Graph Theory HW 7
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1. We prove this using induction on the number of vertices of the directed tree.

Base Case: P(1): trivial to see

Inductive hypothesis: Suppose there exists bijective mapping $f' : V(T) \to \{1, 2, \ldots, n-1\}$ such that for every directed edge $(u \to v) \in E(T)$, $f(u) < f(v)$ for G', prove such relationship holds for G with n vertices.

We know there is a leaf node in the directed tree. Removing the parent leaves a directed tree G' with one less vertex. From our induction hypothesis, G' has a a bijection $f' : V(T) \to \{1, 2, \ldots, n-1\}$ with desired property. Then we can create the bijection $f : V(T) \to \{1, 2, \ldots, n\}$ by setting f(v) = f'(v) when $v \neq x$ and $f(x) = n$.

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2.**Proof:** By Konig-Egervary theorem, we know the size of maximum matching M = size of minimum vertex cover C. Now we need to show $C \geq \dfrac{|E(G)|}{|\Delta(G)|}$ Since even if all vertices are of degree $\Delta(G)$, we still need to cover $|E(G)|$ edges. Thus $C \geq \dfrac{|E(G)|}{|\Delta(G)|}$. Therefore M $\geq \dfrac{|E(G)|}{|\Delta(G)|}$.

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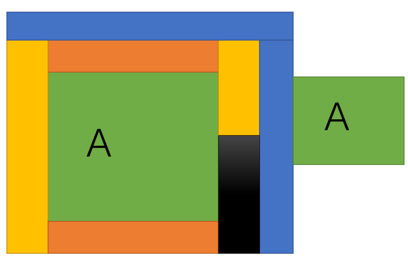
3. **Disprove**. A counter example would be a k_5 attached to a path with length 100000000000000000000000. we then have average degree of 2 with a really small decimal, and let us consider that to be a 3 with out losing generality. 3+1 = 4 but we know $\chi(K_5) = 5$. Thus the statement is false. ■

4. We can have the following example that cannot be 4-colored.



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5. **Proof**. Since we are performing a maximal clique decomposition on G and $\forall v \in V(G), v \in S_i S_j$, there exits H such that $L(H) =$ G with there's no 1 degree vertex in H. Since $\forall S_i \in \{S_1, S_2, ..., S_n\}$, $|S_i|$ is even, we have $\forall v \in H$, $d(v)$ is even since every vertex in H becomes a clique after the maximal clique decomposition. Therefore, we know H is Eulerian. Since H is Eulerian and L(H) = G, we know G is guaranteed to have a Hamiltonian cycle.

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6. **Proof**. In order to prove the line graph of G is Hamiltonian $\iff$ G contains a closed trail T, where the vertices in T form a vertex cover on G, we first show G contains a closed trail T $\implies$ the line graph of G is Hamiltonian. If there exists a closed trail, which is a cycle, that covers all the edges in G, then it connects the cliques in L(G) as a cycle since there is a cycle connecting all the cliques formed in the line graph. Since all cliques must have Hamiltonian paths which can start from any vertices and

end with an arbitrary vertex, then there must be a Hamiltonian path from vertex v connecting to some other clique to the vertex u connecting to another clique Since we know these cliques form a cycle, we know there must exist a Hamiltonian Cycle.

In order to prove the other direction, we can find one such trail by finding the corresponding cliques if we can form cliques in the line graph. We know these vertices will cover the edges since is is a Hamiltonian Cycle.

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7. **Proof:** In order to prove H is Hamiltonian iff G is Hamiltonian, we need to show (i). G is Hamiltonian $\implies$ H is Hamiltonian. (ii). H is Hamiltonian $\implies$ G is Hamiltonian.

It is trivial to prove G is Hamiltonian $\implies$ H is Hamiltonian since adding an edge to a Hamiltonian graph G will not break the Hamiltonian cycle in G. Therefore H is Hamiltonian.

In order to prove H is Hamiltonian $\implies$ G is Hamiltonian, we use induction on the number of vertices $|V(G)|$.

Base Case: $P(3)$. The minimal Hamiltonian graph in this case is C_3. Let S denote some elements in the set (S could also be empty) $\{(u,v) : u, v \in V(C_3) \wedge (u,v) \notin E(C_3)\}$. Thus we know $V(H) = V(C_3)$ and $E(H) = E(C_3) \cup S$. Removing all edges (u,v) with $deg(u) + deg(v) \geq V(C_3)$ will result in the graph C_3, which is Hamiltonian.

Inductive Step: Assume $P(k)$ is valid, where $3 \leq k < n$, prove $P(n)$ is valid.

Let us assume G is not Hamiltonian. Since H is Hamiltonian and we are removing one edge from the graph H to obtain graph G, G is a maximal Hamiltonian graph (adding edge (u,v) must result in a Hamiltonian graph). The other edges in G except for (u,v) must form a Hamiltonian path $v_1v_2...v_n$ with $u = v_1$ and $v = v_n$. For each i where $2 \leq i \leq n$, Consider two possible edges from v_1 to v_i and from v_{i-1} to v_n. At most one of these two edges can be represented in H, otherwise the cycle $v_1v_2..v_{i-1}v_nv_{n-1}v_i$ would be a Hamiltonian cycle. Thus the total number of edges incident to either v_1 or v_n is at most equal to the number of choice $i = n - 1$. Therefore the total number of edges $(degv_1 + degv_n) \geq n$. Since the vertex in G are at most equal to the degrees in H, G must be Hamiltonian.

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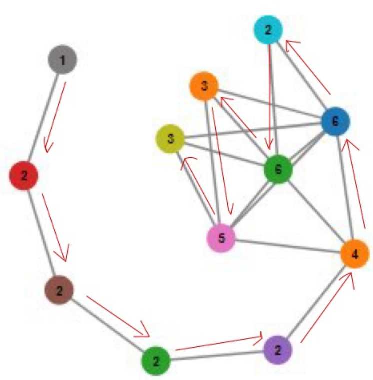
8. We can realize such a graph from the graph sequence given, where the label of the vertex represents the degree of the vertex. We can find one Hamiltonian path following the red Arrow.



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9. **Justification:** The first graph is the graph H. The graph on the right is L(H).

1. 2-connected: break the edge 5 and 4 will isolate vertex b. —V(G)— $\geq$ 3: Trivial

2. H is not Eulerian: $deg(a) = 3$ is odd, and therefore it is not Eulerian.

3. L(H) is Hamiltonian: The smallest degree of all vertices in L(H) is 3. Therefore for any two arbitrary vertices in G we have sum of their degrees $> |V(G)|$. Therefore we know any two non-adjacent vertices in L(H) have sum of degrees $> |V(G)|$. According to **Ore's Theorem**, this is a sufficient condition of L(H) is Hamiltonian.