Homework 7
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1. Suppose $s_1 = 2$ and $s_{n+1} = \dfrac{1}{5}(2s_n + 7)$ for $n \geq 1$.
   (a) Prove that (s_n) is monotone
   (b) Prove that (s_n) is bounded.
   (c) Find $\lim s_n$.

(a). **Proof:** In order to prove (s_n) is monotone, we prove (s_n) is increasing. Suppose $n \in \mathbb{N}$. Let $P(n)$ be the statement $s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$. We prove this by mathematical induction.

We first show $P(1) : s_1 \leq s_2$ is true by substuting $n = 1$ into the LHS and showing $2 \leq \dfrac{1}{5}(2\times2+7) = \dfrac{11}{5}$. Thus $P(1) : s_1 \leq s_2$ holds.

Next we let $k \in \mathbb{N}$ and assume $p(k) : s_k \leq s_{k+1}$ is true and try to prove $p(k+1) : s_{k+1} \leq s_{k+2}$ is true.

$$\begin{aligned} s_{k+1} &\geq s_k \\ 2s_{k+1} &\geq 2s_k \\ 2s_{k+1} + 7 &\geq 2s_k + 7 \\ \frac{1}{5}(2s_{k+1} + 7) &\geq \frac{1}{5}(2s_k + 7) \\ s_{k+2} &\geq s_{k+1} \end{aligned}$$

Thus we have shown $P(n) : s_n \leq s_{n+1}$ for all $n \in \mathbb{N}$ is true.

(b). **Proof:** In order to prove (s_n) is bounded, we need to show there exists constants b and a such that $a \leq s_n \leq b$ for all $n \in \mathbb{N}$. Let s_n be defined by $s_1 = 1$ and $s_{n+1} = \dfrac{1}{5}(2s_n + 7)$, we need to show there exists a and b such that (i).$s_n \geq a$ and (ii).$s_n \leq b$.

Let $a = 0$, we have $s_1 \geq a$ because $2 \geq 0$. Since we know s_n is an increasing sequence in part(a), $s_n \geq s_1 \geq a$ for all $n \in \mathbb{N}$. Thus $s_n \geq a$ for all $n \in \mathbb{N}$.

Let $b = 3$. Suppose $n \in \mathbb{N}$. Let $P(n)$ be the statement $s_n \leq b$ for all $n \in \mathbb{N}$. We prove this by mathematical induction. We first show $P(1) : s_1 \leq b$ is true by substituting $n = 1$ into the LHS and showing $2 \leq 3$. Thus $P(1) : s_1 \leq b$ holds. Next we let $k \in \mathbb{N}$ and assume $p(k) : s_k \leq b$ is true and try to prove $p(k+1) : s_{k+1} \leq b$ is true.

$$\begin{aligned} s_k &\leq 3 \\ 2s_k &\leq 6 \\ 2s_k + 7 &\leq 13 \\ \frac{1}{5}(2s_k + 7) &\leq \frac{13}{5} \\ s_{k+1} &\leq \frac{13}{5} \leq 3 \leq b \end{aligned}$$

Thus we have shown there exists $a = 0$ and $b = 3$ such that $a \leq s_n \leq b$ for all $n \in \mathbb{N}$.

(c). **Proof:** Since we have shown s_n is bounded and monotone, we know by the monotone convergence theorem that there exists $s \in \mathbb{R}$ such that $\lim s_n = s$. Let s_n be defined by $s_1 = 1$ and $s_{n+1} = \frac{1}{5}(2s_n+7)$, then we have

$$\begin{aligned} \lim_{n\to\infty} s_{n+1} &= \lim_{n\to\infty} \frac{1}{5}(2s_n + 7) \\ \lim_{n\to\infty} s_{n+1} &= \frac{2}{5} \lim_{n\to\infty} s_n + \frac{7}{5} \\ s &= \frac{2}{5}s + \frac{7}{5} \\ \frac{3}{5}s &= \frac{7}{5} \\ s &= \frac{7}{3} \end{aligned}$$

Thus $\lim s_n = \frac{7}{3}$.

2. Let S be a bounded infinite set and let $x = \sup(S)$. Prove that if $x \notin S$, then x is an accumulation point of S.

Proof: Our goal here is to prove if $x \notin S$, then x is an accumulation point of S. Let $\gamma > 0$ be given. A deleted neighborhood of x is $N^*(x, \gamma) = (x - \gamma, x) \cup (x, x + \gamma)$ by definition. Since $x = \sup(S)$ and $x \notin S$, we have $x > s$ for all $s \in S$. Because $x - \gamma < x$, there exists $s' \in S$ such that $s' > x - \gamma$. Thus we have $x - \gamma < s < x$ which means $s' \in (x - \gamma, x)$. Therefore $s' \in (x - \gamma, x) \cup (x, x + \gamma)$ and $s' \in S$. Since S is an infinite set, $S \neq \emptyset$. Thus $(x - \gamma, x) \cup (x, x + \gamma) \cap S \neq \emptyset$ which means $N^*(x, \gamma) \cap S \neq \emptyset$. Hence we have shown x is an accumulation point of S if $x \notin S$.
Let (t_n) be a sequence of real numbers.
Define $A = \{t_n : n \in \mathbb{N}\}$ and note that A is the range of (t_n).