Group Contagion

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1 Introduction

2 Notes

2.1 Setup

Suppose there is one group leader plus n group members. Denote by $\eta_0(\omega)$ the default time of the leader, and $\eta_i(\omega), i = 1, ..., n$ the default time of the i-th member due to his/her own investment misjudgement regardless of others. As before, we define

$$\eta_i = \inf\{t \geq 0 \mid A_i(t) \geq Z_i\} \tag{1}$$

where Z_i, a collection of i.i.d. random variables, follow a given distribution, e.g. exp(1). Also, define $A_i(t) = \int_0^t \alpha_i(X_s)ds$ with some non-random default intensity $\alpha_i : \mathbb{R}^d \to [0, \infty)$ and a stochastic process $X : \Omega \to \mathbb{R}^d$ characterizing the market information. Furthermore, assume $\eta_i, i = 0, 1, ..., n$ are conditionally independent given X.

Next, we introduce a contagion factor among the regular group members. If member i defaults because of his/her own investment failure, we suppose he/she may have negative influence on other group members. Denote by $Y_{i,j}$ the indicator of whether member i brings member j down. In order to keep things tractable, we add the following important assumption.

(A1) Group member i may influence others after default only if the default is because of i's own investment failure, not because someone else brought i down.

If we do not assume (A1), then there will be chains of reactions among the group members, and because group members form a network, our analysis below will quickly become intractable.

Furthermore, we say a group defaults if and only if one of the following happens: the group leader defaults or all regular group members default, regardless of the leader. Now, we are interested in computing

$$P(\text{the group has not defaulted at time } T) \tag{2}$$

where T can be the end of a financing cycle. Also, recall $P(\eta_i > T) = E[\exp(-A_i(T))]$. Finally, a note on notation: for an index set I, $\eta_{i\in I}$ means the random vector $(\eta_i)_{i\in I}$; $\eta_{i\in I} \le T$ means $\eta_i \le T$ for all $i \in I$.

2.2 Contagion

The motivation for introducing the idea of contagion is that if one person defaults, the negative influence might propagate to others. And the extent of this negative effect should depend on time: given member i defaults at timestamp η_i, the probability of him/her taking j down before our ending period T should be a function of η_i. Intuitively, if η_i is small, there is a lot of time for the negative effect to take place on j, thus the corresponding probability should be bigger. To be more precise, define $Y_{i,j}$ (the contagion indicator random variable) to satisfy a conditional distribution,

$$\begin{aligned} P(Y_{i,j} = 1 \,|\, \eta_i) &:= q_{i,j}(T - \eta_i) \\ P(Y_{i,j} = 0 \,|\, \eta_i) &:= 1 - q_{i,j}(T - \eta_i) \end{aligned} \tag{3}$$

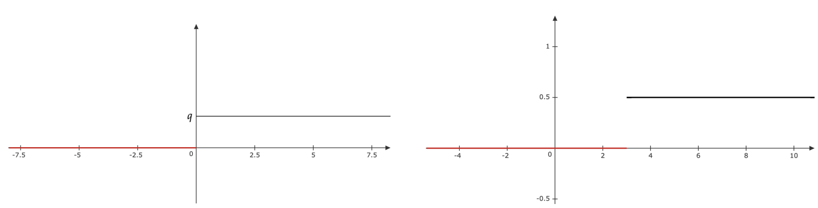
where $q_{i,j}(\cdot)$ is a non-random function of time, not a constant. In addition, we assume

(A2) $\eta_i's$ are all independent

(A3) $\{Y_{i,j}\}$ are independent among each other conditioned on $(\eta_1, ..., \eta_n)$

Because we assume people are doing businesses independently, (A2) is reasonable. For tractability, we put down (A3).

We provide two examples of $q_{i,j}(\cdot)$ below, which represent two special cases: the "immediate contagion" and "no-contagion-up-to-a-point".



(a) Immediate contagion with probability q

(b) Contagion takes place with $q = 0.5$ after time lapse $\delta = 3$

Figure 1: Two examples of $q_{ij}(\cdot)$

Proposition 1. *Given an index set $I \subset \{1, 2, ..., n\}$ representing individuals who default on their own, we have*

$$\begin{aligned} &P(\eta_{i\in I} \le T, \eta_{j\in I^c} > T \text{ and each } j \in I^c \text{ is brought down by some } i \in I) \\ &= \int \mathbb{1}(\eta_{i\in I} \le T) \cdot \prod_{j\in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i\in I} (1 - q_{i,j}(T - \eta_i)) \right] \right) dP_{\eta_{i\in I}} \end{aligned} \tag{4}$$

Proof. First, we compute the desired quantity under conditioning.

$$
\begin{aligned}
& P(\eta_{i\in I} \leq T, \eta_{j\in I^c} > T \text{ and each } j \in I^c \text{ is brought down by some } i \in I \mid \eta_{i\in I}) \\
&= \mathbb{1}(\eta_{i\in I} \leq T) \cdot E(E[\mathbb{1}(\eta_{j\in I^c} > T) \cdot \mathbb{1}(j \in I^c \text{ contagion via } I) \mid \eta_1, ..., \eta_n] \mid \eta_{i\in I}) \\
&= \mathbb{1}(\eta_{i\in I} \leq T) \cdot E[\mathbb{1}(\eta_{j\in I^c} > T) \cdot P(j \in I^c \text{ contagion via } I \,|\, \eta_1, ..., \eta_n) \,|\, \eta_{i\in I}]
\end{aligned} \tag{5}
$$

where we use the facts: $E[E(X\,|\,\mathcal{F}_1)\,|\,\mathcal{F}_2] = E(X\,|\,\mathcal{F}_2)$ if $\mathcal{F}_2 \subset \mathcal{F}_1$, and $E(XY\,|\,\mathcal{F}) = XE(Y\,|\,\mathcal{F})$ if $X \in \mathcal{F}$.

Focus now on the term $P(j \in I^c \text{ contagion via } I \mid \eta_1, ..., \eta_n)$ from above. Member j defaults by contagion when the indicator function $Y_{i,j}$ kicks in. By the independence assumption (A2), this quantity becomes

$$
\begin{aligned}
& \prod_{j\in I^c} P(\text{ j defaults by contagion from } I \mid \eta_1, ...\eta_n) \\
&= \prod_{j\in I^c} [1 - P(Y_{i,j} = 0 \text{ for all } i \in I \mid \eta_1, ..., \eta_n)] \\
&= \prod_{j\in I^c} \left[1 - \prod_{i\in I} P(Y_{i,j} = 0 \,|\, \eta_1, ..., \eta_n)\right] \\
&= \prod_{j\in I^c} \left[1 - \prod_{i\in I} (1 - q_{i,j}(T - \eta_i))\right]
\end{aligned} \tag{6}
$$

where we repeatedly use the fact $E(XY\,|\,\mathcal{F}) = E(X\,|\,\mathcal{F}) \cdot E(Y\,|\,\mathcal{F})$ if $X \perp Y$ conditioned on $\mathcal{F}$. Using (6), (5) then becomes

$$
\begin{aligned}
&= \mathbb{1}(\eta_{i\in I} \leq T) \cdot E\left(\mathbb{1}(\eta_{j\in I^c} > T) \cdot \prod_{j\in I^c} \left[1 - \prod_{i\in I} q_{i,j}(T - \eta_i)\right] \,\Bigg|\, \eta_{i\in I}\right) \\
&= \mathbb{1}(\eta_{i\in I} \leq T) \cdot \prod_{j\in I^c} \left[1 - \prod_{i\in I} q_{i,j}(T - \eta_i)\right] \cdot E(\mathbb{1}(\eta_{j\in I^c} > T) \,|\, \eta_{i\in I}) \\
&= \mathbb{1}(\eta_{i\in I} \leq T) \cdot \prod_{j\in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i\in I} (1 - q_{i,j}(T - \eta_i))\right]\right)
\end{aligned} \tag{7}
$$

where we use the assumption that all η's are independent.

Finally, the iterated expectation gives us the desired result, where notation $P_{\eta_{i\in I}}$ stands for the joint distribution of the natural defaulting time $\{\eta_{i\in I}\}$. $\square$

With the relation laid out in the proposition above, we use the next two propositions to bring the two special cases of $q_{i,j}(\cdot)$ to life. The first special case is called "immediate" contagion because right after someone defaults, there is a positive probability to bring down others. The second case is when contagion is "delayed". We then explore some properties of the "immediate" contagion scenario.

Proposition 2. *Take $q_{i,j}(t) = 0$ for $t < 0$, $q_{i,j}(t) \equiv q_{i,j} \in [0, 1]$ for $t \geq 0$; then*

$$P(\textit{group survives at time } T) = \exp(-A_0(T)) \cdot \left[1 - \sum_{k=1}^{n} \sum_{I_k \in \mathcal{I}_k^n} \beta_{I_k} \cdot \gamma_{I_k}\right] \tag{8}$$

where

$$\begin{cases} \beta_{I_k} = \prod_{i \in I_k} \left(1 - e^{-A_i(T)}\right) \\ \gamma_{I_k} = \prod_{j \in I_k^c} \left\{ e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I_k} (1 - q_{i,j})\right] \right\} \end{cases} \tag{9}$$

(Notation $\mathcal{I}_k^n$ means a collection of index sets each with cardinality k, representing all possible combinations of k out of n elements. For example, if $n = 3$ and $k = 2$, $\mathcal{I}_2^3 = \{\{1, 2\}, \{1, 3\}, \{2, 3\}\}$.)

Proof. Use the previous proposition with $q_{i,j} \equiv q_{i,j}$ to obtain

$$\begin{aligned} & P(\eta_{i \in I} \leq T, \eta_{j \in I^c} > T, \text{ j's still default}) \\ = & \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I} (1 - q_{i,j})\right] \right) \cdot \int_{\eta_{i \in I} \leq T} dP_{\eta_{i \in I}} \\ = & \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I} (1 - q_{i,j})\right] \right) \cdot P(\eta_{i \in I} \leq T) \\ = & \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I} (1 - q_{i,j})\right] \right) \cdot \prod_{i \in I} \left[1 - e^{-A_i(T)}\right] \end{aligned} \tag{10}$$

Next, from the definition of group default, we compute

$$\begin{aligned} & P(\text{group has not defaulted at } T) \\ = & 1 - P(\text{group defaults before or at } T) \\ = & 1 - P(\text{leader defaults}) - P(\text{leader stands, all regular members default}) \\ = & 1 - (1 - e^{-A_0(T)}) - e^{-A_0(T)} \cdot P(\text{all regular members default}) \\ = & \exp\left(-A_0(T)\right) \cdot \left[1 - \sum_{i=1}^{n} \sum_{I_k \in \mathcal{I}_k^n} P(\eta_{i \in I_k} \leq T, \eta_{j \in I_k^c} > T, \text{ j's still default})\right] \end{aligned} \tag{11}$$

The idea for the last equality is that we let a group of people default on their own (what I_k represents), and leave the rest to contagion. Finally, notice each probability term in (11) is already computed in (10). Thus concludes our proof. $\square$

Before we explore a couple properties of immediate contagion, we present the second special case mentioned above: no contagion until δ units of time.

Proposition 3. *Take $q_{i,j} = 0$ for $t < \delta$, $q_{i,j}(t) \equiv q_{i,j} \in [0,1]$ for $t \geq \delta$; then the surviving rate is*

$$\exp(-A_0(T)) \cdot \left\{ 1 - \prod_{i=1}^{n} \left(1 - e^{-A_i(T)}\right) - \sum_{k=1}^{n-1} \sum_{I \in \mathcal{I}_k} \sum_{\substack{I' \subset I \\ I' \neq \emptyset}} \beta_{I,I'} \cdot \gamma_{I,I'} \cdot \tau_{I,I'} \right\} \quad (12)$$

where

$$\begin{cases} \beta_{I'} = \prod_{i \in I'} \left(1 - e^{-A_i(T-\delta)}\right) \\ \gamma_{I,I'} = \prod_{i \in I - I'} \left(e^{-A_i(T-\delta)} - e^{-A_i(T)}\right) \\ \tau_{I,I'} = \prod_{j \in I^c} \left[e^{-A_j(T)} \cdot \left(1 - \prod_{i \in I'} (1 - q_{i,j})\right)\right] \end{cases} \quad (13)$$

Proof. First, notice

$$\{\eta_{i \in I} \leq T\} = \bigsqcup_{I' \subset I} \left\{\eta_{i \in I'} \leq T - \delta, \eta_{i \in I - I'} \in (T - \delta, T]\right\} \quad (14)$$

where I' ranges from all possible subsets of I, and $\bigsqcup$ emphasizes that it is the disjoint union. We name each sub-region $\{\eta_{i \in I} \leq T - \delta, \eta_{i \in I - I'} \in (T - \delta, T]\}$ above as $A_{I,I'}$ for notation simplicity below. The reason for introducing $A_{I,I'}$ is that under this new setting, contagion does not take place until δ units of time has passed. This means only the group member who default early enough, i.e., member $i \in I'$ in our notation, can trigger contagion effect; other $i \in I - I'$ who naturally default cannot bring down j's. Use the exact same reasoning as in Proposition 1, for each sub-region $A_{I,I'}$, we have

$$\begin{aligned} & P(\eta_{i \in I} \text{ satisfy condition of } A_{I,I'}, \eta_{j \in I^c} > T,\ j \in I^c \text{ still default}) \\ = & \int_{A_{I,I'}} \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I'} (1 - q_{i,j}) \right] \right) dP_{\eta_{i \in I}} \\ = & \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I'} (1 - q_{i,j}) \right] \right) \cdot \int_{A_{I,I'}} dP_{\eta_{i \in I}} \\ = & \prod_{j \in I^c} \left(e^{-A_j(T)} \cdot \left[1 - \prod_{i \in I'} (1 - q_{i,j}) \right] \right) \cdot \prod_{i \in I'} \left(1 - e^{-A_i(T-\delta)}\right) \cdot \prod_{i \in I - I'} \left(e^{-A_i(T-\delta)} - e^{-A_i(T)}\right) \end{aligned} \quad (15)$$

The rest is summing over all index sets I and subsets $I' \subset I$, a process similar to the one in previous proposition. □

The product term in (12) represents the case where all group members default on their own. And the triple summation here means: we first select k

members out of n to let them default on their own; and among them, further pick a subset, call it I', to represent those who default on their own before $T-\delta$; this leaves members in $I-I'$ default between time $T-\delta$ and T.

A comment is in place. If $\delta=0$, meaning the negative effects takes place immediately, we go back to the immediate contagion case, and one can check that (12) indeed agrees with (8) because $\gamma_{I,I'}=0$ for all $I'\neq I$. On the other hand, if $\delta=\infty$, effectively meaning the default is *not* contagious, then all terms in the triple summation vanishes (we let $A_i(t)=0$ for $t<0$).

Next, we revisit the immediate contagion case, simplify it a bit further, and explore couple properties.

2.3 Special Cases for Immediate Contagion

Suppose $q_{i,j}=q$ for all $i,j=1,...,n$. Then we can simplify γ_{I_k} in (9) to

$$\gamma_{I_k}=\left[\prod_{j\in I^c}e^{-A_j(T)}\right]\cdot\left[1-(1-q)^k\right]^{n-k} \tag{16}$$

One can check (8), with this simplified γ_{I_k}, is monotone decreasing with respect to q, meaning the group is less robust against default if members have stronger ties among each other. Next, consider two extreme cases where $q=0$ and $q=1$. In the first scenario, (8) becomes

$$e^{-A_0(T)}\cdot\left[1-\prod_{i=1}^{n}\left(1-e^{-A_i(T)}\right)\right] \tag{17}$$

In the second scenario, (8) becomes

$$\prod_{i=0}^{n}e^{-A_i(T)} \tag{18}$$

Actually one can directly compute (17) and (18) because $q=0$ means every group member is independent of each other, so as long one member stands, the whole group is safe; and $q=1$ means anyone falling will bring down the whole group. Because, as previously stated, γ_{I_k} is monotone decreasing with respect to q, (17) and (18) provide the upper and lower bound for (8) under the assumption $q_{i,j}=q$.

Now, suppose we *further* take $A_i(T)=A(T)$ for $i=1,...,n$. Then, (8) is simplified down to

$$e^{-A_0(T)}\cdot\left\{1-\sum_{k=1}^{n}\binom{n}{k}\cdot\left[1-e^{-A(T)}\right]^k\cdot e^{-(n-k)A(T)}\cdot\left[1-(1-q)^k\right]^{n-k}\right\} \tag{19}$$

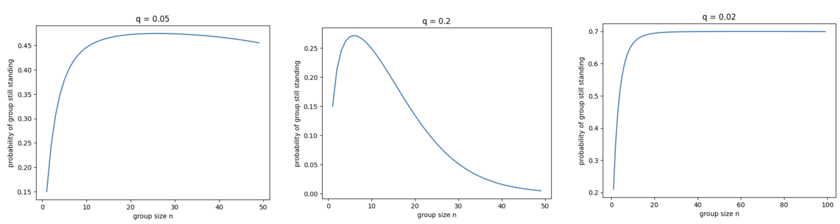
We take a closer look at the relation between individual's surviving rate and the group contagion factor. For notation simplicity, we name $c_1=1-e^{-A(T)}, c_2=$

$e^{-A(T)} = 1 - c_1, c_3 = 1 - q$ from (19), and the summation term above now reads

$$S_n := \sum_{k=1}^{n} \binom{n}{k} \cdot c_1^k \cdot c_2^{n-k} \cdot (1 - c_3^k)^{n-k} \tag{20}$$

Fix any c_2. As c_3 changes from 1 to 0, S_n increases from c_1^n to $1 - c_2^n$. Now, suppose we have two scenarios, characterized by (c_2, c_3) and (c_2', c_3'), where $c_2 < c_2'$ and $c_3 > c_3'$. This means in the first scenario, i.e., (c_2, c_3), each individual is less robust, but the contagion factor q is also smaller. If c_3 is close to 1 and c_3' is close to 0, then S_n in the first case is close to $(1 - c_2)^n$, because S_n is continuous with respect to c_3, which can very well be smaller than $1 - c_2'^n$ as in the second case. This translates to our observation: high individual surviving rate plus high contagion may be worse than low individual surviving rate plus low group contagion. For a concrete example, take $c_2 = 0.2$, $c_2' = 0.9$, $n = 5$, and some $c_3 \approx 1$, $c_3' \approx 0$. Substitute the numbers into (20), and we observe the first scenario corresponds to a smaller S_n, which leads to a higher group surviving rate.

Now, we study the effect of group size on the group's surviving rate. Take $q \in (0, 1)$ since the boundary cases are already addressed above. Note $0 < c_1, c_2, c_3 < 1$. Figure 2 plots the value of group surviving rate (19) against group size n where, for illustration purpose, we picked $e^{-A(T)} = 0.3$ and $e^{-A_0(T)} = 0.5$.



(a) (b) (c)

Figure 2: probability of group still standing as a function of group size n

Proposition 4. *Under the assumption of $q_{i,j} \equiv q$ and $A_i \equiv A$ for all $i, j = 1, ..., n$ (meaning we have a homogeneous group), we have $|S_n - S_{n+1}| \leq a \cdot b^n$ for some $a > 0$ and $b \in (0, 1)$.*

Proof. If we increase n by 1, (20) becomes

$$S_{n+1} = c_1^{n+1} + \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k} \right] \cdot \left[c_2 (1 - c_3^k) \right] \cdot c_1^k c_2^{n-k} (1 - c_3^k)^{n-k} \tag{21}$$

Subtract (20) from (21), we get

$$
\begin{aligned}
S_{n+1} - S_n = c_1^{n+1} &+ \sum_{k=1}^{n} \binom{n}{k} c_1^k c_2^{n-k} (1-c_3^k)^{n-k} [c_2(1-c_3^k) - 1] \\
&+ \sum_{k=1}^{n} \binom{n}{k-1} c_1^k c_2^{n-k} (1-c_3^k)^{n-k} [c_2(1-c_3^k)] \\
= c_1^{n+1} &+ c_1^n [c_2(1-c_3^n) - 1] + \sum_{k=1}^{n-1} \binom{n}{k} c_1^k c_2^{n-k} (1-c_3^k)^{n-k} [c_2(1-c_3^k) - 1] \\
&+ c_1 c_2^n (1-c_3)^n + \sum_{k=1}^{n-1} \binom{n}{k} c_1^{k+1} c_2^{n-k} (1-c_3^{k+1})^{n-k}
\end{aligned}
\tag{22}
$$

where we adjust the index k to prepare combing the two summations. Continuing from above,

$$
\begin{aligned}
S_{n+1} - S_n \geq c_1^{n+1} &+ c_1^n [c_2(1-c_3^n) - 1] + c_1 c_2^n (1-c_3)^n \\
&+ \sum_{k=1}^{n-1} \binom{n}{k} c_1^k c_2^{n-k} (1-c_3^k)^{n-k} [c_2(1-c_3^k) - 1 + c_1]
\end{aligned}
\tag{23}
$$

where we change c_3^{k+1} to c_3^k to get the inequality by $c_3 \in (0,1)$. Carrying on,

$$
\begin{aligned}
\text{RHS of (23)} = c_1^{n+1} &+ c_1^n [c_2(1-c_3^n) - 1] + c_1 c_2^n (1-c_3)^n \\
&- \sum_{k=1}^{n-1} \binom{n}{k} c_1^k c_2^{n-k} (1-c_3^k)^{n-k} (c_2 c_3^k)
\end{aligned}
\tag{24}
$$

where we simplify using $c_1 + c_2 = 1$. Next,

$$
\begin{aligned}
\text{RHS of (24)} &\geq c_1^{n+1} + c_1^n [c_2(1-c_3^n) - 1] + c_1 c_2^n (1-c_3)^n - c_2 \sum_{k=0}^{n} \binom{n}{k} (c_1 c_3)^k c_2^{n-k} \\
&= c_1^{n+1} - c_1^n + c_1^n c_2 (1-c_3^n) + c_1 c_2^n (1-c_3)^n - c_2 (c_1 c_3 + c_2)^n
\end{aligned}
\tag{25}
$$

where we add couple extra terms to the summation and change $1 - c_3^k$ to 1 to get the inequality. Finally,

$$
\text{RHS of (25)} \geq -c_1^n - c_2 (c_1 c_3 + c_2)^n \geq -a_1 \cdot b_1^n \tag{26}
$$

for some suitable $a_1 > 0, b_1 \in (0,1)$, where we drop all the positive terms on the RHS of (26) and use the fact $c_1 c_3 + c_2 < c_1 + c_2 = 1$. One way to select a_1, b_1 is by letting $b_1 = \max(c_1, c_1 c_3 + c_2)$ and $a_1 = 2$.

Next we find an upper bound of $S_{n+1} - S_n$. Again start from (22).

$$\begin{aligned}
S_{n+1} - S_n &\le c_1^{n+1} + \sum_{k=1}^{n} \binom{n}{k} c_1^k c_2^{n-k}(c_2 - 1) + \sum_{k=0}^{n} \binom{n}{k} c_1^{k+1} c_2^{n-k} \\
&= c_1^{n+1} + (c_2 - 1)\left[\sum_{k=0}^{n} \binom{n}{k} c_1^k c_2^{n-k} - c_2^n\right] + c_1(c_1 + c_2)^n \\
&= c_1^{n+1} + (c_2 - 1)[1 - c_2^n] + c_1 \\
&= c_1^{n+1} - c_1(1 - c_2^n) + c_1 \\
&= c_1^{n+1} + c_1 c_2^n \\
&= c_1(c_1^n + c_2^n) \\
&\le a_2 \cdot b_2^n
\end{aligned} \tag{27}$$

where we first drop the term $1 - c_3^k$ to get the inequality, and then adjust index k to get the binomial form, and finally pick $a_2 = 2c_1, b_2 = \max(c_1, c_2) < 1$. Now combine (26) and (27), we get $|S_{n+1} - S_n| \le a \cdot b^n$ with $a = \max(a_1, a_2)$ and $b = \max(b_1, b_2) < 1$. □

Corollary 1. *The sequence $\{S_n\}_{n\ge 1}$ defined above converges.*

Proof. From the previous proposition, for any $\epsilon > 0$, let $N := \left[\log_b \frac{\epsilon(1-b)}{a}\right] + 1$. Then,

$$|S_n - S_m| \le \sum_{n}^{m-1} |S_k - S_{k+1}| \le \sum_{N}^{\infty} ab^n = \frac{ab^N}{1-b} \le \epsilon$$

for any $N < n < m$. This shows S_n is a Cauchy sequence. □

Next, we consider the case where the leader's defaulting rate eventually increases as the group size increases. This is reasonable in reality because when the group contains too many members, it becomes more challenging for the leader to manage all the resources and keep the group functioning smoothly. Recall in our original set up, we defined $e^{-A_0(T)}$ to be the leader's *surviving* probability after time T. Now, in order to reflect the fact that this probability is also influenced by the group size n, we modify it to become $f(n,T)$ (see below in the remark section for couple examples), for generality. And of course we require $f(n,T) \in [0,1]$ for all n and T since it is a probability. We now strengthen the previous proposition as follows.

Proposition 5. *Suppose for all large enough n, we have*

$$f(n,T)/f(n+1,T) \ge 1 + a \cdot b'^n \tag{28}$$

for any $b < b' < 1$, where a, b come from the previous proposition (refer to the very last line in that proposition). Then, at least one of the following statements holds,

- *the group's surviving rate is eventually monotone decreasing*
- *the group's surviving rate converges to* 0

Proof. In the notation of (19) and (20) (where we substitute $e^{-A_0(T)}$ by $f(n,T)$ as mentioned above), the group's surviving rate is

$$f(n,T)\cdot(1-S_n) \tag{29}$$

By the definition of S_n, we know the limit is between $[0,1]$ (limit exists by Corollary 1). First consider the case when the limit is strictly smaller than 1. We claim the group's surviving rate eventually shows a monotone decreasing trend. To be more precise, our goal now is to find some N such that

$$f(n,T)\cdot(1-S_n)\geq f(n+1,T)\cdot(1-S_{n+1})$$

for all $n>N$, which is equivalent to showing

$$\frac{f(n,T)}{f(n+1,T)}\geq\frac{1-S_{n+1}}{1-S_n}=1+\frac{S_n-S_{n+1}}{1-S_n} \tag{30}$$

From proposition 4, the right-side above is less than or equal to $1+(a\cdot b^n)/(1-S_n)$. So, if we can show

$$\frac{f(n,T)}{f(n+1,T)}\geq 1+\frac{a\cdot b^n}{1-S_n} \tag{31}$$

then we automatically obtain

$$\frac{f(n,T)}{f(n+1,T)}\geq 1+\frac{a\cdot b^n}{1-S_n}\geq 1+\frac{S_n-S_{n+1}}{1-S_n}$$

which confirms (30). Now, we turn our attention to establish (31). By our assumption, S_n's limit, call it s, is strictly less than 1, which means that there exists some N_1 such that $S_n\leq s+(1-s)/2=(s+1)/2$ for all $n\geq N_1$ (this directly follows from the definition of limit). Also, we can pick some N_2 such that $(b/b')^n<(1-s)/2$ for all $n\geq N_2$. Define $N=\max(N_1,N_2)$, and appealing to (28), we obtain

$$\frac{f(n,T)}{f(n+1,T)}\geq 1+a\cdot b'^n=1+\frac{a\cdot b^n}{(b/b')^n}\geq 1+\frac{a\cdot b^n}{(1-s)/2}\geq 1+\frac{a\cdot b^n}{1-S_n}$$

for all $n>N$, which concludes our claim.

Next, if $S_n\to 1$, by $f(n,T)\in[0,1]$, we immediately conclude from (29) that the group's surviving rate goes to 0 as the group size approaches infinity. □

Remark. In proposition 4, we only show the fluctuation of the group's survival rate on the tail, i.e., when the group size is large enough, is small; but here we show a general decreasing trend, thus strengthening our result. One thing to note here though, in our simulation, the limit of S_n appears to be always

1. If one can actually prove this result, the monotone decreasing part will become obsolete since the group's surviving rate goes to 0 anyways. Now, we provide two simple examples that satisfy the requirement in this proposition. Take $f(n,T) = 1/n$, then $f(n,T)/f(n+1,T) = 1+1/n$, which is greater than $1 + a \cdot b'^n$ for any $a \in \mathbb{R}^+, b \in (0,1)$ as long n is large enough. As another example, take $f(n,T) = e^{-a \cdot n}$, for any $a > 0$, so it looks similar to our original setup for the group leader behavior, then $f(n,T)/f(n+1,T) = e^a > 1 + a \cdot b'^n$ for large n, because $e^a > 1$ is a constant.

If we look closely at the requirement (28), it is actually not strong: we require $f(n+1,T) - f(n,T) \leq -f(n,T) \cdot \frac{ab'^n}{1+ab'^n} \leq -ab'^n$, because $f(n,T) \in [0,1]$. But the right-side above converges to 0 exponentially fast (by $b' < 1$), meaning $f(n,T)$ just needs to decrease so slow such that the decrease becomes almost like 0 after a few rounds.

What this proposition tells us is that under the setting of homogeneous group, adding too many group members is not a good choice, as it will eventually harm the group performance.