ y_2(t) = te^t$
   b) $ty'' - (t+1)y' + y = t^2e^{2t}; \quad y_1(t) = 1+t,\ y_2(t) = e^t$
   c) $t^2y'' - 3ty' + 4y = t^{5/2}; \quad y_1(t) = t^2,\ y_2(t) = t^2\ln t$

4. The *Bessel equation* of order p is $t^2y'' + ty' + (t^2 - p^2)y = 0$. In this problem assume that $p = \frac{1}{2}$.

   a) Show that $y_1 = \sin t/\sqrt{t}$ and $y_2 = \cos t/\sqrt{t}$ are linearly independent solutions for $0 < t < \infty$.
   b) Use the result from part (a), and the preamble in Exercise 3, to find the general solution of $t^2y'' + ty' + (t^2 - 1/4)y = t^{3/2}\cos t$.

3.10 ▪ Linear Oscillator

The problem considered involves a mass, spring, and dashpot as illustrated in Figure 3.1. The differential equation in this case has the form

$$mu'' + cu' + ku = f(t), \tag{3.43}$$

where $f(t)$ is an external forcing function. In this equation, $u(t)$ is the displacement of the mass from its rest position, with positive in the upward direction. The physical interpretation of the terms in this differential equation, and the basic properties of the solution are described in the following pages.

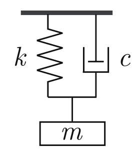

Figure 3.1. *Mass-spring-dashpot system.*

3.10.1 ▪ The Spring Constant

To begin, a spring of length ℓ is suspended as illustrated in Figure 3.2. After this, an object with mass m is attached, which stretches the spring a distance L. The forces on the object in this case are gravity, F_g, and the restoring force from the spring, F_s. The gravitational force is just $F_g = -mg$, where g is the gravitational acceleration constant. The spring force is determined using **Hooke's law**, which states that the restoring force is proportional to how much the spring is stretched. To translate this into a mathematical formula, according to Hooke's law, $F_s = kL$, where k is the proportionally constant, and it is referred to as the **spring constant**. The whole system is at rest, and so, from Newton's second law, we have that $F_s + F_g = 0$. From this we obtain

$$k = \frac{mg}{L}\,. \tag{3.44}$$

3.10.2 ▪ Simple Harmonic Motion

Now, with the mass attached, we set it in motion. For example, as illustrated in Figure 3.2, the mass is pulled down and then released. The equation governing the motion is, again, determined from Newton's second law. As before, the gravitational force is $F_g = -mg$. From Hooke's law, the restoring force due to the spring is $F_s = k(L - u)$, where $u(t)$ is the displacement of the mass from its rest position. Since $F = ma$, and the force in this problem is $F = F_g + F_s$, we get the following differential

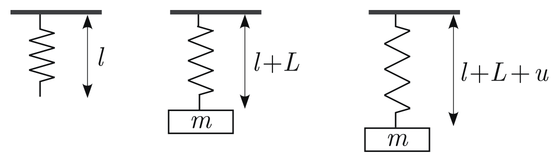

Figure 3.2. *Left: The original spring. Middle: The situation after the mass is attached, and at rest. Right: The mass set in motion.*

equation

$$mu'' + ku = 0. \tag{3.45}$$

To find the general solution of (3.45), from the assumption that $u = e^{rt}$ the characteristic equation is found to be $mr^2 + k = 0$. This produces the roots $r = \pm\omega_0 i$, where

$$\omega_0 = \sqrt{\frac{k}{m}}\,. \tag{3.46}$$

From (3.22), the general solution is

$$u = R\cos(\omega_0 t - \varphi). \tag{3.47}$$

This periodic function corresponds to what is called **simple harmonic motion**. In this context, the coefficient R is the **amplitude**, ω_0 is the **natural frequency**, and the **period** is $T = 2\pi/\omega_0$. The argument $\theta = \omega_0 t - \varphi$ of the cosine is called the phase, and $-\varphi$ is the phase shift, or the phase constant.

In terms of initial conditions, it is usual to specify the initial displacement and the initial velocity. Together, these correspond to

$$u(0) = u_0, \quad \text{and} \quad u'(0) = u_0', \tag{3.48}$$

where u_0 and u_0' are given. To satisfy these, from (3.47), we get that

$$R\cos(\varphi) = u_0, \tag{3.49}$$

$$R\sin(\varphi) = \frac{u_0'}{\omega_0}. \tag{3.50}$$

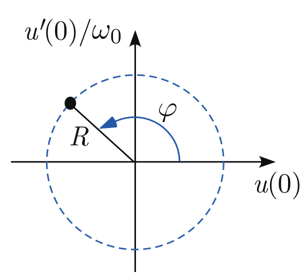

Figure 3.3. *Connection between the initial conditions in (3.48) and the constants in (3.47).*

Finding R: Using the identity $\cos^2(\varphi) + \sin^2(\varphi) = 1$, it follows that

$$R = \sqrt{u_0^2 + \left(\frac{u_0'}{\omega_0}\right)^2}. \tag{3.51}$$

Finding φ: The value for φ depends on whether u_0 and u_0' are positive or negative, as illustrated in Figure 3.3. For example, if $u_0 = 0$, then $\varphi = \pi/2$ if $u_0' > 0$, but $\varphi = 3\pi/2$ if $u_0' < 0$. If $u_0 \neq 0$, then taking the ratio of (3.50) with (3.49) yields

$$\tan(\varphi) = u_0'/(\omega_0 u_0).$$

With this, you can determine φ using the arctan function, making sure the value is consistent with Figure 3.3. Be warned that if you use a calculator, or a program like MATLAB, to determine the numerical value of $\arctan(x)$, it is very likely it will return a value from the interval $-\frac{\pi}{2} \leq \arctan(x) \leq \frac{\pi}{2}$. If so, you will then need to adjust the result so it is consistent with Figure 3.3.

Example 1: Suppose the mass is set in motion by pulling it down 2 cm and then releasing it with an upward velocity of 1 cm/s. Find the resulting simple harmonic motion, assuming that $k = 1\,\mathrm{kg/s^2}$ and $m = 9\,\mathrm{kg}$. In answering this, also determine its period and amplitude.

Answer: The initial conditions are $u(0) = -2$, and $u'(0) = 1$. Also, from (3.46), $\omega_0 = 1/3$, and this means, using (3.47), that the general solution is $u = R\cos(t/3 - \varphi)$. From (3.51), the amplitude is $R = \sqrt{13}$. As for φ, the given initial conditions are positioned in the second quadrant in Figure 3.3. So, we have that $\varphi = -\arctan(3/2)$, with the stipulation that $\pi/2 < \varphi < \pi$. The resulting solution is shown in Figure 3.4. In this figure, $t_1 = \varphi/\omega_0 \approx 6.5$, and the period $T = 2\pi/\omega_0 = 6\pi$. ■

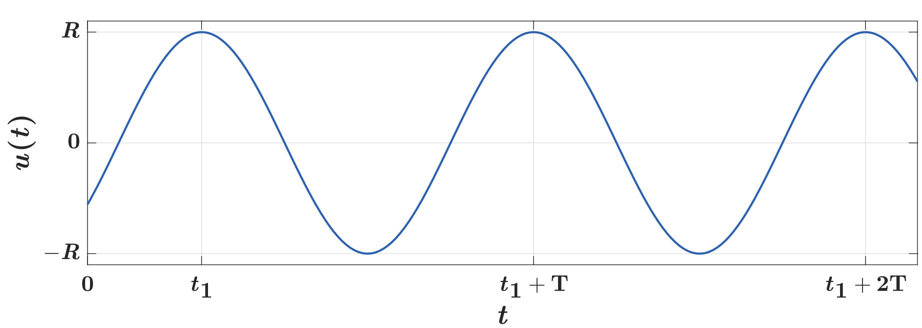

Figure 3.4. *Simple harmonic motion solution for Example 1.*

3.10.3 ▪ Damping

We will now include a damping mechanism. It is assumed that the damping force is proportional to the velocity. This is similar to the assumption made about air resistance in Section 2.3.2, and comes with the same caveat about its limitations. For the mass-spring system the resistance is usually illustrated as a dashpot, as shown in Figure 3.1. Irrespective of exactly what mechanism is involved, the result is that the damping force is $F_d = -cv$, where v is the velocity, and c is the **damping constant** and it is non-negative. Since $v = u'$, then the resulting differential equation is

$$mu'' + cu' + ku = 0. \tag{3.52}$$

Finding the general solution is straightforward. Assuming $u = e^{rt}$, then the resulting characteristic equation is $mr^2 + cr + k = 0$. The roots are

$$r = \frac{1}{2m}\Big(- c \pm \sqrt{c^2 - 4mk}\Big). \tag{3.53}$$

Just as in Section 3.5, there are three cases to consider. The only difference now is that certain terminology is introduced to identity the cases.

Over-damped: This means that the damping constant is large enough that $c^2 > 4mk$. In this case both roots are real-valued, and the resulting general solution is

$$u = c_1 e^{r_1 t} + c_2 e^{r_2 t}, \tag{3.54}$$

where $r_1 = \big(-c+\sqrt{c^2 - 4mk}\big)/(2m)$ and $r_2 = \big(-c-\sqrt{c^2 - 4mk}\big)/(2m)$. It is worth noting that not only are the roots real, they are both negative. Therefore, no matter what the initial conditions,

$$\lim_{t\to\infty} u = 0. \tag{3.55}$$

Critically damped: This means that the damping constant has just that right value that $c^2 = 4mk$. So, there is one root, and the resulting general solution is

$$u = (c_1 + c_2 t)e^{rt}, \tag{3.56}$$

where $r = -c/(2m)$. So, as for the previous case, no matter what the initial conditions, (3.55) holds.

Under-damped: This means that the damping constant is small enough that $c^2 < 4mk$. The roots are complex-valued, and the resulting general solution is

$$u = Re^{\lambda t}\cos(\mu t - \varphi), \tag{3.57}$$

where $\lambda = -c/(2m)$, and $\mu = \sqrt{4mk - c^2}/(2m)$. The solution is not periodic, but it is oscillatory with an amplitude $Re^{\lambda t}$ that decays to zero (assuming, of course, that $c > 0$). Consequently, no matter what the initial conditions, (3.55) holds.

One conclusion coming from the above discussion is that because of damping, no matter what the initial conditions, the solution decays exponentially to zero. The role of damping, and how it affects the solution, is explored in the next examples.

Example 2: Find, and then sketch, the solution of the mass-spring-dashpot problem when $m = 2$, $k = 1$, $c = 1$. The initial conditions are $u(0) = 1$, and $u'(0) = 3/2$.

Answer: From (3.53), the roots are $r = (-1 \pm i\sqrt{7})/4$. So, this is a case of under-damping, with $\lambda = -1/4$ and $\mu = \sqrt{7}/4$. From (3.57), the general solution is

$$u = Re^{-t/4} \cos\left(\frac{1}{4}\sqrt{7}t - \varphi\right). \tag{3.58}$$

To satisfy the initial conditions, we need $R\cos\varphi = 1$, and $R\sin\varphi = \sqrt{7}$. From this we get that $R = 2\sqrt{2}$, and $\varphi = \arctan(\sqrt{7})$.

To sketch the solution, from (3.58), we know that it oscillates between $Re^{-t/4}$ and $Re^{-t/4}$. These are the red dashed curves in Figure 3.5 (the under-damped plot). Since $u'(0) > 0$, then the solution starts out at $u(0) = 1$, and moves upward. From that point on it simply bounces back and forth between the red dashed curves. As for the zero crossings, they occur when $\cos(\mu t - \varphi) = 0$. The first one occurs at the smallest positive value of t that satisfies this equation. Given the value for φ, this is when $\mu t - \varphi = \pi/2$, which means that $t = 2(\pi + \arctan(\sqrt{7}))/\sqrt{7} \approx 4.2$. ■

Example 3: For a given mass-spring-dashpot system, how does the solution change as the damping coefficient changes?

Answer: Taking $m = 2$, $k = 1$, then, from (3.53), $r = \big(-c \pm \sqrt{c^2 - 8}\big)/4$. Using the initial conditions $u(0) = 1$, and $u'(0) = 2$, the resulting solution is shown in Figure 3.5, for different values for the damping constant. The values used give rise to: over-damping ($c = 6$), critically damped ($c = \sqrt{2}$), under-damped ($c = 1$), and weakly damped ($c = 1/40$). To say it is **weakly damped** means that it is under-damped, and c is so small that the solution resembles the periodic solution of an undamped oscillator, at least at the beginning. Eventually, the damping does reduce the amplitude enough to be noticeable.

For over-damping, and critical damping, except near the beginning, the solution simply decays monotonically to zero. In comparison, for both under-damped cases the solution oscillates as it decays to zero. The

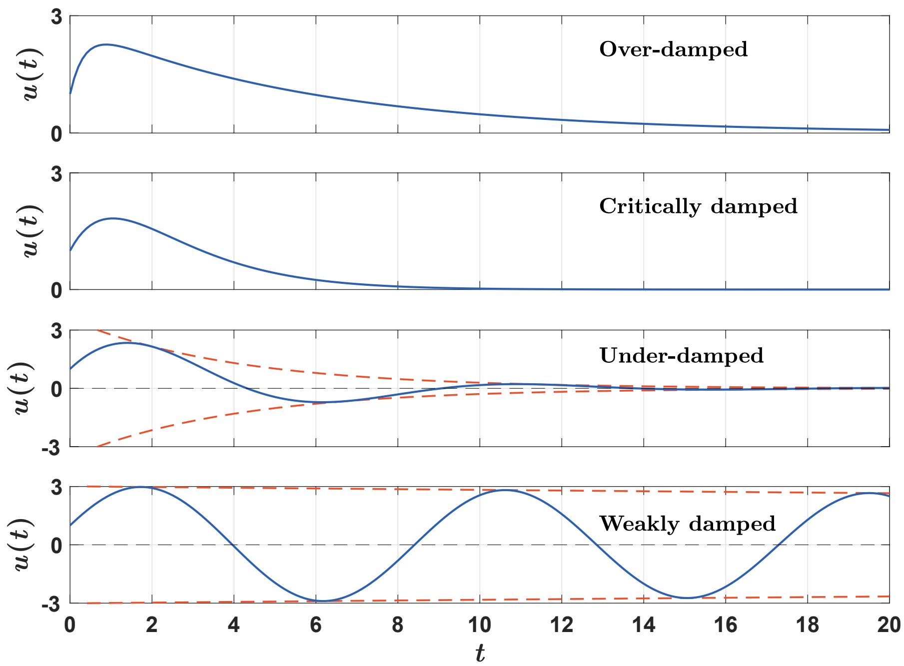

Figure 3.5. *Response of a damped mass-spring system, depending on the strength of damping that is present. The dashed red curves in the two lower graphs are the functions* $\pm Re^{\lambda t}$, *where* $\lambda = -c/(2m)$.

solution, in both cases, bounces back and forth between the two dashed red curves, which are just the functions $Re^{\lambda t}$ and $-Re^{-\lambda t}$. ■

3.10.4 ▪ Resonance

We now consider what happens when a simple harmonic oscillator is forced periodically. The specific equation to solve is

$$mu'' + ku = F\cos\omega t, \tag{3.59}$$

where ω is the driving frequency, and F is the amplitude of the forcing (both ω and F are positive). We have solved similar equations earlier (see Examples 1 and 2 in Section 3.7). The general solution is

$$u = \begin{cases} \dfrac{F}{k - m\omega^2}\cos(\omega t) + u_h(t), & \text{if } \ \omega \neq \omega_0, \\ \dfrac{F}{2\sqrt{km}}\, t\sin(\omega_0 t) + u_h(t), & \text{if } \ \omega = \omega_0, \end{cases} \tag{3.60}$$

where $u_h(t) = d_1\cos(\omega t) + d_2\sin(\omega t)$ and ω_0 is given in (3.46).

What is of interest here is that, when the system is driven at its natural frequency ω_0, the solution contains the term $t\sin(\omega_0 t)$. This is an oscillatory function whose amplitude becomes unbounded as t increases. This happens even though the amplitude of the forcing is constant. This is an example of what is called **resonance**. In contrast, when $\omega \neq \omega_0$ the solution is periodic (and therefore bounded).

Resonance is a particularly important phenomena in science and engineering, and it is often something that is to be avoided. As an example, a wing on an airplane can act like a simple harmonic oscillator, and under certain conditions can start to go into resonance. This is known as flutter, and the resulting large oscillations can lead to the wing breaking off (which can be upsetting to those in the airplane). What is a concern is that this will happen no matter what the value of F, as long as it's nonzero. So, even a very small force, what would normally be considered to be inconsequential, can lead to extremely large oscillations.

One way to avoid resonance is to include a damping mechanism in the system. With the dashpot we introduced earlier, the equation to solve is

$$mu'' + cu' + ku = F\cos\omega t. \tag{3.61}$$

The forcing no longer contains a solution of the associated homogeneous equation, and so resonance will not occur. However, it is often the case that the damping is weak. This means that if $\omega = \omega_0$, then the solution will start out like it's going into resonance, but eventually the damping will stop this, and the solution will simply decay to zero.

This brings us to the question to be considered. Using the flutter example, the question is: we don't want the wings to break off, so just how large do the oscillations get before the damping stops this? To answer this, as seen in (3.60), it is the particular solution that is responsible for the growing oscillations. So, for the weakly damped case we are considering, only the particular solution is considered. To find the particular solution of (3.61), the assumption that $u = A\cos\omega t + B\sin\omega t$ leads to the requirement that A and B satisfy

$$\begin{aligned} m(\omega_0^2 - \omega^2)A + c\omega B &= F, \\ -c\omega A + m(\omega_0^2 - \omega^2)B &= 0. \end{aligned}$$

Solving this yields

$$\begin{aligned} A &= \frac{m(\omega_0^2 - \omega^2)}{c^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2}F, \\ B &= \frac{c\,\omega}{c^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2}F. \end{aligned}$$

Now, to determine the amplitude of the resulting oscillation, it makes things easier to write the solution in the form $u = R\cos(\omega t - \varphi)$. This requires that $R\cos\varphi = A$ and $R\sin\varphi = B$, and therefore

$$R = \sqrt{A^2 + B^2} = \frac{1}{\sqrt{c^2\omega^2 + m^2(\omega_0^2 - \omega^2)^2}}F. \tag{3.62}$$

Example: The amplitude R is plotted in Figure 3.6 as a function of the driving frequency ω. The specific values used to generate these curves are: $F = 1$, $m = 1$, and $k = 1/2$. What is seen is that the smaller the damping coefficient c, the more peaked the response becomes. Also, the peak response occurs at a frequency smaller than the natural frequency ω_0, but this difference decreases as c is reduced. ■

Our flutter question is answered by determining what driving frequency ω gives the largest value for R. Taking the derivative of R with respect to ω, and setting it to zero gives us that $\omega = \omega_M$, where

$$\omega_M = \sqrt{\omega_0^2 - \omega_c^2} \tag{3.63}$$

and $\omega_c = c/(\sqrt{2}m)$. The resulting maximum for R is therefore

$$R_M = \frac{1}{c\sqrt{\omega_0^2 - \frac{1}{2}\omega_c^2}} F. \tag{3.64}$$

Now, suppose that for the flutter problem it is found experimentally that the wings won't break off if the amplitude of the oscillation satisfies $R \leq R_b$. Based on our calculations, this means that the damping coefficient c must be large enough that $R_M \leq R_b$.

Reality Check: The resonance phenomena considered here is not possible for the mass-spring system envisioned in Figure 3.2. As the oscillations grow, as predicted by (3.60), they will eventually get to the point that the mass will start banging up against the upper support. Presumably, as the amplitude grows, our simple linear model is no longer valid, and a more physically realistic, nonlinear, model is necessary. Even so, the simple model is useful as it provides information about the onset of resonance.

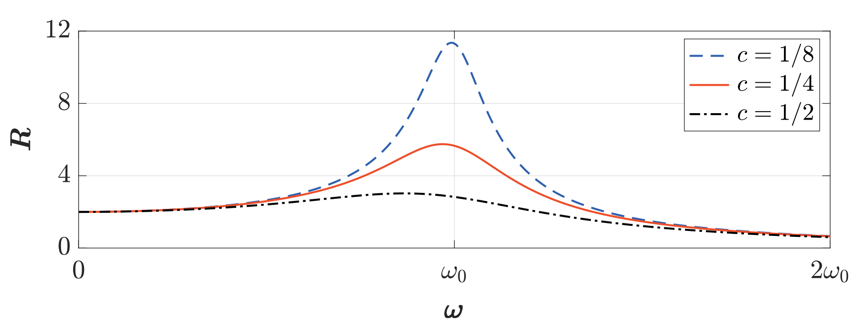

Figure 3.6. *The amplitude (3.62) of the forced, but damped, oscillator, as a function of the driving frequency. Note that ω_0 is the natural frequency of the undamped oscillator.*

Units and Values: In the exercises, the value to use for g is usually stated. If it is not given, then you should leave g unevaluated. Whatever value is used, it is only approximate. If a more physically realistic value is needed, then you should probably use the Somigliana equation. Finally, weight is a force, so for an object that weighs w lbs, its mass can be determined from the equation $w = mg$.

Exercises

In answering the following questions, do not numerically evaluate numbers such as $\sqrt{2}$, $\pi/3$, e^2, $\ln(4/3)$, etc. The exception to this is when the question explicitly asks you to *compute* the answer.

1. A block weighing 2 lb stretches a spring 6 in. Assume that the mass is pulled down an additional 3 in and then released. Assume that $g = 32\,\text{ft/s}^2$.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
   c) What is the natural frequency, period, and amplitude of the motion?
   d) Sketch the solution for $0 \leq t \leq 3T$.
2. A mass of 100 gm stretches a spring $\frac{5}{6}$ m. Assume that the mass is pulled down a distance of 1 m, and then set in motion with an upward velocity of 2 m/s. Assume that $g = 10\,\text{m/s}^2$.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
   c) What is the natural frequency, period, and amplitude of the motion?
   d) Sketch the solution for $0 \leq t \leq 3T$.
3. A mass of 1 kg stretches a spring 10 cm. Assume that the mass is pushed upward a distance of 5 cm, and then set in motion with a downward velocity of 50 cm/s. Assume that $g = 10\,\text{m/s}^2$.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
   c) What is the natural frequency, period, and amplitude of the motion?
   d) Sketch the solution for $0 \leq t \leq 3T$.
4. According to Archimedes' principle, an object that is completely or partially submerged in water is acted on by an upward (buoyant) force equal to the weight of the displaced water. You are to use this for the following situation: A cubic block of wood, with side l and mass density ρ, is floating in water. If the block is slightly depressed and then released, it oscillates in the vertical direction. Derive the differential

equation of motion and determine the period of the motion. In doing this let ρ_0 be the mass density of the water, and assume that $\rho_0 > \rho$.

Damping

5. A block weighing 16 lb stretches a spring 6 in. The mass is attached to a viscous damper with a damping constant of 2 lbs·s/ft. Assume that the mass is set in motion from its equilibrium position with a downward velocity of 4 in/s.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
   c) Sketch the solution.
6. A spring is stretched $\frac{1}{10}$ m by a force of $\frac{1}{2}$ N. A mass of $\frac{1}{2}$ kg is hung from the spring, and dashpot is attached that exerts a force of -3 N when the velocity of the mass is 1 m/s. Assume that the mass is pulled up 1 m from its equilibrium position and given an initial downward velocity of 2 m/s. Assume that $g = 10\,\text{m/s}^2$.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
   c) Sketch the solution.
7. The general solution for the under-damped case is given in (3.57). Suppose the initial conditions are $u(0) = u_0$ and $u'(0) = u'_0$.
   a) Show that
   $$R = \sqrt{u_0^2 + \left(\frac{u'_0 - \lambda u_0}{\mu}\right)^2}.$$
   b) How does Figure 3.3 change?
   c) What are R and φ if $u'_0 = 0$?
8. It is usually stated that negative damping is unstable. For the mass-spring-dashpot system, negative damping means that c is negative. From the solution, explain why the system is unstable for any nonzero initial conditions.

Resonance and Forced Motion

9. A mass weighing 4 lb stretches a spring 1.5 in. The mass is pulled down a distance of 2 in and released from rest. Assume that the mass is acted on by a periodic forcing as in (3.59), with $F = 3$ lb and $\omega = 16$ /sec. Assume that $g = 32\,\text{ft/s}^2$.
   a) What IVP does $u(t)$ satisfy?
   b) What is the solution of the IVP?
10. For a spring-mass system, suppose the mass is at rest but, starting at $t = 0$, it is subjected to a force of $5\cos 3t$ N. Assume that the mass is 2 kg, and the spring constant is $18\,\text{kg/s}^2$..

a) What IVP does $u(t)$ satisfy?

b) What is the solution of the IVP?

11. Suppose the forcing in (3.61) is replaced with $F \sin \omega t$. Does this change (3.62)?

12. This exercise considers what happens when the forcing in (3.61) consists of a combination of driving frequencies.

a) Suppose the forcing is

$$F_0 \cos \omega_0 t + F_1 \cos \omega_1 t + F_2 \cos \omega_2 t,$$

where the F_i's are nonzero, and the ω_i's are all different, with ω_0 given in (3.46). Does resonance still occur?

b) Suppose the forcing is $F_0 \cos \omega_0 t \cos \omega_1 t$, where F_0 is nonzero, $\omega_1 \neq \omega_0$, and ω_0 given in (3.46). Does resonance still occur?

3.11 ▪ Euler Equation

Although second-order equations with constant coefficients are the ones that most often arise in applications, there is a notable exception to this statement. This is the Euler equation, which is

$$x^2 \frac{d^2y}{dx^2} + bx \frac{dy}{dx} + cy = 0, \tag{3.65}$$

where b and c are constants. The reason this equation arises as often as it does is that it comes from using polar coordinates when solving what is known as Laplace's equation.

A complication that arises with (3.65) is that it is not a second-order differential equation when $x = 0$. For this reason, $x = 0$ is referred to as a *singular point* for the equation. This is an issue as it is often the case that the interval used when solving Euler's equation has the form $0 \leq x < L$. What condition, if any, you can impose at $x = 0$ is a question we will consider below.

In what follows it is assumed that $x > 0$. Solving (3.65) is rather easy, as one just assumes a solution of the form $y = x^r$. Since $y' = rx^{r-1}$ and $y'' = r(r-1)x^{r-2}$, then from (3.65) we get that

$$r(r-1) + br + c = 0, \tag{3.66}$$

The solutions of this quadratic equation are

$$r = \frac{1}{2}\Big(1 - b \pm \sqrt{(1-b)^2 - 4c}\Big). \tag{3.67}$$

Just as in Section 3.3, what happens next depends on the values of r obtained from this solution.

Two Real Roots

When there are two real-valued roots, say, r_1 and r_2, then the two corresponding solutions of (3.65) are $y_1 = x^{r_1}$ and $y_2 = x^{r_2}$. It is left as an exercise to show they are independent. Therefore, the resulting general solution of (3.65) is

$$y = c_1 x^{r_1} + c_2 x^{r_2}, \tag{3.68}$$

where c_1 and c_2 are arbitrary constants.

One Real Root

When there is only one root, then you use reduction of order. This means that to find a second solution, assume that $y = w(x)x^r$. Proceeding as in Section 3.3.2, one finds that $w = \ln x$. Therefore, the resulting general solution of (3.65) is

$$y = c_1 x^r + c_2 \ln(x) x^r,$$

where c_1 and c_2 are arbitrary constants.

Complex Roots

In this case, the roots can be written as $r = \lambda \pm i\mu$, where

$$\lambda = \frac{1}{2}(1-b), \tag{3.69}$$

and

$$\mu = \frac{1}{2}\sqrt{4c-(1-b)^2}. \tag{3.70}$$

It is assumed here that $4c > (1-b)^2$. Writing the general solution as in (3.68), and then separating into real and complex parts using Euler's formula, one finds that the resulting general solution can be written as

$$y = d_1 x^\lambda \cos(\mu \ln x) + d_2 x^\lambda \sin(\mu \ln x), \tag{3.71}$$

where d_1 and d_2 are arbitrary constants.

3.11.1 ▪ Examples

Example 1: Find the solution of $x^2y'' + 2xy' - 6y = 0$, for $0 < x < 2$, that is bounded for $0 < x < 2$ and satisfies $y(2) = 1$.

Answer: Substituting in $y = x^r$, one gets the equation $r^2+r-6 = 0$. The solutions of this are $r = -3$, and $r = 2$. So, the general solution of the differential equation is

$$y = c_1 x^{-3} + c_2 x^2.$$

The requirement that y is bounded means that $c_1 = 0$. As for $y(2) = 1$, we need $c_2 = 1/4$. Therefore, the solution is

$$y(x) = \frac{1}{4}x^2. \quad \blacksquare$$

Example 2: Find the general solution of $4x^2y''+17y = 0$, for $0 < x < \infty$.

Answer: Substituting in $y = x^r$, one gets the equation $4r^2 - 4r + 17 = 0$. The solutions of this are $r = \frac{1}{2} \pm 2i$. So, from (3.71), the general solution is

$$y = d_1\sqrt{x}\cos(2\ln x) + d_2\sqrt{x}\sin(2\ln x). \quad \blacksquare$$

Exercises

1. Assuming $x > 0$, find the general solution of the following Euler equations.

   a) $x^2y'' - 3xy' + 4y = 0$
   b) $x^2y'' - 5xy' + 10y = 0$
   c) $6x^2y'' + 7xy' - y = 0$
   d) $x^2y'' + y = 0$
   e) $x^2y'' - 3xy' + 13y = 0$
   f) $5x^2y'' + 12xy' + 2y = 0$
   g) $x^2y'' + xy' = 0$
   h) $x^2y'' - 2xy' = 0$
   i) $x^2y'' - xy' - n(n+2)y = 0$, where n is a positive integer

2. Find the solution of the following problems. Before doing these problems, you might want to review Exercise 3, on page 63.

   a) $x^2y'' - 2xy' + 2y = x^3e^x$, where $y(1) = 0$, and $y'(1) = 0$
   b) $x^2y'' - 4xy' + 4y = -2x^2 + 1$, where $y(1) = 0$, and $y'(1) = 0$
   c) $x^2y'' - xy' + y = \ln x$, where $y(1) = 0$, and $y'(1) = 0$
   d) $xy'' + y' = x$, where $y(1) = 1$, and $y'(1) = -1$
   e) $(x-1)^2y'' + (x-1)y' - y = 0$, where $y(2) = 1$, and $y'(2) = 0$

Chapter 4

Linear Systems

This chapter, and the one that follows, consider problems that involve two or more first-order ordinary differential equations. Together the equations form what is called a first-order system. These are very common. To explain why, it is worth considering a couple of examples.

Example 1: Mechanics

As stated on several occasions earlier in this text, one of the biggest generators of differential equations is Newton's second law, which states that $F = ma$. To demonstrate its connection with a system of differential equations, let $x(t)$ denote the position of an object. The velocity is then $v = x'(t)$, and the acceleration is $a = x''(t)$. So, $F = ma$ can be written as $mv' = F$. Along with the equation $x' = v$, the resulting system is

$$\frac{dx}{dt} = v,$$
$$\frac{dv}{dt} = \frac{1}{m}F.$$

As an example, for a uniform gravitation field, and including air resistance, then $F = -mg - cv$ (see Section 2.3.2). In this case, the system becomes

$$x' = v,$$
$$v' = -g - \frac{c}{m}v.$$

This is a linear first-order system for x and v. It is also inhomogeneous since $x \equiv 0$ and $v \equiv 0$ is not a solution. ■

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Example 2: Epidemics

Epidemics, such as the black death and cholera, have come and gone throughout human history. Given the catastrophic nature of these events there is a long history of scientific study trying to predict how and why they occur. One of particular prominence is the Kermack-McKendrick model for epidemics. This assumes the population can be separated into three groups. One is the population $S(t)$ of those susceptible to the disease, another is the population $I(t)$ that is ill, and the third is the population $R(t)$ of individuals that have recovered. A model that accounts for the susceptible group getting sick, the subsequent increase in the ill population, and the eventual increase in the recovered population is the following set of equations

$$\begin{aligned}\frac{dS}{dt} &= -k_1 SI,\\ \frac{dI}{dt} &= -k_2 I + k_1 SI,\\ \frac{dR}{dt} &= k_2 I.\end{aligned}$$

Given the three groups, and the letters used to designate them, this is an example of what is known as a SIR model in mathematical epidemiology. For us, this is an example of a nonlinear first-order system for S, I, and R. The reason it is nonlinear is the SI term that appears in the first two equations. ■

As you might expect, solving a nonlinear system can be challenging. So, in this chapter, we will concentrate on linear systems. In the next chapter, the nonlinear problems are considered.

4.1 ▪ Linear Systems

To get things started, consider the problem of solving

$$x' = ax + by, \tag{4.1}$$

$$y' = cx + dy. \tag{4.2}$$

This is a first-order, linear, homogeneous system. In these equations, $x(t)$ and $y(t)$ are the dependent variables, and a, b, c, and d are constants. This can be written in system form as

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}.$$

A simpler way to write this is as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.3}$$

where the vector is

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

and the matrix is

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

The equation in (4.3) plays a central role throughout this chapter. Written in this way, we could be dealing with 20 equations, or 200 equations, and not just the two in (4.1) and (4.2).

Before getting into the discussion of how to solve (4.3), it is worth considering what we already know about the solution.

4.1.1 ▪ Example: Transforming to System Form

In Section 3.5, Example 1, we found that for

$$y'' + 2y' - 3y = 0 \tag{4.4}$$

the roots of the characteristic equation are $r_1 = -3$ and $r_2 = 1$. The resulting independent solutions are $y_1 = e^{-3t}$ and $y_2 = e^t$. In this example, the differential equation, along with its solutions, are translated into vector form.

a) Write (4.4) as a linear first-order system as in (4.3).

The standard way to do this is to let $v = y'$, so the differential equation can be written as $v' + 2v - 3y = 0$, or equivalently, $v' = 3y - 2v$. This, along with the equation $y' = v$, gives us the system

$$\begin{aligned} y' &= v, \\ v' &= 3y - 2v. \end{aligned}$$

In other words, we have an equation of the form (4.3), where

$$\mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix}, \qquad \text{and} \qquad \mathbf{A} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}.$$

b) Write the two linearly independent solutions in vector form.

For $y_1 = e^{-3t}$, then $v_1 = y_1' = -3e^{-3t}$. Letting $\mathbf{x}_1$ be the solution vector coming from y_1, then

$$\mathbf{x}_1 = \begin{pmatrix} y_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} e^{-3t} \\ -3e^{-3t} \end{pmatrix} = \begin{pmatrix} 1 \\ -3 \end{pmatrix} e^{-3t} = \mathbf{a}_1 e^{r_1 t},$$

where $r_1 = -3$ and

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix}.$$

Similarly, since $v_2 = y_2' = e^t$, then letting $\mathbf{x}_2$ be the vector version of y_2,

$$\mathbf{x}_2 = \begin{pmatrix} y_2 \\ v_2 \end{pmatrix} = \begin{pmatrix} e^t \\ e^t \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^t = \mathbf{a}_2 e^{r_2 t},$$

where $r_2 = 1$ and

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

c) Write the general solution in vector form.

The general solution for the second-order equation is $y = c_1 y_1 + c_2 y_2$. From this, we get that $v = y' = c_1 y_1' + c_2 y_2'$. Therefore, the general solution vector is

$$\begin{aligned} \mathbf{x} = \begin{pmatrix} y \\ v \end{pmatrix} &= \begin{pmatrix} c_1 y_1 + c_2 y_2 \\ c_1 y_1' + c_2 y_2' \end{pmatrix} = \begin{pmatrix} c_1 y_1 \\ c_1 y_1' \end{pmatrix} + \begin{pmatrix} c_2 y_2 \\ c_2 y_2' \end{pmatrix} \\ &= c_1 \mathbf{x}_1 + c_2 \mathbf{x}_2. \quad \blacksquare \end{aligned} \tag{4.5}$$

There are a few observations to be made about the above example that will be useful when we solve (4.3). For the single linear equation $x' = ax$, there is one linearly independent solution. As the above example shows, for two linear first-order equations there are two linearly independent solutions. Consequently, it should not be a surprise to find out that for n linear first-order equations that there are n linearly independent solutions.

A second useful observation is that the linearly independent solutions have the form $\mathbf{x} = \mathbf{a}e^{rt}$, where $\mathbf{a}$ is a constant vector. In fact, when the time comes to solve (4.3) we will simply assume that $\mathbf{x} = \mathbf{a}e^{rt}$, and then determine how to find r and $\mathbf{a}$.

There is third observation that should be made. As the above example illustrates, except for the Euler equation, *all* of the homogeneous equations considered in Chapter 3 can be written as in (4.3). The similar statement can be made about the respective inhomogeneous equations. Consequently, this chapter contains most of Chapter 3 as a special case. As a student, a natural question at this point would be: why even do Chapter 3, and just do this chapter and be done with it? Some textbooks are actually written this way, often stating that this is the "modern" approach to differential equations. However, there are very important reasons why there is a Chapter 3. One is that second-order equations are a natural consequence of Newton's second law. A second reason is that by being more general, solving (4.3) requires more work. As a case in point, solving the equation in the above example in Chapter 3 took about three lines in this text. Solving it as a system will require about four times that.

4.1.2 ▪ General Version

We are going to consider solving homogeneous linear first-order systems. Assuming there are n dependent variables, then the system can be written as

$$\begin{aligned}
x_1' &= a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n \\
x_2' &= a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n \\
\vdots \quad & \qquad \vdots \qquad\qquad \vdots \\
x_n' &= a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n,
\end{aligned}$$

where the a_{ij}'s are constants. This can be written as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.6}$$

where $\mathbf{A}$ is an $n \times n$ matrix, and $\mathbf{x}$ is an n-vector, given, respectively, as

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

For an initial value problem, an n-vector $\mathbf{x}_0$ would be given, and the condition to be satisfied would be $\mathbf{x}(0) = \mathbf{x}_0$.

Because (4.6) is linear and homogeneous, the principle of superposition holds (see page 5). Therefore, if $\mathbf{x}_1$ and $\mathbf{x}_2$ are solutions of (4.6), then

$$\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$$

is a solution for any values of the constants c_1 and c_2.

As a final comment, the inhomogeneous equation $\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x} + \mathbf{f}$ is not considered in this chapter, but it is considered in Section 6.6.

Exercises

1. Write the following as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, making sure to identify the entries in $\mathbf{x}$ and $\mathbf{A}$. If initial conditions are given, write them as $\mathbf{x}(0) = \mathbf{x}_0$.

a) $u' = u - v$
$v' = 2u - 3v$

b) $2u' = -u$
$3v' = u + v$

c) $u' = u - v + 2w$
$v' = u$
$w' = -u + 5v$

d) $u' = u - v$
$v' = 2u - 3v$
$u(0) = -1,\ v(0) = 0$

e) $u' = 2u - w$
$v' = u + v + w$
$3w' = 2v + 6w$
$u(0) = -1,\ v(0) = 0,\ w(0) = 3$

2. For the following: a) Write the equation in the form $\mathbf{x}' = \mathbf{A}\mathbf{x}$. b) Find the general solution of the second-order equation and then write it in vector form as $\mathbf{x} = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$, where $\mathbf{x}_1 = \mathbf{a}_1 e^{r_1 t}$ and $\mathbf{x}_2 = \mathbf{a}_2 e^{r_2 t}$. Make sure to identify $\mathbf{a}_1$, $\mathbf{a}_2$, r_1 and r_2.

a) $y'' + 2y' - 3y = 0$

b) $4y'' + 3y' - y = 0$

c) $3y'' + 4y' + 3y = 0$

d) $y'' + 4y' = 0$

3. Show that $\mathbf{x}$ is a solution of the given differential equation. Also, what initial condition does $\mathbf{x}$ satisfy?

a)
$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} 4 \\ 2 \end{pmatrix} e^{2t}$$

b)
$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 2 & 2 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = 3\begin{pmatrix} 1 \\ -2 \end{pmatrix} e^{t}$$

c)
$$\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{2t} + \begin{pmatrix} 0 \\ 2 \end{pmatrix} e^{-3t}$$

d)
$$\mathbf{x}' = \begin{pmatrix} \frac{1}{2} & 1 \\ -1 & \frac{1}{2} \end{pmatrix}\mathbf{x}, \quad \mathbf{x} = \begin{pmatrix} \cos t \\ -\sin t \end{pmatrix} e^{t/2}$$

4. This problem considers other possibilities for transforming a second-order equation into a first-order system.

a) Assuming that $b \neq 0$, show that (4.1), (4.2) can be reduced to the second-order linear equation
$$y'' - (a + d)y' + (ad - bc)y = 0.$$

b) Using the result from part (a), transform $y'' + 2y' - 3y = 0$ into a first-order system where none of the entries in $\mathbf{A}$ are zero.

4.2 ▪ General Solution of a Homogeneous Equation

The problem considered here is

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \quad \text{for } t > 0. \tag{4.7}$$

From (4.5), as well as Exercise 2 in the previous section, we have an idea of what the general solution of this equation looks like. Namely, if we are able to find n linearly independent solutions $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, …, $\mathbf{x}_n(t)$, then the general solution can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}_1(t) + c_2\mathbf{x}_2(t) + \cdots + c_n\mathbf{x}_n(t),$$

where c_1, c_2, …, c_n are arbitrary constants.

The requirement to be linearly independent is a simple generalization of the definition given in Section 3.2. Namely, $\mathbf{x}_1(t)$, $\mathbf{x}_2(t)$, …, $\mathbf{x}_n(t)$ are **linearly independent** if, and only if, the only constants c_1, c_2, …, c_n that satisfy

$$c_1\mathbf{x}_1 + c_2\mathbf{x}_2 + \cdots + c_n\mathbf{x}_n = \mathbf{0}, \quad \forall t \geq 0, \tag{4.8}$$

are $c_1 = 0$, $c_2 = 0$, …, $c_n = 0$.

In the last chapter, the Wronskian was used to determine independence. It is possible to also use the Wronskian with (4.7), but this is not particularly useful for larger n. There is an easier way to show independence, and this will be explained in Section 4.4.

The general solution of (4.7) is found by assuming that $\mathbf{x} = \mathbf{a}e^{rt}$, where $\mathbf{a}$ is a constant vector. Differentiating this expression, $\mathbf{x}' = r\mathbf{a}e^{rt}$, and so (4.7) becomes $r\mathbf{a}e^{rt} = \mathbf{A}(\mathbf{a}e^{rt})$. Since e^{rt} is never zero we can divide by it, which gives us the equation

$$\mathbf{A}\mathbf{a} = r\mathbf{a}. \tag{4.9}$$

What we want are nonzero solutions of this equation, and so we require that $\mathbf{a} \neq \mathbf{0}$. This problem for r and $\mathbf{a}$ is called an **eigenvalue problem**, where r is an **eigenvalue**, and $\mathbf{a}$ is an **associated eigenvector**. This is one of the core topics covered in linear algebra. We do not need to know the more theoretical aspects of this problem, but we certainly need to know how to solve it. So, for completeness, the more pertinent aspects of an eigenvalue problem are reviewed next.

4.3 ▪ Review of Eigenvalue Problems

Given an $n \times n$ matrix $\mathbf{A}$, its eigenvalues r and the associated eigenvectors $\mathbf{a}$ are found by solving

$$\mathbf{A}\mathbf{a} = r\mathbf{a}. \tag{4.10}$$

It is required that $\mathbf{a}$ is not the zero vector. There are no conditions placed on r, and it can be real or complex valued.

In preparation for solving the equation, it is first rewritten as $\mathbf{Aa} - r\mathbf{a} = \mathbf{0}$, or equivalently as

$$(\mathbf{A} - r\mathbf{I})\mathbf{a} = \mathbf{0}. \tag{4.11}$$

The $n \times n$ matrix $\mathbf{I}$ is known as the **identity matrix** and it is defined as

$$\mathbf{I} \equiv \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix}.$$

For example, when $n = 2$ and $n = 3$,

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathbf{I} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

In linear algebra, it is shown that for an equation as in (4.11) to have a nonzero solution, it is necessary that the matrix $\mathbf{A} - r\mathbf{I}$ be singular, or non-invertible. What this means is that the determinant of this matrix is zero. This gives rise to the following method for solving the eigenvalue problem.

Eigenvalue Algorithm. *The procedure used to solve the eigenvalue problem consists of two steps:*

1. *Find the r's by solving*

$$\det(\mathbf{A} - r\mathbf{I}) = 0. \tag{4.12}$$

   This is known as the ***characteristic equation****, and the left-hand-side of this equation is an nth degree polynomial in r.*

2. *For each eigenvalue r, find the associated eigenvectors by finding the nonzero solutions of*

$$(\mathbf{A} - r\mathbf{I})\mathbf{a} = \mathbf{0}. \tag{4.13}$$

In this textbook, we are mostly interested in systems involving two equations. For those who might not remember, the determinant of a 2×2 matrix is defined as

$$\det\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \equiv a_{11}a_{22} - a_{12}a_{21}.$$

In the second step, when solving (4.13), we are interested in finding the vectors that can be used to form the general solution of this equation. To say this more mathematically, we want to find linearly independent solutions. For those you might not remember what this is, the definition is given next.

Linearly Independent. *The vectors* $\mathbf{a}_1$, $\mathbf{a}_2$, $\cdots$, $\mathbf{a}_k$ *are linearly independent if, and only if, the only numbers* c_1, c_2, $\ldots$, c_k *that satisfy*

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + \cdots + c_k\mathbf{a}_k = \mathbf{0}, \tag{4.14}$$

are $c_1 = c_2 = \cdots = c_k = 0$. *If there are nonzero numbers so this equation is satisfied, the vectors are linearly dependent.*

In n dimensions, it is not possible to have more than n linearly independent vectors. Consequently, n is the maximum number of linearly independent eigenvectors you can find for an $n \times n$ matrix $\mathbf{A}$. Finally, as will be seen in one of the examples to follow, eigenvectors can contain complex numbers. When this happens, the c_i's in the above equation must be allowed to be complex-valued.

The following examples all involve 2×2 matrices. What is illustrated are the various situations that can arise with eigenvalue problems. In these examples, the eigenvector will be written in component form as

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix}. \tag{4.15}$$

Example 1: Two Real Eigenvalues

For

$$\mathbf{A} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix},$$

we get that

$$\mathbf{A} - r\mathbf{I} = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - r\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 2-r & 1 \\ 1 & 2-r \end{pmatrix}.$$

Since $\det(\mathbf{A} - r\mathbf{I}) = (2-r)^2 - 1 = r^2 - 4r + 3$, then the characteristic equation (4.12) is $r^2 - 4r + 3 = 0$. Solving this we get that the eigenvalues are $r_1 = 3$ and $r_2 = 1$. For r_1, (4.13) takes the form

$$\begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

In component form, we have that

$$\begin{aligned} -a + b &= 0, \\ a - b &= 0. \end{aligned}$$

The solution is $b = a$, and so the eigenvectors are

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ a \end{pmatrix} = a\,\mathbf{a}_1, \tag{4.16}$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}. \tag{4.17}$$

For the second eigenvalue $r_2 = 1$, one finds that the eigenvectors have the form $\mathbf{a} = a\mathbf{a}_2$, where a is an arbitrary nonzero constant and

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

An important observation here is that $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent. To show this, note that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} c_1 + c_2 \\ c_1 - c_2 \end{pmatrix}.$$

So, from (4.14), if $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$, then $c_1 + c_2 = 0$ and $c_1 - c_2 = 0$. From the last equation, $c_1 = c_2$, and inserting this into the first equation yields $2c_2 = 0$. So, $c_2 = 0$, and this also means that $c_1 = 0$. Therefore, $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent. ■

There is an observation that needs to be made here. In the above example, it was shown that eigenvectors for different eigenvalues are linearly independent. **This is always true**. In fact, a collection of eigenvectors, each one corresponding to a different eigenvalue for a matrix $\mathbf{A}$, are linearly independent. This will be useful later when we write down what is stated to be the general solution of the homogeneous equation.

Example 2: One Eigenvalue But Two Independent Eigenvectors

When

$$\mathbf{A} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix},$$

the characteristic equation is $(r-3)^2 = 0$. So the only eigenvalue is $r_1 = 3$. In this case,

$$\mathbf{A} - r\mathbf{I} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

This means that all vectors are solutions of (4.13). In other words, the solutions are

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ b \end{pmatrix} = a\,\mathbf{a}_1 + b\,\mathbf{a}_2,$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

To check on independence, note that

$$c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}.$$

So, if $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$, then we conclude that $c_1 = c_2 = 0$. Therefore, $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent. ■

As you probably noticed, the observation made after Example 1 that eigenvectors corresponding to different eigenvalues must be independent does not apply to the above example. For that reason, independence was shown using the definition.

Example 3: Complex-Valued Eigenvalues

For the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix},$$

the characteristic equation is $r^2 - 2r + 2 = 0$. The resulting eigenvalues are $r = 1 + i$ and $r = 1 - i$. Proceeding as usual, for r_1,

$$\mathbf{A} - r_1\mathbf{I} = \begin{pmatrix} i & 2 \\ -\frac{1}{2} & i \end{pmatrix}.$$

This means that (4.13) requires that $-ia + 2b = 0$, or equivalently, $a = -2ib$. So, the eigenvectors are

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = b\begin{pmatrix} -2i \\ 1 \end{pmatrix} = b\,\mathbf{a}_1,$$

where

$$\mathbf{a}_1 = \begin{pmatrix} -2i \\ 1 \end{pmatrix}.$$

Similarly, for $r_2 = 1 - i$, one finds that the eigenvectors are

$$\mathbf{a} = b\,\mathbf{a}_2,$$

where

$$\mathbf{a}_2 = \begin{pmatrix} 2i \\ 1 \end{pmatrix}.$$

Finally, because $\mathbf{a}_1$ and $\mathbf{a}_2$ are eigenvectors for different eigenvalues, they are independent. ■

There is an observation that needs to be made here. In the above example, the eigenvalues have the form $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, where λ and μ are real numbers, with $\mu \neq 0$. Because of this, the eigenvalues are said to be **complex conjugates**. When a matrix only contains real numbers, as in the last example, and it has complex eigenvalues, they must occur as complex conjugates. Moreover, you should notice that the respective eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$ are also complex conjugates (if you change i to $-i$ in $\mathbf{a}_1$, you get $\mathbf{a}_2$). This is useful information as it means that once you know $\mathbf{a}_1$, you immediately know $\mathbf{a}_2$.

Example 4: Only One Independent Eigenvector

The matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix},$$

has one eigenvalue $r = 3$ (similar to Example 2). Solving (4.13), one finds that $b = 0$. Consequently, the eigenvectors have the form

$$\mathbf{a} = \begin{pmatrix} a \\ 0 \end{pmatrix} = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} = a\,\mathbf{a}_1,$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

In the previous three examples involving 2×2 matrices we found two linearly independent eigenvectors. This matrix is different as there is only one. An $n \times n$ matrix that has fewer than n independent eigenvectors is said to be **defective**. So, the matrix of this example is defective, while the matrices for the three previous examples are not defective. ■

Exercises

1. Determine whether the following pairs of vectors are linearly independent.

   a) $\mathbf{a}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

   b) $\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} -3 \\ 3 \end{pmatrix}$

   c) $\mathbf{a}_1 = \begin{pmatrix} 2 \\ -8 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} -1 \\ 4 \end{pmatrix}$

   d) $\mathbf{a}_1 = \begin{pmatrix} -5 \\ 10 \end{pmatrix}$, $\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

2. The following matrices have two real-valued eigenvalues. Find the eigenvalues, and two linearly independent eigenvectors.

   a) $\begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ b) $\begin{pmatrix} -2 & -7 \\ 1 & 6 \end{pmatrix}$

3. The following matrices have complex-valued eigenvalues. Find the eigenvalues, and two linearly independent eigenvectors.

   a) $\begin{pmatrix} 2 & -4 \\ 1 & 2 \end{pmatrix}$ b) $\begin{pmatrix} 2 & 13 \\ -1 & -4 \end{pmatrix}$

4. Show that the following matrices are defective.

   a) $\begin{pmatrix} 3 & -2 \\ 2 & -1 \end{pmatrix}$ b) $\begin{pmatrix} -1 & 1 \\ -9 & 5 \end{pmatrix}$

4.4 ▪ Solving a Homogeneous Equation

As stated earlier, given an $n \times n$ matrix $\mathbf{A}$, to find the general solution of

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}, \tag{4.18}$$

you start by assuming that $\mathbf{x} = \mathbf{a}e^{rt}$, where $\mathbf{a}$ is a constant vector. Substituting this into the differential equation, and simplifying, leads to the eigenvalue problem

$$\mathbf{A}\mathbf{a} = r\mathbf{a}. \tag{4.19}$$

If $\mathbf{A}$ is not defective, then there are n linearly independent eigenvectors $\mathbf{a}_1$, $\mathbf{a}_2$, …, $\mathbf{a}_n$. Letting r_1, r_2, …, r_n be their respective eigenvalues, then the general solution of (4.18) can be written as

$$\mathbf{x} = c_1\mathbf{a}_1e^{r_1t} + c_2\mathbf{a}_2e^{r_2t} + \cdots + c_n\mathbf{a}_ne^{r_nt}, \tag{4.20}$$

where the c_i's are arbitrary constants.

The vectors $\mathbf{x}_j = \mathbf{a}_j e^{r_j t}$ used in the above formula for the general solution are linearly independent. The reason is that the test for independence in (4.8) must hold at $t = 0$, and for $t = 0$ the equation reduces to (4.14). Since the $\mathbf{a}_j$'s are independent, it follows that the c_j's are all zero. Consequently, the vectors $\mathbf{x}_j$ are linearly independent.

The problems with the fewest complications are those where the eigenvalues are real-valued, and the matrix is not defective. So, we begin with them.

Example 1: General Solution

Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x}.$$

Answer: In the last section (Example 1), we found that the eigenvalues of this matrix are $r_1 = 3$ and $r_2 = 1$, with respective eigenvectors

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

Therefore, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}. \tag{4.21}$$

It is sometimes convenient to write the solution in component form. To do this, letting

$$\mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix},$$

then the general solution in component form is

$$\begin{aligned} x &= c_1 e^{3t} + c_2 e^{t} \\ y &= c_1 e^{3t} - c_2 e^{t}. \quad \blacksquare \end{aligned}$$

Example 2: IVP

Find the solution of

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x},$$

where

$$\mathbf{x}(0) = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

Answer: With the general solution given in (4.21), letting $t = 0$, we need that

$$c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \end{pmatrix}.$$

This can be written in component form as

$$\begin{aligned} c_1 + c_2 &= 0 \\ c_1 - c_2 &= 2. \end{aligned}$$

From the first equation, $c_2 = -c_1$, and using this in the second equation, $c_1 = 1$. Therefore, the solution of the IVP is

$$\mathbf{x} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} - \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}. \quad \blacksquare$$

Example 3: Single Eigenvalue

Find the general solution of

$$\mathbf{x}' = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix} \mathbf{x}.$$

Answer: In the last section (Example 2), we found that the only eigenvalue is $r = 3$, and linearly independent eigenvectors are

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 0 \\ 1 \end{pmatrix} e^{3t}.$$

In component form, the general solution is

$$\begin{aligned} x &= c_1 e^{3t} \\ y &= c_2 e^{3t}. \quad \blacksquare \end{aligned}$$

Much of what we are doing is similar to the approach taken in the last chapter. There are a few differences, and one occurs in the last example. For second-order equations, when there is only one root of the characteristic equation, a second solution of the differential equation is obtained by considering te^{rt} (see Section 3.3.2). This is not the case for linear systems, as shown in Example 2 above. As will be seen shortly, we still have need for a solution containing te^{rt}, but this happens when the matrix is defective.

4.4.1 ▪ Complex-Valued Eigenvalues

As usual, when the roots are complex-valued there are options as to how the general solution can be written. It is certainly possible to just use the expression in (4.20). However, it is often easier to rewrite the solution so as to avoid the use of complex variables. It is easiest to explain how this is done using an example.

Example

The matrix in the differential equation,

$$\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -\frac{1}{2} & 1 \end{pmatrix} \mathbf{x},$$

is the one considered in Example 3 of the previous section. The eigenvalues are $r = 1 + i$ and $r = 1 - i$. Using the eigenvectors found earlier, the general solution can be written as

$$\mathbf{x} = c_1 \begin{pmatrix} -2i \\ 1 \end{pmatrix} e^{(1+i)t} + c_2 \begin{pmatrix} 2i \\ 1 \end{pmatrix} e^{(1-i)t}.$$

Because complex numbers are used for the r's, both c_1 and c_2 must be allowed to be complex-valued.

Given that $\mathbf{x}$ is real-valued, the coefficients c_1 and c_2 must be complex conjugates. In other words, if $c_1 = \alpha + i\beta$, where α and β are real-valued, then it must be that $c_2 = \alpha - i\beta$. We are going to separate the solution into real and imaginary parts, which for the eigenvectors means that

$$\begin{pmatrix} -2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} + i \begin{pmatrix} -2 \\ 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} - i \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

It makes things a bit easier to write these as

$$\begin{pmatrix} -2i \\ 1 \end{pmatrix} = \mathbf{p} + i\mathbf{q} \quad \text{and} \quad \begin{pmatrix} 2i \\ 1 \end{pmatrix} = \mathbf{p} - i\mathbf{q}$$

where

$$\mathbf{p} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{q} = \begin{pmatrix} -2 \\ 0 \end{pmatrix}.$$

Now, using Euler's formula, we have that

$$\begin{aligned}
\mathbf{x} &= (\alpha + i\beta)(\mathbf{p} + i\mathbf{q})e^t(\cos t + i\sin t) + (\alpha - i\beta)(\mathbf{p} - i\mathbf{q})e^t(\cos t - i\sin t) \\
&= d_1(\mathbf{p}\cos t - \mathbf{q}\sin t)e^t + d_2(\mathbf{p}\sin t + \mathbf{q}\cos t)e^t \\
&= d_1\begin{pmatrix} 2\sin t \\ \cos t \end{pmatrix} e^t + d_2\begin{pmatrix} -2\cos t \\ \sin t \end{pmatrix} e^t,
\end{aligned}$$

where $d_1 = 2\alpha$ and $d_2 = -2\beta$ are arbitrary real-valued constants. ■

General Formula

To summarize what was done in the above example, suppose that $\mathbf{A}$ is a 2×2 matrix with complex-valued eigenvalues $r_1 = \lambda + i\mu$ and $r_2 = \lambda - i\mu$, where λ and μ are real-valued with $\mu \neq 0$. Also, assume that their respective eigenvectors are $\mathbf{a}_1 = \mathbf{p} + i\mathbf{q}$ and $\mathbf{a}_2 = \mathbf{p} - i\mathbf{q}$, where $\mathbf{p}$ and $\mathbf{q}$ are vectors containing only real numbers. In this case, instead of writing the general solution as

$$\mathbf{x} = c_1\mathbf{a}_1 e^{r_1 t} + c_2\mathbf{a}_2 e^{r_2 t},$$

it can be written as

$$\mathbf{x}(t) = d_1\mathbf{b}_1 e^{\lambda t} + d_2\mathbf{b}_2 e^{\lambda t},$$

where

$$\begin{aligned}
\mathbf{b}_1 &= \mathbf{p}\cos(\mu t) - \mathbf{q}\sin(\mu t), \\
\mathbf{b}_2 &= \mathbf{p}\sin(\mu t) + \mathbf{q}\cos(\mu t),
\end{aligned}$$

and d_1 and d_2 are arbitrary real-valued constants. As a labor saving observation, it should be noted that $\mathbf{p}$ and $\mathbf{q}$ are known once the eigenvector for r_1 is found, which means you do not also need to find the eigenvector for r_2.

4.4.2 ▪ Defective Matrix

The last case to consider is what to do when there are not enough linearly independent eigenvectors, which means that $\mathbf{A}$ is defective. So, suppose that $\mathbf{A}$ is a 2×2 matrix that has one eigenvalue r, and $\mathbf{a}$ is its associated eigenvector. Based on the way we fixed the single root solution in Chapter 3, you might expect for the vector version you should assume a solution of the form $\mathbf{x} = \mathbf{b}te^{rt}$. However, this does not work, and to find a second independent solution, the assumption is that

$$\mathbf{x} = \mathbf{a}te^{rt} + \mathbf{b}e^{rt}.$$

To find $\mathbf{b}$, the above expression is substituted into the differential equation to obtain

$$\mathbf{Ab} = r\mathbf{b} + \mathbf{a},$$

or equivalently

$$(\mathbf{A} - r\mathbf{I})\mathbf{b} = \mathbf{a}. \tag{4.22}$$

It is useful to know that we don't need all solutions of this equation. Rather, all we need is just one of them. Once this is determined, the general solution is

$$\mathbf{x} = c_1\mathbf{a}e^{rt} + c_2(t\mathbf{a} + \mathbf{b})e^{rt}.$$

Example

We showed earlier that for the matrix in the differential equation

$$\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 0 & 3 \end{pmatrix}\mathbf{x},$$

the only eigenvalue is $r = 3$, and an associated eigenvector is

$$\mathbf{a} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

More importantly, for this example, there are no other linearly independent eigenvectors, so this matrix is defective. To find a second independent solution, from (4.22) we must solve

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{b} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

A solution of this is

$$\mathbf{b} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Therefore, the general solution of the differential equation can be written as

$$\begin{aligned} \mathbf{x}(t) &= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{3t} + c_2\left[\begin{pmatrix} 1 \\ 0 \end{pmatrix}t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right]e^{3t} \\ &= c_1\begin{pmatrix} 1 \\ 0 \end{pmatrix}e^{3t} + c_2\begin{pmatrix} t \\ 1 \end{pmatrix}e^{3t}. \quad \blacksquare \end{aligned}$$

4.5 ▪ Summary for Solving a Homogeneous Equation

Assuming that $\mathbf{A}$ is 2×2, then the general solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is as given below.

- When $\mathbf{A}$ is not defective.
  - If $\mathbf{A}$ has real eigenvalues r_1 and r_2, with respective eigenvectors $\mathbf{a}_1$ and $\mathbf{a}_2$, then
  $$\mathbf{x} = c_1\mathbf{a}_1 e^{r_1 t} + c_2\mathbf{a}_2 e^{r_2 t}. \tag{4.23}$$
  This expression can be used when $r_1 = r_2$ (in this case, just make sure $\mathbf{a}_1$ and $\mathbf{a}_2$ are independent).
  - If $\mathbf{A}$ has complex eigenvalues $r = \lambda \pm i\mu$ (with $\mu \neq 0$), with respective eigenvectors $\mathbf{p} \pm i\mathbf{q}$, then
  $$\mathbf{x}(t) = d_1\mathbf{b}_1 e^{\lambda t} + d_2\mathbf{b}_2 e^{\lambda t}, \tag{4.24}$$
  where
  $$\begin{aligned}\mathbf{b}_1 &= \mathbf{p}\cos(\mu t) - \mathbf{q}\sin(\mu t),\\ \mathbf{b}_2 &= \mathbf{p}\sin(\mu t) + \mathbf{q}\cos(\mu t),\end{aligned}$$
- When $\mathbf{A}$ is defective, with eigenvalue r and eigenvector $\mathbf{a}$, then
$$\mathbf{x} = c_1\mathbf{a}e^{rt} + c_2(t\mathbf{a} + \mathbf{b})e^{rt}, \tag{4.25}$$
where $\mathbf{b}$ is any solution of
$$(\mathbf{A} - r\mathbf{I})\mathbf{b} = \mathbf{a}.$$

Example 1: Find the general solution of
$$\mathbf{x}' = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}\mathbf{x}.$$

Step 1: Find the eigenvalues and eigenvectors. Using the eigenvalue algorithm, from (4.12),
$$\begin{aligned}\det(\mathbf{A} - r\mathbf{I}) = 0 \quad &\Rightarrow \quad \det\begin{pmatrix} -r & 1 \\ 2 & 1-r \end{pmatrix} = 0\\ &\Rightarrow \quad r^2 - r - 2 = 0\\ &\Rightarrow \quad r = -1, 2.\end{aligned}$$

For $r = -1$, then from (4.13),
$$(\mathbf{A} - r\mathbf{I})\mathbf{a} = \mathbf{0} \quad \Rightarrow \quad \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\mathbf{a} = \mathbf{0} \quad \Rightarrow \quad a + b = 0.$$

So, $b = -a$, and this means that

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} a \\ -a \end{pmatrix} = a\,\mathbf{a}_1,$$

where

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

In a similar manner, one finds that for $r = 2$, an eigenvector is

$$\mathbf{a}_2 = \begin{pmatrix} 1 \\ 2 \end{pmatrix}.$$

Step 2: Since this is a non-defective matrix with real eigenvalues, the general solution is

$$\mathbf{x} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{2t}. \quad \blacksquare$$

Example 2: Find the solution of the IVP:

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -2 & 0 \end{pmatrix} \mathbf{x}, \quad \text{where} \quad \mathbf{x}(0) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Step 1: Find the eigenvalues and eigenvectors. Using the eigenvalue algorithm, from (4.12), you find that the eigenvalues are $r_1 = 1 + i$ and $r_2 = 1 - i$. To determine the eigenvector for r_1, we have that

$$\mathbf{A} - r_1\mathbf{I} = \begin{pmatrix} 2-(1+i) & 1 \\ -2 & -(1+i) \end{pmatrix} = \begin{pmatrix} 1-i & 1 \\ -2 & -(1+i) \end{pmatrix}.$$

So, writing $\mathbf{a}$ as in (4.15), then $(\mathbf{A} - r_1\mathbf{I})\mathbf{a} = \mathbf{0}$ can be written in component form as

$$\begin{aligned} (1-i)a + b &= 0 \\ -2a - (1+i)b &= 0. \end{aligned}$$

Both equations lead to the conclusion that $b = -(1-i)a$. So, the eigenvectors are

$$\mathbf{a} = \begin{pmatrix} a \\ b \end{pmatrix} = a \begin{pmatrix} 1 \\ -1+i \end{pmatrix} = a\,\mathbf{a}_1.$$

As explained earlier, it makes things easier to write $\mathbf{a}_1 = \mathbf{p} + i\mathbf{q}$, which means that

$$\begin{pmatrix} 1 \\ -1+i \end{pmatrix} = \mathbf{p} + i\mathbf{q},$$

where

$$\mathbf{p} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \qquad \text{and} \qquad \mathbf{q} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

Moreover, because the eigenvector for $r_2 = 1 - i$ is the complex conjugate of $\mathbf{a}_1$, then $\mathbf{a}_2 = \mathbf{p} - i\mathbf{q}$.

Step 2: Find the general solution. Since there are complex eigenvalues, from (4.24), the general solution is

$$\mathbf{x} = d_1\mathbf{b}_1 e^t + d_2\mathbf{b}_2 e^t,$$

where

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin t,$$

and

$$\mathbf{b}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t.$$

Step 3: Satisfy the initial condition. Setting $t = 0$ in the general solution, we get that

$$d_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + d_2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

This can be written in component form as

$$\begin{aligned} d_1 &= 0 \\ -d_1 + d_2 &= 1. \end{aligned}$$

So, $d_1 = 0$ and $d_2 = 1$.

Step 4: The resulting solution is

$$\mathbf{x}(t) = \left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos t\right] e^t. \quad \blacksquare$$

Exercises

1. Find a general solution of the following differential equations.

a) $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & 0 \end{pmatrix}\mathbf{x}$ b) $\mathbf{x}' = \begin{pmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{pmatrix}\mathbf{x}$

c) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix}\mathbf{x}$

d) $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix}\mathbf{x}$

e) $\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ 0 & -2 \end{pmatrix}\mathbf{x}$

f) $\mathbf{x}' = \begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix}\mathbf{x}$

g) $\mathbf{x}' = \begin{pmatrix} 1 & 5 \\ -\frac{1}{4} & -1 \end{pmatrix}\mathbf{x}$

h) $\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{4} \\ -5 & 0 \end{pmatrix}\mathbf{x}$

i) $\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ -1 & 3 \end{pmatrix}\mathbf{x}$

j) $\mathbf{x}' = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}\mathbf{x}$

2. Find the solution of the initial value problem $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where the differential equation is given in the previous problem, and the initial condition is
$$\mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}.$$

3. A solution of $\mathbf{x}' = \mathbf{A}\mathbf{x}$ is given below. What are the eigenvalues of $\mathbf{A}$, and what are corresponding eigenvectors?

a) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + 2\begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}$

b) $\mathbf{x}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} e^{-5t} + \begin{pmatrix} 1 \\ 3 \end{pmatrix}$

c) $\mathbf{x}(t) = \begin{pmatrix} e^{-2t} \\ 3e^{4t} \end{pmatrix}$

d) $\mathbf{x}(t) = \begin{pmatrix} e^{-8t} - e^{-t} \\ 3e^{-t} \end{pmatrix}$

4. The general solution (4.20), and the eigenvalue algorithm given in Section 4.3, can be used for any dimension n. In this exercise you are to find the general solution for the case of when $n = 3$.

a) $\mathbf{x}' = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}\mathbf{x}$

b) $\mathbf{x}' = \begin{pmatrix} 1 & 1 & 2 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{pmatrix}\mathbf{x}$

c) $\mathbf{x}' = \begin{pmatrix} -2 & 2 & 0 \\ 0 & 1 & 0 \\ 4 & 2 & -1 \end{pmatrix}\mathbf{x}$

d) $\mathbf{x}' = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 2 & -1 \end{pmatrix}\mathbf{x}$

4.6 ▪ Phase Plane

For differential equations involving 2×2 matrices, there are different ways the solution can be portrayed. As an example, earlier, for the differential equation

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \mathbf{x},$$

we found that the general solution can be written in vector form as

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{3t} + c_2 \begin{pmatrix} 1 \\ -1 \end{pmatrix} e^{t}, \tag{4.26}$$

or in component form as

$$\begin{aligned} x(t) &= c_1 e^{3t} + c_2 e^{t}, \\ y(t) &= c_1 e^{3t} - c_2 e^{t}. \end{aligned}$$

Given values for c_1 and c_2, using the component form, graphing the solution simply involves plotting x and y as functions of t. In contrast, with the vector version (4.26), the solution traces out a curve in the x,y-plane, with t being the parameter that generates the curve. The x,y-plane is referred to as the **phase plane**, and the curves that can be generated using (4.26) are known as **integral curves**.

4.6.1 ▪ Examples

Two Positive Eigenvalues.

The solution (4.26) involves two positive eigenvalues, $r_1 = 3$ and $r_2 = 1$. The resulting integral curves generated by (4.26) are shown in Table 4.1(a). Each curve corresponds to a specific choice for c_1 and c_2, and the arrows indicate the direction for increasing t. Together, the integral curves provide what is called a **phase portrait** for the equation. Any equation with two positive eigenvalues will produce a phase portrait that is roughly similar to the one for this example.

To explain how the phase portrait is constructed, the two eigenvectors for this equation are

$$\mathbf{a}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

So, $\mathbf{a}_1$ lies on the line $y = x$, and $\mathbf{a}_2$ lies on the line $y = -x$. These two lines are shown in red in the phase portrait. The vector $c_1\mathbf{a}_1 e^{t}$, when $c_1 > 0$, lies on the line $y = x$ with $x > 0$. Since e^{t} grows as t increases, the solution moves outward from the origin, and so the arrow on the line points outward. When $c_1 < 0$, $c_1\mathbf{a}_1 e^{t}$ lies on the line $y = x$ with $x < 0$. Again, because of the growth in e^{t}, the arrow on the line points outward. The arrows on the other red line point outward for the same reason.

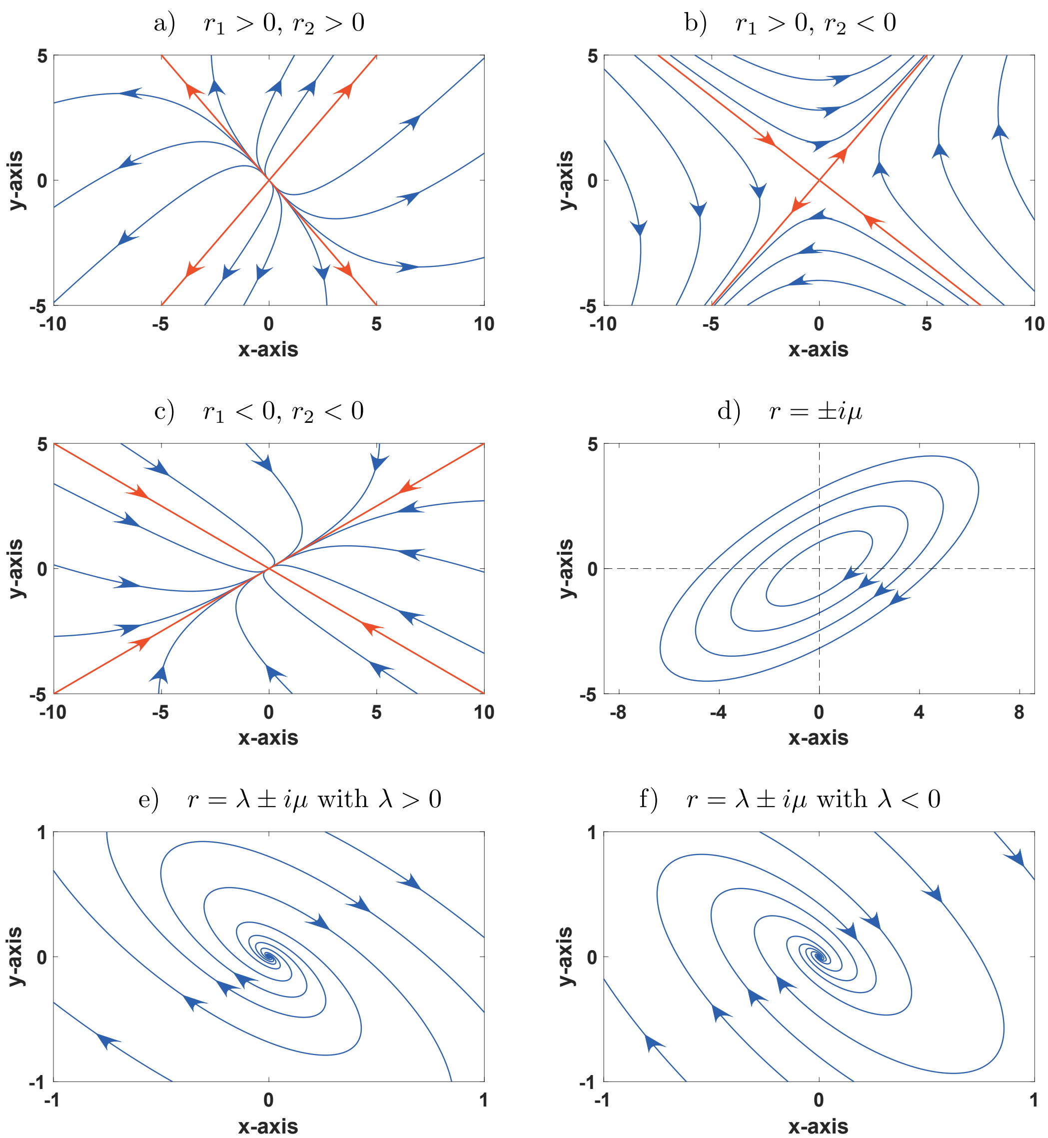

Table 4.1. *Examples of integral curves and how they depend on the eigenvalues of* $\mathbf{A}$*. Each curve corresponds to a specific choice for the constants appearing in the general solution. The arrows indicate the direction for increasing* t*. It is assumed here that* $\mu \neq 0$*.*

The general solution (4.26) consists of the addition of the two components we just considered. So, for any start point in one of the four quadrants determined by the red lines, the solution curve moves outward from the origin. Three representative examples are shown in the phase portrait for each quadrant. ■

Two Negative Eigenvalues.

An example of this arises with the differential equation

$$\mathbf{x}' = \begin{pmatrix} -2 & 2 \\ \frac{1}{2} & -2 \end{pmatrix} \mathbf{x},$$

which has eigenvalues $r_1 = -1$ and $r_2 = -3$. The general solution is found to be

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 2 \\ 1 \end{pmatrix} e^{-t} + c_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix} e^{-3t}. \tag{4.27}$$

The resulting phase portrait is shown in Table 4.1(c). Any equation with two negative eigenvalues will produce a phase portrait that is roughly similar to the one for this example.

The construction of the phase portrait mimics the one used in the last example. The eigenvectors in this case are

$$\mathbf{a}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{a}_2 = \begin{pmatrix} -2 \\ 1 \end{pmatrix}.$$

So, $\mathbf{a}_1$ lies on the line $y = x/2$, and $\mathbf{a}_2$ lies on the line $y = -x/2$. These two lines are shown in red in the phase portrait. The multipliers $c_1 e^{-t}$ and $c_1 e^{-3t}$ both go to zero as t increases. Consequently, the arrows on the respective lines point towards the origin. The same applies to the integral curves in each of the quadrants determined by the red lines. ■

One Positive and One Negative Eigenvalue.

An example of this arises with the differential equation

$$\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \mathbf{x},$$

which has eigenvalues $r_1 = 2$ and $r_2 = -3$. The general solution is found to be

$$\mathbf{x}(t) = c_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} e^{2t} + c_2 \begin{pmatrix} 3 \\ -2 \end{pmatrix} e^{-3t}. \tag{4.28}$$

The resulting integral curves are shown in Table 4.1(b). Any equation with one positive, and one negative, eigenvalue will produce a phase portrait that is roughly similar to the one for this example.

The two lines determined by the eigenvectors are $y = x$ and $y = -2x/3$, and these are shown in red in the phase portrait. Because the eigenvalue associated with $y = x$ is positive, the arrows point outward, and the eigenvalue for $y = -2x/3$ is negative so the arrows point inward.

To explain the other integral curves, the contribution of $c_2\mathbf{a}_2e^{-3t}$ goes to zero as t increases, but $c_1\mathbf{a}_1e^{2t}$ becomes unbounded. A consequence is that a solution curve will asymptotically approach the red line $y = x$. ■

Imaginary Eigenvalues.

When the eigenvalues are imaginary, the integral curves are concentric ellipses centered at the origin (a proof of this statement is given in Section 5.3). To demonstrate this, consider the differential equation

$$\mathbf{x}' = \begin{pmatrix} -2 & 4 \\ -2 & 2 \end{pmatrix}\mathbf{x}.$$

The eigenvalues are $r_1 = 2i$ and $r_2 = -2i$, and the general solution, from (4.24), is

$$\mathbf{x}(t) = d_1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\cos 2t - \begin{pmatrix} 0 \\ 1 \end{pmatrix}\sin 2t\right] + d_2\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\sin 2t + \begin{pmatrix} 0 \\ 1 \end{pmatrix}\cos 2t\right]. \quad (4.29)$$

The ellipses generated by this solution are shown in Table 4.1(d).

The question is, is the movement around each ellipse clockwise, or counter-clockwise? This can be determined from the differential equation. For this example, $x' = -2x+4y$, which means that when the ellipse crosses the x-axis (so $y = 0$), $x' = -2x$. Consequently, along the positive x-axis, $x' < 0$. The direction of the arrows must be consistent with this, and so the rotation is clockwise. ■

Complex Eigenvalues.

It is assumed here that the eigenvalues have nonzero real and imaginary parts. The integral curves in this case are spirals centered at the origin, but the specifics depend on whether the real part is positive or negative. As an example,

$$\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ -10 & 0 \end{pmatrix}\mathbf{x},$$

has eigenvalues are $r_1 = 1 + 3i$ and $r_2 = 1 - 3i$. The general solution, from (4.24), is

$$\mathbf{x}(t) = d_1\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix}\sin 3t\right]e^t + d_2\left[\begin{pmatrix} 1 \\ -1 \end{pmatrix}\sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix}\cos 3t\right]e^t. \quad (4.30)$$

Similarly, for the differential equation

$$\mathbf{x}' = \begin{pmatrix} -2 & 1 \\ -10 & 0 \end{pmatrix}\mathbf{x},$$

the eigenvalues $r_1 = -1 + 3i$ and $r_2 = -1 - 3i$. The general solution is

$$\mathbf{x}(t) = d_1 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \cos 3t - \begin{pmatrix} 0 \\ 3 \end{pmatrix} \sin 3t \right] e^{-t} + d_2 \left[\begin{pmatrix} 1 \\ 1 \end{pmatrix} \sin 3t + \begin{pmatrix} 0 \\ 3 \end{pmatrix} \cos 3t \right] e^{-t}. \tag{4.31}$$

The resulting integral curves are shown in Table 4.1 (lower row). The one on the left comes from (4.30). The outward motion in this case is because the real part of the eigenvalue is positive. The one of the right comes from (4.31), and the inward motion is because the real part of the eigenvalue is negative.

The spiral curves seen in these two graphs are explainable from the formula for the solutions. The solutions contain $\cos \mu t$ and $\sin \mu t$ terms, and these are responsible for the motion around the origin. This is similar to what happens when $r = \pm i\mu$. However, these terms are multiplied by $e^{\lambda t}$, and this causes the radial distance from the origin to either increase, when $\lambda > 0$, or decrease, when $\lambda < 0$. ■

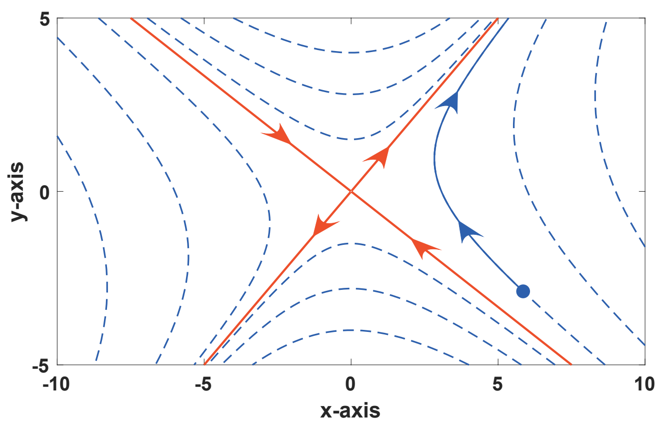

Figure 4.1. *The solid blue curve is the solution (4.33), and the solid blue dot is the location of the initial condition The dashed blue curves, and the red lines, are integral curves for (4.32).*

4.6.2 ▪ Connection with an IVP

To illustrate the role the phase plane can play when solving an initial value problem, suppose the problem to solve is

$$\mathbf{x}' = \begin{pmatrix} -1 & 3 \\ 2 & 0 \end{pmatrix} \mathbf{x}, \tag{4.32}$$

where

$$\mathbf{x}(0) = \begin{pmatrix} 6 \\ -3 \end{pmatrix}.$$

This is the same differential equation used for the phase plane example in Table 4.1 for one positive and one negative eigenvalue, and the general solution is given in (4.28). From the initial condition, the solution is found to be

$$\mathbf{x}(t) = \frac{3}{5}\begin{pmatrix}1\\1\end{pmatrix}e^{2t} + \frac{9}{5}\begin{pmatrix}3\\-2\end{pmatrix}e^{-3t}. \tag{4.33}$$

The plot of this curve in the phase plane is shown in Figure 4.1. The integral curves for the differential equation, which appear in Table 4.1, are also included in the figure. As this shows, the solution of the initial value problem is simply a portion of one of its integral curves. The starting point is determined by the initial condition, and the resulting solution follows the respective integral curve for increasing t.

The above observation is true in general. Namely, the integral curves in Table 4.1 are illustrations of the various solutions you can get with the respective differential equation. Which curve, or how much of the curve, you get depends on the location of the initial condition.

Exercises

1. The eigenvalues for the following equations are real-valued. You are to sketch the phase portrait as follows: (i) Draw the (red) lines that are determined from the eigenvectors, and include the four arrows. (ii) In each of the four quadrants determined by the red lines, include two integrals curves, with arrows.

a) $\mathbf{x}' = \begin{pmatrix}3 & -1\\-1 & 3\end{pmatrix}\mathbf{x}$ c) $\mathbf{x}' = \begin{pmatrix}3 & 2\\-4 & -3\end{pmatrix}\mathbf{x}$

b) $\mathbf{x}' = \begin{pmatrix}6 & 2\\-3 & 1\end{pmatrix}\mathbf{x}$ d) $\mathbf{x}' = \begin{pmatrix}4 & -2\\3 & -3\end{pmatrix}\mathbf{x}$

2. Explain why the integral curves for the differential equation are concentric ellipses centered at the origin. Also, determine if the movement along each ellipse is clockwise or counter-clockwise.

a) $\mathbf{x}' = \begin{pmatrix}-1 & 2\\-2 & 1\end{pmatrix}\mathbf{x}$ b) $\mathbf{x}' = \begin{pmatrix}3 & 6\\-3 & -3\end{pmatrix}\mathbf{x}$

3. The phase portraits for four differential equations are shown in Figure 4.2. Answer the following questions: (i) What properties of the eigenvalues result in the integral curves shown in the respective plot? (ii)

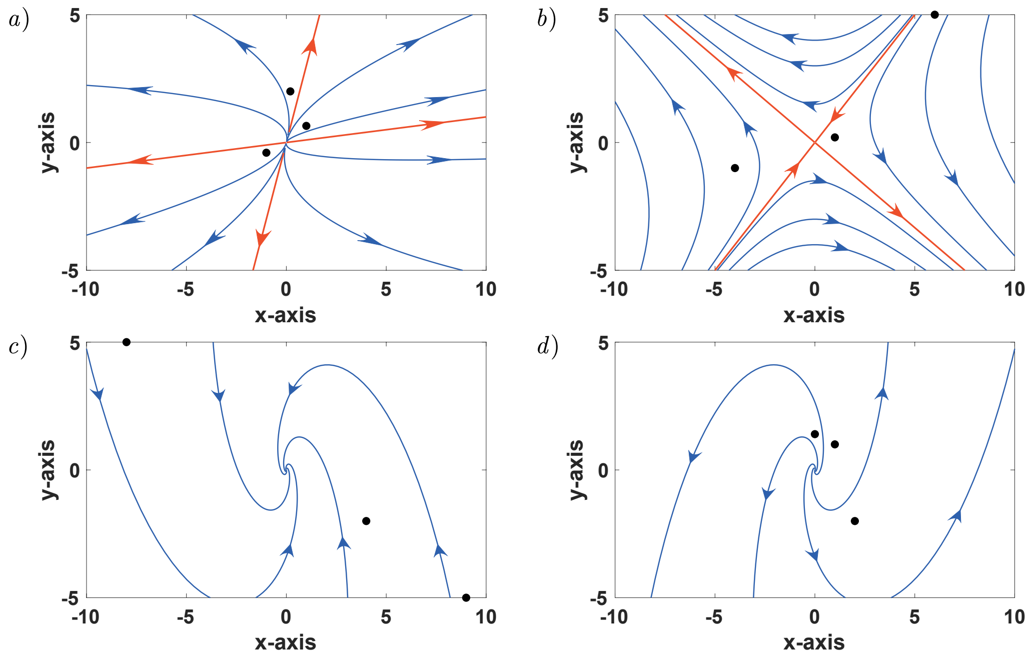

Figure 4.2. *Integral curves, and location of three initial conditions, for Exercise 3.*

Three different initial conditions are shown by the black dots. Sketch the resulting solution for the resulting IVP.

4.7 ▪ Stability

The phase plane can be useful for visualizing stability or instability of a steady state solution. To explain how, recall from Section 2.4 that a **steady state** is a constant that satisfies the differential equation. So, for the equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$, a steady state is a constant vector $\mathbf{x}_s$ that satisfies $\mathbf{A}\mathbf{x}_s = \mathbf{0}$. To avoid complications, it will be assumed that $\mathbf{A}$ is invertible, which means that the only steady state solution is $\mathbf{x}_s = \mathbf{0}$. It is useful to know that $\mathbf{A}$ is invertible if, and only if, $r = 0$ is not an eigenvalue for $\mathbf{A}$.

The definitions of unstable and asymptotically stable are effectively the same as in Section 2.4. Namely, a steady state $\mathbf{x}_s$ is **asymptotically stable** if any initial value $\mathbf{x}(0)$ chosen near $\mathbf{x}_s$ results in

$$\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}_s. \tag{4.34}$$

The steady state is **unstable** if, no matter how close to $\mathbf{x}_s$ you restrict the choice for $\mathbf{x}(0)$, it is always possible to find an initial value $\mathbf{x}(0)$ that results in the solution $\mathbf{x}(t)$ becoming unbounded as t increases.

It is easy to determine stability using the phase plane. For example, in Table 4.1(a), when $r_1 > 0$ and $r_2 > 0$, the arrows on the integral curves indicate movement out away from the origin. Consequently, this is an example of when $\mathbf{x}_s = \mathbf{0}$ is unstable. Conversely, when $r_1 < 0$ and $r_2 < 0$, the flow in towards the origin, and this means $\mathbf{x}_s = \mathbf{0}$ is asymptotically stable. In fact, looking at the various possibilities in Table 4.1, you conclude that if $\mathbf{A}$ has an eigenvalue with $\text{Re}(r) > 0$, then the steady state in unstable. Similarly, if the eigenvalues of $\mathbf{A}$ are both negative, or if $\text{Re}(r) < 0$, then the steady state is asymptotically stable.

The conclusions in the previous paragraph were made using the phase portraits in Table 4.1. For those that prefer more rigorous derivations, then the formulas for the general solutions given in Section 4.5 can be used.

Our classification of a steady state being unstable or asymptotically stable does not include what happens when the eigenvalues are imaginary. As shown in Table 4.1(d), the solution does not decay to zero, or blowup, but simply encircles the origin. In this case, the steady state is said to be **neutrally stable**.

The other case we are missing here is what happens when the matrix is defective. From (4.25), the conclusion we had earlier still holds. Namely, if $r < 0$, then we have asymptotically stability, and if $r > 0$, then we have instability.

The above discussion is summarized in the following theorem.

Stability Theorem for a Linear System. *For $\mathbf{x}' = \mathbf{A}\mathbf{x}$, if $r = 0$ is not an eigenvalue for $\mathbf{A}$, then the following hold:*

1. *If all of the eigenvalues of $\mathbf{A}$ satisfy Re$(r) < 0$, then $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable steady state.*
2. *If $\mathbf{A}$ has one or more eigenvalues with Re$(r) > 0$, then $\mathbf{x}_s = \mathbf{0}$ is an unstable steady state.*
3. *If the eigenvalues of $\mathbf{A}$ are imaginary, then $\mathbf{x}_s = \mathbf{0}$ is a neutrally stable steady state.*

It is worth pointing out that even though the above theorem was derived for 2×2 matrices, it actually holds when $\mathbf{A}$ is $n \times n$.

Example 1: Determine the stability of the steady state $\mathbf{x}_s = \mathbf{0}$ for

$$\mathbf{x}' = \begin{pmatrix} 1 & 3 \\ 2 & -4 \end{pmatrix} \mathbf{x}.$$

Answer: The characteristic equation for the matrix is $r^2 + 3r - 10 = 0$, and from this it follows that the eigenvalues are $r = -5$ and $r = 2$. Given that there is at least one eigenvalue that is positive, $\mathbf{x}_s = \mathbf{0}$ is unstable. ■

Example 2: Determine the stability of the steady state $\mathbf{x}_s = \mathbf{0}$ for

$$\mathbf{x}' = \begin{pmatrix} -1 & -2 \\ 2 & 0 \end{pmatrix}\mathbf{x}.$$

Answer: The characteristic equation for the matrix is $r^2+r+4=0$, and from this it follows that the eigenvalues are $r = \frac{1}{2}(-1 \pm i\sqrt{15})$. Given that both have negative real part, then $\mathbf{x}_s = \mathbf{0}$ is asymptotically stable. ■

Example 3: Find the steady state, and determine its stability for

$$\mathbf{u}' = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 2 \\ 4 \end{pmatrix}. \tag{4.35}$$

Steady State: Since a steady state is a constant vector that satisfies the differential equation, then we require that

$$\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\mathbf{u} = -\begin{pmatrix} 2 \\ 4 \end{pmatrix}.$$

Solving this for $\mathbf{u}$, one finds the steady state

$$\mathbf{u}_s = \begin{pmatrix} -3 \\ 1 \end{pmatrix}.$$

Stability: Letting $\mathbf{u} = \mathbf{u}_s + \mathbf{x}$, and substituting this into the differential equation, one finds that $\mathbf{x}' = \mathbf{A}\mathbf{x}$, where $\mathbf{A}$ is the matrix in (4.35). If $\mathbf{x}_s = \mathbf{0}$ is unstable, then so is $\mathbf{u}_s$. Similarly, if $\mathbf{x}_s = \mathbf{0}$ is asymptotically stable, then $\mathbf{u}_s$ asymptotically stable. Now, the characteristic equation for $\mathbf{A}$ is $r^2 - 2 = 0$. From this, the eigenvalues are found to be $r = \pm\sqrt{2}$. Given that one is positive, $\mathbf{x}_s$ is unstable, and therefore $\mathbf{u}_s$ is unstable. ■

Exercises

1. Determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state for the following differential equations.

a) $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & 0 \end{pmatrix}\mathbf{x}$ b) $\mathbf{x}' = \begin{pmatrix} 1 & 2 \\ -3 & -4 \end{pmatrix}\mathbf{x}$ c) $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}\mathbf{x}$

d) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 3 & 4 \end{pmatrix}\mathbf{x}$ f) $\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{4} \\ -5 & 0 \end{pmatrix}\mathbf{x}$ h) $\mathbf{x}' = \begin{pmatrix} 0 & -9 \\ 1 & 0 \end{pmatrix}\mathbf{x}$

e) $\mathbf{x}' = \begin{pmatrix} -1 & -1 \\ 6 & -6 \end{pmatrix}\mathbf{x}$ g) $\mathbf{x}' = \begin{pmatrix} 2 & 5 \\ -5 & -6 \end{pmatrix}\mathbf{x}$ i) $\mathbf{x}' = \begin{pmatrix} 1 & -4 \\ 1 & -1 \end{pmatrix}\mathbf{x}$

2. Write the following as $\mathbf{x}' = \mathbf{A}\mathbf{x}$, and then determine whether $\mathbf{x}_s = \mathbf{0}$ is an asymptotically stable, unstable, or neutrally stable steady state.
   a) The simple harmonic oscillator given in (3.45).
   b) The damped oscillator given in (3.52).
3. Find the steady state $\mathbf{u}_s$, and determine its stability, for the following differential equations.

   a) $\mathbf{u}' = \begin{pmatrix} 1 & 3 \\ 0 & -1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ c) $\mathbf{u}' = \begin{pmatrix} -3 & -1 \\ 2 & -1 \end{pmatrix}\mathbf{u} - \begin{pmatrix} 1 \\ 2 \end{pmatrix}$

   b) $\mathbf{u}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\mathbf{u} + \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ d) $\mathbf{u}' = \begin{pmatrix} 1 & -1 \\ 4 & 1 \end{pmatrix}\mathbf{u} + \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

4. This exercise contains useful information that will enable you to determine if a steady state is unstable or neutrally stable (without having to calculate eigenvalues). Let
   $$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$
   and assume that $\det(\mathbf{A}) \neq 0$ (i.e., $ad - bc \neq 0$).
   a) Explain why $r = 0$ is not an eigenvalue for $\mathbf{A}$.
   b) Show that if $a + d > 0$, then $\mathbf{x}_s = \mathbf{0}$ is unstable.
   c) The result from part (b) can be used on three of the equations in Exercise 1. Which ones are they?
   d) Show that if $\det(\mathbf{A}) < 0$, then $\mathbf{x}_s = \mathbf{0}$ is unstable.
   e) The result from part (d) can be used on which equations in Exercise 1?
   f) Show that if $a + d = 0$, and $ad - bc > 0$, then $\mathbf{x}_s = \mathbf{0}$ is neutrally stable.

Chapter 5

Nonlinear Systems

This chapter considers problems that involve two nonlinear first-order ordinary differential equations. These problems are usually difficult enough that finding a formula for the solution is not possible. Consequently, most of the chapter does not concern solving these problems, but instead concentrates on developing ways to determine the properties of the solution. What this means exactly will be explained as the methods are derived. We begin with examples that illustrate the problems we will be considering.

Example 1: Pendulum

The equation for the angular deflection of a pendulum is (see Figure 5.1)

$$\ell \frac{d^2\theta}{dt^2} = -g \sin\theta, \tag{5.1}$$

where the initial angle $\theta(0)$ and the initial angular velocity $\theta'(0)$ are assumed to be given. Also, ℓ is the length of the pendulum and g is the gravitational acceleration constant. Introducing the angular velocity $v = \theta'$ then the equation can be written as the first-order system

$$\theta' = v, \tag{5.2}$$

$$v' = -\alpha \sin\theta, \tag{5.3}$$

where $\alpha = g/\ell$. Although(5.2) is linear, (5.3) is nonlinear because of the $\sin\theta$ term. Consequently, together (5.2), (5.3) form a nonlinear first-order system for θ and v. ■

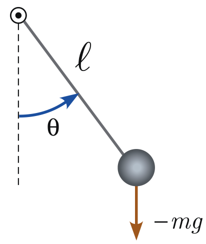

Figure 5.1. *Angular deflection of a pendulum.*

Example 2: Measles

A model for the spread of a disease, like measles, is

$$\frac{dS}{dt} = \alpha N - (\beta I + \alpha)S,$$
$$\frac{dI}{dt} = \beta IS - (\alpha + \gamma)I.$$

In these equations, $S(t)$ is the number of people susceptible to the disease, and $I(t)$ is number that are ill. There is a third group, represented $R(t)$, that is the number of individuals that have recovered or are immune to the disease. There is no differential equation for it as it is just $R = N - (S + I)$, where N is the total number of individuals in the population. ■

The equations for the pendulum and the spread of measles are not solvable using elementary functions. What is possible it to ask questions about the solution that are significant and answerable. As an example, with measles, a reasonable question would be: what would it take to eliminate the disease from the population? This requires that $I \to 0$ as $t \to \infty$ (and the faster this happens the better). In more mathematical terms, we want $I = 0$ to be an asymptotically stable steady state. The values of the k_i's appearing in the equations can be changed using vaccinations, and other actions which limit the propagation of the disease, and how these affect the stability of the $I = 0$ steady-state provide possible solution strategies to quickly eliminate the disease.

A question arising with the pendulum is, does it ever stop moving? Given the physical assumptions used in the derivation of the equation it is reasonable to expect that it does not stop and, in fact, the solution is expected to be periodic. So, we would like to know if it is possible to show that the solution is periodic, and in the process determine the period (without actually solving the problem).

5.1 ▪ Non-Linear Systems

The problems in this chapter can be written in component form as

$$u' = f(u, v), \tag{5.4}$$
$$v' = g(u, v). \tag{5.5}$$

In these equations, $u(t)$ and $v(t)$ are the dependent variables, and f and g are given functions of u and v. It is assumed that the equations are *autonomous*, which means that f and g do not depend explicitly on t.

The vector form of (5.4), (5.5) is

$$\frac{d\mathbf{y}}{dt} = \mathbf{f}(\mathbf{y}), \tag{5.6}$$

where

$$\mathbf{y} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \text{and} \quad \mathbf{f} = \begin{pmatrix} f(u, v) \\ g(u, v) \end{pmatrix}.$$

For an initial value problem, an initial condition of form

$$\mathbf{y}(0) = \begin{pmatrix} u_0 \\ v_0 \end{pmatrix}. \tag{5.7}$$

would also be given.

5.1.1 ▪ Steady-State Solutions

For $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, a **steady state** solution $\mathbf{y}_s$ is a constant vector that satisfies $\mathbf{f}(\mathbf{y}_s) = \mathbf{0}$. In component form, the requirements are that

$$f(u_s, v_s) = 0, \tag{5.8}$$
$$g(u_s, v_s) = 0. \tag{5.9}$$

In terms of solving for the steady states, the graphing method used in Section 2.4 to locate steady states is not useful here. In fact, given that $f(u, v)$ and $g(u, v)$ can be almost anything, there is no method that always works for solving these equations. The recommendation is to pick one of the equations, and use it to solve for u in terms of v, or v in terms of u. The equation to pick for this is usually the one that is easiest to solve. This solution is then substituted into the other equation, and you then have one equation and one unknown (see Example 1 below). It is also not uncommon that you need to be opportunistic, and take advantage of certain terms in the equation to help simply the equations (see Example 2 below).

Example 1: Find the steady states of

$$\frac{dx}{dt} = 3 - x - y - xy,$$
$$\frac{dy}{dt} = xy - 2y.$$

Answer: The equations to solve are

$$3 - x - y - xy = 0,$$
$$xy - 2y = 0.$$

The second equation looks the easiest to work with. Factoring it as $y(x-2) = 0$, we get two solutions: $y = 0$ and $x = 2$. Taking $y = 0$, then from the first equation we get that $x = 3$. For $x = 2$, from the first equation we get that $y = 1/3$. Therefore, we have found two steady states: $(x_s, y_s) = (3, 0)$, and $(x_s, y_s) = (2, 1/3)$. ■

Example 2: Assuming α is a positive constant, find the steady states of

$$\frac{du}{dt} = 1 - (1+\alpha)u + u^2 v,$$
$$\frac{dv}{dt} = u - u^2 v.$$

Answer: The equations to solve are

$$1 - (1+\alpha)u + u^2 v = 0,$$
$$u - u^2 v = 0.$$

It is possible to use the approach from the previous example, but it is easier to look a little closer at these equations. They both contain the term $u^2 v$. In fact, from the second equation $u^2 v = u$. Using this information in the first equation, we get that $u = 1/\alpha$. From the second equation, it follows that $v = \alpha$. Therefore, we have found that the only steady state is: $(u_s, v_s) = (1/\alpha, \alpha)$. ■

Example 3: Find the steady states of

$$x' = x - x^2 - xy,$$
$$y' = 2y - y^2 - 3xy.$$

Answer: The equations to solve are

$$x - x^2 - xy = 0,$$
$$2y - y^2 - 3xy = 0.$$

Factoring the first equation as $x(1-x-y)=0$, then either $x=0$ or $x=1-y$. If $x=0$, then from the second equation $y=0$ or $y=2$, giving us the two steady states $(0,0)$ and $(0,2)$. When $x=1-y$, the second equation reduces to $y(1-2y)=0$, which has solutions $y=0$ and $y=1/2$. This gives us two more steady states, which are $(1/2,1/2)$ and $(1,0)$. ■

Example 4: For the system

$$x' = x - y,$$
$$y' = (x-y)^3,$$

the steady states are any points that satisfy $y = x$. ■

We are going to avoid the situation in Example 4. Specifically, in the problems we will consider, there can be multiple steady states, but they are discrete points as in Examples 1, 2, and 3. The way this will be stated is that the steady states are **isolated**, which means that there is a nonzero distance d so that the distance between any two steady states for the problem is at least d.

Exercises

1. Write the following as $\mathbf{y}' = \mathbf{f}(\mathbf{y})$, making sure to identify the entries in $\mathbf{y}$ and $\mathbf{f}$. If initial conditions are given, write them as $\mathbf{y}(0) = \mathbf{y}_0$.

a) $u' = u^2 - v$
$v' = 2u - 3v$

b) $u' = u^2 + v^2$
$2v' = \sin(u)$

c) $u' = e^u - v$
$v' = uv$
$u(0) = -1,\ v(0) = 0$

d) Van der Pol oscillator
$u'' + (1-u^2)u' + u = 0$

e) Toda oscillator
$u'' + e^u - 1 = 0$

f) Duffing oscillator
$u'' + u + u^3 = 0$
$u(0) = 1,\ u'(0) = -1$

g) Michaelis-Menten system
$S' = -k_1 ES + k_{-1}(E_0 - E),$
$E' = -k_1 ES + (k_2 + k_{-1})(E_0 - E)$
$S(0) = 1,\ E(0) = 2$

h) Predator-prey
$u' = au - buv$
$v' = -cv + duv$

i) Projectile (nonuniform field)
$y'' = -\dfrac{gR^2}{(R+y)^2}$
$y(0) = 0,\ y'(0) = 3$

j) Orbital motion
$r'' = \dfrac{\alpha^2}{r^3} - \dfrac{\mu}{r^2}$
$r(0) = 1,\ r'(0) = 2$

2. Find the steady state solutions of the following.

a) $\begin{cases} x' = 1 - 2x - y - xy \\ y' = 3xy - y \end{cases}$

b) $\begin{cases} x' = y - x^2 \\ y' = y + x^3 \end{cases}$

c) $\begin{cases} u' = 4 - uv^2 \\ v' = -v + uv^2 \end{cases}$

d) $\begin{cases} S' = 2S - S^2 - \frac{2SP}{1+S} \\ P' = \frac{2SP}{1+S} - P \end{cases}$

e) $\begin{cases} S' = -IS + 5 - I - S \\ I' = IS - I \end{cases}$

f) $\begin{cases} s' = c - s^2 \\ c' = 1 + sc \end{cases}$

g) $\begin{cases} u' = \sin(v) + \sin(u) \\ v' = 3v^2 + u^4 \end{cases}$

h) $\begin{cases} u' = uv \\ v' = (2 - u - v)(1 + v) \end{cases}$

5.2 ▪ Stability

The question considered now is central to this chapter, and it is whether a steady state is achievable. What this means is that the steady state is asymptotically stable. To explain how we are going to determine stability, consider the problem of solving

$$x' = x - x^2 - xy, \tag{5.10}$$

$$y' = 2y - y^2 - 3xy. \tag{5.11}$$

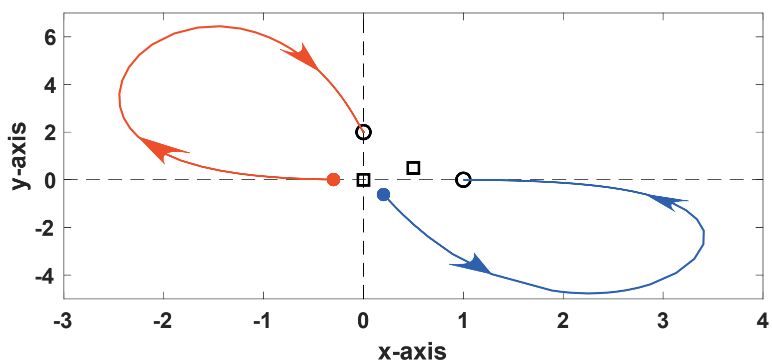

Figure 5.2. *Solution of (5.10),(5.11) for different initial points. The blue curve approaches the steady state* $(1, 0)$*, while the red curve approaches the steady state* $(0, 2)$*.*

This is the problem from Example 3 in the previous section, and we found that there are four steady states: $(0, 0)$, $(0, 2)$, $(1, 0)$, and $(1/2, 1/2)$. One approach for providing insight about stability is to solve the problem numerically. This is easy to do, and two computed solution curves are shown in Figure 5.2. The curves are consistent with what is expected if $(0, 2)$ and $(1, 0)$ are asymptotically stable. Also, since both curves start near

$(0,0)$, yet move away from it, it would appear that $(0,0)$ is an unstable steady state.

Solving the problem numerically is so easy that it possible to solve the problem for many different initial conditions, and check if the solution approaches one of the various steady states. The results from such a calculation are shown in Figure 5.3. What is found is that there are, apparently, two asymptotically stable steady states, $(0,2)$ and $(1,0)$. The calculations also identity the regions for the initial conditions that result in the solution ending up at the respective steady state. The two regions determined from this computation are called the *domain of attraction* for the respective steady state.

Our goal is not to be able to determine the shaded regions shown in Figure 5.3, but, rather, to show that there is a small region around the respective steady state with the same property as the shaded region. Namely, for any initial condition in that small region, the solution of the resulting IVP will end up at the steady state. In this case, the steady state is said to be **asymptotically stable**. What we are doing now is the two dimensional version of what we did in Section 2.4, and the nonlinear version of what was done in Section 4.7.

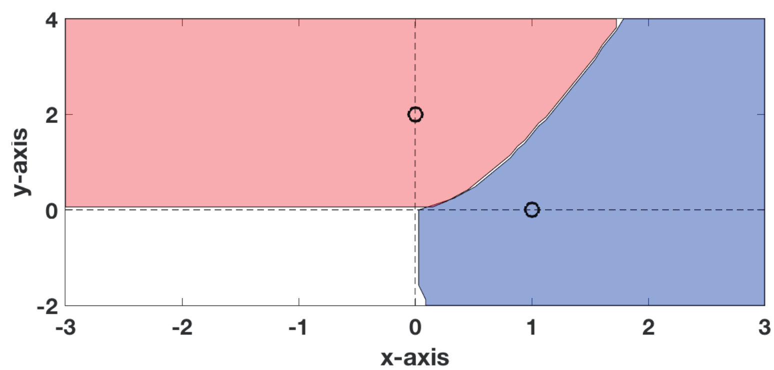

Figure 5.3. *An initial condition $(x(0), y(0))$ located in one of the shaded regions results in the solution of (5.10),(5.11) ending up at the steady state in that shaded region. The two steady states are shown by the dark circles.*

In the section below, the stability conditions are derived, and after that the results are summarized. Also, as mentioned in the previous section, in this chapter it is assumed that the steady states are isolated.

5.2.1 ▪ Derivation of the Stability Conditions

The differential equations are

$$u' = f(u,v), \tag{5.12}$$

$$v' = g(u,v). \tag{5.13}$$

Assume that (u_s, v_s) is a steady-state, which means that u_s and v_s are constants that satisfy

$$\begin{aligned} f(u_s, v_s) &= 0, \\ g(u_s, v_s) &= 0. \end{aligned}$$

The reason for considering stability comes from this question: If we start the solution near (u_s, v_s), what happens? To answer this, it is assumed that the initial position $(u(0), v(0))$ is very close to (u_s, v_s). To determine what happens, we will approximate $f(u, v)$ and $g(u, v)$ for (u, v) near (u_s, v_s) using Taylor's theorem. Using this theorem, we get that

$$\begin{aligned} f(u, v) &= f(u_s, v_s) + (u - u_s) f_u(u_s, v_s) + (v - v_s) f_v(u_s, v_s) \\ &\qquad + \frac{1}{2}(u - u_s)^2 f_{uu} + (u - u_s)(v - v_s) f_{uv} + \frac{1}{2}(v - v_s)^2 f_{vv} + \cdots, \\ g(u, v) &= g(u_s, v_s) + (u - u_s) g_u(u_s, v_s) + (v - v_s) g_v(u_s, v_s) + \cdots \\ &\qquad + \frac{1}{2}(u - u_s)^2 g_{uu} + (u - u_s)(v - v_s) g_{uv} + \frac{1}{2}(v - v_s)^2 g_{vv} + \cdots. \end{aligned}$$

In the above expressions, $f_u = \frac{\partial f}{\partial u}$, $f_v = \frac{\partial f}{\partial v}$, $f_{uu} = \frac{\partial^2 f}{\partial u^2}$, $f_{uv} = \frac{\partial^2 f}{\partial v \partial u}$, etc.

By assumption, $f(u_s, v_s) = 0$ and $g(u_s, v_s) = 0$. Also, if $u - u_s$ is small, then $(u - u_s)^2$ is much smaller. For example, if $u - u_s = 10^{-4}$, then $(u - u_s)^2 = 10^{-8}$. So, we can approximate $f(u, v)$ and $g(u, v)$ as follows:

$$\begin{aligned} f(u, v) &\approx (u - u_s) f_u(u_s, v_s) + (v - v_s) f_v(u_s, v_s), \\ g(u, v) &\approx (u - u_s) g_u(u_s, v_s) + (v - v_s) g_v(u_s, v_s). \end{aligned}$$

With this, (5.12) and (5.13) can be approximated as

$$\begin{aligned} u' &= (u - u_s) f_u(u_s, v_s) + (v - v_s) f_v(u_s, v_s), \\ v' &= (u - u_s) g_u(u_s, v_s) + (v - v_s) g_v(u_s, v_s). \end{aligned}$$

This can be written in system form as

$$\mathbf{u}' = \mathbf{J}(\mathbf{u} - \mathbf{u}_s), \tag{5.14}$$

where

$$\mathbf{u} = \begin{pmatrix} u \\ v \end{pmatrix}, \quad \mathbf{u}_s = \begin{pmatrix} u_s \\ v_s \end{pmatrix},$$

and

$$\mathbf{J} = \begin{pmatrix} f_u(u_s, v_s) & f_v(u_s, v_s) \\ g_u(u_s, v_s) & g_v(u_s, v_s) \end{pmatrix}.$$

The matrix $\mathbf{J}$ is known as the **Jacobian matrix** of $\mathbf{f}$ evaluated at $\mathbf{u}_s$.

To put the problem into the form covered in the last chapter, let $\mathbf{x} = \mathbf{u} - \mathbf{u}_s$. With this, (5.14) becomes

$$\mathbf{x}' = \mathbf{J}\mathbf{x}. \tag{5.15}$$

The general solution of this is given in Section 4.5. For what we are doing it is not necessary to distinguish between real or complex valued eigenvalues. Using the formulas in Section 4.5, and remembering that $\mathbf{u} = \mathbf{u}_s + \mathbf{x}$, we conclude that if $\mathbf{J}$ is not defective, then

$$\mathbf{u} = \mathbf{u}_s + c_1\mathbf{a}_1 e^{r_1 t} + c_2\mathbf{a}_2 e^{r_2 t}, \tag{5.16}$$

and if it is defective, then

$$\mathbf{u} = \mathbf{u}_s + c_1\mathbf{a}e^{rt} + c_2(t\mathbf{a} + \mathbf{b})e^{rt}. \tag{5.17}$$

Whether the e^{rt} terms in (5.16) or (5.17) go to zero, or blow up, as $t \to \infty$, depends on whether $\text{Re}(r)$ is positive or negative. To determine this, it is easiest to go through the various possibilities individually.

- If all of the eigenvalues of $\mathbf{J}$ satisfy $\text{Re}(r) < 0$, then the exponentials in (5.16) and (5.17) go to zero as $t \to \infty$. So, $\mathbf{u}_s$ is asymptotically stable

- If one, or more, of the eigenvalues of $\mathbf{J}$ satisfies $\text{Re}(r) > 0$, then at least one of the exponentials in (5.16) and (5.17) blows up as $t \to \infty$. So, $\mathbf{u}_s$ is unstable.

There is a notable hole in the above list in that there is no conclusion for the case of when the eigenvalues are imaginary. In the theorem in Section 4.7, this is referred to as being neutrally stable. There are neutrally stable steady states for nonlinear systems, but our derivation can not determine this. The reason is that our Taylor series approximation does not include the quadratic and higher terms in the series, and these play an important role when determining neutral stability.

As a final comment, it is possible to carry out a more mathematically rigorous derivation of the above conclusions. In the proof, the usual assumption is that the first and second partial derivatives of $f(u, v)$ and $g(u, v)$ are continuous. Those interested in this should consult Stuart and Humphries [1998] or Perko [2001].

5.2.2 ▪ Summary

For the nonlinear system

$$\begin{aligned} u' &= f(u, v) \\ v' &= g(u, v), \end{aligned}$$

the associated **Jacobian matrix J** is given as

$$\mathbf{J} = \begin{pmatrix} \dfrac{\partial f}{\partial u} & \dfrac{\partial f}{\partial v} \\ \dfrac{\partial g}{\partial u} & \dfrac{\partial g}{\partial v} \end{pmatrix}.$$

The eigenvalues of $\mathbf{J}$ are used to determine stability, as explained in the next theorem.

Linearized Stability Theorem. *Given a steady state $\mathbf{u}_s$, and letting $\mathbf{J}_s$ be the Jacobian matrix evaluated at $\mathbf{u}_s$:*

- *If all of the eigenvalues of $\mathbf{J}_s$ satisfy $Re(r) < 0$, then $\mathbf{u}_s$ is asymptotically stable*
- *If one, or more, of the eigenvalues of $\mathbf{J}_s$ satisfies $Re(r) > 0$, then $\mathbf{u}_s$ is unstable.*

Not every possibility is included in the above theorem. As an example, no conclusion can be made when there are only imaginary eigenvalues. Any case that is not covered by the theorem will be referred to as *indeterminate* in this chapter.

There are two, relatively simple, ways to determine if a steady state is unstable that come from the above theorem. To state these, the formula for the eigenvalues of $\mathbf{J}$ can be written as

$$r_{\pm} = \frac{1}{2}\Big(\text{tr}(\mathbf{J}) \pm \sqrt{[\text{tr}(\mathbf{J})]^2 - 4\det(\mathbf{J})}\Big), \tag{5.18}$$

where the **trace** of $\mathbf{J}$ is given as

$$\text{tr}(\mathbf{J}) = \frac{\partial f}{\partial u} + \frac{\partial g}{\partial v},$$

and the **determinant** of $\mathbf{J}$ is given as

$$\det(\mathbf{J}) = \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} - \frac{\partial f}{\partial v}\frac{\partial g}{\partial u}.$$

With this, the following conclusions can be made:

- If $\text{tr}(\mathbf{J}_s) > 0$, then $\mathbf{u}_s$ is unstable.
- If $\det(\mathbf{J}_s) < 0$, then $\mathbf{u}_s$ is unstable

It is worth pointing out that $\text{tr}(\mathbf{J}_s) < 0$, or $\det(\mathbf{J}_s) > 0$, does not necessarily mean that $\mathbf{u}_s$ is asymptotically stable. Also, the first three bulleted statements hold when you have n equations. The fourth holds when there are n equations, as long as n is even.

5.2.3 ▪ Examples

Example 1: Determine the stability of the steady states of

$$x' = x - x^2 - xy,$$
$$y' = 2y - y^2 - 3xy.$$

Answer: In Section 5.1.1, Example 3, we found that there are four steady states: $(0,0)$, $(0,2)$, $(1/2,1/2)$ and $(1,0)$. To determine their stability, the Jacobian is

$$\mathbf{J} = \begin{pmatrix} 1-2x-y & -x \\ -3y & 2-2y-3x \end{pmatrix}.$$

$(0,0)$: In this case

$$\mathbf{J} = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix},$$

and so $\mathrm{tr}(\mathbf{J}) = 3$. Since this is positive, the steady state is unstable.

$(1,0)$: In this case

$$\mathbf{J} = \begin{pmatrix} -1 & -1 \\ 0 & -1 \end{pmatrix}.$$

There is only one eigenvalue $r = -1$, and it is negative, which means that the steady state is asymptotically stable.

$(0,2)$: In this case

$$\mathbf{J} = \begin{pmatrix} -1 & 0 \\ -6 & -2 \end{pmatrix}.$$

The eigenvalues are $r = -1$ and $r = -2$, and since they are both negative, the steady state is asymptotically stable.

$(1/2,1/2)$: In this case

$$\mathbf{J} = \begin{pmatrix} -1/2 & -1/2 \\ -3/2 & -1/2 \end{pmatrix}.$$

Since $\det(\mathbf{J}) = -1/2$, which is negative, it follows that the steady state is unstable. ■

Example 2: As introduced at the beginning of the chapter, a model for the spread of a disease, like measles, is

$$\frac{dS}{dt} = \alpha N - (\beta I + \alpha)S,$$
$$\frac{dI}{dt} = \beta IS - (\alpha + \gamma)I,$$

where N is the total number of individuals in the population (it is constant). The coefficients, α, β, and γ, are positive constants. It is not hard to show that the two steady states are $(S, I) = (N, 0)$ and $(S, I) = (S_e, I_e)$, where

$$S_e = \frac{\alpha + \gamma}{\beta} \quad \text{and} \quad I_e = \frac{\alpha}{\beta}(N - S_e)\,.$$

The first steady state, $(N, 0)$, corresponds to the case of when the disease is eliminated, and everyone ends up in the S group. The other steady state, (S_e, I_e), is an example of what is known as an epidemic equilibrium, and this is something that is usually avoided if at all possible. Said another way, we want this steady state to be unstable.

To determine the stability of the steady states, note that

$$\mathbf{J} = \begin{pmatrix} -(\beta I + \alpha) & -\beta S \\ \beta I & \beta I - (\alpha + \gamma) \end{pmatrix}.$$

$(S, I) = (N, 0)$: In this case

$$\mathbf{J} = \begin{pmatrix} -\alpha & -\beta N \\ 0 & \beta N - (\alpha + \gamma) \end{pmatrix}.$$

The eigenvalues of this matrix are $-\alpha$ and $\beta(N - S_e)$. Therefore, if $N < S_e$, then this steady state is asymptotically stable, and if $N > S_e$, then it is unstable.

$(S, I) = (S_e, I_e)$: One finds that this steady state is unstable if $N < S_e$, and it is asymptotically stable if $N > S_e$.

Measles: According to the model, to eradicate the disease, which means that (S_e, I_e) is unstable, it is required that

$$N < \frac{\alpha + \gamma}{\beta}\,. \tag{5.19}$$

The parameter α is the birth rate in the population and γ is associated with the rate at which people get well, both of which you can do little to change. As for β, it reflects how contagious the disease is (a larger β means it is more contagious). For measles, $\alpha = 1/50$, $\gamma = 100$, and $\beta = 1800/N$ [Engbert and Drepper, 1994], in which case

$$\frac{\alpha+\gamma}{\beta} \approx \frac{1}{18}N.$$

Clearly, (5.19) is not even close to being satisfied. This is a reflection of that fact that measles is one of the most contagious diseases known. What is needed is to reduce β by a factor of 20 (or more). It is possible to make β smaller with vaccinations, and other actions which limit the propagation of the disease. However, a large percentage of the population must be vaccinated to get β as small as required. This is why there has been a fairly aggressive campaign to vaccinate as many people as possible, with the goal of getting β reduced to the point that $I = 0$ is an asymptotically stable steady state. ■

Exercises

1. For the following find the steady states, and then determine whether they are asymptotically stable, unstable, or indeterminate. Any parameters appearing in the equations should be assumed to be positive.

a) $\begin{cases} x' = 1 - 2x - y - xy \\ y' = 3xy - y \end{cases}$

b) $\begin{cases} x' = y - x^2 \\ y' = y + x^3 \end{cases}$

c) $\begin{cases} u' = 1 + v \\ v' = u + v^3 \end{cases}$

d) $\begin{cases} u' = 4 - uv^2 \\ v' = -v + uv^2 \end{cases}$

e) $\begin{cases} u' = u^2 - v \\ v' = 2u - 3v \end{cases}$

f) $\begin{cases} u' = u^2 + v^2 \\ 2v' = \sin(u) \end{cases}$

g) $\begin{cases} u' = e^u - v \\ v' = uv \end{cases}$

h) $\begin{cases} u' = au - buv \\ v' = -cv + duv \end{cases}$

i) $\begin{cases} S' = 2S - S^2 - \frac{2SP}{1+S} \\ P' = \frac{2SP}{1+S} - P \end{cases}$

j) $\begin{cases} S' = -\frac{1}{2}IS + 1 - I - S \\ I' = \frac{1}{2}IS - I \end{cases}$

k) $\begin{cases} u' = v - u \\ v' = (2 - u - v)(1 + v^2) \end{cases}$

l) $\begin{cases} S' = -aES + b(E_0 - E) \\ E' = -aES + (c + b)(E_0 - E) \end{cases}$

2. Suppose that $y = Y$ is a steady state solution of $y'' + cy' + g(y) = 0$. So, $y = Y$ is a constant and $g(Y) = 0$.

a) Show that Y is unstable if $c < 0$.
b) Show that Y is asymptotically stable if $c > 0$ and $g'(Y) > 0$, and it is unstable if $c > 0$ and $g'(Y) < 0$.

5.3 ▪ Periodic Solutions

With the stability test derived in the previous section, we have a fairly good tool for determining if, and when, the solution of a nonlinear system will come to rest. The next question concerns what can be learned about periodic solutions. This is needed as periodicity plays an important role in our lives, and examples are the sleep-wake cycle and the periodic events associated with the Earth's rotation.

To begin, it's best to define what is meant by periodicity. A solution of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$ is **periodic** if there is a positive T so that

$$\mathbf{y}(t+T) = \mathbf{y}(t), \ \forall t \geq 0. \tag{5.20}$$

The smallest positive T, if it exists, is **the period**.

We will first find a way to determine the solution curve in the phase plane directly from the differential equation and initial conditions. Once that is done, we will then be able to determine the period T, as well as other properties of the solution.

Example: Mass-Spring

In Section 3.10, it was shown that the displacement $u(t)$ of a mass in a spring-mass system satisfies

$$mu'' + ku = 0. \tag{5.21}$$

The general solution of this equation can be written as $u = R\cos(\omega_0 t - \varphi)$, where $\omega_0 = \sqrt{k/m}$. Consequently, the solution is periodic, with period $T = 2\pi/\omega_0$. The same is true for the velocity, which is $v = u' = -\omega_0 R\cos(\omega_0 t - \varphi)$. The key observation here is that, using the identity $\cos^2\theta + \sin^2\theta = 1$,

$$\left(\frac{u}{R}\right)^2 + \left(\frac{v}{\omega_0 R}\right)^2 = 1,$$

or equivalently

$$u^2 + \frac{1}{\omega_0^2}v^2 = R^2. \tag{5.22}$$

This is an equation for an ellipse in the u,v-plane. As an example, suppose that $m = 1$, $k = 4$, and the initial conditions are $u(0) = 1$ and $v(0) = 0$. In this case, $u = \cos(2t)$, $v = -2\sin(2t)$, and from (5.22), the ellipse is

$$u^2 + \frac{1}{4}v^2 = 1. \tag{5.23}$$

This curve is shown in Figure 5.4. Because the period is $T = \pi$, the solution goes around the ellipse and returns to the starting point $(1, 0)$ at $t = \pi,\ 2\pi,\ 3\pi,\ \ldots$.

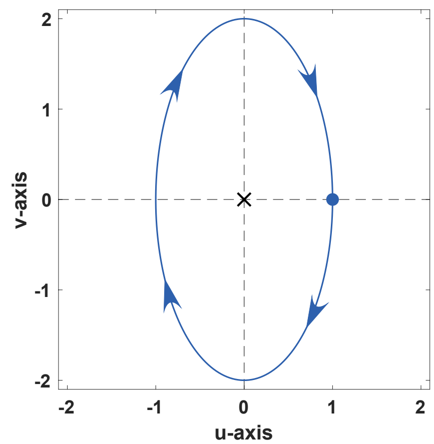

Figure 5.4. *Elliptical path, given in (5.23), that is followed by the solution of the mass-spring IVP. The blue dot is the location of the initial condition.*

To see what can be learned from the vector form of the problem, the equations are

$$\begin{aligned} u' &= v, \\ v' &= -\omega_0^2 u. \end{aligned}$$

This can be used to determine the direction of the arrows in Figure 5.4. Since $v' = -\omega_0^2 u$, using the initial condition given earlier, $v'(0) = -\omega_0^2$. The fact that this is negative means that v must decrease as it leaves the initial point, and so the direction of motion is clockwise around the curve. Note that it is not possible for the solution to reverse direction because this would require that there is a point on the curve where $u' = 0$ and $v' = 0$. Such a point corresponds to a steady state, and the only steady state for this problem is the origin. $\blacksquare$

The important conclusion coming from the above example is that, no matter what time t you select, the solution is located somewhere on the curve shown in Figure 5.4. Having a closed curve like this is a requirement for the solution to be periodic. The reason is that a solution traces out a curve in the phase plane, whether the solution is periodic or not (see Table 4.1 for examples). For the solution to be periodic, it must return to its original position, and that is why a closed curve as in Figure 5.4 is required. What is shown below is how to determine this curve without actually knowing what the solution is.

5.3.1 ▪ Closed Solution Curves and Hamiltonians

It is possible to find the equation for the closed curve without too much trouble if the differential equation comes from Newton's second law, $F = ma$. To explain, if $u(t)$ is the displacement, and F is a function of u, then $F = ma$ gives us the differential equation

$$mu'' = F(u). \tag{5.24}$$

Multiplying this by the velocity u' yields

$$mu'u'' = F(u)u'. \tag{5.25}$$

The key is to observe that the left hand side can be written as

$$\frac{d}{dt}\Big(\frac{1}{2}m\big(u'\big)^2\Big).$$

To do the same for the right hand side, let $V(u)$ be such that $V'(u) = -F(u)$. In this case, the right hand side of (5.25) can be written as $-\frac{d}{dt}V(u)$. What we have done is to rewrite (5.25) as

$$\frac{d}{dt}\Big(\frac{1}{2}m\big(u'\big)^2 + V(u)\Big) = 0. \tag{5.26}$$

Integrating this equation, and letting $v = u'$,

$$\frac{1}{2}mv^2 + V(u) = c. \tag{5.27}$$

The value of the constant c is determined from the initial condition.

What we have shown is that the solution of (5.24) must satisfy (5.27). The fact is that not every forcing function $F(u)$ will result in (5.27) being a closed curve, and an example of this will be given later. Also, it is typical that when $F(u)$ is nonlinear, that not all initial conditions, if any, will yield a closed curve. This happens in the pendulum problem, and this will be discussed in example below.

Assuming the curve determined by (5.27) is closed, the question arises as to the direction of the arrows on the curve, as illustrated in Figure 5.4. The clockwise direction was determined using the initial conditions. You might wonder whether it is possible for the solution to change from a clockwise to a counterclockwise motion (or, visa-versa) on the closed curve. The answer is no, as long as there is not a steady state on the curve. The reason is that for the solution to reverse course, it would first have to come to a stop (i.e., $u' = v' = 0$). The only place it can come to a stop is at a steady state.

There is a physical interpretation of the equation we have derived that is worth knowing about. The left hand side of (5.27) is

$$H(u, v) = \frac{1}{2}mv^2 + V(u). \tag{5.28}$$

This function is a **Hamiltonian** for the problem. In this instance it is the total mechanical energy of the system, and it consists of the sum of the kinetic energy, $\frac{1}{2}mv^2$, and a potential energy, $V(u)$. What we have shown in (5.27) is that the total energy is constant. So, the solution moves along a constant energy curve determined by the Hamiltonian and the initial conditions.

Finally, as is often the case in mathematics, it is not recommended that you memorize the formula given in (5.27). It is better that you remember how it is derived. Namely, you multiply the second-order equation by the velocity, and then rewrite the terms as derivatives.

Example: Mass-Spring Revisited

Starting with the equation

$$mu'' + ku = 0,$$

we multiply by u' and obtain

$$mu'u'' + kuu' = 0.$$

This can be written as

$$\frac{d}{dt}\Big(\frac{1}{2}m\big(u'\big)^2 + \frac{1}{2}ku^2\Big) = 0.$$

This means that

$$\frac{1}{2}m\big(u'\big)^2 + \frac{1}{2}ku^2 = c,$$

where c is an arbitrary constant. Taking, as in the last example, $u(0) = 1$, $v(0) = 0$, $m = 1$, and $k = 4$, and substituting these values into the above equation we find that $c = 2$. Consequently, the above equation becomes

$$u^2 + \frac{1}{4}v^2 = 1. \tag{5.29}$$

This is exactly the same equation (5.23) we derived earlier using the known solution to the problem. What is significant is that we have found this curve without first finding the solution of the problem. ■

It was mentioned earlier that not every forcing function will result in a closed curve. For the above mass-spring problem the spring force is $F = -km$. This is attractive, in the sense that it pulls the mass back towards the rest position $u = 0$. If the force is repelling, so $F = km$, then instead of (5.29), you get $u^2 - \frac{1}{4}v^2 = 1$. This is an equation for a hyperbola, which is not a close curve.

Example: Pendulum

The equation for the angular deflection of a pendulum can be written as

$$\frac{d^2\theta}{dt^2} = -\alpha \sin\theta. \tag{5.30}$$

where $\alpha = g/\ell$. Introducing the angular velocity $v = \theta'$, then we obtain the first-order system

$$\theta' = v, \tag{5.31}$$

$$v' = -\alpha \sin\theta. \tag{5.32}$$

In this example, assume that $\alpha = 4$, and that the initial conditions are $\theta(0) = \pi/4$ and $v(0) = 0$. To determine the closed solution curve, we multiply (5.30) by the velocity θ', giving us

$$\theta'\theta'' = -4\theta' \sin\theta.$$

Writing this as

$$\frac{d}{dt}\frac{1}{2}(\theta')^2 = \frac{d}{dt}\big(4\cos\theta\big),$$

and then integrating gives us the equation

$$\frac{1}{2}v^2 - 4\cos\theta = c.$$

With the initial conditions we find that $c = -2\sqrt{2}$, and so the equation for the curve takes the form

$$v^2 - 8\cos\theta = -4\sqrt{2}. \tag{5.33}$$

The curve obtained from this equation is shown in Figure 5.5.

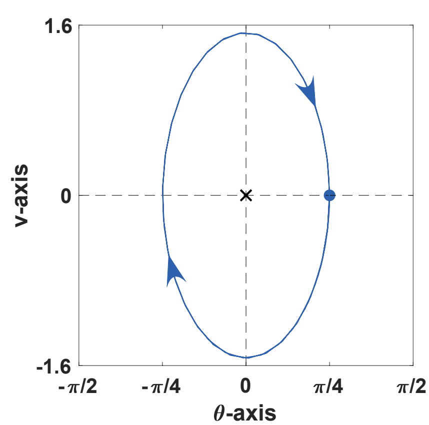

Figure 5.5. *Computed solution of the pendulum example.*

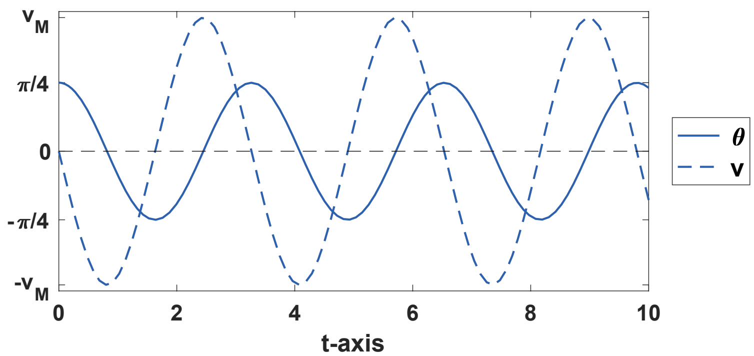

Figure 5.6. *Solution curves for $\theta(t)$ and $v(t)$ for the pendulum solution shown in Figure 5.5.*

To check on the steady states for this problem, from (5.31) and (5.32) the steady states are when $v = 0$ and $\sin\theta = 0$. This results in the values $\theta = 0, \pm\pi, \pm 2\pi, \ldots$. None of the resulting steady states satisfy (5.33).

The direction of the arrows is determined from the v' equation. Namely, since $v'(0) = -\alpha \sin(\theta(0)) = -2\sqrt{2}$, and this is negative, then v must decrease as it leaves the initial point. Therefore, since there are no steady states on the curve, the direction of motion is clockwise around the curve.

It is possible to determine various properties of the solution from (5.33). For example, the maximum velocity v_M occurs when $v' = 0$. Since $v' = -4\sin\theta$, then from Figure 5.5 it is apparent the only solution is $\theta = 0$. In this case, from (5.33), $v^2 = 4(2 - \sqrt{2})$. Therefore, $v_M = 2\sqrt{2 - \sqrt{2}}$.

Finally, to illustrate the periodicity of the individual components of the solution, both θ and v are plotted in Figure 5.6 as functions of t. An interesting question is whether it is possible to determine the period of these functions without knowing the solution. It is, and how this is possible will be explained in the next section ■

Example: Librating and Circulating Orbits

For a pendulum, if the initial velocity is large enough, the mass will go all the way around, pass through $\theta = \pi$ (or, $\theta = -\pi$) and return to where it started. It will continue to do this indefinitely. This motion is periodic, but it does not satisfy the definition of a periodic solution given in (5.20). In mechanics it is called a circulating orbit, while the tick-toc type of periodic motion considered in the previous example is referred to as libration.

The integral curves for the pendulum are shown in Figure 5.7. The closed, solid blue, curves correspond to the periodic solutions discussed earlier. The dashed curves are the possible circulating orbits. On these

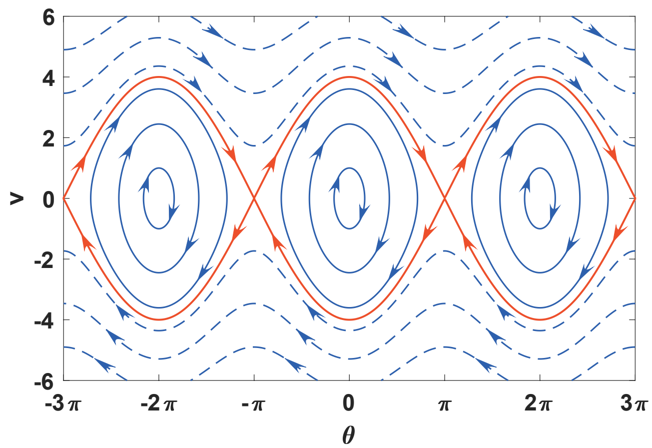

Figure 5.7. *Phase portrait for the pendulum equations (5.31), (5.32), when $\alpha = 4$.*

curves, the angular coordinate θ increases monotonically, if $v > 0$, or decreases if $v < 0$. In the physical plane this corresponds to the mass continually making complete circuits around the pivot point (i.e., it is making a circulating motion).

The red curves in Figure 5.7 form what is known as the separatrix for the pendulum. If you start at a point on the separatrix, the solution will approach the vertical, unstable, steady state. It will also take the solution an infinitely long time to reach this point. ■

5.3.2 ▪ Finding the Period

Once the closed curve formed by the periodic solution is known, it is possible to find the period. As usual, it is easiest to explained how this is done using examples.

Example: Mass-Spring

The equation for the curve is given in (5.29). Solving this for v yields $v = \pm 2\sqrt{1-u^2}$. Which sign you use depends on what part of the curve you are considering. In Figure 5.4 the two u intercepts are $u = \pm 1$. So, for the lower part of the curve connecting $(1, 0)$ to $(-1, 0)$, v is negative, and so $v = -2\sqrt{1-u^2}$. Since $v = u'$, then we have the first-order differential equation

$$\frac{du}{dt} = -2\sqrt{1-u^2}.$$

This equation is separable, which yields

$$\int \frac{du}{\sqrt{1-u^2}} = \int -2dt.$$

Carrying out the integrations,

$$\arcsin(u) = -2t + c.$$

Given that $u = 1$ at $t = 0$, then $c = \pi/2$.

To determine the period, we solve the above equation for t to obtain

$$t = \frac{1}{2}\Big(\frac{\pi}{2} - \arcsin(u)\Big).$$

It is now possible to determine how long it takes for the solution to move along the lower half of the curve and arrive at $(-1, 0)$. Namely, letting $u = -1$ in the above equation we get that

$$t = \frac{1}{2}\Big(\frac{\pi}{2} - \arcsin(-1)\Big) = \frac{\pi}{2}.$$

To compute the time to transverse the upper part of the curve, you can either use the separation of variables approach or you can use the symmetry of the solution curve. Both yield the result that the time is $\pi/2$. Therefore, the period is the sum, which means that $T = \pi$. This agrees with what we found earlier using the exact solution to the problem. ■

Example: Pendulum

The equation for the curve is given in (5.33). Solving this for v yields $v = \pm 2\sqrt{2\cos\theta - \sqrt{2}}$. The lower part of the solution curve, shown in Figure 5.5, goes from $(\pi/4, 0)$ to $(-\pi/4, 0)$. On this part of the curve $v = -2\sqrt{2\cos\theta - \sqrt{2}}$, which gives us the first-order differential equation

$$\frac{d\theta}{dt} = -2\sqrt{2\cos\theta - \sqrt{2}}.$$

This equation is separable, which yields

$$\int \frac{d\theta}{\sqrt{2\cos\theta - \sqrt{2}}} = -2t + c.$$

In anticipation of imposing the initial condition, the above integral is written as

$$\int_{\theta_0}^{\theta} \frac{dr}{\sqrt{2\cos r - \sqrt{2}}} = -2t + c.$$

Now, given that $\theta = \pi/4$ at $t = 0$, then $\theta_0 = \pi/4$ and $c = 0$. The above equation then takes the form

$$\int_{\pi/4}^{\theta} \frac{dr}{\sqrt{2\cos r - \sqrt{2}}} = -2t.$$

The time to reach $\theta = -\pi/4$ is therefore

$$\begin{aligned} t &= -\frac{1}{2}\int_{\pi/4}^{-\pi/4} \frac{dr}{\sqrt{2\cos r - \sqrt{2}}} \\ &= \frac{1}{2}\int_{-\pi/4}^{\pi/4} \frac{dr}{\sqrt{2\cos r - \sqrt{2}}} \,. \end{aligned}$$

Using the separation of variables approach, or using the symmetry of the solution curve, the time to transverse the upper part of the curve is the same as the above value. Therefore, the period T for the pendulum is

$$T = \int_{-\pi/4}^{\pi/4} \frac{dr}{\sqrt{2\cos r - \sqrt{2}}} \,. \tag{5.34}$$

So, we have a formula for the period that does not require knowing the solution. The complication is that it is an improper integral, of the type often referred to in a calculus textbook as "Type II," which means the integrand becomes infinite at the endpoints. It is easy to evaluate it using a computer, and one finds that $T = 3.267\ldots$. ■

We have been able to determine a great deal about the properties of a periodic solution, without actually knowing what the solution is. As stated earlier, this is significant as most of the nonlinear problems that give rise to a periodic solution can not be solved using elementary functions. Consequently, they are almost always solved numerically. Our results complement what can be learned numerically, as we have been able to derive analytical formulas for the period, the closed curve, and other components of the solution. This makes it much easier to determine how the solution changes when the initial conditions, or the parameters appearing in the equations, are changed.

Exercises

1. Find a Hamiltonian function $H(u, v)$ for each of the following:

   a) $2u'' + 3e^{2u} - 3 = 0$

   b) $u'' + \dfrac{u}{1+u^2} = 0$

   c) $4u'' + u^7 = 0$

   d) $5u'' + 7u + 5u^9 = 0$

2. This problem considers periodic, and non-periodic, solutions of $\mathbf{y}' = \mathbf{f}(\mathbf{y})$.

   a) Explain why any steady state is a periodic solution of this equation.

b) Suppose that $\mathbf{y}_b$, given below, is a solution. Is it a periodic solution?

$$\mathbf{y}_b = \begin{pmatrix} \sin t \\ \sin(3t) \end{pmatrix} \qquad \mathbf{y}_c = \begin{pmatrix} \sin t \\ \sin(\pi t) \end{pmatrix}$$

c) Suppose that $\mathbf{y}_c$, given above, is a solution. Is it a periodic solution?

3. The problem concerns a Duffing oscillator, and the differential equation is $u''+u+u^3=0$. Assume the initial conditions are $u(0)=1$ and $v(0)=0$. This equation comes from a mass-spring system, as shown in Figure 3.2, where the restoring force of the spring is nonlinear (specifically, cubic) rather than the linear form assumed using Hooke's law.

   a) The solution in the phase plane is shown in Figure 5.8. Find the equation for this closed curve.
   b) Find the steady-state, show that it is not on the curve you found in part (a).
   c) Draw arrows on the curve indicating the direction of motion. Make sure to explain how you determine this.
   d) What is the maximum velocity?
   e) What is the minimum displacement?
   f) Find a formula, similar to the one in (5.34), for the period.
   g) Find an initial condition that results in the closed curve shown in Figure 5.8, but the solution goes around the curve in the opposite direction found in part (c).

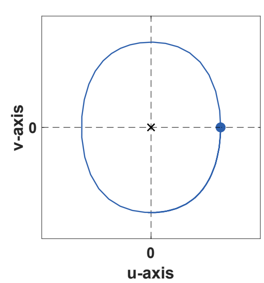

Figure 5.8. *Solution for the Duffing oscillator considered in Exercise 3.*

4. The problem concerns what is known as a Morse oscillator, and the differential equation is

$$u'' + 2\big(1 - e^{-u}\big)e^{-u} = 0.$$

Assume the initial conditions are $u(0)=1$ and $v(0)=0$. The forcing term here arises when considering the oscillatory behavior of a diatomic molecule due to interatomic forces.

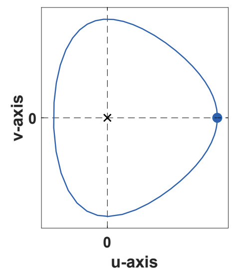

Figure 5.9. *Solution for the Morse oscillator in Exercise 4.*

a) The solution in the phase plane is shown in Figure 5.9. Find the equation for this closed curve.
b) Find the steady-state, show that it is not on the curve you found in part (a).
c) Draw arrows on the curve indicating the direction of motion. Make sure to explain how you determine this.
d) What is the maximum velocity?
e) What is the minimum displacement?
f) Find a formula, similar to the one in (5.34), for the period.
g) Find an initial condition that results in the closed curve shown in Figure 5.9, but the solution goes around the curve in the opposite direction found in part (c).

5. The problem concerns what is known as a Toda oscillator, and the differential equation is $u'' + e^u - 1 = 0$. Assume the initial conditions are $u(0) = 1$ and $v(0) = 0$. The forcing term here arises when modeling intensity fluctuations in solid-state lasers.

a) The solution in the phase plane is shown in Figure 5.10. Find the equation for this closed curve.
b) Find the steady-state, show that it is not on the curve you found in part (a).

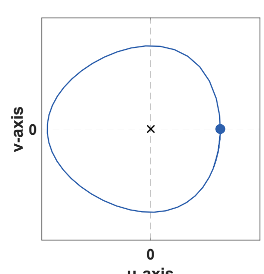

Figure 5.10. *Solution for the Toda oscillator in Exercise 5.*

c) Draw arrows on the curve indicating the direction of motion. Make sure to explain how you determine this.
d) What is the maximum velocity?
e) Letting u_m be the minimum displacement, show that u_m satisfies $e^{u_m} + 1 = u_m + e$.
f) Find a formula, similar to the one in (5.34), for the period.

5.4 ▪ Motion in a Central Force Field

The problem of interest concerns the motion in three dimensions of a particle that is subjected to a radial force $\mathbf{F}$. The specific assumption is that

$$\mathbf{F} = \frac{1}{r} f(r)\mathbf{x}, \tag{5.35}$$

where $\mathbf{x}(t)$ is the position of the particle and $r = ||\mathbf{x}||$. From Newton's second law, the resulting differential equation is

$$m\mathbf{x}'' = \frac{1}{r} f(r)\mathbf{x}, \tag{5.36}$$

where m is the mass of the particle. As for the initial conditions, it is assumed that the initial position $\mathbf{x}(0) = \mathbf{x}_0$ and the initial velocity $\mathbf{x}'(0) = \mathbf{v}_0$ are given. To avoid some uninteresting situations, it is assumed that $\mathbf{x}_0 \times \mathbf{v}_0 \neq \mathbf{0}$.

The force $\mathbf{F}$ can be thought of as coming from the interaction with a particle located at the center. . For example, if the force is gravity, then $f(r) = -k/r^2$, where $k = GMm$. In contrast, if the particles are charged and the force is electrostatic, then $f(r) = -k/r^2$, where $k = -qQ/4\pi\varepsilon_0$. The definition of the various constants making up k is not important here, other than to know that it is possible for k to be positive or negative. In particular, it is positive for a gravitational force, and for an electrostatic force if the charges of the particles are opposite. It is negative for an electrostatic force if the charges of the particles are the same.

The solution of (5.35) can be shown to lie in a plane that has a normal vector $\mathbf{n}$ that is parallel to $\mathbf{p} = \mathbf{x}_0 \times \mathbf{v}_0$ (see Exercise 4). We will orient the coordinate system so the z-axis is in the $\mathbf{n}$ direction, which means that the solution of (5.36) is confined to the x,y-plane. To take advantage of this, we will use polar coordinates and write $x(t) = r(t)\cos\theta(t)$ and $y(t) = r(t)\sin\theta(t)$. After some routine change of variables calculations one finds that (5.36) reduces to

$$m\big[r'' - r(\theta')^2\big] = f(r), \tag{5.37}$$

$$\frac{d}{dt}\big[r^2(\theta')\big] = 0. \tag{5.38}$$

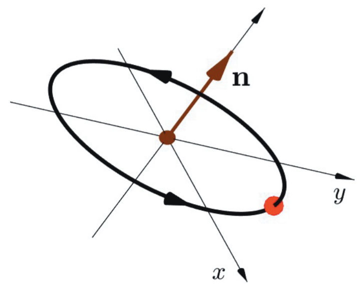

Figure 5.11. *A particle, the red dot, orbits a particle located at the origin. The orbit curve lies in a plane containing the origin and has normal* $\mathbf{n}$*, where* $\mathbf{n}$ *is parallel to* $\mathbf{p} = \mathbf{x}_0 \times \mathbf{v}_0$.

The last equation gives us that $r^2\theta' = p$ is constant, and this means that the first equation reduces to

$$mr'' = f(r) + \frac{mp^2}{r^3} . \tag{5.39}$$

This is a force balance equation, where $f(r)$ is the force introduced earlier and mp^2/r^3 is an outward directed force due to angular momentum.

The second-order differential equation (5.39) be written as a first-order nonlinear system by letting $v = r'$, giving

$$r' = v, \tag{5.40}$$

$$v' = \frac{1}{m} f(r) + \frac{p^2}{r^3}. \tag{5.41}$$

It is worth knowing that in the derivation of (5.39), it is found that $p = ||\mathbf{x}_0 \times \mathbf{v}_0||$. So, p is a positive constant that is known from the initial conditions.

5.4.1 ▪ Steady States

The steady states, if there are any, satisfy $v = 0$ and $r^3 f(r) + mp^2 = 0$. For example, if $f(r) = -k/r^2$, then to be a steady state it is required that $kr = mp^2$. This means we need $k > 0$, and the resulting steady state is $r = mp^2/k$. Since $r^2\theta' = p$, then $\theta = \omega t + \theta_0$, where $\omega = k^2/m^2p^3$. The corresponding solution is a circular orbit in the x,y-plane, with radius $r = mp^2/k$ and period $2\pi/\omega$.

To check on the stability, note that

$$\mathbf{J} = \begin{pmatrix} 0 & 1 \\ \frac{2k}{mr^3} - 3\frac{p^2}{r^4} & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -p^2/r^4 & 0 \end{pmatrix}.$$

From this one finds that the eigenvalues are $\pm ip/r^2$, which means that the stability of the steady state is indeterminate. However, as shown in Exercise 6, it is possible to show that the steady state is actually neutrally stable.

5.4.2 ▪ Periodic Orbit

The next question is whether the solution is periodic. Said another way, we would like to know if the particle orbits the particle that is located at the origin. To find the closed curve formed by the solution, if there is one, we multiply (5.39) by r'. From this, and remembering that we have taken $f(r) = -k/r^2$, it is found that

$$\frac{1}{2}mv^2 + \frac{mp^2}{2r^2} - \frac{k}{r} = c, \tag{5.42}$$

where c is a constant determined by the initial conditions. Completing the square, we get that

$$v^2 + p^2\Big(\frac{1}{r} - \frac{k}{mp^2}\Big)^2 = c_0^2 \tag{5.43}$$

where $c_0^2 = v_0^2 + p^2\Big(1/r_0 - k/mp^2\Big)^2$, $r(0) = r_0$, and $v(0) = v_0$.

To answer the question about a periodic orbit, it makes things easier to let $u = 1/r$, so (5.43) takes the form

$$v^2 + p^2\Big(u - \frac{k}{mp^2}\Big)^2 = c_0^2. \tag{5.44}$$

This is an equation for an ellipse in the u,v-plane with center $(u, v) = (k/mp^2, 0)$. Two representative elliptical paths obtained from this equation are shown in Figure 5.12. Since $u = 1/r$, then u must be positive. This means that the dashed portion of the ellipse on the left is not possible physically. To obtain an ellipse which only has positive u values, we can use the u intercepts. Setting $v = 0$ in (5.44) yields

$$u_\pm = \frac{k}{mp^2} \pm \frac{1}{p}c_0.$$

For these to be positive it is required that $k > 0$. To guarantee that u_- is positive we need $k > mpc_0$. As shown in Exercise 5, this will hold as long as the initial velocity is not too large. In this case, the particle will orbit the larger particle.

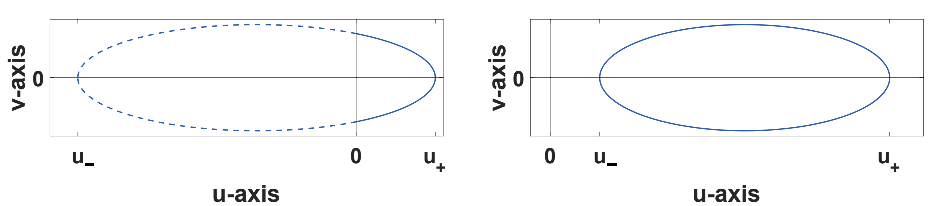

Figure 5.12. *Two possible elliptical curves coming from (5.44). The u-intercepts for each ellipse are u_- and u_+.*

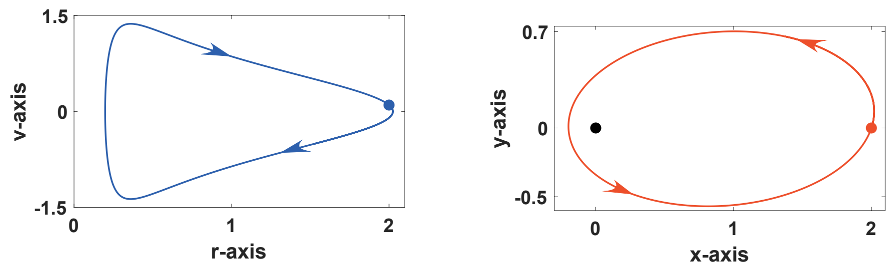

Figure 5.13. *Numerical solution of (5.39) in the case of when the solution is periodic. The initial position, and direction of motion, are shown on each curve.*

As an example of a periodic solution, the numerical solution of the central force problem in (5.39) is shown in Figure 5.13. The orbital path in the r,v-plane is on the left. The physical path, in the x,y plane, is shown on the right.

The question arises as what happens when you get an ellipse like the one on the left in Figure 5.12. Irrespective of which point you start at on the solid curve, and no matter which direction you go on the curve, u approaches zero. In other words, $r \to \infty$ as $t \to \infty$. Physically, what is happening is that the angular momentum is so large that an orbit is not possible, and the particle simply escapes whatever hold the particle at the origin might have on it. It is also evident from Figure 5.12, contrary to what is often shown in cartoons, that the particle does not make several orbits around the origin before escaping. In fact, the particle is incapable of making even one complete orbit.

Exercises

1. Suppose the law of gravity resulted in $f(r) = -k/r^3$, where $k > 0$. In this case the angular momentum term in (5.39) is unaffected.

   a) Are there any steady state solutions? If so, check on their stability.
   b) Assuming there is a periodic solution, determined its equation in the u,v-plane.
   c) Use your result from part (b) to explain why there is no periodic solution of this problem.

2. Suppose the law of gravity resulted in $f(r) = -k/r^5$, where $k > 0$. In this case the angular momentum term in (5.39) is unaffected.

   a) Are there any steady state solutions? If so, check on their stability.
   b) Assuming there is a periodic solution, determined its equation in the u,v-plane.

c) Use your result from part (b) to explain why there is no periodic solution of this problem.

3. This exercise explores the usefulness of making the change of variables from r, t to u, τ, where $u(\tau) = 1/r$ and $\tau = \theta(t)$. This is an approach often used in physics textbooks.

   a) Show that $r'(t) = -pu'(\tau)$, and $r''(t) = -p^2u^2u''(\tau)$.

   b) The mathematical requirement for the change of variables to be valid is that $\theta(t)$ is a strictly monotonic function of t. Explain why this holds in this problem.

   c) Using the results from part (a), show that (5.39) takes the form
$$u'' + u = -\frac{1}{mp^2u^2} f\left(\frac{1}{u}\right).$$

   d) Assuming that $f(r) = -k/r^2$, find the general solution of the resulting differential equation in part (c). In doing this, use (3.22) when writing down the general solution of the associated homogeneous equation. Also, what is the resulting formula for r?

   e) Use the results from Exercise 5 below to show that $u(0) = 1/x_0$ and $u'(0) = -x_0'/(x_0y_0')$. Use these to find the two arbitrary constants in your solution in (d).

4. Let $\mathbf{p} = \mathbf{x} \times \mathbf{x}'$. In this exercise you will likely need to review the properties of the cross product you learned in calculus.

   a) Show that $\mathbf{p}' = \mathbf{0}$. This means that $\mathbf{p}$ is a constant vector, and so, from the initial conditions, $\mathbf{p} = \mathbf{x}_0 \times \mathbf{v}_0$. It is assumed that $\mathbf{x}_0 \times \mathbf{v}_0 \neq \mathbf{0}$.

   b) Explain why $\mathbf{p} \cdot \mathbf{x} = 0$ and $\mathbf{p} \cdot \mathbf{x}' = 0$. Why does this mean that $\mathbf{x}$ and $\mathbf{x}'$ are in the plane that is perpendicular to $\mathbf{p}$, and which contains the origin?

   c) Assuming $\mathbf{x} = (r(t)\cos\theta(t), r(t)\sin\theta(t), 0)$, show that $\mathbf{p} = r^2\theta'\mathbf{k}$, where $\mathbf{k}$ is the unit vector pointing in the positive z-direction.

   d) The plane has normal $\mathbf{p}$ as well as normal $-\mathbf{p}$. Which one is used when orientating the positive z-axis in such a way that $r^2\theta' > 0$?

5. This problem determines how the initial conditions for (5.35) contribute to the reduced problem for the orbit. Assume that $\mathbf{x}_0 = (x_0, 0, 0)^T$ and $\mathbf{v}_0 = (x_0', y_0', 0)^T$, where x_0, x_0', and y_0' are given with x_0 and y_0' both positive. The superscript T indicates transpose. Also, assume that $f(r) = -k/r^2$ and $\theta(0) = 0$.

   a) Show that $p = x_0y_0'$.

   b) Show that the initial conditions for (5.39) are $r(0) = x_0$ and $r'(0) = x_0'$.

   c) Show that $u_- > 0$ reduces to the requirement that $k > mx_0(y_0')^2$.

6. A steady state is *neutrally stable* if it is not asymptotically stable, but for any initial condition selected very near the steady state, the resulting solution stays near the steady state. Explain how (5.44) can be used to show that the steady state found in Section 5.4.1 is neutrally stable.

Chapter 6

Laplace Transform

We have found that to solve $y'' + by' + cy = 0$ you assume that $y = e^{rt}$, and for $\mathbf{x}' = \mathbf{A}\mathbf{x}$ you assume that $\mathbf{x} = \mathbf{a}e^{rt}$. What is notable here is the exponential dependence of the solution on t. It is possible to extend this assumption in such a way that it is possible to solve a wide variety of more complicated problems, such as those involving partial differential equations. The extension we are going to consider is called the Laplace transform.

It is recommended that if you are a bit fuzzy on integration by parts, or partial fractions, that you spend some time reviewing those integration methods as one or both are used in the majority of the examples and exercises in this chapter.

6.1 ▪ Definition

The generalization we are interested in called the Laplace transform, and its definition is given next.

Laplace Transform. *Given a function $y(t)$, for $0 \le t < \infty$, its Laplace transform $Y(s)$ is defined as*

$$Y(s) \equiv \int_0^\infty y(t)e^{-st}dt. \tag{6.1}$$

It will be useful to have a more compact notation for the integral in this expression, and this will be done by writing the above formula as

$$Y(s) \equiv \mathcal{L}(y). \tag{6.2}$$

The Laplace variable s is analogous to the r used in the assumption $y = e^{rt}$ or $\mathbf{x} = \mathbf{a}e^{rt}$. Like r, s can be complex-valued.

Introduction to Differential Equations, M. H. Holmes, 2019

6.1.1 ▪ Restrictions

It is relatively easy to use the Laplace transform to solve differential equations, but there are mathematical requirements, and conditions, that must be discussed before doing so.

One has to do with the restrictions on s. To explain, as we will show below, if $y(t) = e^{3t}$, then

$$Y(s) = \frac{1}{s-3}, \tag{6.3}$$

and if $y(t) = \sin(2t)$, then

$$Y(s) = \frac{2}{s^2+4}. \tag{6.4}$$

Both of these functions have singular points. For (6.3) it is at $s = 3$, and for (6.4) is it is when $s^2 + 4 = 0$, which means that $s = \pm 2i$. The requirement imposed on the Laplace transform is that, in the complex plane, the s values that are used are to the right of any singular point that $Y(s)$ has. So, for (6.3) it is required that $\mathrm{Re}(s) > 3$, and for (6.4) it is required that $\mathrm{Re}(s) > 0$. This requirement gives rise to what is known as the **half-plane of convergence** for the Laplace transform.

To illustrate how, and why, this condition is needed we consider an example.

Example: If $y(t) = e^{3t}$, find $Y(s)$.

Answer: Using (6.1),

$$\begin{aligned} Y(s) &= \int_0^\infty e^{3t} e^{-st} dt \\ &= \int_0^\infty e^{-(s-3)t} dt \\ &= -\frac{1}{s-3} e^{-(s-3)t} \Big|_{t=0}^{\infty}. \end{aligned}$$

The issue now is what is the limiting value of $e^{-(s-3)t}$ as $t \to \infty$. If $\mathrm{Re}(s) < 3$, then the limit does not exist, while if $s = 3$, then $1/(s-3)$ is not defined. Consequently, for $Y(s)$ to be defined it is required that $\mathrm{Re}(s) > 3$. In this case, we get that $Y(s) = 1/(s-3)$. ■

The second mathematical requirement concerns $y(t)$, and the improper integral in (6.1). For the problems considered in this textbook, it is enough to assume that $y(t)$ is continuous for $0 \le t < \infty$, except for a finite number of jump discontinuities. What a **jump discontinuity** means is that $y(t)$ is not continuous at the point, but the limits of

y from the left and right are defined. A simple example, with a jump discontinuity at $t = 2$, is

$$y(t) = \begin{cases} 3 & \text{if} \quad 0 \le t \le 2, \\ -1 & \text{if} \quad 2 < t. \end{cases} \tag{6.5}$$

It is also necessary to impose the restriction that $y(t)$ has *exponential order*. This means that y grows no faster than a linear exponential function as $t \to \infty$. The specific requirement is that there is a constant α so that

$$\lim_{t\to\infty} y(t)e^{\alpha t} = 0.$$

As examples, any polynomial function in t, or any linear combination of $\sin(\omega t)$ and $\cos(\omega t)$, have exponential order. On the other hand, e^{t^2} and e^{t^3} do not.

6.1.2 ▪ Examples

With the technical details out of the way, we consider a few examples. As you will see, finding the Laplace transform of a function provides ample opportunity to practice using integration by parts.

Example 1: If $y(t) = \sin 2t$, find $Y(s)$.

Answer: Using (6.1), and integration by parts,

$$\begin{aligned} Y(s) &= \int_0^\infty \sin(2t)e^{-st}dt \\ &= -\frac{1}{2}\cos(2t)e^{-st}\Big|_{t=0}^{\infty} - \frac{s}{2}\int_0^\infty \cos(2t)e^{-st}dt \\ &= \frac{1}{2} - \frac{s}{2}\int_0^\infty \cos(2t)e^{-st}dt. \end{aligned}$$

To guarantee that $\cos(2t)e^{-st}$ has a finite limit as $t \to \infty$, it has been assumed that $\mathrm{Re}(s) > 0$. Using integration by parts again gives us

$$\begin{aligned} Y(s) &= \frac{1}{2} - \frac{s}{2}\left[\frac{s}{2}\sin(2t)e^{-st}\Big|_{t=0}^{\infty} + \int_0^\infty \sin(2t)e^{-st}dt\right] \\ &= \frac{1}{2} - \frac{s^2}{4}\int_0^\infty \sin(2t)e^{-st}dt \\ &= \frac{1}{2} - \frac{s^2}{4}Y(s). \end{aligned}$$

Solving for Y, we get that $Y = 2/(s^2+4)$. Using the $\mathcal{L}(y)$ notation, we have that

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}. \quad \blacksquare$$

Example 2: If $y(t)$ is the jump function given in (6.5), find $Y(s)$.

Answer: Using the additive property of integrals,

$$\begin{aligned} Y(s) &= \int_0^\infty y(t)e^{-st}dt \\ &= \int_0^2 y(t)e^{-st}dt + \int_2^\infty y(t)e^{-st}dt \\ &= \int_0^2 3e^{-st}dt - \int_2^\infty e^{-st}dt \\ &= -\frac{3}{s}e^{-st}\Big|_{t=0}^{2} + \frac{1}{s}e^{-st}\Big|_{t=2}^{\infty} \\ &= -\frac{4}{s}e^{-2s} + \frac{3}{s}\,. \end{aligned}$$

To guarantee that $\frac{1}{s}e^{-st}$ has a defined limit as $t \to \infty$, it has been assumed that $\text{Re}(s) > 0$. ■

In the above two examples, the condition on s so $Y(s)$ is defined is stated explicitly. In the remained of the chapter this will not be done, and it is assumed that the condition is obvious from the derivation.

Example 3: If $y(t) = 3t - \sin 2t$, find $Y(s)$.

Answer: Using integration by parts, and the result from Example 1,

$$\begin{aligned} Y(s) &= \int_0^\infty 3te^{-st}dt - \int_0^\infty \sin 2te^{-st}dt \\ &= -\frac{3t}{s}e^{-st}\Big|_{t=0}^{\infty} + \frac{3}{s}\int_0^\infty e^{-st}dt - \frac{2}{s^2+4} \\ &= \frac{3}{s^2} - \frac{2}{s^2+4}\,. \end{aligned}$$

Using the $\mathcal{L}(y)$ notation, we have found that

$$\mathcal{L}(3t - \sin 2t) = 3\mathcal{L}(t) - \mathcal{L}(\sin 2t)\,,$$

where $\mathcal{L}(t) = 1/s$ and $\mathcal{L}(\sin 2t) = 2/(s^2+4)$. ■

The last example is an illustration of a property we will use frequently. Namely, if c_1 and c_2 are constants, then

$$\mathcal{L}\big(c_1y_1(t) + c_2y_2(t)\big) = c_1\mathcal{L}(y_1(t)) + c_2\mathcal{L}(y_2(t)). \tag{6.6}$$

Another way to write this is to let $y(t) = c_1 y_1(t) + c_2 y_2(t)$, in which case

$$Y(s) = c_1 Y_1(s) + c_2 Y_2(s), \tag{6.7}$$

where Y_1 and Y_2 are the Laplace transforms for y_1 and y_2, respectively. Because the Laplace transform has this property, it is said to be a *linear operator*. The usefulness of the linearity of the Laplace transform is why it is listed first in Table 6.1.

It is worth pointing out that you know several other linear operators. One is a matrix, because it satisfies $\mathbf{A}(c_1\mathbf{y}_1 + c_2\mathbf{y}_2) = c_1\mathbf{A}\mathbf{y}_1 + c_2\mathbf{A}\mathbf{y}_2$. A second is differentiation, as it satisfies $\frac{d}{dt}(c_1 y_1(t) + c_2 y_2(t)) = c_1\frac{d}{dt}y_1(t) + c_2\frac{d}{dt}y_2(t)$.

Exercises

1. Find the Laplace transform of the following functions.

   a) $y = -e^{5t}$
   b) $y = 3 + 4t$
   c) $y = 2t + 4e^{-t}$
   d) $y = e^{-2t} - e^{7t}$
   e) $y = 4t^2$
   f) $y = (t-3)^2$
   g) $y = 4(t+1)^2$
   h) $y = te^{-t}$
   i) $y = 2\sin(3\pi t + 4)$
   j) $y = e^{2t} + 4\cos(2t)$

2. Find the Laplace transform of the following functions.

   a)
   $$y(t) = \begin{cases} 0 & \text{if } 0 \le t \le 2, \\ 5 & \text{if } 2 < t. \end{cases}$$
   b)
   $$y(t) = \begin{cases} -1 & \text{if } 0 \le t \le 1, \\ 1 & \text{if } 1 < t < 2, \\ 0 & \text{if } 2 \le t \end{cases}$$
   c)
   $$y(t) = \begin{cases} 0 & \text{if } 0 \le t \le 12, \\ 2 & \text{if } 12 < t < 15, \\ 0 & \text{if } 15 \le t \end{cases}$$
   d)
   $$y(t) = \begin{cases} t & \text{if } 0 \le t \le 2, \\ 2 & \text{if } 2 < t. \end{cases}$$
   e)
   $$y(t) = \begin{cases} t & \text{if } 0 \le t \le 1, \\ e^{-t+1} & \text{if } 1 < t. \end{cases}$$
   f)
   $$y(t) = \begin{cases} 5-t & \text{if } 0 \le t \le 3, \\ t-1 & \text{if } 3 < t. \end{cases}$$

3. One way to avoid using integration by parts is to use the formulas $\cos x = \frac{1}{2}\left(e^{ix} + e^{-ix}\right)$ and $\sin x = \frac{1}{2i}\left(e^{ix} - e^{-ix}\right)$ (see Section 3.4.1). Use these to find the Laplace transform of $y(t)$.

a) $y(t) = \cos(3t)$
b) $y(t) = 4\sin(7t)$
c) $y(t) = e^{-t}\cos(\pi t)$
d) $y(t) = e^{2t}\sin(5t)$
e) $y(t) = \cos(t)\sin(2t)$
f) $y(t) = \cos(2t)\cos(4t)$

6.2 ▪ Inverse Laplace Transform

As will be seen when we get around to solving differential equations, we will use the Laplace transform to change the problem from solving for y to solving for Y. It is actually fairly easy to do this. Once Y is known, it is then necessary to determine y. This requires us to know how to find the inverse Laplace transform.

Using the $\mathcal{L}(y)$ notation, the inverse Laplace transform is written as $\mathcal{L}^{-1}(Y)$. As an example, earlier we found that

$$\mathcal{L}(\sin 2t) = \frac{2}{s^2+4}\,.$$

The inverse is therefore

$$\mathcal{L}^{-1}\Big(\frac{2}{s^2+4}\Big) = \sin 2t\,.$$

The caveat here is that if $Y = \mathcal{L}(y)$, it is not always true that $y = \mathcal{L}^{-1}(Y)$. It is true for the above example, and this is because the original function $y(t)$ is continuous. What happens when $y(t)$ has a jump discontinuity will be discussed later.

In Section 6.4, the first differential equation we will solve using a Laplace transform is $y' + 3y = e^{2t}$. We will find that $Y = \frac{1}{s+3}(2 - \frac{1}{2-s})$, and this will mean that to find y we will need to determine $\mathcal{L}^{-1}(Y)$. There is a general formula for the inverse Laplace transform, which involves a line integral in the complex plane. Although this can provide some entertaining mathematical challenges, most find the inverse transform by using tables. Table 6.1 is one such example, and it is one that is used in this text. Note that the first six entries are general properties for the transform. The first one listed is the linearity property, and how this applies to the transform is given in (6.6). Writing it in terms of the inverse transform, we have that

$$\begin{aligned}\mathcal{L}^{-1}\big(c_1Y_1(s) + c_2Y_2(s)\big) &= c_1\mathcal{L}^{-1}(Y_1(s)) + c_2\mathcal{L}^{-1}(Y_2(s))\\ &= c_1y_1(t) + c_2y_2(t).\end{aligned}$$

This is used, and needed, for most of the examples to follow.

As demonstrated in the next example, using a table, finding an inverse transform can be very easy.

	$Y(s) = \mathcal{L}(y)$	$\mathcal{L}^{-1}(Y)$
1.	$aY(s) + bV(s)$	$ay(t) + bv(t)$
2.	$V(s)Y(s)$	$\int_0^t v(t-r)y(r)dr$
3.	$sY(s)$	$y'(t) + y(0)$
4.	$\frac{1}{s}Y(s)$	$\int_0^t y(r)dr$
5.	$e^{-as}Y(s)$	$y(t-a)H(t-a)$ for $a > 0$
6.	$Y(s-a)$	$e^{at}y(t)$
7.	$\frac{1}{(s+a)^n}$	$\frac{1}{(n-1)!}t^{n-1}e^{-at}$ for $n = 1, 2, 3, \ldots$
8.	$\frac{bs+c}{(s+a)^2+\omega^2}$	$e^{-at}\Big(b\cos(\omega t) + \frac{c-ab}{\omega}\sin(\omega t)\Big)$
9.	$\frac{cs+d}{(s+a)(s+b)}$	$\frac{1}{b-a}\big((bc-d)e^{-bt} - (ac-d)e^{-at}\big)$
10.	$\frac{1}{(s+a)(s+b)(s+c)}$	$\frac{1}{(b-a)(c-a)}e^{-at} + \frac{1}{(c-b)(a-b)}e^{-bt} + \frac{1}{(a-c)(b-c)}e^{-ct}$
11.	$\frac{p(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)}$	see (6.9)
12.	$\frac{1}{\sqrt{s+a}}$	$\frac{1}{\sqrt{\pi t}}e^{-at}$
13.	$\frac{1}{\sqrt{s}(\sqrt{s}+a)}$	$e^{a^2 t}\mathrm{erfc}(a\sqrt{t})$
14.	$\frac{1}{s}e^{-as}$	$H(t-a)$ for $a > 0$
15.	$e^{-a\sqrt{s}}$	$\frac{a}{2\sqrt{\pi}}t^{-3/2}e^{-a^2/(4t)}$ for $a > 0$
16.	$\frac{1}{\sqrt{s}}e^{-a\sqrt{s}}$	$\frac{1}{\sqrt{\pi t}}e^{-a^2/(4t)}$ for $a > 0$

Table 6.1. *Inverse Laplace transforms. The Heaviside step function $H(x)$ is defined in (6.14). The constants in Y are assumed not to result in a divide-by-zero in the inverse formula. So, for example, in Property 9, it is required that $b \neq a$.*

Example: If $Y(s) = \frac{3}{s^2} - \frac{7s}{s^2+25}$, find $y(t)$.

Answer: Using the linearity property,

$$\begin{aligned} \mathcal{L}^{-1}(Y) &= \mathcal{L}^{-1}\Big(\frac{3}{s^2} - \frac{7s}{s^2+25}\Big) \\ &= 3\mathcal{L}^{-1}\Big(\frac{1}{s^2}\Big) - 7\mathcal{L}^{-1}\Big(\frac{s}{s^2+25}\Big). \end{aligned}$$

From Property 7 in the table, with $n = 2$ and $a = 0$,

$$\mathcal{L}^{-1}\Big(\frac{1}{s^2}\Big) = t.$$

Similarly, from Property 8 in the table, with $b = 1$, $c = 0$, $a = 0$, and $\omega = 5$,

$$\mathcal{L}^{-1}\Big(\frac{s}{s^2+25}\Big) = \cos(5t).$$

Therefore,

$$y(t) = 3t - 7\cos(5t). \quad \blacksquare$$

6.2.1 ▪ Useful Special Case

There are numerous special cases where formulas can be found for finding the inverse Laplace transform. One that deserves to be mentioned is the case of when $Y(s)$ can be written in the form

$$Y(s) = \frac{p(s)}{(s-a_1)(s-a_2)\cdots(s-a_n)}\,, \tag{6.8}$$

where $p(s)$ is a polynomial of degree less than n. The a_i's can be complex-valued but they must be distinct, which means that $a_i \neq a_j$ if $i \neq j$. In this case, using something called residue theory, it can be shown that the inverse transform $\mathcal{L}^{-1}(Y)$ is

$$y(t) = \sum_{i=1}^{n} \frac{p(a_i)}{q'(a_i)} e^{a_i t}, \tag{6.9}$$

where $q(s) = (s-a_1)(s-a_2)\cdots(s-a_n)$.

Example: Find the inverse transform of

$$Y = \frac{2}{(s^2+1)(s^2+4)}\,.$$

Answer: The straightforward approach would be to use a partial fractions decomposition of Y, and then find the inverse transform of the terms in the decomposition. However, it is easier to factor the denominator, and write

$$Y = \frac{2}{(s+i)(s-i)(s-2i)(s+2i)}\,.$$

Setting $q(s) = (s+i)(s-i)(s-2i)(s+2i)$, then

$$\begin{aligned}\frac{d}{ds}q(s) &= (s-i)(s-2i)(s+2i) + (s+i)(s-2i)(s+2i)\\ &\quad + (s+i)(s-i)(s+2i) + (s+i)(s-i)(s-2i).\end{aligned}$$

So, $q'(-i) = (-2i)(-3i)(i) = -6i$, $q'(i) = -6i$, $q'(2i) = -12i$, and $q'(-2i) = 12i$. With this, from (6.9) we get that

$$\begin{aligned}y(t) &= \sum_{i=1}^{4} \frac{2}{q'(a_i)} e^{a_i t}\\ &= \frac{2}{-6i}e^{-it} + \frac{2}{6i}e^{it} + \frac{2}{-12i}e^{2it} + \frac{2}{12i}e^{-2it}\\ &= \frac{1}{3i}\big(e^{it} - e^{-it}\big) - \frac{1}{6i}\big(e^{2it} - e^{-2it}\big).\end{aligned}$$

Since $\frac{1}{i} = -i$ and $e^{\theta i} - e^{-\theta i} = 2i\sin\theta t$, then we get that

$$y(t) = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t. \quad \blacksquare \tag{6.10}$$

6.2.2 ▪ Jump Discontinuities

At points t where the original function $y(t)$ has a jump discontinuity, then $\mathcal{L}^{-1}(Y)$ equals the average in the jump in y. The formula is, for $t > 0$,

$$\mathcal{L}^{-1}(Y) = \frac{1}{2}\big[y(t^+) + y(t^-)\big]. \tag{6.11}$$

To illustrate, in Example 2 above we found that if

$$y(t) = \begin{cases} 3 & \text{if} \quad 0 \le t \le 2, \\ -1 & \text{if} \quad 2 < t, \end{cases} \tag{6.12}$$

then

$$Y(s) = -\frac{4}{s}e^{-2s} + \frac{3}{s}\,.$$

Taking the inverse transform of this, if $t \neq 2$, then $y = \mathcal{L}^{-1}(Y)$. At $t = 2$, the average in the jump in $y(t)$ is $\frac{1}{2}\big[y(2^+) + y(2^-)\big] = \frac{1}{2}(3-1) = 1$. Therefore, the inverse transform is

$$\mathcal{L}^{-1}(Y) = \begin{cases} 3 & \text{if} \quad 0 \le t < 2, \\ 1 & \text{if} \quad t = 2, \\ -1 & \text{if} \quad 2 < t. \end{cases} \tag{6.13}$$

It is convenient to use what is called the **Heaviside step function** $H(x)$ when jumps occur. This is defined as

$$H(x) \equiv \begin{cases} 0 & \text{if } \; x < 0, \\ \frac{1}{2} & \text{if } \; x = 0, \\ 1 & \text{if } \; 0 < x. \end{cases} \tag{6.14}$$

Note that this has built into its definition the value at a jump that is needed for the inverse Laplace transform.

We obtained (6.13) using the fact that we knew y to start with. In the next example, the inverse is determined using Y. This situation is more typical, and it gives us a chance to practice using $H(x)$.

Example: If $Y = -\frac{4}{s}e^{-2s} + \frac{3}{s}$, find y.

Answer: Using the linearity property,

$$\begin{aligned} \mathcal{L}^{-1}(Y) &= \mathcal{L}^{-1}\Big(-\frac{4}{s}e^{-2s} + \frac{3}{s}\Big) \\ &= -4\mathcal{L}^{-1}\Big(\frac{1}{s}e^{-2s}\Big) + 3\mathcal{L}^{-1}\Big(\frac{1}{s}\Big). \end{aligned}$$

From Property 14 in the table, with $a = 2$,

$$\mathcal{L}^{-1}\Big(\frac{1}{s}e^{-2s}\Big) = H(t-2).$$

Similarly, from Property 7 in the table, with $n = 1$ and $a = 0$,

$$\mathcal{L}^{-1}\Big(\frac{1}{s}\Big) = 1.$$

Therefore,

$$y(t) = -4H(t-2) + 3. \tag{6.15}$$

To check that this agrees with (6.13), if $0 \le t < 2$, then $H(t-2) = 0$, and this means that $y = 3$. When $t = 2$, $H(t-2) = 1/2$, and in this case $y = 1$. Finally, when $t > 2$, $H(t-2) = 1$, and this means that $y = -1$. So, it is indeed true that the compact form for y given above agrees with the more expansive version given in (6.13). ■

For those who are picky about doing things correctly, there is a mild case of notation abuse in the last example that should be mentioned. Because the function $y(t)$ has a jump discontinuity, and we only know its Laplace transform, it is not possible to determine the value of $y(t)$ at the jump. The function given in (6.15) is the answer that is consistent with the formula determined using the inverse Laplace transform.

The next example contains the derivation of Property 5 in the table.

Example: Derive Property 5 in Table 6.1.

Answer: This will be done by showing that the Laplace transform of the entry in the third column gives the result in the second column. So,

$$\begin{aligned}\mathcal{L}(y(t-a)H(t-a)) &= \int_0^\infty y(t-a)H(t-a)e^{-st}dt \\ &= \int_a^\infty y(t-a)e^{-st}dt.\end{aligned}$$

Making the change of variables $r = t - a$,

$$\begin{aligned}\mathcal{L}(y(t-a)H(t-a)) &= \int_0^\infty y(r)e^{-s(r+a)}dr \\ &= e^{-sa}\int_0^\infty y(r)e^{-sr}dr \\ &= e^{-sa}Y(s). \qquad (6.16)\end{aligned}$$

Now, the table states that $\mathcal{L}^{-1}(e^{-sa}Y(s)) = y(t-a)H(t-a)$. Our result (6.16) verifies this at points where $y(t-a)H(t-a)$ is continuous. Since $y(t)$ is assumed to be continuous, and $H(x)$ is continuous except when $x = 0$, then we have verified the property everywhere except when $t = a$. At $t = a$, the inverse transform equals the average in the jump of the function at $t = a$. This holds, because, by definition, $H(t-a)$ provides the needed factor of 1/2 at $t = a$. ■

Exercises

1. Sketch the following functions and then find its Laplace transform.

a) $y = H(t-6)$
b) $y = tH(t-1)$
c) $y = H(t-1) + 2H(t-7)$
d) $y = H(t-1)H(t-5)$
e) $y = [H(t-2)]^2$
f) $y = [H(t-1)]^3 + H(t-7)$
g) $y = H(t) - H(t-1) + H(2-t) - H(t-3)$
h) $y = H(t) + H(t-2) - H(4-t) - H(t-8)$

2. Find the inverse Laplace transform of the following functions.

a) $Y = \frac{2}{s^2+9}$
b) $Y = \frac{3}{(s+4)^2}$
c) $Y = \frac{1}{s^2+3s-4}$
d) $Y = \frac{s+1}{s^2+2s+5}$

e) $Y = \frac{2s-3}{s^2-4}$

f) $Y = \frac{2s-3}{s^2+2s+10}$

g) $Y = \frac{s+1}{s(s^2+5s+10)}$

h) $Y = \frac{4s-1}{(s^2+1)(s^2-3s+2)}$

i) $Y = \frac{2}{s^2+4} - \frac{3}{s^2+9}$

j) $Y = \frac{5}{s^2} - \frac{3}{s-2}$

k) $Y = \frac{s+1}{(s+1)^2+9}e^{-3s}$

l) $Y = \left(\frac{1}{s^2} - \frac{1}{s^3}\right)e^{-2s}$

m) $Y = \frac{1}{s}\left(e^{-s} - e^{-2s} + e^{-3s}\right)$

n) $Y = \frac{2}{s} - \frac{1}{s^2} + \frac{4}{s^3} - \frac{7}{s^4}$

o) $Y = \frac{5s+1}{s^2}e^{-5s}$

p) $Y = \frac{5s+1}{s^2+1}e^{-6s}$

3. Suppose that $y(t)$ is periodic with period $T > 0$. Consequently, $y(t + T) = y(t)$ for all $t \geq 0$. It is also assumed that $y(t)$ is continuous, except perhaps for a finite number of jump discontinuities, for $0 \leq t \leq T$. Show that
$$\mathcal{L}(y) = \frac{1}{1 - e^{-sT}} \int_0^T y(t)e^{-st}dt.$$

4. The following functions are periodic with period T. Sketch the function for $0 \leq t \leq 3T$, and then use the result of the pervious exercise to find the Laplace transform. Also, provide an explanation for where the name of the wave comes from.

   a) Square wave: $T = 2$, and $y(t) = H(t) - H(t-1)$, for $0 \leq t < 2$.

   b) Sawtooth wave: $T = 1$, and $y(t) = t$, for $0 \leq t < 1$.

   c) Triangle wave: $T = 2$, and $y(t) = tH(t) - 2(t-1)H(t-1)$, for $0 \leq t < 2$.

   d) Bang-bang wave: $T = 2$, and $y(t) = H(t) - 2H(t-1)$, for $0 \leq t < 2$.

6.3 ▪ Properties of the Laplace Transform

What follows is the derivation of Properties 3 and 4 in Table 6.1. They are important as they will be needed when solving differential equations.

6.3.1 ▪ Transformation of Derivatives

One of the hallmarks of the Laplace transform, as with most integral transforms, is that it converts differentiation into multiplication. To explain what this means, we use integration by parts to obtain the following

$$\begin{aligned}\mathcal{L}(y'(t)) &= \int_0^\infty y'(t)e^{-st}dt \\ &= ye^{-st}\Big|_{t=0}^{\infty} + s\int_0^\infty ye^{-st}dt \\ &= -y(0) + s\mathcal{L}(y). \qquad (6.17)\end{aligned}$$

This formula can be used to find the transform of higher derivatives, and as an example

$$\begin{aligned}\mathcal{L}(y'') &= -y'(0) + s\mathcal{L}(y')\\ &= -y'(0) + s\big(-y(0) + s\mathcal{L}(y)\big)\\ &= s^2\mathcal{L}(y) - y'(0) - sy(0).\end{aligned} \tag{6.18}$$

Generalizing this to higher derivatives

$$\mathcal{L}(y^{(n)}) = s^n\mathcal{L}(y) - y^{(n-1)}(0) - sy^{(n-2)}(0) - \cdots - s^{n-1}y(0). \tag{6.19}$$

6.3.2 ▪ Convolution Theorem

A common integral that arises when solving differential equations is a convolution integral of the form

$$y(t) = \int_0^t g(t-\tau)v(\tau)d\tau. \tag{6.20}$$

Taking the Laplace transform of this equation we obtain

$$\begin{aligned}\mathcal{L}(y) &= \int_0^\infty \int_0^t g(t-\tau)v(\tau)e^{-st}d\tau dt\\ &= \int_0^\infty \int_\tau^\infty g(t-\tau)v(\tau)e^{-st}dtd\tau\\ &= \int_0^\infty \int_0^\infty g(r)v(\tau)e^{-s(r+t)}drd\tau\\ &= \int_0^\infty \left(v(\tau)e^{-s\tau}\int_0^\infty g(r)e^{-sr}dr\right)d\tau = V(s)G(s).\end{aligned}$$

Using the inverse transform this can be written as

$$\mathcal{L}^{-1}(V(s)G(s)) = \int_0^t g(t-\tau)v(\tau)d\tau. \tag{6.21}$$

This is Property 2, in Table 6.1, and it is known as the convolution theorem.

6.4 ▪ Solving Differential Equations

The examples to follow illustrate how to use the Laplace transform to solve a linear initial value problem. In each case, however, the methods covered in Chapters 2 and 3 are easier to use. In the subsequent sections, examples are considered where the Laplace transform is a recommenced way to find the solution. It can also be effective for solving certain partial differential equations and delay equations.

Example 1: Solve $y' + 3y = e^{2t}$, where $y(0) = 2$.

Answer: The first step is to take the Laplace transform of the differential equation, which gives

$$\mathcal{L}(y' + 3y) = \mathcal{L}(e^{2t}).$$

Using the linearity of the transform, and the derivative formula (6.17),

$$\begin{aligned}\mathcal{L}(y' + 3y) &= \mathcal{L}(y') + \mathcal{L}(3y)\\ &= -y(0) + s\mathcal{L}(y) + 3\mathcal{L}(y)\\ &= -2 + (s+3)Y(s).\end{aligned}$$

Also,

$$\begin{aligned}\mathcal{L}(e^{2t}) &= \int_0^\infty e^{2t}e^{-st}dt = \int_0^\infty e^{(2-s)t}dt\\ &= -\frac{1}{2-s}\,.\end{aligned}$$

The transformed problem is therefore $-2 + (s+3)Y = -1/(2-s)$, and from this we get that

$$Y = \frac{1}{s+3}\Big(2 - \frac{1}{2-s}\Big).$$

Consequently, using Table 6.1 (Properties 7 and 9),

$$\begin{aligned}y &= \mathcal{L}^{-1}(Y)\\ &= \mathcal{L}^{-1}\Big(\frac{2}{s+3}\Big) + \mathcal{L}^{-1}\Big(\frac{1}{(s+3)(s-2)}\Big)\\ &= 2e^{-3t} - \frac{1}{5}\big(-e^{2t} + e^{-3t}\big)\\ &= \frac{9}{5}e^{-3t} + \frac{1}{5}e^{2t}. \quad \blacksquare\end{aligned}$$

Example 2: Solve $y'' + y = \sin 2t$, where $y(0) = 2$ and $y'(0) = 1$.

Answer: In Example 1 we used the Laplace transform to the solve the IVP directly. It is usually a bit easier to use the Laplace transform to find a particular solution, and then solve the IVP after that. This is the approach taken in this example. So, when deriving the particular solution it is assumed that $y(0) = y'(0) = 0$ (see Section 3.6.1). Taking the Laplace transform of the differential equation gives $\mathcal{L}(y'' + y) = \mathcal{L}(\sin 2t)$. Using the linearity of the transform, and the derivative formula (6.18), we get that

$$s^2Y - sy(0) - y'(0) + Y = \frac{2}{s^2+4}\,.$$

From the assumed initial conditions, and solving for Y,

$$Y = \frac{2}{(s^2+1)(s^2+4)}\,.$$

The inverse transform is given in (6.10), and so

$$y_p = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t.$$

Now, the general solution of the associated homogeneous equation is $y_h = c_1 \sin t + c_2 \cos t$. With this, the general solution of the differential equation is $y = y_h + y_p$. From the initial conditions, and since $y_p(0) = y_p'(0) = 0$, we require that $y_h(0) = 2$ and $y_h'(0) = 1$. From this one finds that $c_1 = 1$ and $c_2 = 2$. Therefore, the solution of the initial value problem is

$$y = 2\cos t + \frac{5}{3}\sin t - \frac{1}{3}\sin 2t. \quad \blacksquare$$

Example 3: Solve $y' + 3y = g(t)$, where $y(0) = 1$ and $g(t)$ is continuous.

Answer: Taking the Laplace transform of the differential equation gives $\mathcal{L}(y' + 3y) = \mathcal{L}(g)$. Using the derivative formula (6.17), and writing $\mathcal{L}(g) = G(s)$, we get that

$$sY - y(0) + 3Y = G\,.$$

From the assumed initial condition, and solving for Y,

$$Y = \frac{1}{s+3}\big(1 + G\big)\,.$$

Now, using Property 7, $\mathcal{L}^{-1}(1/(s+3)) = e^{-3t}$. Also, using the convolution formula (6.21) with $V = 1/(s+3)$, we have that

$$\mathcal{L}^{-1}\Big(\frac{1}{s+3}G\Big) = \int_0^t g(t-\tau)e^{-3\tau}d\tau.$$

Therefore, the solution is

$$y = e^{-3t} + \int_0^t g(t-\tau)e^{-3\tau}d\tau. \tag{6.22}$$

It is not hard to show that this is the same solution we obtained using an integrating factor, as given in (2.21). $\blacksquare$

6.4.1 ▪ Comments and Limitations on Using the Laplace Transform

In using the Laplace transform to solve a differential equation we paid little attention to the restrictions given in Section 6.1.1. The prominent one in this case is the requirement that y have exponential order. When solving a differential equation the solution is unknown, so you usually have no idea if the requirement is satisfied. This did not stop us from using the transform. This often happens in applied mathematics, and if there is some concern about using the transform you can just check that your solution satisfies the differential equation. As an example, it is very possible that you can find a function $g(t)$ appearing in (6.22) so that y does not have exponential order. Nevertheless, (6.22) does satisfy the differential equation and initial condition. It is, therefore, the solution irrespective of whether y has exponential order or not. In other words, from a mathematical point of view, the end justifies the means.

It is useful to know some of the limitations on using the Laplace transform to solve a differential equation. To explain what they are, the above examples have two things in common. First, they are linear differential equations. To be able to use the Laplace transform to solve a differential equation it must be linear. As the examples illustrate, the Laplace transform can be used irrespective of the order of the equation. It can also be used to solve partial differential equations, delay equations, and integral equations. However, in all cases, the equations are linear.

The second thing common to the above examples is that they have constant coefficients. Examples with non-constant coefficients are $y' + e^t y = 0$ and $y'' + (1+t)^2 y' + e^t y = 0$. Except in special cases, the Laplace transform method will not work on equations with non-constant coefficients. Just in case you are interested, an example where it does work is Airy's equation $y'' + ty = 0$. You might try finding the Laplace transform of this equation to see why the coefficients are "just right" so that the method works.

Exercises

1. Use the Laplace transform to find the solution of the IVP.

   a) $2y' + y = 1, \quad y(0) = 2$
   b) $3y' = -y + e^{-t}, \quad y(0) = 0$
   c) $y'' + y' - 2y = 0, \quad y(0) = 0,\ y'(0) = -1$
   d) $2y'' + 3y' - 2y = 0, \quad y(0) = -1,\ y'(0) = 0$
   e) $5y'' - y' = 0, \quad y(0) = -1,\ y'(0) = -1$
   f) $4y'' + y = 0, \quad y(0) = -1,\ y'(0) = -1$
   g) $y'' - 2y' + 2y = 0, \quad y(0) = -1,\ y'(0) = -1$

2. Use the Laplace transform to find a particular solution of the differential equation (as explained in Example 2). After that, find the solution of the IVP.

   a) $y'' + y' - 2y = 3t$, $y(0) = 0$, $y'(0) = 1$
   b) $y'' + 4y = t^2$, $y(0) = 1$, $y'(0) = 0$
   c) $y'' - y' = \sin t$, $y(0) = 1$, $y'(0) = -1$
   d) $y'' + 3y' = 3t - 1$, $y(0) = 1$, $y'(0) = 0$
   e) $4y'' - y = e^{-t}$, $y(0) = 0$, $y'(0) = -1$
   f) $y'' - y' + y = t + 1$, $y(0) = 0$, $y'(0) = 1$

6.5 ▪ Solving Differential Equations with Non-Smooth Forcing

As stated earlier, the last three examples should not be solved using the Laplace transform because there are easier methods that can be used on those problems. The purpose was to illustrate how the Laplace transform can be used to solve a differential equation. The next examples are different in that the Laplace transform is the recommended method for finding the solution.

The next example considers how to solve a differential equation with a discontinuous forcing function. This is a situation that is not uncommon in applications.

Example: Solve $y'' + 3y' + 2y = f(t)$, where $y(0) = 1$, $y'(0) = -1$, and

$$f(t) = \begin{cases} 2 & \text{if} \quad 0 \le t \le 3, \\ 0 & \text{if} \quad 3 < t. \end{cases}$$

Answer: The Laplace transform is going to be used to find a particular solution, after which we will use it to solve the initial value problem. So, in deriving the particular solution it is assumed that $y(0) = y'(0) = 0$ (see Section 3.6.1). Now,

$$\begin{aligned} \mathcal{L}(f) &= \int_0^3 2e^{-st}dt \\ &= \frac{2}{s}\Big(1 - e^{-3s}\Big). \end{aligned}$$

Taking the Laplace transform of the differential equation gives us

$$(s^2 + 3s + 2)Y = \frac{2}{s}\Big(1 - e^{-3s}\Big),$$

which means that

$$Y = \frac{2}{s(s+2)(s+1)}\Big(1 - e^{-3s}\Big).$$

To determine the inverse transform, using Property 10 from Table 6.1,

$$\mathcal{L}^{-1}\Big(\frac{2}{s(s+2)(s+1)}\Big) = 1 + e^{-2t} - 2e^{-t},$$

and from Properties 5 and 10,

$$\mathcal{L}^{-1}\Big(\frac{2}{s(s+2)(s+1)}e^{-3s}\Big) = \Big(1 + e^{-2(t-3)} - 2e^{-(t-3)}\Big)H(t-3).$$

The resulting particular solution is

$$y_p = 1 + e^{-2t} - 2e^{-t} - \Big(1 + e^{-2(t-3)} - 2e^{-(t-3)}\Big)H(t-3).$$

Now, the general solution of the associated homogeneous equation is

$$y_h = c_1 e^{-2t} + c_2 e^{-t}.$$

The general solution of the differential equation is $y = y_h + y_p$. From the initial conditions, and since $y_p(0) = y_p'(0) = 0$, we require that $y_h(0) = 1$ and $y_h'(0) = -1$. From this one finds that $c_1 = 0$ and $c_2 = 1$. Therefore, the solution of the initial value problem is

$$y = 1 + e^{-2t} - e^{-t} - \Big(1 + e^{-2(t-3)} - 2e^{-(t-3)}\Big)H(t-3). \quad \blacksquare$$

A comment needs to be made about the mathematical correctness of the solution we just derived. Namely, $y(t)$ and $y'(t)$ are defined and continuous for $0 \le t < \infty$, but $y''(t)$ is not defined at $t = 3$ (it is, however, defined and continuous everywhere else). This throws into question whether the differential equation $y'' + 3y' + 2y = f(t)$ is defined at $t = 3$. The way this needs to be interpreted is that the differential equation holds for $0 < t < 3$, and then again for $3 < t < \infty$. The discontinuity in the forcing function effectively resets the problem at $t = 3$. One approach to dealing with this is to break the problem into two IVPs, one for $0 < t < 3$, and another for $3 < t < \infty$. By using the Laplace transform we have been able to avoid having to do this.

6.5.1 ▪ Impulse Forcing

It often happens that the impulse forcingforcing is fairly intense but it occurs over a short time interval. Writing the interval as $t_0 - \varepsilon < t < t_0 + \varepsilon$, we are considering the situation of when the forcing has the form

$$f(t) = \begin{cases} 0 & \text{if } \; 0 \le t \le t_0 - \varepsilon, \\ d(t) & \text{if } \; t_0 - \varepsilon < t < t_0 + \varepsilon, \\ 0 & \text{if } \; t_0 + \varepsilon \le t. \end{cases} \tag{6.23}$$

With this, the solution of $y' = f$, where $y(0) = 0$, is

$$y = \begin{cases} 0 & \text{if} \quad 0 \le t \le t_0 - \varepsilon, \\ \int_{t_0-\varepsilon}^{t} d(r)dr & \text{if} \quad t_0 - \varepsilon < t < t_0 + \varepsilon, \\ D & \text{if} \quad t_0 + \varepsilon \le t, \end{cases} \tag{6.24}$$

where

$$D = \int_{t_0-\varepsilon}^{t_0+\varepsilon} d(r)dr.$$

We are assuming that the forcing interval is very short, but D is large enough to be physically meaningful.

There is a mathematical idealization for a concentrated force that makes solving the problem easier than trying to use a formulation as in (6.23). This is done by introducing what is known as the delta function $\delta(t)$.

Delta Function. *The delta function $\delta(t)$ is defined to have the following properties:*

1. *Given any t_0,*
$$\delta(t - t_0) = 0, \quad \text{if } t \neq t_0, \tag{6.25}$$

2. *Given any continuous function $g(t)$, and assuming $a < t_0 < b$:*
$$\int_a^b \delta(t - t_0)g(t)dt = g(t_0), \tag{6.26}$$
and
$$\int_a^{t_0} \delta(t - t_0)g(t)dt = \int_{t_0}^b \delta(t - t_0)g(t)dt = \frac{1}{2}g(t_0)\,. \tag{6.27}$$

As an example of how the delta function is used, instead of using (6.23), the forcing is assumed to have the form $f(t) = D\delta(t - t_0)$. This means we are assuming that there is a delta forcing at t_0 with strength D. With this, the differential equation becomes

$$y' = D\delta(t - t_0).$$

where $y(0) = 0$. The solution of this IVP is

$$y = \int_0^t D\delta(r - t_0)dr.$$

To evaluate this, first note that if $0 \le t < t_0$, then from (6.25), $y(t) = 0$. If $t = t_0$, then from (6.27), $y(t_0) = D/2$. Lastly, when $t_0 < t$, then from (6.26), $y(t) = D$. Consequently, the solution is

$$y = \begin{cases} 0 & \text{if} \quad 0 \le t < t_0, \\ \frac{1}{2}D & \text{if} \quad t = t_0, \\ D & \text{if} \quad t_0 < t. \end{cases} \tag{6.28}$$

Except for the very small time interval $t_0 - \varepsilon < t < t_0 + \varepsilon$, this solution is the same as the one in (6.24).

Example 1: If $f(t) = -2\delta(t-7)$, find $\mathcal{L}(f)$.

Answer: Using (6.1), and (6.26),

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty -2\delta(t-7)e^{-st}dt \\ &= -2\int_0^\infty \delta(t-7)e^{-st}dt \\ &= -2e^{-7s}. \quad \blacksquare \end{aligned}$$

Example 2: If $f(t) = 3\delta(t-1) - 2\delta(t-2) + \delta(t-3)$, find $\mathcal{L}(f)$.

Answer: Using (6.1), and (6.26),

$$\begin{aligned} \mathcal{L}(f) &= \int_0^\infty \big[3\delta(t-1) - 2\delta(t-2) + \delta(t-3)\big]e^{-st}dt \\ &= 3\int_0^\infty \delta(t-1)e^{-st}dt - 2\int_0^\infty \delta(t-2)e^{-st}dt \\ &\quad + \int_0^\infty \delta(t-3)e^{-st}dt \\ &= 3e^{-s} - 2e^{-2s} + e^{-3s}. \quad \blacksquare \end{aligned} \tag{6.29}$$

Example 3: Solve $y'' + y = 2\delta(t-15)$, where $y(0) = 0$ and $y'(0) = 0$.

Answer: Taking the Laplace transform of the differential equation gives $\mathcal{L}(y''+y) = \mathcal{L}(2\delta(t-15))$. Using the linearity of the transform, and the derivative formula (6.18), we get that

$$s^2Y - sy(0) - y'(0) + Y = 2e^{-15s}\,.$$

From the given initial conditions, and solving for Y,

$$Y = \frac{2}{s^2+1}e^{-15s}\,.$$

Since

$$\mathcal{L}^{-1}\Big(\frac{2}{s^2+1}\Big) = 2\sin t,$$

then using Property 5, from Table 6.1,

$$y = 2\sin(t-15)H(t-15). \quad \blacksquare$$

Mathematical Tidbits

As you likely noticed, $\delta(t)$ is not actually a function. The more accurate statement is that it is a *distribution*, or a *generalized function*. There are various ways to obtain a mathematically rigorous definition of $\delta(t)$, using limits or test functions. This is useful as it provides a way to derive the properties of $\delta(t)$, such as to how to find its derivative or to find what are called eigenfunction expansions for it. Most of this is not needed in this text, but the result in the next example is worth knowing about.

Example 4: What function is $p(t) = \int_{-\infty}^{t} \delta(r)dr$?

Answer: This can be answered by considering three cases. First, if $t < 0$, then $\delta(r) = 0$ for $-\infty < r \le t$, and we conclude that $p(t) = 0$. If $t > 0$, then from (6.26) with $g(t) = 1$, we get that $p(t) = 1$. Finally, when $t = 0$, from (6.27) we get that $p(0) = 1/2$. What this means is that $p(t)$ is the Heaviside step function $H(t)$ defined in (6.14). Therefore, we have shown that

$$H(t) = \int_{-\infty}^{t} \delta(r)dr.$$

In this sense we can write that

$$H'(t) = \delta(t). \quad \blacksquare$$

To summarize, we have taken something that is not a function, even though we refer to it as such, and used it to compute the derivative of a function that is not differentiable. Moreover, we have done this with a “function” that is zero everywhere except at one point, and to make matters worse, the area under the curve for this “function” is not necessarily zero. In other words, we have violated almost every rule you learned in calculus. What you are seeing here is how it is possible to loosen some, but not all, of the rules in calculus. This sort of generalization is often done in higher level mathematics courses, and it can be the source of some interesting, and occasionally profound, insights into a problem.

A reasonable question at this point is, what exactly is permitted when using the delta function? As demonstrated in Example 2, linear combinations of delta functions are allowed. It is also possible to both differentiate and integrate a delta function. What is not allowed, generally, involves nonlinear operations. So, expressions such as $\delta(t-1)\delta(t-2)$, $\delta(t-1)/\delta(t-2)$, and $\sin(\delta(t-1))$ are not allowed.

As you might expect, given how it seems to violate numerous rules from calculus, the introduction of the delta function generated considerable discussion in the mathematics and physics communities. An outcome of this was a *six* volume series of books, totaling some 2,165 pages, entitled "Generalized Functions," by Gel'fand, Shilov, Vilenkin, and Pyatetskii-Shapiro. For most, this is overkill, and Wikipedia is often sufficient for answering the usual questions about the delta function.

Exercises

1. Use the Laplace transform to find the solution of the IVP.

   a) $y' + 4y = 3H(t-1), \quad y(0) = -1$
   b) $3y' - y = 1 - H(t-4), \quad y(0) = -1$
   c) $y' + y = 2\delta(t-3), \quad y(0) = -1$
   d) $y' - 4y = 2H(t-2) - \delta(t-1), \quad y(0) = 0$
   e) $y'' - y' - 6y = 3H(t-5), \quad y(0) = 0,\ y'(0) = 0$
   f) $y'' + 4y = 3H(t-4) - 3H(t-2), \quad y(0) = 0,\ y'(0) = 0$
   g) $y'' - 4y' = 3\delta(t-1), \quad y(0) = 0,\ y'(0) = 0$
   h) $y'' + y = \delta(t-3) - 2\delta(t-2), \quad y(0) = 0,\ y'(0) = 0$

2. Show that the following identities hold for the delta function. Do this by showing that when the left and right sides of the equation are inserted into (6.25)-(6.27), that they produce the same result.

   a) $\delta(a(t-t_0)) = \frac{1}{a}\delta(t-t_0), \quad \text{for} \quad a > 0,$
   b) $\delta(-(t-t_0)) = \delta(t-t_0)$,
   c) $(t-t_0)\delta(t-t_0) = 0$,
   d) If g(t) is continuous, then $g(t)\delta(t-t_0) = g(t_0)\delta(t-t_0)$.

3. In quantum physics there are occasions when the coefficients of the differential equation contain delta functions. The point of this exercise is to demonstrate that care is needed in such situations.

   a) Consider the problem of solving

   $$y'(t) = \delta(t-t_0)y(t), \text{ for } t > 0,$$

   where $t_0 > 0$ and $y(0) = 1$. Using separation of variables, and Example 4, find the solution. Make sure to determine its value for $0 \le t < t_0$, for $t = t_0$, and for $t_0 < t$. For the record, this is the correct solution of this problem.

b) By simply integrating the differential equation in part (a), and then using the initial condition, one gets that

$$y(t) = 1 + \int_0^t y(r)\delta(r - t_0)dr.$$

Not thinking too hard about the situation, and using (6.25)-(6.27), explain how you might conclude that

$$y = \begin{cases} 1 & \text{if} \quad 0 \leq t < t_0, \\ 2 & \text{if} \quad t = t_0, \\ 3 & \text{if} \quad t_0 < t. \end{cases}$$

This differs from the solution for part (a). Where is the error made in the derivation of the above solution?

6.6 ▪ Solving Linear Systems

The Laplace transform can be used to solve a linear system of differential equations, and this is often the approach taken for what are known as state space models in engineering. It is relatively easy to do this, and to explain why, suppose we want to solve

$$x' = ax + by + f(t), \tag{6.30}$$
$$y' = cx + dy + g(t), \tag{6.31}$$

where $x(0) = x_0$ and $y(0) = y_0$. Taking the Laplace transform of each equation, and using (6.17), we get

$$sX - x_0 = aX + bY + F, \tag{6.32}$$
$$sY - y_0 = cX + dY + G, \tag{6.33}$$

where X, Y, F, and G are the Laplace transforms of x, y, f, and g, respectively. The next step is to solve for X and Y, and then attempt to find the inverse transforms. How hard it is to find the inverse transforms depends on f and g.

Example 1: Using the Laplace transform, solve

$$x' = x - y,$$
$$y' = 4x - 2y,$$

where $x(0) = 1$ and $y(0) = -1$.

Answer: From (6.32) and (6.33) the transformed equations are

$$sX - 1 = X - Y,$$

$$sY + 1 = 4X - 2Y,$$

From the first equation, $Y = 1 + (1 - s)X$, and after substituting this into the second equation, and simplifying, one finds that

$$X = \frac{s+3}{s^2+s+2}\,.$$

Since $s^2 + s + 2 = (s + 1/2)^2 + 7/4$, then from Property 8 of Table 6.1,

$$x = e^{-t/2}\left(\cos(\omega t) + \frac{5}{2\omega}\sin(\omega t)\right),$$

where $\omega = \sqrt{7}/2$. To find y we can either find the inverse transform for Y, or we can use the first differential equation. The latter option is easiest, and so

$$\begin{aligned} y &= x - x' \\ &= e^{-t/2}\left(-\cos t + \frac{11}{2\omega}\sin t\right). \quad \blacksquare \end{aligned}$$

Problems like the last example were solved in Chapter 4 by finding the eigenvalues and eigenvectors of the coefficient matrix. The Laplace transform approach, in contrast, avoids this and solves the problem directly. What it does require is the inverse transform, but for the above example that was easy to find.

Example 2: Using the Laplace transform, solve

$$\begin{aligned} x' &= 3x - 6y + f(t), \\ y' &= x - 4y + g(t), \end{aligned}$$

where $x(0) = 0$ and $y(0) = 0$. Also, $f(t)$ and $g(t)$ are continuous functions.

Answer: From (6.32) and (6.33) the transformed equations are

$$\begin{aligned} sX &= 3X - 6Y + F, \\ sY &= X - 4Y + G, \end{aligned}$$

From the second equation, $X = (s+4)Y - G$, and after substituting this into the first equation, and simplifying, one finds that

$$Y = \frac{s-3}{s^2+s-6}G(s) + \frac{1}{s^2+s-6}F(s)\,.$$

The convolution theorem is going to be used in finding the inverse transform, and in preparation for this note that, using Property 8 of Table 6.1,

$$\begin{aligned}\mathcal{L}^{-1}\Big(\frac{1}{s^2+s-6}\Big) &= \mathcal{L}^{-1}\Big(\frac{1}{(s-2)(s+3)}\Big)\\ &= \frac{1}{5}\Big(e^{2t}-e^{-3t}\Big),\end{aligned}$$

and

$$\mathcal{L}^{-1}\Big(\frac{s-3}{s^2+s-6}\Big) = \frac{1}{5}\Big(6e^{-3t}-e^{2t}\Big).$$

So, using the convolution theorem, which is Property 2 of Table 6.1,

$$\mathcal{L}^{-1}\Big(\frac{1}{s^2+s-6}F(s)\Big) = \int_0^t \frac{1}{5}\Big(e^{2(t-r)}-e^{-3(t-r)}\Big)f(r)dr,$$

and

$$\mathcal{L}^{-1}\Big(\frac{s-3}{s^2+s-6}G(s)\Big) = \int_0^t \frac{1}{5}\Big(6e^{-3(t-r)}-e^{2(t-r)}\Big)g(r)dr.$$

Therefore, the solution is

$$\begin{aligned}y(t) = \frac{1}{5}&\int_0^t \big(e^{2(t-r)}-e^{-3(t-r)}\big)f(r)dr\\ &+\frac{1}{5}\int_0^t \big(6e^{-3(t-r)}-e^{2(t-r)}\big)g(r)dr.\end{aligned}$$

To find x you can either find the inverse transform for X, or you can use the second differential equation (similar to what was done in the previous example). ■

6.6.1 ▪ General Formulation

It is worth considering what happens in the general case. Assuming there are n unknowns, then the problem can be written as

$$\frac{d}{dt}\mathbf{x} = \mathbf{A}\mathbf{x}+\mathbf{f}, \tag{6.34}$$

where $\mathbf{A}$ is an $n\times n$ constant matrix. Also, assume that $\mathbf{x}(0)=\mathbf{x}_0$.

In preparation of taking the Laplace transform of this equation, note that if

$$\mathbf{x} = \begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_n\end{pmatrix},$$

then

$$\mathbf{X} = \mathcal{L}(\mathbf{x}) = \begin{pmatrix} \mathcal{L}(x_1) \\ \mathcal{L}(x_2) \\ \vdots \\ \mathcal{L}(x_n) \end{pmatrix} = \begin{pmatrix} X_1 \\ X_2 \\ \vdots \\ X_n \end{pmatrix}.$$

Also, $\mathcal{L}(\mathbf{Ax}) = \mathbf{A}\mathcal{L}(\mathbf{x}) = \mathbf{AX}$. Consequently, after taking the Laplace transform of (6.34), we get that

$$s\mathbf{X} - \mathbf{x}_0 = \mathbf{AX} + \mathbf{F}. \tag{6.35}$$

This can be written as

$$(\mathbf{A} - s\mathbf{I})\mathbf{X} = -\mathbf{F} - \mathbf{x}_0,$$

where $\mathbf{I}$ is the $n \times n$ identity matrix. Consequently,

$$\mathbf{X} = -\mathbf{R}(\mathbf{F} + \mathbf{x}_0). \tag{6.36}$$

where $\mathbf{R} = (\mathbf{A} - s\mathbf{I})^{-1}$ is referred to as the resolvent of $\mathbf{A}$. In control systems, $-\mathbf{R}$ is called the transfer matrix. Given that the formula for $\mathbf{X}$ involves the inverse of an $n \times n$ matrix, finding the inverse transform can be a formidable task. One possibility is to introduce a series expansion of $\mathbf{R}$, and then use a matrix factorization to reduce the problem. This requires certain assumptions to hold, and pursuing this is beyond the scope of this textbook.

Exercises

1. Use the Laplace transform to find the solution of the following IVPs.

a) $\mathbf{x}' = \begin{pmatrix} -1 & 6 \\ 1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

b) $\mathbf{x}' = \begin{pmatrix} 0 & \frac{1}{4} \\ 1 & 0 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

c) $\mathbf{x}' = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

d) $\mathbf{x}' = \begin{pmatrix} 2 & 1 \\ 6 & 3 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

e) $\mathbf{x}' = \begin{pmatrix} 2 & 0 \\ -1 & 2 \end{pmatrix} \mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$

f) $\mathbf{x}' = \begin{pmatrix} 1 & \frac{1}{4} \\ -5 & 0 \end{pmatrix}\mathbf{x}, \quad \mathbf{x}(0) = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

2. This exercise uses the Laplace transform to solve

$$\begin{aligned} x' &= ax + by, \\ y' &= cx + dy, \end{aligned}$$

where $x(0) = x_0$ and $y(0) = y_0$.

a) Taking the Laplace of the above equations, and then solving for X and Y, show that

$$\begin{aligned} X &= \frac{1}{(s-r_1)(s-r_2)}\big((s-d)x_0 + by_0\big), \\ Y &= \frac{1}{(s-r_1)(s-r_2)}\big(cx_0 + (s-a)y_0\big). \end{aligned}$$

b) What values of a, b, c, and d result in r_1 and r_2 being real and $r_1 \neq r_2$? Assuming this is the case, use the inverse Laplace transform to find x and y.

c) What values of a, b, c, and d result in r_1 and r_2 being complex (with a nonzero imaginary part)? Assuming this is the case, use the inverse Laplace transform to find x and y.

d) What values of a, b, c, and d result in $r_1 = r_2$? Assuming this is the case, use the inverse Laplace transform to find x and y.

Chapter 7

Partial Differential Equations

A partial differential equation is simply a differential equation in which the dependent variable depends on more than one independent variable. It is typical that the independent variables are time (t) and space (x). If $u(x,t)$ is the dependent variable, then examples of partial differential equations (PDEs) are

$$\begin{aligned} \text{Advection Equation:} \quad & u_t + a u_x = 0 \\ \text{Diffusion Equation:} \quad & u_t = D u_{xx} \\ \text{Wave Equation:} \quad & u_{tt} = c^2 u_{xx}. \end{aligned}$$

These equations have names as they so often arise in applications. Mathematically, they are linear and homogeneous. Also, the advection equation is first order, while the other two are second order.

Subscripts are used in the above PDEs to indicate partial differentiation. There are two other ways this can be done that are very common. First, there is the form used in calculus, and examples are

$$\frac{\partial u}{\partial t}, \quad \frac{\partial u}{\partial x}, \quad \frac{\partial^2 u}{\partial t^2}, \quad \frac{\partial^2 u}{\partial t \partial x}, \quad \frac{\partial^2 u}{\partial x^2}.$$

A more contemporary notation is to abbreviate the above expressions, and write

$$\partial_t u, \quad \partial_x u, \quad \partial_t^2 u, \quad \partial_t \partial_x u, \quad \partial_x^2 u.$$

In this chapter the method of separation of variables is used to solve PDEs. This is usually limited to problems where the spatial interval is bounded, and for that reason we only consider problems with $0 < x < L$. For situations where $0 < x < \infty$, or $-\infty < x < \infty$, other methods should be used, and one possibility is the Laplace transform.

Introduction to Differential Equations, M. H. Holmes, 2019

7.1 ▪ Balance Laws

The PDEs listed above are the mathematical consequence of a balance law, much like the ODEs obtained for simple harmonic motion and the various modeling examples in Section 2.3. For example, the wave equation describes the vertical displacement $u(x,t)$ of an elastic string. The PDE is a force balance equation coming from Newton's second law $F = ma$. In this case, the acceleration is $a = u_{tt}$, and F is the restoring force in the string due to its being stretched.

The diffusion and advection equations apply to objects moving along the x-axis. The balance law in this case is the requirement that the total number of objects is constant, which means that if one region experiences an increase, then this is balanced by a decrease in other regions. What distinguishes the diffusion and advection equations is the reason for the motion. As an example, the diffusion equation is obtained when the objects move due to Brownian motion. In contrast, the advection equation is obtained when the objects are all moving with velocity a.

Explaining the physical and mathematical assumptions underlying the derivation of PDEs is outside the purview of this textbook. If you are interested in this you should consult Holmes [2009].

7.2 ▪ Separation of Variables

The solution method will be introduced by using it to solve a problem involving the diffusion equation. This requires a correctly formulated problem, and the one considered is to find the function $u(x,t)$ that satisfies

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \quad \text{for} \quad \begin{cases} 0 < x < L, \\ 0 < t. \end{cases} \tag{7.1}$$

In this equation, the positive constant D is the called the diffusion coefficient. To complete the formulation we will prescribe the values of u at the two endpoints, where $x = 0$ and $x = L$, and at the beginning, when $t = 0$. Specifically, for boundary conditions it is assumed that

$$u(0,t) = 0, \tag{7.2}$$

and

$$u(L,t) = 0. \tag{7.3}$$

For the initial condition, it is assumed that

$$u(x,0) = g(x), \quad \text{for} \ \ 0 < x < L, \tag{7.4}$$

where $g(x)$ is a given function.

When using the method of separation of variables, you first find all possible nonzero solutions of the PDE that satisfy the boundary conditions. It is important to note that $u = 0$ is a possible solution of (7.1) that also satisfies (7.2) and (7.3). What we want are the nonzero ones.

A consequence is that for separation of variables to work it is required that $u = 0$ is a solution of the PDE, and it also satisfies the boundary conditions. So, if the left boundary condition is changed to, say, $u(0, t) = 1$, or the PDE is changed to, say, $Du_{xx} = u_t + x$, then separation of variables will not work. What is necessary in these cases is to first transform the problem into one where $u = 0$ is a solution of the PDE and boundary conditions. How this is done is considered in Sections 7.5 and 7.6.

7.2.1 ▪ Separation of Variables Assumption

The assumption is simply that

$$u(x, t) = F(x)G(t). \tag{7.5}$$

Substituting this into the PDE (7.1) gives $DF''(x)G(t) = F(x)G'(t)$. Separating variables yields

$$D\frac{F''(x)}{F(x)} = \frac{G'(t)}{G(t)}\,. \tag{7.6}$$

Now comes the key observation. The only way a function of x can equal a function of t, since x and t are independent, is that the function of x is a constant, the function of t is a constant, and the constants are equal. In other words, there is a constant λ so that

$$D\frac{F''(x)}{F(x)} = \lambda\,,$$

and

$$\frac{G'(t)}{G(t)} = \lambda\,.$$

These can be rewritten as

$$DF''(x) = \lambda F(x)\,. \tag{7.7}$$

and

$$G'(t) = \lambda G(t)\,. \tag{7.8}$$

The λ appearing here is called, not unsurprisingly, the **separation constant**.

In this textbook we are going to make two assumptions to simplify the reduction: (i) the separation constant is real-valued, and (ii) $u(x, t)$ is a bounded function of t. It is certainly possible to find problems where one, or both, assumptions do not hold. However, this just makes the reduction more time consuming, and the majority of useful problems do satisfy these assumptions.

7.2.2 ▪ Finding $G(t)$

The general solution of (7.8) is

$$G(t) = ae^{\lambda t}, \tag{7.9}$$

where a is an arbitrary constant. Because we have assumed a bounded solution in t, and λ is real-valued, it is required that $\lambda \leq 0$. Also, the function $G(t)$ is not required to satisfy the initial condition (7.4). We will satisfy that condition once we have found the general solution of the PDE and boundary conditions.

7.2.3 ▪ Finding $F(x)$ and λ

The separation of variables assumption must be applied to the boundary conditions. So, to have $u(0,t) = 0$, we need $F(0)G(t) = 0$. For this to happen, and u not be identically zero, we require that $F(0) = 0$. Similarly, we need $F(L) = 0$. Consequently, all-together, the function $F(x)$ must satisfy

$$DF''(x) = \lambda F(x)\,, \quad \text{for } \ 0 < x < L, \tag{7.10}$$

where

$$F(0) = 0 \quad \text{and} \quad F(L) = 0, \tag{7.11}$$

Because the solution is required to satisfy boundary conditions, rather than initial conditions, this is an example of what is called a **boundary value problem** (BVP).

The differential equation (7.10) is second-order, linear, and homogeneous. We will solve it the same way we found the general solution of this type of differential equation in Chapter 3. In fact, solving BVPs was considered in Exercises 5 and 6 in Section 3.8.

The solution of the BVP depends on whether λ is zero or negative. So, we have two cases to consider.

$\lambda = 0:$ In this case (7.10) is $F'' = 0$, and so $F(x) = a + bx$. To satisfy $F(0) = 0$ we need $a = 0$, and for $F(L) = 0$ we need $b = 0$. So, we just get the zero solution in this case.

$\lambda < 0:$ Assuming $F(x) = e^{rx}$, and setting $\lambda = -k^2$, where $k > 0$, then (7.10) reduces to $Dr^2 = -k^2$. This means that $r = \pm ik/\sqrt{D}$. The resulting general solution of (7.10) is

$$F(x) = a\cos(kx/\sqrt{D}) + b\sin(kx/\sqrt{D}).$$

To satisfy $F(0) = 0$ we need $a = 0$. To satisfy $F(L) = 0$ we need $b\sin(kL/\sqrt{D}) = 0$. To obtain a *nonzero* solution for $F(x)$ we take k so that $\sin(kL/\sqrt{D}) = 0$. This holds if any one of the following values are used:

$$kL/\sqrt{D} = \pi, 2\pi, 3\pi, \ldots,$$

or equivalently

$$k = \frac{\pi\sqrt{D}}{L}, \frac{3\pi\sqrt{D}}{L}, \frac{3\pi\sqrt{D}}{L}, \ldots. \tag{7.12}$$

The conclusion is that the nonzero solutions of (7.10) and (7.11) are

$$F_n(x) = b_n \sin\left(\frac{n\pi x}{L}\right), \tag{7.13}$$

and

$$\lambda_n = -D\left(\frac{n\pi}{L}\right)^2, \tag{7.14}$$

for $n = 1, 2, 3, \ldots$. Also, b_n is an arbitrary constant.

7.2.4 ▪ The General Solution

We have shown that for any given n, the function $u_n(x,t) = F_n(x)G_n(t)$ is a solution of the PDE that satisfies the boundary conditions. Because the PDE and boundary conditions are homogeneous, and the problem is linear, the principle of superposition can be used (see page 5). Therefore, the resulting general solution, that satisfies the PDE and boundary conditions, is

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t),$$

or equivalently

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin\left(\frac{n\pi x}{L}\right), \tag{7.15}$$

where b_n is an arbitrary constant, and λ_n is given in (7.14). In writing this down, the constant a in (7.9) has been absorbed into the b_n.

7.2.5 ▪ Satisfying the Initial Condition

It remains to satisfy the initial condition, which is $u(x,0) = g(x)$. According to our solution in (7.15), we need

$$\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right) = g(x). \tag{7.16}$$

This is the equation that is used to determine the b_n's. However, the left-hand-side is an example of what is known as a Fourier series. More specifically, it is an example of a Fourier sine series. There are some significant mathematical questions that arise here, one of which is whether the series converges. This, and some related questions, are addressed in the next section. For the moment, we simply state the conclusion. If $g(x)$ is continuous, except perhaps for a few jump discontinuities, then

$$b_n = \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx. \tag{7.17}$$

7.2.6 ▪ Examples

Example 1: Suppose that $D = 1$, $L = 2$, and $g(x) = 3\sin(\pi x)$. In this case, from (7.14), $\lambda_n = -(n\pi/2)^2$, and the resulting general solution (7.15) is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t/4}\sin\Big(\frac{n\pi x}{2}\Big).$$

To satisfy the initial condition, it helps to notice that $g(x)$ is one of the sine functions in the series. To make this more evident, the requirement that $u(x,0) = g(x)$ means that we need

$$\sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{2}\Big) = 3\sin(\pi x).$$

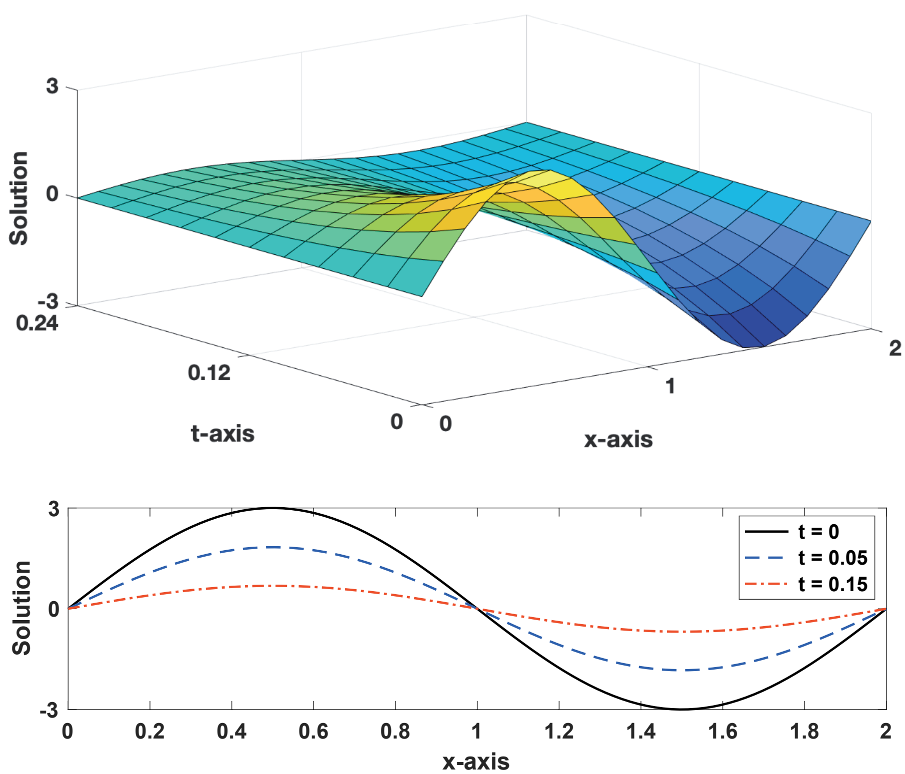

Figure 7.1. *Solution of the diffusion equation in Example 1. Shown is the solution surface as well as the solution profiles at specific time values.*

To satisfy this, take $b_2 = 2$ and set all the other b_n's to zero. Therefore, the solution is

$$u(x,t) = 3e^{-\pi^2 t}\sin(\pi x). \tag{7.18}$$

This solution is shown in Figure 7.1, both as time slices and as the solution surface for $0 \le t \le 0.24$. ■

Example 2: Suppose that in the previous example,

$$g(x) = 3\sin\Big(\frac{\pi x}{2}\Big) - 4\sin\Big(\frac{3\pi x}{2}\Big) + 5\sin(2\pi x).$$

This is an example of when $g(x)$ involves the sum of three of the sine functions in the series. The requirement is that

$$\sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{2}\Big) = 3\sin\Big(\frac{\pi x}{2}\Big) - 4\sin\Big(\frac{3\pi x}{2}\Big) + 5\sin(2\pi x).$$

To satisfy this we take $b_1 = 3$, $b_3 = -4$, $b_4 = 5$, and all the other b_n's are zero. The resulting solution is

$$u(x,t) = 3e^{-\pi^2 t/4}\sin\Big(\frac{\pi x}{2}\Big) - 4e^{-9\pi^2 t/4}\sin\Big(\frac{3\pi x}{2}\Big) + 5e^{-4\pi^2 t}\sin(2\pi x). \quad ■$$

Example 3: Suppose that $D = 1$, $L = 1$, and

$$g(x) = \begin{cases} 1 & \text{if } \frac{1}{3} \le x \le \frac{2}{3}, \\ 0 & \text{otherwise.} \end{cases} \tag{7.19}$$

In this case, it is necessary to use (7.17) to find the b_n's. Carrying out the integration

$$\begin{aligned} b_n &= 2\int_{1/3}^{2/3} \sin(n\pi x)dx \\ &= -\frac{2}{n\pi}\cos(n\pi x)\Big|_{x=1/3}^{2/3} \\ &= \frac{2}{n\pi}\Big(\cos(n\pi/3) - \cos(2n\pi/3)\Big). \end{aligned}$$

As for the solution, since $\lambda_n = -(n\pi)^2$, then

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t}\sin(n\pi x). \tag{7.20}$$

This solution is shown in Figure 7.2 for $0 \le t \le 0.1$. ■

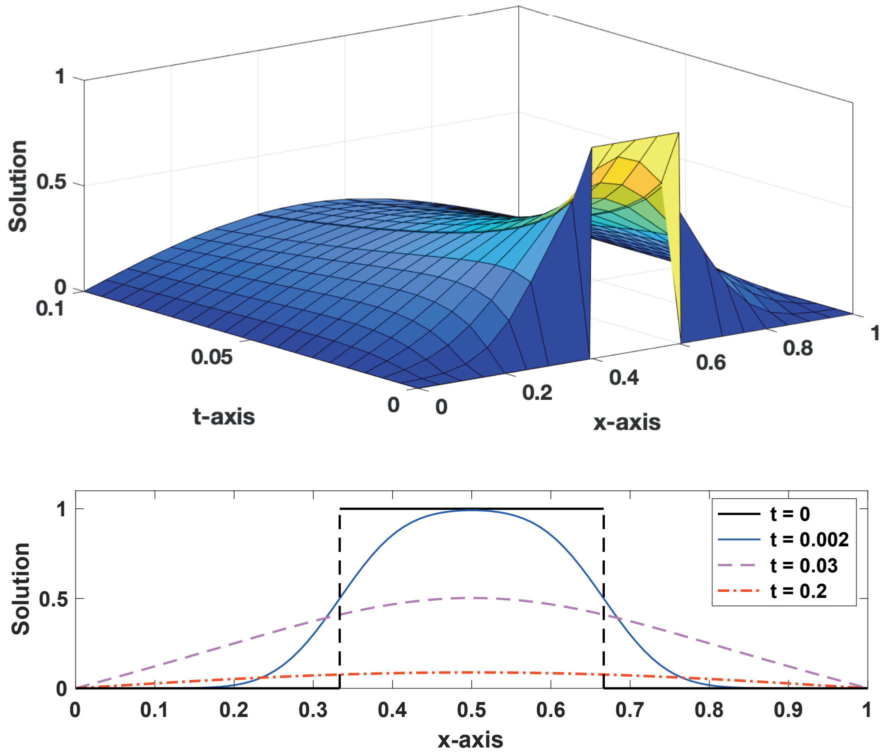

Figure 7.2. *Solution of the diffusion equation in Example 3. Shown is the solution surface as well as the solution profiles at specific time values.*

Exercises

1. You are to find the solution of the diffusion problem for the following initial conditions. Assume that $L = 1$ and $D = 3$. Note that you should be able to answer this question without using integration.

   a) $g(x) = -4\sin(5\pi x)$.

   b) $g(x) = 6\sin(11\pi x)$.

   c) $g(x) = \sin(\pi x) + 8\sin(4\pi x) - 10\sin(7\pi x)$.

   d) $g(x) = -\sin(3\pi x) + 7\sin(8\pi x) + 2\sin(15\pi x)$.

   e) $g(x) = 2\sin(2\pi x)\cos(\pi x)$.

   f) $g(x) = -3\cos(2\pi x)\sin(\pi x)$.

   g) $g(x) = 4\cos(2\pi x)\sin(3\pi x) - 4\sin(2\pi x)\cos(3\pi x)$.

2. You are to find the solution of the diffusion problem for the following initial conditions. Assume that $L = 2$ and $D = 4$.

a) $g(x) = 1$
b) $g(x) = 2 + x$
c) $g(x) = \begin{cases} -1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$
d) $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{3} \\ 2 & \text{otherwise} \end{cases}$
e) $g(x) = -5x$
f) $g(x) = \cos(x)$
g) $g(x) = \begin{cases} 3 & \text{if } 1 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$
h) $g(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \le x \le \frac{3}{2} \\ 0 & \text{otherwise} \end{cases}$

3. Find the general solution of the following.
   a) $u_{xx} = u_t$, for $0 < x < 1$, with the boundary conditions $u(0,t) = 0$ and $u_x(1,t) = 0$.
   b) $4u_{xx} = u_t$, for $0 < x < 1$, with the boundary conditions $u_x(0,t) = 0$ and $u(1,t) = 0$.
   c) $(1+t)u_{xx} = u_t$, for $0 < x < 1$, with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$.
   d) $u_{xx} = u_t + e^{-t}u$, for $0 < x < 1$, with the boundary conditions $u(0,t) = 0$ and $u(1,t) = 0$.
4. Using the general solution from Exercise 3, find the solution of the problem when the initial condition is

$$g(x) = \begin{cases} 1 & \text{if } \frac{1}{2} \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

5. The following are BVPs similar to what is obtained using separation of variables. Find the values λ, and functions $F(x)$, that satisfy the given problem. You can assume that $\lambda \le 0$.
   a) $F''(x) = \lambda F(x)$, for $0 < x < 1$, with the boundary conditions $F(0) = 0$ and $F'(1) = 0$.
   b) $F''(x) = \lambda F(x)$, for $0 < x < 4$, with the boundary conditions $F'(0) = 0$ and $F'(4) = 0$.
   c) $F''(x) + F'(x) = \lambda F(x)$, for $0 < x < 1$, with the boundary conditions $F(0) = 0$ and $F(1) = 0$. Hint: $4\lambda < -1$.
   d) $F''(x) = \lambda F(x)$, for $0 < x < 1$, with the boundary conditions $F(0) = F(1)$ and $F'(0) = F'(1)$.

7.3 ▪ Sine and Cosine Series

To satisfy the initial condition, given $g(x)$, we were required to find the b_n's so that

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right), \quad \text{for} \quad 0 < x < L. \tag{7.21}$$

This is an example of a Fourier sine series. Finding the b_n's is not hard. However, this requires knowing what restrictions must be placed on $g(x)$, and so, this is where we begin.

One of the requirements is that $g(x)$ is **piecewise continuous** for $0 \leq x \leq L$. This means that $g(x)$ is continuous on the interval except, possibly, for a finite number of jump discontinuities. What a **jump discontinuity** means is that $g(x)$ is not continuous at the point, but the limits of $g(x)$ from the left, $g(x^-)$, and from the right, $g(x^+)$, are defined. This is the requirement when $0 < x < L$. For $x = 0$, then $g(0^+)$ must be defined, but it is not required to equal $g(0)$. Similarly, for $x = L$, $g(L^-)$ must be defined, but it is not required to equal $g(L)$. An example of a function with two jumps is given in (7.19). Also, all of the functions in Exercise 2 in the previous section are piecewise continuous.

A consequence of the assumption that $g(x)$ is piecewise continuous is that the integral in (7.17) is well-defined.

7.3.1 ▪ Finding the b_n's

The working hypothesis is that the sine series converges, and we can integrate it term-by-term. The reason for this assumption is that the key for finding the coefficients is the integration formula: if m and n are positive integers, then

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} \dfrac{L}{2} & \text{if } m = n, \\ 0 & \text{if } m \neq n. \end{cases} \tag{7.22}$$

To illustrate how this is used, suppose we want to find the value for, say, b_7. Multiplying (7.21) by $\sin(7\pi x/L)$, and then integrating yields

$$\int_0^L g(x) \sin\left(\frac{7\pi x}{L}\right) dx = \sum_{n=1}^{\infty} b_n \int_0^L \sin\left(\frac{7\pi x}{L}\right) \sin\left(\frac{n\pi x}{L}\right) dx.$$

According to (7.22), all of the integrals on the right are zero except when $n = 7$. Consequently,

$$\int_0^L g(x) \sin\left(\frac{7\pi x}{L}\right) dx = \frac{L}{2} b_7,$$

or equivalently

$$b_7 = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{7\pi x}{L}\right) dx.$$

A similar result is obtained for the other b_n's, and the resulting formula is

$$b_n = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{7.23}$$

7.3.2 ▪ Convergence Theorem

Proving a sine series converges requires more than just using the ratio test, which is the way you prove a power series converges. The proof is beyond the scope of this textbook, but the result is important for our using a sine series when solving PDEs.

Sine Series Convergence Theorem. *Assume that $g(x)$ and $g'(x)$ are piecewise continuous for $0 \le x \le L$, and the b_n's are given in (7.23).*

For $0 < x < L$: *If $g(x)$ is continuous at x, then*

$$g(x) = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big), \tag{7.24}$$

and if $g(x)$ has a jump discontinuity at x, then

$$\frac{1}{2}\big[g(x^+) + g(x^-)\big] = \sum_{n=1}^{\infty} b_n \sin\Big(\frac{n\pi x}{L}\Big). \tag{7.25}$$

At $x = 0$ or $x = L$: *The sine series is zero when $x = 0$ or $x = L$.*

In words, the theorem states that the sine series equals the function $g(x)$ at points in the interval where $g(x)$ is continuous, and it equals the average in the jump of $g(x)$ at a jump discontinuity. At the endpoints, no matter what the value of $g(0)$ or $g(L)$, the series sums to zero.

7.3.3 ▪ Cosine Series

Using separation of variables, it is not uncommon to end up with a cosine series rather than a sine series. In this case, the initial condition requires finding the a_n's that satisfy

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big), \ \text{ for } \ 0 < x < L. \tag{7.26}$$

The convergence theorem for this is very similar to the one for the sine series. First, the needed integration formula is, if m and n are integers,

$$\int_0^L \cos\Big(\frac{n\pi x}{L}\Big)\cos\Big(\frac{m\pi x}{L}\Big)dx = \begin{cases} L & \text{if } \ m = n = 0, \\ \dfrac{L}{2} & \text{if } \ m = n \neq 0, \\ 0 & \text{if } \ m \neq n. \end{cases}$$

This formula is used in the same way the one for the sine series was used. Namely, if you want to determine, say, a_4, you multiply (7.26) by

$\cos(4\pi x/L)$ and then integrate over the interval $0 \le x \le L$. The resulting formula, for general n, is

$$a_n = \frac{2}{L}\int_0^L g(x)\cos\Big(\frac{n\pi x}{L}\Big)dx. \tag{7.27}$$

This brings us to the next result.

Cosine Series Convergence Theorem. *Assume that $g(x)$ and $g'(x)$ are piecewise continuous for $0 \le x \le L$, and the a_n's are given in (7.27). If $g(x)$ is continuous at x, then*

$$g(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big). \tag{7.28}$$

If $g(x)$ has a jump discontinuity at x, and $0 < x < L$, then

$$\frac{1}{2}\big[g(x^+) + g(x^-)\big] = \frac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n \cos\Big(\frac{n\pi x}{L}\Big). \tag{7.29}$$

At $x = 0$, the series sums to $g(0^+)$, and at $x = L$, the series sums to $g(L^-)$.

In words, the theorem states that the cosine series equals the function $g(x)$ at points in the interval where $g(x)$ is continuous, and it equals the average in the jump of $g(x)$ at a jump discontinuity. At the endpoints, it sums to the respective limit of $g(x)$ at the endpoint.

7.3.4 ▪ Examples

Finding a sine or cosine series is rather uneventful as it is simply a matter of evaluating the given formulas. The only concern is how hard it is to evaluate the given integrals to find the coefficients. So, in the examples below, a more practical question is also considered. Namely, how many terms of the series do you have to add together to obtain an accurate approximation of the function $g(x)$? As will be seen, the answer depends on whether the function is continuous, and whether it has the right values at the endpoints.

Example 1: Taking $L = 3$, what function does the sine series for $g(x)$, given below, converge to for $0 \le x \le 3$?

$$g(x) = \begin{cases} x & \text{if } \ 0 \le x \le 2\,, \\ 1 & \text{if } \ 2 < x \le 3. \end{cases} \tag{7.30}$$

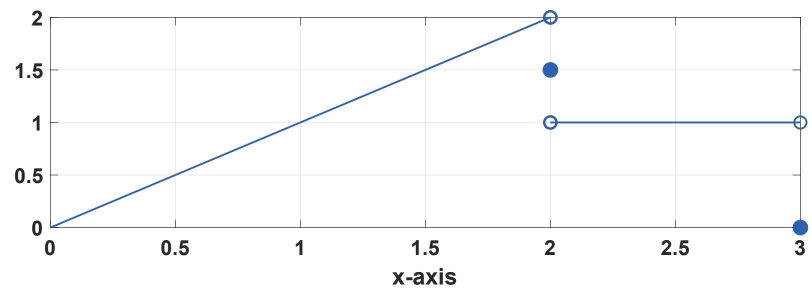

Figure 7.3. *The function the sine series sums to for $g(x)$ given in (7.30).*

Answer: For $0 < x < 3$, $g(x)$ is continuous except at $x = 2$. So, the series converges to $g(x)$ for $0 < x < 2$ and for $2 < x < 3$. At $x = 2$, it converges to $\frac{1}{2}[g(2^+) + g(2^-)] = \frac{1}{2}(1+2) = \frac{3}{2}$. Finally, at $x = 0$, and at $x = 3$, the series sums to zero. The sketch of this function is given in Figure 7.3. ■

Example 2: For $0 \le x \le 1$, find the sine series of

$$g(x) = x(1-x).$$

Answer: Using (7.23), and integrating by parts twice,

$$\begin{aligned} b_n &= 2\int_0^1 x(1-x)\sin(n\pi x)dx \\ &= \frac{4}{n^3\pi^3}\big(1-\cos(n\pi)\big) \\ &= \frac{4}{n^3\pi^3}\big(1-(-1)^n\big). \end{aligned}$$

Because $g(x)$ is continuous, and $g(0) = g(1) = 0$, we have that

$$x(1-x) = \sum_{n=1}^{\infty} \frac{4}{n^3\pi^3}\big(1-(-1)^n\big)\sin(n\pi x), \quad \text{for} \quad 0 \le x \le 1. \tag{7.31}$$

When computing the value of the sine series it is necessary to pick an N, and then use the approximation

$$x(1-x) \approx \sum_{n=1}^{N} \frac{4}{n^3\pi^3}\big(1-(-1)^n\big)\sin(n\pi x). \tag{7.32}$$

The accuracy of this is shown in Figure 7.4. It is evident that the approximation is not bad for $N = 1$, and is rather good at $N = 5$. ■

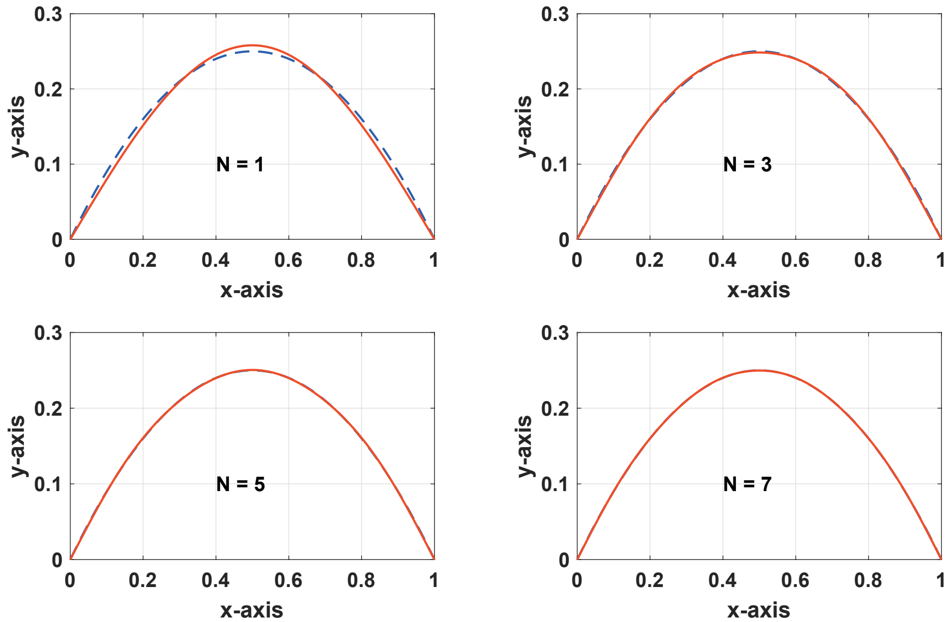

Figure 7.4. *Comparison between the function $g(x) = x(1-x)$, shown with the dashed blue curve, and the approximation in (7.34), shown using a solid red curve.*

Example 3: For $0 \leq x \leq 1$, find the sine series of

$$g(x) = \begin{cases} 1 & \text{if } \; 0 \leq x \leq \frac{1}{4}\,, \\ 0 & \text{otherwise.} \end{cases}$$

Answer: Using (7.23),

$$\begin{aligned} b_n &= 2\int_0^{1/4} \sin(n\pi x)dx \\ &= \frac{2}{n\pi}\big(1-\cos(n\pi/4)\big). \end{aligned}$$

From this we have that, except for $x = 0$ and $x = 1/4$,

$$g(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\big(1-\cos(n\pi/4)\big)\sin(n\pi x). \tag{7.33}$$

At $x = 0$ the series is zero, and at $x = 1/4$ the series sums to $1/2$, which is the average in the jump if $g(x)$ at this point.

The resulting approximation is, given N,

$$g(x) \approx \sum_{n=1}^{N} \frac{2}{n\pi}\big(1-\cos(n\pi/4)\big)\sin(n\pi x). \tag{7.34}$$

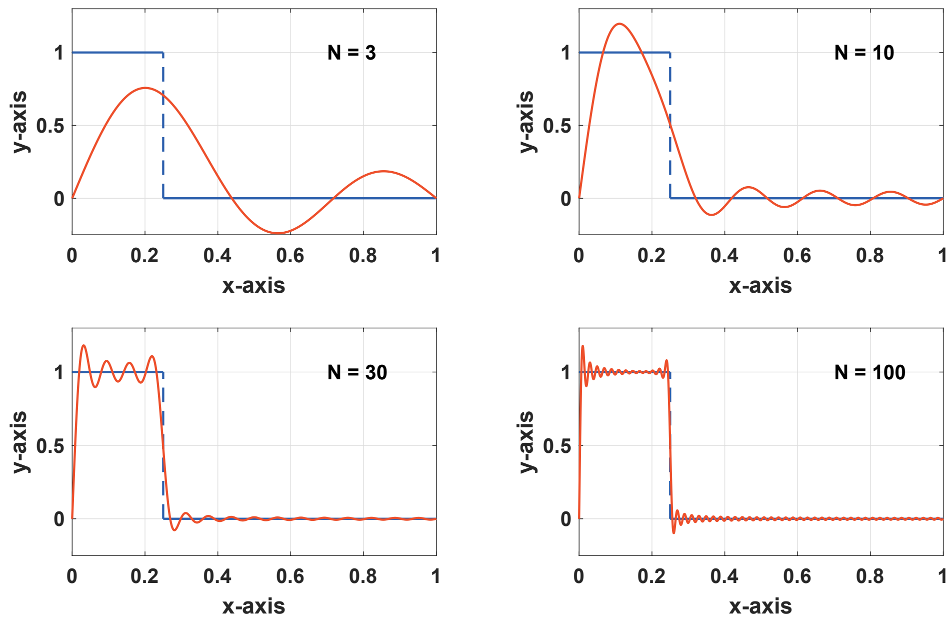

Figure 7.5. *Comparison between the function $g(x)$ for Example 3, shown with the blue curve, and the approximation in (7.34), shown using the red curve.*

The accuracy of this is shown in Figure 7.5. Because of the jump in the function, the sine series requires a larger value of N than needed in Example 1 to provide an accurate approximation. However, even with a larger N, the series has difficulty in the immediate vicinity of the jump. It has the same problem near $x = 0$ since the series is zero at $x = 0$ but $g(0) = 1$. The larger oscillations near the jump points are associated with what is called **Gibbs phenomenon**. As can be seen in the figure, the region where these oscillations occur can be reduced by taking larger values of N. However, the maximum overshoot and undershoot on either side of the jump do not go to zero, but instead they both approach a value that is equal to about 9% of the jump in the function. Because jump discontinuities arise so often in applications, there has been considerable research into how to remove the over and under shoots in the Fourier series solution. One of the more well known methods involves filtering them out, and an example is Fejér summation. More about this can be found in Jerri [1998]. ■

Example 4: For $0 \le x \le 1$, find the cosine series of

$$g(x) = \begin{cases} x+1 & \text{if } \ 0 \le x \le \frac{1}{2}\,, \\ 2 & \text{if } \ \frac{1}{2} < x \le 1. \end{cases}$$

Answer: Using (7.27), if $n \neq 0$,

$$\begin{aligned} a_n &= 2\int_0^1 g(x)\cos(n\pi x)dx \\ &= 2\int_0^{1/2}(x+1)\cos(n\pi x)dx + 2\int_{1/2}^1 2\cos(n\pi x)dx \\ &= \frac{2}{n^2\pi^2}\Big(\cos\big(\frac{n\pi}{2}\big)-1\Big) - \frac{1}{n\pi}\sin\big(\frac{n\pi}{2}\big), \end{aligned}$$

and when $n = 0$, $a_0 = 13/4$. From this we have that, except for $x = 1/2$,

$$g(x) = \frac{13}{8} + \sum_{n=1}^{\infty}\Big[\frac{2}{n^2\pi^2}\Big(\cos\big(\frac{n\pi}{2}\big)-1\Big) - \frac{1}{n\pi}\sin\big(\frac{n\pi}{2}\big)\Big]\cos(n\pi x). \tag{7.35}$$

At $x = 1/2$ the series sums to the average in the jump in $g(x)$, and so it equals $7/4$.

The resulting approximation is, given N,

$$g(x) \approx \frac{13}{8} + \sum_{n=1}^{N}\Big[\frac{2}{n^2\pi^2}\Big(\cos\big(\frac{n\pi}{2}\big)-1\Big) - \frac{1}{n\pi}\sin\big(\frac{n\pi}{2}\big)\Big]\cos(n\pi x). \tag{7.36}$$

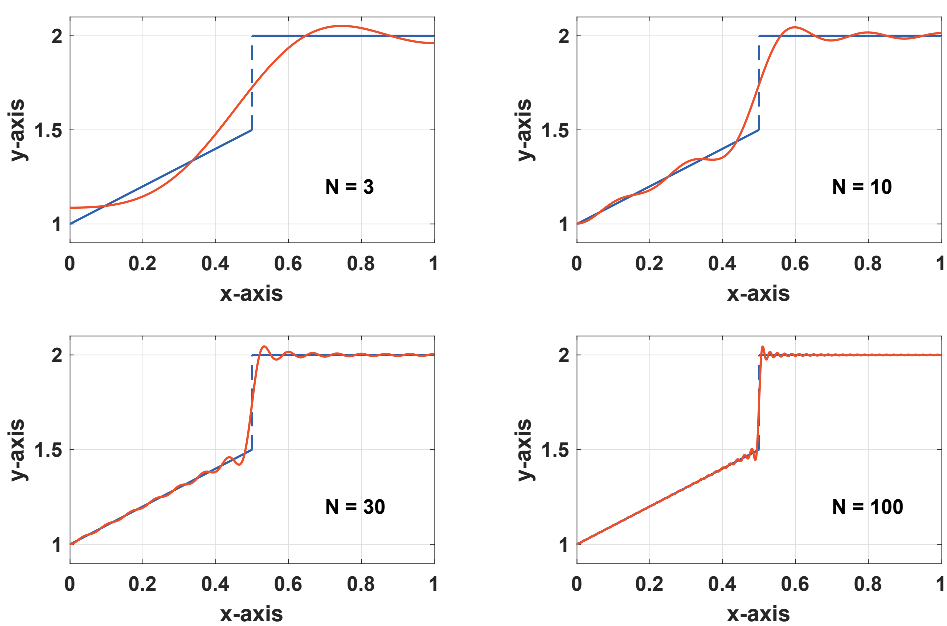

Figure 7.6. *Comparison between the function $g(x)$ for Example 4, shown with the blue curve, and the approximation in (7.36), shown using the red curve.*

The accuracy of this is shown in Figure 7.6. As happened with the sine series, in the immediate vicinity of the jump the series oscillates. However, unlike Example 3, there are no oscillations at the endpoints. ■

You might have noticed that the coefficients in the above three examples can be written in a reduced form. To illustrate, in Example 1, the resulting series in (7.31) contains the term $\big(1-(-1)^n\big)$. This is zero when n is even and it is equal to two for odd n. Consequently, it is possible to write the series as

$$x(1-x) = \sum_{\substack{n=1 \\ n \text{ odd}}}^{\infty} \frac{8}{n^3\pi^3}\sin(n\pi x),$$

or as

$$x(1-x) = \sum_{k=1}^{\infty} \frac{8}{(2k-1)^3\pi^3}\sin((2k-1)\pi x).$$

In this text, simplifications such as $\sin(n\pi)=0$ and $\cos(n\pi)=(-1)^n$ are used, and expected. Your instructor will tell you if they expect anything more than this.

7.3.5 ▪ Differentiability

In using a sine or cosine series when solving a PDE, it is implicitly assumed you can differentiate the series term-by-term. What this means is that it is assumed that

$$\frac{d}{dx}\sum_{n=1}^{\infty} p_n(x) = \sum_{n=1}^{\infty}\frac{d}{dx}p_n(x).$$

With this in mind, in Example 2, if you try this with (7.33), you get

$$g'(x) = \sum_{n=1}^{\infty} 2\big(1-\cos(n\pi/4)\big)\cos(n\pi x). \tag{7.37}$$

As you should recall, if an infinite series $\sum a_n$ converges, then it must be true that $a_n \to 0$ as $n \to \infty$. The above series for $g'(x)$ does not satisfy this condition, and therefore it does not converge. In other words, you can not differentiate (7.33) term-by-term. In contrast, for Example 1 you can differentiate the series term-by-term. The theorem that explains this states that if $g(x)$ is continuous, and $g'(x)$ is piecewise continuous, for $0 \le x \le L$, then you can differentiate the cosine series term-by-term, but to do this for a sine series you need an additional assumption [Tolstov and Silverman, 1976]. An easy to use version of the needed assumption is that $g(0)=g(L)=0$. This holds for Example 1, and that is why term-by-term differentiation can be done with that sine series. For both the cosine and sine series, if $g(x)$ is not continuous, then term-by-term differentiation is not possible without additional assumptions. Those interested in pursuing this issue a bit further should look at Exercise 9.

The situation for term-by-term integration is better. Specifically, if $g(x)$ satisfies the requirements of the convergence theorem, its sine, and cosine, series can be integrated term-by-term.

The next question is whether the potential non-differentiability of a sine series means that we can not use them to solve the diffusion equation. To explain why this is not a problem, consider the solution (7.35), which is

$$u(x,t) = \sum_{n=1}^{\infty} b_n e^{-n^2\pi^2 t} \sin(n\pi x).$$

As long as $t > 0$, the coefficients of this series are exponentially decreasing functions of n^2. This, along with the fact that the series for $g(x)$ converges, guarantee that you can differentiate the series term-by-term without reservation, as long as $t > 0$.

Exercises

1. Sketch the graph of the given function. Also, determine whether $f(x)$ is continuous, piecewise continuous, or neither for $0 \le x \le 1$.

a) $f(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \\ 2x & \text{if } \frac{1}{2} < x \le \frac{3}{4} \\ \frac{3}{2} & \text{if } \frac{3}{4} < x \le 1 \end{cases}$ c) $f(x) = \begin{cases} 0 & \text{if } x = 0 \\ \ln x & \text{if } 0 < x < 1 \\ 1 & \text{if } x = 1 \end{cases}$

b) $f(x) = \begin{cases} 1 & \text{if } x = \frac{1}{4}, \frac{1}{2}, \frac{3}{4}, 1 \\ 2 & \text{otherwise} \end{cases}$ d) $f(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2} \\ \frac{1}{2x-1} & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

2. In the following $L = 2$. Sketch the function to which the sine series converges, for $0 \le x \le 2$.

a) $g(x) = x$

b) $g(x) = e^x$

c) $g(x) = \cos(\pi x)$

d) $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{2} \\ -3 & \text{if } \frac{1}{2} < x \le 2 \end{cases}$

e) $g(x) = \begin{cases} 2 - x & \text{if } 0 \le x \le 1 \\ 0 & \text{if } 1 < x \le 2 \end{cases}$

f) $g(x) = \begin{cases} 3 & \text{if } x = 0, \frac{1}{2}, 1, 2 \\ x & \text{otherwise} \end{cases}$

g) $g(x) = \begin{cases} -1 & \text{if } x = 0, \frac{1}{3}, 2 \\ e^x & \text{otherwise} \end{cases}$

h) $g(x) = \begin{cases} 1 & \text{if } 0 \le x \le \frac{1}{3} \\ 0 & \text{if } \frac{1}{3} < x \le \frac{4}{3} \\ 3 & \text{if } \frac{4}{3} < x \le 2 \end{cases}$

3. Find the sine series for the functions in Exercise 2.
4. For the functions in Exercise 2, sketch the function to which the cosine series converges, for $0 \leq x \leq 2$.
5. Find the cosine series for the functions in Exercise 2.
6. For any given x from the interval $0 \leq x \leq 1$, use the comparison test to show that the series in (7.31) converges absolutely.
7. In this exercise let $g(x) = x^2$, for $0 \leq x \leq 1$.
   a) Find the cosine series for $g(x)$.
   b) For any given x from the interval $0 \leq x \leq 1$, use the comparison test to show that the series in part (a) converges absolutely.
   c) Using your result from part (a), show that
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2} = \frac{\pi^2}{12}\,.$$
8. In this exercise let $g(x) = x$, for $0 \leq x \leq 1$.
   a) Find the sine series for $g(x)$.
   b) Using your result from part (a), show that
$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots\,.$$
9. This exercise deals with the restriction on term-by-term differentiability of the sine series. This requires you to have read Section 6.5.1. If the observation made in this exercise interests you, you might want to look at Stakgold [2000].
   a) Write the function $g(x)$ in Example 3 in terms of the Heaviside function $H(x)$.
   b) Using Example 4, from Section 6.5.1, what is $g'(x)$?
   c) Using your result from part (b), what is the sine series for $g'(x)$? How does this differ from the result in (7.37)?

7.4 ▪ Wave Equation

The problem involves finding the function $u(x,t)$ that satisfies

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}\,, \quad \text{for} \quad \begin{cases} 0 < x < L, \\ 0 < t, \end{cases} \tag{7.38}$$

where c is a positive constant. This PDE is known as the wave equation. It applies, for example, to the vertical displacement $u(x,t)$ of an elastic string. This provides an interesting interpretation of the terms in the sine series solution, and this is discussed in Section 7.4.2.

To complete the problem, the boundary conditions are

$$u(0,t) = 0, \tag{7.39}$$

and

$$u(L,t) = 0. \tag{7.40}$$

For the initial conditions, it is assumed that

$$u(x,0) = g(x), \quad \text{for } \ 0 < x < L, \tag{7.41}$$

and

$$u_t(x,0) = h(x), \quad \text{for } \ 0 < x < L, \tag{7.42}$$

where $g(x)$ and $h(x)$ are given functions. To avoid the complication with differentiability, as described in Section 7.3.5, it is assumed that $g(x)$ and $h(x)$ are smooth functions that satisfy the boundary conditions. It is also assumed that $g''(0) = g''(L) = 0$.

As with the diffusion problem, separation of variables will be used to find the general solution of the PDE and boundary conditions. After that, the initial conditions will be satisfied. Also, you should notice, as with the diffusion problem, the PDE and boundary conditions are homogeneous. This is required for separation of variables to work.

Separation of Variables Assumption

Assuming

$$u(x,t) = F(x)G(t), \tag{7.43}$$

and then substituting this into the PDE gives us

$$c^2 \frac{F''(x)}{F(x)} = \frac{G''(t)}{G(t)} \,. \tag{7.44}$$

Since the left-hand-side is only a function of x, and the right-hand-side is only a function of t, we can conclude that there is a constant λ so that

$$c^2 F''(x) = \lambda F(x) \,. \tag{7.45}$$

and

$$G''(t) = \lambda G(t) \,. \tag{7.46}$$

Finding $G(t)$

Assuming that $G(t) = e^{rt}$, then from (7.46) we get that $r^2 = \lambda$. This means that the general solution for G will contain the functions $e^{\sqrt{\lambda}t}$ and $e^{-\sqrt{\lambda}t}$. If $\lambda > 0$ then $e^{\sqrt{\lambda}t}$ is unbounded. By assumption, the solution is bounded, and so it must be that $\lambda \leq 0$. Before writing down the formula for $G(t)$, we will first determine the values for λ, and for this we need to consider the problem for $F(x)$.

Finding $F(x)$ and λ

The separation of variables assumption must be used on the boundary conditions. So, to have $u(0,t) = 0$, we need $F(0)G(t) = 0$. For this to happen, and u not be identically zero, we require that $F(0) = 0$. Similarly, we need $F(L) = 0$. Consequently, all-together, the function $F(x)$ must satisfy

$$c^2 F''(x) = \lambda F(x)\,, \tag{7.47}$$

where

$$F(0) = 0 \quad \text{and} \quad F(L) = 0, \tag{7.48}$$

The only difference between the above BVP, and the one for the diffusion equation, is that we now have the coefficient c^2 instead of D. Consequently, from (7.13) and (7.14), the nonzero solutions of (7.47) and (7.48) are

$$F_n(x) = \bar{b}_n \sin\Big(\frac{n\pi x}{L}\Big), \tag{7.49}$$

and

$$\lambda_n = -c^2\Big(\frac{n\pi}{L}\Big)^2, \tag{7.50}$$

for $n = 1, 2, 3, \ldots$. Also, $\bar{b}_n$ is an arbitrary constant.

Finding $G(t)$ Redux

Now that we know λ, (7.46) takes the form

$$G''(t) = -c^2\Big(\frac{n\pi}{L}\Big)^2 G(t)$$

Assuming that $G(t) = e^{rt}$, we get that $r^2 = -(cn\pi/L)^2$. So, $r = \pm icn\pi/L$, and from this we get the general solution

$$G_n(t) = a_n \cos(\omega_n t) + b_n \sin(\omega_n t), \tag{7.51}$$

where

$$\omega_n = \frac{cn\pi}{L}\,, \tag{7.52}$$

and a_n and b_n are arbitrary constants.

The General Solution

We have shown that for any given n, the function $u_n(x,t) = F_n(x)G_n(t)$ is a solution of the PDE that satisfies the boundary conditions. The resulting general solution, that satisfies the PDE and boundary conditions, is, therefore,

$$u(x,t) = \sum_{n=1}^{\infty} u_n(x,t),$$

or equivalently

$$u(x,t) = \sum_{n=1}^{\infty} \big[a_n \cos(\omega_n t) + b_n \sin(\omega_n t)\big] \sin\Big(\frac{n\pi x}{L}\Big), \tag{7.53}$$

where a_n and b_n are arbitrary constants, and ω_n is given in (7.52). In writing this down, the constant $\bar{b}_n$ in (7.49) has been absorbed into the a_n and b_n.

Satisfying the Initial Conditions

To satisfy (7.41), we need

$$\sum_{n=1}^{\infty} a_n \sin\Big(\frac{n\pi x}{L}\Big) = g(x). \tag{7.54}$$

From (7.23), this means that

$$a_n = \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx. \tag{7.55}$$

To satisfy (7.42), we need

$$\sum_{n=1}^{\infty} \omega_n b_n \sin\Big(\frac{n\pi x}{L}\Big) = h(x). \tag{7.56}$$

Letting $B_n = \omega_n b_n$, then the above equation takes the form

$$\sum_{n=1}^{\infty} B_n \sin\Big(\frac{n\pi x}{L}\Big) = h(x). \tag{7.57}$$

This is the same problem we had in Section 7.2.5, except that the coefficient is being denoted as B_n instead of b_n. So, from (7.17),

$$B_n = \frac{2}{L}\int_0^L h(x)\sin\Big(\frac{n\pi x}{L}\Big)dx.$$

Since $b_n = B_n/\omega_n$, the conclusion is that

$$b_n = \frac{2}{cn\pi}\int_0^L h(x)\sin\Big(\frac{n\pi x}{L}\Big)dx. \tag{7.58}$$

7.4.1 ▪ Examples

Example 1: Suppose that $c = 1$, $L = 2$, $g(x) = 3\sin(\pi x)$, and $h(x) = 0$. In this case, from (7.52), $\omega_n = n\pi/2$. The resulting general solution (7.53) is

$$u(x,t) = \sum_{n=1}^{\infty} \Big[a_n \cos\big(\frac{n\pi}{2}t\big) + b_n \sin\big(\frac{n\pi}{2}t\big)\Big] \sin\Big(\frac{n\pi x}{2}\Big).$$

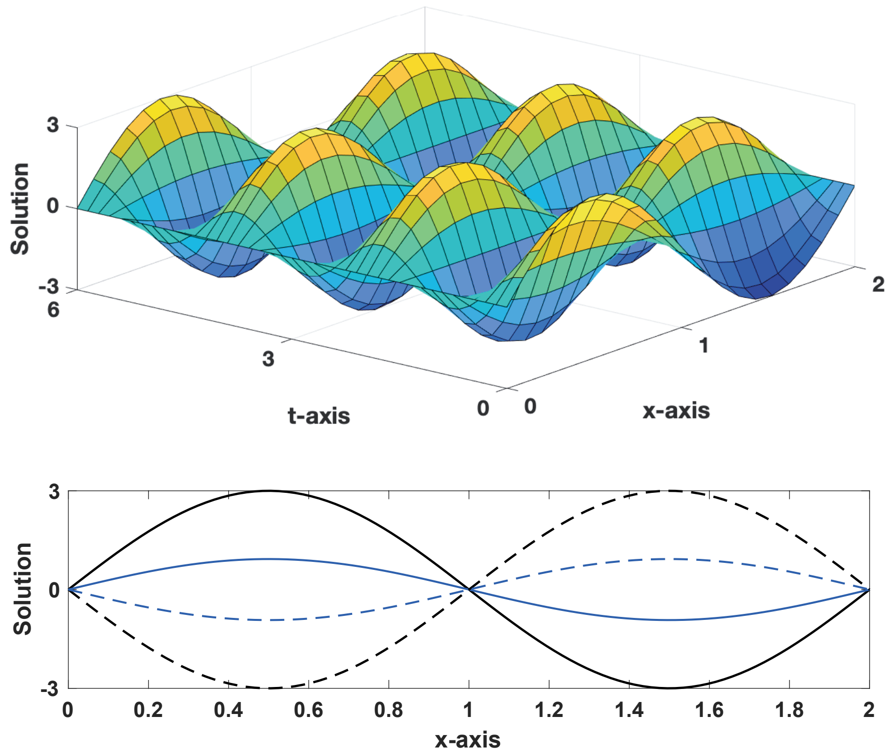

Figure 7.7. *Solution of the wave equation in Example 1. Shown is the solution surface as well as the solution profiles at specific time values.*

To satisfy the initial condition, since $h(x) = 0$ then, from (7.58), the b_n's are all zero. As for the a_n's, note that $g(x)$ is one of the sine functions in the series. Namely, it is the one when $n = 2$. This enables us to avoid the integral in (7.55). To satisfy (7.54) we simply take $a_2 = 3$, and all the other b_n's are zero. Therefore, the solution is

$$u(x,t) = 3\cos(\pi t)\sin(\pi x). \tag{7.59}$$

This solution is shown in Figure 7.7, both as time slices and as the solution surface for $0 \leq t \leq 3T$, where $T = 2$ is the period of oscillation. ■

Example 2: Suppose that in the previous example, the initial conditions are $g(x) = 0$ and

$$h(x) = 3\sin\left(\frac{\pi x}{2}\right) - 4\sin\left(\frac{3\pi x}{2}\right) + 5\sin(2\pi x).$$

This consists of the sum of three of the sine functions in (7.57): $n = 1$, $n = 3$, and $n = 4$. To satisfy (7.57) we take $B_1 = 3$, $B_3 = -4$, $B_4 = 5$, and all the other B_n's are zero. With this,

$b_1 = 3/\omega_1 = 6/\pi$, $b_3 = -4/\omega_3 = -8/(3\pi)$, $b_4 = 5/\omega_4 = 5/(2\pi)$. Also, since $g(x) = 0$, then from (7.55), all the a_n's are zero. The resulting solution is

$$u(x,t) = \frac{3}{\pi}\sin\Big(\frac{\pi t}{2}\Big)\sin\Big(\frac{\pi x}{2}\Big) - \frac{4}{3\pi}\sin\Big(\frac{3\pi t}{2}\Big)\sin\Big(\frac{3\pi x}{2}\Big) + \frac{5}{2\pi}\sin\Big(2\pi t\Big)\sin(2\pi x). \quad \blacksquare$$

7.4.2 ▪ Natural Modes and Frequencies

We found that the general solution of the wave equation, with the given boundary conditions, is the sum of functions of the form

$$u_n(x,t) = \big[a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\big]\sin\Big(\frac{n\pi x}{L}\Big), \tag{7.60}$$

where

$$\omega_n = \frac{cn\pi}{L}\,. \tag{7.61}$$

The expression in the square brackets is a periodic function of t, with period $2\pi/\omega_n$. For this reason, u_n is said to be a natural mode for the problem, having the natural frequency ω_n. What this means is that any set of initial conditions that involve only $\sin(n\pi x/L)$, will result in a solution of the form $A(t)\sin(n\pi x/L)$, where $A(t)$ is periodic with period $2\pi/\omega_n$.

Exercises

1. You are to find the solution of the wave equation problem for the following initial conditions. Assume that $L = 1$ and $c = 4$. Note that you should be able to answer this question without using integration.
   a) $g(x) = \sin(3\pi x)$, and $h(x) = 0$
   b) $g(x) = 0$, and $h(x) = -2\sin(8\pi x)$
   c) $g(x) = -\sin(\pi x) + 4\sin(3\pi x) - 3\sin(5\pi x)$, and $h(x) = 0$
   d) $g(x) = 0$, and $h(x) = -\sin(\pi x) + 2\sin(8\pi x) + 3\sin(12\pi x)$.
   e) $g(x) = 2\sin(2\pi x)\cos(\pi x)$, and $h(x) = -2\sin(8\pi x)$
   f) $g(x) = 0$, and $h(x) = -3\cos(2\pi x)\sin(\pi x)$
   g) $g(x) = 4\cos(2\pi x)\sin(3\pi x) - 4\sin(2\pi x)\cos(3\pi x)$, and $h(x) = -3\cos(2\pi x)\sin(\pi x)$
2. Find the general solution of the following.
   a) $u_{xx} = u_{tt}$, for $0 < x < 1$, with the boundary conditions $u(0,t) = 0$ and $u_x(1,t) = 0$.

b) $4u_{xx} = u_{tt}$, for $0 < x < 1$, with the boundary conditions $u_x(0,t) = 0$ and $u(1,t) = 0$.

c) $u_{xx} = 4u_{tt}$, for $0 < x < 1$, with the boundary conditions $u(0,t) = u(1,t)$ and $u_x(0,t) = u_x(1,t)$.

7.5 ▪ Inhomogeneous Boundary Conditions

Solving the diffusion and wave equations using separation of variables required the boundary conditions to be homogeneous. What is of interest is how to solve them when the boundary conditions are inhomogeneous, and have the form

$$u(0,t) = \alpha, \tag{7.62}$$

and

$$u(L,t) = \beta, \tag{7.63}$$

where α and β are constants. The method used to find the solution is to write the solution as

$$u(x,t) = w(x) + v(x,t),$$

where we pick $w(x)$ so it satisfies the given boundary conditions. In other words, so that $w(0) = \alpha$ and $w(L) = \beta$. Pretty much any smooth function can be used, but it makes things easier if w comes from the steady state equation. What this entails is explained below.

7.5.1 ▪ Steady State Solution

The steady state problem is the one that comes from the PDE and boundary conditions when assuming the solution is independent of t. Assuming we are solving the diffusion equation (7.1), then we are looking for the function u that satisfies

$$\frac{d^2u}{dx^2} = 0, \quad \text{for} \ \ 0 < x < L,$$

where u satisfies (7.62) and (7.63). The resulting solution is

$$u = \alpha + \frac{\beta - \alpha}{L}x.$$

7.5.2 ▪ Transformed Problem

Now that we know the steady state solution, we write the solution of the original diffusion problem as

$$u(x,t) = \alpha + \frac{\beta - \alpha}{L}x + v(x,t). \tag{7.64}$$

Since $u_{xx} = v_{xx}$ and $u_t = v_t$, then from the diffusion equation (7.1) we have that

$$D\frac{\partial^2 v}{\partial x^2} = \frac{\partial v}{\partial t}, \quad \text{for} \quad \begin{cases} 0 < x < L, \\ 0 < t. \end{cases} \tag{7.65}$$

At $x = 0$, from (7.64), $v(0,t) = u(0,t) - \alpha = 0$. This also happens at the other endpoint. So, the boundary conditions are

$$v(0,t) = 0, \tag{7.66}$$

and

$$v(L,t) = 0. \tag{7.67}$$

Finally, if the initial condition is $u(x,0) = g(x)$, then the resulting initial condition for v is

$$v(x,0) = g(x) - \alpha - \frac{\beta - \alpha}{L}x, \quad \text{for } \ 0 < x < L. \tag{7.68}$$

The above problem for v has the same form as the one for u, as given in (7.1)-(7.4), except for a slightly different looking initial condition. Consequently, we can use the solution as given in (7.15) and (7.17) if we make the appropriate adjustments. In particular,

$$v(x,t) = \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin\left(\frac{n\pi x}{L}\right), \tag{7.69}$$

where

$$b_n = \frac{2}{L}\int_0^L \left(g(x) - \alpha - \frac{\beta - \alpha}{L}x\right) \sin\left(\frac{n\pi x}{L}\right) dx. \tag{7.70}$$

The solution for the original diffusion problem is therefore

$$u(x,t) = \alpha + \frac{\beta - \alpha}{L}x + \sum_{n=1}^{\infty} b_n e^{\lambda_n t} \sin\left(\frac{n\pi x}{L}\right),$$

where b_n is given in (7.70).

Exercises

1. Find the steady state solution of the following problems.
   a) $u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u(0,t) = 1$ and $u_x(L,t) = -1$.
   b) $4u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u_x(0,t) = 2$ and $u(L,t) = 1$.
   c) $(1 + t)u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u_x(0,t) = -1$ and $u(L,t) = 2$.

d) $u_{xx} = u_t + u$, for $0 < x < L$, with the boundary conditions $u(0,t) = 1$ and $u(L,t) = 2$.
e) $u_{xx} - u_x = u_t$, for $0 < x < L$, with the boundary conditions $u(0,t) = -1$ and $u(L,t) = 1$.

2. Find the general solution of the following.
   a) $u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u(0,t) = 1$ and $u_x(L,t) = -1$.
   b) $4u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u_x(0,t) = 2$ and $u(L,t) = 1$.
   c) $(1+t)u_{xx} = u_t$, for $0 < x < L$, with the boundary conditions $u_x(0,t) = -1$ and $u(L,t) = 2$.
3. Solve the wave equation (7.38), using the inhomogeneous boundary conditions (7.62) and (7.63), along with the initial conditions (7.41) and (7.42).

7.6 ▪ Inhomogeneous PDEs

It is not unusual in applications to have a PDE that is not homogeneous. To explain how to solve such a problem, suppose the PDE is

$$D\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + p(x,t)\,, \quad \text{for} \quad \begin{cases} 0 < x < L, \\ 0 < t, \end{cases} \tag{7.71}$$

where $p(x,t)$ is a given smooth function of x and t. It is assumed the boundary conditions are homogeneous, and so,

$$u(0,t) = 0, \tag{7.72}$$

and

$$u(L,t) = 0. \tag{7.73}$$

We found earlier that the general solution of the $p = 0$ case consists of the superposition of functions containing $\sin(n\pi x/L)$. We also know, from the Sine Convergence Theorem on page 177, that you can use these functions to expand smooth functions. In particular, we can write

$$p(x,t) = \sum_{n=1}^{\infty} p_n(t) \sin\Big(\frac{n\pi x}{L}\Big), \tag{7.74}$$

where

$$p_n(t) = \frac{2}{L}\int_0^L p(x,t)\sin\Big(\frac{n\pi x}{L}\Big)dx \tag{7.75}$$

are known functions of t. We can also expand u in this way, and can write

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t) \sin\Big(\frac{n\pi x}{L}\Big). \tag{7.76}$$

In this expression, the $w_n(t)$'s are going to be determined from the PDE. In preparation for this, and assuming the series can be differentiated term-by-term, we get that

$$u_{xx} = -\sum_{n=1}^{\infty}\Big(\frac{n\pi}{L}\Big)^2 w_n(t)\sin\Big(\frac{n\pi x}{L}\Big) \quad \text{and} \quad u_t = \sum_{n=1}^{\infty} w_n'(t)\sin\Big(\frac{n\pi x}{L}\Big).$$

Introducing these into (7.71), as well as using (7.74), we have that

$$-D\sum_{n=1}^{\infty}\Big(\frac{n\pi}{L}\Big)^2 w_n(t)\sin\Big(\frac{n\pi x}{L}\Big) = \sum_{n=1}^{\infty}\big(w_n'(t)+p_n\big)\sin\Big(\frac{n\pi x}{L}\Big).$$

Multiplying this equation by $\sin(m\pi x/L)$, where m is a positive integer, and integrating, we use (7.22) to conclude that

$$-D\Big(\frac{n\pi}{L}\Big)^2 w_n = w_n'(t) + p_n$$

or equivalently,

$$w_n' + \kappa_n w_n = -p_n, \tag{7.77}$$

where

$$\kappa_n = D\Big(\frac{n\pi}{L}\Big)^2. \tag{7.78}$$

This is a first order linear differential equation for w_n, which can be solved using an integrating factor. The integrating factor in this case is, from (2.18), $\mu = e^{\kappa_n t}$. So, from (2.21), we get the general solution

$$w_n(t) = e^{-\kappa_n t}\Big[-\int_0^t p_n(s)e^{\kappa_n s}ds + w_n(0)\Big]. \tag{7.79}$$

To solve the PDE problem it remains to satisfy the initial condition

$$u(x,0) = g(x), \quad \text{for } \ 0 < x < L. \tag{7.80}$$

From (7.17), and since $b_n = w_n(0)$, we require that

$$w_n(0) = \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx.$$

7.6.1 ▪ Summary

To summarize our findings, the solution of the diffusion problem (7.71)-(7.73), which satisfies the initial condition (7.80), is

$$u(x,t) = \sum_{n=1}^{\infty} w_n(t)\sin\Big(\frac{n\pi x}{L}\Big), \tag{7.81}$$

where

$$w_n(t) = e^{-\kappa_n t}\Big[-\int_0^t p_n(s)e^{\kappa_n s}ds + w_n(0)\Big], \tag{7.82}$$

$$w_n(0) = \frac{2}{L}\int_0^L g(x)\sin\Big(\frac{n\pi x}{L}\Big)dx,$$

$$p_n(t) = \frac{2}{L}\int_0^L p(x,t)\sin\Big(\frac{n\pi x}{L}\Big)dx, \tag{7.83}$$

and $\kappa_n = D(n\pi/L)^2$.

7.6.2 ▪ Example

Suppose the problem to solve is

$$4\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t} + 3\sin 2t \sin \pi x\,, \quad \text{for} \quad \begin{cases} 0 < x < 1, \\ 0 < t, \end{cases} \tag{7.84}$$

where

$$u(0,t) = 0, \tag{7.85}$$

$$u(1,t) = 0, \tag{7.86}$$

and

$$u(x,0) = 0, \quad \text{for} \ \ 0 < x < 1. \tag{7.87}$$

First note that $D = 4$ and $L = 1$ in this problem. Since $p = 3\sin 2t \sin \pi x$, and $\sin \pi x$ corresponds to $n = 1$ in (7.83), then $p_1(t) = 3\sin 2t$ and all the other p_n's are zero. Also, since $g(x) = 0$ then $w_n(0) = 0$, for all n This leaves the integral in (7.82), and so

$$\begin{aligned}\int_0^t p_1(s)e^{\kappa_1 s}ds &= \int_0^t 3\sin 2s e^{\kappa_1 s}ds \\ &= 3\frac{2+\kappa_1 e^{\kappa_1 t}\sin(2t) - 2e^{\kappa_1 t}\cos(2t)}{\kappa_1^2+4}\,.\end{aligned}$$

Since $\kappa_1 = 4\pi^2$, then

$$w_1(t) = \frac{3}{2(1+4\pi^4)}\Big[\cos(2t) - 2\pi^2\sin(2t) - e^{-4\pi^2 t}\Big].$$

Therefore, the solution of the diffusion problem is

$$u(x,t) = \frac{3}{2(1+4\pi^4)}\Big[\cos(2t) - 2\pi^2\sin(2t) - e^{-4\pi^2 t}\Big]\sin(\pi x).$$

Exercises

1. You are to find the solution of the diffusion equation (7.71), where $u(0,t) = 0$, $u(1,t) = 0$, $u(x,0) = 0$, and $p(x,t)$ is given below. Assume that $D = 4$ and $L = 1$.
   a) $p(x,t) = -4\cos(t)\sin(5\pi x)$.
   b) $p(x,t) = e^{-2t}\sin(3\pi x)$.
   c) $p(x,t) = 3 + 5e^{-t}\sin(4\pi x)$.
2. There is a simpler way to solve an inhomogeneous PDE when the forcing function does not dependent on t. In this problem assume that $p(x,t) = x^2$.
   a) Find the steady state solution of (7.71), that satisfies (7.72) and (7.73).
   b) Letting $u(x,t) = w(x) + v(x,t)$, where $w(x)$ is the steady state solution you found in part (a), find the PDE, boundary conditions, and initial condition satisfied by $v(x,t)$.
   c) Find $v(x,t)$, and from this determine the solution of the original diffusion problem.

Bibliography

G. Bonani, S. Ivy, W. Wolfli, M. Broshi, I. Carmi, and J. Strugnell. Radiocarbon dating of fourteen dead sea scrolls. *Radiocarbon*, 34(3): 843–849, 1992. doi: 10.1017/S0033822200064158.

A. Coddington and R. Carlson. *Linear Ordinary Differential Equations*. Society for Industrial and Applied Mathematics, 1997. ISBN 9780898713886.

D. Eigler. Quantum Corral. Website, 2019. `http://www.nisenet.org/catalog/scientific-image-quantum-corral-top-view`.

R. Engbert and F. Drepper. Chance and chaos in population biology; models of recurrent epidemics and food chain dynamics. *Chaos, Solutions, and Fractals*, 4:1147–1169, 1994.

M. H. Holmes. *Introduction to the Foundations of Applied Mathematics*. Texts in Applied Mathematics. Springer, 2009. ISBN 9780387877495.

A. Jerri. *The Gibbs Phenomenon in Fourier Analysis, Splines and Wavelet Approximations*. Springer, 1998. doi: 10.1007/978-1-4757-2847-7.

NASA. Drag of a sphere. Website, 2019. `https://www.grc.nasa.gov/WWW/K-12/airplane/dragsphere.html`.

NRC. Radioactive atom. Website, 2019. `https://www.nrc.gov`.

L. Perko. *Differential Equations and Dynamical Systems*. Texts in Applied Mathematics. Springer, 2001. ISBN 9780387951164.

F. W. Roos and W. W. Willmarth. Some experimental results on sphere and disk drag. *AIAA Journal*, 9(2):285–291, 1971. doi: 10.2514/3.6164.

I. Stakgold. *Boundary Value Problems of Mathematical Physics (Vol. 2)*. Society for Industrial and Applied Mathematics, Philadelphia, PA, 2000. ISBN 978-0-89871-456-2.

A. M. Stuart and A. R. Humphries. *Dynamical Systems and Numerical Analysis*. Cambridge Monographs on Applied and Computational Mathematics. Cambridge University Press, 1998. ISBN 9780521645638.

G. P. Tolstov and R. A. Silverman. *Fourier Series*. Dover Publications, 1976. ISBN 9780486633176.

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