1.

Let us consider the optimal coloring $\chi(G) = k$. Let us consider the folling Coloring classes $C_i = \{v \in V(G)|c(v) = i\}$. Let $k_i = \{C_1 \cup C_2 \cup ... \cup C_i\}$ for $1 \leq i \leq k_i$. Let k_i be 0. Thus we can arrange the vertices such that $C_i = \{v_{k_{i-1}+1}, v_{k_{-1}+2}, ...v_{k_i}\}$. Thus given this coloring, we will be able to cover the Graph with k colors since $1 \leq i \leq k$.

■

2.Let us prove $\chi(G) \leq \Delta(G) + 1$ using induction on graph G that has fixed $|V(G)| = n$ and $\Delta(G) = k$.

Base case:

We have $V(G) = n$ and $\Delta(G) = 0$. In other words, $\forall u \in V(G), \forall v \in V(G), \nexists (u, v) \in E(G)$. Therefore, one color is sufficient for coloring this graph. Thus we have $\chi(G) = 1 \leq 0 + 1 \leq \Delta(G) + 1$. Therefore the base case holds.

Inductive Hypothesis: Assume $\chi(G) \leq \Delta(G) + 1$ holds for graph G which has $|V(G)| = n$ and $\Delta(G) = k - 1$.

Inductive Step: Assume the inductive hypothesis holds and prove I.H. holds for case $V(G) = n$ and $\Delta(G) = k$ holds.

We connect one more vertex v_1 to the vertex that has the maximum degree v_2 in graph G to construct a G' such that $V(G') = n$ and $\Delta(G') = k$. Consider 3 exhaustive cases:

1. $c(v_1) \neq c(v_2)$.

   In this case, We do not need to modify anything, and we have $\chi(G') = \chi(G) \leq k + 1 \leq \Delta(G) + 1$

2. $c(v_1) = c(v_2)$ and $\exists v_j \in V(G' - v_1 - v_2)$ such that $c(v_j) = c(v_1) = c(v_2)$.

   In this case, we replace $c(v_1)$ with a new color so that $\nexists v_j \in V(G' - v_1 - v_2)$ such that $v_j = v_1$. Thus we have $\chi(G') = \chi(G) + 1 \leq ((k - 1) + 1) + 1 \leq k + 1 \leq \Delta(G') + 1$.

3. $c(v_1) = c(v_2)$ and $\nexists v_j \in V(G' - v_1 - v_2)$ such that $c(v_j) = c(v_1) = c(v_2)$.

   In this case, we replace $c(v_1)$ with a new color so that $\nexists v_j \in N(v_1)$ such that $c(v_j) = c(v_1)$ and $c(v_1) \neq c(v_2)$. Thus we have $\chi(G') = \chi(G) + 1 \leq ((k - 1) + 1) + 1 \leq k + 1 \leq \Delta(G') + 1$.

Thus we have shown the Inductive step holds for all cases. Combining with the base case, we have proved $\chi(G) \leq \Delta(G) + 1$.

■

3. To construct G' with $\chi(G') = 2$ from a graph G that contains no odd cycles, we first rearrange the vertices in G to form two vertex sets X and Y where $\forall u \in X, \forall v \in X, \nexists (u, v) \in E(G)$ and $\forall u \in Y, \forall v \in Y, \nexists (u, v) \in E(G)$. Note here the rearrangement is feasible since G contains no odd cycles $\implies$ G is bipartite. Then we add the vertex set $U = \{u_1, ..., u_{||X|-|Y||}\}$ to the set with smaller size between X and Y. After adding the set, we have $|X| = |Y|$. Then $\forall u \in X$, we add an edge between u and all vertices $v \in Y$ to construct G'. Therefore $\forall v \in G', d(v) \geq \dfrac{|V(G')| - 1}{2}$ since every vertex is connected to $\dfrac{|V(G')|}{2}$ vertices in the other part of the bipartite graph. G' is still 2 colorable since it is still a bipartite graph, and all bipartite graph are 2 colorable.

■

4. Based on the previous 2 problems, we can claim $\chi(G) \leq \Delta(G) + 1$ for a general graph. A connected graph always $k - colorable$ where $k \geq 2$.

■

5. In order to prove the chromatic polynomial for K_n is $\chi(K_n, k) = k(k-1)\dots(k-n+1)$, we use induction on the number of vertices of K_n.

Base Case:

For non-trivial graph K_1 with $|V(G)| = 1$, we know it can be colored with k colors. Thus $\chi(G) = k..(k-1+1) = k(k-1)\dots(k-n+1)$. Therefore the base case holds.

Inductive Step:

Assume graph K_n has chromatic polynomial $\chi(K_n, k) = k(k-1)\dots(k-n+1)$. Prove graph K_{n+1} has chromatic polynomial $\chi(K_{n+1}, k) = k(k-1)\dots(k-n+1)k(n-1)$.

To construct graph K_{n+1}, we add v_n vertex to the graph K_n and $\forall v_i \in V(K_n - v_n)$, we connect v_i with v_n. Since k_n is a clique, we have used n colors and thus v_n has $(k-n)$ color options. From our Inductive hypothsis we know $\chi(K_n, k) = k(k-1)\dots(k-n+1)$. Since v_n has $(k-n)$ color options, we know K_{n+1} has chromatic polynomial $\chi(K_{n+1}, k) = k(k-1)\dots(k-n+1)k(n-1)$. Therefore the Inductive step holds.

Since both the base case and inductive step hold, we have proved $\chi(K_n, k) = k(k-1)\dots(k-n+1)$.

■

6. Claim: $2 \leq \chi(G) \leq 3$.

For the lowerbound: A graph with more than 1 vertex is at least 2-colorable. For the upperbound: Consider the cut vertex $v_c \in V(G)$ such that the number of components in $G - v_c$ is 4. Since G is biconnected, we know $d(v_c) = 4$ and v_c is in 4 different cycles since every biconnected graph of G is ismporphic to C_n. We first give v_c color c_1, then we can color C_n with 3 colors if n is odd and 2 colors if n is even. Thus we can color all the cycles with $\max 3, 2 - 1 = 2$ extra colors. We substract one since v_c is in all the cycles and we have already counted that color. Since the coloring in one cycle will not affect the coloring in another cycle (they only have one common vertex v_c), we can color all the vertices with $2 + 1$(the color for v_c) $= 3$ colors. Since the bound holds for the cut vertex v_c that results in the maximum number of components in $G - v_c$, it holds for all the other cut vertices. Thus we have $\chi(G) \leq 3$.

■

7. In order to prove G is k-color-critical for $\chi(G) = k = 3$ if and only if G is an odd cycle, we need to prove (i). G is k-color-critical for $\chi(G) = k = 3 \implies G$ is an odd cycle. (ii). G is an odd cycle $\implies G$ is k-color-critical for $\chi(G) = k = 3$.

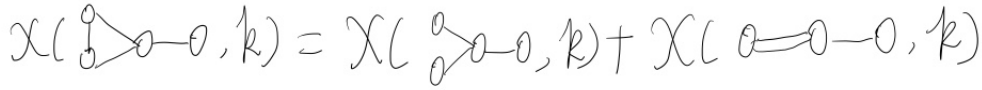
(i). If G is k-color-critical for $\chi(G) = k = 3$, then G is not 2 colorable. Therefore G must contain an odd cycle. Let C be the smallest odd cycle of graph G. If $G \neq C$, then C is a proper induced subgraph with $\chi(C) = \chi(G) = 3$. Thus it must be the case $G = C$ since G is 3-color critical. Otherwise we would have a contradiction. Since G is 3-color critical and $G = C$, we can conclude G is k-color-critical for $\chi(G) = k = 3 \implies G$ is an odd cycle.

(ii). If G is an odd cycle, then G is 3 colorable and is not 2 colorable. Removing any vertex in G breaks the cycle and thus become 2 colorable. Thus we can conclude G is an odd cycle $\implies G$ is k-color-critical for $\chi(G) = k = 3$.

Since we have proved the statement holds for both direction, we can prove G is k-color-critical for $\chi(G) = k = 3$ if and only if G is an odd cycle.

■

8.

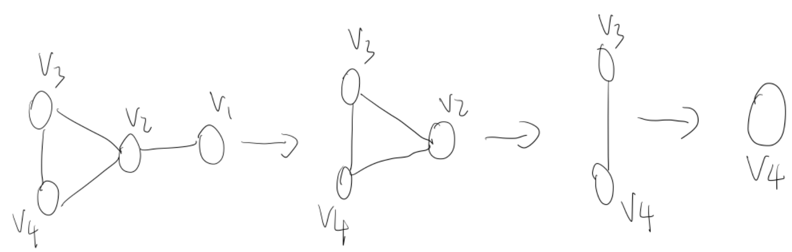


$$= k(k-1)^3 - k(k-1)^2$$

tree   tree

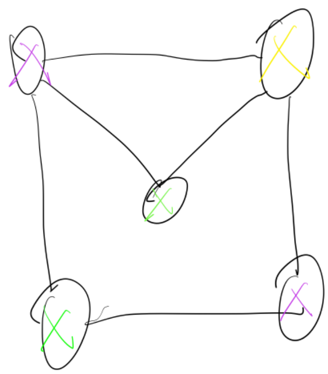
In the second recurrence relation, the number of edges between two vertices does not matter.To determine the Chromatic number $\chi(G)$, we set $\chi(G,k) = k(k-1)^3 - k(k-1)^2 > 0$. Thus we have $k > 2$ (said by WolframAlpha). Therefore we have $\chi(G) = 3$.

9.



The simplicial elimination ordering is $\{v_1, v_2, v_3, v_4\}$. Since G has a simplicial elimination odering, it is a chordal graph. Since G is a chordal graph, it is a perfect graph.

10.



As we can see, $\chi(G) = \omega(G) = 3$. Howeverk, there's a properly induced subgraph k_3 wihich has $\chi(k_3) = 3$ and $\omega(k_3)$. Thus G is not perfect.