Participation4

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Given

$$argmin_{x\in\mathbf{R}}\frac{1}{2}(x-\alpha)^2+\lambda|x|$$

with $\lambda > 0$ is a nonnegative constant.

1 Argue that this is a convex optimization problem, and it has a unique solution, given any α.

To argue that $argmin_{x\in\mathbf{R}}\frac{1}{2}(x-\alpha)^2+\lambda|x|$ is a convex optimization problem and has a unique solution, we need to argue $\frac{1}{2}(x-\alpha)^2+\lambda|x|$ is a strictly convex function. We can prove this by proving $(x-\alpha)^2$ is strictly convex and $|x|$ is convex.
To prove $\frac{1}{2}(x-\alpha)^2$ is strictly convex, we prove the hessian is positive definite.

$$\nabla f(x) = (x-\alpha)$$
$$\nabla^2 f(x) = 1 > 0$$

Thus $\frac{1}{2}(x-\alpha)^2$ is strictly convex.
The simplified first order condition always holds as we have $x \neq y$.
Therefore $(x-\alpha)^2$ is strictly convext. Since nonnegative multiple of strictly convex function is convex.
To prove $|x|$ is convex, we prove by definition.
$|\alpha x+(1-\alpha)y| \leq |\alpha x|+|(1-\alpha)y| = t|x|-(1-t)|y|$ from the triangular inequality. Therefore $|x|$ is convex.
Therefore the sum of those functions is strictly convex function and therefore is convex and has a unique solution.

2 Let $s_\lambda(a)$ be the unique solution to this optimization problem, given an α. State Fermat's optimality condition as concisely as you can, using our rules for subdifferential manipulation.

By the sum rule, we have:
$\partial s_{\lambda(\alpha)} = \partial(\frac{1}{2}(s_\lambda(a) - \alpha)^2) + \lambda|s_\lambda(a)|$
The derivative of $\partial(\frac{1}{2}(x-\alpha)^2)$ is: $(x-a)$
The subdifferential of $\lambda|s_\lambda(a)|$ is:

$$\partial\lambda|s_\lambda(a)| = \begin{cases} \lambda, & \text{if } s_\lambda(a) > 0 \\ -\lambda, & \text{if } s_\lambda(a) < 0 \\ \lambda[-1,1], & \text{if } s_\lambda(a) = 0 \end{cases}$$

Therefore the subdifferential for the whole function is:

$$\partial f(s_\lambda(\alpha)) = \begin{cases} s_\lambda(a) - \alpha + \lambda, & \text{if } s_\lambda(a) > 0 \\ s_\lambda(a) - \alpha - \lambda, & \text{if } s_\lambda(a) < 0 \\ s_\lambda(a) - \alpha + \lambda[-1,1], & \text{if } s_\lambda(a) = 0 \end{cases}$$

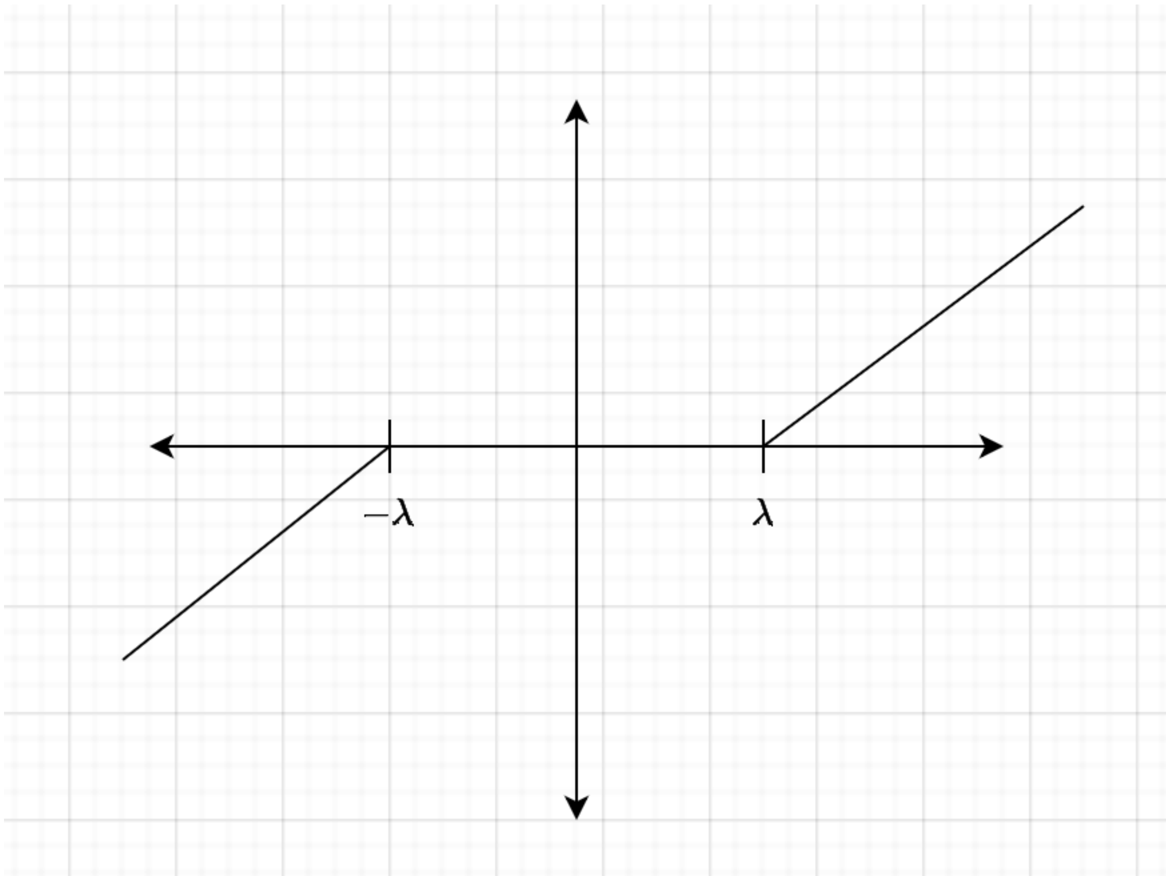
The Fermat's optimality condition therefore is: $s_\alpha \in argmin_x f(x)$

$$\text{iff } 0 \in \partial f(s_\lambda(\alpha)) = \begin{cases} s_\lambda(a) - \alpha + \lambda, & \text{if } s_\lambda(a) > 0 \\ s_\lambda(a) - \alpha - \lambda, & \text{if } s_\lambda(a) < 0 \\ s_\lambda(a) - \alpha + \lambda[-1,1], & \text{if } s_\lambda(a) = 0 \end{cases}$$

3 Use Fermat's optimality condition to find an expression for $s_\lambda(\alpha)$, and draw a plot of $s_\lambda(\alpha)$

By setting $\partial f(s_\lambda(\alpha))$ to 0, we have $s_\lambda(\alpha) = a - \lambda$, $s_\lambda(\alpha) = a + \lambda$, and $s_\lambda(\alpha) = a - \lambda[-1,1]$ accordingly. Therefore we have:

$$s_\lambda(\alpha) = \begin{cases} \alpha - \lambda, & \text{if } \alpha < -\lambda \\ \alpha + \lambda, & \text{if } \alpha > \lambda \\ 0, & \text{if } \alpha \in \lambda[-1,1] \end{cases}$$



$-\lambda$
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