Homework 6
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1. Use the formal definition of convergent sequence (and theorem 4.1.8) to prove that the sequence $\left(\frac{5n^2+3n}{4n^2-2n}\right)$ converges to $\frac{5}{4}$.

   Proof: Let $\epsilon > 0$ be given. By the archmedian property we can choose $N \in \mathbb{N}$ so $N > \frac{11}{\epsilon}$.
   If $n \in \mathbb{N}$ and $n \geq N$ then $n \geq \max\{N, 2\}$ which means $n \geq N$ and $n \geq 2$ so $\frac{11}{n} < \epsilon$.
   So if $n \in \mathbb{N}$ and $n \geq N$ and $n \geq 2$ then $|S_n - S| = \left|\frac{5n^2+3n}{4n^2-2n} - \frac{5}{4}\right| = \left|\frac{20n^2+12n}{4(4n^2-2n)} - \frac{20n^2-10n}{4(4n^2-2n)}\right| = \left|\frac{22n}{4(4n^2-2n)}\right|$. Since $n \geq 2$, then $n^2 \geq 2n$ and $4n^2 - 2n \geq 3n^2$. Therefore $\left|\frac{22n}{4(4n^2-2n)}\right| \leq \frac{22n}{4(3n^2)} = \frac{11}{6n} \leq \frac{11}{n} \leq \frac{11}{N} < \epsilon$.

2. Suppose that $\lim s_n = s$ with $s > 0$. Prove there exists $N \in \mathbb{N}$ such that $s_n > 0$ for all $n \geq N$.

   Proof: Our goal here is to show there exists $N \in \mathbb{N}$ such that $s_n > 0$ for all $n \geq N$. Since $\lim s_n = s$ with $s > 0$, by definition we have for all $\epsilon > 0$, there exists $N_1 \in \mathbb{N}$ such that for all $n \in \mathbb{N}$, $n \geq N_1$ implies $|s_n - s| < \epsilon$. Assume there exists an $N = N_1$ and we need to show $s_n > 0$ for all $n \geq \mathbb{N}$. We consider three exhuastive cases. (i). $s_n > s$ for all $n \geq N$ (ii). $s_n < s$ for all $n \geq N$.

   Since $s_n > s$ for all $n \geq N$ and $s > 0$, $s_n > 0$ for all $n \geq N$.

   If $s_n < s$ for all $n \geq N$, then $|s_n - s| = s - s_n < \epsilon$. Rearranging the terms gives us $s_n > s + \epsilon$. since $s > 0$ and $\epsilon > 0$, $s + \epsilon > 0$. Thus $s_n > s + \epsilon > 0$.

   If $s_n = s$, since $s > 0$, $s_n > 0$

   Thus we have shown there exists an $N = N_1$ such that $s_n > 0$ for all $n \geq \mathbb{N}$.

3. Use the definition of a sequence (s_n) diverging to $-\infty$ to prove that $\lim\left(\frac{2+n-n^2}{2+3n}\right) = -\infty$.

   Proof: Given any $M \in \mathbb{R}$, take $N > \max\{1, \frac{2}{M}\}$. Then $n \geq N$ implies that $n > 1$ and $n > \frac{2}{M}$. Since $n > 1$, we have $n - n^2 < 0$. Thus for $n \geq N$ we have

$$\frac{2+n-n^2}{2+3n} \leq \frac{2+0}{3n} \leq \frac{2}{n} < M$$

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   Hence $\lim\left(\frac{2+n-n^2}{2+3n}\right) = -\infty$.