Homework 1
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There are three problems in this homework set. The expectation is that students will submit a very high quality proof for each problem. The problem solutions must be written in LaTeX. Follow the instructions on LMS for how to submit the homework assignment.

1. Prove the following statement. Suppose a, b, c, d are real numbers. If $a < b < c < d$ then $(b, c) \subseteq (a, c) \cap (b, d)$.

   Assume (b, c) is a non-empty set, else the result follows because the empty set is a subset of every set. Let $x \in (b, c)$.Since we have $x \in (b, c)$ we have $b < x < c$. Because $a < b$ we know that $a < x < c$. Hence $x \in (a, c)$. Likewise since $c < d$ we have that $b < x < d$. Hence $x \in (b, d)$ and $x \in (a, c)$, thus we know that $x \in (a, c) \cap (b, d)$. Since our choice of x is arbitrary, this holds for any $x \in (b, c)$. Thus we have shown $(b, c) \subseteq (a, c) \cap (b, d)$.

2. Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

   In order to prove $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, we need to show that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   In order to prove $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$, assume $A \cap (B \cup C)$ is a non-empty set, else the result follows and let $x \in A \cap (B \cup C)$ which means $x \in A$ and $x \in (B \cup C)$. Since $x \in (B \cup C)$, $x \in B$ or $x \in C$. Consider two exhaustive cases.

   If $x \in B$, then $x \in A$ and $x \in B$.
   If $x \in C$, then $x \in A$ and $x \in C$.

   Therefore we have shown that if $x \in A \cap (B \cup C)$, then ($x \in A$ and $x \in B$) or ($x \in A$ and $x \in C$) which means $x \in (A \cap B) \cup (A \cap C)$. Hence $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$.

   In order to prove $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$, assume $(A \cap B) \cup (A \cap C)$ is a non-empty set, else the result follows and let $x \in (A \cap B) \cup (A \cap C)$. Then $x \in A \cap B$ or $x \in A \cap C$. Consider two exhaustive cases.

   If $x \in A \cap B$, then $x \in A$ and $x \in B$. Therefore $x \in A$ and $x \in B \cup C$.
   If $x \in A \cap C$, then $x \in A$ and $x \in C$. Therefore $x \in A$ and $x \in B \cup C$.

   Therefore for all $x \in (A \cap B) \cup (A \cap C)$, we have shown that if $x \in (A \cap B) \cup (A \cap C)$ then $x \in A$ and $x \in B \cup C$ which means $x \in A \cap (B \cup C)$. Hence we have proved that $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$.

   We have proved that $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$ and $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$. Therefore $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

3. Let A and B be subsets of a universal set U. Prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

In order to prove $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$, we need to show $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ and $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

In order to prove $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$, assume $(A \setminus B) \cup (B \setminus A)$ is a non-empty set, else the result follows and let $x \in (A \setminus B) \cup (B \setminus A)$ which means $x \in (A \setminus B)$ or $x \in (B \setminus A)$. Consider two exhaustive cases.

If $x \in (A \setminus B)$, then $x \in A$ and $x \notin B$. Since $x \in A$, $x \in A \cup B$. In order to prove $x \in A \cup B$ and $x \notin A \cap B$, we need to prove $x \notin A \cap B$.

Suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \notin B$, there's a contradiction. Therefore $x \notin A \cap B$.

Hence $x \in A \cup B$ and $x \notin A \cap B$ which means $x \in (A \cup B) \setminus (A \cap B)$ when $x \in (A \setminus B)$.

If $x \in (B \setminus A)$, then $x \in B$ and $x \notin A$. Since $x \in B$, $x \in A \cup B$. In order to prove $x \in A \cup B$ and $x \notin A \cap B$, we need to prove $x \notin A \cap B$.

Suppose $x \in A \cap B$, then $x \in A$ and $x \in B$. Since $x \notin A$, there's a contradiction. Therefore $x \notin A \cap B$.

Hence $x \in A \cup B$ and $x \notin A \cap B$ which means $x \in (A \cup B) \setminus (A \cap B)$.

Therefore we have prove that if $x \in (A \setminus B) \cup (B \setminus A)$, then $x \in (A \cup B) \setminus (A \cap B)$. Hence we have showed that $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$.

In order to prove $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$, assume $(A \cup B) \setminus (A \cap B)$ is a non-empty set, else the result follows and let $x \in (A \cup B) \setminus (A \cap B)$ which means $x \in (A \cup B)$ and $x \notin (A \cap B))$. Because $x \in (A \cup B)$, $x \in A$ or $x \in B$. Consider two exhaustive cases.

If $x \in A$, we need to prove $x \notin B$.

suppose $x \in B$, then $x \in A$ and $x \in B$. Hence $x \in A \cap B$ which leads to a contradiction because $x \notin A \cap B$. Therefore $x \notin B$.

Hence if $x \in A$ then $x \in A$ and $x \notin B$.
If $x \in B$, we need to prove $x \notin A$.

suppose $x \in A$, then $x \in A$ and $x \in B$. Hence $x \in A \cap B$ which leads to a contradiction because $x \notin A \cap B$. Therefore $x \notin A$.

Hence if $x \in B$ then $x \in B$ and $x \notin A$.

Therefore we have proved that for all $x \in (A \cup B)$, we have ($x \in A$ and $x \notin B$) or ($x \notin A$ and $x \in B$) which means $x \in (A \setminus B) \cup (B \setminus A)$. Hence $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$.

Therefore, we have proved $(A \setminus B) \cup (B \setminus A) \subseteq (A \cup B) \setminus (A \cap B)$ and $(A \cup B) \setminus (A \cap B) \subseteq (A \setminus B) \cup (B \setminus A)$. Hence $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

Note that proofs involving sets in this class must be done on an element level as presented both in lecture and in the book.