FOCS Quiz 1
Wang Xinshi

Exercise 1

(a) We will prove the contrapositive.

Suppose $I - T$ is not invertible. There exists non-zero $v \in V$ such that $Iv - Tv = 0$, which implies $Tv = v$. Applying T to both sides of this last equation, yields $TTv = Tv = v$. Continuing this, we see that for any positive integer n, $T^n v = v$. Therefore T^n can never be 0.

Now suppose $T^n = 0$. We have

$$\begin{aligned}(I-T)(I+T+\cdots+T^{n-1}) &= (I-T)\sum_{k=0}^{n-1} T^k \\ &= \sum_{k=0}^{n-1} T^k - \sum_{k=0}^{n-1} T^{k+1} \\ &= I + \sum_{k=1}^{n-1} T^k - \sum_{k=1}^{n} T^k \\ &= I + \sum_{k=1}^{n-1} T^k - T^n - \sum_{k=1}^{n-1} T^k \\ &= I - T^n \\ &= I\end{aligned}$$

Therefore $I - T$ is an inverse of $I + T + \cdots + T^{n-1}$, which implies the desired result.

(b) I wouldn't.

Exercise 2

Suppose λ is an eigenvalue of T and v a corresponding eigenvector. Then

$$\begin{aligned}0 &= (T-2I)(T-3I)(T-4I)v \\ &= (T^3 - 9T^2 + 26T - 24I)v \\ &= T^3 v - 9T^2 v + 26T - 24v \\ &= \lambda^3 v - 9\lambda^2 v + 26\lambda v - 24v \\ &= (\lambda^3 - 9\lambda^2 + 26\lambda - 24)v\end{aligned}$$

Therefore, because $v \neq 0$, $\lambda^3 - 9\lambda^2 + 26\lambda + 24 = 0$. But we can factor this to $(\lambda-2)(\lambda-3)(\lambda-4) = 0$. Thus λ is either 2, 3 or 4.

Exercise 3

Let $v \in V$. Then

$$0 = (T^2 - I^2)v = (T+I)(T-I)v$$

which implies $(T - I)v \in \operatorname{null}(T + I)$. We but need to prove that $T + I$ is injective and we will have $Tv = v$. Let $u, w \in V$ such that $(T+I)u = (T+I)w$. Then $Tu + u = Tw + w$ which implies $T(u-w) = w - u = -1(u-w)$. But -1 is not an eigenvalue of T, thus $u - w = 0$, implying $u = w$, as desired.

Exercise 4

Suppose $v \in V$. We have $v = (I-P)v + Pv$. Because $P - P^2 = 0$, it follows that $P(I-P)v = 0$. Therefore $(I-P)v \in \operatorname{null} P$. Obviously $Pv \in \operatorname{range} P$, thus $v \in \operatorname{null} P + \operatorname{range} P$ and we have $V \subset \operatorname{null} P + \operatorname{range} P$. The inclusion in the opposite direction is clearly true, therefore $V = \operatorname{null} P + \operatorname{range} P$.

To see why this is a direct sum, suppose $u \in \operatorname{null} P \cap \operatorname{range} P$. Because $u \in \operatorname{range} P$, there exists w such that $Pw = u$. But $u \in \operatorname{null} P$, so we must have

$$0 = Pu = P^2 w = Pw = u$$

Therefore $\operatorname{null} P \cap \operatorname{range} P = \{0\}$ and by 1.45 it is direct sum.

Exercise 5

Note that, for non-negative k, $(STS^{-1})^k = ST^kS^{-1}$, because the S's and S^{-1}'s cancel out (this can easily me proven by induction on k). Let $n = \deg p$ and $a_0, a_1, \ldots, a_n \in \mathbb{F}$ be the coefficients of p. Then

$$
\begin{aligned}
p(STS^{-1}) &= \sum_{k=0}^{n} a_k (STS^{-1})^k \\
&= \sum_{k=0}^{n} a_k ST^k S^{-1} \\
&= S(\sum_{k=0}^{n} a_k T^k) S^{-1} \\
&= Sp(T)S^{-1}
\end{aligned}
$$

Exercise 6

It is easy to see that $T^k u \in U$ for $u \in U$ and non-negative k (this can easily be proven by induction on k). Since $p(T)u$ is a sum of terms like this, multiplied by the coefficients of p, and U is closed under addition and scalar multiplication, then $p(T)u \in U$.

Exercise 7

Suppose 9 is an eigenvalue of T^2. Let v be a corresponding eigenvector. Then $(T^2 - 9I)v = 0$, which implies $(T - 3I)(T + 3I)v = 0$. If $(T + 3I)v = 0$ then -3 is an eigenvalue of T. Otherwise, $(T + 3I)v$ is an eigenvector of T and 3 is a corresponding eigenvalue.

Conversely, suppose ± 3 is an eigenvalue of T. Let v be a corresponding eigenvector. Then $T^2 v = T(\pm 3v) = (\pm 3)^2 v = 9v$, showing that 9 is an eigenvalue of T^2.

Exercise 8

Define $T \in \mathcal{L}(\mathbb{R}^2)$ by

$$
T(x, y) = \frac{1}{\sqrt{2}}(x - y, x + y)
$$

Then, for $(a, b) \in \mathbb{R}^2$, we have

$$
\begin{aligned}
T^4(a, b) &= \frac{1}{\sqrt{2}} T^3(a - b, a + b) \\
&= \frac{1}{2} T^2(-2b, 2a) \\
&= \frac{1}{2\sqrt{2}} T(-2a - 2b, 2a - 2b) \\
&= \frac{1}{4}(-4a, -4b) \\
&= -(a, b)
\end{aligned}
$$

Exercise 9

Let $c(z - \lambda_n) \cdots (z - \lambda_1)$ be a factorization of p. Then $c(T - \lambda_n I) \cdots (T - \lambda_1 I)$ is a factorization of $p(T)$. Since p is the polynomial of smallest degree such that $p(T)v = 0$, it follows that $(T - \lambda_j I) \cdots (T - \lambda_1 I)v \neq 0$, for $j < n$. Therefore, we have that $(T - \lambda_{n-1} I) \cdots (T - \lambda_1 I)v \neq 0$ is an eigenvector of T and λ_n the corresponding eigenvalue. Note that, by 5.20, the order of factorization can be changed, placing any other factor $(T - \lambda_j)$ in the beginning. This implies that all λ's are indeed eigenvalues of T.

Exercise 10

Note that $T^k v = \lambda^k v$. Let $n = \deg p$ and $a_0, a_1, \ldots, a_n \in \mathbb{F}$ be the coefficients of p. Then

$$
\begin{aligned}
p(T)v &= (\sum_{k=0}^{n} a_k T^k) v \\
&= \sum_{k=0}^{n} a_k T^k v \\
&= \sum_{k=0}^{n} a_k \lambda^k v \\
&= (\sum_{k=0}^{n} a_k \lambda^k) v \\
&= p(\lambda) v
\end{aligned}
$$

Exercise 11

Suppose α is an eigenvalue of $p(T)$. Let $c(z-\lambda_1)\cdots(z-\lambda_n)$ be a factorization of $p(z)-\alpha$. We have

$$p(T)-\alpha I = c(T-\lambda_1 I)\cdots(T-\lambda_n I)$$

Because $p(T)-\alpha I$ is not injective, it follows that, for some j, $T-\lambda_j I$ is not injective. Therefore λ_j is an eigenvalue of T. Since λ_j is a root of $p(z)-\alpha$, we have that $p(\lambda_j)=\alpha$.

The converse is the same as Exercise 10.

Exercise 12

Define $T \in \mathcal{L}(\mathbb{R}^4)$ by

$$T(x_1,x_2,x_3,x_4) = (x_2,x_3,x_4,-x_1)$$

Let $p \in \mathcal{P}(R)$ such that $p(x)=x^4$. Then -1 is an eigenvalue of $p(T)$, but p is always positive, therefore no eigenvalue λ of T satisfies $p(\lambda)=-1$.

Exercise 13

By 5.21, W is either $\{0\}$ or infinite-dimensional. Let U be a subspace of W invariant under T. Then $T|_U$ also has no eigenvalues. But $T|_U$ is also an operator on a complex vector space, therefore U is either $\{0\}$ or infinite-dimensional.

Exercise 14

Suppose V is finite-dimensional vector space. Let $v_1,\ldots,v_n$ be a basis of V. Define $T \in \mathcal{L}(V)$ by

$$\begin{aligned} Tv_j &= v_{j+1}, \text{ for } j=1,\ldots,n-1 \\ Tv_n &= v_1 \end{aligned}$$

T is clearly invertible, but $\mathcal{M}(T)$ with respect to same basis only has zeros in the diagonal.

Exercise 15

Suppose V is finite-dimensional vector space. Let $v_1,\ldots,v_n$ be a basis of V. Define $T \in \mathcal{L}(V)$ by

$$Tv_j = v_1 + \cdots + v_n$$

$\mathcal{M}(T)$ contains 1's in all its entries, but T is clearly not inveritible.

Exercise 16

Let $n = \dim V$. Define $\Psi \in \mathcal{L}(\mathcal{P}_n(\mathbb{C}), V)$ by

$$\Psi(p) = (p(T))v$$

for $p \in \mathcal{P}_n(\mathbb{C})$. One can easily verify that Ψ is linear. Since $\dim \mathcal{P}_n(\mathbb{C}) > \dim V$, by 3.23, there exists $p \in \mathcal{P}_n(\mathbb{C})$ such that $0 = \Psi(p) = (p(T))v$. The rest follows exactly as 5.21.

Exercise 17

This is almost the same as Exercise 16.

Exercise 18

Note that f can only output integer values. Thus, if f is not constant, there will be a jump discontinuity at some point. We will prove f is not constant.

If T is invertible, then the existence of an eigenvalue of T (guaranteed by 5.21) implies that $T-\lambda I$ is not surjective for some $\lambda \in \mathbb{F}$. Hence $f(0) = \dim \operatorname{range} T > \dim \operatorname{range}(T-\lambda I) = f(\lambda)$.

If T is not invertible, choose λ such that it is not an eigenvalue of T. Then, for any non-zero $v \in V$, $(T-\lambda I)v \neq 0$, showing that $T-\lambda I$ is injective and, therefore, surjective. Hence $f(0) = \dim \operatorname{range} T < \dim \operatorname{range}(T-\lambda I) = f(\lambda)$.

Exercise 19

5.20 implies that any two operators in $\{p(T) : p \in \mathcal{P}(\mathbb{F})\}$ commute. But this is obviously not true for $\mathcal{L}(V)$, because $\dim V > 1$. For example, let $v_1,\ldots,v_n$ be a basis of V. Define $S, R \in L(V)$ by

$$\begin{aligned} Sv_1 = v_2, Sv_j = 0 \text{ for } j=2,\ldots,n \\ Rv_1 = 0, Rv_j = v_j \text{ for } j=2,\ldots,n. \end{aligned}$$

Then $SRv_1 = 0$ but $RSv_1 = v_2$. Thus $SR \neq RS$.

Exercise 20

This follows directly from 5.27 and 5.26. $\hat{y}_i = 0$