Assignment 1 of MATP4820

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Problem 1

Compute the gradient and Hessian matrix of the function $f(x_1, x_2) = 25(x_2 - x_1^2)^2 + (1 - x_1)^2$. Prove that $(1, 1)$ is the only local minimizer of this function, and that the Hessian matrix at $(1, 1)$ is positive definite.

To compute the gradient of $f(x_1, x_2)$, we compute its partial derivative with respect to x_1 and x_2.

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_1} &= 50(x_2 - x_1^2) \cdot -2x_1 + 2(1 - x_1) \cdot -1 \\ &= -100x_2(x_2 - x_1^2) - 2(1 - x_1) \\ &= -100x_1x_2 + 100x_1^3 - 2 + 2x_1\end{aligned}$$

$$\begin{aligned}\frac{\partial f(x_1, x_2)}{\partial x_2} &= 50(x_2 - x_1^2) \\ &= 50x_2 - 50x_1^2\end{aligned}$$

Therefore, the gradient is:

$$\nabla f(x) = \begin{bmatrix} -100x_1x_2 + 100x_1^3 - 2 + 2x_1 \\ 50x_2 - 50x_1^2 \end{bmatrix}$$

To compute the Hessian matrix. we compute the 2^{nd} degree partial derivative.

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1^2} = -100x_2 + 300x_1^2 + 2$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_1 x_2} = \frac{\partial^2 f(x_1, x_2)}{\partial x_2 x_1} = -100x_1$$

$$\frac{\partial^2 f(x_1, x_2)}{\partial x_2^2} = 50$$

Therefore, the Hessian matrix is:

$$\nabla^2 f(x) = \begin{bmatrix} -100x_2 + 300x_1^2 + 2 & -100x_1 \\ -100x_1 & 50 \end{bmatrix}$$

To find all the possible local minimizers, we let $\nabla f(x) = 0$. Thus we have the following:

$$\begin{cases} -100x_1x_2 + 100x_1^3 - 2 + 2x_1 & = 0 \\ 50x_2 - 50x_1^2 & = 0 \end{cases}$$

Since $50x_2 - 50x_1^2 = 0$, we have $50(x_2 - x_1^2) = 0$. Thus $x_1 = x_2 = 0$ or $x_1 = x_2 = 1$

if $x_1 = x_2 = 0$, $-100x_1x_2 + 100x_1^3 - 2 + 2x_1 = -2 \neq 0$. Thus we only have point $(1, 1)$ and $-100 + 100 - 2 + 2 = 0$. Therefore the 1^{st} order condition is satisfied. For the sufficient second-order condition, we have:

$$\nabla^2 f(1, 1) = \begin{bmatrix} 202 & -100 \\ -100 & 50 \end{bmatrix}$$

$$\det(\begin{bmatrix} 202 - \lambda & -100 \\ -100 & 50 - \lambda \end{bmatrix}) = 0$$

$$\lambda^2 - 252\lambda + 100 = 0$$

$$(\lambda - 126)^2 = 15776$$

$$\lambda = 126 \pm 4\sqrt{986} > 0$$

Therefore Hessian $= \nabla^2 f(x)$ is positive definite.

Since $\nabla f(x) = 0$ and $\nabla^2 f(x)$ is positive definite, we have point $(1, 1)$ as our local minimizer.

Problem 2

Recall that a set $X \subseteq \mathbb{R}^n$ is convex if

$$\lambda\mathbf{x} + (1-\lambda)\mathbf{y} \in X,\ \forall\, \mathbf{x},\mathbf{y} \in X,\ \forall\, \lambda \in [0,1],$$

and that a function $f : \mathbb{R}^n \to \mathbb{R}$ is convex if

$$f(\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \le \lambda f(\mathbf{x}) + (1-\lambda) f(\mathbf{y}),\ \forall\, \mathbf{x},\mathbf{y} \in \mathbb{R}^n,\ \forall\, \lambda \in [0,1].$$

[**Only for 4820 students**] Let $f(x_1, x_2) = x_1^2 + cx_1x_2 + \frac{1}{2}x_2^2$ where c is a real number. Give a value of c such that f is not convex, and give a value of c such that f is convex. [*Hint*: for a quadratic function $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{a}^\top\mathbf{x}$, where $\mathbf{Q}$ is a symmetric matrix, it is convex if and only if its Hessian matrix $\nabla^2 f(\mathbf{x}) = \mathbf{Q}$ is positive semidefinite.]

Let $Q = \begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}$ and $a = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$. Let $f(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\top \mathbf{Q}\mathbf{x} + \mathbf{a}^\top\mathbf{x}$, we can get the following by expansion.

$$\begin{aligned}
f(x) &= \frac{1}{2}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} q_1 & q_2 \\ q_3 & q_4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} a_1 & a_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\
&= \frac{1}{2}\begin{bmatrix} q_1x_1 + q_3x_2 & q_2x_1 + q_4x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + a_1x_1 + a_2x_2 \\
&= \frac{1}{2}(q_1x_1^2 + q_3x_1x_2 + q_2x_1x_2 + q_4x_2^2) + a_1x_1 + a_2x_2
\end{aligned}$$

By setting the above equation to $x_1^2 + c_1x_1x_2 + \frac{1}{2}x_2^2$, we have $q_1 = 2$, $q_2 + q_3 = 2c$, $q_4 = 1$, and $a_1 = a_2 = 0$.

Let us assume Q is a symmetric matrix, to satisfy the condition that $q_2 + q_3 = 2c$, we let:

$$Q = \begin{bmatrix} 2 & c \\ c & 1 \end{bmatrix}$$

Since $\nabla^2 f(\mathbf{x}) = \mathbf{Q}$ and has to be positive semidefinite if $f(x)$ is convex, the eigenvalues of Q must be greater than or equal to 0.

$$
\begin{aligned}
\det(\begin{bmatrix} 2-\lambda & c \\ c & 1-\lambda \end{bmatrix}) &= 0 \\
2 - (\frac{c}{2} - \lambda)^2 &= 0 \\
(2-\lambda)(1-\lambda) - c^2 &= 0 \\
\lambda^2 - 3\lambda + 2 - c^2 &= 0 \\
\lambda &= \frac{3 \pm \sqrt{1+4c^2}}{2}
\end{aligned}
$$

Therefore $f(x)$ is convex if $\frac{3-\sqrt{1+4c^2}}{2} \geq 0$. $f(x)$ is convex when c is 0. $f(x)$ is not convex when c is 4.

Problem 3

Is the sequence $x^{(k)} = 2 + (\frac{3}{2})^{-2^k}$ for $k = 1, 2, \ldots,$ Q-quadratically convergent to a finite real number? If yes, give the limit and prove it; if not, explain why.

Yes. It converges to a finite real number Q-quadratically.
$Z^* = \lim_{k\to\infty} x^{(k)} = 2$ since $\lim_{k\to\infty}(\frac{3}{2})^{-2^k} = 0$
$\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*)||^2} = \frac{2+(\frac{3}{2})^{-2^{k+1}}-2}{(2+(\frac{3}{2})^{-2^k}-2)^2} = \frac{1}{\frac{3}{2}^{2^{k+1}}} \cdot \frac{1}{\frac{3}{2}^{-2^{k+1}}} = \frac{\frac{3}{2}^{2^{k+1}}}{\frac{3}{2}^{k+1}} = 1$
Thus $\lim_{k\to\infty} \frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = 1$
Let $M = 1$, we then have $\{x^{(k)}\}_{k=1}^{\infty}$ Q-quadratically converges.

Problem 4

Consider the sequence $\{x^{(k)}\}_{k=1}^{\infty}$ defined by

$$x^{(k)} = \begin{cases} (\frac{3}{2})^{-2^k}, & \text{if } k \text{ is odd} \\ \frac{1}{k+1}x^{(k-1)}, & \text{if } k \text{ is even} \end{cases}$$

1. Is this sequence Q-quadratically convergent? Justify your answer.

   We first check Z^*. $\lim_{k\to\infty} x^k = \begin{cases} 0, & \text{if } k \text{ is odd} \\ \frac{1}{k+1}\cdot 0 = 0, & \text{if } k \text{ is even} \end{cases}$
   Therefore $Z^* = 0$
   If k is odd, we have:
   $\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = \frac{\frac{1}{k+1+1}x^{(k+1-1)}}{\frac{3}{2}^{-2^k}} = \frac{\frac{1}{k+1+1}\cdot(\frac{3}{2})^{-2^k}}{\frac{3}{2}^{-2^k}} = \frac{\frac{1}{k+2}}{(\frac{3}{2})^{-2^k}} = \frac{(\frac{3}{2})^{2^k}}{k+2}$
   Therefore $\lim_{k\to\infty} \frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = \infty$.
   If k is even, we have:
   $\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = \frac{\frac{3}{2}^{-2^{k+1}}}{\frac{1}{k+1}x^{(k-1)}} = \frac{1}{(\frac{1}{k+1})^2}\cdot\frac{(\frac{3}{2})^{-2^{k+1}}}{(\frac{3}{2})^{-2k}} = (k+1)^2\cdot(\frac{3}{2})^{-2^k}$
   Thus $lim_{k\to\infty}\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = 0$

   Therefore it is not Q-quadratically convergent.

2. Is this sequence R-quadratically convergent? Justify your answer.

   We have
   $$x^{(k)} = \begin{cases} (\frac{3}{2})^{-2^k}, & \text{if } k \text{ is odd} \\ \frac{1}{k+1}(\frac{3}{2})^{-2^{k-1}}, & \text{if } k \text{ is even} \end{cases}$$

   As k increases, $(\frac{3}{2})^{2^k}$ increases. Therefore $(\frac{3}{2})^{-2^k}$ decreases. Thus we have $x^{(k)}$ upper-bounded by $(\frac{3}{2})^{-2^{k-1}}$ since $(\frac{3}{2})^{-2^{k-1}} \geq \frac{1}{k+1}(\frac{3}{2})^{-2^{k-1}}$ and $(\frac{3}{2})^{-2^{k-1}} \geq (\frac{3}{2})^{-2^k}$.

   Let the sequence $\{v^{(k)}\}_{k=1}^{\infty}$ defined by $v^{(k)} = (\frac{3}{2})^{-2^{k-1}}$. Then we prove $\{v^{(k)}\}_{k=1}^{\infty}$ Q-quadratically converges.

   $Z^* = \lim_{k\to\infty} v^{(k)} = 0$.

   $\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*)||^2} = \frac{(\frac{3}{2})^{-2^k}}{((\frac{3}{2})^{-2^{k-1}})^2} = \frac{1}{\frac{3}{2}^{2^k}}\cdot\frac{1}{\frac{3}{2}^{-2^k}} = \frac{\frac{3}{2}^{2^k}}{\frac{3}{2}^k} = 1$
   Thus $\lim_{k\to\infty}\frac{||Z^{k+1}-Z^*||}{||Z^k-Z^*||^2} = 1$
   Let $M = 1$, we then have $\{v^{(k)}\}_{k=1}^{\infty}$ Q-quadratically converges. Therefore x^k R-quadratically converges.