Homework 8
Wang Xinshi
RIN 6612975305

1. Let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, denoted $\mathbb{R}^3$. Let $\mathbf{x} = (x_1, x_2, x_3)$ and $\mathbf{y} = (y_1, y_2, y_3)$ be elements of X and defined $d : X \times X \to \mathbb{R}$ by $d(\mathbf{x}, \mathbf{y}) = max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\}$.

   (a) Prove that d is a metric.
   Proof: In order to prove d is a metric space, we need to show (i).$d(\mathbf{x}, \mathbf{y}) \geq 0$ (ii).$d(\mathbf{x}, \mathbf{y}) = 0$ iff $\mathbf{x} = \mathbf{y}$. (iii). $d(\mathbf{x}, \mathbf{y}) = d(\mathbf{y}, \mathbf{x})$ (iv)$d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$.

   (i). Consider three exhaustive cases, $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$, $d(\mathbf{x}, \mathbf{y}) = |x_2 - y_2|$, and $d(\mathbf{x}, \mathbf{y}) = |x_3 - y_3|$. For those three cases, we have all of them greater than or equal to 0 since they are in the absolute value sign. Thus we have $d(\mathbf{x}, \mathbf{y}) \geq 0$

   (ii). We need to show (i). $d(\mathbf{x}, \mathbf{y}) = 0$ implies $\mathbf{x} = \mathbf{y}$ and (ii). $\mathbf{x} = \mathbf{y}$ implies $d(\mathbf{x}, \mathbf{y}) = 0$.
   (i). Consider three exhuastive Cases: (a). $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$ (b). $d(\mathbf{x}, \mathbf{y}) = |x_2 - y_2|$ (c). $d(\mathbf{x}, \mathbf{y}) = |x_3 - y_3|$. In (a), we have $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| = 0$. Thus we have $x_1 = y_1$. Likewise we have $x_2 = y_2$ and $x_3 = y_3$. Therefore we have $\mathbf{x} = \mathbf{y}$ if $d(\mathbf{x}, \mathbf{y}) = 0$.
   (ii). Consider three exhuastive Cases: (a). $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$ (b). $d(\mathbf{x}, \mathbf{y}) = |x_2 - y_2|$ (c). $d(\mathbf{x}, \mathbf{y}) = |x_3 - y_3|$. If $\mathbf{x} = \mathbf{y}$, we have $x_1 = y_1$, $x_2 = y_2$ and $x_3 = y_3$. Thus $d(\mathbf{x}, \mathbf{y}) = 0$ for all three cases. Thus we have $d(\mathbf{x}, \mathbf{y}) = 0$ if $\mathbf{x} = \mathbf{y}$.

   (iii)Consider three exhuastive Cases: (a). $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$ (b). $d(\mathbf{x}, \mathbf{y}) = |x_2 - y_2|$ (c). $d(\mathbf{x}, \mathbf{y}) = |x_3 - y_3|$. In case (a) we have $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1| = |-(x_1 - y_1)| = |y_1 - x_1|$. Likewise, we can apply to x_2, y_2 and x_3, y_3. Thus $d(\mathbf{x}, \mathbf{y}) = max\{|y_1 - x_1|, |y_2 - x_2|, |y_3 - x_3|\} = d(\mathbf{y}, \mathbf{x})$.

   (iv)We have $d(\mathbf{x}, \mathbf{y}) = \max\{|x_1 - y_1|, |x_2 - y_2|, |x_3 - y_3|\} \leq \max\{|x_1 - z_1| + |z_1 - y_1|, |x_2 - z_2| + |z_2 - y_2|, |x_3 - z_3| + |z_3 - y_3|\}$. Consider three exhuastive Cases: (a). $d(\mathbf{x}, \mathbf{y}) = |x_1 - y_1|$ (b). $d(\mathbf{x}, \mathbf{y}) = |x_2 - y_2|$ (c). $d(\mathbf{x}, \mathbf{y}) = |x_3 - y_3|$. In case (a), we have $|x_1 - z_1| \leq \max\{|x_1 - z_1|, |x_2 - z_2|, |x_3 - z_3|\} = d(\mathbf{x}, \mathbf{z})$. $|z_1 - y_1| \leq \max\{|z_1 - y_1|, |z_2 - y_2|, |z_3 - y_3|\} = d(\mathbf{z}, \mathbf{y})$. Thus we have $|x_1 - z_1| + |z_1 - y_1| = d(\mathbf{x}, \mathbf{z}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$. Likewise, we can apply to x_2, y_2 and x_3, y_3. Thus $d(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{z}) + d(\mathbf{z}, \mathbf{y})$ holds.

   Since all of the properties holds, d is a metric space.

   (b) For the metric d, describe $N((0, 0, 0); 1)$ as a geometric shape. *Description must be typed out in sentence(s). Additionally, you may choose to include a hand-drawn sketch of the neighborhood.*

   It is a cube centered at the origin in three dimensional space, with each edge having a distance of 1 from the corresponding axis.

2. Suppose that $X = Y = \mathbb{R}^2$ and the metric d_1 is defined on X by

$$d_1((x_1, x_2), (y_1, y_2)) = max\{|x_1 - y_1|, |x_2 - y_2|\}$$

   and the metric d_2 is defined on Y by

$$d_2((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|.$$

Suppose f is a function that maps the metric space (X, d_1) to the metric space (Y, d_2) is defined by

$$f\left((x_1, x_2)\right) = \left(3x_1 - 4x_2, 5x_1 + 9x_2\right).$$

Use the definition of continuity on metric spaces to prove that f is continuous on X.

Proof: Let $\epsilon > 0$ be given, we pick $\delta = \dfrac{\epsilon}{18}$ such that for all $z \in X$, $d_1((x_1, x_2), (z_1, z_2)) < \delta$. Thus we have $|x_1 - z_1| < \delta$ and $|x_2 - z_2| < \delta$. Then it follows that

$$\begin{aligned}
d_2(f(x_1, x_2), f(z_1, z_2)) &= d_2((3x_1 - 4x_2, 5x_1 + 9x_2), (3z_1 - 4z_2, 5z_1 + 9z_2)) \\
&= |3x_1 - 4x_2 - 3z_1 + 4z_2| + |5x_1 + 9x_2 - 5z_1 - 9z_2| \\
&= |3(x_1 - z_1) - 4(x_2 - z_2)| + |5(x_1 - z_1) + 9(x_2 - z_2)| \\
&< 4|(x_1 - z_1) - (x_2 - z_2)| + 9|(x_1 - z_1) + (x_2 - z_2)| \\
&< 9(|(x_1 - z_1) - (x_2 - z_2)| + |(x_1 - z_1) + (x_2 - z_2)|) \\
&\leq 9(|(x_1 - z_1)| + | - (x_2 - z_2)| + |(x_1 - z_1)| + |(x_2 - z_2)|) \\
&< 9(2(|(x_1 - z_1)| + |(x_2 - z_2)|)) \\
&< 18\delta \\
&< \epsilon
\end{aligned}$$

Since our choice of ϵ is arbitrary, we have proved f is continuous on X.