Semidefinite Programming

Applications in approximating NP-Complete problems & Matrix Completetion

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Presentation Overview

Reviewing TSP

The Traveling Salesman Problem (TSP) is an optimization problem in which the objective is to find the shortest possible route for a salesman to visit a given set of cities, passing through each city exactly once, and returning to the starting city. It is a well-known NP-hard problem.

Semidefinite Programming Methods for the Symmetric Traveling Salesman Problem , 1999

Let $C \in \mathbb{R}^{n\times n}$ denote the matrix of edge costs. Let J denote the all-ones matrix, and e denote the all-ones vector.

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\text{trace}(CX) \\ \text{subject to} & Xe = 2e \\ & X_{ii} = 0, \quad i = 1, \ldots, n \\ & 0 \leq X_{ij} \leq 1, \quad i, j = 1, \ldots, n \\ & 2I - X + (2 - 2\cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0 \\ & X \text{ is a real, symmetric } n \times n \text{ matrix.} \end{array}$$

X is a fractional adjacency matrix, meaning for $e = \{i, j\}$, $x_{ij} = x_{ji}$ is the proportion of edge e used.

Integrality Gap And Running Time

# Of Nodes	SDP Time	BF Time	SDP Objective Value	BF Objective Value	Integrity Gap	Time Ratio
10	0.7101	0.0156	53224.4854	53228.3976	0.9999	45.519
15	0.6776	0.8224	65753.5934	67299.5625	0.9770	0.8239
20	1.2271	97.2059	69558.9865	76199.4928	0.9129	0.0126
21	1.3689	266.7778	73969.6527	77373.6362	0.9560	0.0051
22	5.4774	657.7847	66459.7265	68245.9576	0.9738	0.0083

Visualization

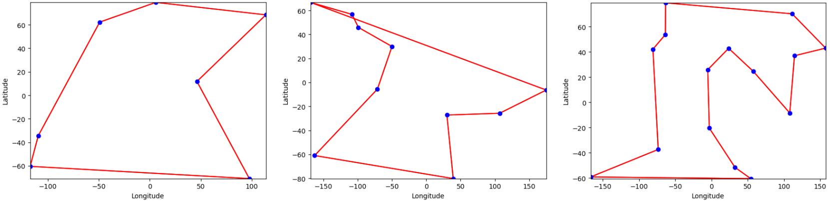

Figure: reasonable solution

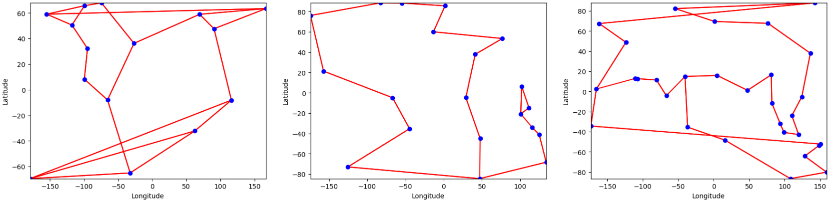

Figure: unreasonable solution

Low rank matrices

Given an incomplete matrix, can we recover the missing values?

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1
1	1	-1	1	-1

Yes! Given:

- The matrix is low rank
- We have enough sample data
- and more ... (needle in haystack)

Why is this useful

1. **Netflix** has an incomplete set of user preferences based off their past watch history. They
2. **Recommendation Engine**
3. **Images**

Relaxing Matrix Completetion to SDP

Suppose we have a low rank matrix $\mathbf{M}$. We have a set of location Ω describing our sampling. That is, if $(i,j) \in \Omega$, we observe entry M_{ij}. Given $\mathbf{M}$ is low rank, it seems resonable that we would like to solve the following optimization problem

$$\begin{array}{ll} \text{minimize} & \text{rank}(\mathbf{X}) \\ \text{subject to} & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\ & \mathbf{X} \in \mathbb{R}^{n \times n} \end{array}$$

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But...

Rank is not a convex. This turns out to be an NP-Hard Problem.

Introduce the nuclear norm

Nuclear Norm

The nuclear norm is a close approximation of the rank.

The nuclear norm of a matrix **X** is defined as the sum of the eigenvalues.

$$\|\mathbf{X}\|_* = \sum_{k=1}^{n} \sigma_k \mathbf{X}$$

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For a symmetric positive semi-definite (SPSD) matricies, the nuclear norm is equal to the trace.

A better relaxation

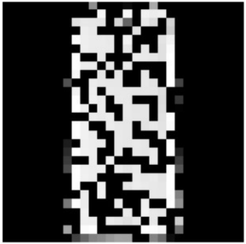
What if our matrix is not SPSD

- We introduce two matricies $\mathbf{W}_1$ and $\mathbf{W}_2$

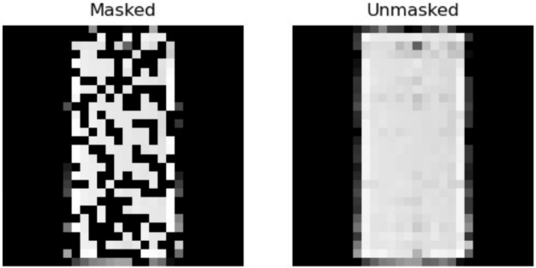
A better relaxation

$$\begin{array}{ll} \text{minimize} & \text{trace}(\mathbf{W}_1) + \text{trace}(\mathbf{W}_2) \\ \text{subject to} & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\ & \begin{bmatrix} \mathbf{W}_1 & \mathbf{X} \\ \mathbf{X}^\top & \mathbf{W}_2 \end{bmatrix} \succeq 0 \end{array}$$

Fashion-MNIST



Fashion-MNIST



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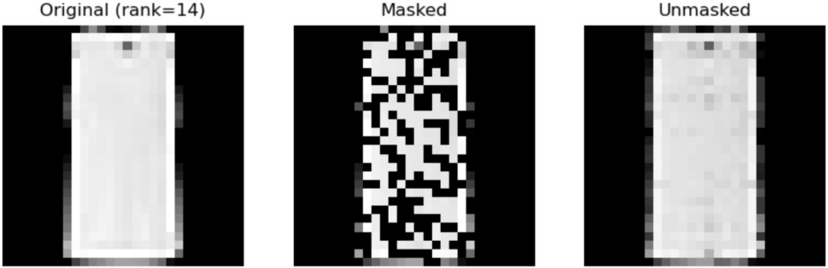

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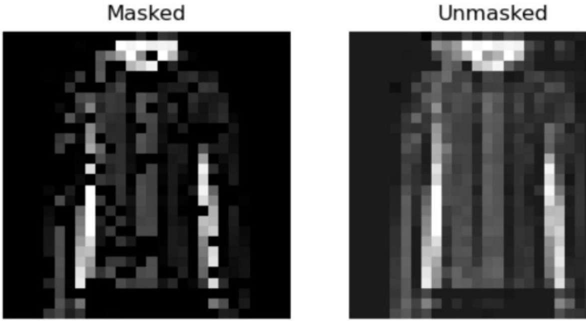
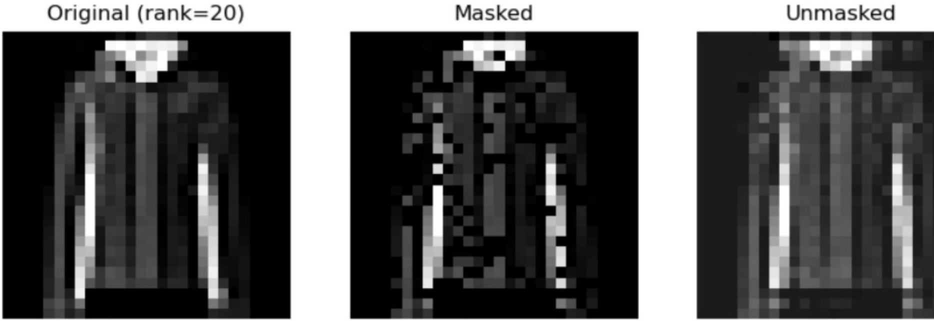


Fashion-MNIST



Lists

Bullet Points and Numbered Lists

- Lorem ipsum dolor sit amet, consectetur adipiscing elit
- Aliquam blandit faucibus nisi, sit amet dapibus enim tempus
  - Lorem ipsum dolor sit amet, consectetur adipiscing elit
  - Nam cursus est eget velit posuere pellentesque
- Nulla commodo, erat quis gravida posuere, elit lacus lobortis est, quis porttitor odio mauris at libero

1. Nam cursus est eget velit posuere pellentesque
2. Vestibulum faucibus velit a augue condimentum quis convallis nulla gravida

Citing References

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2022, Kennedy, 2023].

References

John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.

Annabelle Kennedy (2023)
Publication title
Journal Name 12(3), 45 – 678.

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