CSCI-4968: ML and Optimization Spring 2023

Semidefinite Programming

For approximating Max Cut and Travelling Salesman

1 Motivation

TODO: Update this section (matrix completion)

1.1 Max Cut

TODO: Fix We would like to partition our graph into two set of vertices such that the sum of the edge weights between the two partitiions is maximized.

Applications include examples: Clustering - Intuition: The distances between the two partitiions are the greatest so they have a high likelihood of belonging to two different categories.

1.2 Travelling Salesman Problem

TODO: Fix Imagine you are prospective student touring RPI. You have a list of buildings you want to visit. You have on hand the distance between each pair of buildings. Is it possible to find a path such that each buidling is visited exactly once? If so, what is the shortest possible distance you will need to travel. Since you toruing the college, it is likely you have some method of transportation to came from and need to come back to . We have the added constraint that you start at the parking lot and will need to come back to the parking lot. This is know as the travelling salesman. As our goal is to find an *efficient* route, we will be satisfied with an approximation. Or a series of buidlings to visit that is close the optimal.

2 Semidefinite Programming

2.1 Recap of Linear Programming

To give background on semi-definite programming we first start with a brief recap of linear programming. Suppose that you have control over a set of nonnegative variables and you are attempting to find a selection for each of these variables such that some linear combination is maximized / minimized. To make this more complicated suppose that these variables each have a constraint that must be met.

Formally we can write this out as the following. Suppose that we have n variables that we have control over which we can represent as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

Say for each of these n variables we have an associated coefficient $\vec{c} = \begin{bmatrix} c_1, c_2, \dots, c_n \end{bmatrix}^\top$ and in the end we want to pick $\vec{x}$ such that the linear combinatrion of the two $(\vec{c} \cdot \vec{x})$ is

$$\min \vec{c} \cdot \vec{x}$$

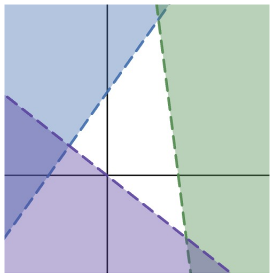
Say for each of these n variables has an associated coefficient $\vec{c} = \begin{bmatrix} c_1, c_2, ..., c_n \end{bmatrix}^\top$. We would like to find $\vec{x}$ such that the linear combinatrion of the two $(\vec{c} \cdot \vec{x})$ is minimized

$$\min \quad \vec{c} \cdot \vec{x}$$

We can extend this idea to a system of equations. We define $\vec{a}_i$ and $\vec{b}$ to vectors in $\mathbb{R}^n$. We would like to find $\vec{x}$ such that

$$\begin{aligned}\vec{a}_1 \cdot \vec{x} &= b_1\\ \vec{a}_2 \cdot \vec{x} &= b_2\\ &\vdots\\ \vec{a}_n \cdot \vec{x} &= b_n\end{aligned}$$

Graphically, each constraint creates a *half-space.* The intersection of the halfspaces (the white space) is our solution set. Our goal is the find the minimize value of the intersections.



We can gather our $\vec{a}_i$ vectors into a single matrix. Let $\mathbf{A} = [\vec{a_1}^\top \cdots \vec{a_n}^\top]$. Solving the above systems of equations is equivalent to solving: $\ldots$
$_n \cdot \vec{x} = b_n$

We can gather our $\vec{a}_i$ vectors into a single matrix

$$\mathbf{A} = \begin{bmatrix} \ldots \vec{a}_1 \ldots \\ \ldots \vec{a}_2 \ldots \\ \vdots \\ \ldots \vec{a}_n \ldots \end{bmatrix}$$

so that we only need to solve:

$$\mathbf{A}\vec{x} = \vec{b}$$

In summary, a linear programming asks for an optimal solution, $\vec{x}$, given a set of linear constraints

$$\begin{array}{ll}\text{minimize} & \vec{c} \cdot \vec{x}\\ \text{subject to} & \mathbf{A}\vec{x} = \vec{b}\vec{x} \geq 0\end{array}$$

As an example of this, suppose that you

We note that our constraints form a convex set. Thus, linear programming is a subset of a convex optimization problem Finding a solution to an instance of linear programming can be done with a variety of algorithms including *simplex algorithm.*

Many real world problems can be modelled as a set of linear contraints. One use case arises in manufacturing. Given a set of products, how much of each kind should be produced to either minimize cost or maximize profits?

2.2 Formalization of Semidefinite Programming

Instead of having just n variables, suppose that we have n^2 variables. Thus instead of finding a $\vec{x}$ we are trying to find $\mathbf{X}$. With more variables comes more constraint so suppose that we n equations to satisfy. For each equation we have a matrix $\mathbf{A}_i$ and a scalar value b_i defining our constraint. Then for a given constraint we have:

It is needed that $\vec{x}$ is nonnegative ($\vec{x} \geq 0$). This makes sense given our constraint problem as it w

What is nice about linear programming is that it is a convex optimization problem. Finding a solution to an instance of linear programming can be done with a variety of algorithms including the simplex algorithm.

Linear programming has a large number of applications as it is a general approach of taking a constrained linear optimization problem into a form where a solution can easily be found.

2.3 Formalization of Semidefinite Programming

Instead of having just n variables that we have a matrix $\mathbf{X}$ of n^2 elements. Thus instead of finding an optimal $\vec{x}$ we are trying to find an optimal $\mathbf{X} \in \mathbb{R}^{n \times n}$ that is symmetric positive semidefinite (SPSD) such that

$$\begin{gathered} \min = \sum_{i=1}^{n} \sum_{j=1}^{n} x_{i,j} c_{i,j} \\ \mathbf{X} \succeq 0 \end{gathered}$$

Where $c_{i,j}$ is the coefficient corresponding to the $x_{i,j}$ variable and $\mathbf{X} \succeq 0$ ensures that $\mathbf{X}$ is SPSD. We can gather all n^2 coefficients into a matrix $\mathbf{C}$ and then rewrite our minimization term as:

$$\min \mathbf{C} \bullet \mathbf{X}$$

Where the $\bullet$ operator is simply taking the sum of all elements after doing element wise multiplication on two matricies $\mathbf{C}$ and $\mathbf{X}$ with the same shape. Formally we can write it as:

$$\mathbf{C} \bullet \mathbf{X} = \sum_{i=1}^{n} \sum_{j=1}^{n} \mathbf{C}_{i,j} \mathbf{X}_{i,j}$$

Of important note is the fact that $\mathbf{C} \bullet \mathbf{X} = \text{trace}(\mathbf{C}^\top \mathbf{X})$ where trace is the sum of the diagonal terms for a matrix. Calculating the trace of the matrix multiplication is not as efficient as the $\bullet$ operator but the notation comes up quite frequently within the literature so it is important to know.

Returning back to our formalization of SDP, we note that with more variables comes with more constraints. Suppose that we have n equations to satisfy. For each equation we have a matrix $\mathbf{A}_i$ and a scalar value b_i defining our constraint. Then for a given constraint we have:

$$\mathbf{A}_i \bullet \mathbf{X} = b_i$$

This formally defines SDP. In total we can write out a rigorous form of semidefinite programming.

$$\begin{aligned} \min \mathbf{C} \bullet \mathbf{X} & \\ \mathbf{A}_i \bullet \mathbf{X} &= b_i \text{ for } i = 1, 2, ..., n \\ \mathbf{X} &\succeq 0 \end{aligned}$$

The parallels between semidefinite programming and linear programming should be quite obvious. In many ways, semidefinite programming is linear programming with more variables of control and more constraints.

3 Relaxation

3.1 Proofs

- performance guarantees - proof of relaxations (how accurate + how it works)
Integrality Gap: ratio btwen optimal / relaxation

4 Experimental Results

4.1 Visualization

- table of result - for small examples, we find the acutal solution (brute force), or a solution (integer programming) and check how good our bound is

vhttps://www.cs.cmu.edu/ anupamg/adv-approx/lecture14.pdf used ”randomized rounding” to find a max cut. i.e. if p_{ij} is close to -1, then we should cut it. We attempt do do something similar

5 Discussion

- can we prove the runtime - can we prove how good of a lower bound we have? - what are some easy things to prove?

5.1 Scope and limitations

6 Appendix

6.1 Problem Set

6.2 References

https://www.uit.edu.mm/storage/2020/09/WWM-2.pdf