Semidefinite Programming

Applications in approximating NP-Complete problems & Matrix Completetion

Dimitri Lopez   Xinshi Wang   Jenny Gao

Rensselaer Polytechnic Institute

April 17, 2023

Presentation Overview

Motivation for Semidefinite Programming

- Linear Programming is a common constrained optimization technique with uses in:
  - Math
  - Computer Science
  - Economics
  - Business
- Wide applicability combined with fast runtimes makes linear programming quite popular

Motivation for Semidefinite Programming

- Linear Programming is a common constrained optimization technique with uses in:
  - Math
  - Computer Science
  - Economics
  - Business
- Wide applicability combined with fast runtimes makes linear programming quite popular

---

- Semidefinite programming (SDP) expands upon the ideas of linear programming
  - SDP can solve everything that linear programming can and more
  - Widely used in combinatorial optimzation problems
  - Many NP-Hard problems can be approximated well with SDP
- Both linear programming and semidefinite programming are convex problems

Formalization of Linear Programming

- Suppose that we have control over a set of variables: $\vec{x} = [x_1, x_2, \ldots, x_n]^\top$

Formalization of Linear Programming

- Suppose that we have control over a set of variables: $\vec{x} = [x_1, x_2, \ldots, x_n]^\top$
- Furthermore we often constraint $\vec{x}$ to be non-negative: $\vec{x} \geq 0$.

Formalization of Linear Programming

- Suppose that we have control over a set of variables: $\vec{x} = [x_1, x_2, \ldots, x_n]^\top$
- Furthermore we often constraint $\vec{x}$ to be non-negative: $\vec{x} \geq 0$.
- Where for each of these n variables we have an associated coefficient $\vec{c} = [c_1, c_2, \ldots, c_n]^\top$

Formalization of Linear Programming

- Suppose that we have control over a set of variables: $\vec{x} = [x_1, x_2, \ldots, x_n]^\top$
- Furthermore we often constraint $\vec{x}$ to be non-negative: $\vec{x} \geq 0$.
- Where for each of these n variables we have an associated coefficient $\vec{c} = [c_1, c_2, \ldots, c_n]^\top$
- In the end we want to find the optimal value for the following equation:

$$\min \vec{c} \cdot \vec{x}$$

Formalization of Linear Programming

$$\min \vec{c} \cdot \vec{x}$$

- The problem is quite easy as is. What if each of these n variables corresponds to how much of a given product that we want to buy?
  - This would add additional constraints to what values $\vec{x}$ could take on.

Formalization of Linear Programming

$$\min \vec{c} \cdot \vec{x}$$

- The problem is quite easy as is. What if each of these *n* variables corresponds to how much of a given product that we want to buy?
  - This would add additional constraints to what values $\vec{x}$ could take on.
- Each variable must satisfy some constraint, we can express this as:

$$\begin{gathered} \vec{a}_1 \cdot \vec{x} \geq b_1 \\ \vec{a}_2 \cdot \vec{x} \geq b_2 \\ \vdots \\ \vec{a}_n \cdot \vec{x} \geq b_n \end{gathered}$$

Formalization of Linear Programming

$$\min \vec{c} \cdot \vec{x}$$

- The problem is quite easy as is. What if each of these *n* variables corresponds to how much of a given product that we want to buy?
  - This would add additional constraints to what values $\vec{x}$ could take on.
- Each variable must satisfy some constraint, we can express this as:

$$\vec{a}_1 \cdot \vec{x} \geq b_1$$
$$\vec{a}_2 \cdot \vec{x} \geq b_2$$
$$\vdots$$
$$\vec{a}_n \cdot \vec{x} \geq b_n$$

- We can gather this into a single expression using a matrix

$$\mathbf{A}\vec{x} \geq \vec{b}$$

Formalization of Linear Programming

$$\min \vec{c} \cdot \vec{x}$$

- The problem is quite easy as is. What if each of these *n* variables corresponds to how much of a given product that we want to buy?
    - This would add additional constraints to what values $\vec{x}$ could take on.
- Each variable must satisfy some constraint, we can express this as:

$$\begin{aligned} \vec{a}_1 \cdot \vec{x} &\geq b_1 \\ \vec{a}_2 \cdot \vec{x} &\geq b_2 \\ &\vdots \\ \vec{a}_n \cdot \vec{x} &\geq b_n \end{aligned}$$

- We can gather this into a single expression using a matrix

$$\mathbf{A}\vec{x} \geq \vec{b}$$

Standard Form for a Linear Program

$$\begin{aligned} \text{minimize} \quad & \vec{c} \cdot \vec{x} \\ \text{subject to} \quad & \mathbf{A}\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

Standard Form for a Linear Program

$$\begin{aligned} \text{minimize} \quad & \vec{c} \cdot \vec{x} \\ \text{subject to} \quad & \mathbf{A}\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

- Where $\vec{x}$ are the variables that we have control over
  - Example: Each variable of $\vec{x}$ represents how much of a certain product to purchase

Standard Form for a Linear Program

$$\begin{aligned} \text{minimize} \quad & \vec{c} \cdot \vec{x} \\ \text{subject to} \quad & \mathbf{A}\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

- Where $\vec{x}$ are the variables that we have control over
  - Example: Each variable of $\vec{x}$ represents how much of a certain product to purchase
- $\vec{c}$ is a set of corresponding coefficients for the variables of $\vec{x}$
  - Example: Each variable of $\vec{c}$ corresponds to how much each product costs

Standard Form for a Linear Program

$$\begin{aligned} \text{minimize} \quad & \vec{c} \cdot \vec{x} \\ \text{subject to} \quad & \mathbf{A}\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{aligned}$$

- Where $\vec{x}$ are the variables that we have control over
  - Example: Each variable of $\vec{x}$ represents how much of a certain product to purchase
- $\vec{c}$ is a set of corresponding coefficients for the variables of $\vec{x}$
  - Example: Each variable of $\vec{c}$ corresponds to how much each product costs
- $\min \vec{c} \cdot \vec{x}$ is the function we are trying to optimize
  - Example: We want to minimize the cost of buying our products

Standard Form for a Linear Program

$$
\begin{array}{ll}
\text{minimize} & \vec{c} \cdot \vec{x} \\
\text{subject to} & \mathbf{A}\vec{x} \geq \vec{b} \\
 & \vec{x} \geq 0
\end{array}
$$

- Where $\vec{x}$ are the variables that we have control over
  - Example: Each variable of $\vec{x}$ represents how much of a certain product to purchase
- $\vec{c}$ is a set of corresponding coefficients for the variables of $\vec{x}$
  - Example: Each variable of $\vec{c}$ corresponds to how much each product costs
- $\min \vec{c} \cdot \vec{x}$ is the function we are trying to optimize
  - Example: We want to minimize the cost of buying our products
- $\mathbf{A}\vec{x} \geq \vec{b}$ constraints on our solutions. Defines the feasible region.
  - Example: This could represent the minimum amount of products that we must buy

Standard Form for a Linear Program

$$\begin{array}{ll} \text{minimize} & \vec{c} \cdot \vec{x} \\ \text{subject to} & \mathbf{A}\vec{x} \geq \vec{b} \\ & \vec{x} \geq 0 \end{array}$$

- Where $\vec{x}$ are the variables that we have control over
  - Example: Each variable of $\vec{x}$ represents how much of a certain product to purchase
- $\vec{c}$ is a set of corresponding coefficients for the variables of $\vec{x}$
  - Example: Each variable of $\vec{c}$ corresponds to how much each product costs
- $\min \vec{c} \cdot \vec{x}$ is the function we are trying to optimize
  - Example: We want to minimize the cost of buying our products
- $\mathbf{A}\vec{x} \geq \vec{b}$ constraints on our solutions. Defines the feasible region.
  - Example: This could represent the minimum amount of products that we must buy

Linear Programming to Semidefinite Programming

- Semidefinite programming (SDP) takes the concept that linear programming has with vectors and generalizes it to matrices.
- The : operator is the Frobenius inner product which is the sum of element wise multiplication on vectors: $\mathbf{C} : \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_{i,j}$
- $\mathbf{X} \succeq 0$ means that $\mathbf{X}$ is positive semi-definite (PSD)

Linear Programming	Semidefinite Programming
$\vec{x} \in \mathbb{R}^n$	$\mathbf{X} \in \mathbb{R}^{n \times n}$
$\vec{x} \geq 0$	$\mathbf{X} \succeq 0$
$\vec{c} \in \mathbb{R}^n$	$\mathbf{C} \in \mathbb{R}^{n \times n}$
$\min \vec{c} \cdot \vec{x}$	$\min \mathbf{C} : \mathbf{X}$
$\mathbf{A} \in R^{n \times n}, \vec{b} \in \mathbb{R}^n$	$\mathbf{A}_i \in R^{n \times n}, \vec{b} \in \mathbb{R}^m$
$\mathbf{A}\vec{x} \geq \vec{b}$	$\mathbf{A}_i : \mathbf{X} = b_i : i = 1, \ldots, m$

Linear Programming to Semidefinite Programming

- Semidefinite programming (SDP) takes the concept that linear programming has with vectors and generalizes it to matrices.
- The $:$ operator is the Frobenius inner product which is the sum of element wise multiplication on vectors: $\mathbf{C} : \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} c_{i,j} x_{i,j}$
- $\mathbf{X} \succeq 0$ means that $\mathbf{X}$ is positive semi-definite (PSD)

Linear Programming	Semidefinite Programming
$\vec{x} \in \mathbb{R}^n$	$\mathbf{X} \in \mathbb{R}^{n \times n}$
$\vec{x} \geq 0$	$\mathbf{X} \succeq 0$
$\vec{c} \in \mathbb{R}^n$	$\mathbf{C} \in \mathbb{R}^{n \times n}$
$\min \vec{c} \cdot \vec{x}$	$\min \mathbf{C} : \mathbf{X}$
$\mathbf{A} \in R^{n \times n}, \vec{b} \in \mathbb{R}^n$	$\mathbf{A}_i \in R^{n \times n}, \vec{b} \in \mathbb{R}^m$
$\mathbf{A}\vec{x} \geq \vec{b}$	$\mathbf{A}_i : \mathbf{X} = b_i : i = 1, \ldots, m$

Linear Programming to Semidefinite Programming

- In total we define a semidefinite program with the following:

$$
\begin{array}{ll}
\text{minimize} & \vec{c} \cdot \vec{x} \\
\text{subject to} & \mathbf{A}\vec{x} \geq \vec{b} \\
 & \vec{x} \geq 0
\end{array}
$$

- In total we define a semidefinite program with the following:

$$
\begin{array}{lll}
\text{min} & \mathbf{C} \bullet \mathbf{X} & \\
\text{subject to} & \mathbf{A}_i \bullet \mathbf{X} \geq b_i & i = 1, \ldots, m \\
 & \mathbf{X} \succeq 0 &
\end{array}
$$

Semidefinite Programming Duality

- It is important to note the dual of an SDP problem which is:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} z_i b_i \\ \text{such that} & \sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C} \\ & \mathbf{S} \succeq 0 \end{array}$$

Semidefinite Programming Duality

- It is important to note the dual of an SDP problem which is:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} z_i b_i \\ \text{such that} & \sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C} \\ & \mathbf{S} \succeq 0 \end{array}$$

- We are trying to find a set of scalars $z_1, z_2, \ldots, z_m$
- Where our objective function is $\vec{z} \cdot \vec{b}$
- We also satisfy the constraint $\sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C}$ where $\mathbf{A}_i$ and $\mathbf{C}$ are from before.

Semidefinite Programming Duality

- It is important to note the dual of an SDP problem which is:

$$\begin{array}{ll} \max & \sum_{i=1}^m z_i b_i \\ \text{such that} & \sum_{i=1}^m z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C} \\ & \mathbf{S} \succeq 0 \end{array}$$

- We are trying to find a set of scalars $z_1, z_2, \ldots, z_m$
- Where our objective function is $\vec{z} \cdot \vec{b}$
- We also satisfy the constraint $\sum_{i=1}^m z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C}$ where $\mathbf{A}_i$ and $\mathbf{C}$ are from before.
- We know that $\mathbf{S} \succeq 0$ which allows us to get the more intuitive:

$$\mathbf{C} - \sum_{i=1}^{m} z_i \mathbf{A}_i \succeq 0$$

- Pulling it all together:

$$\begin{array}{ll} \max & \sum_{i=1}^m z_i b_i \\ \text{such that} & \mathbf{C} - \sum_{i=1}^m z_i \mathbf{A}_i \succeq 0 \end{array}$$

Semidefinite Programming Runtime

- SDPs can be solved in polynomial time which makes them qutie useful.
- One algorithm to solve them is Alizadeh's interior point method which runs in:

$$\tilde{O}(n^{3.5})$$

Reviewing TSP

The Traveling Salesman Problem (TSP) is an optimization problem in which the objective is to find the shortest possible route for a salesman to visit a given set of cities, passing through each city exactly once, and returning to the starting city. It is a well-known NP-hard problem.

Semidefinite Programming Methods for the Symmetric Traveling Salesman Problem , 1999

Let $C \in \mathbb{R}^{n \times n}$ denote the matrix of edge costs. Let J denote the all-ones matrix, and e denote the all-ones vector.

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\text{trace}(CX) \\ \text{subject to} & Xe = 2e \\ & X_{ii} = 0, \quad i = 1, \ldots, n \\ & 0 \leq X_{ij} \leq 1, \quad i, j = 1, \ldots, n \\ & 2I - X + (2 - 2\cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0 \\ & X \text{ is a real, symmetric } n \times n \text{ matrix.} \end{array}$$

X is a fractional adjacency matrix, meaning for $e = \{i, j\}$, $x_{ij} = x_{ji}$ is the proportion of edge e used.

Integrality Gap And Running Time

# Of Nodes	SDP Time	BF Time	SDP Objective Value	BF Objective Value	Integrity Gap	Time Ratio
10	0.7101	0.0156	53224.4854	53228.3976	0.9999	45.519
15	0.6776	0.8224	65753.5934	67299.5625	0.9770	0.8239
20	1.2271	97.2059	69558.9865	76199.4928	0.9129	0.0126
21	1.3689	266.7778	73969.6527	77373.6362	0.9560	0.0051
22	5.4774	657.7847	66459.7265	68245.9576	0.9738	0.0083

Visualization

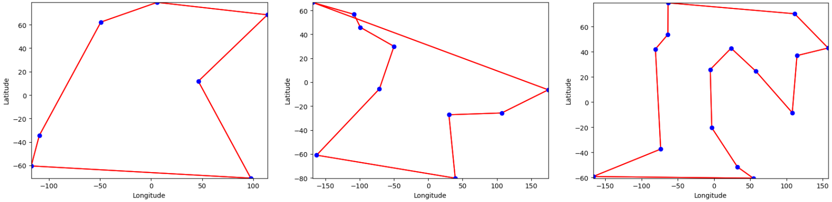


Figure: reasonable solution

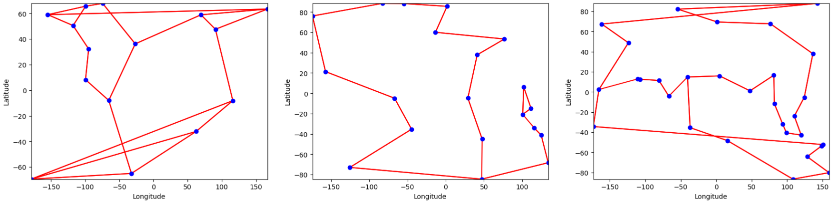


Figure: unreasonable solution

Low rank matrices

Given an incomplete matrix, can we recover the missing values?

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1

Yes!

Given:

- The matrix is low rank*
- We have enough sample data

Note: This does not apply to *all* low-rank matrices. But most.

Yes!

Given:

- The matrix is low rank*
- We have enough sample data

Note: This does not apply to *all* low-rank matrices. But most.

Why low-rank matrices?

Why is this useful

1. **Netflix** has an incomplete set of user preferences based off their past watch history. Can they use this information to recommend new movies?
2. **Recommendation Engine:** The netflix problem can be extended to general recommendation engines where a vendor knows some of the user preferences.
3. **Images:** We will give an example of recovering a corrupted image using matrix completetion

Relaxing Matrix Completetion to SDP

Suppose we have a low rank matrix $\mathbf{M}$. We have a set of location Ω describing our sampling. That is, if $(i, j) \in \Omega$, we observe entry M_{ij}. Given $\mathbf{M}$ is low rank, it seems resonable that we would like to solve the following optimization problem

Relaxing Matrix Completetion to SDP

Suppose we have a low rank matrix **M**. We have a set of location Ω describing our sampling. That is, if $(i,j) \in \Omega$, we observe entry M_{ij}. Given **M** is low rank, it seems resonable that we would like to solve the following optimization problem

$$\begin{aligned} \text{minimize} \quad & \operatorname{rank}(\mathbf{X}) \\ \text{subject to} \quad & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\ & \mathbf{X} \in \mathbb{R}^{n \times n} \end{aligned}$$

Relaxing Matrix Completetion to SDP

Suppose we have a low rank matrix $\mathbf{M}$. We have a set of location Ω describing our sampling. That is, if $(i, j) \in \Omega$, we observe entry M_{ij}. Given $\mathbf{M}$ is low rank, it seems resonable that we would like to solve the following optimization problem

$$\begin{array}{ll} \text{minimize} & \text{rank}(\mathbf{X}) \\ \text{subject to} & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\ & \mathbf{X} \in \mathbb{R}^{n \times n} \end{array}$$

But...

Rank is not a convex. This turns out to be an NP-Hard Problem.

Introduce the nuclear norm

Nuclear Norm

The nuclear norm is a close approximation of the rank.

The nuclear norm of a matrix **X** is defined as the sum of the eigenvalues.

$$\|\mathbf{X}\|_* = \sum_{k=1}^{n} \sigma_k \mathbf{X}$$

Introduce the nuclear norm

Nuclear Norm

The nuclear norm is a close approximation of the rank.

The nuclear norm of a matrix **X** is defined as the sum of the eigenvalues.

$$\|\mathbf{X}\|_* = \sum_{k=1}^{n} \sigma_k \mathbf{X}$$

For a symmetric positive semi-definite (SPSD) matricies, the nuclear norm is equal to the trace.

A better relaxation

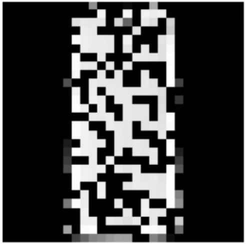
What if our matrix is not SPSD

- We introduce two matricies $\mathbf{W}_1$ and $\mathbf{W}_2$

A better relaxation

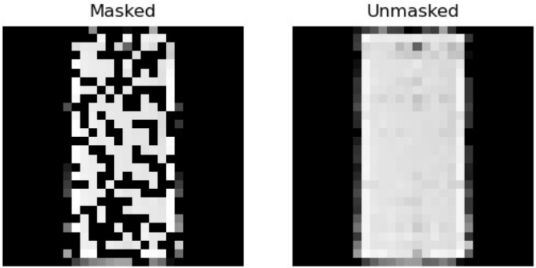
$$\begin{aligned} \text{minimize} \quad & \operatorname{trace}(\mathbf{W}_1) + \operatorname{trace}(\mathbf{W}_2) \\ \text{subject to} \quad & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\ & \begin{bmatrix} \mathbf{W}_1 & \mathbf{X} \\ \mathbf{X}^\top & \mathbf{W}_2 \end{bmatrix} \succeq 0 \end{aligned}$$

55% of data



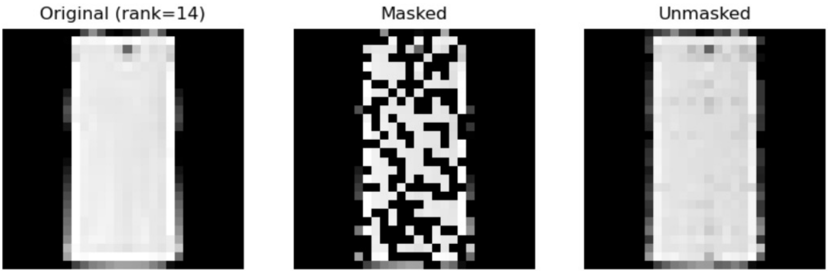
Fashion-MNIST

55% of data



Fashion-MNIST

55% of data

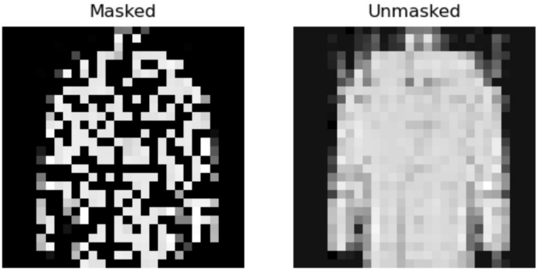


Fashion-MNIST

50% of data

Fashion-MNIST

50% of data



50% of data

Citing References

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2022, Kennedy, 2023].

References

John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.

Annabelle Kennedy (2023)
Publication title
Journal Name 12(3), 45 – 678.

Acknowledgements

Smith Lab

- Alice Smith
- Devon Brown

Cook Lab

- Margaret
- Jennifer
- Yuan

Funding

- British Royal Navy
- Norwegian Government