Xinshi Wang
661975305
hw04

1.
Proof:
G is bipartite $\implies$ every block of G is bipartite.

This direction is trivial. Since every block of G is maximal biconnected subgraphs of G, we know every block is bipartite since it is a subgraph of G and it will not contain an odd cycle.

Every block of G is bipartite $\implies$ G is bipartite.

We prove this statment by proving its contrapositive; G is not bipartite $\implies$ there exists a block of G that is not bipartite.

If G is not bipartite, then G contains an odd cycle. Since the odd cycle it self has no cut vertices, it will appear in a block of G by definition of maximal biconnected subgraph. Thus there exists a block that contains odd cycle, which means there exists a block that is not bipartite.

■

2.
Proof: k We first prove if for graph G, $\forall v \in V(G) : d(v)$ is even then every maximal biconnected component $B_i \in G$, $\forall u \in V(B_i) : d(u)$ is even.

Assume there exists a graph G such that $\forall v \in V(G) : d(v)$ is even and $\exists u \in V(B_i) : d(u)$ is odd. By definition, we know there exists at most one common vertex u between B_i and B_j where $i \neq j$. Since B_i and B_j are subgraphs of G, all vertices except for u have even degrees. Consider $u \in V(B_i)$ has an odd degree and all other vertices in $V(B_i)$have even degrees, the sum of degrees is odd. By the handshake theorem, this is impossible. Thus we have derived a contradiction. Therefore $\nexists u \in V(B_i) : d(u)$ is odd, which means $\forall u \in V(B_i) : d(u)$ is even.

We next prove if for every maximal biconnected component $B_i \in G$, $\forall u \in V(B_i) : d(u)$ is even, then for graph G, $\forall v \in V(G) : d(v)$ is even.

We prove this using induction. Let P(n) denote the statement if for every maximal biconnected component B_i, $\forall u \in V(B_i) : d(u)$ is even, then for graph G composed of n biconnected component, $\forall v \in V(G) : d(v)$ is even.

Base case: P(1). The biconnected component is the graph G. Thus $\forall u \in V(B_i) : d(u)$ is even $\implies$ $\forall v \in V(G) : d(v)$ is even.

Inductive step: Assume $P(n)$ is true, prove $p(n+1)$ is true.

By adding one biconnected component B_j to G will result a new graph G'. Let us consider one common vertex $u \in V(B_i) \cup V(B_j)$. Since the case $u \in V(B_i)$ has an even degree by the inductive hypothesis and the case $u \in V(B_j)$ also have an even degree since we are adding a biconnected component B_j such that $\forall u \in V(B_j) : d(u)$ is even. Thus $u \in V(B_i) \cup V(B_j)$ have an even degree. This holds for all verices that is also in other blocks in B_j. Thus vertices that are only in $V(B_j)$ have even degrees and *vertices* that are also in other blocks have even degrees. Thus $\forall v \in V(G) : d(v)$ is even.

■

3.

Claim: $1 \leq \kappa(G) \leq 1 \implies \kappa(G) = 1$, $1 \leq \kappa'(G) \leq 2$.

Proof:

Since there is a non-empty set of articulation vertices, we can remove one vertex to disconnect graph G. Thus $\kappa(G) = 1$.

For $\kappa'(G)$, consider 3 exhaustive cases for an articulation vertex u. Since the min degree is 3 and the max degree is 5, u can have degree 3, 4, or 5.

(1). u have degree 3 and $u \in V(B_i)$ and $V(B_j)$.

There are two exhaustive cases: (1). $d(u) = 1$ when $u \in V(B_i)$ and $d(u) = 2$ when $u \in V(B_j)$. (2). $d(u) = 2$ when $u \in V(B_i)$ and $d(u) = 1$ when $u \in V(B_j)$. Then $\kappa_1(G) \leq \max(\min d(u)$ in these 2 cases$) = 2$ since cutting all the edges of u in one block will disconnect this block with another block. Taking the maximum of all the possibilities gives us the upperbound for $\kappa_1(G)$.

(2). have degree 4 and $u \in V(B_i)$ and $V(B_j)$.

There are three exhaustive cases: (1). $d(u) = 1$ when $u \in V(B_i)$ and $d(u) = 3$ when $u \in V(B_j)$. (2). $d(u) = 2$ when $u \in V(B_i)$ and $d(u) = 2$ when $u \in V(B_j)$. (3). $(u) = 3$ when $u \in V(B_i)$ and $d(u) = 1$ when $u \in V(B_j)$. Then $\kappa_2(G) \leq \max(\min d(u)$ in these 3 cases$) = 2$ since cutting all the edges of u in one block will disconnect this block with another block. Taking the maximum of all the possibilities gives us the upperbound for $k_2(G)$.

(3). u have degree 5 and $u \in V(B_i)$ and $V(B_j)$.

There are four exhaustive cases: (1). $d(u) = 1$ when $u \in V(B_i)$ and $d(u) = 4$ when $u \in V(B_j)$. (2). $d(u) = 2$ when $u \in V(B_i)$ and $d(u) = 3$ when $u \in V(B_j)$. (3). $(u) = 3$ when $u \in V(B_i)$ and $d(u) = 2$ when $u \in V(B_j)$. (4). $(u) = 4$ when $u \in V(B_i)$ and $d(u) = 1$ when $u \in V(B_j)$ Then $\kappa_3(G) \leq \max(\min d(u)$ in these 4 cases$) = 2$ since cutting all the edges of u in one block will disconnect this block with another block. Taking the maximum of all the possibilities gives us the upperbound for $\kappa_3(G)$

Therefore the upperbound for $\kappa'(G)$ is 2. For the lowerbound, there exists cases as we dicussed above that could disconnect two blocks by moving one edge. Thus $\kappa'(G) \geq 1$. Therefore $1 \leq \kappa'(G) \leq 2$. ■

4.

Proof:

First we replace each $e = (u, v) \in E(G)$ with directed edges $f = (u \to v)$ and $h = (v \to u)$ and consider a network flow from x to y. Let x denote the source vertex and y denote the sink vertex. Let us assign weight of 1 to each edge on G.

Let F denote the maximal flow on G. Since the capacity of each edge on G is 1, units of flow correspond to pairwise edge-disjoint x,y path in G. Then a flow of value k correspnds to a set of k iedps.

Let the source and sink partions be S and T. Deleteing them will make it impossible to reach from x to y. The size of the set = cap(S,T)

By the Max flow Min cut Theorem we have

$$\lambda'(x, y) \geq maxval(f) = mincap(S, T) \geq k'(x, y)$$

Since $k'(x, y) \geq \lambda'(x, y)$ always hold, then equality must hold.

5.

Proof:

In order to derive $König - Egerváry's$ theorem from Menger's theorem, let us consider the bipartite graph G consisted of X and Y. Let us add vertex x to Y and add vertex y to X. Then connect edges in X to x and edges in Y to y. In order to prove $König - Egerváry's$ theorem, we need to show $\alpha'(G)$ (the size of maximum match) corresponds to $\lambda(x,y)$, and $\beta(G)$ (the size of the smallest vertex cover) corresponds to $\kappa(x,y)$.

Let us first consider $\alpha'(G)$ and $\lambda(x,y)$. If we remove the endpoints of all the x,y internally disjoint paths, we obtain a set of edges that share no common endpoints (by the definition of internally disjoint paths). The set of edges that share no common endpoints is a matching of graph G. Since the size of a maximum matching is greater than or equal to any other matching in G, we have $\alpha'(G) \geq \lambda(x,y)$.

Next let us consider $\beta(G)$ and $\kappa(x,y)$. In order to break all x-y internally disjoint paths, an x-y separator needs to contain at least an end-point of an edge. Thus the size of an x-y separator is at least the size of an vertex cover of G. Then we have $\kappa(x,y) \geq \beta(G)$.

Therefore we have the following inequality with Menger's theorem:

$$\alpha'(G) \geq \lambda(x,y) = \kappa(x,y) \geq \beta(G)$$

Since edges in the Maximum cover are disjoint, no two edges share an endpoint. Thus each vertex v covers at most one edge. Thus $\alpha'(G) \leq \beta(G)$. Since we have derive $\alpha'(G) \geq \beta(G)$ above, we have shown $\alpha'(G) = \beta(G)$ ■

6.

A giant component emerges when $np \to c > 1$. Then we have $p > \frac{1}{n} > \frac{1}{1000} = \frac{1}{999}$.

7.

A giant component emerges when $np \to c > 1$. Then we have $10^{-3} > \frac{1}{n} \implies n > \frac{1}{10^{-3}} = 1001$. Thus we need to have at least 1001 vertices.

8.

We have the following formula for β:

$$C_v(\beta) \approx C_v(0) \times (1-\beta)^3$$

Since $C_v(0) \approx \frac{3}{4}$, we have $\frac{1}{3} = \frac{3}{4} \times (1-\beta)^3$. Thus we have $\beta \approx 0.237$.

Then we have $|V| = 2000000000$ and $|E| = 500000000000$. Thus we have the sum of degree = 250000000000. Then we have the average degree for each node $k = 125$, and $N = |V| = 2000000000$ by definition. 9.

First we compute the proportion of infected population at time infinity with assumption $s_0 = 1$:

$$s(\infty) - s(0) = \frac{log(s(\infty))}{R_0}, s(0) = 1, R_0 = 2.5$$

$$s(\infty) = \frac{log(s(\infty))}{2.5} + 1$$

We have $s(\infty) \approx 0.10735$. Thus we have $r(\infty) \approx 0.893$. Thus there will be 893 people be infected.

10.

If there are only 1 person died and the death rate is 1%, we have in total 100 people being infected. Thus we have $r(\infty) = \dfrac{100}{1000} = 0.1$. Thus $s(\infty) = 0.9$. Then we have

$$R_0 = \frac{log(\frac{s(\infty)}{s(0)})}{s(\infty) - s(0)}$$

Thus $R_0 \approx 1.0536$.