

# Semidefinite Programming

Applications in approximating NP-Complete problems & Matrix Completion

<https://github.com/Xinshi0726/SDP-Relaxations-With-Applications>

April 2023

## 1 Motivation

Linear programming is a common constrained optimization problems with uses in several fields including math, computer science, economics, and business. Several real world problems can be modelled as a set of linear constraints, making linear programming a ubiquitous construct. We are able to find solutions to a linear program in reasonable time.

Semidefinite programming (SDP) expands on the ideas used within linear programming and generalizes them so that they can be applied to even more cases. SDPs are widely used for combinatorial optimization - a class of problems that are of abundance in network science. The notorious NP-Hard problems can be approximated quite well with SDPs. The flexibility and speed of finding solutions of what can be represented as a SDP problem is what gives it its power.

We will go over how to approximate a classic NP-Complete problem, the Travelling Salesman and how it can be used to fill in the data given a sample of a low rank. We then provide guide on how it can be used to approximate another NP-Complete Problem, Max Cut.

## 2 Semidefinite Programming

### 2.1 Overview of Linear Programming

To give background on semi-definite programming, we begin with a brief recap of linear programming. Suppose that you have control over a set of variables and you are attempting to find a selection for each of these variables such that some linear combination is either maximized or minimized. Formally we can write this out as the following. Suppose that we have  $n$  variables that we have control over. This is represented as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

This problem is made trivial by setting  $\vec{x}$  to be arbitrarily large or small value depending on the optimization direction. The solution becomes more meaningful if the domain of each variable is constrained.

Say for each of these  $n$  variables has an associated coefficient  $\vec{c} = [c_1, c_2, \dots, c_n]^T$ . We would like to find  $\vec{x}$  such that  $\vec{c} \cdot \vec{x}$  is minimized

$$\min \quad \vec{c} \cdot \vec{x}$$

Suppose we have vectors  $\vec{a}_i$  and  $\vec{b}$  in  $\mathbb{R}^n$ . Extending the above idea to a system of equations, we would like to find  $\vec{x}$  such that

$$\begin{aligned} \vec{a}_1 \cdot \vec{x} &\geq b_1 \\ \vec{a}_2 \cdot \vec{x} &\geq b_2 \\ &\vdots \\ \vec{a}_n \cdot \vec{x} &\geq b_n \end{aligned}$$

Graphically, each constraint creates a *half-space*. The intersection of the halfspaces (the white space) is our solution set. Our goal is the find the minimize value of the intersections.

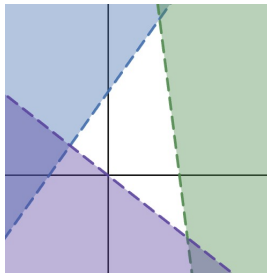


Figure 1: Graphical representation of linear inequalities

We can gather our  $\vec{a}_i$  vectors into a single matrix. Let  $\mathbf{A} = [\vec{a}_1^\top \cdots \vec{a}_n^\top]$ . so that we only need to solve:

$$\mathbf{A}\vec{x} \geq \vec{b}$$

It is often the case we would like  $\vec{x}$  to be nonnegative.

$$\vec{x} \geq 0$$

In summary, a linear programming is a technique used to find an optimal solution,  $\vec{x}$  when we are given a set of linear constraints.

$$\begin{aligned} &\text{minimize} && \vec{c} \cdot \vec{x} \\ &\text{subject to} && \mathbf{A}\vec{x} \geq \vec{b} \\ &&& \vec{x} \geq 0 \end{aligned}$$

We note that our constraints form a convex set. Thus, linear programming is a subset of a convex optimization problem. Finding a solution to an instance of linear programming can be done with a variety of algorithms including the *simplex algorithm*.

## 2.2 Formalization of Semidefinite Programming

With linear programming, we optimize over  $n$  variables. Now, suppose we have  $n^2$  elements. Thus instead of finding an optimal  $\vec{x} \in \mathbb{R}$ , we'd like to find an optimal  $\mathbf{X} \in \mathbb{R}^{n \times n}$ . A common interpretation of positivity for matrices is *positive semi-definite* (PSD). Positive semidefinite matrices are symmetric and when operated upon, will output a positive number. That is, for arbitrary  $\vec{v}$ ,  $\vec{v}^\top \mathbf{X} \vec{v} > 0$ . We constraint  $\mathbf{X}$  to be positive semi-definite or  $\mathbf{X} \succeq 0$ . Let  $\mathbf{C} \in \mathbb{R}^{n \times n}$  be a matrix where  $c_{i,j}$  is the coefficient corresponding to the  $x_{i,j}$  variable. Generalizing ideas introduced in linear programming, we have

$$\begin{aligned} \min & \sum_{i=1}^n \sum_{j=1}^n x_{i,j} c_{i,j} \\ \text{s.t} & \mathbf{X} \succeq 0 \end{aligned}$$

We introduce the  $\bullet$  operator. This is shorthand for performing an element wise matrix multiplication on its two operands and subsequently summing each element. Formally,

$$\mathbf{C} \bullet \mathbf{X} = \sum_{i=1}^n \sum_{j=1}^n c_{i,j} x_{i,j}$$

Of important note is the fact that  $\mathbf{C} \bullet \mathbf{X} = \text{trace}(\mathbf{C}^\top \mathbf{X})$  where trace is the sum of the diagonal terms for a matrix. Calculating the trace of the matrix multiplication is not as efficient as the  $\bullet$  operator but the notation comes up quite frequently within the literature so it is important to know.

Returning back to our formalization of SDP, we note that with more variables comes with more constraints. Suppose that we have  $m$  equations to satisfy. For each equation we have a matrix  $\mathbf{A}_i$  and a scalar value  $b_i$  defining our constraint. Then for a given constraint we have:

$$\mathbf{A}_i \bullet \mathbf{X} \geq b_i$$

This is all that is needed for SDP. We can write this out formally as:

$$\begin{array}{ll} \min & \mathbf{C} \bullet \mathbf{X} \\ \text{subject to} & \mathbf{A}_i \bullet \mathbf{X} \geq b_i \quad i = 1, \dots, m \\ & \mathbf{X} \succeq 0 \end{array}$$

The parallels between semidefinite programming and linear programming should be quite apparent. In many ways, semidefinite programming is linear programming with more variables and more constraints.

## 2.3 Semidefinite Programming Duality

It won't be explained in depth here, but it is also useful to know what the dual of SDP looks like. From before we know that SDP is defined as

$$\begin{array}{ll} \min & \mathbf{C} \bullet \mathbf{X} \\ \text{subject to} & \mathbf{A}_i \bullet \mathbf{X} \geq b_i \quad i = 1, \dots, m \\ & \mathbf{X} \succeq 0 \end{array}$$

We note that we have to satisfy  $m$  equations. Then the dual is defined as:

$$\begin{array}{ll} \max & \sum_{i=1}^m z_i b_i \\ \text{such that} & \sum_{i=1}^m z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C} \\ & \mathbf{S} \succeq 0 \end{array}$$

Where we are trying to find a set of scalar values  $z_1, z_2, \dots, z_m$  such that the dot product  $\vec{z} \cdot \vec{b}$  is maximized while also satisfying that  $\sum_{i=1}^m z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C}$  where  $\mathbf{A}_i$  and  $\mathbf{C}$  are from before. We know that  $\mathbf{S} \succeq 0$  which gives us the slightly more intuitive:

$$\mathbf{C} - \sum_{i=1}^m z_i \mathbf{A}_i \succeq 0$$

## 2.4 Semidefinite Programming Running Time

The only reason why SDP would be useful at all in the first place is if it was able to be solved in polynomial time. There are a variety of algorithms that can solve SDPs. One such algorithm is Alizadeh's interior point method which runs in:

$$\tilde{O}(n^{3.5})$$

## 3 Travelling Salesman Relaxation

Imagine you are prospective student touring RPI. You have a list of buildings you want to visit. You have on hand the distance between each pair of buildings. Is it possible to find a path such that each building is visited exactly once? If so, what is the shortest possible distance you will need to travel. Since you touring the college, it is likely you have some method of transportation to come from and need to come back to . We have the added constraint that you start at the parking lot and will need to come back to the parking lot. This is know as the euclidean travelling salesman. We say euclidean as the distances between two points are distances is the two norm. As our goal is to find an *efficient* route, we will be satisfied with an approximation. Or a series of buidlings to visit that is close the optimal.

### 3.1 Relaxation For Traveling Salesman

Let  $C \in \mathbb{R}^{n \times n}$  denote the matrix of edge costs. Let  $J$  denote the all-ones matrix, and  $e$  denote the all-ones vector.

$$\begin{aligned}
 & \text{minimize} && \frac{1}{2} \text{trace}(CX) \\
 & \text{subject to} && Xe = 2e \\
 & && X_{ii} = 0, \quad i = 1, \dots, n \\
 & && 0 \leq X_{ij} \leq 1, \quad i, j = 1, \dots, n \\
 & && 2I - X + (2 - 2 \cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0 \\
 & && X \text{ is a real, symmetric } n \times n \text{ matrix.}
 \end{aligned}$$

$X$  is a fractional adjacency matrix, meaning for  $e = \{i, j\}$ ,  $x_{ij} = x_{ji}$  is the proportion of edge  $e$  used.

#### 3.1.1 Explanation For Each Constraint

1.  $Xe = 2e$  means the sum of each row in the adjacency matrix should equal to 2. This is trivial as we have an origin and destination as 2 suggests.
2.  $X_{ii} = 0$  is trivial as the distance from a city to itself is always 0.
3.  $0 \leq X_{ij} \leq 1, \quad i, j = 1, \dots, n$  is a relaxation that does not enforce an exact solution. i.e, we can have a fractional value for each edge.
4.  $2I - X + (2 - 2 \cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0$  is the tricky part of the formulation. An understanding is not required but an explanation is given below.

The constraint  $2I - X + (2 - 2 \cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0$  is related to the idea of graph spectrum, Laplacian of Hamiltonian cycle, and algebraic connectivity. Here are some facts needed in order to understand the meaning of the constraint:

1. Let  $G_C = (V, E, C)$  denote a C-edge-weighted graph with nonnegative edge values, Laplacian  $L(G_C)$  is defined as  $L(G_C) = \text{diag}(r_1, \dots, r_n) - C$ , where  $r_i$  is the sum of the i-th row of  $C$ .
2.  $L(G_C)$  is symmetric, positive semidefinite with the smallest eigenvalue  $\lambda_1 = 0$  and the corresponding eigenvector  $e$ .
3. The algebraic connectivity  $\alpha(G_C)$  is the second smallest eigenvalue of  $L(G_C)$ . There are two main properties: (1).  $\alpha(G_C) = \min_{x \in S} x^T L(G_C) x$  (2).  $\alpha(G_C) \geq 0$ , and  $\alpha(G_C) > 0$  iff  $G_C$  is connected.
4. Algebraic connectivity of a Hamiltonian circuit is well known in the theory of graph spectra. The Laplacian of a circuit with  $n$  vertices has the spectrum

$$2 - 2\cos(2\pi j/n), j = 1, \dots, n$$

and the second smallest eigenvalue is obtained for  $j = 1$  and  $j = n - 1$ , i.e.  $\lambda_2 = \lambda_3 = 2 - 2\cos(2\pi/n)$ . This value will be denoted by  $h_n$ , i.e.  $h_n = 2 - 2\cos(2\pi/n)$ .

The constraint is making sure weighted graph corresponding to  $X$  (as a weighted adjacency matrix) is at least as connected as a cycle graph, with respect to algebraic connectivity.  $2I - x$  gives us the Laplacian matrix corresponding to the weighted graph according to entries of  $x$ .  $J$  makes sure we ignore trivial eigenvalues of Laplacian  $\lambda = 0$ .  $2 - 2\cos(2\pi/n)$  makes sure that the non-trivial eigenvalues is greater than or equal to the second smallest eigenvalue of a connected Hamiltonian Cycle.

### 3.2 Performance Guarantees

The integrality gap is the ratio between our estimated solution and the optimal solution. Due to properties of hamiltonain cycles we will not delve into, the integrality gap is unbounded. The time complexity is polynomial. We will prove it empirically in the Experimental Results.

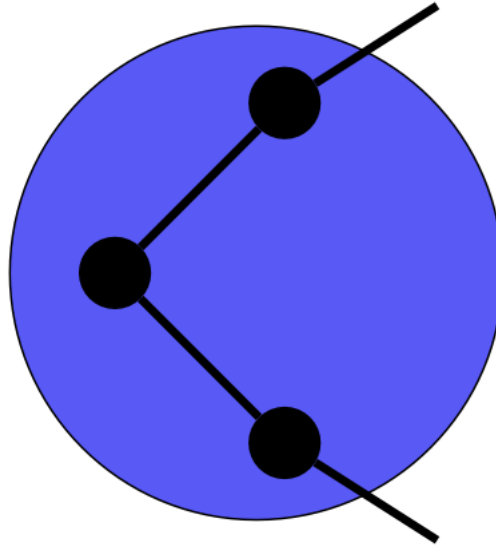


Figure 2: Caption: A node in the Hamiltonian Cycle

### 3.3 Integrality Gap And Running Time

Table 1: SDP/Brute Force Methods Comparison

# Of Nodes	SDP Time	BF Time	SDP Objective Value	BF Objective Value	Integrity Gap	Time Ratio
10	0.7101	0.0156	53224.4854	53228.3976	0.9999	45.519
15	0.6776	0.8224	65753.5934	67299.5625	0.9770	0.8239
20	1.2271	97.2059	69558.9865	76199.4928	0.9129	0.0126
21	1.3689	266.7778	73969.6527	77373.6362	0.9560	0.0051
22	5.4774	657.7847	66459.7265	68245.9576	0.9738	0.0083

1. BF denotes the brute force method and was implemented using dynamic programming
2. Integrity Gap is the ratio between SDP Objective Value and BF Objective Value

Our SDP solver grows much more slowly as compared to the brute force approach. The objective values of SDP is lower than the true optimal solution (as indicated in BF Objective Value). For our small examples, we are able to find a good lower bound.

### 3.4 Visualization

Our SDP relaxation will yield an objective value representative of the lower bound. The final  $\mathbf{X}$  matrix is a fractional adjacency matrix. As such, for an edge with a high value, it can be interpreted as being a high probability of being an edge in the Hamiltonian cycle. To find the Hamiltonian cycle, we set the edge that has value greater than or equal to the  $3^{rd}$  largest value in that row to 1 and else 0 to find the adjacency matrix. Then we use the adjacency matrix and coordinates of points to plot the results.

We find that it is easy to break the Hamiltonian cycle, especially as the number of nodes increases.

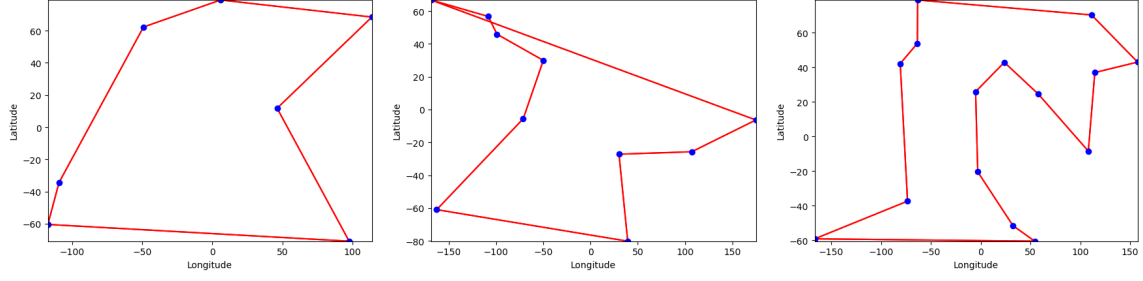


Figure 3: Sets of the Problems that SDP Relaxation is able to find a reasonable solution

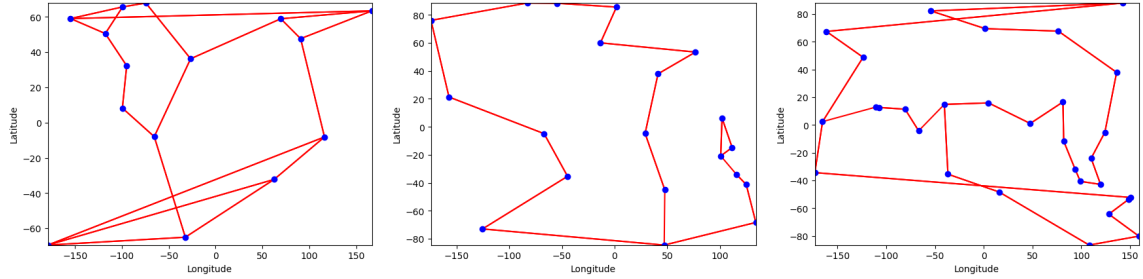


Figure 4: Sets of the Problems that SDP Relaxation is unable to find a resonable solution

## 4 Matrix Completetion

Suppose we have a incomplete sampling of some data. Are we able to fill in the missing values? When dealing with low dimensional data, given a representative sampling, it is possible to recover the missing entires. Low rank matrices are special in that they have a sparse set of singular values. This means much of the data can be described in few eigenvector. Eigenvectors describes the variance of the data. Eigenvectors associated with larger eigenvalues retains a larger variance, while eigenvectors of the smaller eigenvalues do not hold as much value.

We will walk through how to formulate a low rank matrix completion problem into a semidefinite program problem. We then show how it performs with some low rank images.

### 4.1 Relaxation

Suppose we have a low rank matrix  $\mathbf{M}$ . We have a set of location  $\Omega$  describing our sampling. That is, if  $(i, j) \in \Omega$ , we observe entry  $M_{ij}$ . Given  $\mathbf{M}$  is low rank, it seems resonable that we would like to solve the following optimization problem

$$\begin{aligned} & \text{minimize} && \text{rank}(\mathbf{X}) \\ & \text{subject to} && X_{ij} = M_{ij} \quad (i, j) \in \Omega \\ & && \mathbf{X} \in \mathbb{R}^{n \times n} \end{aligned}$$

We make the assumption that  $|\Omega|$  is sufficiently large and  $\mathbb{X}$  is uniformly distrubited. This makes it reasonable to assume there exists only one low rank matrix under the constraints. However, the rank is not a convex. The above problem is very hard to solve. In fact, it is NP-hard.

We instead rely on the nuclear norm, a close approximation of the rank. The nuclear norm of a matrix  $\mathbf{X}$  is defined as the sum of the eigenvalues.

$$\|\mathbf{X}\|_* = \sum_{k=1}^n \sigma_k \mathbf{X}$$

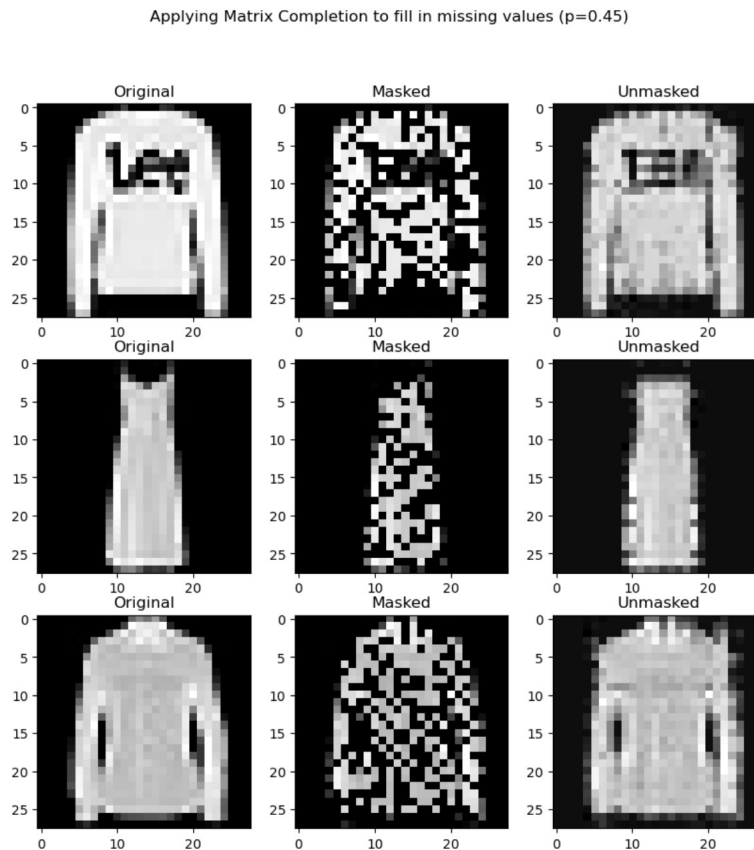
For a symmetric positive semi-definite (SPSD) matrices, the nuclear norm is equal to the trace. However, the matrix we would like to recover,  $\mathbf{M}$  is not necessarily SPSP. As this is quite the constraint on the input data, we augment  $\mathbf{X}$  with two matrices  $\mathbf{W}_1$  and  $\mathbf{W}_2$  to create a SPSP matrix. Solving the following will yield an  $\mathbf{X}$  that recovers  $\mathbf{M}$  given the entries specified by  $\Omega$ .

$$\begin{aligned} & \text{minimize} && \text{trace}(\mathbf{W}_1) + \text{trace}(\mathbf{W}_2) \\ & \text{subject to} && X_{ij} = M_{ij} \quad (i, j) \in \Omega \\ & && \begin{bmatrix} \mathbf{W}_1 & \mathbf{X} \\ \mathbf{X}^\top & \mathbf{W}_2 \end{bmatrix} \succeq 0 \end{aligned}$$

## 4.2 Image recovery

Suppose you have a corrupted image. You have knowledge which pixels remains uncorrupted. Can we use matrix completion to fill in the corrupted portion?

We test how matrix completion work with the Fashion-MNIST data. Each image is a  $28 \times 28$  grayscale image. We corrupt the images by masking a percentage of the pixels. Next, we used the relaxation described above to recover the original image.



We find that we are still able to recover a good representation of the original given 55% of the data (remove  $p = 45\%$ ). We note for large images, a smaller percentage data is needed to recover to  $\mathbf{M}$ . For smaller matrices, each entry is weighted more heavily. Errors would also be magnified.

## 5 Appendix

### 5.1 Problem Set

We will now give one application of semidefinite programming in approximating an NP-Hard problem. We will first formalize the SDP and then use to find the approximation.

Consider the max-cut problem where for graph  $G$  the goal is to split the vertices into two disjoint sets  $S$  and  $\bar{S}$  such that the sum of the edges between the two sets is maximum. Thus we know that  $S \cup \bar{S} = V(G)$  and  $e(i, j)$  is the weight of the edge between vertex  $v_i$  and vertex  $v_j$ . Thus our optimization problem is:

$$\max \sum \{e(i, j) : v_i \in S \text{ and } v_j \in \bar{S}\}$$

We will collect all our edge weights into matrix  $\mathbf{W}$  where  $w_{i,j}$  is the edge weight between vertex  $v_i$  and  $v_j$ .

We will work our way towards a formalization of SDP.

#### 5.1.1 Question 1.

For all of our vertices we have a corresponding variable  $x_i$  such that:

$$\begin{aligned} x_i &\in \{-1, 1\} \\ \text{if } x_i &\in S \text{ then } x_i = 1 \\ \text{if } x_i &\in \bar{S} \text{ then } x_i = -1 \end{aligned}$$

Then for two vertices  $v_i, v_j \in V(G)$  what is the intuition behind the following equation:

$$w_{i,j}(1 - x_i x_j)$$

Answer: If  $v_i$  and  $v_j$  are in the same set the above equation will be zero otherwise  $w_{i,j}$  will be returned. The weight will only get returned if the edge between  $v_i$  and  $v_j$  is a cut edge for the sets  $S$  and  $\bar{S}$ .

#### 5.1.2 Question 2.

How can we turn  $x_i x_j$  from above into a matrix given that  $\vec{x} = [x_1, x_2, \dots, x_n]$ . This matrix should be square with the number of rows equal to the number of vertices in our graph.  $\mathbf{X}_{i,j} = x_i x_j$ : Answer:  $\mathbf{X} = \vec{x} \vec{x}^\top$

#### 5.1.3 Question 3.

What can you say about the values on the diagonal of  $\mathbf{X}$ ? Is  $\mathbf{X}$  symmetric?

#### 5.1.4 Question 4.

Before we found that our optimization problem was:

$$\max \sum \{e(i, j) : v_i \in S \text{ and } v_j \in \bar{S}\}$$

And in question 1. we found that for a single vertex pair we have:

$$w_{i,j}(1 - x_i x_j)$$

How can we rewrite our optimization problem to account for all vertex pairs using  $\mathbf{X}$  and  $\mathbf{W}$  and the  $\bullet$  operator. Ensure that in your solution that no edge is counted twice:



Answer:

$$\begin{aligned}
 &= \max \frac{1}{2} \left( \sum_{i=1}^n \sum_{j=1}^n [w_{i,j}(1 - x_i x_j)] \right) \\
 &= \max \sum_{i=1}^n \sum_{j=1}^n [w_{i,j} - w_{i,j} x_i x_j] \\
 &= \max \frac{1}{2} \left( \sum_{i=1}^n \sum_{j=1}^n [w_{i,j}] - \mathbf{W} \bullet \mathbf{X} \right)
 \end{aligned}$$

The  $\frac{1}{2}$  comes from the fact that  $\mathbf{X}$  and  $\mathbf{W}$  are symmetric and thus each cut edge is counted twice.

### 5.1.5 Question 5.

We have formalized the value that we want to maximize in the previous question. One of our constraints for SDP is that  $\mathbf{X}$  has to be SPSD. Thus we know that  $\mathbf{X} \succeq 0$ . Currently our constraints are that for  $x_{i,j} \in \mathbf{X}$  that  $x_{i,j} \in \{-1, 1\}$ . Such a hard constraint is more specific than SDP can solve. We can relax our problem such that  $\mathbf{X} \succeq 0$  and  $x_{i,i} = 1$  and dropping our previous  $x_{i,j} \in \{-1, 1\}$ . What is the intuition behind having  $x_{i,i} = 1$ ?

Answer:  $x_{i,i} = 1$  ensures that vertex  $v_i$  is only in one set. This is represented by  $x_i x_i = 1$ .

Formally write out the SDP problem:

$$\begin{aligned}
 &\max \frac{1}{2} \left( \sum_{i=1}^n \sum_{j=1}^n [w_{i,j}] - \mathbf{W} \bullet \mathbf{X} \right) \\
 &\quad \mathbf{X}_{i,i} = 1 \\
 &\quad \mathbf{X} \succeq 0
 \end{aligned}$$

### 5.1.6 Question 6.

We have now formalized an SDP problem for an approximation of the max-cut problem. Solving this will give us a lower bound for the max-cut problem. However we can get an approximate solution. In python generate  $n$  random 2-dimensional points. Find the distance between each pair of points. Each point will be a vertex in our graph and the pairwise distances between points will be our edges. `scipy.spatial.distance.cdist` is useful for this

Answer:

```

n = 20
points = np.random.rand(n, 2)
adj_matrix = cdist(points, points)
W = adj_matrix

```

Using the cvxpy example as a reference (<https://www.cvxpy.org/examples/basic/sdp.html>) setup the SDP problem and solve it. Your solution should look something like the following (here we set  $n = 5$ ):

```

>>> print(X.value)
[[ 1.          1.00000039 -1.00000052 -1.00000054  1.00000065]
 [ 1.00000039  1.          -1.00000055 -1.00000057  1.00000072]
 [-1.00000052 -1.00000055  1.          1.00000033 -1.00000081]
 [-1.00000054 -1.00000057  1.00000033  1.          -1.00000083]
 [ 1.00000065  1.00000072 -1.00000081 -1.00000083  1.          ]]

```

What is the lower bound for the max-cut problem?

**5.1.7 Question 7.**

For a value  $x_{i,j}$  if  $x_{i,j} < 0$  then  $v_i$  and  $v_j$  are in different sets. Determine these two sets, plot the points of these two sets using two different colors and show the result:

**5.1.8 Question 8.**

What is the value of the max-cut for the two sets that you found?

**5.1.9 Extra Credit**

- The SDP problem that we formulated was an approximation for the max-cut problem - not the optimal solution. If we were able to set  $x_{i,j} \in \{-1, 1\}$  then we would have an optimal solution. We will try and see how well the SDP solver did:
- Create a brute force max-cut solver and compare the results to what the SDP found.
- Note that such a brute force solver will have a  $O(2^n)$  runtime so be careful for large values of  $n$ .
- Using a simulation approach, how well of a bound does the SDP problem set?
- In question 7. we got a potential max-cut. How well does this compare to the actual optimal max-cut?

## 5.2 References

- [https://users.math.msu.edu/users/iwenmark/Teaching/MTH995/Papers/SDP\\_notes\\_Marina\\_Epelman\\_UM.pdf](https://users.math.msu.edu/users/iwenmark/Teaching/MTH995/Papers/SDP_notes_Marina_Epelman_UM.pdf)
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