Homework 9
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1. Let $\overline{N}(x;\epsilon) = \{y \in X : d(x,y) \leq \epsilon\}$ denote a **closed neighborhood** in X.
   Using the lecture definitions of open and closed sets, prove that $\overline{N}(x;\epsilon)$ is a closed set in X.

   Proof: In order to show $\bar{N}(x;\epsilon)$ is a closed set in X, we need to show $X \setminus \bar{N}(x;\epsilon)$ is an open set in X.

   Let $x_1 \in X \setminus \bar{N}(x;\epsilon)$. Then $x_1 \in X$ and $x_1 \notin \bar{N}(x;\epsilon)$ which means $d(x,x_1) > \epsilon$. We pick $\epsilon_1 = d(x,x_1) - \epsilon$ and note $\epsilon_1 > 0$ since $d(x,x_1) > \epsilon$. Next, we need to prove $N(x_1;\epsilon_1) \subseteq X \setminus \bar{N}(x;\epsilon)$.

   Let $x_2 \in N(x_1;\epsilon_1)$, then we have $x_2 \in X$ and $d(x_1,x_2) < \epsilon_1$. By the triangle inequality we have

$$\begin{aligned} d(x,x_1) &\leq d(x,x_2) + d(x_2,x_1) \\ d(x,x_2) &\geq d(x,x_1) - d(x_1,x_2) \\ d(x,x_2) &> d(x,x_1) - \epsilon_1 \\ &> d(x,x_1) - d(x,x_1) + \epsilon \\ &> \epsilon \end{aligned}$$

   Thus we have $d(x,x_2) > \epsilon$ which implies $x_2 \notin \bar{N}(x;\epsilon)$. Therefore, $x_2 \in X$ and $x_2 \notin \bar{N}(x;\epsilon)$ which means $x_2 \in X \setminus \bar{N}(x;\epsilon)$. Hence $N(x_1;\epsilon_1) \subseteq X \setminus \bar{N}(x;\epsilon)$.

   Since our choice of x_1 and x_2 are arbitrary, we have proved $X \setminus \bar{N}(x;\epsilon)$ is an open set in X, and therefore $\bar{N}(x;\epsilon)$ is a closed set in X.

2. Let (X,d) be a non-empty metric space and let $A \subseteq X$ be non-empty. Prove that $X \setminus \text{int}(A) = \text{cl}(X \setminus A)$.

   In order to prove $X \setminus \text{int}(A) = \text{cl}(X \setminus A)$, we need to show $(i).X \setminus \text{int}(A) \subseteq \text{cl}(X \setminus A)$ $(ii).\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$.

   First we need to prove $X \setminus \text{int}(A) \subseteq \text{cl}(X \setminus A)$. Let $x \in X \setminus \text{int}(A)$. Thus we have $x \in X$ and $x \notin \text{int}(A)$. Consider two exhaustive cases, $x \in A$ or $x \notin A$. If $x \in A$, we have $x \in bd(A) \cup \text{int}(A)$ since $A \subseteq bd(A) \cup \text{int}(A)$. Since $x \notin \text{int}(A)$, $x \in bd(A)$. Thus $N(x;\epsilon) \cap A \neq \emptyset$ and $N(x;\epsilon) \cap (X \setminus A) \neq \emptyset$. Hence $x \in bd(X \setminus A)$. Thus $x \in (X \setminus A) \cup bd(X \setminus A)$ which means $X \in \text{cl}(A)$ if $x \in A$. If $x \notin A$, then $x \in X \setminus A$ since $x \in X$. Thus $x \in (X \setminus A) \cup bd(A)$ which means $X \in \text{cl}(A)$ if $x \notin A$. Therefore, we have $X \in \text{cl}(A)$ if $x \in X \setminus \text{int}(A)$. Thus we have shown $X \setminus \text{int}(A) \subseteq \text{cl}(X \setminus A)$.

   Next we need to prove $\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$. Let $x \in \text{cl}(X \setminus A)$ and we have $x \in (X \setminus A) \cup bd(X \setminus A)$. Consider two exhaustive cases, $x \in (X \setminus A)$ or $x \in bd(X \setminus A)$. If $x \in (X \setminus A)$, we have $x \in X$ and $x \notin A$. Since $\text{int}(A) \subseteq A$, we have $x \notin \text{int}(A)$. Thus $x \in X \setminus \text{int}(A)$. Therefore we have $\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$ if $x \in (X \setminus A)$. If $x \in bd(X \setminus A)$, we have $x \in X \setminus A$. With the same reasoing in case 1 above, we have $\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$ if $x \in bd(X \setminus A)$. Thus we have shown $\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$.

Since we have shown $X \setminus \text{int}(A) \subseteq \text{cl}(X \setminus A)$ and $\text{cl}(X \setminus A) \subseteq X \setminus \text{int}(A)$, we have therefore proved $X \setminus \text{int}(A) = \text{cl}(X \setminus A)$.

3. Consider the metric space $(\mathbb{R}^2, d)$ where $d = |x_1 - y_1| + |x_2 - y_2|$. Use the definition of compactness to prove that the set
$$S = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}$$
is not a compact subset of $\mathbb{R}^2$.

Proof: In order to show the set S is not a compact subset of $\mathbb{R}^2$, we need to prove there exists an open cover of S that has no finite subcovers.

Let $\mathscr{F}$ be the set $\{N((0,0), \sqrt{2} - \frac{1}{n}) | n \in \mathbb{N}\}$. We need to prove $\mathscr{F}$ is an open cover of S, which means $S \subseteq \cup_{j \in \mathscr{F}} A_j = N((0,0), \sqrt{2})$.

Let $(x_1, x_2) \in S$, then we have $x_1^2 + x_2^2 < 1$. We need to prove $(x_1, x_2) \in N((0,0), \sqrt{2})$. Let us assume $x \notin N((0,0), \sqrt{2})$. Then we know $|x_1| + |x_2| \geq 2$ by definition. Then we have $\sqrt{|x_1|^2 + |x_2|^2} \leq \sqrt{||x_1| + |x_2||^2}$ by Minkowski's inequality. Thus we have $|x_1| + |x_2| \geq \sqrt{|x_1 + x_2|^2} \geq \sqrt{2}$. Consider two exhaustive cases: $|x_1| \geq 1$ or $|x_1| < 1$. If $x_1 \geq 1$, then $|x_1|^2 + |x_2|^2 = (x_1)^2 + (x_2)^2 \geq 1$. Thus we found a contradiction when $x_1 \geq 1$. If $|x_1| < 1$, we have $|x_2| = \sqrt{2} - |x_1| \geq 1$. Thus $|x_1|^2 + |x_2|^2 = (x_1)^2 + (x_2)^2 \geq 1$. Then we found a contradiction when $x < 1$. Therefore it must be the case $(x_1, x_2) \in N((0,0), \sqrt{2})$. Thus $S \subseteq \cup_{j \in \mathscr{F}} A_j = N((0,0), \sqrt{2})$ and $\mathscr{F}$ is an open cover for S.

Next, we prove there does not exist a finite subcover $\mathscr{F}'$ for $\mathscr{F}$ by contradiction. We need to find a point $(x_1, x_2) \in S$ such that is not covered by $\mathscr{F}'$. There exists a largest numgber n such that $N((0,0), \sqrt{2} - \frac{1}{n}) \in \mathscr{F}'$. Thus for all $N \leq n$, we have $N((0,0), \sqrt{2} - \frac{1}{N}) \in \mathscr{F}'$ and $\cup_{j \in \mathscr{F}'} A_j = N((0,0), \sqrt{2} - \frac{1}{N})$. Since $N \leq n$, we have $N((0,0), \sqrt{2} - \frac{1}{N}) \subset N((0,0), \sqrt{2} - \frac{1}{n})$ which means $\mathscr{F}'$ is a subset of $\mathscr{F}$. let (x_1, x_2) be the point $(\frac{\sqrt{2}}{2} - \epsilon, \frac{\sqrt{2}}{2})$ and $\epsilon < \frac{1}{N}$, Then we have $x_1^2 + x_2^2 = \frac{2}{4} + \frac{2}{4} + \epsilon^2 < 1$. Thus $(x_1, x_2) \in S$. However, $(x_1, x_2) \notin \mathscr{F}'$ since $|x_1| + |x_2| = \sqrt{2} - \epsilon > \sqrt{2} - \frac{1}{N}$. Thus we have point $(x_1, x_2) \in S$ but $(x_1, x_2) \notin \mathscr{F}'$.

Since our choice of (x_1, x_2) is arbitarry and we have shown there exists an open cover of S that has no finite subcovers, S is not compact