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hw03

1.
Proof: Assume there exists M_{P1} and M_{P2} such that $M_{P1} \neq M_{P2}$. Consider one leaf node v for tree T. Since T has a perfect match, the parent vertex of v denoted as u could only have one childb(Otherwise some leaf vertices will not be saturated). Then we remove all the leaf vertices which gives us $\{e_1, e_2, ...e_n\}$. We repeat this process untill we have no vertices. Since in every iteration the edges between the leaf vertices and parent vertices are unique, the set $e_1, e_2, ..., e_n$ which is the set of edges in perfect match, is unique. Thus M_{p1} must equal to M_{p2}. ■

2.
Proof: Since the match $|M| = \frac{|V(G)|}{2}$, we know M is a perfect match. We know G has a perfect match M iff $o(G - S) \leq |S|, \forall S \subseteq V(G)$ from the Tutte condition. In this case, S is a single vertex v. Since graph G has a perfect match, $o(G - v) \leq 1$. There are two exhuastive cases, $o(G - v) = 0$ or $o(G - v) = 1$. If $o(G - v) = 0$, there are no odd componenets, which means all components are even. Thus we have $|V(G)| - 1$(since we removed one vertex) being even. Since we know $|V(G)|$ is even, $|V(G)| - 1$ must be odd, which forms a contradiction. Thus it must be the case $o(G - v) = 1$, which means G has exactly one odd component. ■

3.
Proof: We need to prove two statements. (1). S is a vertex cover $\implies$ $\bar{S}$ is an independent set. (2). $\bar{S}$ is an independent set $\implies$ S is a vertex cover.

Let us first prove S is a vertex cover $\implies$ $\bar{S}$ is an independent set. Let us assume that $\exists(u, v) \in E(\bar{S})$, then $\nexists(u, v) \in E(S)$ by the definition of a complement graph. Since $u \in V(S)$, $v \in V(S)$, and $\nexists(u, v) \in E(S)$, S is not an vertex cover which leads to a contradiction. Thus $\forall u, v \in \bar{S}$, $\nexists(u, v) \in E(\bar{S})$. In other words, no vertices in $\bar{S}$ are connected. Thus $\bar{S}$ is an vertex cover.

Let us next prove $\bar{S}$ is an independent set of graph $G = (V, E)$ $\implies$ S is a vertex cover. If $\bar{S}$ is an independent set, then $\forall(u, v) \in E(G)$, at most one vertex is in $\bar{S}$ otherwise it is not an independent set. Therefore at least one vertex is in S. Therefore every pair of vertices has at least one endpoint in S, which implies S is an vertex cover.

■

4. $|8| \leq |\text{vetex cover}| \leq 21$

From the $Konig$ theorem, we know the cardinality of minimum vertex cover of G equals the cardinality of the maximum match of G. Thus minimum vertex cover $= |M| = 9 - 1$(one vertex is unsaturated). The maximum vertex cover is the the cardinality of the vertex set since all vertices must cover the edge set of G.

■

5. $0 < |M| \leq 12$.

From the $Konig$ theorem, we know the cardinality of minimum vertex cover of G equals the cardinality of the maximum match of G. Thus the upper bound for the matching of G is 12. Thus $|M| \leq 12$. Also, since the graph is connected, the matching is non-empty

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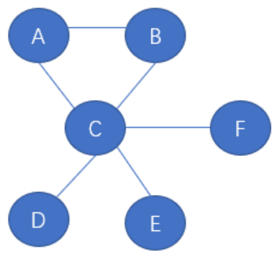
6.

Let us draw the following graph One match here is edge set $\{AB, CE\}$. The maximum match here



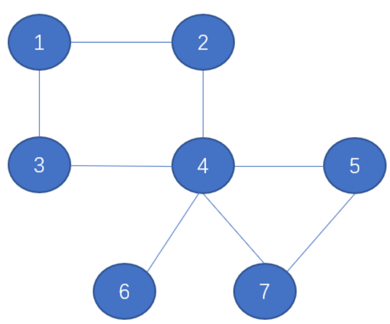
has cardinality of 2. We prove this using tutte's condition.

Since there are 6 vertices, $|M_p| = \dfrac{|V(G)|}{2} = 3$. We proceed to prove the graph does not have a perfect match, thus $|M_{max}| < 3$.

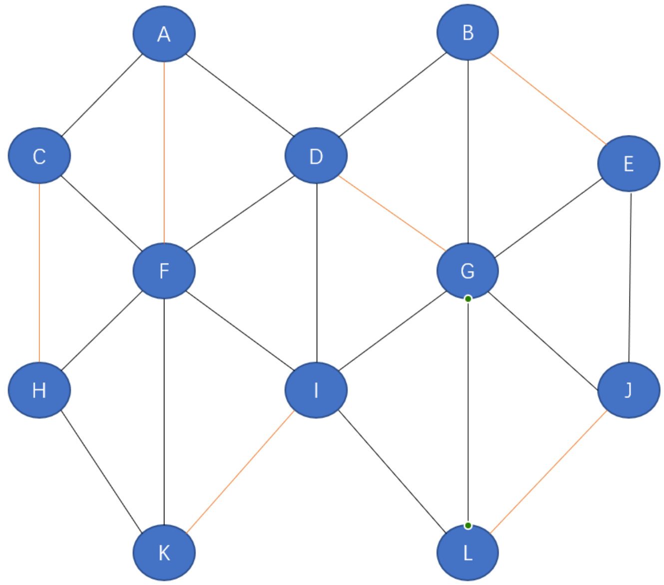
If there exists a perfect match for graph G, then $o(G - S) \leq |S|, \forall S \subseteq G$. However, if we remove the vertex c, the odd components are $\{F\}, \{D\}, \{E\}$. Thus we have $o(G - S) = 3 > 1 > |S|$. Thus the graph does not have a perfect match. Since we have found a match M such that $|M| = 2$ and proved $|M_{max}| < 3$, the match we found above is the maximum match.

■

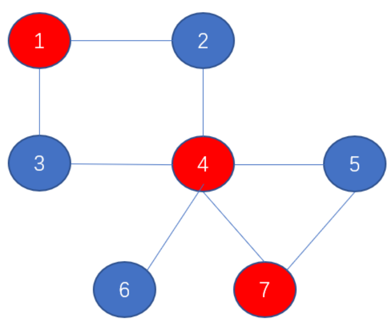
7. (a). Let $S = 4$, then if we remove S, the odd components are $\{1, 2, 3\}, \{6\}$. Thus $o(G - S) = 2 > 1 > |S|$. Thus Tutte's condition does not hold and perfect match does not exist.



(b). Let us draw the following graph
Note here it is a perfect match since all vertices have been saturated.



8. (a). Vertex cover



(b). Edge cover