Homework 4

There are three homework problems. The expectation is that students will submit very high quality proofs. The problem solutions must be written in LaTeX and compiled to a pdf file. Follow the instructions on LMS for how to submit the problem.

1. Prove that $14^n - 5^n$ is a multiple of 9 for all $n \in \mathbb{N}$. *hint: show that* $14^n - 5^n = 9m$ *for some integer* m.

   Proof. Suppose $n \in \mathbb{N}$. Let $P(n)$ be the statement $14^n - 5^n$ is a multiple of 9 for all $n \in \mathbb{N}$. In other words there exists an $m \in \mathbb{Z}$ such that $14^n - 5^n = 9m$ for all $n \in \mathbb{N}$. We prove this by mathematical induction.

   We first show that $P(1)$ is true by substituting $n = 1$ into the LHS formula and showing it is equivalent to the RHS formula.

   $$14^1 - 5^1 = 9 = 9(1)$$

   Therefore we have shown the LHS is equivalent to RHS when $n = 1$.

   Next we let $k \in \mathbb{N}$ and assume $P(k) : 14^k - 5^k = 9m, m \in \mathbb{Z}$ is true and try to prove $P(k+1) : 14^{k+1} - 5^{k+1} = 9m, m \in \mathbb{Z}$ is true. This follows by adding $13(14^n) - 4(5^n)$ to both sides of $P(k)$ and simplify to get $P(k+1)$.

   $$\begin{aligned}
   14^n - 5^n + 13(14^n) - 4(5^n) &= 9m + 13(14^n) - 4(5^n) \\
   14^{n+1} - 5^{n+1} &= 9m + 13(14^n) - 4(5^n) \\
   \because 14^k - 5^k &= 9m \\
   \therefore 14^k &= 9m + 5^k \\
   14^{n+1} - 5^{n+1} &= 9m + 13(14^n) - 4(5^n) = 9m + 13(9m + 5^k) - 4(5^n) \\
   14^{n+1} - 5^{n+1} &= 9m + 13(9m) - 4(5^n) + 13(5^n) - 4(5^n) \\
   14^{n+1} - 5^{n+1} &= 9m + 9(13m) + 9(5^n) \\
   Let \quad m_1 \quad be \quad & m + 13m + 5^n, \\
   14^{n+1} - 5^{n+1} &= 9m_1, m_1 \in \mathbb{Z}.
   \end{aligned}$$

   Therefore we have shown $P(1)$ is true and if $P(k)$ is true $P(k+1)$ is true. Thus we have proved $P(n)$ is true.

2. *typo fixed at 11 PM on 10/6*** Prove that $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all $n \in \mathbb{N}$ with $n \geq 2$.

   Proof. Suppose $n \in \mathbb{N}$. Let $P(n)$ be the statement $\left(1 - \frac{1}{2^2}\right)\left(1 - \frac{1}{3^2}\right)\left(1 - \frac{1}{4^2}\right)\cdots\left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}$ for all $n \in \mathbb{N}$ with $n \geq 2$. We prove this by mathematical induction.

   We first show that $P(2)$ is true by substituting $n = 2$ into the LHS formula and showing it is equivalent to the RHS formula.

$$\left(1-\frac{1}{2^2}\right)=\frac{3}{4}=\frac{2+1}{2\times 2}$$

Therefore we have shown the LHS is equivalent to RHS when $n=2$.

Next we let $k\in\mathbb{N}$ and assume $P(k):\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{k^2}\right)=\frac{k+1}{2k}$ is true and try to prove $P(k+1):\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)=\frac{(k+1)+1}{2(k+1)}$ is true. This follows by multiplying $\left(1-\frac{1}{(k+1)^2}\right)$ to both sides of $P(k)$ and simplify to get $P(k+1)$.

$$\begin{aligned}
\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)&=\frac{k+1}{2k}\times\left(1-\frac{1}{(k+1)^2}\right)\\
\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)&=\frac{k+1}{2k}-\frac{1}{2k}\cdot\frac{1}{k+1}\\
\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)&=\frac{k^2+2k+1}{2k(k+1)}-\frac{1}{2k(k+1)}\\
\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)&=\frac{k^2+2k}{2k(k+1)}\\
\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{3^2}\right)\left(1-\frac{1}{4^2}\right)\cdots\left(1-\frac{1}{(k+1)^2}\right)&=\frac{(k+1)+1}{2(k+1)}
\end{aligned}$$

Therefore we have shown $P(1)$ is true and if $P(k)$ is true $P(k+1)$ is true. Thus we have proved $P(n)$ is true.

3. Define $H_j=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{j}=\sum_{j=1}^{n}\frac{1}{j}$. Use mathematical induction to prove that $H_{2^n}\geq 1+\frac{n}{2}$ is true for all $n\in\mathbb{N}$.

Proof. Suppose $n\in\mathbb{N}$. Let $P(n)$ be the statement $1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^n}\geq 1+\frac{n}{2}$ for all $n\in\mathbb{N}$. We prove this by mathematical induction.

We first show that $P(1)$ is true by substituting $n=1$ into the LHS formula and showing it is equivalent to the RHS formula.

$$1+\frac{1}{2}\geq 1+\frac{1}{2}$$

Therefore we have shown the LHS is equivalent to RHS when $n=1$.

Next we let $k\in\mathbb{N}$ and assume $P(k):1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}\geq 1+\frac{k}{2}$ is true and try to prove $P(k+1):1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^{k+1}}\geq 1+\frac{(k+1)}{2}$ is true. This follows by adding $\frac{1}{2^{k+1}}$ to both sides of $P(k)$ and simplify to get $P(k+1)$.

$$
\begin{aligned}
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &\geq 1+\frac{k}{2}+\frac{1}{2^{k+1}} \\
\because \frac{1}{2^{k+1}} &\leq \frac{1}{2}, \forall k \in \mathbb{N} \\
\therefore 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &\geq 1+\frac{k}{2}+\frac{1}{2} \\
1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{2^k}+\frac{1}{2^{k+1}} &\geq 1+\frac{k+1}{2}
\end{aligned}
$$

Therefore we have shown $P(1)$ is true and if $P(k)$ is true $P(k+1)$ is true. Thus we have proved $P(n)$ is true.

Note: Do not cite axioms of real numbers (sec 3.2) on each step of your proof.