Replicating Ross Recovery Theorem with SPX Options

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1 Introduction and Purpose

In 2015, a professor from MIT named Steven Ross wrote a paper called the Ross Recovery Theorem[2]. The idea of the paper was two fold, to use index options to “recover” the natural probabilities of state to state transitions of the index and the market’s current risk aversion and as a non parameterized test for the Efficient Market Hypothesis. The first use of the theorem is why he gained notoriety for this specific research. He made three main assumptions that allowed him to accurately create his theorem. The largest of them is the idea that the data is made from a singular representative investor. Followed by the time homogeneous constraint and the delta constant. Both of these assumptions were made to allow for solutions that were mathematical. Time homogeneity allows for the same transition to occur from state i to state j regardless of where you are in the time i.e. a day after an option is priced or the correlating one hundredth day. Therefore he could use a state transition matrix that was also restricted to only positive values as they would be probabilities in a Markovian chain. Many fellow researchers viewed these assumptions as unrealistic and needed to be relaxed to have any viable application in the realm of quantitative finance. Therefore, we have provided an extended section in our report detailing the relaxations, other assumptions and ultimately the outcomes that different researchers have added to the basic Ross Recovery Theorem literature and why they are important moving forward.

The United States and other developed nations have enough actively trading investors to find market clearing prices or prices that are perceived to be efficient, the market’s fair value for an asset given all known information. However there has always been speculation around the extent of what is considered “known” information, that ponderance has led to the development of EMH (the Efficient Market Hypothesis). EMH has weak, strong and semi-strong versions. The standard test for EMH requires a high amount of parameterization. For Ross, EMH mixed with the idea of proving which information set is the truth was a key to the development of the Ross Recovery Theorem as a way to test EMH without having to parameterize anything, only bold assumptions in which people have continued to relax in other papers on the topic. Although EMH confirmation is a key portion of the importance of the Ross Recovery Theorem, it stands to be outside the scope of our paper. It was highlighted to bring more clarity into the importance of the Ross Recovery Theorem and the background of its development by Professor Ross.

The significance of the Ross Recovery Theorem stems from its ability to reflect market beliefs (probabilities) through a forward looking asset. Which is why some academics refer to a correct market belief recovery as the holy grail for trading. All of these combine to create a very challenging situation. The data necessary to achieve the theorem is very important and it is unfortunately expensive or unlikely to be obtained due to the extensive amount of strikes and maturities for a quoted option price. We were able to obtain on singular days data from a yahoo finance API that was sufficient and then a year of options chains produced from Capstone Investment Advisors. A shortened overlook of our process: create a blended implied volatility surface from puts and calls, extract a singular call price surface, and then extract the resulting probabilities and pricing kernel to be analyzed to understand what can be taken from the results. Due to the nature of the data that is very discretized, it is necessary for us to use machine learning techniques to interpolate surfaces of implied volatility and call prices that would be “quoted”.

Looking into the different advancements into the Ross Recovery Theorem, there are consistent adaptations to limit the assumptions, or look further into anomalies produced by the standard recovery theorem. An initial review of the theorem from Borovička, Hansen & Scheinkman (2016, Misspecified Recovery), stated that the results had hidden martingale components within its pricing kernels and therefore the theorem yielded distorted results. While a paper produced in 2020 by Jackwerth and

Menner dove deep into the practicality of the empirical results from thorough testing on large scale options data, finding through many techniques unstable transition prices that lead to the theorem failing to hold up under empirical tests. These brief introductions to just a few in the extensive set of published researched papers related to the theorem. The Literature Review section will provide the insight needed to fully understand not only the Ross Recovery Theorem, but follow the development of the research on a mathematical level as well as the overarching results.

As our project is focused on the replication of the theorem and to be able to obtain a functional way of processing options pricing data into an actionable Ross Recovery Theorem. Our structure allows for modifications to be easily added to relate to other research papers within the topic. Since replication is our purpose, our actual findings are relatively limited. However, we are able to identify common trends among the usage of options from the pricing kernel and its relative shape, while reading into the natural probabilities and risk neutral probabilities.

2 Literature Review

3 Data

4 Econometric Methods

4.1 Core Formula

Following Ross (2015)[2], the recovered transition operator F is given by

$$F = \left(\frac{1}{\delta}\right) DPD^{-1}. \tag{1}$$

The components of this formula can be described as follows:

- P is the Arrow–Debreu price matrix of dimension $m \times m$, where each element p_{ij} reflects the price of a state-contingent payoff.
- F is the $m \times m$ matrix representing the recovered transition probabilities under the real (or natural) probability measure.
- δ is the discount factor, often written as $\delta = e^{-r}$ with r denoting the risk-free interest rate.
- D is a diagonal matrix whose entries correspond to the marginal rates of substitution (scaled by δ), which link consumption-based utility to state pricing.
- D^{-1} is the inverse of this diagonal matrix, completing the similarity transformation.

This representation shows that the recovered transition matrix F can be constructed from three ingredients: the observed state price matrix P, the discount factor δ, and the substitution effects encoded in D. In essence, the equation provides a way to map from risk-neutral pricing information to the real-world probabilities implied by investors' preferences.

4.2 Constructing the State Price Matrix

Since we do not directly observe a full set of multi-state Arrow–Debreu securities, we follow the approach of Breeden and Litzenberger (1978)[1] and recover them from option prices. Specifically, for a given strike K and maturity T, the Arrow–Debreu security price is obtained as

$$p(K,T) = C''(K,T), \tag{2}$$

where $C(K,T)$ denotes the European call option price with strike K and maturity T, and $C''(K,T)$ is the second derivative with respect to the strike price.

In practice, since option prices are observed on a discrete grid of strikes, we approximate the second derivative numerically as

$$C''(K,T) \approx \frac{C(K+\Delta K,T) - 2C(K,T) + C(K-\Delta K,T)}{(\Delta K)^2}. \tag{3}$$

Thus, in our implementation, the Arrow–Debreu prices $p(K,T)$ are constructed from observed S&P 500 option prices by applying this finite difference approximation.

4.2.1 Cleaning the Implied Volatility Data

To construct a consistent state price matrix P, we require a dense and smooth representation of option prices. Directly smoothing the call price surface $C(K,T)$ is problematic, since the payoff introduces a kink at the money. Instead, we first smooth the implied volatility (IV) surface and then map it back to call prices via the Black–Scholes model.

A key difficulty arises around the moneyness region ($K \approx$ Spot), where implied volatilities can be unstable. To address this, we adopt the following strategy:

- For strikes $K \geq 1.02 \times$ Spot, we use call option implied volatilities.
- For strikes $K \leq 0.98 \times$ Spot, we use put option implied volatilities.
- For the middle region $0.98 \times \text{Spot} < K < 1.02 \times \text{Spot}$, we construct a blended implied volatility as a weighted average:

$$\text{IV}_{blend} = w_{call} \cdot \text{IV}_{call} + w_{put} \cdot \text{IV}_{put},$$

where the weights are determined by open interest:

$$w_{call} = \frac{\text{OI}_{call}}{\text{OI}_{call} + \text{OI}_{put}}, \qquad w_{put} = \frac{\text{OI}_{put}}{\text{OI}_{call} + \text{OI}_{put}}.$$

Here, open interest (OI) refers to the number of outstanding option contracts that are currently open and not yet settled. It is often interpreted as a proxy for market liquidity and investor participation. By weighting implied volatilities according to open interest, the blended IV gives more importance to the side of the market (calls or puts) that carries greater trading activity.

4.2.2 Learning a Smooth IV Surface with an MLP

To obtain a dense and smooth implied volatility (IV) surface, we fit a fully connected multilayer perceptron (MLP) that maps strike and maturity to implied volatility:

$$f_\theta : \ (K,T) \ \mapsto \ \widehat{\text{IV}}.$$

The network has two hidden layers with 64 units each and a single scalar output. To ensure smooth and continuous predictions—needed for stable finite–difference second derivatives with respect to K—we use a twice–differentiable activation softplus in the hidden layers. We train the model by minimizing the mean–squared error between market IVs and $f_\theta(K,T)$ over the observed grid of (K,T) quotes. The learned surface $\widehat{\mathrm{IV}}(K,T)$ is then converted to a smoothed call price surface via the Black–Scholes formula, after which we compute Arrow–Debreu prices $p(K,T)$ using discrete second differences in K.

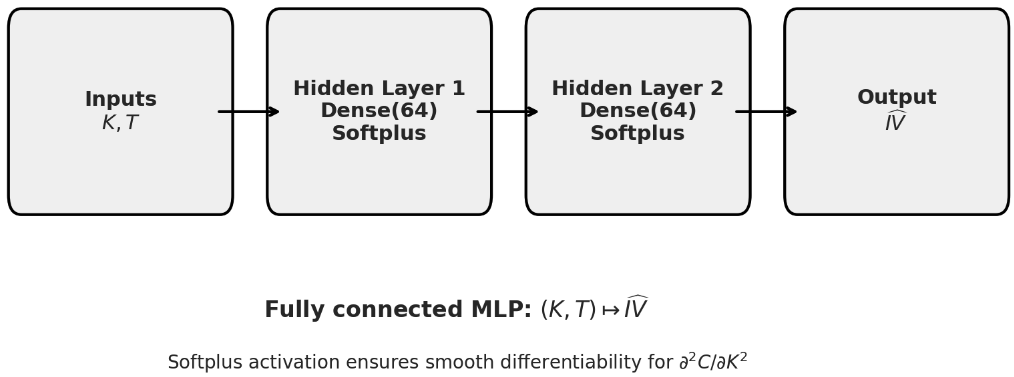

Figure 1: MLP used to smooth the implied volatility surface: inputs (K,T), two hidden layers with 64 units each (twice–differentiable activations), and a scalar output $\widehat{\mathrm{IV}}$.

4.3 Estimating the State-Price Transition Matrix

Let $p^t \in \mathbb{R}^{1\times m}$ denote the row vector of Arrow–Debreu state prices at horizon t (constructed from $C''(K,T)$ on an m-state grid). We apply a light post-estimation normalization that row sums equals to discount factor at t. Following the forward relation $p^{t+1} = p^t P$ (Ross, 2015, Eq. (86)), we estimate the transition matrix $P \in \mathbb{R}_+^{m\times m}$ by solving the regularized nonnegative least squares problem

$$\min_{P\geq 0} \sum_{t=1}^{T-1} \left\| p^t P - p^{t+1} \right\|_2^2 \; + \; \lambda \, \|P\|_F^2, \tag{4}$$

where $\lambda \geq 0$ controls Tikhonov (ridge) regularization. Nonnegativity enforces absence of static arbitrage in state prices.

4.4 Recovery Theorem and Perron–Frobenius Argument

By Theorem 1 of Ross (2015)[2], if the Arrow–Debreu price matrix P is irreducible and generated by a transition-independent kernel, then the system admits a unique positive solution for the natural probability matrix F, together with a unique pricing kernel.

From the Perron–Frobenius theorem, an irreducible nonnegative matrix P possesses a unique largest real eigenvalue λ with a strictly positive associated eigenvector z. In our setting, this eigenvalue is identified with the discount factor δ^{-1}, and the corresponding eigenvector provides the elements required to construct the diagonal matrix D.

Formally, if z denotes the unique positive eigenvector of P corresponding to the eigenvalue δ^{-1}, then the kernel can be written as

$$\phi_i = d_{ii} = \frac{1}{z_i}, \tag{5}$$

and thus

$$D = \text{diag}\left(\frac{1}{z_1}, \frac{1}{z_2}, \ldots, \frac{1}{z_m}\right). \tag{6}$$

4.5 Constructing the Recovered Transition Probabilities

Element by element, the entries of the recovered transition matrix F can be expressed as follow by Ross(2015)[2]

$$f_{ij} = \left(\frac{1}{\delta}\right)\frac{\phi_i}{\phi_j}p_{ij} = \left(\frac{1}{\delta}\right)\frac{U_i'}{U_j'}p_{ij} = \left(\frac{1}{\lambda}\right)\frac{z_j}{z_i}p_{ij}, \tag{7}$$

where p_{ij} are the Arrow–Debreu state prices, ϕ_i are the marginal rates of substitution, and z is the Perron–Frobenius eigenvector of P corresponding to the largest eigenvalue $\lambda = \delta^{-1}$.

Hence, the full transition matrix F can be constructed compactly as

$$F = \left(\frac{1}{\delta}\right) DPD^{-1}, \tag{8}$$

where $D = \text{diag}(1/z_1, \ldots, 1/z_m)$.

5 Empirical Results

1) Obtain Raw Options Data for SPX (Call/Put) 2) Cleaned using the weighted average open interest of a blended call/put structure 3) Using exogenous FRED Data, we construct the IV Surface using Double Hidden Layer ML method

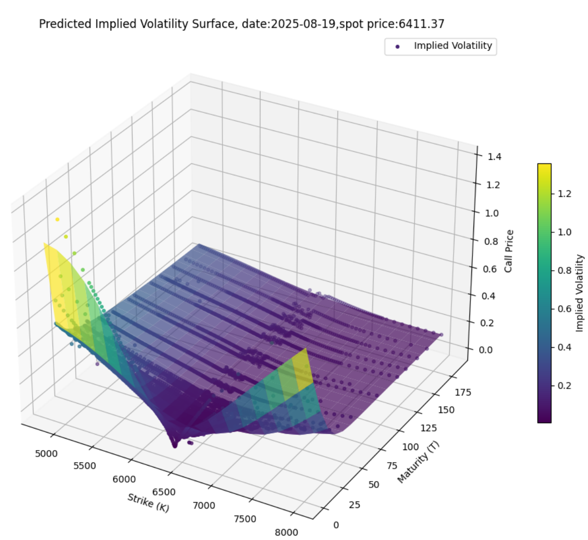

Figure 2: This is the Surface Constructed from the Raw Option Prices

4) Normalized Recovered Call Price at a Discrete Interval

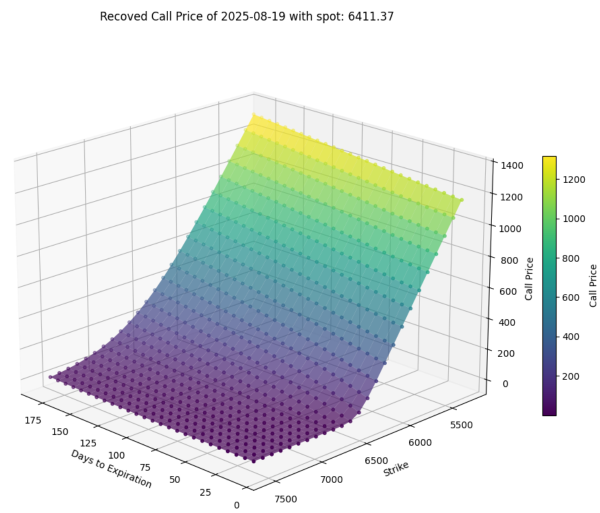

Figure 3: Recovered Call Option Surface

5) Recovered Risk-Neutral Distributions via the Breeden–Litzenberger approach.

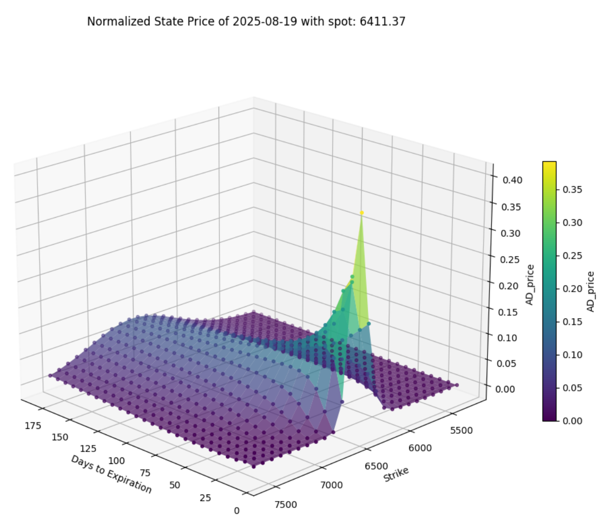

Figure 4: Normalized State Price Surface

6) From the Normalized State Price Surface, we obtain the Transition Surface, giving us the Natural Probability and the Pricing Kernel

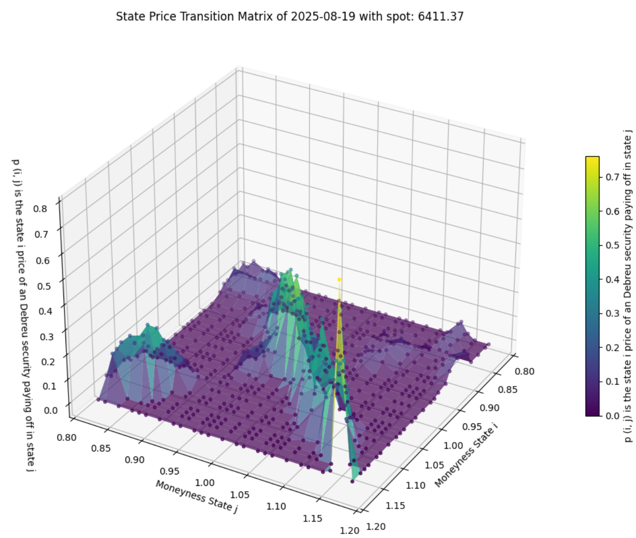

Figure 5: Transition Surface

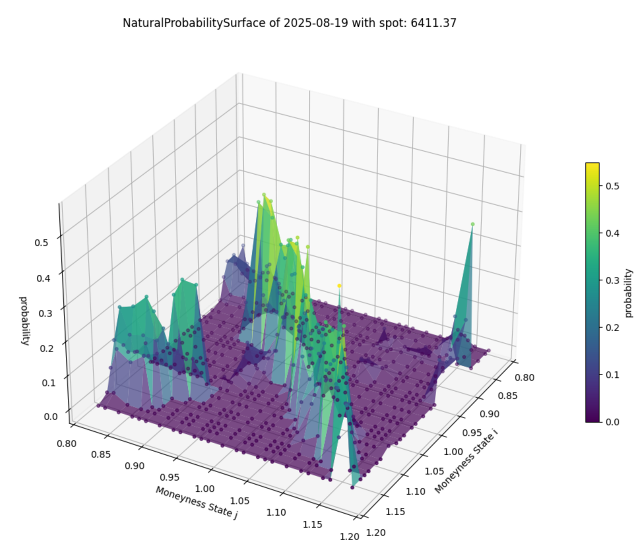

Figure 6: Natural Probability Surface

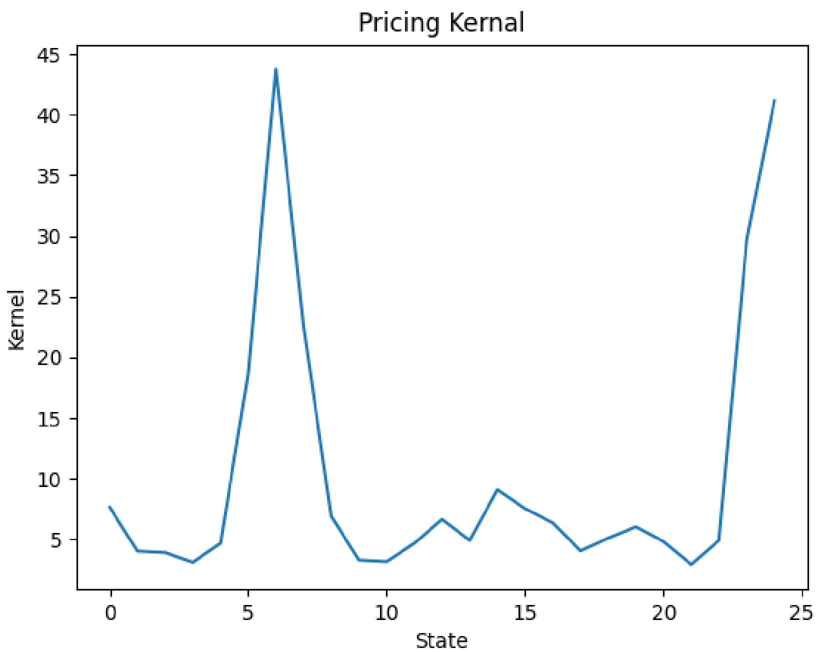

Figure 7: Pricing Kernel

6 Evaluation

7 Conclusion

References

[1] Douglas T. Breeden and Robert H. Litzenberger. Prices of state-contingent claims implicit in option prices. *The Journal of Business*, 51(4):621–651, 1978.

[2] Stephen A. Ross. The recovery theorem. *The Journal of Finance*, 70(2):615–648, 2015.