Graph Theory HW 6
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1.**Proof:** Since $|V(G)| \geq 6$ and G is 3-connected, there must exists at least one vertex $v_p \in V(G)$(even if v_p is in the subdivision of K_5) connecting to 3 vertices $v_a, v_b, v_c \in V(K_5)$; otherwise we could cut one or two edges between v_p and $v_i \in V(K_5)$ to disconnect the graph so it is not 3-connected. Then Let us denote the vertex set $X = \{v_a, v_b, v_c\}$ and $Y = v_p \cup V(K_5) \setminus \{v_a, v_b, v_c\}$. Since every vertex in K_5 is connected to each other and v_p is connected to all $v \in \{v_a, v_b, v_c\}$. Thus we have $\forall v_i \in X, \forall v_j \in Y, \exists (v_i, v_j)$ such that $(v_i, v_j) \in V(G)$. Therefore by definition we have a $K_{3,3}$.

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2.**Proof:** In order to prove $\exists u, v \in V(G) : d(u) \leq 5, d(v) \leq 5$, we need to show (i).it cannot be the case that $\forall u \in V(G) : d(u) > 5$. (ii). it cannot be the cast that there exists only one v_p such that $\forall u \in V(G) : d(u) > 5$ and $v_p \leq 5$.

In order to show it cannot be the case that $\forall u \in V(G) : d(u) > 5$, we use the necessary condition of a planar graph derived from Euler's formula: $m \leq 3n - 6$, where $m = |E(G)|$ and $n = |V(G)|$. Thus we have $|E(G)| \leq 3|V(G)| - 6$. Let us assume $|V(G)| = n$. From the handshake theorem, we know

$$\begin{aligned} \frac{\sum_{v \in V(G)} d(v)}{2} &\leq 3|V(G)| - 6 \\ \frac{6n}{2} &\leq 3n - 6 \qquad\qquad \text{Note: } \forall u \in V(G) : d(u) > 5 \\ 3n &\leq 3n - 6 \\ 0 &\leq -6 \end{aligned}$$

Therefore we have arrived a contradiction. Thus it cannot be the case that $\forall u \in V(G) : d(u) > 5$.

In order to show it cannot be the cast that there exists only one v_p such that $\forall u \in V(G) : d(u) > 5$ and $v_p \leq 5$, we use the necessary condition of a planar graph derived from Euler's formula with the handshake theorem. Assume $\exists v \in V(G)$ s.t. $d(v) = k <= 5$. Thus we have

$$\begin{aligned} \frac{\sum_{v \in V(G)} d(v)}{2} &\leq 3|V(G)| - 6 \\ \frac{6(n-1) + k}{2} &\leq 3n - 6 \\ 3n - 3 + k &\leq 3n - 6 \\ k &\leq -3 \end{aligned}$$

Therefore we have arrived a contradiction. Thus it cannot be the case that there exists only one v_p such that $\forall u \in V(G) : d(u) > 5$ and $v_p \leq 5$.

Since we have eliminated the case where no vertex in G such that degree is less than or equal to 5 and there exists only one vertex in G such that degree is less than or equal to 5, there must have at least 2 vertices in G such that degree is less than or equal to 5.

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3. **Definition:** Given graph G, we can construct a graph G' with $V(G') = V(G) \cup \{v_{new}\}$. We then connect v_{new} with all vertex $v \in V(G)$.

Proof: In order to show G is outer-planar $\iff$ G does not contain a K_4 or $K_{2,3}$ subdivision, we show (i). G is outer-planar $\iff$ G' is planar. (ii). $K_4 \subseteq G \iff K_5 \subseteq G'$. (iii). $K_{2,3} \subseteq G \iff k_{3,3} \subseteq G'$. If the 3 claims above are true, it is obvious we can show the if direction by

$$K_{2,3} \nsubseteq G \wedge K_4 \nsubseteq G \implies K_{3,3} \nsubseteq G' \wedge K_5 \nsubseteq G' \implies \text{G' is planar} \implies \text{G is outer-planar} \quad (1)$$

Similarly, we can prove the only if direction by

$$\text{G is outer-planar} \implies \text{G' is planar} \implies K_5 \nsubseteq G \wedge K_{3,3} \nsubseteq G \implies K_4 \nsubseteq G' \wedge K_{3,3} \nsubseteq G' \quad (2)$$

Claim 1: G is outer-planar $\iff$ G' is planar.

If G is outer-planar, we can we can draw v_{new} and its adjacent edges into the outer face, obtaining a planar drawing of G'. Since G is outer-planar, thus no edges would cross after adding v_{new}. Thus G' is planar. If G' is planar, then we are left with only the contour of G by the definition of our construction of G'. Thus G is outer-planar.

Claim 2: $K_4 \subseteq G \iff K_5 \subseteq G'$

If $K_4 \subseteq G$, then by the definition of our construction $\exists H \subseteq G'$ with $V(H) = V(K_4) \cup \{v_{new}\}$ and edges connecting every pair of vertices in $V(H)$. Thus by definition we have a K_5.

If $K_5 \subseteq G'$, then removing v_{new} from G' decrease the degree of all vertices by one. Thus $\exists H \subseteq G$ with $V(H) = V(K_5) \setminus \{v_{new}\}$ and edges connecting every pair of vertices in $V(H)$. Thus by definition we have a K_4. Thus we have proved the claim is valid.

Claim 3: $K_{2,3} \subseteq G \iff k_{3,3} \subseteq G'$

If $K_{2,3} \subseteq G$, we can let $X = \{v_1, v_2\} \subset H$ denote the independent set with 2 vertices and $Y = \{v_3, v_4, v_5\} \subset H$ denote the independent set with 3 vertices. Since $\forall v \in G$, $\exists (u_{new}, v) \in E(G')$ and $\forall u_1 \in X$, $\forall v_1 \in Y$, $\exists (u_1, v_1) \in E(G')$, we can form an independent set $X' = \{v_1, v_2, v_3\}$ so $\forall u \in X'$, $\forall v \in Y$, $\exists (u, v) \in E(G')$.

If $K_{3,3} \subseteq G'$, we know $\exists X \subset G = \{v_1, v_2, v_{new}\} \wedge \exists Y \subset G = \{v_4, v_5, v_6\}$ so $\forall u \in X$, $\forall v \in Y$, $\exists (u, v) \in E(G')$. After removing v_{new} from X, we get $X' = \{v_1, v_2\}$. We know $\forall u \in Y$, $\forall v \in X'$, $\exists (u, v) \in G$ and vice versa by definition of our construction and $K_{3,3}$. Thus $\exists K_{2,3} \subseteq G$.

Since we have proved these three claims are true, following the path (1) and (2) we showed above gives us the conclusion that G is outer-planar $\iff$ G does not contain a K_4 or $K_{2,3}$ subdivision.

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4. **Proof:** We prove the 2 color theorem using induction on the number of lines n.

Base Case: $P(1)$. This is trivial to see. Since a line divides a plane into at most 2 maps, we can obviously color the 2 maps with 2 colors.

Inductive Step: Assume $P(k < n) =$ "a planed divided by k lines are 2 colorable" is valid, show $P(n)$ is valid.

Since adding a line to a plane will only divide blocks with 1 color to 2 separate blocks, let us denote the set of blocks being divided after adding new line l as $\{B_1, B_2, ..., B_n\}$. We pick an arbitrary block $B_i \in \{B_1, B_2, ..., B_n\}$ and show with the addition of the new block we can still color the map with 2 colors.

Let us denote the left plane after adding l as H_l and the right plane as H_r. We then swap the color of H_l and leave the coloring for H_r unchanged. Since we are dividing block B_i into 2 parts and B_i has 1 color before adding l, reversing the color on one side makes sure the 2 parts for B_i has opposite color. Since our choice of B_i is arbitrary, we can do the same thing for all blocks. Thus we can still color the map after adding l. Therefore we have proved any such map is 2-colorable.

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5. In order to prove for a maximum planar graph G, $\forall v \in V(G) : d(v) = \text{even} \iff \chi(G) \leq 3$ holds, we need to show (i). $\forall v \in V(G) : d(v) = \text{even} \implies \chi(G) \leq 3$. (ii). $\chi(G) \leq 3 \implies \forall v \in V(G) : d(v) = \text{even}$.

Let us first prove (i) is valid. If $\forall v \in V(G) : d(v) = \text{even}$, we know G is a even graph, which means G is Eulerian. We then prove this using induction on the number of faces in G. Let P(n) denote the statement maximal planar even graph G with n faces has $\chi(G) = 3$.

Base Case: $P(1)$ is valid since we can cover one triangle(result from maximal planar) in 3 colors.

Inductive hypothesis: Assume $P(1 \leq k < n) =$ maximal planar even graph G with n faces has $\chi(G) = 3$ is still valid. Prove $P(n)$ is valid.

Let e=xy be an edge on the exterior face of G. There must be an internal face containing xy. Let z be the common neighbor of x and y forming the internal face xyz. Consider 3 exhaustive cases:

1. G is the triangle xyz. This is trivial to see.

2. z is on the exterior boundary of G. In this case one among x and y - say x must have degree 2, and xz must be an edge on the exterior face of G. Consequently, $G \setminus \{x\}$ must be a 2-connected near triangulation with fewer internal faces, and must thereby admit an inductive 3-coloring. Put x back and a free color will be available to color x.

3. z is an internal vertex. As G is a near triangulation, there must be a wheel, say W(z) in G with z at the center and an even number of vertices around z. As there are three vertex disjoint paths in W(z) between x and y, we can see that $G \setminus e$ is a 2-connected near triangulation with fewer faces. Consequently,$G \setminus e$ must be 3-colorable (by induction hypothesis). The path from x to y in $G \setminus e$ along the rim of W(z) contains even number of vertices. Further, the vertices in the path must be 2-colored (the color assigned to z cannot be assigned to them). Consequently, x and y must be colored differently in $G \setminus e$. Put the edge e back and we can color the graph with 3 colors.

Let us next prove (ii) is valid. Suppose $\exists v \in V(G)$ with odd degree. Now look at $N(v)$. Since G is maximal planar, we know $N(v) \geq 3$, and they must be connected in a wheel graph with v at the center. The chromatic number of a wheel graph with an odd length outer cycle is 4. Thus we have a contradiction. Thus $\forall v \in V(G) : d(v) = \text{even}$. ■

6.

(a). Since b_1, b_2, and b_3 are planar, we can make the 3 blocks on the same line. Since there must exists some vertex that is not contained in any face in b_1 and there must exists some vertex that is not contained in any face in b_3, we can connect those 2 vertices to get G'.

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(b). Since G' is a minimal non-planar graph, removing any any from G' makes G' a planar graph. Therefore $G = G' - e$ is a planar graph.

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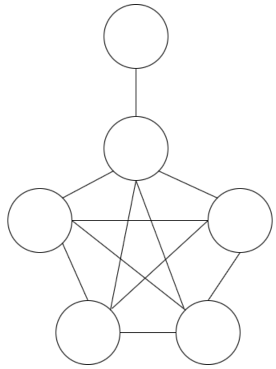
(c). The graph has exactly $|V(G)| = 6$ and $|E(G)| = 11$. It is not planar because K_5 is clearly a subgraph of G.



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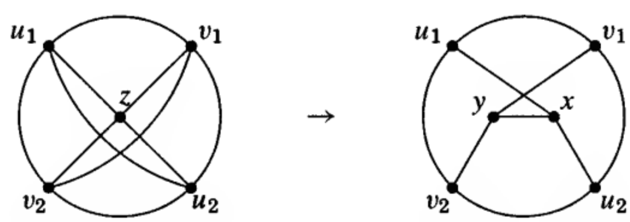
(d). In order to prove G' is planar $\implies$ $G = G' \cdot e$, where $e \in E(G')$, is planar, we can prove the contrapositve statement $G = G' \cdot e$ has kuratowski subgraph $\implies$ G' has kuratowski subgraph.

Let z be the vertex of G e obtained by contracting e = xy. If z is not in H, then H itself is a Kuratowski subgraph of G. If z ∈ V (H) but z is not a branch vertex of H, then we obtain a Kuratowski subgraph of G from H by replacing z with x or y or with the edge xy.

Similarly, if z is a branch vertex in H and at most one edge incident to z in H is incident to x in G, then expanding z into xy lengthens that path, and y is the corresponding branch vertex for a Kuratowski subgraph in G.

In the remaining case (shown below), H is a subdivision of K_5 and z is a branch vertex, and the four edges incident to z in H consist of two incident to x and two incident to y in G. In this case, let u_1, u_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to x in G, and let v_1 , v_2 be the branch vertices of H that are at the other ends of the paths leaving z on edges incident to y in G. By deleting the u_1, u_2 -path and v_1, v_2-path from H, we obtain a subdivision of $K_{3,3}$ in G, in which y, u_1, u_2 are the branch vertices for one partite set and x, v_1 , v_2 are the branch vertices of the other.



K_5 subdivision in G' **$K_{3,3}$ subdivision in G**

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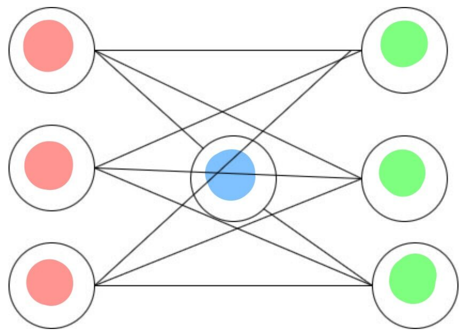
(e). As we can see, The graph contains a $K_{3,3}$ subdivison and it has $\chi(G) = 3$. Thus it is not planar.



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