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Quiz04

1. Proof: Let n denote the vertices of the tree, and P(n) denote the center of the tree is K_1 or K_2.

Base Case: Show $P(k \leq 2)$ is valid.

Consider two cases. 1. $k = 1$. Its center is just a point and thus is K_1. 2. $k = 2$. Its center is a line segment connecting to each other thus is K_2.

Inductive step: Assume $P(1 \leq k < n)$ holds, show $P(n)$ holds.

Let us denote the tree T.

First we delete all the leaf nodes in tree T and mark the result tree T'. We know that T' is still a tree with at least one vertex since T has at least 3 vertices.

We know that $\forall v \in V(T)$, v is a leaf node when we have $\max dist(u, v)$ where $u \in V(T)$. Since there are no more leaves in T' , we have $\epsilon_{T'}(u) = \epsilon_T(u) - 1, \forall u \in V(T')$. Also, forall $leaf \in V(T)$, we have $\epsilon_T(leaf) > \epsilon_T(Neighbor(leaf))$. Therefore the vertices that makes $\epsilon_T(u)$ attain its minimum value are the same vertices that makes $\epsilon'_T(u)$. Thus we have shown T and T' has the same center.

Since T and T' has the same center, T has more vertices than T', and the inductive hypothesis, we can conclude $P(n)$ =the center of the tree is K_1 or K_2 is valid.

2. Proof. We need to show $S = \{d_1, d_2, ..., d_n\}$ realizes a tree $\iff d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2n - 2)$.

We first prove $S = \{d_1, d_2, ..., d_n\}$ realizes a tree $\implies d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2n - 2)$. We know that $\sum_{i=1}^{n} d_i = 2|E|$ and in a tree $|E| = |V| - 1 = n - 1$. Thus $\sum_{i=1}^{n} d_i = 2(n - 1) = 2n - 2$.

We then prove $d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2n - 2) \implies S = \{d_1, d_2, ..., d_n\}$ realizes a tree. We prove it using induction on the number of vertices n.

Let $P(n)$ denote the statement $d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2n - 2) \implies S = \{d_1, d_2, ..., d_n\}$.

Base Case: n = 2, since n = 1 is trivial. Since the sum of degrees is 2, it could only be the case that two vertices are connecting with each other. Inductive step: Assume $P(1 \leq k < n)$ is valid, prove $P(n)$ holds.

Proof: Let k = n-1. Thus $d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2(n - 1) - 2) \implies S' = \{d_1, d_2, ..., d_{n-1}\}$ realizes a tree T' with n-1 vertices. We show $d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = 2(n - 2) \implies S = \{d_1, d_2, ..., d_n\}$ realizes a tree T with n vertices. Since $\sum_{i=1}^{n} d_i = (2n - 2) - \sum_{i=1}^{n} d_i = 2((n - 1) - 2) = 2$. We need to increase the degree of the tree T' by 2. We can add one leaf node to T'. Since adding an edge increases the number of degree each of the two nodes by 1, the total degree is increased by 2. By the inductive hypothesis, T is a tree with one more leaf node than T'. Thus we have shown $P(n)$ is valid.

Thus we have proved $S = \{d_1, d_2, ..., d_n\}$ realizes a tree $\iff d_i \geq 1 \wedge \sum_{i=1}^{n} d_i = (2n - 2)$.