Semidefinite Programming

Applications in approximating NP-Complete problems & Matrix Completetion

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Presentation Overview

Motivation for Semidefinite Programming

- Linear Programming is a common constrained optimization technique with uses in:
  - Math
  - Computer Science
  - Economics
  - Business
- Wide applicability combined with fast (polynomial) runtimes has made linear programming quite popular

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- Semidefinite programming (SDP) expands upon the ideas of linear programming
  - SDP can solve everything that linear programming can and more
  - Widely used in combinatorial optimzation problems
  - Many NP-Hard problems can be approximated well with SDP
- Both linear programming and semidefinite programming are convex problems

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- Where for each of these n variables we have an associated coefficient $\vec{c} = [c_1, c_2, \ldots, c_n]^\top$
- In the end we want to find the optimal value for the following equation:

$$\min \vec{c} \cdot \vec{x}$$

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- The problem is quite easy as is. What if each of these *n* variables corresponds to how much of a given product that we want to buy?
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$$\vec{a}_1 \cdot \vec{x} \leq b_1$$
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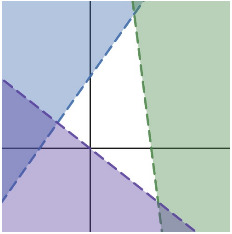
Graphical Representation of Linear Inequalities

- Each of the n constraints create a *half-space* - the viable region that $\vec{x}$ can take on.
- The intersection of the half-spaces is our solution set / feasible region. This is given by the white space below:



- Our goal is to find the value of $\vec{x}$ that optimizes $\min \vec{c} \cdot \vec{x}$ and is within the feasible region

Converting Inequalities to Equalities

- Both linear programming and semidefinite programming are often give with inequalities
- We can transform an inequality to an equality by introducing a new x_{n+1} variable. We note that $x_{n+1} \geq 0$. As an example:

$$\vec{a}_k \cdot \vec{x} \leq b_k \Longrightarrow \vec{a}_k \cdot \vec{x} + x_{n+1} = b_k$$

- We often prefer these equalities instead of the inequalities. Thus our constraints are:

$$\mathbf{A}\vec{x} = \vec{b}$$

Standard Form for a Linear Program

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Linear Programming to Semidefinite Programming

- Semidefinite programming (SDP) takes the concept that linear programming has with vectors and generalizes it to matrices.
- The $\langle\rangle_{\mathrm{F}}$ operator is the Frobenius inner product which is the sum of element wise multiplication on vectors: $\langle \mathbf{C}, \mathbf{X}\rangle_{\mathrm{F}} = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{i,j}x_{i,j} = trace(\mathbf{C}^\top \mathbf{X})$
- $\mathbf{X} \succeq 0$ means that $\mathbf{X}$ is positive semi-definite (PSD)

Linear Programming	Semidefinite Programming
$\vec{x} \in \mathbb{R}^n$	$\mathbf{X} \in \mathbb{R}^{n\times n}$
$\vec{x} \geq 0$	$\mathbf{X} \succeq 0$
$\vec{c} \in \mathbb{R}^n$	$\mathbf{C} \in \mathbb{R}^{n\times n}$
$\min \vec{c} \cdot \vec{x}$	$\min \langle \mathbf{C}, \mathbf{X}\rangle_{\mathrm{F}}$
$\mathbf{A} \in \mathbb{R}^{n\times n}, \vec{b} \in \mathbb{R}^n$	$\mathbf{A}_i \in \mathbb{R}^{n\times n}, \vec{b} \in \mathbb{R}^m$
$\mathbf{A}\vec{x} = \vec{b}$	$\langle \mathbf{A}_i, \mathbf{X}\rangle_{\mathrm{F}} = b_i : i = 1, \ldots, m$

Linear Programming to Semidefinite Programming

- A linear program is defined as:

$$\begin{array}{ll} \text{minimize} & \vec{c} \cdot \vec{x} \\ \text{subject to} & \mathbf{A}\vec{x} = \vec{b} \\ & \vec{x} \geq 0 \end{array}$$

- A semidefinite program is defined as:

$$\begin{array}{lll} \text{min} & \langle \mathbf{C}, \mathbf{X} \rangle_{\mathrm{F}} & \\ \text{subject to} & \langle \mathbf{A}_i, \mathbf{X} \rangle_{\mathrm{F}} = b_i & i = 1, \ldots, m \\ & \mathbf{X} \succeq 0 & \end{array}$$

Semidefinite Programming Duality

- It is important to note the dual of an SDP problem which is:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} z_i b_i \\ \text{such that} & \sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C} \\ & \mathbf{S} \succeq 0 \end{array}$$

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- We are trying to find a set of scalars $z_1, z_2, \ldots, z_m$
- Where our objective function is $\vec{z} \cdot \vec{b}$
- We also satisfy the constraint $\sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C}$ where $\mathbf{A}_i$ and $\mathbf{C}$ are from before.

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- We also satisfy the constraint $\sum_{i=1}^{m} z_i \mathbf{A}_i + \mathbf{S} = \mathbf{C}$ where $\mathbf{A}_i$ and $\mathbf{C}$ are from before.
- We know that $\mathbf{S} \succeq 0$ which allows us to get the more intuitive:

$$\mathbf{C} - \sum_{i=1}^{m} z_i \mathbf{A}_i \succeq 0$$

- Pulling it all together:

$$\begin{array}{ll} \max & \sum_{i=1}^{m} z_i b_i \\ \text{such that} & \mathbf{C} - \sum_{i=1}^{m} z_i \mathbf{A}_i \succeq 0 \end{array}$$

Semidefinite Programming Runtime

- SDPs can be solved in polynomial time which makes them qutie useful.
- One algorithm to solve them is Alizadeh's interior point method which runs in:

$$\tilde{O}(n^{3.5})$$

Reviewing TSP

The Traveling Salesman Problem (TSP) is an optimization problem in which the objective is to find the shortest possible route for a salesman to visit a given set of cities, passing through each city exactly once, and returning to the starting city. It is a well-known NP-hard problem.

Semidefinite Programming Methods for the Symmetric Traveling Salesman Problem , 1999

Let $C \in \mathbb{R}^{n \times n}$ denote the matrix of edge costs. Let J denote the all-ones matrix, and e denote the all-ones vector.

$$\begin{array}{ll} \text{minimize} & \frac{1}{2}\text{trace}(CX) \\ \text{subject to} & Xe = 2e \\ & X_{ii} = 0, \quad i = 1, \ldots, n \\ & 0 \leq X_{ij} \leq 1, \quad i, j = 1, \ldots, n \\ & 2I - X + (2 - 2\cos\left(\frac{2\pi}{n}\right))(J - I) \succeq 0 \\ & X \text{ is a real, symmetric } n \times n \text{ matrix.} \end{array}$$

X is a fractional adjacency matrix, meaning for $e = \{i, j\}$, $x_{ij} = x_{ji}$ is the proportion of edge e used.

Integrality Gap And Running Time

# Of Nodes	SDP Time	BF Time	SDP Objective Value	BF Objective Value	Integrity Gap	Time Ratio
10	0.7101	0.0156	53224.4854	53228.3976	0.9999	45.519
15	0.6776	0.8224	65753.5934	67299.5625	0.9770	0.8239
20	1.2271	97.2059	69558.9865	76199.4928	0.9129	0.0126
21	1.3689	266.7778	73969.6527	77373.6362	0.9560	0.0051
22	5.4774	657.7847	66459.7265	68245.9576	0.9738	0.0083

Visualization

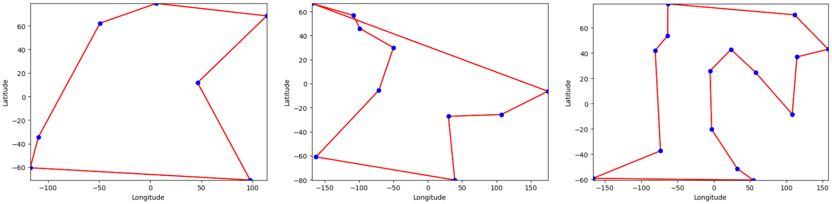

Figure: reasonable solution

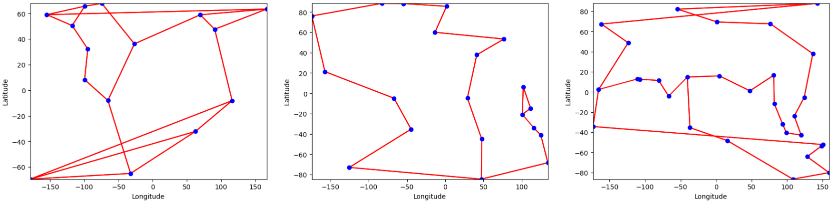

Figure: unreasonable solution

Low rank matrices

Given an incomplete matrix, can we recover the missing values?

		-1		
			1	
1	1	-1	1	-1
1				-1
		-1		

1	1	-1	1	-1
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Yes!

Given:

- The matrix is low rank*
- We have enough sample data

Note: This does not apply to *all* low-rank matrices. But most.

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Why low-rank matrices?

Why is this useful

1. **Netflix** has an incomplete set of user preferences based off their past watch history. Can they use this information to recommend new movies?
2. **Recommendation Engine:** The netflix problem can be extended to general recommendation engines where a vendor knows some of the user preferences.
3. **Images:** We will give an example of recovering a corrupted image using matrix completetion

Relaxing Matrix Completetion to SDP

Suppose we have a low rank matrix **M**. We have a set of location Ω describing our sampling. That is, if $(i, j) \in \Omega$, we observe entry M_{ij}. Given **M** is low rank, it seems resonable that we would like to solve the following optimization problem

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$$\begin{array}{ll} \text{minimize} & \operatorname{rank}(\mathbf{X}) \\ \text{subject to} & X_{ij} = M_{ij} \quad (i, j) \in \Omega \\ & \mathbf{X} \in \mathbb{R}^{n \times n} \end{array}$$

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But...

Rank is not a convex. This turns out to be an NP-Hard Problem.

Introduce the nuclear norm

Nuclear Norm

The nuclear norm is a close approximation of the rank.

The nuclear norm of a matrix $\mathbf{X}$ is defined as the sum of the eigenvalues.

$$\|\mathbf{X}\|_* = \sum_{k=1}^{n} \sigma_k \mathbf{X}$$

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For a symmetric positive semi-definite (SPSD) matricies, the nuclear norm is equal to the trace.

A better relaxation

What if our matrix is not SPSD

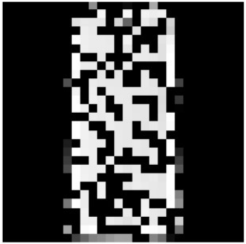
- We introduce two matricies $\mathbf{W}_1$ and $\mathbf{W}_2$

A better relaxation

$$
\begin{array}{ll}
\text{minimize} & \operatorname{trace}(\mathbf{W}_1) + \operatorname{trace}(\mathbf{W}_2) \\
\text{subject to} & X_{ij} = M_{ij} \quad (i,j) \in \Omega \\
& \begin{bmatrix} \mathbf{W}_1 & \mathbf{X} \\ \mathbf{X}^\top & \mathbf{W}_2 \end{bmatrix} \succeq 0
\end{array}
$$

Fashion-MNIST

55% of data



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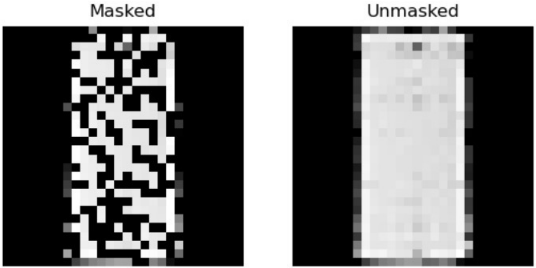

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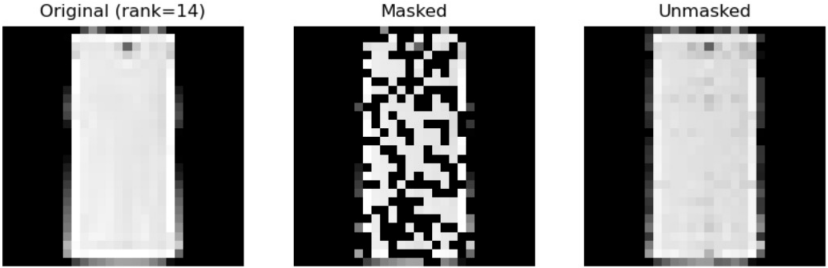

Fashion-MNIST

50% of data

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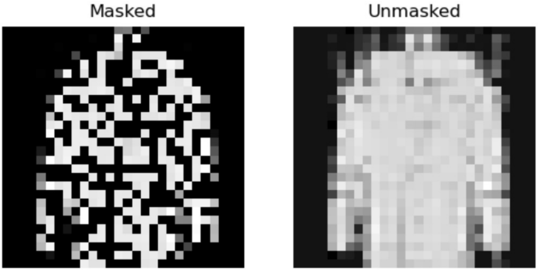

Fashion-MNIST

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Citing References

An example of the `\cite` command to cite within the presentation:

This statement requires citation [Smith, 2022, Kennedy, 2023].

References

John Smith (2022)
Publication title
Journal Name 12(3), 45 – 678.

Annabelle Kennedy (2023)
Publication title
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