Replicating Ross Recovery Theorem with SPX Options

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August 26, 2025

1 Introduction and Purpose

In 2015, a professor from MIT named Steven Ross wrote a paper called the Ross Recovery Theorem[10]. The idea of the paper was two fold, to use index options to "recover" the natural probabilities of state to state transitions and the market's current risk aversion and as a non parameterized test for the Efficient Market Hypothesis. The first use of the theorem is why Ross gained notoriety for this specific research. He made three main assumptions that allowed him to accurately create his theorem. The largest of them is the idea that the data is made from a singular representative investor. Followed by the time homogeneous constraint and the delta constant. Both of these assumptions were made to allow for solutions that were mathematical. Time homogeneity allows for the same transition to occur from state i to state j regardless of where you are in the time i.e. a day after an option is priced or the correlating one hundredth day. Therefore he could use a state transition matrix that was also restricted to only positive values as they would be probabilities in a Markovian chain. Many fellow researchers viewed these assumptions as unrealistic and needed to be relaxed to have any viable application in the realm of quantitative finance. Therefore, we have provided an extended section in our report detailing the relaxations, other assumptions and ultimately the outcomes that different researchers have added to the basic Ross Recovery Theorem literature and why they are important moving forward.

The United States and other developed nations have enough actively trading investors to find market clearing prices or prices that are perceived to be efficient, the market's fair value for an asset given all known information. However there has always been speculation around the extent of what is considered "known" information, that ponderance has led to the development of EMH (the Efficient Market Hypothesis). EMH has weak, strong and semi-strong versions. The standard test for EMH requires a high amount of parameterization. For Ross, EMH mixed with the idea of proving which information set is the truth was a key to the development of the Ross Recovery Theorem as a way to test EMH without having to parameterize anything, only bold assumptions in which people have continued to relax in other papers on the topic. Although EMH confirmation is a key portion of the importance of the Ross Recovery Theorem, it stands to be outside the scope of our paper. It was highlighted to bring more clarity into the importance of the Ross Recovery Theorem and the background of its development by Professor Ross.

The significance of the Ross Recovery Theorem stems from its ability to reflect market beliefs (probabilities) through a forward looking asset. Which is why some academics refer to a correct market belief recovery as the holy grail for trading. All of these combine to create a very challenging situation. The data necessary to achieve the theorem is very important and it is unfortunately expensive or unlikely to be obtained due to the extensive amount of strikes and maturities for a quoted option price. We were able to obtain on a singular day's data from the yahoo finance API that was sufficient and then a year of options chains produced from Capstone Investment Advisors. A shortened overlook of our process: create a blended implied volatility surface from puts and calls, extract a singular call price surface, and then extract the resulting probabilities and pricing kernel to be analyzed to understand what can be taken from the results. Due to the nature of the data that is very discretized, it is necessary for us to use machine learning techniques to interpolate surfaces of implied volatility and call prices that would be "quoted".

Looking into the different advancements into the Ross Recovery Theorem, there are consistent adaptations to limit the assumptions, or look further into anomalies produced by the standard recovery theorem. An initial review of the theorem from the Misspecified Recovery[3], stated that the results had hidden martingale components within its pricing kernels and therefore the theorem yielded distorted results. While a paper produced in 2020 by Jackwerth and Menner[5] dove deep into the practicality of the empirical results from thorough testing on large scale options data, finding through many techniques unstable transition prices that lead to the theorem failing to hold up under empirical tests. These are a brief introduction to the extensive set of published researched papers related to the theorem. The Literature Review section will provide the insight needed to fully understand not only the Ross Recovery Theorem, but follow the development of the research on a mathematical level as well as the overarching results.

As our project is focused on the replication of the theorem and to be able to obtain a functional way of processing options pricing data into an actionable Ross Recovery Theorem. Our structure allows for modifications to be easily added to relate to other research papers within the topic. Since replication is our purpose, our actual findings are relatively limited. However, we are able to identify common trends among the usage of options from the pricing kernel and its relative shape, while reading into the natural probabilities and risk neutral probabilities.

2 Literature Review

2.1 Defining the Recovery Problem

The main empirical question we are interested in addressing in this research is that of whether the physical transition probabilities and SDF can be retrieved from state prices, implied by options. The formal state prices (away from the options' markets) are return real-world probabilities mutiplied by the SDF (pricing kernel) in each state. Stephen Ross's Recovery Theorem[10] discovered that, under some extreme conditions, it was feasible to undo. Specifically, Ross posited a discrete, time-homogeneous Markov chain for the price returns on the underlying and a transition-independent distribution for the pricing kernel (that is, the SDF distribution does not depend on the initial one) and positivity for the SDF. Under these conditions, it was enough to know a one-time snapshot for option prices (then risk-neutral state prices) to uniquely recover the physical transition matrix (the real-world probabilities to move from one to another) and the SDF, dependent on states, at the same time. Radical contrast between the two results remains, as it suggests a possible way for arriving at market risk premia and investors' risk aversion without assuming the form of a utility function based on the option prices by direct observation.

In practice, the empirical question is to utilize observed option prices on an underlying asset (e.g., an equity index) to decompose what the market expects to occur (physical probabilities) from how the market prices risk (the SDF) at each state. That's because it would provide a lot of useful information for investors and policy makers: it strips away what the market is saying about future returns and the price of risk. Recovery of the Natural Density Ross assumes the validity of this question and our project follows the lead and applies Ross's recovery protocol to empirically observed market data, namely we attempt to recover the natural (physical) distribution of future returns and the implied pricing kernel using the option-implied "state" price density. The method involves the construction of a state price transition matrix (SPTM) from option prices, and the use of spectral decomposition (Perron-Frobenius theory) to recover the physical transition probabilities and SDF. In this work we answer the empirical question whether such a recovery can be practically and reliably performed from real data since the conditions of Ross might only approximately hold (or be violated) in practice.

2.2 Benchmark Models and Extensions in the Literature

There has been an increasing body of work examining the Recovery Theorem's consequences, loosening its assumptions, examining its empirical validity, and adapting it in different directions. Here we summarize salient benchmark models and results from previous studies comprising theoretical generalizations, empirical performance and stability, and extensions to more than one variable.

(a) Theoretical Generalizations and Critiques.

The original model by Ross assumed a finite state space and a special structure for the pricing kernel. Subsequent work has generalized the assumptions or disputed them. Borovička, Hansen, and Scheinkman[3] mounted a critique, showing that if the SDF has a non-trivial martingale component (i.e., not solely state-dependent), then the recovery does not recover correct physical probabilities at all. Indeed, they show that Ross's method actually assumes away the martingale component by setting the martingale component equal to one, thus recovering what are in essence the long-run risk-neutral probabilities rather than actual physical probabilities. The implication was an identification problem: unless additional structure intervenes, the separation of risk aversion from correct expected probabilities will fail. Likewise, Qin and Linetsky[8] derived sufficient conditions for recovery in continuous-time models. By a spectral analysis of Markovian pricing operators, they showed that under quite general processes (right-continuous Borel processes), a situation called recurrence suffices to ensure recovery. Later, Walden[13] discovered, for the special but important narrower class of diffusion processes (with unbounded state space), that recurrence is not necessary, Walden found necessary and sufficient conditions for unique recovery in an unbounded continuous state framework. Walden's results show that recovery can find work in the most significant number of realistic continuous models, given the right growth conditions on the diffusion are in force. The subsequent theoretical advances extend the Recovery Theorem beyond the finite-state result by Ross, telling us when recovery mathematically works. The other important extension to multi-period frameworks is by Jensen, Lando, and Pedersen[6] designing a multi-horizon Generalized Recovery framework, that (alongside a variant by Jackwerth & Menner [5] has preserved simply the SDF assumption independent of transition but no longer the complete Ross structure. Remarkably, the Generalized Recovery and the so-called "Ross-Stable" approach (Jackwerth & Menner[5]) show a much more stable-looking pricing kernel, implying differences between the recovered real-world measure and the base risk-neutral measure are pretty small. This suggests that if the recovery problem is corrected by appropriate regularization or recasting, then the massive deviations found initially by the solution by Ross might diminish and bring the recovered distribution in line with the observed risk-neutral distribution.

(b) Empirical Performance and Robustness.

Of particular importance here is the question of whether recovered physical distributions enjoy predictive validity in base markets. The results to date are mixed. On the one hand, Audrino, Huitema & Ludwig[1] deploy a non-parametric recovery to S&P 500 option data and report encouraging results. For the years 2000-2012, they report that market-timing strategies based on recovered distribution moments (e.g., recovered expected returns) significantly outperform strategies that are based on risk-neutral moments derived from the option surface. That is, the additional information that was gleaned by recovery appeared to enjoy additive predictive value over and above the raw option-implied (risk-neutral) densities. Such an encouraging result would imply that Ross's recovery can, at the very least, in some contexts reveal the market's accurate expectations and improve forecasts. On the other hand, Jackwerth and Menner[5] provide a discouraging empirical analysis. With S&P 500 index options again, they find the recovered "physical" distributions are inconsistent with later-observed returns and are not successful in predicting future returns or one-step-out-of-sample realized variances. Their negative results are even weightier once they introduce reasonable economic constraints (such as bounded risk aversion) to the recovery. Indeed, Jackwerth & Menner[5] show that parsimonious benchmark models, such as assuming a representative individual with constant relative risk aversion (power utility) or even the return distribution historical, predict roughly equally well the recovered distribution. Moreover, they show an eye-catching lack of robustness: different approaches to the estimate of Ross's theorem (for instance, if one uses directly observed Martingale transform prices or interim estimated transition prices) can produce enormously distinct recovered probability distributions from the same option data. Such a lack of robustness serves to call into question the real-world usefulness of the very same recovery method. The contrast here between Audrino et al.'s study and Jackwerth & Menner's[5] has been suggestive for additional study. Some recent studies suggest that the contrast stems from differences in research methodology: for instance, Bakshi, Chabi-Yo & GaoBakshi et al. [2] reported that an important implicit assumption in Ross (the unity Martingale component) does not hold in the data, supporting Borovička et al.'s critique. Conversely, the "Ross-Stable" method referenced above (similar to the generalized method by Jensen et al.) determines that once the recovery is adjusted or regularized, the recovered pricing kernel becomes almost flat (i.e., approaches the risk-neutral kernel). This would then mean that the significant discrepancies and predictive "alpha" reported in some initial research works could potentially be model misspecification or overfitting artifacts. Overall, the literature tests at the empirical level have found that Ross's recovered distributions are very sensitive to the details of implementation and might need extra constraints or information (e.g., utilizing the term structure of the interest rates) to yield stable, reliable results. As a matter of fact, Martin and RossMartin and Ross [7] demonstrated theoretically that the recovery solution has a close relationship to the term structure of the risk-free rates, implying that the exclusion of the dynamics in the interest rates (as the fundamental theorem does) would result in ill-conditioned estimates. Because of this observation, researchers make efforts to utilize the term structure in recovery exercises in the hopes of yielding stability.

(c) Multivariate and Extended Models.

Yet another thread in the literature takes the Recovery Theorem beyond the one-state-variable framework. The original formulation by Ross was univariate (one state variable evolving as a Markov chain). However, various risk factors determine stock prices, and neglecting these can result in biased or inaccurate recovery. Sanford[11] introduced a Multivariate Recovery Theorem in the form of a two-dimensional Markov chain for the S&P 500 index involving the index level along with its implied volatility as the state variables. The idea is that the chance of transitioning to "far" states relies on volatility–i.e., given high volatility, large leaps in the index are relatively probable. By including the implied volatility (which serves as a proxy for the market expectation regarding future variance) in the definition of the states, Sanford accounts for an important cause of the transition between states overlooked by the univariate model. The multivariate version significantly enhanced the quality of out-of-sample performance: by utilizing data from 1996–2015, the out-of-sample R^2 in predicting future returns in the index improved by about 30% compared to approximately 12% in the univariate recovery. That is a vast improvement, showing that by controlling for an additional state variable (volatility), the recovered distribution becomes much more descriptive of future states. Some other extensions are models that relate recovery to interest rate dynamics or to multiple assets. For instance, Qin, Linetsky, and Nie[9] construct a two-factor term structure model in which recovery applies to the distribution over the joint dynamics in bond yields, extending forward probabilities to bond risk premiums and so extending recovery to the markets for interest rates. Moreover, researchers addressed the ill-posed nature of the recovery inversion by incorporating regularization procedures. Van Appel and Mare[12] bring to life a regularized multivariate mixture distribution strategy, explicitly responding to the challenge in the recovery for the real-world distribution to solve two ill-posed problems (estimating the state prices and inverting to probabilities). By adding a penalty for extreme or volatile solutions, their technique enhances the stability in the recovered distribution along with the economic reasonableness. Overall, the extensions in the literature work to expand the scope for the recovery (applied to continuous states, to multiple factors, to interest rates) and to make it less problematic in the results it produces. These benchmarks constitute the framework for the empirical project here, pointing to the possible value in recovery while specifying the danger to be avoided.

2.3 The Team Implementation in its Context

This development of the Recovery Theorem within the context of our work extends these observations and corrects several of the deficiencies found in the benchmark studies. We have implemented a full coding pipeline for inferring physical probabilities, the SDF and option data empirically. This pipeline includes several steps in order to retain some numerical stability and meaningful results:

- **State-price matrix.** Construct a discrete state price density (SPD) surface from implied information in options based on liquid options in the underlying. This involves some intelligent interpolation and extrapolation of option prices across strikes and maturities in order to obtain a smooth risk-neutral state price distribution for a specific horizon. Adopting the best practices (such as spline smoothing in Sanford [11]), we make the state-price surface arbitrage-free and smooth enough to suppress noise originating from sparse strikes or missing values.

- **Transition Matrix Estimation.** Under the SPD for the current state and time-homogeneity assumptions, we estimate the transition matrix for the states in the risk-neutral measure. This is Ross's transition prices matrix for contingent claims. We then apply the reasoning in the Recovery Theorem (which, as it happens, is a Perron–Frobenius eigen-decomposition) to this matrix to solve for two components: the physical transition probabilities matrix and the stochastic discount factor (as a matrix, the diagonal pricing kernel). This step becomes significant, as we perform it for numerical precision and to address any feasible constraints that ensure positivity and normalization of the probabilities.

- **Procedures for Interpolation and Extrapolation.** Since real-world option data is discrete, we expend considerable effort on interpolation (and extrapolation) in the strike and maturity directions. By calibrating an implied volatility surface to be well-behaved and then transforming it to state prices, we reconcile the completeness assumption in Ross (i.e., we construct missing options to estimate a complete market). Our interpolation procedures make use of known procedures to estimate over data gaps. This process amounts to a data smoothing/regularization, so that the input state prices do not create spurious artifacts in the recovery (a remedy to the ill-posed nature of the inversion).

- **Back-testing predictive performance.** We extensively back test the estimated physical distributions and implied moments against empirical market realizations. On each date in our sample, we first obtain the physical forecast for the distribution of future returns, and then construct the time series for the forward-looking statistics (for example, expected return, variance, and tail risk measures) based on it, and then we investigate how informative these statistics as forecasts are for realized returns and volatility. This question is related to that asked by Jackwerth & Menner[5] whether the recovery give incremental predictive power. We can evaluate the economic value of recovered probabilities by comparing the out-of-sample performance of recession forecasts based on recovery to those based on simple benchmarks (historical averages or risk-neutral forecasts).

- **Smoothing and Stability Improvements.** At various points along the pipeline, we introduce smoothing to lower noise in the estimates. For instance, we can smooth the time series for recovered distributions or introduce weak regularization penalties to prevent radical oscillations in the SDF or probability estimates from one period to the next. This approach directly answers the literature: the instability noted by Jackwerth & Menner[5] is addressed by introducing a structure that suppresses fluky solutions, akin to the "Ross-Stable" method. Doing so, we can yield a stable pricing kernel and probability measure in the long run that are time-robust (with echoes to the stable results for generalized recovery). We also incorporate the interest rate term structure in the construction of the state price vector (applying the correct discount factors for each maturity in the options), to align the recovery with the market yield curve prevailing at the time–an area noted by Martin & Ross[7] to be important for consistency. In short, our implementation advances previous models by focusing on practical robustness and multi-dimensionality. The construction and interpolation steps in the state-price matrix ensure that we satisfy the data richness and completeness conditions required by Ross's theorem ideally (something Sanford[11] also addressed by employing rich option data).

Estimating the transition matrix by eigen-decomposition is then performed under select constraints to prevent the identification failures raised by Borovička et al.Borovička et al. [3]. (we test that no important "martingale component" goes unchecked).

Back testing the recovered outcomes is a new empirical contribution linking our findings to the Audrino et al. vs. Jackwerth & Menner[5] debate over the existence of predictive gains. Lastly, the smoothing and the regularization parts of the pipeline serve as protection measures, taking on board the lesson from the regularized multivariate approaches in the literature. Correcting for known limitations—markets' incompleteness, noise sensitivity, and model misspecification—our project approach aims to give a firmer version of the Recovery Theorem implementation, shedding new light on the question of whether the recovered physical probabilities and the SDF can be beneficially extracted from prices of options for quantitative asset management.

3 Data

3.1 Data Sources

The data in this study are mainly divided into Option chain data and Interest rate data:

Option chain data: We used two types of data for comparison: cleaned data from Yahoo Finance (accessed through the yfinance interface). Downloaded through Yahoo Finance's yfinance interface, this data includes option chains for dates in August 19th 2025. The data covers both call and put options, including strike price, expiration date, last price, implied volatility, open interest, and volume. This data was used to construct option price surfaces, derive the Arrow–Debreu state price distribution, and recover natural probabilities.

Interest rate data: We obtain the US Treasury yield curve from FRED (Federal Reserve Economic Data) that is consistent with the option data date, so that the interest rate data is consistent with the option data date. At the same time, the obtained data is used in the subsequent calculation of the discount factor. ($\delta = e^{-rT}$), and combines the risk-neutral pricing framework with the Arrow–Debreu state price matrix.

On trading day, we use the spot price of the S&P 500 (6411.37 on August 19th 2025) as the benchmark for the option's relative strike price (moneyness).

3.2 Data Cleansing and Conversion

Given that the raw option chain consists of tens of thousands of contracts, many of which are not traded all, are not liquid and are "far" out of the money, systematic cleaning is required. We implement a function (`clean_df`) to perform the following steps:

1. **Convert the base date and calculate the time to maturity:** We first calculate the time to maturity (in days): $T = \texttt{expirationDate} - \texttt{DATE}$. This yields the number of days to maturity for each contract, T, which provides a time dimension for subsequent pricing and discounting.

2. **Standardization of variables:** We standardize field names: Strike Price $\rightarrow K$, Option Price $\rightarrow$ Price, Implied Volatility $\rightarrow$ IV. This facilitates formulas and numerical modeling.

3. **Liquidity Screening:** We only retain options with Open Interest ≥ 1 and Volume ≥ 1. We exclude contracts with negligible trading to remove price noise or manipulation.

4. **Strike Price Bandwidth Screening:** We retain contracts with strike prices within $\pm 25\%$ of the spot price, i.e., $0.75 \times \text{SPOT} \leq K \leq 1.25 \times \text{SPOT}$. We also exclude options that are deeply in-the-money (ITM) and deeply out-of-the-money (OTM) to avoid introducing extreme noise when fitting the implied volatility surface.

5. **Time-to-expiration Screening:** We restrict expiration times to $T \leq 200$ days (approximately 6.5 months). Contracts with expiration times exceeding 200 days often have sparse trading, large quote variance, and are incomparable to short- and medium-term contracts.

6. **Implied Volatility Validation Test:** Finally, we exclude options with an IV ≤ 0.001. These are often missing values or invalid quotes and cannot be used for stable modeling.

After cleaning the data, we successfully obtained a sample of underlying assets and options, including both call and put options, with strike prices ranging within $\pm 25\%$ of the spot price and expiration dates not exceeding 200 days. We also enhanced data quality, keeping only contracts which had good liquidity and which contained valid quotation data whose maturity/duration is the same to keep input data stable and comparable. We also used zero-coupon Treasury bond rates from FRED to compute robust discount factors.

4 Econometric Methods

4.1 Core Formula

Following Ross (2015)[10], the recovered transition operator F is given by

$$F = \left(\frac{1}{\delta}\right) DPD^{-1}. \tag{1}$$

The components of this formula can be described as follows:

- P is the Arrow–Debreu price matrix of dimension $m \times m$, where each element p_{ij} reflects the price of a state-contingent payoff.
- F is the $m \times m$ matrix representing the recovered transition probabilities under the real (or natural) probability measure.
- δ is the discount factor, often written as $\delta = e^{-r}$ with r denoting the risk-free interest rate.
- D is a diagonal matrix whose entries correspond to the marginal rates of substitution (scaled by δ), which link consumption-based utility to state pricing.
- D^{-1} is the inverse of this diagonal matrix, completing the similarity transformation.

This representation shows that the recovered transition matrix F can be constructed from three ingredients: the observed state price matrix P, the discount factor δ, and the substitution effects encoded in D. In essence, the equation provides a way to map from risk-neutral pricing information to the real-world probabilities implied by investors' preferences.

4.2 Constructing the State Price Matrix

Since we do not directly observe a full set of multi-state Arrow–Debreu securities, we follow the approach of Breeden and Litzenberger (1978)[4] and recover them from option prices. Specifically, for a given strike K and maturity T, the Arrow–Debreu security price is obtained as

$$p(K,T) = C''(K,T), \tag{2}$$

where $C(K,T)$ denotes the European call option price with strike K and maturity T, and $C''(K,T)$ is the second derivative with respect to the strike price.

In practice, since option prices are observed on a discrete grid of strikes, we approximate the second derivative numerically as

$$C''(K,T) \approx \frac{C(K+\Delta K,T) - 2C(K,T) + C(K-\Delta K,T)}{(\Delta K)^2}. \tag{3}$$

Thus, in our implementation, the Arrow–Debreu prices $p(K,T)$ are constructed from observed S&P 500 option prices by applying this finite difference approximation.

4.2.1 Cleaning the Implied Volatility Data

To construct a consistent state price matrix P, we require a dense and smooth representation of option prices. Directly smoothing the call price surface $C(K,T)$ is problematic, since the payoff introduces a kink at the money. Instead, we first smooth the implied volatility (IV) surface and then map it back to call prices via the Black–Scholes model.

A key difficulty arises around the moneyness region ($K \approx$ Spot), where implied volatilities can be unstable. To address this, we adopt the following strategy:

- For strikes $K \geq 1.02 \times$ Spot, we use call option implied volatilities.
- For strikes $K \leq 0.98 \times$ Spot, we use put option implied volatilities.
- For the middle region $0.98 \times \text{Spot} < K < 1.02 \times \text{Spot}$, we construct a blended implied volatility as a weighted average:

$$\text{IV}_{blend} = w_{call} \cdot \text{IV}_{call} + w_{put} \cdot \text{IV}_{put},$$

  where the weights are determined by open interest:

$$w_{call} = \frac{\text{OI}_{call}}{\text{OI}_{call} + \text{OI}_{put}}, \qquad w_{put} = \frac{\text{OI}_{put}}{\text{OI}_{call} + \text{OI}_{put}}.$$

Here, open interest (OI) refers to the number of outstanding option contracts that are currently open and not yet settled. It is often interpreted as a proxy for market liquidity and investor participation. By weighting implied volatilities according to open interest, the blended IV gives more importance to the side of the market (calls or puts) that carries greater trading activity.

4.2.2 Learning a Smooth IV Surface with an MLP

To obtain a dense and smooth implied volatility (IV) surface, we fit a fully connected multilayer perceptron (MLP) that maps strike and maturity to implied volatility:

$$f_\theta : \ (K, T) \ \mapsto \ \widehat{\text{IV}}.$$

The network has two hidden layers with 64 units each and a single scalar output. To ensure smooth and continuous predictions, needed for stable finite–difference second derivatives with respect to K, we use a twice–differentiable activation softplus in the hidden layers. We train the model by minimizing the mean–squared error between market IVs and $f_\theta(K, T)$ over the observed grid of (K, T) quotes. The learned surface $\widehat{\text{IV}}(K, T)$ is then converted to a smoothed call price surface via the Black–Scholes formula, after which we compute Arrow–Debreu prices $p(K, T)$ using discrete second differences in K.

4.3 Estimating the State-Price Transition Matrix

Let $p^t \in \mathbb{R}^{1\times m}$ denote the row vector of Arrow–Debreu state prices at horizon t (constructed from $C''(K, T)$ on an m-state grid). We apply a light post-estimation normalization that row sums equals to discount factor at t. Following the forward relation $p^{t+1} = p^t P$ (Ross, 2015, Eq. (86)), we estimate the transition matrix $P \in \mathbb{R}_+^{m\times m}$ by solving the regularized nonnegative least squares problem

$$\min_{P \geq 0} \sum_{t=1}^{T-1} \left\| p^t P - p^{t+1} \right\|_2^2 \ + \ \lambda \left\| P \right\|_F^2, \tag{4}$$

where $\lambda \geq 0$ controls Tikhonov (ridge) regularization. Nonnegativity enforces absence of static arbitrage in state prices.

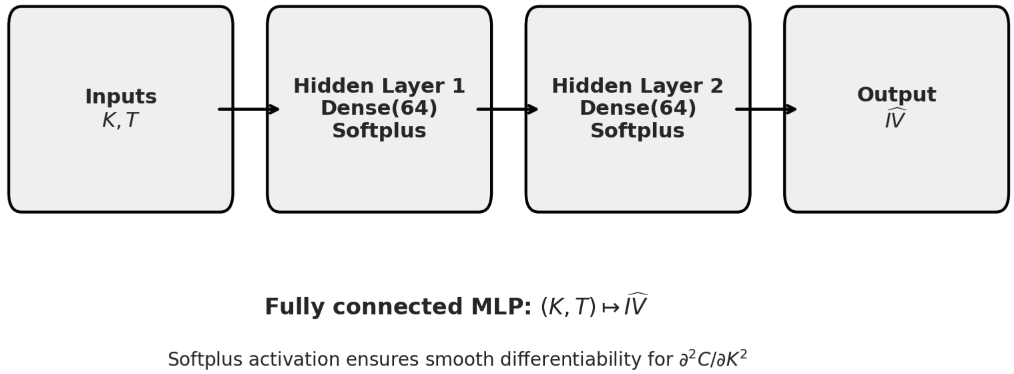

Figure 1: MLP used to smooth the implied volatility surface: inputs (K, T), two hidden layers with 64 units each (twice–differentiable activations), and a scalar output $\widehat{\mathrm{IV}}$.

4.4 Recovery Theorem and Perron–Frobenius Argument

By Theorem 1 of Ross (2015)[10], if the Arrow–Debreu price matrix P is irreducible and generated by a transition-independent kernel, then the system admits a unique positive solution for the natural probability matrix F, together with a unique pricing kernel.

From the Perron–Frobenius theorem, an irreducible nonnegative matrix P possesses a unique largest real eigenvalue λ with a strictly positive associated eigenvector z. In our setting, this eigenvalue is identified with the discount factor δ^{-1}, and the corresponding eigenvector provides the elements required to construct the diagonal matrix D.

Formally, if z denotes the unique positive eigenvector of P corresponding to the eigenvalue δ^{-1}, then the kernel can be written as

$$\phi_i = d_{ii} = \frac{1}{z_i}, \tag{5}$$

and thus

$$D = \mathrm{diag}\left(\frac{1}{z_1}, \frac{1}{z_2}, \dots, \frac{1}{z_m}\right). \tag{6}$$

4.5 Constructing the Recovered Transition Probabilities

Element by element, the entries of the recovered transition matrix F can be expressed as follow by Ross(2015)[10]

$$f_{ij} = \left(\frac{1}{\delta}\right)\frac{\phi_i}{\phi_j}p_{ij} = \left(\frac{1}{\delta}\right)\frac{U_i'}{U_j'}p_{ij} = \left(\frac{1}{\lambda}\right)\frac{z_j}{z_i}p_{ij}, \tag{7}$$

where p_{ij} are the Arrow–Debreu state prices, ϕ_i are the marginal rates of substitution, and z is the Perron–Frobenius eigenvector of P corresponding to the largest eigenvalue $\lambda = \delta^{-1}$.

Hence, the full transition matrix F can be constructed compactly as

$$F = \left(\frac{1}{\delta}\right) DPD^{-1}, \tag{8}$$

where $D = \mathrm{diag}(1/z_1, \dots, 1/z_m)$.

5 Empirical Results

1) Obtain Raw Options Data for SPX (Call/Put) 2) Cleaned using the weighted average open interest of a blended call/put structure 3) Using Double Hidden Layer ML method, we construct the smoothed implied volatility surface using

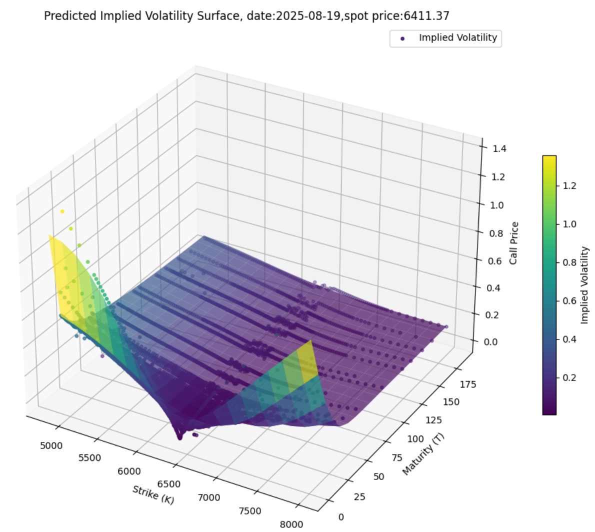

Figure 2: This is the Surface Constructed from the Raw Option Prices

4) Recovered and smoothed Call Price using Black–Scholes from the IV surface

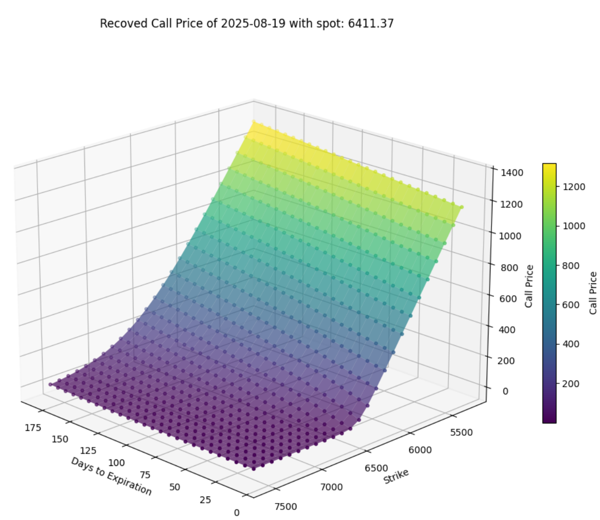

Figure 3: Recovered Call Option Surface

5) Normalized State Price Surface via the Breeden–Litzenberger approach.

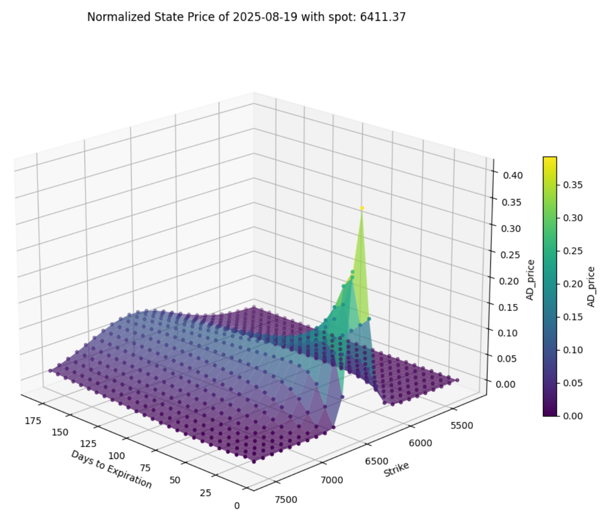

Figure 4: Normalized State Price Surface

6) From the Normalized State Price Surface, we obtain the Transition Surface, giving us the Natural Probability and the Pricing Kernel

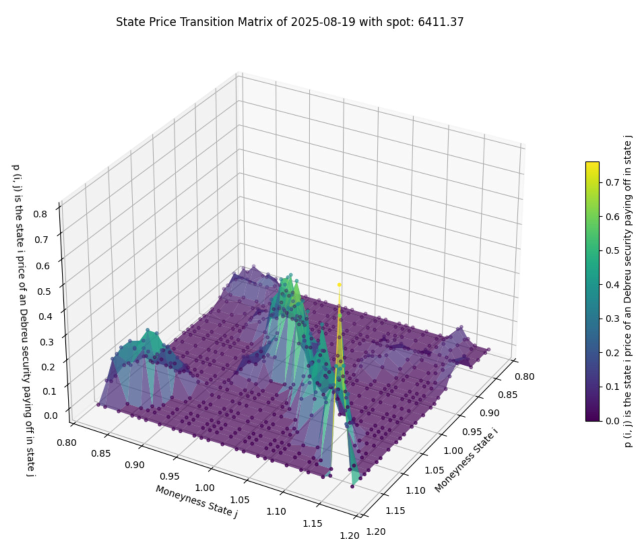

Figure 5: Transition Surface

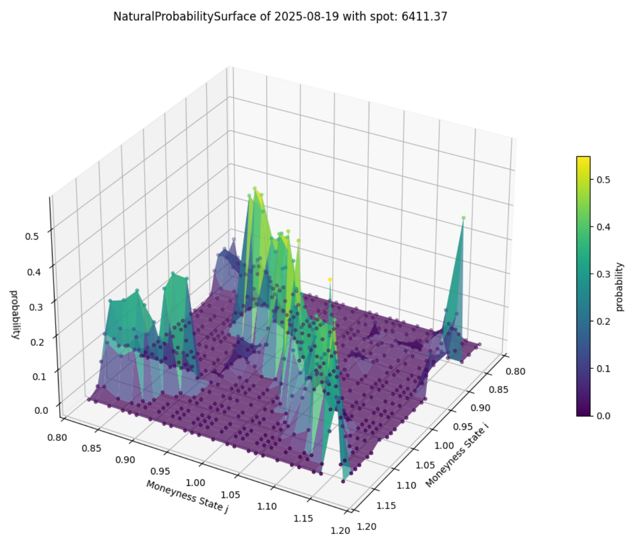

Figure 6: Natural Probability Surface

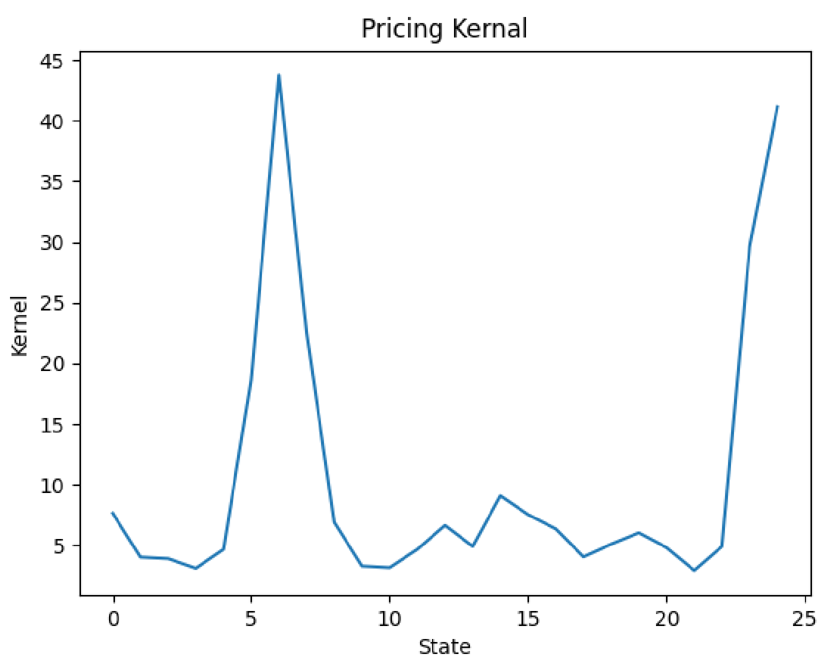

Figure 7: Pricing Kernel

6 Evaluation

Although there has been some empirical evidence against the validity of a standard recovery process, there are still some inferences that can be made based on the results. Starting with the natural probabilities, we can see a small spread in the transition probabilities from any state i has a small range in which it is likely to fall in that is typically 5 percent either side of state i for the majority of the probability mass for the given state. However, the tails exhibit different behavior which is common amongst finance's most studied problems. In the upper state tail of our analysis, we can see a larger than 0 probability of transitions to the lowest tails. A possible conclusion is based on the market, investors could be positioning themselves in a way that reflects a bubble, AI is currently not reflecting any profits and there are the beginnings of whispers of an AI bubble within wall street. On the other hand, in the lower state tail, there is a large spike in what would be considered a recovery from the lower state, reflecting that investors would believe that a dip would be short-lived. While there are many stories that can match these transition probabilities, it is the basis of information that the Ross Recovery Theorem can provide.

Moving on to the pricing kernel, there is a clear U shape presented, which is a typical result given the research of Jackwerth and Menner[5]. Their conclusion for such a result is the heterogeneity of investor beliefs. Our result is a reflection of the biggest flaw in the assumption of a representative investor in the theorem as options specifically index options are used in many different uses which would reflect multiple differing views. Jackwerth's research continues on to describe what types of activities could reflect the pricing kernel. Short sellers may counter their bets with OTM calls, or standard investors are hedging their strategies with OTM puts, there are also strategies around jumps in prices. Since the pricing kernel represents marginal utility which is a proxy to the worth of an additional dollar or additional payoff in that state. The large spike in the downside represents the level at which most investors are positioning themselves for puts, the slight curve upwards after the drastic drop represents an increased amount of puts that are taken in that area. The large spike at the right side would come from either large hedges against short selling or large directional bets at that level for calls. Overall, the results do not have predictive value, however they are a good starting point to determine how investors on wall street are currently planning and behaving.

7 Conclusion

The Ross Recovery Theorem was the start of a long line of mathematically intensive research. Our goal was always to replicate the original theorem, however, the extended research helped us through the process and enlightened us to ways to possibly enhance the continuation of this research at a later date. While there is limited value in the results produced, there are still many key findings both positive and negative from our process. Our results are confirmation of previous research. This project has allowed us to take advanced mathematical concepts into actionable insights through coding. Even with advanced Machine Learning techniques, our output was inconsistent due to the fickle nature of the theorem and how small changes specifically in the tails create large changes in the output. Leading to a struggle to make consistent results from data given from Capstone Investment Advisors. Decision makers can look at the results to have a starting point for understanding market beliefs, how investors are positioned. They can then make decisions based on their own hypothesis to take positions that can exploit these positionings. In the end, we were successful in our attempt to replicate the Ross Recovery Theorem.

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