CSCI-4968: ML and Optimization Spring 2023

Semidefinite Programming

For approximating Max Cut and Travelling Salesman

1 Motivation

TODO: Update this section (matrix completion)

1.1 Max Cut

TODO: Fix We would like to partition our graph into two set of vertices such that the sum of the edge weights between the two partitiions is maximized.

Applications include examples: Clustering - Intuition: The distances between the two partitiions are the greatest so they have a high likelihood of belonging to two different categories.

1.2 Travelling Salesman Problem

TODO: Fix Imagine you are prospective student touring RPI. You have a list of buildings you want to visit. You have on hand the distance between each pair of buildings. Is it possible to find a path such that each buidling is visited exactly once? If so, what is the shortest possible distance you will need to travel. Since you toruing the college, it is likely you have some method of transportation to came from and need to come back to . We have the added constraint that you start at the parking lot and will need to come back to the parking lot. This is know as the travelling salesman. As our goal is to find an *efficient* route, we will be satisfied with an approximation. Or a series of buidlings to visit that is close the optimal.

2 Semidefinite Programming

2.1 Recap of Linear Programming

To give background on semi-definite programming, we begin with a brief recap of linear programming. Suppose that you have control over a set of variables and you are attempting to find a selection for each of these variables such that some linear combination is either maximized or minimized. To give background on semi-definite programming we first start with a brief recap of linear programming. To make this more complicated suppose that these variables each have a constraint that must be met.

Formally we can write this out as the following. Suppose that we have n variables that we have control over which we can represent as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

This problem is made trivial by setting $\vec{x}$ to be arbitrarily large or small value depending on the optimization direction. The solution becomes more meaningful if the domain of each variable is constrainted.

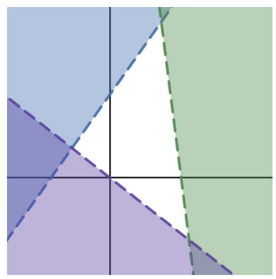
Say for each of these n variables has an associated coefficient $\vec{c} = \begin{bmatrix} c_1, c_2, ..., c_n \end{bmatrix}^\top$. We would like to find $\vec{x}$ such that the linear combinatrion of the two $(\vec{c} \cdot \vec{x})$ is minimized

$$\min \quad \vec{c} \cdot \vec{x}$$

We can extend this idea to a system of equations. We define $\vec{a}_i$ and $\vec{b}$ to vectors in $\mathbb{R}^n$. We would like to find $\vec{x}$ such that

$$\begin{aligned} \vec{a}_1 \cdot \vec{x} &= b_1 \\ \vec{a}_2 \cdot \vec{x} &= b_2 \\ &\vdots \\ \vec{a}_n \cdot \vec{x} &= b_n \end{aligned}$$

Graphically, each constraint creates a *half-space*. The intersection of the halfspaces (the white space) is our solution set. Our goal is the find the minimize value of the intersections.



We can gather our $\vec{a}_i$ vectors into a single matrix. Let $\mathbf{A} = [\vec{a_1}^\top \cdots \vec{a_n}^\top]$. so that we only need to solve:

$$\mathbf{A}\vec{x} = \vec{b}$$

In summary, a linear programming asks for an optimal solution, $\vec{x}$, given a set of linear constraints

$$\begin{array}{ll} \text{minimize} & \vec{c} \cdot \vec{x} \\ \text{subject to} & \mathbf{A}\vec{x} = \vec{b} \\ & \vec{x} \geq 0 \end{array}$$

It is needed that $\vec{x}$ is nonnegative ($\vec{x} \geq 0$). We note that our constraints form a convex set. Thus, linear programming is a subset of a convex optimization problem. Finding a solution to an instance of linear programming can be done with a variety of algorithms including the *simplex algorithm*.

Many real world problems can be modelled as a set of linear contraints. One use case arises in manufacturing. Given a set of products, how much of each kind should be produced to either minimize cost or maximize profits? Linear programming is a general approach of taking a constrained linear optimization problem into a form where a solution can easily be found.

2.2 Formalization of Semidefinite Programming

Instead of having just n variables that we have a matrix $\mathbf{X}$ of n^2 elements. Thus instead of finding an optimal $\vec{x}$ we are trying to find an optimal $\mathbf{X} \in \mathbb{R}^{n \times n}$ that is symmetric positive semidefinite (SPSD) such that

$$\min = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} c_{i,j}$$
$$\mathbf{X} \succeq 0$$

Where $c_{i,j}$ is the coefficient corresponding to the $x_{i,j}$ variable and $\mathbf{X} \succeq 0$ ensures that $\mathbf{X}$ is SPSD. We can gather all n^2 coefficients into a matrix $\mathbf{C}$ and then rewrite our minimization term as:

$$\min \mathbf{C} \bullet \mathbf{X}$$

Where the $\bullet$ operator is simply taking the sum of all elements after doing element wise multiplication on two matricies $\mathbf{C}$ and $\mathbf{X}$ with the same shape. Formally we can write it as:

$$\mathbf{C} \bullet \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{C}_{i,j} \mathbf{X}_{i,j}$$

Of important note is the fact that $\mathbf{C} \bullet \mathbf{X} = \text{trace}(\mathbf{C}^\top \mathbf{X})$ where trace is the sum of the diagonal terms for a matrix. Calculating the trace of the matrix multiplication is not as efficient as the $\bullet$ operator but the notation comes up quite frequently within the literature so it is important to know.

Returning back to our formalization of SDP, we note that with more variables comes with more constraints. Suppose that we have n equations to satisfy. For each equation we have a matrix $\mathbf{A}_i$ and a scalar value b_i defining our constraint. Then for a given constraint we have:

$$\mathbf{A}_i \bullet \mathbf{X} = b_i$$

This is all that is needed for SDP. We can write this out formally as:

$$\begin{aligned} &\min \mathbf{C} \bullet \mathbf{X} \\ &\quad \mathbf{A}_i \bullet \mathbf{X} = b_i \text{ for } i = 1, 2, ..., n \\ &\quad \mathbf{X} \succeq 0 \end{aligned}$$

The parallels between semidefinite programming and linear programming should be quite obvious. In many ways, semidefinite programming is linear programming with more variables of control and more constraints.

3 Max Cut Relaxation

3.1 Proofs

- performance guarantees - proof of relaxations (how accurate + how it works)

Integrality Gap: ratio btwen optimal / relaxation

4 Experimental Results

4.1 Visualization

- table of result - for small examples, we find the acutal solution (brute force), or a solution (integer programming) and check how good our bound is

vhttps://www.cs.cmu.edu/ anupamg/adv-approx/lecture14.pdf used "randomized rounding" to find a max cut. i.e. if p_{ij} is close to -1, then we should cut it. We attempt do do something similar

5 Discussion

- can we prove the runtime - can we prove how good of a lower bound we have? - what are some easy things to prove?

5.1 Scope and limitations

6 Appendix

6.1 Problem Set

We will now give one application of semidefinite programming in approximating an NP-Hard problem. We will first formalize the SDP and then use to find the approximation.

Consider the max-cut problem where for graph G the goal is to split the vertices into two disjoint sets S and $\overline{S}$ such that the sum of the edges between the two sets is maximum. Thus we know that $S \cup \overline{S} = V(G)$ and $e(i,j)$ is the weight of the edge between vertex v_i and vertex v_j. Thus our optimization problem is:

$$\max \sum \{e(i,j) : v_i \in S \text{ and } v_j \in \overline{S}\}$$

We will collect all our edge weights into matrix $\mathbf{W}$ where $w_{i,j}$ is the edge weight between vertex v_i and v_j.

We will work our way towards a formalization of SDP.

6.1.1 Question 1.

For all of our vertices we have a corresponding variable x_i such that:

$$x_i \in \{-1, 1\}$$
$$\text{if } x_i \in S \text{ then } x_i = 1$$
$$\text{if } x_i \in \overline{S} \text{ then } x_i = -1$$

Then for two vertices $v_i, v_j \in V(G)$ what is the inuition behind the following equation:

$$w_{i,j}(1 - x_i x_j)$$

Answer: If v_i and v_j are in the same set the above equation will be zero otherwise $w_{i,j}$ will be returned. The weight will only get returned if the edge between v_i and v_j is a cut edge for the sets S and $\overline{S}$.

6.1.2 Question 2.

How can we turn $x_i x_j$ from above into a matrix given that $\vec{x} = [x_1, x_2, \ldots, x_n]$. This matrix should be square with the number of rows equal to the number of vertices in our graph. $\mathbf{X}_{i,j} = x_i x_j$: Answer:$\mathbf{X} = \vec{x}\vec{x}^\top$

6.1.3 Question 3.

What can you say about the values on the diagonal of $\mathbf{X}$? Is $\mathbf{X}$ symmetric?

6.1.4 Question 4.

Before we found that our optimization problem was:

$$\max \sum \{e(i,j) : v_i \in S \text{ and } v_j \in \overline{S}\}$$

And in question 1. we found that for a single vertex pair we have:

$$w_{i,j}(1 - x_i x_j)$$

How can we rewrite our optimization problem to account for all vertex pairs using $\mathbf{X}$ and $\mathbf{W}$ and the $\bullet$ operator. Ensure that in your solution that no edge is counted twice:

Answer:

$$= \max \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[w_{i,j}(1 - x_i x_j)]\right)$$

$$= \max \sum_{i=1}^{n}\sum_{j=1}^{n}[w_{i,j} - w_{i,j} x_i x_j]$$

$$= \max \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[w_{i,j}] - \mathbf{W} \bullet \mathbf{X}\right)$$

The $\frac{1}{2}$ comes from the fact that $\mathbf{X}$ and $\mathbf{W}$ are symmetric and thus each cut edge is counted twice.

6.1.5 Question 5.

We have formalized the value that we want to maximize in the previous question. One of our constraints for SDP is that $\mathbf{X}$ has to SPSD. Thus we know that $\mathbf{X} \succeq 0$. Currently our constraints are that for $x_{i,j} \in \mathbf{X}$ that $x_{i,j} \in \{-1, 1\}$. Such a hard constraint is more specific than SDP can solve. We can relax our problem such that $\mathbf{X} \succeq 0$ and $x_{i,i} = 1$ and dropping our previous $x_{i,j} \in \{-1, 1\}$. What is the inuition behind having $x_{i,i} = 1$?

Answer: $x_{i,i} = 1$ ensures that vertex v_i is only in one set. This is represented by $x_i x_i = 1$.

Formally write out the SDP problem:

$$\max \frac{1}{2}\left(\sum_{i=1}^{n}\sum_{j=1}^{n}[w_{i,j}] - \mathbf{W} \bullet \mathbf{X}\right)$$
$$\mathbf{X}_{i,i} = 1$$
$$\mathbf{X} \succeq 0$$

6.1.6 Question 6.

We have now formalized an SDP problem for an approximation of the max-cut problem. Solving this will give us a lower bound for the max-cut problem. However we can get an approximate solution. In python generate n random 2-dimensional points. Find the distance between each pair of points. Each point will be a vertex in our graph and the pairwise distances between points will be our edges. scipy.spatial.distance.cdist is useful for this

Answer:

```
n = 20
points = np.random.rand(n, 2)
adj_matrix = cdist(points, points)
W = adj_matrix
```

Using the cvxpy example as a reference (https://www.cvxpy.org/examples/basic/sdp.html) setup the SDP problem and solve it. Your solution should look something like the following (here we set $n = 5$):

```
>>> print(X.value)
[[ 1.          1.00000039 -1.00000052 -1.00000054  1.00000065]
 [ 1.00000039  1.         -1.00000055 -1.00000057  1.00000072]
 [-1.00000052 -1.00000055  1.          1.00000033 -1.00000081]
 [-1.00000054 -1.00000057  1.00000033  1.         -1.00000083]
 [ 1.00000065  1.00000072 -1.00000081 -1.00000083  1.        ]]
```

What is the lower bound for the max-cut problem?

6.1.7 Question 7.

For a value $x_{i,j}$ if $x_{i,j} < 0$ then v_i and v_j are in different sets. Determine these two sets, plot the points of these two sets using two different colors and show the result:

6.1.8 Question 8.

What is the value of the max-cut for the two sets that you found?

6.1.9 Extra Credit

- The SDP problem that we formulated was an approximation for the max-cut problem - not the optimal solution. If we were able to set $x_{i,j} \in \{-1, 1\}$ then we would have an optimal solution. We will try and see how well the SDP solver did:
- Create a brute force max-cut solver and compare the results to what the SDP found.
- Note that such a brute force solver will have a $O(2^n)$ runtime so be careful for large values of n.
- Using a simulation approach, how well of a bound does the SDP problem set?
- In question 7. we got a potential max-cut. How well does this compare to the actual optimal max-cut?

6.2 References

https://www.uit.edu.mm/storage/2020/09/WWM-2.pdf