MMF1928 PRICING THEORY LECTURE NOTES

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1 Lecture 1

1.1 Setup

Financial Market $\Rightarrow$ Time Series $\Rightarrow$ Information Technology

Space: $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$

1. Probability Triple:
   (1) Ω: Sample space, all possible outcomes
   (2) $\mathcal{F}$: σ-algebra, all information, collection of event sets
   (3) $\mathbb{P}$: Probability measure
2. $\{\mathcal{F}_t\}$: A collection of event sets, indexed by t.
   (1) $\mathcal{F}_t$: σ-algebra, information being available at time t, $\forall s < t$, $s, t, \in \mathcal{T}$, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$.
   (2) $\mathcal{T}$ = Time Set
   $\mathcal{T} = \{0, T\}$: Single-period model
   $\mathcal{T} = \{0, 1, 2, \ldots, T\}$: Multi-period model
   $\mathcal{T} = \{0, 1, 2, \ldots\}$: Discrete-time model
   $\mathcal{T} = [0, T]$: Continuous-time model
3. $\{\mathcal{F}\}_{t\in\mathcal{T}}$: filtration, i.e., increasing collection of σ-algebra

1.2 Financial Assets

1. Cash: Corporate/Government bonds, interest rates (r). Usually no risk, deterministic, money market (numéraire).
2. Stock: Usually very risky, stochastic, S_t, adaptive to $\mathcal{F}_t$.
3. Option: Maturity T, strike price K. European call at $T : (S_T - K)^+$. American call before $T : (S_t - K)^+$
   (1) X_t: Wealth of portfolio (to replicate the payoff of option).
   (2) Δ_t: Number of shares in risky assets. $\Delta_t > 0$: Long position. $\Delta_t < 0$: Short position
   (Assumption in this course: continuous, short selling allowed.)

1.3 Single Period Model

Question: $X_0 = V_0 \Longrightarrow \forall\omega \in \Omega, X_T(\omega) = V_T(\omega)$

Key feature: no extra input/output, self-financing strategy (Δ_t)

Definition 1.3.1. A *self-financing strategy* $\{\Delta_t\}_{t\in\{0,T\}}$ is an adaptive stochastic process such that

i. $\{\Delta_t\}_{t\in\{0,T\}}$ is adaptive, Δ_t is $\mathcal{F}_t$-measurable

ii. $X_0 = \Delta_0 S_0 + (X_0 - \Delta_0 S_0)$

Note: As time goes on, it becomes $X_1(\omega) = \Delta_0 S_1(\omega) + (1 + r)(X_0 - \Delta_0 S_0) = \Delta_1(\omega)S_1(\omega) + X_1(\omega) - \Delta_1(\omega)S_1(\omega)$

Example 1.3.2. Assume $r = 0$.

At time $t = 0$, $\Delta_0 = 10$ and $S_0 = \$100$, we have $X_0 = 10 \times \$100 + \$1000 - 10 \times \$100 = \$1000$. At this time point, money in stock is $10 \times \$100 = \$1000$ and money in cash is $\$1000 - 10 \times \$100 = \$0$.

As time goes to $t = 1$, $\Delta_1 = 9$ and $S_1 = \$105$, we have $X_1 = 10 \times \$105 + \$1000 - 10 \times \$100 = \$1050$ and $X_1 = 9 \times \$105 + \$1050 - 9 \times \$105 = \$1050$. At this time point, money in stock is $9 \times \$105 = \$945$ and money in cash is $\$1050 - 9 \times \$105 = \$105$.

Definition 1.3.3. An *arbitrage strategy* $\{\Delta_t\}_{t\in\{0,T\}}$ is a self-financing strategy such that

i. $X_0 = 0$ portfolio at time 0

ii. $\mathbb{P}[(X_T \geq 0] = 1$ and $\mathbb{P}[(X_T > 0] > 0$, X_T portfolio at T

1.4 Binomial Model

Space: $(u, d, r, p = \mathbb{P}[\text{outcome} = H])$

1. Stock: The initial stock price is S_0. If the stock price goes up, then $S_1(H) = uS_0 \Rightarrow u = \frac{S_1(H)}{S_0}$; if the stock price goes down, then $S_2(H) = dS_0 \Rightarrow d = \frac{S_2(T)}{S_0}$, where $u \neq d$.
2. Cash: Compounding at the interest rate of r.
3. Arbitrage: Suppose $d < u \leq 1 + r$, then we have

   $X_0 = 0 = -\Delta_0 S_0 + \Delta_0 S_0$

   $X_1(H) = -\Delta_0 S(H) + (1 + r)\Delta_0 S_0 = (1 + r - u)\Delta_0 S_0$, where $1 + r - u \geq 0$

   $X_1(T) = -\Delta_0 S(T) + (1 + r)\Delta_0 S_0 = (1 + r - d)\Delta_0 S_0$, where $1 + r - d > 0$

   $\mathbb{P}(X_1 \geq 0) = 1, \mathbb{P}(X_1 > 0) = \mathbb{P}(\text{outcome} = T) > 0$

   To prevent arbitrage, we need $p > 0$, $u > 1 + r > d$.

   For example, $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $\mathbb{P}(\text{outcome} = H) = \frac{1}{2}$.

1.5 Risk-Neutral Pricing

Under what measure is the discount stock price being a martingale?

Idea: every option is compounding with respect to the interest rate r.

Consider the conditional expectation, we have

$$\begin{aligned}
&\tilde{\mathbb{E}}[S_1 \mid S_0] = (1 + r)S_0 \\
\Rightarrow\ &\tilde{\mathbb{E}}[S_1 \mid S_0] = \tilde{p}uS_0 + \tilde{q}dS_0 \\
\Rightarrow\ &\begin{cases} \tilde{p} + \tilde{q} = 1 \\ u\tilde{p} + d\tilde{q} = 1 + r \end{cases} \Rightarrow \begin{cases} \tilde{p} = \frac{1+r-d}{u-d} \\ \tilde{q} = \frac{u-(1+r)}{u-d} \end{cases}
\end{aligned}$$

Since $d < 1 + r < u$, then we know $\tilde{p}, \tilde{q} \in (0, 1)$.

Recall a self-financing strategy $\{\Delta_t\}_{t\in\{0,1\}}$, we know

$$X_0 = x = \Delta_0 S_0 + x - \Delta_0 S_0 \Rightarrow X_1 = \Delta_0 S_1 + (1 + r)(x - \Delta_0 S_0)$$

then we can derive that, in $\mathbb{Q}$-world,

$$\tilde{\mathbb{E}}[X_1 \mid S_0, X_0 = x] = \Delta_0 \tilde{\mathbb{E}}[S_1 \mid S_0] + (1 + r)(x - \Delta_0 S_0) = \Delta_0 S_0 + (1 + r)(x - \Delta_0 S_0) = (1 + r)x$$

Example 1.5.1. Assume $S_0 = 4$, $u = 2$, $d = \frac{1}{2}$, $r = \frac{1}{4}$, $V_1 = (S_1 - K)^+$, $K = 4$.

We calculate $\tilde{p} = \frac{1+r-d}{u-d} = \frac{1+\frac{1}{4}-\frac{1}{2}}{2-\frac{1}{2}} = \frac{1}{2}$ and $\tilde{q} = 1 - \tilde{p} = \frac{1}{2}$.

To replicate the portfolio, for some unknown Δ_0, X_0, we have

$$\begin{cases} X_1(H) = V_1(H) = \Delta_0 S_1(H) + (1+r)(X_0 - \Delta_0 S_0) = 4 \\ X_1(T) = V_1TH) = \Delta_0 S_1(T) + (1+r)(X_0 - \Delta_0 S_0) = 0 \end{cases} \Rightarrow \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)} = \frac{4}{8-2} = \frac{2}{3}$$

Therefore, the delta-hedging formula is $\Delta_0 = \frac{V_1(H)-V_1(T)}{S_1(H)-S_1(T)}$.

1.6 Fundamental Theorem of Asset Pricing I (FTAP I)

Theorem 1.6.1. *(Fundamental Theorem of Asset Pricing I (FTAP I)).* No arbitrage in a financial market $\Leftrightarrow$ There exists a risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ such that the discounted stock price is a martingale.

Remark 1.6.2. $\mathbb{Q} \sim \mathbb{P}$ means that $\forall A \in \mathcal{F}, \mathbb{Q}[A] = 0 \Leftrightarrow \mathbb{P}[A] = 0$

Theorem 1.6.3. *(First fundamental theorem of asset pricing from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.4.7).* If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Example 1.6.4. Assume $u = 2$, $d = \frac{1}{2}$, $r = 0$, $S_0 = \$4$. If the stock price goes up, $S_u = \$8$. If the stock price remains the same, $S_n = \$4$. If the stock price goes down, $S_d = \$2$. Let p denote the probability of stock price going up and q denote the probability of stock pricing going down. Assume the probability of going up and going down are $\tilde{p} = \frac{1}{4}$ and $\tilde{q} = \frac{1}{4}$. Then we have $\Pr[S_1 = \$4] = 1 - \frac{1}{4} - \frac{1}{4} = \frac{1}{2}$.

For $A = \varnothing$, we have $\mathbb{P}[A] = \mathbb{Q}[A] = 0$. For $A = \{\varnothing, \{S_1 = \$4\}\}$, $\hat{p} = \hat{q} = \frac{1}{2}$, however, we know that $\tilde{p} = \frac{1}{4}$ and $\tilde{q} = \frac{1}{4}$, which is not equivalent. We also calculate

$$\tilde{\mathbb{E}}[S_1 \mid S_0] = \frac{1}{4} \times \$8 + \frac{1}{4} \times \$2 + \frac{1}{2} \times \$4 = \$4.5 \neq \$4$$

Therefore, it is not a risk-neutral measure.

If $\tilde{p}^{\mathbb{Q}} = \frac{1}{6}$, $\tilde{q}^{\mathbb{Q}} = \frac{1}{3}$, $\Pr^{\mathbb{Q}}[S_1 = \$4] = \frac{1}{2}$, then

$$\tilde{\mathbb{E}}^{\mathbb{Q}}[S_1 \mid S_0] = \frac{1}{6} \times \$8 + \frac{1}{3} \times \$2 + \frac{1}{2} \times \$4 = \$4 = \$4$$

Therefore, it is a risk-neutral measure.

If $\tilde{p}^{\mathbb{Q}} = \frac{1}{5}$, $\tilde{q}^{\mathbb{Q}} = \frac{2}{5}$, $\Pr^{\mathbb{Q}}[S_1 = \$4] = \frac{2}{5}$, then

$$\tilde{\mathbb{E}}^{\mathbb{Q}}[S_1 \mid S_0] = \frac{1}{5} \times \$8 + \frac{2}{5} \times \$2 + \frac{2}{5} \times \$4 = \$4 = \$4$$

Therefore, it is a risk-neutral measure.

1.7 Fundamental Theorem of Asset Pricing II (FTAP II)

Theorem 1.7.1. *(Fundamental Theorem of Asset Pricing II (FTAP II)).* There exists a unique risk-neutral measure $\Leftrightarrow$ The financial market is complete, i.e., for every option there exists a replicating portfolio.

Theorem 1.7.2. *(Second fundamental theorem of asset pricing from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.4.9).* Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

2 Lecture 2

2.1 Review of FTAP I and FTAP II

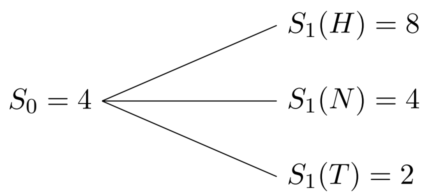
Theorem 2.1.1. *(FTAP I).* There is no arbitrage in the financial market if and only if there exists a probability measure $\mathbb{Q} \sim \mathbb{P}$ such that under $\mathbb{Q}$, the discount asset prices are martingales, i.e., $\mathbb{Q} \sim \mathbb{P}$ is a risk-neutral measure.

Theorem 2.1.2. *(FTAP II).* There exists a unique risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$ if and only if the market is complete, i.e., all options have a replicating portfolio.

Remark 2.1.3. "$\sim$" denotes the equivalence of probability measures.

Remark 2.1.4. The time of a martingale is "today".

Example 2.1.3. *(Application of FTAP I and FTAP II).* On a probability space of $(\Omega, \mathcal{F}, \mathbb{P})$, we have $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_1 = \mathcal{F}$. For $\Omega = \{H, N, T\}$, $\omega \in \Omega$, and $S_1(\omega)$, we have $N_\Omega = 3$, $\mathcal{F} = \{\varnothing, \Omega, \{H\}, \{N\}, \{T\}, \{H, N\}, \{H, T\}, \{N, T\}\}$, and $N_{\mathcal{F}} = 2^{N_\Omega} = 2^3 = 8$.



For a physical measure $\mathbb{P}$, $\mathbb{P}[\Omega] = 1, \mathbb{P}[H] = \mathbb{P}[N] = \mathbb{P}[T] = \frac{1}{3}, \mathbb{P}[\{H, N\}] = \mathbb{P}[S_1 \neq 2] = \frac{2}{3} = \mathbb{P}[\{H, T\}] = \mathbb{P}[\{N, T\}]$, and $\mathbb{P}[\varnothing] = 0$. Are the following portfolios risk-neutral? Assume that $r = 0$.

(1) For a new probability measure $\tilde{\mathbb{P}}$, $\tilde{\mathbb{P}}[H] = \tilde{\mathbb{P}}[T] = \frac{1}{4}$ and $\tilde{\mathbb{P}}[N] = \frac{1}{2}$:

For any $A \in \mathcal{F}$, we can easily get $\tilde{\mathbb{P}}[A] = 0 \Leftrightarrow A = \varnothing$, so $\tilde{\mathbb{P}} \sim \mathbb{P}$.

Then we calculate the discounted asset price as $\tilde{\mathbb{E}}[S_1 \mid S_0] = \frac{1}{4} \cdot 8 + \frac{1}{4} \cdot 2 + \frac{1}{2} \cdot 4 = 4.5 \neq S_0$.

Therefore, $\tilde{\mathbb{P}}$ is not a risk-neutral measure.

(2) For a new probability measure $\hat{\mathbb{P}}$, $\hat{\mathbb{P}}[H] = \hat{\mathbb{P}}[T] = \frac{1}{2}$ and $\hat{\mathbb{P}}[N] = 0$:

For $\{N\} = \{S_1 = 4\} \in \mathcal{F}$, we have $\hat{\mathbb{P}}[N] = 0$, however, $\mathbb{P} = \frac{1}{3}$, so $\hat{\mathbb{P}}$ is not equivalent to $\mathbb{P}$.

(3) For a new probability measure $\mathbb{Q}$, $\mathbb{Q}[H] = \frac{1}{6}$, $\mathbb{Q}[T] = \frac{1}{3}$, and $\mathbb{Q}[N] = \frac{1}{2}$:

For any $A \in \mathcal{F}$, we can easily get $\mathbb{Q}[A] = 0 \Leftrightarrow A = \varnothing$, so $\mathbb{Q} \sim \mathbb{P}$.

Then we calculate the discounted asset price as $\tilde{\mathbb{E}}[S_1 \mid S_0] = \frac{1}{6} \cdot 8 + \frac{1}{3} \cdot 2 + \frac{1}{2} \cdot 4 = 4 = S_0$.

Therefore, $\mathbb{Q}$ is a risk-neutral measure.

Note: The standard mechanism is

(1) Check $\mathbb{Q} \sim \mathbb{P}$

(2) Check if discounted asset prices are martingale

$\mathbb{Q}$ is not unique since $\hat{\mathbb{Q}}[H] = \frac{2}{5}$, $\hat{\mathbb{Q}}[T] = \frac{1}{5}$, and $\hat{\mathbb{Q}}[N] = \frac{2}{5}$ is also a risk-neutral measure. Then by FTAP II, there exists unhedgable options in the market.

To find an unhedgable option in the market, consider the following option

$$
V_0 \begin{cases} V_1(H) = (S_1(H) - 3)^+ = 5 = X_0 - \Delta_0 S_0 + S_1(H)\Delta_0 = X_0 + 4\Delta_0 \\ V_1(N) = (S_1(N) - 3)^+ = 1 = X_0 - \Delta_0 S_0 + S_1(N)\Delta_0 = X_0 \\ V_1(T) = (S_1(T) - 3)^+ = 0 = X_0 - \Delta_0 S_0 + S_1(T)\Delta_0 = X_0 - 2\Delta_0 \end{cases}
$$

There is no solution to these equations, therefore, $V_1 = (S_1 - 3)^+$ is unhedgable.

2.2 Martingale

Definition 2.2.1. On space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{T}}, \mathbb{P})$, a stochastic process $\{X_t\}_{t\in\mathcal{T}}$ is a *martingale* provided

i. $\{X_t\}_{t\in\mathcal{T}}$ is adaptive to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$

ii. $\mathbb{E}[|X_t|] < \infty$

iii. $\forall s < t, t, s \in \mathcal{T}, \mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ almost surely

Example 2.2.2. Let $X \sim N(\mu, \sigma^2)$ and we only know X at time 1.

Define $\{\mathcal{F}_t\}_{t\in[0,1]}$ and $X_t = \tilde{\mathbb{E}}[X \mid \mathcal{F}_t]$. Then, $\forall 0 \le s < t \le 1$, we have

$$
\begin{aligned}
\mathbb{E}\left[X_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[\mathbb{E}[X \mid \mathcal{F}_t] \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[X \mid \mathcal{F}_s\right] \quad \text{by tower property} \\
&= X_s \quad \text{almost surely}
\end{aligned}
$$

Therefore, $\{X_t = \mathbb{E}[X \mid \mathcal{F}_t]\}_{t\in[0,1]}$ is a martingale with respect to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$ by definition of martingale.

Definition 2.2.3. On space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathcal{F}}, \mathbb{P})$, a stochastic process $\{X_t\}_{t\in\mathcal{T}}$ is a *sub-martingale* provided

i. $\{X_t\}_{t\in\mathcal{T}}$ is adaptive to $\{\mathcal{F}_t\}_{t\in\mathcal{T}}$

ii. $\mathbb{E}[|X_t|] < \infty$

iii. $\forall s < t, t, s \in \mathcal{T}, \mathbb{E}[X_t \mid \mathcal{F}_s] \ge X_s$ almost surely

2.3 Multi-period Binomial Model

A multi-period binomial model is

$$
S_t(\omega_1\omega_2\ldots\omega_t) \begin{cases} S_{t+1}(\omega_1\omega_2\ldots\omega_t H) = uS_t(\omega_1\omega_2\ldots\omega_t) \\ S_{t+1}(\omega_1\omega_2\ldots\omega_t T) = dS_t(\omega_1\omega_2\ldots\omega_t) \end{cases}
$$

where u, d, r does not depend on $t, \omega_i \in \{H, T\}, i = 1, 2, \ldots, t$. For an N-period binomial model, $\omega = \omega_1\omega_2\cdots\omega_N \in \Omega$.

Example 2.3.1. *(2-Period).* Consider a 2-period binomial model. Assume $r = \frac{1}{4}$ and $u = \frac{1}{2} = \frac{1}{d}$.

$$
\begin{array}{ccccc}
 & & & & S_2(HH) = 16 \\
 & & S_1(H) = 8 & & \\
S_0 = 4 & & & & S_2(HT) = S_2(TH) = 4 \\
 & & S_1(T) = 2 & & \\
 & & & & S_2(TT) = 1
\end{array}
$$

Under the physical measure, the branching probabilities are $p = \frac{1}{2} = q$.

Under the risk-neutral measure, the discounted stock price is a martingale.

The probability sets are $\Omega = \{HH, HT, TH, TT\}$, $N_\Omega = 4$, $\mathcal{F} = \{\varnothing, \Omega, \{HH\}, \{HT\}, \{TH\}, \{TT\}, \{HH, HT\}, \{HH, TH\}, \{HH, TT\}, \{HT, TH\}, \{HT, TT\}, \{TH, TT\}, \{HH, HT, TH\}, \{HH, TH, TT\}, \{HT, TH, TT\}\}$, and $N_\mathbb{F} = 2^4 = 16$.

Then we know $\mathcal{F}_0 = \{\varnothing, \Omega\}$, $\mathcal{F}_1 = \{\varnothing, \Omega, \{HH, HT\}, \{TH, TT\}\}$, $\mathcal{F}_2 = \mathcal{F}$. Therefore, $\mathcal{F}_0 \subset \mathcal{F}_1 \subset \mathcal{F}_2$.

From $t = 0$ to $t = 1$, we have

$$\begin{cases} \tilde{p}_1 + \tilde{q}_1 = 1 \\ \tilde{\mathbb{E}}\left[\frac{S_1}{1+r} \mid S_0\right] = \frac{1}{1+r}\left[\tilde{p}_1 S_1(H) + \tilde{q}_1 S_1(T)\right] \end{cases}$$
$$\Rightarrow \begin{cases} \tilde{p}_1 + \tilde{q}_1 = 1 \\ \frac{S_0}{1+r}(u\tilde{p}_1 + d\tilde{q}_1) = S_0 \end{cases}$$
$$\Rightarrow \begin{cases} \tilde{p}_1 = \frac{1+r-d}{u-d} > 0 \\ \tilde{q}_1 = \frac{u-(1+r)}{u-d} > 0 \end{cases} \quad \text{with } d < 1 + r < u$$

From $t = 1$ to $t = 2$, we have

$$\begin{cases} \tilde{p}_{2H} + \tilde{q}_{2H} = 1 \\ \tilde{\mathbb{E}}\left[\frac{S_2}{(1+r)^2} \mid S_1(H)\right] = \frac{S_1(H)}{(1+r)^2}(u\tilde{p}_{2H} + d\tilde{q}_{2H}) \end{cases} \Rightarrow \begin{cases} \tilde{p}_{2H} = \frac{1+r-d}{u-d} \\ \tilde{q}_{2H} = \frac{u-(1+r)}{u-d} \end{cases}$$

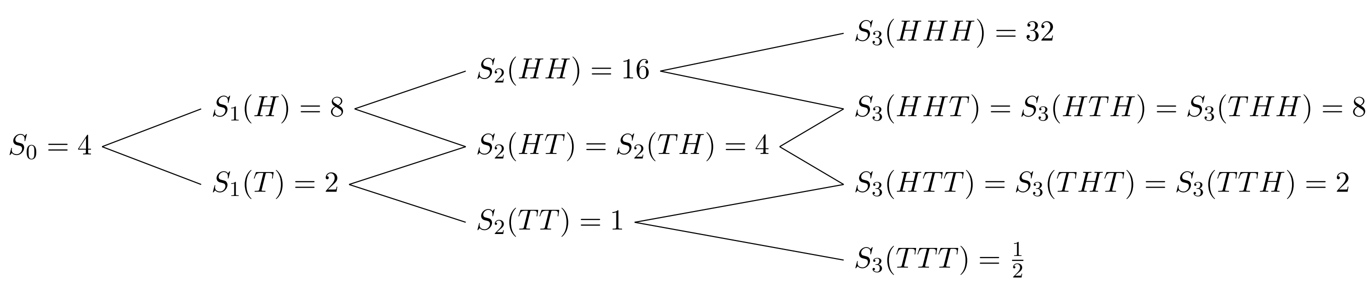
and

$$\begin{cases} \tilde{p}_{2T} + \tilde{q}_{2T} = 1 \\ \tilde{\mathbb{E}}\left[\frac{S_2}{(1+r)^2} \mid S_1(T)\right] = \frac{S_1(T)}{(1+r)^2}(u\tilde{p}_{2T} + d\tilde{q}_{2T}) \end{cases} \Rightarrow \begin{cases} \tilde{p}_{2T} = \frac{1+r-d}{u-d} \\ \tilde{q}_{2T} = \frac{u-(1+r)}{u-d} \end{cases}$$

Theorem 2.3.2. *(Risk-Neutral Probabilities).* For each of the branch, $d < 1 + r < u$, the risk-neutral probabilities for the next period are $\tilde{p} = \frac{1+r-d}{u-d}$ and $\tilde{q} = \frac{u-(1+r)}{u-d}$.

Remark 2.3.3. u, d, r remain unchanged with respect to t. This risk-neutral measure is unique.

Example 2.3.4. *(3-Period European Put Option).* Consider an European put option whose option payoff is calculated as $V_t = (K - S_t)^+$ with $u = 2 = \frac{1}{2}$, $r = \frac{1}{4}$, and $K = 6$.



Then we know $\tilde{p} = \frac{1}{2} = \tilde{q}$. The option payoffs at $t = 3$ are

$$V_3(HHH) = (6 - 32)^+ = 0$$
$$V_3(HHT) = V_3(HTH) = V_3(THH) = (6 - 8)^+ = 0$$
$$V_3(HTT) = V_3(THT) = V_3(TTH) = (6 - 2)^+ = 4$$
$$V_3(TTT) = (6 - \frac{1}{2})^+ = 5.5$$

We discount the option value back to $t = 2$: $\tilde{\mathbb{E}}\left[\frac{V_3}{1+r} \mid \mathcal{F}_2\right]$

$$
\begin{aligned}
V_2(HH) &= \frac{1}{1+r}\left(\tilde{p}V_3(HHH) + \tilde{q}V_3(HHT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 0\right) = 0 \\
V_2(HT) &= \frac{1}{1+r}\left(\tilde{p}V_3(HTH) + \tilde{q}V_3(HTT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 4\right) = 1.6 \\
V_2(TH) &= \frac{1}{1+r}\left(\tilde{p}V_3(THH) + \tilde{q}V_3(THT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 4\right) = 1.6 \\
V_2(TT) &= \frac{1}{1+r}\left(\tilde{p}V_3(TTH) + \tilde{q}V_3(TTT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 4 + \frac{1}{2}\cdot 5.5\right) = 3.8
\end{aligned}
$$

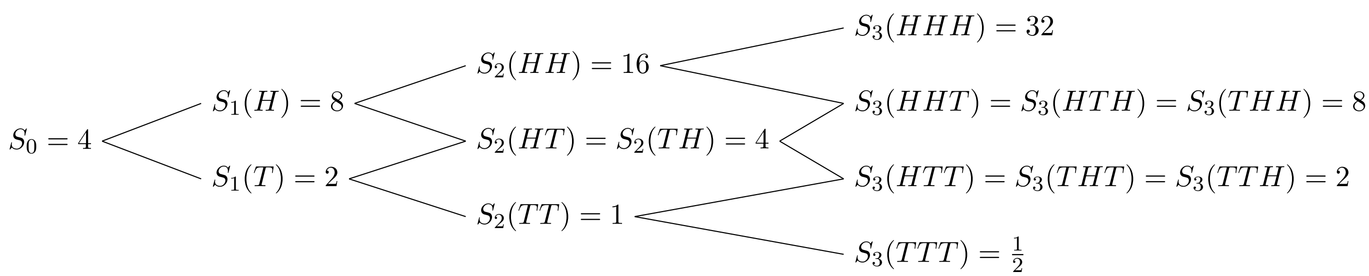
We discount the option value back to $t = 1$: $\tilde{\mathbb{E}}\left[\frac{V_2}{1+r} \mid \mathcal{F}_1\right]$

$$
\begin{aligned}
V_1(H) &= \frac{1}{1+r}\left(\tilde{p}V_2(HH) + \tilde{q}V_2(HT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 1.6\right) = 0.64 \\
V_1(T) &= \frac{1}{1+r}\left(\tilde{p}V_2(TH) + \tilde{q}V_2(TT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 1.6 + \frac{1}{2}\cdot 3.8\right) = 2.16
\end{aligned}
$$

We discount the option value back to $t = 0$: $\tilde{\mathbb{E}}\left[\frac{V_1}{1+r} \mid \mathcal{F}_0\right]$

$$
V_0 = \frac{1}{1+r}\left(\tilde{p}V_1(H) + \tilde{q}V_1(T)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0.64 + \frac{1}{2}\cdot 2.16\right) = 1.12
$$

Example 2.3.5. *(3-Period Lookback Option).* Consider a lookback option whose option payoff is calculated as $V_t = \max_{0<t\leq T} S_t - S_T$ with $u = 2 = \frac{1}{2}$ and $r = \frac{1}{4}$.



Then we know $\tilde{p} = \frac{1}{2} = \tilde{q}$. The option payoffs at $t = 3$ are

$$
\begin{aligned}
V_3(HHH) &= 32 - 32 = 0 \\
V_3(HHT) &= 16 - 8 = 8 \\
V_3(HTH) &= 8 - 8 = 0 \\
V_3(HTT) &= 8 - 2 = 6 \\
V_3(THH) &= 8 - 8 = 0 \\
V_3(THT) &= 4 - 2 = 2 \\
V_3(TTH) &= 4 - 2 = 2 \\
V_3(TTT) &= 4 - 0.5 = 3.5
\end{aligned}
$$

We discount the option value back to $t = 2$: $\tilde{\mathbb{E}}\left[\frac{V_3}{1+r} \mid \mathcal{F}_2\right]$

$$
\begin{aligned}
V_2(HH) &= \frac{1}{1+r}\left(\tilde{p}V_3(HHH) + \tilde{q}V_3(HHT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 8\right) = 3.2 \\
V_2(HT) &= \frac{1}{1+r}\left(\tilde{p}V_3(HTH) + \tilde{q}V_3(HTT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 6\right) = 2.4 \\
V_2(TH) &= \frac{1}{1+r}\left(\tilde{p}V_3(THH) + \tilde{q}V_3(THT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0 + \frac{1}{2}\cdot 2\right) = 0.8 \\
V_2(TT) &= \frac{1}{1+r}\left(\tilde{p}V_3(TTH) + \tilde{q}V_3(TTT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 2 + \frac{1}{2}\cdot 3.5\right) = 2.2
\end{aligned}
$$

We discount the option value back to $t = 1$: $\tilde{\mathbb{E}}\left[\frac{V_2}{1+r} \mid \mathcal{F}_1\right]$

$$
\begin{aligned}
V_1(H) &= \frac{1}{1+r}\left(\tilde{p}V_2(HH) + \tilde{q}V_2(HT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 3.2 + \frac{1}{2}\cdot 2.4\right) = 2.24 \\
V_1(T) &= \frac{1}{1+r}\left(\tilde{p}V_2(TH) + \tilde{q}V_2(TT)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 0.8 + \frac{1}{2}\cdot 2.2\right) = 1.2
\end{aligned}
$$

We discount the option value back to $t = 0$: $\tilde{\mathbb{E}}\left[\frac{V_1}{1+r} \mid \mathcal{F}_0\right]$

$$
V_0 = \frac{1}{1+r}\left(\tilde{p}V_1(H) + \tilde{q}V_1(T)\right) = \frac{1}{1+\frac{1}{4}}\left(\frac{1}{2}\cdot 2.24 + \frac{1}{2}\cdot 1.2\right) = 1.376
$$

Then we replicate the portfolio.

From $t = 0$ to $t = 1$, we compute

$$
\begin{cases}
V_1(H) = (1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(H) \\
V_1(T) = (1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(T)
\end{cases}
\Rightarrow \Delta_0 = \frac{V_1(H) - V_1(T)}{S_1(H) - S_1(T)}
$$

To check $V_0 = X_0$, we calculate

$$
\begin{aligned}
V_0 &= \frac{1}{1+r}\left[\tilde{p}V_1(H) + \tilde{q}V_1(T)\right] \\
&= \frac{1}{1+r}\left[\tilde{p}\left((1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(H)\right) + \tilde{q}\left((1+r)(X_0 - \Delta_0 S_0) + \Delta_0 S_1(T)\right)\right] \\
&= \frac{1}{1+r}\left[(1+r)(X_0 - \Delta_0 S_0) + \Delta_0(\tilde{p}S_1(H) + \tilde{q}S_1(T))\right] \\
&= \frac{1}{1+r}\left[(1+r)(X_0 - \Delta_0 S_0) + \Delta_0(1+r)S_0\right] \\
&= X_0 - \Delta_0 S_0 + \Delta_0 S_0 \\
&= X_0
\end{aligned}
$$

From $t = 1$ to $t = 2$, we compute

$$
\begin{cases}
V_2(HH) = (1+r)(X_1(H) - \Delta_1(H)S_1(H)) + \Delta_1(H)S_2(HH) \\
V_2(HT) = (1+r)(X_1(H) - \Delta_1(H)S_1(H)) + \Delta_1(H)S_2(HT)
\end{cases}
\Rightarrow \Delta_0 = \frac{V_2(HH) - V_2(HT)}{S_2(HH) - S_2(HT)}
$$

$$
\begin{cases}
V_2(TH) = (1+r)(X_1(T) - \Delta_1(T)S_1(T)) + \Delta_1(T)S_2(TH) \\
V_2(TT) = (1+r)(X_1(T) - \Delta_1(T)S_1(T)) + \Delta_1(T)S_2(TT)
\end{cases}
\Rightarrow \Delta_0 = \frac{V_2(TH) - V_2(TT)}{S_2(TH) - S_2(TT)}
$$

3 Lecture 3

3.1 Brownian Motion

Definition 3.1.1. On space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, we say $\{B_t\}_{t\geq 0}$ is a *Brownian Motion (B.M.)* provided

i. $B_0 = 0$

ii. $B_t(\omega)$ has continuous path, $\forall \omega \in \Omega$

iii. $B_t - B_s \sim N(0, t-s), \forall t \geq s \geq 0$

iv. $B_t - B_s$ is independent of $B_u - B_v, \forall 0 \leq s \leq t \leq v \leq u$

Remark 3.1.2. For simulation, we have $dB_t = \Delta B_t \overset{\text{i.i.d.}}{\sim} N(0, \Delta t)$, where Δt is the length of time discretization.

Remark 3.1.3. In *Steve Shreve, Stochastic Calculus for Finance Volume II*, the notation is the Wiener process $\{W_t\}_{t\geq 0}$. In this course, Brownian Motion is the same as Wiener process.

Remark 3.1.4. The mean of a B.M. process is $\mathbb{E}\left[B_t\right] = 0$. The variance of a B.M. process is $\text{Var}[B_t] = \mathbb{E}\left[B_t^2\right] = t$, i.e., growing linearly with respect to time.

Remark 3.1.5. B_t has continuous sample path with $+\infty$ variation.

Claim: $\sum_{t=0}^{N-1} \left|B_{t_{i+1}}(\omega) - B_{t_i}(\omega)\right| \xrightarrow{\max|t_{i+1}-t_i|\to 0} +\infty$

Aside: For $Z \sim N(0,1)$, the moments are $\mathbb{E}[Z] = 0$, $\mathbb{E}[Z^2] = 1$, $\mathbb{E}[Z^3] = 0$, $\mathbb{E}[Z^4] = 3$. The moment generating function for a standard normal variable is $\mathbb{E}[e^{xZ}] = e^{\frac{1}{2}x^2}$.

Proof. For discretization ($t_0 = 0 < t_1 < \cdots < t_N = T$), ($B_{t_0} = B_0 = 0, B_{t_1}, B_{t_2}, \ldots, B_{t_\omega} = B_T$) follows a joint normal distribution. Then the moments are

$$\mathbb{E}\left[B_{t_{i+1}} - B_{t_i}\right] = 0, \quad \mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)^2\right] = t_{i+1} - t_i,$$
$$\mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)^3\right] = 0, \quad \mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)^4\right] = 3\left(t_{i+1} - t_i\right)^2$$

Let $\text{QV}(t_0, \ldots, t_N) := \sum_{i=0}^{N-1}\left(B_{t_{i+1}} - B_{t_i}\right)^2$ be a random variable and its variance is

$$\begin{aligned}
\mathbb{E}\left[(\text{QV}(t_0, \ldots, t_N) - T)^2\right] &= \mathbb{E}\left[\sum_{i=0}^{N-1}\left((B_{t_{i+1}} - B_{t_i})^2 - (t_{i+1} - t_i)\right)^2\right] \quad \text{since } T = \sum_{i=0}^{N-1}(t_{i+1} - t_i) \\
&= \sum_{i=0}^{N-1}\mathbb{E}\left[\left(B_{t_{i+1}} - B_{t_i}\right)^4 - 2(t_{i+1} - t_i)(B_{t_{i+1}} - B_{t_i})^2 + (t_{i+1} - t_i)^2\right] \\
&= \sum_{i=0}^{N-1}\left[3(t_{i+1} - t_i)^2 - 2(t_{i+1} - t_i)^2 + (t_{i+1} - t_i)^2\right] \\
&= 2\sum_{i=0}^{N-1}(t_{i+1} - t_i)^2 \\
&\leq 2 \cdot \max|t_{i+1} - t_i| \cdot \sum_{i=0}^{N-1}(t_{i+1} - t_i) \\
&= 2T \max|t_{i+1} - t_i|
\end{aligned}$$

Therefore, as $\max|t_{i+1} - t_i| \to 0$, $\text{QV}(t_0, \ldots, t_N) \to T$.

We can easily get $0 < \text{QV}(t_0, \ldots, t_N) = \sum_{i=0}^{N-1} \left(B_{t_{i+1}} - B_{t_i}\right)^2 \leq \max|B_{t_{i+1}} - B_{t_i}| \sum_{i=0}^{N-1} |B_{t_{i+1}} - B_{t_i}|$.

We also know that, as $\max|t_{i+1} - t_i| \to 0$, $\text{QV}(t_0, \ldots, t_N) \to T$ but $\max|B_{t_{i+1}} - B_{t_i}| \to 0$ due to continuity of B.M., then $\sum_{i=0}^{N-1} |B_{t_{i+1}} - B_{t_i}|$ must goes to $+\infty$.

Remark 3.1.6. B_t has a continuous path that is no where differentiable.

Remark 3.1.7. $(dB_t)^2 = dt$ due to the Quadrative Variation (QV) that $\langle B, B\rangle_t = [B, B]_t = \int_0^t (dB_u)^2 = t$.

Note that QV for B.M. is deterministic.

Remark 3.1.8. $dB_t \sim O(\sqrt{dt})$

3.2 Itô's Formula for Brownian Motion

Theorem 3.2.1. *(Itô's Formula for Brownian Motion).* For a function $f :\to \mathbb{R}$, f is twice differentiable, then we have

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

Proof. Using Taylor's expansion of $f(B_t + dB_t)$, we have

$$\begin{aligned}
df(B_t) &= f(B_t + dB_t) - f(B_t) \\
&= f(B_t) + dB_t f'(B_t) + \frac{1}{2}(dB_t)^2 f''(B_t) + O((dB_t)^2) - f(B_t) \\
&= f(B_t) + dB_t f'(B_t) + \frac{1}{2}dt f''(B_t) + O((dB_t)^2) - f(B_t) \\
&= f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt
\end{aligned}$$

3.3 Stochastic Differential Equations (SDEs)

1. The stochastic differential equation (SDE) for a *diffusion* process X_t is $dX_t = \mu_t dt + \sigma_t dB_t$.

   The solution is $X_t = X_0 + \int_0^t \mu_t du + \int_0^t \sigma_u dB_u$.

2. The SDE for a *Bachelier* process S_t is $dS_t = \mu_t dt + \sigma_t dB_t$.

   The solution is $S_t = S_0 + \mu t + \sigma dB_t$.

3. The SDE for a *Geometric Brownian Motion (GBM)* process S_t is $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$, where $\tilde{B}_t$ represents a B.M. process under risk-neutral measure.

   The solution is $S_t = S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}$.

3.4 Itô's Formula for SDE

Theorem 3.4.1. *(Itô's Formula for SDE).* Suppose X_t is a stochastic process defined by the SDE:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dB_t$$

Let $f(t, X_t)$ be a function that is twice differentiable. The differential $df(t, X_t)$ is given by

$$\begin{aligned} df(t, X_t) &= f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2 && (1) \\ &= \left[f_t(t, X_t) + f_x(t, X_t)\mu(t, X_t) + \frac{1}{2}f_{xx}(t, X_t)\sigma^2(t, X_t)\right]dt + f_x(t, X_t)\sigma(t, X_t)dB_t && (2) \end{aligned}$$

Proof. Note that since $(dt)^2 = 0$, $(dt)(dB_t) = 0$, and $(dB_t)^2 = dt$, then we have

$$\begin{aligned} df(t, X_t) =& f_t(t, X_t)dt + f_x(t, X_t)dX_t + \frac{1}{2}f_{xx}(t, X_t)(dX_t)^2 \\ =& f_t(t, X_t)dt + f_x(t, X_t)\left[\mu(t, X_t)dt + \sigma(t, X_t)dB_t\right] + \frac{1}{2}f_{xx}(t, X_t)\left[\mu(t, X_t)dt + \sigma(t, X_t)dB_t\right]^2 \\ =& f_t(t, X_t)dt + f_x(t, X_t)\mu(t, X_t)dt + f_x(t, X_t)\sigma(t, X_t)dB_t + \\ & \frac{1}{2}f_{xx}(t, X_t)\left[(\mu(t, X_t)dt)^2 + (\mu(t, X_t)dt\sigma(t, X_t)dB_t) + (\sigma(t, X_t)dB_t)^2\right] \\ =& f_t(t, X_t)dt + f_x(t, X_t)\mu(t, X_t)dt + f_x(t, X_t)\sigma(t, X_t)dB_t + \frac{1}{2}f_{xx}(t, X_t)(\sigma^2(t, X_t)dt \\ =& \left[f_t(t, X_t) + f_x(t, X_t)\mu(t, X_t) + \frac{1}{2}f_{xx}(t, X_t)\sigma^2(t, X_t)\right]dt + f_x(t, X_t)\sigma(t, X_t)dB_t \end{aligned}$$

Example 3.4.2. Consider the SDE for a GBM process, $dS_t = rS_tdt + \sigma S_t d\tilde{B}_t$, $\mu(t, S_t) = rS_t$, $\sigma(t, S_t) = \sigma S_t$, $f(x) = \log x$.

We calculate the first and second derivatives for $f(x)$: $f'(x) = \frac{1}{x}$, $f''(x) = -\frac{1}{x^2}$.

By Itô's Formula for SDE (1), we have

$$\begin{aligned} df(t, S_t) &= f_t(t, S_t)dt + f_s(t, S_t)dS_t + \frac{1}{2}f_{ss}(t, dS_t)(dS_t)^2 \\ \Rightarrow d\log S_t &= 0 + \frac{1}{S_t}\left(rS_tdt + \sigma S_t d\tilde{B}_t\right) + \frac{1}{2}\left(-\frac{1}{S_t^2}\right)\left(r^2S_t^2(dt)^2 + 2\cdot rS_tdt\cdot\sigma S_t d\tilde{B}_t + \sigma^2 S_t^2\left(d\tilde{B}_t\right)^2\right) \\ d\log S_t &= rdt + \sigma d\tilde{B}_t - \frac{1}{2}\sigma^2 S_t dt \\ d\log S_t &= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma d\tilde{B}_t \\ \Rightarrow \log S_t &= \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{B}_t \\ \Rightarrow S_t &= S_0\exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{B}_t\right\} \end{aligned}$$

By Itô's Formula for SDE (2), we have

$$\begin{aligned}
df(t,S_t) &= \left[f_t(t,S_t) + f_s(t,S_t)\mu(t,S_t) + \frac{1}{2}f_{ss}(t,S_t)\sigma^2(t,S_t)\right]dt + f_s(t,S_t)\sigma(t,S_t)dB_t \\
\Rightarrow d\log S_t &= \left[\frac{1}{S_t}rS_t + \frac{1}{2}\left(-\frac{1}{S_t}\right)\sigma^2 S_t^2\right]dt + \frac{1}{S_t^2}\sigma S_t d\tilde{B}_t \\
d\log S_t &= \left(r - \frac{1}{2}\sigma^2\right)dt + \sigma d\tilde{B}_t \\
\Rightarrow \log S_t &= \log S_0 + \left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{B}_t \\
\Rightarrow S_t &= S_0 \exp\left\{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma\tilde{B}_t\right\}
\end{aligned}$$

Example 3.4.3. Derive the differential for $e^{-qt}S_t$ under physical measure, where $dS_t = \mu S_t dt + \sigma S_t dB_t$.

First, we know that $\mu(t,S_t) = \mu S_t$ and $\sigma(t,S_t) = \sigma S_t$.

We calculate the derivatives for $f(t,S_t) = e^{-qt}S_t$:

$$\begin{aligned}
f_t(t,S_t) &= -qe^{-qt}S_t \\
f_s(t,S_t) &= e^{-qt} \\
f_{ss}(t,S_t) &= 0
\end{aligned}$$

Substituting into Itô's Formula (2), we have

$$\begin{aligned}
df(t,S_t) &= \left[f_t(t,S_t) + f_s(t,S_t)\mu(t,S_t) + \frac{1}{2}f_{ss}(t,S_t)\sigma^2(t,S_t)\right]dt + f_s(t,S_t)\sigma(t,S_t)dB_t \\
&= \left(-qe^{-qt}S_t + e^{-qt}\mu S_t + 0\cdot\sigma^2 S_t^2\right)dt + e^{-qt}\sigma S_t dB_t \\
&= e^{-qt}S_t\left[(\mu - q)\,dt + \sigma\right]dB_t
\end{aligned}$$

Example 3.4.4. Derive the differential for $S_t = S_0\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}$.

First we calculate the derivatives for $S_t = f(t,B_t) = S_0\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\}$:

$$\begin{aligned}
f_t(t,B_t) &= \left(\mu - \frac{1}{2}\sigma^2\right)S_0\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} = \left(\mu - \frac{1}{2}\sigma^2\right)f(t,B_t) \\
f_b(t,B_t) &= \sigma S_0\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} = \sigma f(t,B_t) \\
f_{bb}(t,B_t) &= \sigma^2 S_0\exp\left\{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma B_t\right\} = \sigma^2 f(t,B_t)
\end{aligned}$$

Substituting into Itô's Formula (1), we have

$$\begin{aligned}
dS_t = df(t,B_t) &= f_t(t,B_t)dt + f_b(t,B_t)dB_t + \frac{1}{2}f_{bb}(t,dB_t)(dB_t)^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)f(t,B_t)dt + \sigma f(t,B_t)dB_t + \frac{1}{2}\sigma^2 f(t,B_t)(dB_t)^2 \\
&= \left(\mu - \frac{1}{2}\sigma^2\right)S_t dt + \sigma S_t dB_t + \frac{1}{2}\sigma^2 S_t dt \\
&= \mu S_t dt + \sigma S_t dB_t
\end{aligned}$$

3.5 Martingale in Continuous Time

Definition 3.5.1. An *adaptive* stochastic process $\{X_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, is a *martingale* provided

i. $\mathbb{E}[|X_t|] < +\infty$

ii. $\mathbb{E}[X_t \mid \mathcal{F}_s] = X_s$ a.s. $\forall 0 \leq s \leq t$

$\Leftrightarrow$ A *diffusion* process $dX_t = \mu_t dt + \sigma_t dB_t$ is a *martingale* provided

i. $\mu_t = 0$ a.s. $\forall t \geq 0$

ii. $\mathbb{E}\left[\int_0^T \sigma_t^2 dt\right] < +\infty$

Remark 3.5.2. The *moment generating function (MGF)* for $X \sim N(\mu, \sigma^2)$ is $\mathbb{E}\left[e^{\lambda x}\right] = e^{\lambda\mu + \frac{1}{2}\lambda^2\sigma^2}$.

Exercise 3.5.3. Show that the discounted stock price, $e^{-rt}S_t$, where $S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_t}$, is a martingale. For an interest rate r, $dS_t = rS_t dt + \sigma S_t dB_t$, where $\tilde{B}_t$ is a B.M. under risk-neutral measure.

Proof. Method 1: First we need to show that $\mathbb{E}\left[\left|e^{-rt}S_t\right|\right] < +\infty$:

$$\begin{aligned}
\mathbb{E}\left[\left|e^{-rt}S_t\right|\right] &= \mathbb{E}\left[e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_t}\right] \\
&= \mathbb{E}\left[S_0 e^{-\frac{1}{2}\sigma^2 t+\sigma B_t}\right] \\
&= S_0 e^{-\frac{1}{2}\sigma^2 t}\mathbb{E}\left[e^{\sigma B_t}\right] \\
&= S_0 e^{-\frac{1}{2}\sigma^2 t} e^{\frac{1}{2}\sigma^2 t} \quad \text{by MGF and since } B_t \sim N(0,t) \\
&= S_0 < +\infty
\end{aligned}$$

Then for all $t \geq s \geq 0$, we have

$$\begin{aligned}
\mathbb{E}\left[e^{-rt}S_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_t}\middle|\mathcal{F}_s\right] \\
&= e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t}\mathbb{E}\left[e^{\sigma B_t+\sigma B_s-\sigma B_s}\middle|\mathcal{F}_s\right] \\
&= e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_s}\mathbb{E}\left[e^{\sigma(B_t-B_s)}\middle|\mathcal{F}_s\right] \\
&= e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_s}\mathbb{E}\left[e^{\sigma(B_t-B_s)}\right] \quad \text{since } B_t - B_s \perp \mathcal{F}_s \\
&= e^{-rt}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma B_s} e^{\frac{1}{2}\sigma^2(t-s)} \quad \text{by MGF and since } B_t - B_s \sim N(0, t-s) \\
&= S_0 e^{-\frac{1}{2}\sigma^2 s+\sigma B_s} \\
&= e^{-rs}S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)s+\sigma B_s} \\
&= e^{-rs}S_s
\end{aligned}$$

Therefore, $e^{-rt}S_t$ is a martingalge by definition. $\square$

Method 2: By Itô's Formula, we calculate the differential for $f(t, S_t) = e^{-rt}S_t$:

$$\begin{aligned}
df(t, S_t) &= f_t(t, S_t)dt + f_s(t, S_t)dS_t + \frac{1}{2}f_{ss}(t, dS_t)(dS_t)^2 \\
&= -re^{-rt}S_t dt + e^{-rt}(rS_t dt + \sigma S_t dB_t) + \frac{1}{2}\cdot 0dt \\
&= \sigma e^{-rt}S_t dB_t
\end{aligned}$$

Thus, $\mu_t = 0$ for all $t \geq 0$.

Then we need to show that $\mathbb{E}\left[\int_0^T \sigma_t^2\, dt\right] = \mathbb{E}\left[\int_0^T \sigma^2 e^{-2rt} S_t^2\, dt\right] < +\infty$:

$$\begin{aligned}
\mathbb{E}[S_t^2] &= \mathbb{E}\left[S_0^2 e^{2rt-\sigma^2 t+2\sigma B_t}\right] \\
&= S_0^2 e^{2rt-\sigma^2 t}\,\mathbb{E}\left[e^{2\sigma B_t}\right] \\
&= S_0^2 e^{2rt-\sigma^2 t+\frac{1}{2}\cdot 4\sigma^2 t} \quad \text{by MGF and since } B_t \sim N(0,t) \\
&= S_0^2 e^{2rt+\sigma^2 t} \\
\mathbb{E}\left[\int_0^T \sigma^2 e^{-2rt} S_t^2\, dt\right] &= \int_0^T \sigma^2 e^{-2rt}\,\mathbb{E}[S_t^2]\, dt \\
&= \int_0^T \sigma^2 e^{-2rt} S_0 e^{2rt+\sigma^2 t}\, dt \\
&= \int_0^T \sigma^2 S_0 e^{\sigma^2 t}\, dt \\
&= \sigma^2 S_0 \int_0^T e^{\sigma^2 t}\, dt \\
&= \sigma^2 S_0 \left[\frac{e^{\sigma^2 t}}{\sigma^2}\right]_0^T \\
&= S_0 e^{\sigma^2 T} < +\infty
\end{aligned}$$

Therefore, $e^{-rt} S_t$ is a martingale by definition. □

4 Lecture 4

4.1 Review of Brownian Motion

Definition 4.1.1. On space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, we say $\{B_t\}_{t\geq 0}$ is a *Brownian Motion (B.M.)* provided

i. $B_0 = 0$

ii. $B_t(\omega)$ is continuous, $\forall \omega \in \Omega$

iii. $B_t - B_s \sim N(0, t-s), \forall 0 \leq s \leq t$

iv. $B_t - B_s$ is independent of $\mathcal{F}_s$. In particular, $B_t - B_s$ is independent of $B_u - B_v$, $\forall 0 \leq s \leq t \leq v \leq u$

Remark 3.1.2. $\{B_t\}_{t\geq 0}$ is a Gaussian process with $\mathbb{E}[B_t] = 0$ and $\text{Var}[B_t] = t$.

Remark 3.1.3. $B_t(\omega)$ is continuous but nowhere differentiable for all $\omega \in \Omega$.

Remark 3.1.4. $\{B_t\}_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$.

Proof. To prove that the B.M. process $\{B_t\}_{t\geq 0}$ is a martingale, we need to show the two conditions by the definition of martingale.

First, by Cauchy-Schwarz Inequality, we have $\mathbb{E}[|B_t|] \leq \sqrt{\mathbb{E}[B_t^2]} = \sqrt{\text{Var}(B_t)} = \sqrt{t} < +\infty$

Then we evaluate

$$\begin{aligned}
\mathbb{E}\left[B_t \mid \mathcal{F}_s\right] &= \mathbb{E}\left[B_t - B_s + B_s \mid \mathcal{F}_s\right] \\
&= \mathbb{E}\left[B_t - B_s\right] + B_s \quad \text{since } B_t - B_s \perp \mathcal{F}_s \\
&= B_s \quad \text{since } B_t - B_s \sim N(0,1)
\end{aligned}$$

Therefore, a Brownian Motion $\{B_t\}_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$. $\square$

Example 3.1.5. Prove that $\{B_t^2 - t\}_{t\geq 0}$ is a martingale on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$.

Proof. To prove $\{B_t^2 - t\}_{t\geq 0}$ is a martingale, we need to show the two conditions in definition of martingale.

First, we can show that $\mathbb{E}[|B_t^2 - t|] \leq \mathbb{E}[|B_t^2|] + |t| = 2t < +\infty$. Then we calculate

$$\begin{aligned}
\mathbb{E}[B_t^2 \mid \mathcal{F}_s] &= \mathbb{E}[(B_t - B_s + B_s)^2 \mid \mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s)^2 + 2B_s(B_t - B_s) + B_s^2 \mid \mathcal{F}_s] \\
&= \mathbb{E}[(B_t - B_s)^2] + 2B_s\mathbb{E}[B_t - B_s] + B_s^2 \\
&= t - s + B_s^2 \quad \text{since } B_t - B_s \sim N(0, t-s) \\
\Rightarrow \mathbb{E}[B_t^2 - t \mid \mathcal{F}_s] &= B_s^2 - s
\end{aligned}$$

Therefore, $\{B_t^2 - t\}$ is a martingale. $\square$

Remark 3.1.6. Quadratic Variation: $\langle B\rangle_t = [B]_t = t = \int_0^t (dB_s)^2$, $(dB_s)^2 \sim dt$.

4.2 Risk-Neutral Asset Pricing - Probability Approach

- Pros: Closed-form
- Cons: Restricted (GBM-based)

4.2.1 Black-Scholes Equation for European Call Option

Define call option as $C(T, S_T) = (S_T - K)^+$.

Suppose $\{\tilde{B}_t\}_{t\geq 0}$ is the Brownian Motion under risk-neutral measure $\mathbb{Q}$.

Suppose $dS_t = S_t(rdt + \sigma d\tilde{B}_t)$, where r is the interest rate.

We have proved in the last lecture that the discounted stock price, $\{e^{-rt}S_t\}_{t\geq 0}$, is a martingale under $\mathbb{Q}$. Specially with GBM, we have $S_t = S_0 e^{\left(r-\frac{1}{2}\sigma^2\right)t+\sigma\tilde{B}_t}$.

To realize risk-neutral pricing, we need to prove $\{e^{-rt}C(t, S_t)\}_{t\in[0,T]}$ is a martingale, i.e., to calculate $C(t, S_t) = e^{-r(T-t)}\mathbb{E}\left[(S_T - K)^+ \mid \mathcal{F}_t\right]$.

Recall that for $Z \sim N(0, 1)$, the probability density function (pdf) is $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}z^2}$, and the cumulative distribution function (CDF) is $N(z) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{z} e^{-\frac{1}{2}y^2}dy$.

Let $Y = Z + c$. Then we have $\mathbb{P}[Y \geq y] = \mathbb{P}[Z \geq y - c] = N(c - y)$, and we calculate

$$\begin{aligned}
\mathbb{E}\left[Z\mathbb{1}_{[Z\geq c]}\right] &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \cdot y\mathbb{1}_{[y\geq c]}\, dy \\
&= \int_{c}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \cdot y\, dy \\
&= \frac{1}{\sqrt{2\pi}}\left[e^{-\frac{1}{2}y^2}\cdot\right]_{+\infty}^{c} \quad \text{since } \frac{\partial}{\partial y}e^{-\frac{1}{2}y^2} = -ye^{-\frac{1}{2}y^2} \\
&= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}c^2} \quad \text{for any } c \geq 0 \\
\mathbb{E}\left[e^Z\mathbb{1}_{[Z\geq c]}\right] &= \int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}y^2} \cdot e^y\mathbb{1}_{[y\geq c]}\, dy \\
&= \int_{c}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-2y)}\, dy \\
&= \int_{c}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y^2-2y+1)+\frac{1}{2}}\, dy \\
&= e^{\frac{1}{2}}\int_{c}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}(y-1)^2}\, dy \\
&= \sqrt{e}\int_{c-1}^{+\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}x^2}\, dx \quad \text{by changing variable of } x = y - 1 \\
&= \sqrt{e}N(-(c-1)) \\
&= \sqrt{e}N(-c+1)
\end{aligned}$$

Then we calculate the call option price:

$$\begin{aligned}
C(t, S_t) &= e^{-r(T-t)}\mathbb{E}[(S_T - K)^+ \mid \mathcal{F}_t] \\
&= e^{-r(T-t)}\mathbb{E}[(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(B_T-B_t)} - K)^+ \mid \mathcal{F}_t] \quad \text{by substituting } S_T = S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma(B_T-B_t)} \\
&= e^{-r(T-t)}\mathbb{E}[(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K)^+ \mid \mathcal{F}_t]
\end{aligned}$$

since $B_T - B_t \sim N(0, T-t)$, then $B_T - B_t = \sqrt{T-t}z$, where $Z \sim N(0, 1)$. Also, since $B_T - B_t \perp \mathcal{F}_t$, then $z \perp \mathcal{F}_t$.

Then we calculate the stock price part and strike price part inside the expectation:

$$
\begin{aligned}
\left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right)^+ &= \left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right) \mathbb{1}_{\left[S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} \geq K\right]} \\
&= \left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right) \mathbb{1}_{\left[\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z \geq \log\left(\frac{K}{S_t}\right)\right]} \\
&= \left(S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} - K\right) \mathbb{1}_{\left[z \geq \frac{\log\frac{K}{S_t} - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]}
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[K \mathbb{1}_{\left[z \geq \frac{\log\frac{K}{S_t} - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right]} \middle| \mathcal{F}_t\right] &= KN\left(-\frac{\log\frac{K}{S_t} - \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= KN\left(\frac{\log\frac{S_t}{K} + \left(r-\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}\right) \\
&= KN(d_-(T-t, S_t))
\end{aligned}
$$

where $d_-(z,x) = \frac{\log\frac{x}{K} + \left(r-\frac{1}{2}\sigma^2\right)z}{\sigma\sqrt{z}}$.

Then we calculate the other part:

$$
\begin{aligned}
&\mathbb{E}\left[S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)+\sigma\sqrt{T-t}z} \mathbb{1}_{[z\geq -d_-(T-t,S_t)]} \middle| \mathcal{F}_t\right] \\
&= S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)} \int_{-d_-(T-t,S_t)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}y^2} e^{\sigma\sqrt{T-t}y}\, dy \\
&= S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)} \int_{-d_-(T-t,S_t)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2 - 2\sigma\sqrt{T-t}y + \sigma^2(T-t)\right) + \frac{1}{2}\sigma^2(T-t)}\, dy \\
&= S_t e^{\left(r-\frac{1}{2}\sigma^2\right)(T-t)} e^{\frac{1}{2}\sigma^2(T-t)} \int_{-d_-(T-t,S_t)}^{+\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}\left(y^2 - 2\sigma\sqrt{T-t}y + \sigma^2(T-t)\right)}\, dy \\
&= S_t e^{r(T-t)} N(d_+(T-t, S_t))
\end{aligned}
$$

where $d_+(z,x) = d_-(z,x) + \sigma\sqrt{z} = \frac{\log\frac{x}{K} + \left(r+\frac{1}{2}\sigma^2\right)z}{\sigma\sqrt{z}}$.

Substiting to the original call option price formula, we get the Black-Scholes-Merton formula for European *call* option price as $C(t, S_t) = S_t N(d_+(T-t, S_t)) - Ke^{-r(T-t)}N(d_-(T-t, S_t))$. Aside, the Black-Scholes-Merton formula for European *put* option price is $P(t, S_t) = Ke^{-r(T-t)}N(-d_-(T-t, S_t)) - S_t N(-d_+(T-t, S_t))$.

4.2.2 Put-Call Parity

We use the put-call parity to derive put option price:

$$
\begin{aligned}
&(S_T - K)^+ - (K - S_T)^+ = S_T - K \\
&e^{-rt}\left(C(t, S_t) - P(t, S_t)\right) = \tilde{\mathbb{E}}\left[e^{-rT}\left((S_T-K)^+ - (K-S_T)^+\right) \middle| \mathcal{F}_t\right] \\
&e^{-rt}\left(C(t, S_t) - P(t, S_t)\right) = e^{-rt}S_t - e^{-rT}K
\end{aligned}
$$

Therefore, the put option price is $P(t, S_t) = C(t, S_t) - S_t + e^{-r(T-t)}K$.

4.2.3 The Greeks

For the call option formula $C(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x))$, the Greeks are

1. Delta (Δ): $C_x(t,x) = N(d_+(T-t,x)) \geq 0$
2. Theta (Θ): $C_t(t,x) = -\frac{\sigma x}{2\sqrt{T-t}}N'(d_+(T-t,x)) - rKe^{-r(T-t)}N(d_-(T-t,x))$
3. Gamma (Γ): $C_{xx}(t,x) = \frac{1}{\sigma\sqrt{T-t}x}N'(d_+(T-t,x)) \geq 0$

4.3 Self-Financing and Arbitrage

Definitions:

1. Self-Financing Strategy: $V_t^\varphi = \varphi_t S_t + V_t^\varphi - \varphi_t S_t$
   (1) Continuous: $dV_t^\varphi = \varphi_t dS_t + r(V_t^\varphi - \varphi_t S_t)dt$
   (2) Discrete: $\Delta V_t\varphi = \varphi(S_{t+1} - S_t) + r(V_t^\varphi - \varphi_t S_t)$
2. Arbitrage Strategy:
   (1) $\{\varphi_t\}_{t\geq 0}$ is self-financing and $V_0^\varphi = 0$
   (2) $\exists t \in [0, +\infty)$ s.t. $\mathbb{P}[V_t^\varphi \geq 0] = 1$ and $\mathbb{P}[V_t^\varphi > 0] > 0$

Example 4.3.1. Suppose a risk-neutral measure $\mathbb{Q} \sim \mathbb{P}$, $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$.

Show that a discounted self-financing process is a martingale.

Let $\{\varphi_t\}_{t\geq 0}$ be a self-financing process such that

$$\begin{aligned}
dV_t^\varphi &= \varphi_t dS_t + r(V_t^\varphi - \varphi_t S_t)dt \quad \text{from the continuous model of self-financing strategy} \\
&= \varphi_t(rS_t dt + \sigma S_t d\tilde{B}_t) + (rV_t^\varphi - r\varphi_t S_t)dt \quad \text{by substituting } dS_t \\
&= rV_t^\varphi dt + \sigma S_t \varphi_t d\tilde{B}_t \\
\Rightarrow de^{-rt}V_t^\varphi &= -re^{-rt}V_t^\varphi dt + e^{-rt}dV_t^\varphi + \frac{1}{2}\cdot 0 \cdot dt \\
&= -re^{-rt}V_t^\varphi dt + e^{-rt}(rV_t^\varphi dt + \sigma S_t\varphi_t d\tilde{B}_t) \quad \text{by substituting } dV_t^\varphi \\
&= e^{-rt}\sigma S_t \varphi_t d\tilde{B}_t
\end{aligned}$$

Therefore, the discounted wealth $\{e^{-rt}V_t^\varphi\}_{t\geq 0}$ is a martingale under the risk-neutral measure $\mathbb{Q}$.

Example 4.3.2. Replicate the portfolio for an European call, where $C(t, S_t) = V_t^\varphi$, in particular, $V_T\varphi = (S_T - K)^+$.

Recall the Black-Scholes equation for an European call is

$$C(t, S_t) = S_t N(d_+(T-t, S_t)) - Ke^{-r(T-t)}N(d_-(T-t, S_t))$$

Then we calculate the differentials for the call option price and the discounted call option price

$$\begin{aligned}
dC(t,S_t) &= C_t(t,S_t)dt + C_s(t,S_t)dS_t + \frac{1}{2}C_{ss}(t,S_t)(dS_t)^2 \\
de^{-rt}C(t,S_t) &= (-re^{-rt}C(t,S_t) + e^{-rt}C_t(t,S_t))dt + e^{-rt}C_s(t,S_t)(rS_t dt + \sigma S_t d\tilde{B}_t) + \frac{1}{2}e^{-rt}C_{ss}(t,S_t)(\sigma^2 S_t^2 dt) \\
&= e^{-rt}\left(-rC(t,S_t) + C_t(t,S_t) + rS_t C_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t)\right)dt + e^{-rt}\sigma S_t C_s(t,S_t)d\tilde{B}_t \\
&= e^{-rt}\sigma S_t C_s(t,S_t)d\tilde{B}_t \quad \text{according to formula of Greeks}
\end{aligned}$$

Therefore, from Example 4.3.1, we know $\varphi_t = C_s(t, S_t)$, replicating the portfolio and also indicating the delta hedging.

4.4 Risk-Neutral Asset Pricing - PDE Approach

PDE Approach:

- $\{e^{-rt}C(t, S_t)\}_{t\geq 0}$ is a martingale under risk-measure $\mathbb{Q}$.
- PDE Approach: Using Black-Scholes PDE

Pros and Cons:

- Pros: General
- Cons: Limited closed-form solution

Theorem 4.4.1. *(4-Step Procedure to Find a Maringale (General)).* Suppose we have the SDE

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)d\tilde{B}_t$$

and $Y_T = h(X_T)$. We want to find $Y_t := \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$. Then we follow the following steps:

(1) Assume that there exists $g \in C^2$, $g(t, x)$ such that $g(t, X_t) = Y_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$.
Then $\{g(t, X_t)\}_{t\geq 0}$ is a martingale.

(2) To find $g(t, X_t)$, compute its differential via Itô's Formula:

$$\begin{aligned} dg(t, X_t) &= g_t(t, X_t)dt + g_x(t, X_t)dX_t + \frac{1}{2}g_{xx}(t, X_t)(dX_t)^2 \\ &= (g_t(t, X_t) + g_x(t, X_t)\mu(t, X_t) + \frac{1}{2}g_{xx}(t, X_t)\sigma^2(t, X_t))dt + g_x(t, X_t)\sigma(t, X_t)d\tilde{B}_t \end{aligned}$$

(3) Set dt term equal to 0:

$$g_t(t, X_t) + g_x(t, X_t)\mu(t, X_t) + \frac{1}{2}g_{xx}(t, X_t)\sigma^2(t, X_t) = 0$$

(4) Replace (t, X_t) to (t, x) and write down the terminal formula

$$\begin{cases} g_t(t, x) + g_x(t, x)\mu(t, x) + \frac{1}{2}g_{xx}\sigma^2(t, x) = 0 \\ g(T, x) = h(x) \end{cases}$$

Example 4.4.2. Apply the 4-step procedure to find a martingale to the discounted call option price.

Suppose we have the SDE

$$dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$$

and $e^{-rT}C(T, S_T) = e^{-rT}(S_T - K)^+$.

We want to find $e^{-rt}C(t, S_t) = \mathbb{E}[e^{-rT}(S_T - K)^+ \mid \mathcal{F}_t]$. Then we follow the following steps.

Step 1: Assume that there exists $e^{-rt}C(t, s)$ such that $e^{-rt}C(t, S_t) = \mathbb{E}[e^{-rT}(S_T - K)^+ \mid \mathcal{F}_t]$.

Then $\{e^{-rt}C(t, S_t)\}_{t\geq 0}$ is a martingale.

Step 2: To find $e^{-rt}C(t,S_t)$, we compute its differential via Itô's Formula:

$$\begin{aligned} de^{-rt}C(t,S_t) &= (-re^{-rt}C(t,S_t) + e^{-rt}C_t(t,S_t))dt + e^{-rt}C_s(t,S_t)dS_t + \frac{1}{2}e^{-rt}C_{ss}(t,S_t)(dS_t)^2 \\ &= e^{-rt}\left(-rC(t,S_t) + C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t)\right)dt + e^{-rt}\sigma S_t C_s(t,S_t)d\tilde{B}_t \end{aligned}$$

Step 3: Setting dt term equal to 0, we have

$$\begin{aligned} e^{-rt}\left(-rC(t,S_t) + C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t)\right) &= 0 \\ -rC(t,S_t) + C_t(t,S_t) + rS_tC_s(t,S_t) + \frac{1}{2}\sigma^2 S_t^2 C_{ss}(t,S_t) &= 0 \end{aligned}$$

Step 4: Replacing (t,S_t) to (t,x), the terminal value is

$$\begin{cases} -rC(t,x) + C_t(t,x) + rxC_x(t,x) + \frac{1}{2}\sigma^2 x^2 C_{xx}(t,x) = 0 \\ C(T,x) = (x-K)^+ \end{cases}$$

by using Black-Scholes PDE.

5 Lecture 5

5.1 Review of Black-Scholes Equation for European Call Option

1. Expression: $C(t, S_t) = S_t N(d_+(T-t, S_t)) - Ke^{-r(T-t)} N(d_-(T-t, S_t))$, where
   (1) $N(\cdot)$ is the CDF of a standard normal distribution.
   (2) $d_+(z, x) = \frac{\log \frac{x}{K} + \left(r + \frac{1}{2}\sigma^2\right) z}{\sigma\sqrt{z}}$
   (3) $d_-(z, x) = \frac{\log \frac{x}{K} + \left(r - \frac{1}{2}\sigma^2\right) z}{\sigma\sqrt{z}}$
2. Assumptions:
   (1) No arbitrage $\Leftrightarrow$ There exists risk-neutral measure (FTAP I), and the B.M. process $\{\tilde{B}_t\}_{t\geq 0}$ is under risk-neutral measure.
   (2) Under risk-neutral measure, stock price dynamic follows a GBM, i.e., $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$.
   (3) The interest r is constant.
3. Properties:
   (1) Discounted option price is a martingale under the risk-neutral measure.
   That is, $\tilde{\mathbb{E}}[e^{-rt} C(t, S_t) \mid \mathcal{F}_u] = e^{-ru} C(u, S_u)$, for all $- \leq u \leq t \leq T$ (or $de^{-rt} C(t, S_t) = e^{-rt} \sigma S_t C_s(t, S_t) d\tilde{B}_t$).
   In particular, $\tilde{\mathbb{E}}[e^{-rT}(S_T - K)^+ \mid \mathcal{F}_t] = e^{-rt} C(t, S_t)$.
   (2) If no trading costs, no liquidity impact, market is frictionless.
   The replicating portfolio is $X_t = \Delta_t S_t + (X_t - \Delta_t S_t) = C(t, S_t)$.
   Delta-hedging for continuous model is

$$
\begin{aligned}
de^{-rt} X_t &= de^{-rt} C(t, S_t) \\
e^{-rt} \sigma S_t \Delta_t d\tilde{B}_t &= e^{-rt} \sigma S_t C_s(t, S_t) d\tilde{B}_t \quad \text{from Example 4.3.1 and 4.3.2} \\
\Delta_t &= C_s(t, S_t) = N(d_+(T-t, S_t)) \geq 0 \quad \text{for European call, by option Greeks}
\end{aligned}
$$

   Recall that delta-hedging for discrete model is

$$
\begin{aligned}
\Delta_t &= \frac{C(t+1, H) - C(t+1, T)}{S(t+1, H) - S(t+1, T)} \quad \text{and} \\
C_s(t, S_t) &\approx \frac{C(t + \Delta t, S_{t+\Delta t}(H)) - C(t + \Delta t, S_{t+\Delta t}(T))}{S_{t+\Delta t}(H) - S_{t+\Delta t}(T)}
\end{aligned}
$$

   (3) Put-call parity: $P(t, S_t) = C(t, S_t) + e^{-r(T-t)} K - S_t$
   At time $t = T$, it becomes $P(T, S_T) - C(T, S_T) = K - S_T$
4. Parameters: T, K, r, σ, t, stock price S_t treated as a random variable.
   In Reality: T, K, r, t, $C(0, S_0)$, σ unobservable in the market. The Black-Scholes equation in reality is

$$
C(0, S_0; T, K) = S_0 N(d_+(T, S_0; K)) - Ke^{-rT} N(d_-(T, S_0; K))
$$

5.2 Dupire's Formula

To arrive at the Dupire's formula, we need the first and second order partial derivatives of Black-Scholes equation for European call option with respect to K and the first-order derivative with respect to T.

1. The first-order derivative with respect to K is

$$\begin{aligned}\frac{\partial C(0,S_0;T,K)}{\partial K} =& S_0 N'(d_+(T,S_0;K))\frac{\partial d_+(T,S_0;K)}{\partial K} - e^{-rT}N(d_-(T,S_0;K)) \\ &- Ke^{-rT}N'(d_-(T,S_0;K))\frac{\partial d_-(T,S_0;K)}{\partial K}\end{aligned}$$

First, we calculate the $\frac{\partial d_+(T,S_0;K)}{\partial K}$ and $\frac{\partial d_-(T,S_0;K)}{\partial K}$ parts:

$$\frac{\partial d_+(T,S_0;K)}{\partial K} = \frac{\partial d_-(T,S_0;K)}{\partial K} = \frac{\partial}{\partial K}\left(\frac{\log\frac{S_0}{K} + \left(r \pm \frac{1}{2}\sigma^2\right)T}{\sigma\sqrt{T}}\right) = -\frac{1}{\sigma\sqrt{T}K} \qquad (1)$$

Secondly, we verify that $Ke^{-rT}N'(d_-(T,S_0;K)) = S_0N'(d_+(T,S_0;K))$:

$$\begin{aligned}Ke^{-rT}N'(d_-(T,S_0;K)) &= e^{-rT}K\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}d_-^2(T,S_0;K)} \\ &= e^{-rT}K\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\frac{\left(\log\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right)^2}{\sigma^2 T}} \\ &= e^{-rT}K\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2T}\left[\left(\log\frac{S_0}{K}+\left(r-\frac{1}{2}\sigma^2\right)T\right)^2-\left(\log\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)T\right)^2+\left(\log\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)T\right)^2\right]}\end{aligned}$$

Using the identity $A^2 - B^2 = (A-B)(A+B)$, we get

$$\begin{aligned}\left(\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right) - \left(\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right) &= -\sigma^2 T \\ \left(\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right) + \left(\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right) &= 2\left(\log\frac{S_0}{K} + rT\right) \\ \Rightarrow \left(\log\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right)^2 - \left(\log\frac{S_0}{K} + \left(r + \frac{1}{2}\sigma^2\right)T\right)^2 &= -2\sigma^2 T\left(\log\frac{S_0}{K} + rT\right)\end{aligned}$$

Thus, we continue to calculate that

$$\begin{aligned}Ke^{-rT}N'(d_-(T,S_0;K)) &= e^{-rT}K\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2\sigma^2T}\left[-2\sigma^2T\left(\log\frac{S_0}{K}+rT\right)+\left(\log\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)T\right)^2\right]} \\ &= K\frac{1}{\sqrt{2\pi}}e^{\log\frac{S_0}{K}}e^{-\frac{1}{2}d_+^2(T,S_0;K)} \\ &= K\frac{S_0}{K}N'(d_+(T,S_0;K)) \\ &= S_0N'(d_+(T,S_0;K)) \qquad (2)\end{aligned}$$

Hence, together by equation (1) and (2), the equation is only left with

$$\frac{\partial C(0,S_0;T,K)}{\partial K} = -e^{-rT}N(d_-(T,S_0;K)) \qquad (*)$$

2. The second-order derivative with respect to K is

$$\frac{\partial^2}{\partial K^2}C(0,S_0;T,K) = -e^{-rT}N'(d_-(T,S_0;K))\frac{\partial}{\partial K}d_-(T,S_0;K) = \frac{e^{-rT}}{\sigma\sqrt{T}K}N'(d_-(T,S_0;K)) \qquad (**)$$

3. The first-order derivative with respect to T is

$$\begin{aligned}\frac{\partial C(0,S_0;T,K)}{\partial T} =& S_0 N'(d_+(T,S_0;K))\frac{\partial d_+(T,S_0;K)}{\partial T} + rKe^{-rT}N(d_-(T,S_0;K)) \\ &- Ke^{-rT}N'(d_-(T,S_0;K))\frac{\partial d_-(T,S_0;K)}{\partial T}\end{aligned}$$

From equation (2), we only needs to calculate the $\frac{\partial d_+(T,S_0;K)}{\partial T} - \frac{\partial d_-(T,S_0;K)}{\partial T}$ parts:

$$\begin{aligned}\frac{\partial d_+(T,S_0;K)}{\partial T} - \frac{\partial d_-(T,S_0;K)}{\partial T} &= \frac{\partial}{\partial T}\left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0}{K} + \left(r+\frac{1}{2}\sigma\right)\sqrt{T} - \left(\frac{1}{\sigma\sqrt{T}}\log\frac{S_0}{K} + \left(r-\frac{1}{2}\sigma\right)\sqrt{T}\right)\right) \\ &= \frac{\partial}{\partial T}\sigma\sqrt{T} \\ &= \frac{\sigma}{2\sqrt{T}}\end{aligned}$$

Then we continue to calculate that

$$\frac{\partial C(0,S_0;T,K)}{\partial T} = Ke^{-rT}N'(d_-(T,S_0;K))\frac{\sigma}{2\sqrt{T}} + rKe^{-rT}N(d_-(T,S_0;K)) \qquad (***)$$

Combining steps 1, 2, and 3, we can observe that

$$\begin{aligned}\frac{\partial C(0,S_0;T,K)}{\partial T} &= Ke^{-rT}N'(d_-(T,S_0;K))\frac{\sigma}{2\sqrt{T}} + rKe^{-rT}N(d_-(T,S_0;K)) \\ \frac{\partial}{\partial T}C(0,S_0;T,K) &= \frac{1}{2}K^2\sigma^2\frac{\partial^2}{\partial K^2}C(0,S_0;T,K) - rK\frac{\partial}{\partial K}C(0,S_0;T,K) \\ \sigma^2_{\text{BS}} &= \frac{\frac{\partial}{\partial T}C(0,S_0;T,K) + rK\frac{\partial}{\partial K}C(0,S_0;T,K)}{\frac{1}{2}K^2\frac{\partial^2}{\partial K^2}C(0,S_0;T,K)} \quad \text{by substituting equations } (*) \text{ and } (**) \\ &= \frac{Ke^{-rT}N'(d_-(T,S_0;K))\frac{\sigma}{2\sqrt{T}}}{\frac{1}{2}K^2\frac{e^{-rT}}{\sigma\sqrt{T}K}N'(d_-(T,S_0;K))} \quad \text{verifying} \\ &= \sigma^2\end{aligned}$$

Thus, the implied volatility σ^2_{imp} is to find the solution of σ such that $C(0,S_0;T,K) = \text{observation}$.

Note that the assumptions for Dupire's formula are: 1) under Black-Scholes, 2) r constant, and 3) GBM for S.

5.3 Different Volatility Models

1. Physical Measure: $dS_t = \mu_t dt + \sigma_t dB_t$, where S_t is the observed stock price at time t.
2. Market Volatility: $\hat{\sigma}_t = \frac{\text{std}(S_t)}{S_t}$ within a short time horizon, usually around 20%. ($dt \sim (dB_t)^2 \Rightarrow dB_t = \sqrt{dt} >> dt$)

   Note that in $\mu_t = \frac{S_{t+\Delta t} - S_t}{\Delta t}$, where Δt is relatively large.
3. Implied Volatility (1 data): $C(0,S_0;T,K) = S_0N(d_+(T,S_0;K)) - Ke^{-rT}N(d_-(T,S_0;K))$, which is obtained from reverse calculation from Black-Scholes pricing formula in reality:
4. Black-Scholes Volatility (>1 data): $\sigma^2_{\text{BS}} = \frac{\frac{\partial}{\partial T}C(0,S_0;T,K) + rK\frac{\partial}{\partial K}C(0,S_0;T,K)}{\frac{1}{2}K^2\frac{\partial^2}{\partial K^2}C(0,S_0;T,K)}$, which is obtained from Dupire's formula.
5. Local Volatility: $dS_t = rS_t dt + \sigma(t,S_t)S_t d\tilde{B}_t$
6. Stochastic Volatility:

   (1) Cox-Ingersoll-Ross (CIR) Model: $dS_t = rS_t dt + \sqrt{v_t}S_t d\tilde{B}_t$

   (2) Heston Model: $dV_t = b(t,v_t)dt + \alpha(t,v_t)d\tilde{B}_t + \beta(t,v_t)(d\tilde{B}_t)^2$

5.4 Review of Risk-Neutral Asset Pricing Approaches

1. Probability Approach:
   (1) Find the distribution of S_T under the risk-neutral measure.
   (2) Calculate $\tilde{\mathbb{E}}[e^{-r(T-t)}f(S_T) \mid \mathcal{F}_t]$.
2. PDE Approach:
   (1) Assume that there exists $g \in C^2$, $g(t,x)$ such that $g(t,X_t) = Y_t = \mathbb{E}[h(X_T) \mid \mathcal{F}_t]$.
   Then $\{g(t,X_t)\}_{t\geq 0}$ is a martingale.

   (2) To find $g(t,X_t)$, compute its differential via Itô's Formula:

$$\begin{aligned} dg(t,X_t) &= g_t(t,X_t)dt + g_x(t,X_t)dX_t + \frac{1}{2}g_{xx}(t,X_t)(dX_t)^2 \\ &= (g_t(t,X_t) + g_x(t,X_t)\mu(t,X_t) + \frac{1}{2}g_{xx}(t,X_t)\sigma^2(t,X_t))dt + g_x(t,X_t)\sigma(t,X_t)d\tilde{B}_t \end{aligned}$$

   (3) Set dt term equal to 0:

$$g_t(t,X_t) + g_x(t,X_t)\mu(t,X_t) + \frac{1}{2}g_{xx}(t,X_t)\sigma^2(t,X_t) = 0$$

   (4) Replace (t,X_t) to (t,x) and write down the terminal formula

$$\begin{cases} g_t(t,x) + g_x(t,x)\mu(t,x) + \frac{1}{2}g_{xx}\sigma^2(t,x) = 0 \\ g(T,x) = h(x) \end{cases}$$

Exercise 5.4.1. Consider the Asian option of the stock S_t with maturity T, and the terminal payoff is given by $\left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+$. Using the 4-step procedure to establish the PDF for the price of the option.

Let $Y_t = \int_0^t S_u\,du \Rightarrow dY_t = S_t dt$, where $Y_0 = 0$.

The stock price S_t follows $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$ under risk-neutral measure.

Step 1: Assume there exists a martingale:

$$g(t,S_t,Y_t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}\int_0^T S_t\,dt - K\right)^+ \middle| \mathcal{F}_t\right] = \tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(\frac{1}{T}Y_T - K\right)^+ \middle| \mathcal{F}_t\right]$$

Step 2: Apply Itô's formula for discounted option price:

$$\begin{aligned} dg(t,S_t,Y_t) &= g_t(t,S_t,Y_t)dt + g_s(t,S_t,Y_t)dS_t + \frac{1}{2}g_{ss}(t,S_t,Y_t)(dS_t)^2 + g_y(t,S_t,Y_t)dY_t + \frac{1}{2}g_{yy}(t,S_t,Y_t)(dY_t)^2 \\ &\quad + g_{sy}(t,S_t,Y_t)(dS_t dY_t) \quad \text{by Itô's formula for } C^2 \text{ function} \\ &= g_t(t,S_t,Y_t)dt + g_s(t,S_t,Y_t)(rS_t dt + \sigma S_t d\tilde{B}_t) + \frac{1}{2}g_{ss}(t,S_t,Y_t)(rS_t dt + \sigma S_t d\tilde{B}_t)^2 \\ &\quad + g_y(t,S_t,Y_t)(S_t dt) + \frac{1}{2}g_{yy}(t,S_t,Y_t)(S_t dt)^2 + g_{sy}(t,S_t,Y_t)(rS_t dt + \sigma S_t d\tilde{B}_t)(S_t dt) \\ &= \left(g_t(t,S_t,Y_t) + rS_t g_s(t,S_t,Y_t) + \frac{1}{2}\sigma^2 S_t^2 g_{ss}(t,S_t,Y_t) + S_t g_y(t,S_t,Y_t)\right)dt + \sigma S_t g_s(t,S_t,Y_t)d\tilde{B}_t \\ de^{-rt}g(t,S_t,Y_t) &= e^{-rt}(-rg(t,S_t,Y_t)dt + dg(t,S_t,Y_t)) \quad \text{by product rule} \\ &= e^{-rt}\left(-rg(t,S_t,Y_t) + g_t(t,S_t,Y_t) + rS_t g_s(t,S_t,Y_t) + \frac{1}{2}\sigma^2 S_t^2 g_{ss}(t,S_t,Y_t) + S_t g_y(t,S_t,Y_t)\right)dt \\ &\quad + e^{-rt}\sigma S_t g_s(t,S_t,Y_t)d\tilde{B}_t \end{aligned}$$

Step 3: Set dt term equal to 0:

$$-rg(t,S_t,Y_t)+g_t(t,S_t,Y_t)+rS_tg_s(t,S_t,Y_t)+\frac{1}{2}\sigma^2S_t^2g_{ss}(t,S_t,Y_t)+S_tg_y(t,S_t,Y_t)=0$$

Step 4: Replace (t,S_t,Y_t) to (t,x,y):

$$\begin{cases}-rg(t,x,y)+g_t(t,x,y)+rxg_x(t,x,y)+\frac{1}{2}\sigma^2x^2g_{xx}(t,x,y)+xg_y(t,x,y)=0\\ g(T,x,y)=(\frac{1}{T}y-K)^+\end{cases}$$

Exercise 5.4.2. Suppose there is an option of the stock S_t with maturity T, and the terminal payoff is given by $\left(S_T+\int_0^T S_u\,du-K\right)^+$. Using the 4-step procedure to establish the PDE for the price of the option. Can we still have a delta-hedging formula similar to the discrete-time binomial model?

Let $Y_t=\int_0^t s_u\,du \Rightarrow dY_t=S_tdt$.

The stock price follows $dS_t=rS_tdt+\sigma S_td\tilde{B}_t$.

Step 1: Assume there exists a martingale:

$$g(t,S_t,Y_t)=\tilde{\mathbb{E}}\left[e^{-r(T-t)}\left(S_T+\int_0^T S_u\,du-K\right)^+\middle|\mathcal{F}_t\right]=\tilde{\mathbb{E}}[e^{-r(T-t)}(S_T+Y_T-K)^+\mid\mathcal{F}_t]$$

Step 2: Apply Itô's formula for discounted option price:

$$\begin{aligned}dg(t,S_t,Y_t)&=g_t(t,S_t,Y_t)dt+g_s(t,S_t,Y_t)dS_t+\frac{1}{2}g_{ss}(t,S_t,Y_t)(dS_t)^2+g_y(t,S_t,Y_t)dY_t+\frac{1}{2}g_{yy}(t,S_t,Y_t)(dY_t)^2\\&\quad+g_{sy}(t,S_t,Y_t)(dS_tdY_t)\quad\text{by Itô's formula for } C^2 \text{ function}\\&=g_t(t,S_t,Y_t)dt+g_s(t,S_t,Y_t)(rS_tdt+\sigma S_td\tilde{B}_t)+\frac{1}{2}g_{ss}(t,S_t,Y_t)(rS_tdt+\sigma S_td\tilde{B}_t)^2\\&\quad+g_y(t,S_t,Y_t)(S_tdt)+\frac{1}{2}g_{yy}(t,S_t,Y_t)(S_tdt)^2+g_{sy}(t,S_t,Y_t)(rS_tdt+\sigma S_td\tilde{B}_t)(S_tdt)\\&=\left(g_t(t,S_t,Y_t)+rS_tg_s(t,S_t,Y_t)+\frac{1}{2}\sigma^2S_t^2g_{ss}(t,S_t,Y_t)+S_tg_y(t,S_t,Y_t)\right)dt+\sigma S_tg_s(t,S_t,Y_t)d\tilde{B}_t\\de^{-rt}g(t,S_t,Y_t)&=e^{-rt}(-rg(t,S_t,Y_t)dt+dg(t,S_t,Y_t))\quad\text{by product rule}\\&=e^{-rt}\left(-rg(t,S_t,Y_t)+g_t(t,S_t,Y_t)+rS_tg_s(t,S_t,Y_t)+\frac{1}{2}\sigma^2S_t^2g_{ss}(t,S_t,Y_t)+S_tg_y(t,S_t,Y_t)\right)dt\\&\quad+e^{-rt}\sigma S_tg_s(t,S_t,Y_t)d\tilde{B}_t\end{aligned}$$

Step 3: Set dt term equal to 0:

$$-rg(t,S_t,Y_t)+g_t(t,S_t,Y_t)+rS_tg_s(t,S_t,Y_t)+\frac{1}{2}\sigma^2S_t^2g_{ss}(t,S_t,Y_t)+S_tg_y(t,S_t,Y_t)=0$$

Step 4: Replace (t,S_t,Y_t) to (t,x,y):

$$\begin{cases}-rg(t,x,y)+g_t(t,x,y)+rxg_x(t,x,y)+\frac{1}{2}\sigma^2x^2g_{xx}(t,x,y)+xg_y(t,x,y)=0\\ g(T,x,y)=(x+y-K)^+\end{cases}$$

Yes, we have delta-hedging $\Delta_t=g_x$.

5.5 Numerical Methods

Finite Difference Methods (Pricing/PDE in Deep Learning):

1. For the time derivative:

$$
\begin{aligned}
g_t(t,x,y) &\approx \frac{g(t+\Delta t,x,y)-g(t,x,y)}{\Delta t} \quad \text{(forward difference)} \\
&\approx \frac{g(t,x,y)-g(t-\Delta t,x,y)}{\Delta t} \quad \text{(backward difference)} \\
&\approx \frac{g(t+\Delta t,x,y)-g(t-\Delta t,x,y)}{2\Delta t} \quad \text{(central difference)}
\end{aligned}
$$

2. For the spatial derivatives:

$$
\begin{aligned}
g_x(t,x,y) &\approx \frac{g(t,x+\Delta x,y)-g(t,x-\Delta x,y)}{2\Delta x} \\
g_y(t,x,y) &\approx \frac{g(t,x,y+\Delta y)-g(t,x,y-\Delta y)}{2\Delta y} \\
g_{xx}(t,x,y) &\approx \frac{g(t,x+\Delta x,y)+g(t,x-\Delta x,y)-2g(t,x,y)}{(\Delta x)^2}
\end{aligned}
$$

3. Start at T:
   (1) Set $g(T,x,y)$ for all x,y.
   (2) Calculate $g_x(T,x,y)$, $g_{xx}(T,x,y)$, $g_y(T,x,y)$, and use the PDE to obtain $g_t(T,x,y)$.
   (3) Step backward in time to find $g(T-\Delta t,x,y)$.

6 Lecture 6

6.1 Connections between SDE and PDE

The stock price dynamic under risk-neutral measure is $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$, which is the Black-Scholes Model.

The generalized Black-Scholes Model is $dS_t = rS_t dt + S_t \sigma(t, S_t) d\tilde{B}_t$, where $\sigma(t, S_t)$ is a local volatility.

An European-type option, i.e., can only exercise at time T of the underlying asset has terminal payoff $G(T)$. For example, European call payoff is $G(S_T) = (S_T - K)^+$.

Let $g(t, S_t)$ denote the time t price of an option, then by 4-step procedure, we have

(1) $g(t, S_t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)} G(S_T) \mid \mathcal{F}_t\right]$, where $\{e^{-rt} g(t, S_t)\}_{0 \le t \le T}$ is a martingale.

(2) $de^{-rt} g(t, S_t)$ via Itô's formula is

$$de^{-rt} g(t, S_t) = e^{-rt}\left(-rg(t, S_t) + g_t(t, S_t) + rS_t g_t(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 g_{ss}(t, S_t)\right) dt + e^{-rt} \sigma S_t g_s(t, S_t) d\tilde{B}_t$$

(3) Set dt term to 0.

(4) Replace (t, S_t) to (t, x) and write down PDE system together with the terminal condition

$$\begin{cases} -rg(t, x) + g_t(t, x) + rxg_x(t, x) + \frac{1}{2}x^2\sigma^2(t, x) g_{xx}(t, x) = 0 \\ g(T, x) = G(x) \end{cases}$$

which is a generalized Black-Scholes PDE.

Therefore, from a stochastic process martingale, we can have the PDE system, and from the PDE system, we can derive SDE, which returns back to a stochastic process.

6.2 Feynman-Kac Theorem

Theorem 6.2.1. *(Feynman-Kac Theorem).* Let $g(t, x)$ be a C^2 function satisfying the following PDE:

$$\begin{cases} g_t(t, x) + a(t, x) g_x(t, x) + \frac{1}{2}b^2(t, x) g_{xx}(t, x) = c(t, x) g(t, x) \\ g(T, x) = G(x) \end{cases}$$

where $a(t, x)$, $b(t, x)$, and $c(t, x)$ are known functions.

The solution $g(t, x)$ of the PDE can be represented as

$$g(t, x) = \mathbb{E}\left[e^{-\int_t^T c(u, X_u)\, du} G(X_T) \middle| \mathcal{F}_t, X_t = x\right]$$

Then a stochastic process $\{X_t\}_{t \ge 0}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \ge 0}, \mathbb{P})$ satisfies the SDE

$$dX_t = a(t, X_t) dt + b(t, X_t) dB_t, X_0 = x$$

where $\{B_t\}_{t \ge 0}$ is a B.M. process on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \ge 0}, \mathbb{P})$.

Example 6.2.2. The SDE $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$, $S_0 = s$ has the solution $S_t = se^{\left(r - \frac{1}{2}\sigma^2\right)t + \sigma \tilde{B}_t}$.

Corollary 6.2.3. 4-step approach (PDE) $\overset{\text{Feynman-Kac Formula}}{\Longleftrightarrow}$ calculating conditional expectation (SDE)

6.3 Radon-Nikodym Derivative

For discrete-time model: $\mathbb{Q} \sim \mathbb{P}$, $\tilde{p}, \tilde{q}$ s.t. $\tilde{p} + \tilde{q} = 1$.

For continuous-time model: $\mathbb{Q} \sim \mathbb{P}$, $\{\tilde{B}_t\}_{t\geq 0}$ under $\mathbb{Q}$.

For every continuous adaptive process $\{X_t\}_{t\geq 0}$ on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ with the SDE $dX_t = \mu_t dt + \sigma_t dB_t$, where $\{B_t\}_{t\geq 0}$ is a B.M. on $\mathbb{P}$, then we can apply the martingale representation theorem.

Under physical measure $\mathbb{P}$, we have the SDE

$$dS_t = S_t\mu(t, S_t)dt + S_t\sigma(t, S_t)dB_t$$

and the bank account

$$dX_t = rX_t dt$$

By subtracting two equations, we get the Sharpe ratio, $\frac{\mu(t,S_t)-r}{\sigma(t,S_t)}$. Aggregating into $\mathbb{Q}$, we have

$$dS_t = rS_t dt + S_t\sigma(t, S_t)\left(dB_t - \frac{\mu(t, S_t) - r}{\sigma(t, S_t)}dt\right)$$

Note that $dB_t - \frac{\mu(t,S_t)-r}{\sigma(t,S_t)}dt = d\tilde{B}_t$, where $\{\tilde{B}_t\}$ is a B.M. under $\mathbb{Q}$, and the Sharpe ratio is an adjusted drift term w.r.t. dt.

Example 6.3.1. Let $X \sim N(0, 1)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{P}[X \leq x] = N(x)$.

Let $Y = X + \theta \sim N(0, 1)$, $\theta \in \mathbb{R}$, $\mathbb{P}[Y \leq y] = \mathbb{P}[X \leq y - \theta] = N(y - \theta)$.

Try to find $\mathbb{Q}$ ($\tilde{\mathbb{P}}$) s.t.

(1) $\tilde{\mathbb{P}}[Y \leq y] = N(y)$

(2) $\tilde{\mathbb{P}} \sim \mathbb{P}$

Solution. First, we let $Z = e^{-\theta X - \frac{1}{2}\theta^2} > 0$, $\mathbb{P}[Z = 0] = 0$. $\forall A \in \mathcal{F}$, we define $\tilde{\mathbb{P}}[A] = \tilde{\mathbb{E}}[\mathbb{1}_A] = \mathbb{E}[Z\mathbb{1}_A] = \int_A Z d\mathbb{P}$.

Then we calculate (1):

$$\begin{aligned}
\tilde{\mathbb{P}}\left[Y \leq y\right] &= \tilde{\mathbb{P}}\left[X \leq y - \theta\right] \\
&= \mathbb{E}\left[Z\mathbb{1}_{[X\leq y-\theta]}\right] \\
&= \mathbb{E}\left[e^{-\theta X - \frac{1}{2}\theta^2}\mathbb{1}_{[X\leq y-\theta]}\right] \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y-\theta} e^{-\theta x - \frac{1}{2}\theta^2 - \frac{1}{2}x^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y-\theta} e^{-\frac{1}{2}(x+\theta)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-\frac{1}{2}z^2} dz \\
&= N(y)
\end{aligned}$$

which shows that Y is a standard normal random variable under the probability measure $\mathbb{P}$. And Z is the *Radon-Nikodym derivative* of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$.

Secondly, we evaluate the equivalence. On one hand, $\forall A \in \mathcal{F}$,$\mathbb{P}[A] = 0 \Rightarrow \tilde{\mathbb{P}}[A] = \mathbb{E}\left[Z\mathbb{1}_A\right] = 0$.

On the other hand, $\tilde{\mathbb{P}}[A] = 0 = \mathbb{E}\left[Z\mathbb{1}_A\right]$. Since $Z(\omega) > 0, \forall\omega \in \Omega$, suppose $\mathbb{P}[A] > \epsilon \Rightarrow \tilde{\mathbb{P}}[A] = \mathbb{Z}\mathbb{1}_{\mathbb{A}} > \tilde{\epsilon}$, then $Z\mathbb{1}_A > \tilde{\epsilon}$, which is a contradiction against $\tilde{\mathbb{P}} = 0$, so $\mathbb{P}[A] = 0$ proved by contradiction. Therefore, $\tilde{\mathbb{P}} \sim \mathbb{P}$.

When $Z > 0 \Rightarrow \frac{1}{Z} > 0$, we have $\hat{\mathbb{P}}[A] = \hat{\mathbb{E}}[\mathbb{1}_A] = \tilde{\mathbb{E}}[\frac{1}{Z}\mathbb{1}_A] = \mathbb{E}[\frac{1}{Z}\mathbb{1}_A Z] = \mathbb{P}[A], \forall A \in \mathcal{F} \Rightarrow \frac{1}{Z} = \frac{d\mathbb{P}}{d\tilde{\mathbb{P}}}$.

Note that for random variable ξ, $\mathbb{E}[\xi] = \tilde{\mathbb{E}}[\frac{\xi}{Z}] \nRightarrow \tilde{\mathbb{E}}[\xi] = \mathbb{E}[Z\xi]$.

For continuous-time model, $Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$, we need to find that for all $t \geq 0$ on space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ such that $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$. Note that $\mathbb{E}$ is with respect to $\mathbb{P}$.

Definition 6.3.2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space.

Let $\tilde{\mathbb{P}}$ be another probability measure on $(\Omega, \mathcal{F})$ that is equivalent to $\mathbb{P}$, i.e., $\tilde{\mathbb{P}} \sim \mathbb{P}$.

Let Z be an almost surely positive random variable that relates $\mathbb{P}$ and $\tilde{\mathbb{P}}$ via $\tilde{\mathbb{P}}(A) = \int_A Z(\omega) d\mathbb{P}(\omega)$.

Then Z is called the *Radon-Nikodym derivative* of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$, and we write

$$Z = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}}$$

Definition 6.3.3. On probability space space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, for $\mathcal{F}_s \subset \mathcal{F}$, we say that $\mathbb{E}[X \mid \mathcal{F}_s]$ is the *conditional expectation* of X on $\mathcal{F}_s$ provided $\forall A \in \mathcal{F}_s$,

$$\int_A X \, d\mathbb{P} = \int_A \mathbb{E}[X \mid \mathcal{F}_s] \, d\mathbb{P}$$

Lemma 6.3.3. Let Z be a Radon-Nikodym derivative of $\tilde{\mathbb{P}}$ and $\mathbb{P}$ on space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, then

(1) $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$, $\{Z_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale.

(2) For an adaptive process $\{Y_t\}_{t\geq 0}$ on space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, for $0 \leq s \leq t$,

$$\tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Y_t Z_t \mid \mathcal{F}_s]$$

In particular, when $s = 0$, $\tilde{\mathbb{E}}[Y_t] = \mathbb{E}[Y_t Z_t]$.

(3) Let $\{Y_t\}_{t\geq 0}$ be a $\tilde{\mathbb{P}}$-martingale. Then $\{Y_t Z_t\}$ is a $\mathbb{P}$-martingale.

Proof.

(1) Since $\mathbb{E}[Z] = \mathbb{E}[Z\mathbb{1}_\Omega] = \tilde{\mathbb{P}}[\Omega] = 1$ and $Z \geq 0$ by definition of the Radon-Nikodym derivative, we have

$$0 \leq \mathbb{E}[|Z|] = \mathbb{E}[Z] = 1 < +\infty$$

For $0 \leq s \leq t \leq T$, we calculate

$$\mathbb{E}[Z_t \mid \mathcal{F}_s] = \mathbb{E}[\mathbb{E}[Z \mid \mathcal{F}_t] \mid \mathcal{F}_s] = \mathbb{E}[Z \mid \mathcal{F}_s] = Z_s$$

Therefore, $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$, $\{Z_t\}_{t\geq 0}$ is a $\mathbb{P}$-martingale.

(2) Consider when $s = 0$, $\tilde{\mathbb{E}}[Y_t] = \mathbb{E}[Y_t Z_t]$ and $\mathbb{E}[Y_t Z \mid \mathcal{F}_t] = Y_t \mathbb{E}[Z \mid \mathcal{F}_t] = Y_t Z_t$ by taking out what is known.

Then we have $\tilde{\mathbb{E}}[Y_t] = \mathbb{E}[\mathbb{E}[Y_t Z \mid \mathcal{F}_t]] = \mathbb{E}[Y_t Z_t]$ by tower property of conditional expectation.

For $\mathcal{F}_t$-measurable random variable, we only need to know Z_t to change measure.

For $0 \le s \le t$, consider $\forall A \in \mathcal{F}_s$, $Y_t \mathbb{1}_A$. By definition of conditional expectation, we have

$$\tilde{\mathbb{E}}[Y_t \mathbb{1}_A] = \int_A Y_t \, d\tilde{\mathbb{P}} = \int_A \tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s] \, d\tilde{\mathbb{P}} \quad (1)$$

$$\mathbb{E}[Y_t Z_t \mathbb{1}_A] = \int_A Y_t Z_t \, d\mathbb{P} = \int_A \mathbb{E}[Y_t Z_t \mid \mathcal{F}_s] \, d\mathbb{P} \quad (2)$$

When $s = 0$, above two equations are equivalent.

Let $\eta_s := \tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s]$. Using similar idea as the unconditional expectation result, we obtain that

$$\begin{aligned} \tilde{\mathbb{E}}[Y_t \mathbb{1}_A] &= \int_A \eta_s \, d\tilde{\mathbb{P}} \quad \text{from equation (1)} \\ &= \tilde{\mathbb{E}}[\eta_s \mathbb{1}_A] \\ &= \mathbb{E}[Z_s \mathbb{1}_A \eta_s] \quad \text{by change of measure} \\ &= \int_A Z_s \eta_s \, d\mathbb{P} \end{aligned}$$

By definition of conditional expectation and definition of η_s, we have

$$\mathbb{E}[Y_t Z_t \mid \mathcal{F}_s] = Z_s \eta_s = Z_s \tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s] \Rightarrow \tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s] = \frac{1}{Z_s} \mathbb{E}[Y_t Z_t \mid \mathcal{F}_s] \quad (3)$$

(3) Let $\{Y_t\}_{t \ge 0}$ be a $\tilde{\mathbb{P}}$-martingale and $Z_t = \mathbb{E}[Z \mid \mathcal{F}_t]$. Then we have

$$\begin{aligned} Y_s = \tilde{\mathbb{E}}[Y_t \mid \mathcal{F}_s] &= \frac{1}{Z_s} \mathbb{E}[Y_t Z_t \mid \mathcal{F}_s] \quad \text{from equation (3)} \\ \Rightarrow Y_s Z_s &= \mathbb{E}[Y_t Z_t \mid \mathcal{F}_s] \end{aligned}$$

Since $\mathbb{E}[Y_t Z_t \mid \mathcal{F}_s]$ is well-defined, then $\mathbb{E}[|Y_t Z_t|] < +\infty$.

Therefore, $\{Y_t Z_t\}_{t \ge 0}$ is a $\mathbb{P}$-martingale.

6.4 Girsanov's Theorem for a Single Brownian Motion

Fact 6.4.1. For an adaptive process $\{X_t\}_{t \ge 0}$ with

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t X_u^2 \, du}\right] < +\infty,$$

then we have

$$Y_t := e^{\int_0^t X_u \, dB_u - \frac{1}{2}\int_0^t X_u^2 \, du}$$

is a *martingale*.

Fact 6.4.2. For adaptive processes $\{X_t^{(1)}\}_{t \ge 0}$, $\{X_t^{(2)}\}_{t \ge 0}$ with

$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^t \left(X_u^{(1)}\right)^2 + \left(X_u^{(2)}\right)^2 du}\right] < +\infty,$$

then we have

$$Y_t := e^{\int_0^t X_u^{(1)} \, dB_u + i \int_0^t X_u^{(2)} \, dB_u - \frac{1}{2}\int_0^t \left(\left(X_u^{(1)}\right)^2 + i\left(X_u^{(2)}\right)^2\right) du}$$

is a *martingale*.

Theorem 6.4.3. *(Girsanov's Change of Measure (1-dim)).* Let $\{B_t\}_{t\geq 0}$ be a Brownian Motion on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$ and $\{\Theta_t\}_{t\geq 0}$ be an adaptive process on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$.

Focus on $[0, T]$ and assume $\mathbb{E}[e^{\frac{1}{2}\int_0^T \Theta_u^2\, du}] < +\infty$. Define

$$\tilde{B}_t = B_t + \int_0^t \Theta_u\, du$$
$$Z_t = e^{-\int_0^t \Theta_u\, dB_u - \frac{1}{2}\int_0^t \Theta_u^2\, du}$$

Then for $\tilde{\mathbb{P}}$ defined by $\tilde{\mathbb{E}}[X] := \mathbb{E}[Z_T X]$, for all X being $\mathcal{F}_T$-measurable, we have that $\{\tilde{B}_t\}_{t\geq 0}$ is a Brownian Motion under $\tilde{\mathbb{P}}$-measure.

Proof. Since $\tilde{B}_t = B_t + \int_0^t \Theta_u\, du$, then we have

$$\tilde{B}_0 = B_0 + 0 = 0$$
$$\tilde{B}_t(\omega) = B_t(\omega) + \int_0^t \Theta_u(\omega)\, du$$

where $\int_0^t \Theta_u(\omega) du$ is continuous, thus satisfying the first two conditions in definition of Brownian Motion.

To arrive at the third and fourth condition in the Brownian Motion definition, we consider $\tilde{B}_t - \tilde{B}_s = B_t - B_s + \int_s^t \Theta_u\, du\ (*)$ and use the characteristic function to capture the probability distribution of $\tilde{B}_t - \tilde{B}_s$:

$$\tilde{\mathbb{E}}[e^{ik(\tilde{B}_t - \tilde{B}_s)} \mid \mathcal{F}_s] = \frac{1}{Z_s}\mathbb{E}[Z_t e^{ik(\tilde{B}_t - \tilde{B}_s)} \mid \mathcal{F}_s] \quad \text{by Lemma 6.3.3 (2)}$$

Taking out Z_t, we have

$$\begin{aligned}
\frac{Z_t}{Z_s} e^{ik(\tilde{B}_t - \tilde{B}_s)} &= e^{ik(B_t - B_s) + ik\int_s^t \Theta_u\, du - \int_s^t \Theta_u\, dB_u - \frac{1}{2}\int_s^t \Theta_u^2\, du} && \text{from equation } (*) \text{ and definition of } Z_t \\
&= e^{\int_s^t (ik - \Theta_u)\, dB_u - \frac{1}{2}\int_s^t (\Theta_u^2 - ik\Theta_u)\, du} && \text{by taking } B_t - B_s \text{ into integral} \\
&= e^{\int_s^t (ik - \Theta_u)\, dB_u - \frac{1}{2}\int_s^t (\Theta_u^2 - ik\Theta_u + (ik\Theta_u)^2 - (ik\Theta_u)^2)\, du} && \text{to create a square term of } \Theta_u - ik \\
&= e^{\int_s^t (ik - \Theta_u)\, dB_u - \frac{1}{2}\int_s^t (\Theta_u - ik)^2\, du - \frac{1}{2}k^2(t-s)}
\end{aligned}$$

Factor 6.4.1 yields that $Y_t := e^{\int_0^t (ik - \Theta_u) dB_u - \frac{1}{2}\int_0^t (\Theta_u - ik)^2\, du}$ is a martingale. Then we can calculate

$$\begin{aligned}
\mathbb{E}[Y_t \mid \mathcal{F}_s] &= Y_s \\
\Rightarrow \frac{1}{Y_s}\mathbb{E}[Y_t \mid \mathcal{F}_s] &= 1 \\
\Rightarrow \mathbb{E}[\frac{Y_t}{Y_s} \mid \mathcal{F}_s] &= 1 \\
\Rightarrow \mathbb{E}\left[e^{\int_s^t (ik - \Theta_u) dB_u - \frac{1}{2}\int_s^t (\Theta_u - ik)^2\, du}\right] &= 1 \quad (**)
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
\tilde{\mathbb{E}}[e^{ik(\tilde{B}_t - \tilde{B}_s)} \mid \mathcal{F}_s] &= \frac{1}{Z_s}\mathbb{E}[Z_t e^{ik(\tilde{B}_t - \tilde{B}_s)} \mid \mathcal{F}_s] \\
&= \mathbb{E}\left[\frac{Z_t}{Z_s} e^{ik(\tilde{B}_t - \tilde{B}_s)} \mid \mathcal{F}_s\right] \\
&= e^{-\frac{1}{2}k^2(t-s)}\mathbb{E}\left[e^{\int_s^t (ik - \Theta_u) dB_u - \frac{1}{2}\int_s^t (\Theta_u - ik)^2\, du}\right] \\
&= e^{-\frac{1}{2}k^2(t-s)} \quad \text{from equation } (**)
\end{aligned}$$

$$\Rightarrow \tilde{B}_t - \tilde{B}_s \mid \mathcal{F}_s \sim N(0, t-s)$$

$$\Rightarrow \tilde{B}_t - \tilde{B}_s \sim N(0, t-s), \tilde{B}_t - \tilde{B}_s \perp \mathcal{F}_s \quad \text{satisfying third and fourth conditions in Brownian Motion definition}$$

7 Lecture 7

7.1 Self-Financing and Arbitrage

Recall:

- 1 risky asset, $dS_t = \alpha_t S_t dt + \sigma_t S_t dB_t$
- 1 safe asset, interest rate r

1. Self-Financing Strategy:
   (1) $\{\varphi_t\}_{t\geq 0}$, $V_0 = 0$
   (2) $V_t^\varphi = V_t^\varphi - \varphi_t S_t + \varphi_t S_t$
   (3) $dV_t^\varphi = r(V_t^\varphi - \varphi_t S_t)dt + \varphi_t dS_t$: instantaneous change in stock
2. Arbitrage Strategy:
   (1) $\{\varphi_t\}_{t\geq 0}$ self-financing, $V_0 = 0$
   (2) $\exists t \in [0, +\infty)$, $\mathbb{P}[V_t^\varphi \geq 0] = 1$, $\mathbb{P}[V_t^\varphi] > 0$
   (3) $dV_t^\varphi = r(V_t^\varphi - \varphi_t S_t)dt + \varphi_t dS_t$

Example 7.1.1. r interest rate, $dS_t = \alpha S_t dt + 0 \cdot S_t dB_t$, $\alpha > r$.

$V_0 = 0$, $V_t^\varphi = V_t^\varphi - \varphi_t S_t + \varphi_t S_t$. Calculate differential for discounted self-financing strategy.

Solution.

$$\begin{aligned}
dV_t^\varphi &= r(V_t^\varphi - \varphi_t S_t)dt + \varphi_t dS_t \quad \text{from instantaneous change in stock} \\
&= r(V_t^\varphi - \varphi_t S_t)dt + \varphi_t(\alpha S_t dt) \\
&= (rV_t^\varphi + (\alpha - r)\varphi_t S_t)dt \\
de^{-rt}V_t^\varphi &= -re^{-rt}V_t^\varphi dt + e^{-rt}dV_t^\varphi \quad \text{by Chain Rule} \\
&= -re^{-rt}V_t^\varphi dt + e^{-rt}(rV_t^\varphi + (\alpha - r)\varphi_t S_t)dt \quad \text{by substituting } dV_t^\varphi \\
&= e^{-rt}(\alpha - r)\varphi_t S_t dt
\end{aligned}$$

Since $\alpha > r$ and $\varphi_t S_t > 0$, then the discounted self-financing strategy is positive, thus having arbitrage if we

- borrow $\varphi_t S_t$ from bank
- invest all $\varphi_t S_t$ into stock

7.2 Girsanov's Change of Measure (1-dim)

Example 7.2.1. Suppose 2 risky assets:

$$\begin{aligned}
dS_t^{(1)} &= \alpha^{(1)} S_t^{(1)} dt + \sigma^{(1)} S_t^{(1)} dB_t, \quad \text{number of stock 1} : \varphi_t^{(1)} \\
dS_t^{(2)} &= \alpha^{(2)} S_t^{(2)} dt + \sigma^{(2)} S_t^{(2)} dB_t, \quad \text{number of stock 2} : \varphi_t^{(2)}
\end{aligned}$$

and 1 safe asset with interest rate r.

Suppose $\alpha^{(1)} > r$, $\alpha^{(2)} > r$, $\frac{\alpha^{(1)} - r}{\sigma^{(1)}} > \frac{\alpha^{(2)} - r}{\sigma^{(2)}}$.

Solution.

$$
\begin{aligned}
V_0 &= 0 \\
V_t^\varphi &= V_t^\varphi - \varphi_t^{(1)} S_t^{(1)} - \varphi_t^{(2)} S_t^{(2)} + \varphi_t^{(1)} S_t^{(1)} + \varphi_t^{(2)} S_t^{(2)} \\
dV_t^\varphi &= r(V_t^\varphi - \varphi_t^{(1)} S_t^{(1)} - \varphi_t^{(2)} S_t^{(2)})dt + \varphi_t^{(1)} dS_t^{(1)} + \varphi_t^{(2)} dS_t^{(2)} \quad \text{from instantaneous change in stocks} \\
&= r(V_t^\varphi - \varphi_t^{(1)} S_t^{(1)} - \varphi_t^{(2)} S_t^{(2)})dt + \varphi_t^{(1)}(\alpha^{(1)} S_t^{(1)} dt + \sigma^{(1)} S_t^{(1)} dB_t) + \varphi_t^{(2)}(\alpha^{(2)} S_t^{(2)} dt + \sigma^{(2)} S_t^{(2)} dB_t) \\
&= (rV_t^\varphi + (\alpha^{(1)} - r)\varphi_t^{(1)} S_t^{(1)} + (\alpha^{(2)} - r)\varphi_t^{(2)} S_t^{(2)})dt + (\sigma^{(1)}\varphi_t^{(1)} S_t^{(1)} + \sigma^{(2)}\varphi_t^{(2)} S_t^{(2)})dB_t
\end{aligned}
$$

For arbitrage, setting dB_t term $= 0$ and dt term > 0:

$$
\begin{aligned}
&\sigma^{(1)}\varphi_t^{(1)} S_t^{(1)} + \sigma^{(2)}\varphi_t^{(2)} S_t^{(2)} = 0 \Rightarrow \varphi_t^{(2)} S_t^{(2)} = -\frac{\sigma^{(1)}}{\sigma^{(2)}}\varphi_t^{(1)} S_t^{(1)} \quad (*) \\
&rV_t^\varphi + (\alpha^{(1)} - r)\varphi_t^{(1)} S_t^{(1)} + (\alpha^{(2)} - r)\varphi_t^{(2)} S_t^{(2)} > 0 \\
&rV_t^\varphi + \left[(\alpha^{(1)} - r) - \frac{\sigma^{(1)}}{\sigma^{(2)}}(\alpha^{(2)} - r)\right]\varphi_t^{(1)} S_t^{(1)} > 0 \quad \text{by substituting } (*) \\
&rV_t^\varphi + \sigma^{(1)}\left[\frac{\alpha^{(1)} - r}{\sigma^{(2)}} - \frac{\alpha^{(2)} - r}{\sigma^{(2)}}\right]\varphi_t^{(1)} S_t^{(1)} > 0 \\
\Rightarrow &\frac{\alpha^{(1)} - r}{\sigma^{(2)}} - \frac{\alpha^{(2)} - r}{\sigma^{(2)}} > 0 \\
&V_0 = 0 = \left(-\varphi_t^{(1)} S_0^{(1)} + \frac{\sigma^{(1)}}{\sigma^{(2)}}\varphi_0^{(1)} S_0^{(1)}\right) + \varphi_t^{(1)} S_0^{(1)} - \frac{\sigma^{(1)}}{\sigma^{(2)}}\varphi_0^{(1)} S_0^{(1)}
\end{aligned}
$$

Theorem 7.2.2. *(Girsanov's Change of Measure (1-dim)).*

$\mathbb{P} \to \mathbb{Q}, \mathbb{Q} \sim \mathbb{P}$.

Let $\{B_t\}_{t\geq 0}$ be a B.M. on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$

Focusing on $[0, T]$ and assuming for adaptive $\{\Theta_t\}_{0\leq t\leq T}$ with $\mathbb{E}[e^{\frac{1}{2}\int_0^T \Theta_u^2\, du}] < +\infty$, define

$$
\begin{aligned}
\tilde{B}_t &= B_t + \int_0^t \Theta_u\, du \\
Z_t &= e^{-\int_0^t \Theta_u\, dB_u - \frac{1}{2}\int_0^t \Theta_u^2\, du}
\end{aligned}
$$

Then for $\tilde{\mathbb{P}}(\mathbb{Q})$ defined by $\mathbb{P}[A] := \mathbb{E}[Z_T \mathbb{1}_A], \forall A \in \mathcal{F}$,

we have $\{\tilde{B}_t\}_{t\geq 0}$ is a B.M. under $\tilde{\mathbb{P}}$.

Exercise 7.2.3. How are the Brownian Motions related between risky asset and risk-free asset.
Solution.

$$
\begin{aligned}
dS_t = \alpha_t S_t dt + \sigma_t S_t dB_t &= rS_t dt + \sigma_t S_t d\tilde{B}_t \\
\Rightarrow d\tilde{B}_t - dB_t &= \frac{\alpha_t - r}{\sigma_t} dt \\
\Rightarrow d\tilde{B}_t &= dB_t + \frac{\alpha_t - r}{\sigma_t} dt \\
\Rightarrow \tilde{B}_t &= B_t + \int_0^t \frac{\alpha_u - r}{\sigma_u} du
\end{aligned}
$$

7.3 Girsanov's Change of Measure (d-dim)

Example 7.3.1. m different risky assets, interest rate r.

Under $\mathbb{P}$,

$$
\begin{aligned}
dS_t^{(i)} &= \alpha_t^{(i)} S_t^{(i)} dt + \sigma_t^{(i)} S_t^{(i)} dB_t^{(i)} \quad i = 1, \dots, m \\
\Theta_t^{(i)} &= \frac{\alpha_t^{(i)} - r}{\sigma_t^{(i)}} \\
dB_t^{(i)} \cdot dB_t^{(j)} &= \rho_{ij} dt \quad i \neq j, 0 \leq |\rho_{ij}| \leq 1 \\
\Rightarrow dS_t^{(i)} &= a_t^{(i)} S_t^{(i)} dt + S_t^{(i)} \sum_{j=1}^{d} \sigma_t^{ij} dB_t^j, \quad \{B_t^j\}_{t \geq 0} \text{ are independent B.M.} \\
\left(\sigma_t^{(i)}\right)^2 &= \sum_{j=1}^{d} \left(\sigma_t^{(ij)}\right)^2 \\
V_t^{\varphi} &= \left(V_t^{\varphi} - \varphi_t^{(1)} S_t^{(1)} - \varphi_t^{(2)} S_t^{(2)} - \varphi_t^{(m)} S_t^{(m)}\right) + \varphi_t^{(1)} S_t^{(1)} + \varphi_t^{(2)} S_t^{(2)} + \dots + \varphi_t^{(m)} S_t^{(m)} \\
\varphi_t &= \begin{pmatrix} \varphi_t^{(1)} \\ \vdots \\ \varphi_t^{(m)} \end{pmatrix}, \quad S_t = \begin{pmatrix} S_t^{(1)} \\ \vdots \\ S_t^{(m)} \end{pmatrix}, \quad \sum_{k=1}^{m} \varphi_t^{(k)} S_t^{(k)} = \varphi_t^{\top} S_t, \quad dB_t = \begin{pmatrix} dB_t^1 \\ \vdots \\ dB_t^d \end{pmatrix} \\
dV_t^{\varphi} &= r(V_t^{\varphi} - \varphi_t^{\top} S_t) dt + \varphi_t^{\top} dS_t \\
dS_t &= \begin{pmatrix} \alpha_t^{(1)} S_t^{(1)} \\ \vdots \\ \alpha_t^{(m)} S_t^{(m)} \end{pmatrix} dt + \begin{pmatrix} S_t^{(1)} \sigma_t^{11} & S_t^{(1)} \sigma_t^{12} & \dots & S_t^{(1)} \sigma_t^{1d} \\ \vdots & \vdots & & \vdots \\ S_t^{(m)} \sigma_t^{m1} & S_t^{(m)} \sigma_t^{m2} & \dots & S_t^{(m)} \sigma_t^{md} \end{pmatrix} dB_t \\
dV_t^{\varphi} &= r(V_t^{\varphi} - \varphi_t^{\top} S_t) dt + \varphi_t^{\top} \begin{pmatrix} dS_t^{(1)} \\ \vdots \\ dS_t^{(m)} \end{pmatrix} dt + \varphi_t^{\top} \begin{pmatrix} S_t^{(1)} \sigma_t^{11} & S_t^{(1)} \sigma_t^{12} & \dots & S_t^{(1)} \sigma_t^{1d} \\ \vdots & \vdots & & \vdots \\ S_t^{(m)} \sigma_t^{m1} & S_t^{(m)} \sigma_t^{m2} & \dots & S_t^{(m)} \sigma_t^{md} \end{pmatrix} dB_t
\end{aligned}
$$

Theorem 7.3.2. *(Girsanov's Change of Measure (d-dim)).*

$\mathbb{P} \to \mathbb{Q}$, $\mathbb{Q} \sim \mathbb{P}$.

Let $\{B_t\}_{t \geq 0}$ be a d-dim independent B.M. on $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t \geq 0}, \mathbb{P})$

Focusing on $[0, T]$ and assuming for adaptive d-dim process $\{\Theta_t\}_{t \geq 0}$ with $\|\Theta_t\|^2 = \sum_{j=1}^{d} \left(\Theta_t^j\right)^2$ and $\mathbb{E}[e^{\frac{1}{2} \int_0^T \|\Theta_u\|^2 \, du}] < +\infty$, define

$$
\begin{aligned}
\tilde{B}_t &= B_t + \int_0^t \Theta_u \, du \\
Z_t &= e^{-\int_0^t \Theta_u^T \, dB_u - \frac{1}{2} \int_0^t \|\Theta_u\|^2 \, du}
\end{aligned}
$$

Then for $\tilde{\mathbb{P}}(\mathbb{Q})$ defined by $\mathbb{P}[A] := \mathbb{E}[Z_T \mathbb{1}_A], \forall A \in \mathcal{F}$,

we have $\{\tilde{B}_t\}_{0 \leq t \leq T}$ is a B.M. under $\tilde{\mathbb{P}}$.

Example 7.3.3. How to find risk-neutral measure?

Since Under $\mathbb{P}: B_t = \begin{pmatrix} B_t^1 \\ \vdots \\ B_t^d \end{pmatrix}$, Under $\tilde{\mathbb{P}}: \tilde{B}_t = \begin{pmatrix} \tilde{B}_t^1 \\ \vdots \\ \tilde{B}_t^d \end{pmatrix}$, then we have

$$dS_t = \begin{pmatrix} \alpha_t^{(1)} S_t^{(1)} \\ \vdots \\ \alpha_t^{(m)} S_t^{(m)} \end{pmatrix} dt + \begin{pmatrix} S_t^{(1)} \sigma_t^{11} & \cdots & S_t^{(1)} \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ S_t^{(m)} \sigma_t^{m1} & \cdots & S_t^{(m)} \sigma_t^{md} \end{pmatrix} dB_t$$

$$\Rightarrow \frac{dS_t}{S_t} = \begin{pmatrix} \frac{dS_t^{(1)}}{S_t^{(1)}} \\ \vdots \\ \frac{dS_t^{(m)}}{S_t^{(m)}} \end{pmatrix} = \begin{pmatrix} \alpha_t^{(1)} \\ \vdots \\ \alpha_t^{(m)} \end{pmatrix} dt + \begin{pmatrix} \sigma_t^{11} & \cdots & \sigma_t^{1d} \\ \vdots & \ddots & \vdots \\ \sigma_t^{m1} & \cdots & \sigma_t^{md} \end{pmatrix} dB_t$$

$$\Rightarrow \mathbb{P}: \frac{dS_t}{S_t} = \alpha_t \, dt + \sigma_t \, dB_t \quad (1) \quad \text{where } \alpha_t = \begin{pmatrix} \alpha_t^{(1)} \\ \vdots \\ \alpha_t^{(m)} \end{pmatrix}, \sigma_t = (\sigma_t^{ij}), i = 1, \ldots, m, j = 1, \ldots, d$$

$$\Rightarrow \mathbb{P}: dS_t^{(i)} = r S_t^{(i)} \, dt + S_t^{(i)} \sum_{j=1}^{d} \sigma_t^{ij} \, d\tilde{B}_t^j$$

$$\frac{dS_t}{S_t} = r \mathbb{1}_m \, dt + \sigma_t \, dB_t \quad (2)$$

From equation (1) and (2), we get

$$\alpha_t \, dt + \sigma_t \, dB_t = \frac{dS_t}{S_t} = r \mathbb{1}_m \, dt + \sigma_t \, d\tilde{B}_t$$

Assume $\tilde{B}_t = B_t + \int_0^t \Theta_u \, du \Rightarrow d\tilde{B}_t = dB_t + \Theta_t \, dt$, then we have

$$\sigma_t \, d\tilde{B}_t = \sigma_t (dB_t + \Theta_t \, dt)$$
$$\Rightarrow \sigma_t \, dB_t + (\alpha_t - r \mathbb{1}_m) \, dt = \sigma_t \, d\tilde{B}_t + \sigma_t \, \Theta_t \, dt$$
$$(\alpha_t - r \mathbb{1}_m) \, dt = \sigma_t \, \Theta_t \, dt$$
$$\Rightarrow \alpha_t^{(i)} - r = \sum_{j=1}^{d} \sigma_t^{ij} \Theta_t^j$$

There are d unknowns Θ_u^j and m equations $\alpha_t^{(i)} - r = \sum_{j=1}^{d} \sigma_t^{ij} \Theta_t^j$.

Linear equations:

- 0 solutions for Θ_t
- Exist 1 solution for Θ_t
- Exist ∞ solutions for Θ_t

7.4 Review of FTAP I and FTAP II

Theorem 7.4.1. *(First fundamental theorem of asset pricing from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.4.7).* If a market model has a risk-neutral probability measure, then it does not admit arbitrage.

Theorem 7.4.2. *(Second fundamental theorem of asset pricing from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.4.9).* Consider a market model that has a risk-neutral probability measure. The model is complete if and only if the risk-neutral probability measure is unique.

Definition 7.4.3. *(Uniqueness of risk-neutral measure from Steve Shreve, Stochastic Calculus for Finance Volume II Definition 5.4.4).* A market model is complete if every derivative security can be hedged.

Example 7.4.3. Solve the equation for 2-dim and 3-dim.

$$\begin{aligned} dS_t^{(1)} &= \alpha^{(1)} S_t^{(1)} dt + \sigma^{(1)} S_t^{(1)} dB_t^1 \\ dS_t^{(2)} &= \alpha^{(2)} S_t^{(2)} dt + \sigma^{(2)} S_t^{(2)} \rho dB_t^1 + \sigma^{(2)} S_t^{(2)} c dB_t^2 \quad \text{where } B_t^1 \perp B_t^2 \end{aligned}$$

The Black-Scholes volatility of $S_t^{(2)}$ is $\sigma^{(2)}$, thus

$$\begin{aligned} &\left(\sigma^{(2)}\right)^2 \rho^2 + \left(\sigma_t^{(2)}\right)^2 c^2 = \left(\sigma^{(2)}\right)^2 \Rightarrow c = \sqrt{1-\rho^2} \\ &\frac{d\langle S_t^{(1)}, S_t^{(2)}\rangle}{S_t^{(1)}, S_t^{(2)}} = \frac{dS_t^{(1)}, S_t^{(2)}}{S_t^{(1)}, S_t^{(2)}} = \frac{\rho\sigma^{(1)}\sigma^{(2)} S_t^{(1)} S_t^{(2)}}{S_t^{(1)} S_t^{(2)}} (dB_t^1)^2 = \rho\sigma^{(1)}\sigma^{(2)} dt \end{aligned}$$

Assume $\frac{\alpha^{(1)}-r}{\sigma^{(1)}} > \frac{\alpha^{(2)}-r}{\sigma^{(2)}}$, then from Example 7.3.3 we have

$$\begin{aligned} &\begin{cases} \alpha^{(1)} - r = \sigma^{(1)}\Theta_1 \\ \alpha^{(2)} - r = \sigma^{(2)}\Theta_2 = \sigma^{(2)}\rho\Theta_1 + \sigma^{(2)}\sqrt{1-\rho^2}\Theta_2 \end{cases} \\ \Rightarrow &\begin{cases} \Theta_1 = \frac{\alpha^{(1)}-r}{\sigma^{(1)}} \\ \Theta_2 = \frac{\alpha^{(2)}-r-\sigma^{(2)}\rho\Theta_t^1}{\sigma^{(2)}\sqrt{1-\rho^2}} = \frac{1}{\sqrt{1-\rho^2}}\left(\frac{\alpha^{(2)}-r}{\sigma^{(2)}} - \rho\frac{\alpha^{(1)}-r}{\sigma^{(1)}}\right) \end{cases} \end{aligned}$$

For a 3-dim case, we have

$$\begin{aligned} dS_t^{(1)} &= \alpha^{(1)} S_t^{(1)} dt + \sigma^{(1)} S_t^{(1)} dB_t^1 \\ dS_t^{(2)} &= \alpha^{(2)} S_t^{(2)} dt + \sigma^{(2)} S_t^{(2)} \left(\rho dB_t^1 + \tilde{\rho} dB_t^2 + \sqrt{1-\rho-\tilde{\rho}} dB_t^3\right) \end{aligned}$$

The unknowns are $\Theta_1, \Theta_2, \Theta_3$.

The equations are

$$\begin{cases} \alpha^{(1)} - r = \sigma^{(1)}\Theta_1 \\ \alpha^{(2)} - r = \sigma^{(2)}\rho\Theta_1 + \sigma^{(2)}\tilde{\rho}^2\Theta_2 + \sigma^{(2)}\sqrt{1-\rho^2-\tilde{\rho}^2}\Theta_3 \end{cases}$$

Thus, we have infinite solutions of (Θ^2, Θ^3), given by $\Theta_1 = \frac{\alpha^{(1)}-r}{\sigma^{(1)}}$.

Therefore, not all risk/derivatives are hedgable by FTAP II/Definition 7.4.3.

Example 7.4.4. Assume that the interest rate is not constant, and for the adaptive process $\{R_t\}_{t\geq 0}$, $R_t \geq 0$.

The discount factor is $D(t,T) := e^{-\int_t^T R_u du}$, or $D_t = e^{-\int_0^t R_u du}$.

Let $X_t = \int_0^t R_u du \Rightarrow dX_t = R_t dt$. Let $f(x) = e^{-x}$. Then $dD_t = f'(X_t)dX_t = -e^{-X_t}dX_t = -D_t dX_t = -D_t R_t dt$.

The discounted stock price is $D_t S_t$.

In the 1-dim case, $dS_t = R_t S_t dt + \sigma_t S_t d\tilde{B}_t$, where $\tilde{B}$ is risk-neutral. Then we have

$$\begin{aligned} dD_t S_t &= S_t dD_t + D_t dS_t + dD_t dS_t \\ &= -S_t D_t R_t dt + D_t(R_t S_t dt + \sigma_t S_t d\tilde{B}_t) \\ &= D_t \sigma_t S_t d\tilde{B}_t \end{aligned}$$

In the d-dim case, $\{B_t\}_{t\geq 0}$ is independent for $B_t^j, j = 1, \ldots, d$.

For m risky assets, $\frac{dS_t}{S_t} = \alpha_t dt + \sigma_t dB_t, \alpha_t \in \mathbb{R}^{m\times 1}, \sigma_t \in \mathbb{R}^{m\times d}$.

d unknowns are $\begin{pmatrix} \Theta_t^{(1)} \\ \vdots \\ \Theta_t^d \end{pmatrix}$.

m equations are $\alpha_t - R_t \mathbb{1}_m = \sigma_t \Theta_t$ for each $t \in [0, T]$.

8 Lecture 8

R_t is the interest rate, which can be constant, deterministic, or stochastic.

In the market where

- 1 stock S_t with interest rate R_t, there exists a risk-neutral $\tilde{\mathbb{P}} \sim \mathbb{P}$
- Zero-coupon bond with all maturities can be traded.

then by FTAP I, there is no arbitrage.

Discount factor : $D_t = e^{-\int_0^t R_u du}$

- is known at time t
- $D_t X_t$ is the discounted value of X_t

To prove D_t is a martingale, for $0 \le s \le t$, we compute

$$\begin{aligned}
\tilde{\mathbb{E}}[D_t \mid \mathcal{F}_s] &= \tilde{\mathbb{E}}\left[e^{-\int_0^t R_u du} \middle| \mathcal{F}_s\right] \\
&= \tilde{\mathbb{E}}\left[e^{-\int_0^s R_u du} \cdot e^{-\int_s^t R_u du} \middle| \mathcal{F}_s\right] \\
&= D_s \tilde{\mathbb{E}}\left[e^{-\int_s^t R_u du} \middle| \mathcal{F}_s\right] \\
&= D_s
\end{aligned}$$

since expectation of discounted value equal to current value under risk-neutral measure, i.e., $\tilde{\mathbb{E}}\left[e^{-\int_s^t R_u du} \middle| \mathcal{F}_s\right] = 1$.

By definition of a risk-neutral measure, $D_t S_t$ is a martingale since $dD_t S_t$ has no drift term from Example 7.4.4.

8.1 Zero-Coupon Bond

Definition 8.2.1. *Zero-coupon bond* is represented as

$$B(t,T) = \tilde{\mathbb{E}}\left[e^{-\int_t^T R_u du} \middle| \mathcal{F}_t\right] = \tilde{\mathbb{E}}\left[\frac{D_T}{D_t} \middle| \mathcal{F}_t\right] = \frac{1}{D_t}\tilde{\mathbb{E}}\left[D_T \mid \mathcal{F}_t\right]$$

Exercise 8.1.2 Prove that $D_t B(t,T)$ is also a martingale.
By definition of zero-coupon bond, we have

$$B(t,T) = \frac{1}{D_t}\tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t] \Rightarrow D_t B(t,T) = \tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t] \quad (*)$$

Since $D_T \le 1$, then we have

$$\tilde{\mathbb{E}}[|D_t B(t,T)|] = \tilde{\mathbb{E}}[|\tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t]|] \le 1 < +\infty$$

For $0 \le s \le t \le T$, we compute

$$\begin{aligned}
\tilde{\mathbb{E}}\left[D_t B(t,T) \mid \mathcal{F}_s\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}\left[D_T \mid \mathcal{F}_t\right] \middle| \mathcal{F}_s\right] \quad \text{by tower property} \\
&= \tilde{\mathbb{E}}\left[D_T \mid \mathcal{F}_s\right] \quad \text{since } D_t \text{ is a martingale} \\
&= D_s B(s,T) \quad \text{from equation } (*)
\end{aligned}$$

8.2 Forward Contract

Definition 8.2.1. *(Forward contract from Steve Shreve, Stochastic Calculus for Finance Volume II Definition 5.6.1).* A *forward contract* is an agreement to pay a specified price K at a delivery date T, where $0 \le T \le \bar{T}$, for the asset whose price at time t is S_t.

- have to pay K at time T
- over-the-counter

Definition 8.2.2. *(Forward contract from Steve Shreve, Stochastic Calculus for Finance Volume II Definition 5.6.1).* The T-forward price $\text{For}_S(t, S)$ of this asset at time t, where $0 \le t \le T \le \bar{T}$, is the value of K that makes the forward contract have no-arbitrage price zero at time t.

Theorem 8.2.3. *(Forward contract price formula from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.6.2).* Assume the zero-coupon bonds of all maturities can be traded. Then

$$\text{For}_S(t, T) = \frac{S_t}{B(t, T)}, \quad 0 \le t \le \bar{T}$$

Remark 8.2.4. Only need risk-neutral measure $\tilde{\mathbb{P}}$

Remark 8.2.5. No assumptions on dS_t

Remark 8.2.6. No assumptions on R_t

Example 8.2.7. Let $X_T = S_T - K$ and $X_t = 0$.

Since the discounted portfolio is a martingale, i.e., $D_t X_t = \tilde{\mathbb{E}}[D_T X_T \mid \mathcal{F}_t]$, then we have

$$\begin{aligned}
0 = X_t &= \frac{1}{D_t}\tilde{\mathbb{E}}\left[D_T(S_T - K) \mid \mathcal{F}_t\right] \quad \text{by multiplying the discounted factor to the portfolio} \\
&= \frac{1}{D_t}\left(D_t S_t - K\tilde{\mathbb{E}}\left[D_T \mid \mathcal{F}_t\right]\right) \quad \text{taking out known} \\
&= S_t - K\tilde{\mathbb{E}}\left[\frac{D_T}{D_t}\middle|\mathcal{F}_t\right] \\
\Rightarrow K &= \frac{S_t}{B(t, T)} \quad \text{by definition of zero-coupon bond}
\end{aligned}$$

Example 8.2.8. At time $u > t$, $\text{For}_S(u, T) = \frac{S_u}{B(t,T)}$ is the time u value of constant $\text{For}_S(t, T)$ to delivery at time T. Long forward position pays $S_T - \text{For}_S(t, T)$ at time T. To check if a long forward position is a martingale, we compute

$$\begin{aligned}
\tilde{\mathbb{E}}\left[S_T - \text{For}_S(t, T) \middle| \mathcal{F}_u\right] &= \tilde{\mathbb{E}}\left[\frac{D_T}{D_u}S_T - \frac{D_T}{D_u}\text{For}_S(t, T)\middle|\mathcal{F}_u\right] \\
&= S_u - \tilde{\mathbb{E}}\left[\frac{D_T}{D_u}\frac{S_t}{B(t, T)}\middle|\mathcal{F}_u\right] \\
&= S_u - S_t\tilde{\mathbb{E}}\left[\frac{D_T}{D_u}\frac{D_t}{D_T}\middle|\mathcal{F}_u\right] \\
&= S_u - \frac{D_t}{D_u}S_t \quad \Rightarrow \tilde{\mathbb{E}}\left[D_u S_u \middle| \mathcal{F}_t\right] = D_t S_t \\
&\ne S_u - \frac{S_u}{B(u, T)}
\end{aligned}$$

Forward price $\text{For}_S(t,T)$ is not the price of a forward contract.

8.3 Futures Contract

Futures prices: $\text{Fut}_S(t,T)$, $\{\text{Fut}_S(u,T)\}_{t\le u\le T}$, $\text{Fut}_{S_u}(t_{k-1},t_k) = \tilde{\mathbb{E}}[S_T \mid \mathcal{F}_{k-1}]$, $\text{Fut}_S(T,T) = S_T$.

Long position in futures (t_k, t_{k+1}): receive $\text{Fut}_S(t_{k+1},T) - \text{Fut}_S(t_k,T)$.

From $(0,T)$: $\sum_{k=0}^{K} \text{Fut}_S(t_{k+1},T) - \text{Fut}_S(t_k,T) = S_T - \text{Fut}_S(0,T)$

Definition 8.3.1. *(Futures contract from Steve Shreve, Stochastic Calculus for Finance Volume II Definition 5.6.4).* The *futures price* of an asset whose value at time T is S_T is given by the formula

$$\text{Fut}_S(t,T) = \tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t], \quad 0 \le t \le T$$

A *long position in the futures contract* is an agreement to receive as a cash flow the changes in the futures price (which may be negative as well as positive) during the time the position is held. A *short position in the futures contract* received the opposite cash flow.

Theorem 8.3.2. *(Futures price is a martingale from Steve Shreve, Stochastic Calculus for Finance Volume II Theorem 5.6.5).* The futures price is a martingale under the risk-neutral measure $\tilde{\mathbb{P}}$, it satisfies $\text{Fut}_S(T,T) = S_T$, and the value of a long (or a short) futures position to be held over an interval of time is always zero.

Proof. For $0 \le t \le u \le T$, assume $\tilde{\mathbb{E}}[|S_T|] < +\infty$, and we evaluate

$$\begin{aligned}
\tilde{\mathbb{E}}\left[|\text{Fut}_S(t,T)|\right] &= \tilde{\mathbb{E}}\left[\left|\tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t]\right|\right] \\
&\le \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[|S_T| \mid \mathcal{F}_t]\right] \quad \text{by Jensen's Inequality of Expectation} \\
&= \tilde{E}\left[|S_T|\right] \quad \text{by tower property} \\
&< +\infty \\
\tilde{\mathbb{E}}\left[\text{Fut}_S(u,T) \mid \mathcal{F}_t\right] &= \tilde{\mathbb{E}}\left[\tilde{\mathbb{E}}[S_T \mid \mathcal{F}_u]\middle|\mathcal{F}_t\right] \quad \text{by definition of futures price} \\
&= \tilde{\mathbb{E}}\left[S_T \mid \mathcal{F}_t\right] \quad \text{by tower property} \\
&= \text{Fut}_S(t,T)
\end{aligned}$$

Therefore, futures price is a martingale under risk-neutral measure.

8.4 Forward-Futures Spread

$\text{For}_S(t,T) = \frac{S_t}{B(t,T)}$, $\text{Fut}_S(t,T) = \tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t]$

1. Interest rate is constant r:

$$\begin{aligned}
B(t,T) = e^{-r(T-t)} &\Rightarrow \text{For}_S(t,T) = e^{r(T-t)}S_t \\
\tilde{\mathbb{E}}[e^{-rT}S_T \mid \mathcal{F}_t] = e^{-rt}S_t &\Rightarrow \tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t] = e^{r(T-t)}S_t \\
&\Rightarrow \text{Fut}_S(t,T) = e^{r(T-t)}S_t
\end{aligned}$$

2. Interest rate R_t is deterministic, $R(t)$:

   Then $D_T = D(T) = e^{-\int_0^T R(u)du}$ is deterministic, thus

$$\begin{aligned}
&\tilde{\mathbb{E}}[D(T)S_T \mid \mathcal{F}_t] = D(T)\tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t] \\
&\text{For}_S(t,T) = \text{Fut}_S(t,T)
\end{aligned}$$

3. Interest rate R_t is stochastic: Note that the following does not hold in general

$$\tilde{\mathbb{E}}[D_T S_T \mid \mathcal{F}_t] \neq D_T \tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t]$$
$$\tilde{\mathbb{E}}[D_T S_T \mid \mathcal{F}_t] \neq \tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t]\tilde{\mathbb{E}}[S_T \mid \mathcal{F}_t]$$

Considering $[0, T]$, we have

$$\begin{aligned}
\text{For}_S(0,T) &= \frac{S_0}{B(0,T)} = \frac{S_0}{\tilde{\mathbb{E}}[D_T]}, B[0,T] = \tilde{\mathbb{E}}[D_T], S_0 = \tilde{\mathbb{E}}[D_T S_T] \\
\text{Fut}_S(0,T) &= \tilde{\mathbb{E}}[S_T] \\
\Rightarrow \text{For}_S(0,T) - \text{Fut}_S(0,T) &= \frac{\tilde{\mathbb{E}}[D_T S_T]}{\tilde{\mathbb{E}}[D_T]} - \tilde{\mathbb{E}}[S_T] \\
&= \frac{\tilde{\mathbb{E}}[D_T S_T] - \tilde{\mathbb{E}}[D_T]\tilde{\mathbb{E}}[S_T]}{\tilde{\mathbb{E}}[D_T]} \\
&= \frac{Cov(D_T, S_T)}{\tilde{\mathbb{E}}[D_T]}
\end{aligned}$$

Conclusion:

(1) If $\tilde{\text{Cov}}(D_T, S_T) \neq 0$, then $\text{For}_S(0,T) = \text{Fut}_S(0,T)$.
(2) $\tilde{\text{Cov}}(D_T, S_T) < 0$
(3) $\tilde{\text{Cov}}(D_T, S_T) > 0$

8.5 Dynamics of Futures Price

1. Suppose $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$ under risk-neutral measure.
   $\text{Fut}_S(t,T) = e^{r(T-t)}S_t = g(t, S_t)$
   $g(t,x) = e^{r(T-t)}x$, $g_t(t,x) = -rg(t,x)$, $g_x(t,x) = e^{r(T-t)}$, $g_{xx} = 0$

$$\begin{aligned}
d\text{Fut}_S(t,T) &= dg(t,S_t) \\
&= g_t dt + g_x dS_t + \frac{1}{2} g_{xx}(dS_t)^2 \\
&= -rg(t,S_t)dt + e^{r(T-t)}dS_t \\
&= -re^{r(T-t)}S_t dt + e^{r(T-t)}(rS_t dt + \sigma S_t d\tilde{B}_t) \\
&= \sigma e^{r(T-t)} S_t d\tilde{B}_t
\end{aligned}$$

$$\mathbb{E}[\int_0^t \sigma^2 e^{r(T-t)} S_t^2 dt] < +\infty$$

2. Replicating portfolio:
   Suppose $dS_t = rS_t dt + \sigma S_t d\tilde{B}_T$ under risk-neutral measure

$$\begin{aligned}
X_t &= X_t - \Delta S_t + \Delta_t S_t \\
dX_t &= r(X_t - \Delta_t S_t)dt + \Delta_t dS_t \\
&= r(X_t - \Delta_t S_t)dt + \Delta_t(rS_t dt + \sigma S_t d\tilde{B}_t) \\
&= rX_t dt + \Delta_t \sigma S_t d\tilde{B}_t \\
\Leftrightarrow d\text{Fut}_S(t,T) &= \sigma e^{r(T-t)} S_t d\tilde{B}_t
\end{aligned}$$

$$\Rightarrow \begin{cases} \Delta_t = e^{r(T-t)}, \quad X_0 = 0 \\ X_0 = 0 = -\Delta_0 S_0 + \Delta_0 S_0, \quad \Delta_0 = e^{rT} \end{cases}$$

$$\begin{aligned} \Delta_0 = e^{rT} \to \Delta_t = e^{r(T-t)}, &\quad \text{selling stock at rate } -r\Delta_t \\ d\Delta_t &= -r\Delta_t dt \\ de^{-rt}X_t &= e^{-rt}\Delta_t \sigma_t S_t d\tilde{B}_t \quad (\tilde{\mathbb{E}}[e^{-rt}X_t] = 0 \text{ since } X_0 = 0) \end{aligned}$$

3. Selling 1 futures contract at time 0, Δ_t shares of stock, constant interest r, $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$ under risk-neutral measure, we have

$$\begin{aligned} X_t &= X_t - \Delta_t S_t + \Delta_t S_t - \text{Fut}_S(t,T) \\ dX_t &= r(X_t - \Delta_t S_t)dt + \Delta_t dS_t - d\text{Fut}_S(t,T) \\ &= r(X_t - \Delta_t S_t)dt + \Delta_t(rS_t dt + \sigma S_t d\tilde{B}_t) - \sigma e^{r(T-t)} S_t d\tilde{B}_t \\ &= rX_t dt + (\Delta\sigma S_t - \sigma e^{r(T-t)} S_t)d\tilde{B}_t \end{aligned}$$

9 Lecture 9

9.1 Interest Rate Model

1. Interest rate is R_t, and discount factor is $D_t = e^{-\int_0^t R_u du}$. Under risk-neutral measure, $B(t,T) = \frac{1}{D_t}\tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t]$.
2. Bond Price = Face Value $\times e^{-\text{Yield}\times\text{Time to Maturity}}$.
3. Modeling of R_t short rate
$$dR_t = \beta(t,R_t)dt + \gamma(t,R_t)d\tilde{B}_t, \quad (*)$$
which is the dynamic of R_t under risk-neutral measure.
4. 1-factor model $\Leftrightarrow$ 1-dim Brownian Motion.
5. We are interested in the risk-neutral measure that
$$\begin{aligned} B(t,T) &= \frac{1}{D_t}\tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t] \\ &= e^{\int_0^t R_u du}\tilde{\mathbb{E}}[e^{-\int_0^T R_u du} \mid \mathcal{F}_t], \end{aligned}$$
which can use 2 methods to solve:
   (1) Risk-neural pricing method: $\tilde{\mathbb{E}}[e^{-\int_0^T R_u du} \mid \mathcal{F}_t]$
      (a) Need $\int_0^T R_t dt$ distributional information
      (b) Very hard for general dynamic
   (2) PDE method: 4-step approach

9.2 PDE Approach

Step 1: Assume there exists a martingale:

$g(t,R_t) = B(t,T) = \frac{1}{D_t}\tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t]$, where $g(T,R_T) = 1$.

Since $D_t B(t,T) = \tilde{\mathbb{E}}[D_T \mid \mathcal{F}_t]$, then $\{D_t B(t,T)\}_{0\le t\le T}$ is a martingale under risk-neutral measure

Step 2: Apply Itô's formula for discounted option price:

$$\begin{aligned} dg(t,R_t) &= g_t(t,R_t)dt + g_r(t,R_t)dR_t + \frac{1}{2}g_{rr}(t,R_t)(dR_t)^2 \\ &= \left(g_t(t,R_t) + g_r(t,R_t)\beta(t,R_t) + \frac{1}{2}\gamma^2(t,R_t)g_{rr}(t,R_t)\right)dt + g_r(t,R_t)\gamma(t,R_t)d\tilde{B}_t \quad \text{from equation } (*) \\ dD_t g(t,R_t) &= D_t dg(t,R_t) + g(t,R_t)dD_t + dg(t,R_t)dD_t \\ &= D_t\left[\left(g_t(t,R_t) + g_r(t,R_t)\beta(t,R_t) + \frac{1}{2}\gamma^2(t,R_t)g_{rr}(t,R_t)\right)dt + g_r(t,R_t)\gamma(t,R_t)d\tilde{B}_t\right] + g(t,R_t)(-R_t D_t dt) \\ &= D_t\left[\left(-R_t g(t,R_t) + g_t(t,R_t) + g_r(t,R_t)\beta(t,R_t) + \frac{1}{2}\gamma^2(t,R_t)g_{rr}(t,R_t)\right)\right]dt + D_t g_r(t,R_t)\gamma(t,R_t)d\tilde{B}_t \end{aligned}$$

Step 3: Set dt term equal to 0:

$$-R_t g(t,R_t) + g_t(t,R_t) + g_r(t,R_t)\beta(t,R_t) + \frac{1}{2}\gamma^2(t,R_t)g_{rr}(t,R_t) = 0$$

Step 4: Replace (t,R_t) to (t,r):

$$\begin{cases} -rg(t,r) + g_t(t,r) + g_r(t,r)\beta(t,r) + \frac{1}{2}\gamma^2(t,r)g_{rr}(t,r) = 0 \\ g(T,r) = 1 \end{cases}$$

for all $0 \le t \le T$, for all possible values of r.

9.3 Mean-Reverting Processes

9.3.1 Ornstein-Uhlenbeck (OU) Process

$$dR_t = \kappa(\theta - R_t)dt + \sigma d\tilde{B}_t$$

where

- $\beta(t, r) = \kappa(\theta - r)$
- $\gamma(t, r) = \sigma$
- $\kappa > 0$: mean-reverting speed
- θ: mean-reverting level

Let $g(t, R_t) = e^{\kappa t} R_t$, then

$g_t(t, R_t) = \kappa e^{\kappa t} R_t = \kappa g(t, R_t)$

$g_r(t, R_t) = e^{\kappa t}$

$g_{rr}(t, R_t) = 0$

We calculate the dynamic of $e^{\kappa t} R_t$:

$$\begin{aligned}
de^{\kappa t} R_t = dg(t, R_t) &= g_t(t, R_t)dt + g_r(t, R_t)dR_t + \frac{1}{2} g_{rr}(t, R_t)(dR_t)^2 \\
&= \kappa e^{\kappa t} R_t dt + e^{\kappa t}(\kappa(\theta - R_t)dt + \sigma d\tilde{B}_t) \\
&= \kappa\theta e^{\kappa t} dt + \sigma e^{\kappa t} d\tilde{B}_t \\
\Rightarrow e^{\kappa t} R_t - R_0 &= \int_0^t \kappa\theta e^{\kappa u} du + \sigma \int_0^t e^{ku} d\tilde{B}_u
\end{aligned}$$

The expectation R_t is:

$$\begin{aligned}
\tilde{\mathbb{E}}[e^{\kappa t} R_t] &= R_0 + \int_0^t \kappa\theta e^{\kappa u} du \quad \text{since } \sigma^2 \int_0^t e^{2\kappa u} du < +\infty \\
\Rightarrow e^{\kappa t} \tilde{\mathbb{E}}[R_t] &= R_0 + \theta(e^{\kappa t} - 1) \\
\Rightarrow \tilde{\mathbb{E}}[R_t] &= e^{-\kappa t} R_0 + \theta(1 - e^{-\kappa t}) \overset{t \to +\infty}{\longrightarrow} \theta \quad \text{which is the long-term mean}
\end{aligned}$$

The variance of R_t is

$$\begin{aligned}
\widetilde{\text{Var}}(e^{\kappa t} R_t) &= \tilde{\mathbb{E}}\left[\left(\sigma \int_0^t e^{\kappa u} d\tilde{B}_t\right)^2\right] \\
&= \sigma^2 \tilde{\mathbb{E}}\left[\int_0^t e^{2\kappa u} du\right] \quad \text{since } \widetilde{\text{Var}}(\tilde{B}_t) = t \\
&= \frac{\sigma^2}{2\kappa}(e^{2\kappa t} - 1) \\
\Rightarrow \widetilde{\text{Var}}(R_t) &= e^{-2\kappa t} \frac{\sigma^2}{2\kappa}(e^{2\kappa t} - 1) \\
&= \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})
\end{aligned}$$

Therefore, $R_t \sim \left(e^{-\kappa t} R_0 + \theta(1 - e^{-\kappa t}), \frac{\sigma^2}{2\kappa}(1 - e^{-2\kappa t})\right)$.

9.3.2 Cox-Ingersoll-Ross (CIR) Process

$$dR_t = \kappa(\theta - R_t)dt + \sigma\sqrt{R_t}d\tilde{B}_t$$

- $\kappa > 0, \theta > 0, R_t \geq 0$
- Continuous process, $R_t = 0 \Rightarrow$ volatility $= 0$, hit 0 but never goes down below 0
- Not Gaussian
- $R_t \geq 0$
- Still mean reverting

9.4 Diffusion Process

$$dX_t = \mu_t dt + \sigma_t dB_t, \quad X_0 = X, \quad B_t \text{ is } \mathbb{P}\text{-martingale}$$

1. Martingale for Diffusion Process: $\{X_t\}_{0\leq t\leq T}$ is a $\mathbb{P}$-martingale $\Longleftrightarrow$
   (1) $\mu_t = 0$ for all $t \geq 0$
   (2) $\mathbb{E}\left[\int_0^T \sigma_u^2 du\right] < +\infty$
2. Remark: Suppose $\mathbb{E}\left[\int_0^T \sigma_u^2 du\right] < +\infty$
   Then $Y_t = \int_0^t \sigma_u dB_u, Y_0 = 0 \Longleftrightarrow dY_t = \sigma_t dB_t, Y_0 = 0$.
   By definition of martingale for diffusion process, $\{Y_t\}_{0\leq t\leq T}$ is a $\mathbb{P}$-martingale.
   Therefore, $\mathbb{E}[Y_t \mid \mathcal{F}_s] = Y_s, \forall 0 \leq t \leq T$.
   In particular, $Y_0 = 0 = \mathbb{E}[Y_t \mid \mathcal{F}_0] = \mathbb{E}[Y_t]$.

$$\begin{aligned} dX_t &= \mu_t dt + \sigma_t dB_t = \mu_t dt + dY_t \\ \Rightarrow X_t &= X_0 + \int_0^t \mu_u du + Y_t \\ \Rightarrow \mathbb{E}[X_t] &= X_0 + \mathbb{E}\left[\int_0^t \mu_u du\right] \quad \text{since } \mathbb{E}[Y_t] = 0 \end{aligned}$$

9.5 Famous Interest Rate Models

1. Ho-Lee Model: $dR_t = \mu dt + \sigma d\tilde{B}_t$
2. Vasicek Model: $dR_t = \kappa(\theta - R_t)dt + \sigma d\tilde{B}_t$
3. Hull-White Model: $dR_t = \kappa(\theta(t) - R_t)dt + \sigma d\tilde{B}_t$
4. Cox-Ingersoll-Ross (CIR) Model: $dR_t = \kappa(\theta - R_t)dt + \sigma\sqrt{R_t}d\tilde{B}_t$

All leads to affine (linear) yield curve: $B(t,T) = e^{-\text{yield}\cdot T} \Rightarrow$ yield $= -\frac{\partial}{\partial T}\log B(t,T)$

Example 9.5.1 Hull-White Model

Recall

$$\begin{cases} -rg(t,r) + g_t(t,r) + \beta(t,r)g_r(t,r) + \frac{1}{2}\gamma^2(t,r)g_{rr}(t,r) = 0 \\ g(T,r) = 1, \forall r \end{cases}$$

For a Hull-White model, $\beta(t,r) = \kappa(\theta(t) - r), \gamma(t,r) = \sigma$. Therefore the equations for the Hull-White model are

$$\begin{cases} -rg(t,r) + g_t(t,r) + \kappa(\theta(t) - r)g_r(t,r) + \frac{1}{2}\sigma^2 g_{rr}(t,r) = 0 \\ g(T,r) = 1, \forall r \end{cases}$$

Guess $g(t,r) = e^{-C(t;T)r - A(t;T)}$ since the yield curve is affine. Then we have

$g_t(t,r) = \left(-\frac{\partial C(t;T)}{\partial t} r - \frac{\partial A(t;T)}{\partial t}\right) e^{-C(t;T)r - A(t;T)} = -\left(\frac{\partial C(t;T)}{\partial t} r + \frac{\partial A(t;T)}{\partial t}\right) g(t,r)$

$g_r(t,r) = -C(t;T)e^{-C(t;T)r - A(t;T)} = -C(t;T)g(t,r)$

$g_{rr}(t,r) = C^2(t;T)g(t,r)$

To solve the PDE equations, we first calculate

$$\begin{aligned}
&-rg(t,r) + g_t(t,r) + \kappa(\theta(t) - r)g_r(t,r) + \frac{1}{2}\sigma^2 g_{rr}(t,r) = 0 \\
&-rg(t,r) - \left(\frac{\partial C(t;T)}{\partial t} r + \frac{\partial A(t;T)}{\partial t}\right) g(t,r) - \kappa(\theta(t) - r)C(t;T)g(t,r) + \frac{1}{2}\sigma^2 C^2(t;T)g(t,r) = 0 \\
&-r - \frac{\partial C(t;T)}{\partial t} r - \frac{\partial A(t;T)}{\partial t} - \kappa(\theta(t) - r)C(t;T) + \frac{1}{2}\sigma^2 C^2(t;T) = 0 \quad \text{since } g(t,r) > 0 \\
&-r\left(1 + \frac{\partial C(t;T)}{\partial t} - \kappa C(t;T)\right) - \frac{\partial A(t;T)}{\partial t} - \kappa\theta(t)C(t;T) + \frac{1}{2}\sigma^2 C^2(t;T) = 0 \\
\Rightarrow &\begin{cases} 1 + \frac{\partial C(t;T)}{\partial t} - \kappa C(t;T) = 0 \\ -\frac{\partial A(t;T)}{\partial t} - \kappa\theta(t)C(t;T) + \frac{1}{2}\sigma^2 C^2(t;T) = 0 \end{cases} \qquad \text{since PDE holds for all } r
\end{aligned}$$

From $g(T,r) = 1, \forall r$, we know $C(T;T) = 0$ and $A(T;T) = 0$. Then we can transfer PDE to ODE:

$$\begin{cases} 1 + \frac{\partial C(T;T)}{\partial t} - \kappa C(t;T) = 0 \\ C(T;T) = 0 \end{cases} \qquad \begin{cases} -\frac{\partial A(t;T)}{\partial t} - \kappa\theta(t)C(t;T) + \frac{1}{2}\sigma^2 C^2(t;T) = 0 \\ A(T;T) = 0 \end{cases}$$

For $C(t;T)$, we have

$$\begin{aligned}
\frac{\partial}{\partial t} C(t;T) &= \kappa C(t;T) - 1 \\
\Rightarrow \frac{\partial}{\partial t} e^{-\kappa t} C(t;T) &= e^{-\kappa t} \frac{\partial}{\partial t} C(t;T) - \kappa e^{-\kappa t} C(t;T) \\
&= e^{-\kappa t}\kappa C(t;T) - e^{-\kappa t} - \kappa e^{-\kappa t} C(t;T) \\
&= e^{-\kappa t} \\
\Rightarrow \int_t^T \frac{\partial}{\partial t} e^{-\kappa t} C(t;T) &= \int_t^T e^{-\kappa t} \\
e^{-\kappa T} C(T;T) - e^{-\kappa t} C(t;T) &= -\int_t^T e^{-\kappa u} du \\
-e^{-\kappa t} C(t;T) &= \frac{1}{\kappa}(e^{-\kappa T} - e^{-\kappa t}) \\
\Rightarrow C(t;T) &= \frac{1}{\kappa}(1 - e^{-\kappa(T-t)}), \quad \frac{\partial}{\partial T} C(t;T) = e^{-\kappa(T-t)}
\end{aligned}$$

For $A(t;T)$, we have

$$\begin{aligned}
-\frac{\partial}{\partial t}A(t;T) &= \kappa\theta(t)C(t;T) - \frac{1}{2}\sigma^2 C^2(t;T) \\
A(t;T) - A(T;T) &= \int_t^T \kappa\theta(u)C(t;T)du - \frac{1}{2}\sigma^2 \int_t^T C^2(u;T)du \quad \text{by taking integrals on both sides} \\
A(t;T) &= \int_t^T \kappa\theta(u)C(t;T)du - \frac{1}{2}\sigma^2 \int_t^T C^2(u;T)du \quad \text{since } A(T;T) = 0 \\
\frac{\partial}{\partial T}A(t;T) &= \kappa\theta(u)C(T;T) - \int_t^T \kappa\theta(u)\frac{\partial}{\partial T}C(u;T)du - \frac{1}{2}\sigma^2 C(T;T) + \frac{1}{2}\sigma^2 \int_t^T \frac{\partial}{\partial T}C^2(u;T)du \\
&= -\int_t^T \kappa\theta(u)\frac{\partial}{\partial T}C(u;T)du + \frac{1}{2}\sigma^2 \int_t^T 2C(u;T)\frac{\partial}{\partial T}C(u;T)du \\
&= \underbrace{-\int_t^T \kappa\theta(u)e^{-\kappa(T-u)}du}_{\text{mean-reverting level}} + \underbrace{\sigma^2 \int_t^T C(u;T)e^{-\kappa(T-u)}du}_{\text{volatility of } R_t}
\end{aligned}$$

With both $C(t;T)$ and $A(t;T)$, we are able to give the bond price as

$$\begin{aligned}
B(t,T) &= g(t,R_t) = e^{-C(t;T)R_t - A(t;T)} \\
B(0,T) &= e^{-C(0,;T)R_0 - A(0;T)} \\
\text{yield} &= -\frac{\partial}{\partial T}\log B(0,T) \\
&= \frac{\partial}{\partial T}(C(0;T)R_0 + A(0;T)) \\
&= \underbrace{e^{-\kappa T}}_{\text{slope}} R_0 + \underbrace{\frac{\partial}{\partial T}A(0;T)}_{\text{intercept}}
\end{aligned}$$

9.6 Derivatives on Interest Rate

9.6.1 Forward Interest Rate

Consider a contract that pays R_T at time T, P_t for price of the contract at time t, $0 \le t \le T$.

$$\begin{aligned}
D_t P_t &= \tilde{\mathbb{E}}[D_T P_T \mid \mathcal{F}_t] \\
&= \tilde{\mathbb{E}}[D_T R_T \mid \mathcal{F}_t] \quad \text{since } P_T = R_T \text{ at maturity} \\
P_t &= \tilde{\mathbb{E}}\left[R_T e^{-\int_t^T R_u du} \middle| \mathcal{F}_t\right] \quad \text{by dividing } D_t \text{ on both sides and by definition of } B(t,T) \\
&= -\frac{\partial}{\partial T}\tilde{\mathbb{E}}\left[e^{-\int_t^T R_u du} \middle| \mathcal{F}_t\right] \quad \text{since } \frac{\partial}{\partial T}e^{-\int_t^T R_u du} = -R_T e^{-\int_t^T R_u du} \\
&= -\frac{\partial}{\partial T}B(t,T) \quad \text{by definition of } B(t,T)
\end{aligned}$$

9.6.2 Forward Interest Rate

$\text{Fut}_R(t,T) = \tilde{\mathbb{E}}[R_T \mid \mathcal{F}_t]$

Under Hull-White model, we have

$$\begin{aligned} dR_t &= \kappa(\theta(t) - R_t)dt + \sigma d\tilde{B}_t \\ \Rightarrow R_T &= R_0 + \int_0^T \kappa(\theta(u) - R_u)du + \sigma \tilde{B}_T \end{aligned}$$

$$\begin{aligned} de^{\kappa t}R_t &= e^{\kappa t}dR_t + \kappa e^{\kappa t}R_t dt \\ &= e^{\kappa t}\left[\kappa(\theta(t) - R_t)dt + \sigma d\tilde{B}_t\right] + \kappa e^{\kappa t}R_t dt \\ &= e^{\kappa t}\kappa\theta(t)dt + e^{\kappa t}\sigma d\tilde{B}_t \\ \Rightarrow e^{\kappa T}R_T &= R_0 + \int_0^T e^{\kappa t}\kappa\theta(t)dt + \sigma\int_0^T e^{\kappa t}d\tilde{B}_t \end{aligned}$$

Since $\mathbb{E}[\int_0^T \sigma^2 e^{2\kappa t}dt] \leq \frac{\sigma^2}{2\kappa}e^{2\kappa T} < +\infty$, we have

$$\begin{aligned} \tilde{\mathbb{E}}[e^{\kappa T}R_T \mid \mathcal{F}_t] &= R_t + \int_t^T e^{\kappa t}\kappa\theta(t)dt \\ \text{Fut}_R(t,T) = \tilde{\mathbb{E}}[R_T \mid \mathcal{F}_t] &= e^{-\kappa T}R_t + \kappa\int_t^T e^{-\kappa(T-t)}\theta(t)dt \end{aligned}$$

10 Lecture 10

10.1 Girsanov's Change of Measure (1-dim)

Theorem 10.1.1. *(Girsanov's Change of Measure (1-dim)).*

On $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \mathbb{P})$, let $\{B_t\}_{t\geq 0}$ be a Brownian Motion and $\{\Theta_t\}_{t\geq 0}$ be an adaptive process.

Focusing $[0,T]$ and assuming $\mathbb{E}[e^{\frac{1}{2}\int_0^T \Theta_t^2\, dt}] < +\infty$, define

$$\tilde{B}_t = B_t + \int_0^t \Theta_u\, du \Longleftrightarrow d\tilde{B}_t = dB_t + \Theta_t dt$$
$$Z_t = e^{-\int_0^t \Theta_u\, dB_u - \frac{1}{2}\int_0^t \Theta_u^2\, du} \Longleftrightarrow dZ_t = -Z_t\Theta_t dB_t$$

Then for $\tilde{\mathbb{P}}$ defined by $\tilde{\mathbb{E}}[X] = \mathbb{E}[XZ_T]$, for all random variables X,

we have that $\{\tilde{B}_t\}_{t\geq 0}$ is a Brownian Motion under $\tilde{\mathbb{P}}$ on On $(\Omega, \mathcal{F}, \{\mathcal{F}\}_{t\geq 0}, \tilde{\mathbb{P}})$.

Example 10.1.2. Show that $dZ_t = -Z_t\Theta_t dB_t$.

Let $g(t) = -\int_0^t \Theta_u\, dB_u - \frac{1}{2}\int_0^t \Theta_u^2$. Then we have $dg(t) = -\frac{1}{2}\Theta_t^2 dt - \Theta_t dB_t$.

Let $f(t, g(t)) = e^{g(t)} = Z_t$. Then we have $f_g(t, g(t)) = e^{g(t)} = Z_t$ and $f_{gg}(t, g(t)) = e^{g(t)} = Z_t$.

Applying Itô's formula, we have

$$\begin{aligned}
dZ_t = df(t, g(t)) &= \left[f_g(t, g(t))\mu_t + \frac{1}{2} f_{gg}(t, g(t))\sigma_t^2 \right] dt + f_g(t, g(t))\sigma_t dB_t \\
&= \left(-Z_t \frac{1}{2}\Theta_t^2 + \frac{1}{2} Z_t(-\Theta_t)^2 \right) dt - Z_t\Theta_t dB_t \\
&= -Z_t\Theta_t dB_t
\end{aligned}$$

Exercise 10.1.3 In a Black-Scholes market model, let $S = \{S_t\}_{t\geq 0}$ denote the price process satisfying

$$dS_t = S_t(\mu + \sigma dW_t),$$

under the physical measure $\mathbb{P}$ with interest rate $r = 0$. Using the FTAP and Girsanov's Theorem, determine the Radon-Nikodym derivative from $\mathbb{P}$ to a risk-neutral measure $\tilde{\mathbb{P}}$. Further, write down the stock dynamics under the risk-neutral measure $\tilde{\mathbb{P}}$ and determine the relationship between the Brownian motions under measure $\mathbb{P}$ and $\tilde{\mathbb{P}}$.

Solution.

Under physical measure $\mathbb{P}$, $dS_t = \mu S_t dt + \sigma S_t dB_t$, where $\{B_t\}_{t\geq 0}$ is a Brownian Motion under $\mathbb{P}$ (usually $\mu > r$).

Under risk-neutral measure $\tilde{\mathbb{P}}$, $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$, where $\{\tilde{B}_t\}_{t\geq}$ is a Brownian Motion under $\tilde{\mathbb{P}}$.

Equating $dS_t = \mu S_t dt + \sigma S_t dB_t$ and $dS_t = rS_t dt + \sigma S_t d\tilde{B}_t$, we have

$$\begin{aligned}
\mu S_t dt + \sigma S_t dB_t &= rS_t dt + \sigma S_t d\tilde{B}_t \\
d\tilde{B}_t - dB_t &= \frac{\mu - r}{\sigma} dt \\
\Rightarrow d\tilde{B}_t &= dB_t + \frac{\mu - r}{\sigma} dt
\end{aligned}$$

where $\frac{\mu - r}{\sigma} > 0$ and $\sigma > 0$.

By Girsanov's Change of Measure, define $\Theta_t = \frac{\mu - r}{\sigma}$, then the Radon-Nikodym derivative from physical measure $\mathbb{P}$ to risk-neutral measure $\tilde{\mathbb{P}}$ is

$$\begin{aligned} Z_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} &= e^{-\int_0^t \frac{\mu-r}{\sigma} dB_u - \frac{1}{2}\int_0^t \left(\frac{\mu-r}{\sigma}\right)^2 du} \\ &= e^{-\frac{\mu-r}{\sigma}B_t - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 t} \quad \text{since } \frac{\mu-r}{\sigma} \text{ is constant} \end{aligned}$$

Note that when $r = 0$, $Z_t = \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = e^{-\frac{\mu}{\sigma}B_t - \frac{1}{2}\left(\frac{\mu}{\sigma}\right)^2 t}$.

Aside:

Note that MGF is $\mathbb{E}[e^{\lambda Z}] = e^{\lambda\mu + \frac{1}{2}\lambda^2\sigma^2}$, where $Z \sim N(\mu, \sigma^2)$.

The expectations of strike price and final stock price by change of measure (from physical measure $\mathbb{P}$ to risk-neutral measure $\tilde{\mathbb{P}}$) are

$$\tilde{\mathbb{E}}[K] = \mathbb{E}[KZ_T] = \mathbb{E}\left[Ke^{-\frac{\mu-r}{\sigma}B_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}\right] = K \quad \text{since } K \text{ is constant and } \mathbb{E}[Z_T] = 1$$

$$\begin{aligned} \tilde{\mathbb{E}}[S_T] = \mathbb{E}[S_T Z_T] &= \mathbb{E}\left[S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T} e^{-\frac{\mu-r}{\sigma}B_T - \frac{1}{2}\left(\frac{\mu-r}{\sigma}\right)^2 T}\right] \quad \text{where } \mathbb{E}[S_T] = \mathbb{E}\left[S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)T + \sigma B_T}\right] = e^{\mu T} \text{ by MGF} \\ &= S_0 \mathbb{E}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)B_T - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu-r}{\sigma}\right)^2\right)T + \mu T}\right] \\ &= S_0 \mathbb{E}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)B_T - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu-r}{\sigma}\right)^2 - 2(\mu-r)\right)T - (\mu-r)T + \mu T}\right] \\ &= S_0 \mathbb{E}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)B_T - \frac{1}{2}\left(\sigma^2 + \left(\frac{\mu-r}{\sigma}\right)^2 - 2(\mu-r)\right)T + rT}\right] \\ &= S_0 e^{rT} \mathbb{E}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)B_T - \frac{1}{2}\left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T}\right] \\ &= S_0 e^{rT} \mathbb{E}\left[e^{\left(\sigma - \frac{\mu-r}{\sigma}\right)B_T}\right] e^{-\frac{1}{2}\left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T} \\ &= S_0 e^{rT} e^{\frac{1}{2}\left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T} e^{-\frac{1}{2}\left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T} \quad \text{by MGF and } \left(\sigma - \frac{\mu-r}{\sigma}\right)B_T \sim N\left(0, \left(\sigma - \frac{\mu-r}{\sigma}\right)^2 T\right) \\ &= S_0 e^{rT} \quad \text{which aligns with the definition of risk-neutral measure} \end{aligned}$$

10.2 Black-Scholes Equation for European Call Option by Change of Measure

Recall $\tilde{\mathbb{E}}\left[e^{-rT}(S_T - K)^+\right] = \tilde{\mathbb{E}}\left[e^{-rT}S_T \mathbb{1}_{[S_T \geq K]}\right] - e^{-rT}K\tilde{\mathbb{E}}\left[\mathbb{1}_{[S_T \geq K]}\right]$

(1) For $\tilde{\mathbb{E}}\left[\mathbb{1}_{[S_T \geq K]}\right] = \tilde{\mathbb{P}}[S_T \geq K]$, since $S_T = S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{B}_T}$, we calculate

$$\begin{aligned} S_T &\geq K \\ S_0 e^{\left(r - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{B}_T} &\geq K \\ \left(r - \frac{1}{2}\sigma^2\right)T + \sigma\tilde{B}_T &\geq -\ln\frac{S_0}{K} \\ \sigma\tilde{B}_T &\geq -\left(\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right) \\ -\frac{\tilde{B}_T}{\sqrt{T}} &\leq \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K} + \left(r - \frac{1}{2}\sigma^2\right)T\right) \end{aligned}$$

where $-\frac{\tilde{B}_T}{\sqrt{T}}$ is a standard normal variable. Therefore, $\tilde{\mathbb{P}}[S_T \geq K] = N(d_-(T, S_0))$.

(2) For $\tilde{\mathbb{E}}\left[e^{-rT}S_T\mathbb{1}_{[S_T\geq K]}\right] = \tilde{\mathbb{E}}\left[e^{-rT}S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\tilde{B}_T}\mathbb{1}_{[S_T\geq K]}\right] = \tilde{\mathbb{E}}\left[S_0e^{\sigma\tilde{B}_T-\frac{1}{2}\sigma T}\mathbb{1}_{[S_T\geq K]}\right]$, we define $\Theta_t = -\sigma$ since $e^{\sigma\tilde{B}_T-\frac{1}{2}\sigma T}$ looks like a Radon-Nikodym derivative.

By Girsanov's Change of Measure, the Radon-Nikodym derivative from risk-neutral measure $\tilde{\mathbb{P}}$ to the measure $\mathbb{S}$ and the Brownian Motion under $\mathbb{S}$-measure are

$$
\begin{aligned}
Z_t &= \frac{d\mathbb{S}}{d\tilde{\mathbb{P}}} = e^{-\int_0^t\Theta_u\,d\tilde{B}_u-\frac{1}{2}\int_0^t\Theta_u^2\,du} = e^{\sigma\tilde{B}_t-\frac{1}{2}\sigma^2 t} \\
B_t^{\mathbb{S}} &= \tilde{B}_t + \int_0^t\Theta_u du = \tilde{B}_t - \sigma t
\end{aligned}
$$

since σ is constant. Then we can write

$$
\begin{aligned}
&\tilde{\mathbb{E}}\left[e^{-rT}S_T\mathbb{1}_{[S_T\geq K]}\right] = \tilde{\mathbb{E}}\left[S_0e^{\sigma\tilde{B}_T-\frac{1}{2}\sigma T}\mathbb{1}_{[S_T\geq K]}\right] = \mathbb{E}^{\mathbb{S}}\left[S_0\mathbb{1}_{[S_T\geq K]}\right] = S_0\mathbb{P}^{\mathbb{S}}[S_T\geq K] \quad \text{and} \\
&S_T = S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma\tilde{B}_T} = S_0e^{\left(r-\frac{1}{2}\sigma^2\right)T+\sigma(B_T^{\mathbb{S}}+\sigma T)} = S_0e^{\left(r+\frac{1}{2}\sigma^2\right)T+\sigma B_T^{\mathbb{S}}}
\end{aligned}
$$

by change of measure. Then we have

$$
\begin{aligned}
S_T &\geq K \\
S_0e^{\left(r+\frac{1}{2}\sigma^2\right)T+\sigma B_T^{\mathbb{S}}} &\geq K \\
\sigma B_T^{\mathbb{S}} &\geq -\left(\ln\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)T\right) \\
-\frac{B_T^{\mathbb{S}}}{\sqrt{T}} &\leq \frac{1}{\sigma\sqrt{T}}\left(\ln\frac{S_0}{K}+\left(r+\frac{1}{2}\sigma^2\right)T\right)
\end{aligned}
$$

Therefore, $\mathbb{P}^{\mathbb{S}}[S_T\geq K] = N(d_+(T,S_0))$.

Therefore, we have arrived at $\tilde{\mathbb{E}}\left[e^{-rT}(S_T-K)^+\right] = S_0N(d_+(T,S_0)) - e^{-rT}KN(d_-(T,S_0))$, which is the Black-Scholes equation for European call.

Example 10.2.1 Use the change of measure $\Theta_t = -\sigma$ to solve $\mathbb{E}\left[e^{\sigma B_T-\frac{1}{2}\sigma^2 T}B_T^4\right]$.

Solution. Since $\Theta_t = -\sigma$, we have $Z_t = e^{\sigma B_T-\frac{1}{2}\sigma^T}$ and $\hat{B}_t = B_t - \sigma t \Rightarrow B_t = \hat{B}_t + \sigma t$. Then we can calculate

$$
\begin{aligned}
B_T^4 &= \left(\hat{B}_t+\sigma t\right)^4 \\
&= \binom{4}{0}(\hat{B}_T)^4(\sigma T)^0 + \binom{4}{1}(\hat{B}_T)^3(\sigma T)^1 + \binom{4}{2}(\hat{B}_T)^2(\sigma T)^2 + \binom{4}{3}(\hat{B}_T)^1(\sigma T)^3 + \binom{4}{4}(\hat{B}_T)^0(\sigma T)^4 \\
&= \hat{B}_T^4 + 4\sigma T\hat{B}_T^3 + 6\sigma^2T^2\hat{B}_T^2 + 4\sigma^3T^3\hat{B}_T + \sigma^4T^4 \\
\mathbb{E}\left[e^{\sigma B_T-\frac{1}{2}\sigma^2 T}B_T^4\right] &= \hat{\mathbb{E}}\left[B_T^4\right] \quad \text{by change of measure} \\
&= \hat{\mathbb{E}}\left[\hat{B}_T^4 + 4\sigma T\hat{B}_T^3 + 6\sigma^2T^2\hat{B}_T^2 + 4\sigma^3T^3\hat{B}_T + \sigma^4T^4\right] \\
&= \hat{\mathbb{E}}\left[\hat{B}_T^4\right] + 6\sigma^2T^2\hat{\mathbb{E}}\left[\hat{B}_T^2\right] + \sigma^4T^4 \quad \text{since } \hat{\mathbb{E}}\left[\hat{B}_T\right] = 0 \text{ and } \hat{\mathbb{E}}\left[\hat{B}_T^3\right] = 0 \\
&= 3T^2 + 6\sigma^2T^3 + \sigma^4T^4 \quad \text{since } \hat{\mathbb{E}}\left[\hat{B}_T^2\right] = T \text{ and } \hat{\mathbb{E}}\left[\hat{B}_T^4\right] = 3T^2
\end{aligned}
$$

Remark 10.2.2 Moments of Brownian Motion B_t:

(1) Mean: $\mathbb{E}[B_t] = 0$.

(2) Second Moment: $\mathbb{E}[B_t^2] = t$.

(3) Third Moment: $\mathbb{E}[B_t^3] = 0$.

(4) Fourth Moment: $\mathbb{E}[B_t^4] = 3t^2$.

10.3 Girsanov's Change of Measure (d-dim) in Currencies

CAD\$/USD\$

EUR€/USD\$

Consider the Brownian Motions

- $(\tilde{B}_t^{(1)}, \tilde{B}_t^{(2)}, \ldots, \tilde{B}_t^{(d)})$ under risk-neutral measure $\tilde{\mathbb{P}}$ in the American market (\$), and
- $(\hat{B}_t^{(1)}, \hat{B}_t^{(2)}, \ldots, \hat{B}_t^{(d)})$ under risk-neutral measure $\tilde{\mathbb{P}}$ in the European market (€).

Changing from \$ to €, we have

$$
\begin{aligned}
d\hat{B}_t &= \begin{pmatrix} d\hat{B}_t^{(1)} \\ \vdots \\ d\hat{B}_t^{(d)} \end{pmatrix}, \quad d\hat{B}_t = \begin{pmatrix} \hat{B}_t^{(1)} \\ \vdots \\ \hat{B}_t^{(d)} \end{pmatrix}, \quad \Theta_t = \begin{pmatrix} \Theta_t^{(1)} \\ \vdots \\ \Theta_t^{(d)} \end{pmatrix} \\
d\hat{B}_t &= d\tilde{B}_t + \Theta_t dt \\
Z_t &= e^{-\int_0^t \Theta_u^T \, dB_u - \frac{1}{2}\int_0^t \|\Theta_u\|^2 \, du}
\end{aligned}
$$

Let $\{S_t\}_{t\geq 0}$ and $\{N_t\}_{t\geq 0}$ be the prices of two assets determined in the same currency,

$$
\begin{aligned}
dS_t &= R_t S_t dt + \sigma_t^{(1)} S_t d\tilde{B}_t^{(1)} + \sigma_t^{(2)} S_t d\tilde{B}_t^{(2)} + \cdots + \sigma_t^{(d)} S_t d\tilde{B}_t^{(d)} \quad \text{, and} \\
dN_t &= R_t N_t dt + \nu_t^{(1)} N_t d\tilde{B}_t^{(1)} + \nu_t^{(2)} N_t d\tilde{B}_t^{(2)} + \cdots + \nu_t^{(d)} N_t d\tilde{B}_t^{(d)},
\end{aligned}
$$

where the discount factor is $D_t = e^{-\int_0^t R_u \, du}$. For $\sigma_t = (\sigma_t^{(1)}, \ldots, \sigma_t^{(d)})$ and $\nu_t = (\nu_t^{(1)}, \ldots, \nu_t^{(d)})$, we define

$$
\begin{aligned}
\sigma_t d\tilde{B}_t &= \sum_{i=1}^d \sigma_t^{(i)} d\tilde{B}_t \quad \text{and} \quad \nu_t d\tilde{B}_t = \sum_{i=1}^d \nu_t^{(i)} d\tilde{B}_t \\
dD_t S_t &= D_t S_t \sigma_t d\tilde{B}_t \quad \text{and} \quad dD_t N_t = D_t N_t \nu_t d\tilde{B}_t
\end{aligned}
$$

Here, we take N_t as the numéraire.

To verify, we write $D_t S_t = D_0 S_0 e^{\int_0^t \sigma_u d\tilde{B}_u - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)})^2 du} = D_0 S_0 e^{\sum_{i=1}^d \int_0^t \sigma_u^{(i)} d\tilde{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)})^2 du}$.

By letting $f(t) = \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)})^2 du$ and $M_t = \sum_{i=1}^d \int_0^t \sigma_u^{(i)} d\tilde{B}_u^{(i)} - f(t)$, we calculate

$$
\begin{aligned}
df(t) &= \frac{1}{2}\sum_{i=1}^d (\sigma_t^{(i)})^2 dt \\
dM_t &= \sum_{i=1}^d \sigma_t^{(i)} d\tilde{B}_t^{(i)} - \frac{1}{2}\sum_{i=1}^d (\sigma_t^{(i)})^2 dt \\
dD_t S_t &= D_t S_t \left(dM_t + \frac{1}{2} d[M]_t \right) \\
&= D_t S_t \left(\sum_{i=1}^d \sigma_t^{(i)} d\tilde{B}_t^{(i)} - \frac{1}{2}\sum_{i=1}^d (\sigma_t^{(i)})^2 dt + \frac{1}{2}\sum_{i=1}^d (\sigma_t^{(i)})^2 dt \right) \\
&= D_t S_t \sum_{i=1}^d \sigma_t^{(i)} d\tilde{B}_t^{(i)}
\end{aligned}
$$

To continue the previous steps, we have

$$
\begin{cases}
D_t S_t = D_0 S_0 e^{\sum_{i=1}^d \int_0^t \sigma_u^{(i)} d\tilde{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)})^2 du} \\
D_t S_t = D_0 S_0 e^{\sum_{i=1}^d \int_0^t \nu_u^{(i)} d\tilde{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d (\nu_u^{(i)})^2 du}
\end{cases}
$$

$$\begin{aligned}\frac{D_t S_t}{D_t N_t} = \frac{S_t}{N_t} &= \frac{S_0}{N_0} e^{\sum_{i=1}^d \int_0^t (\sigma_u^{(i)} - \nu_u^{(i)}) d\tilde{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d ((\sigma_u^{(i)})^2 du - (\nu_u^{(i)})^2) du} \\ &= \frac{S_0}{N_0} e^{\sum_{i=1}^d \int_0^t (\sigma_u^{(i)} - \nu_u^{(i)}) d\tilde{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)} - \nu_u^{(i)})^2 du - \int_0^t \sum_{i=1}^d (\sigma_u^{(i)} - \nu_u^{(i)}) \nu_u^{(i)} du} \\ &= \frac{S_0}{N_0} e^{\sum_{i=1}^d \int_0^t (\sigma_u^{(i)} - \nu_u^{(i)}) (d\tilde{B}_u^{(i)} - \nu_u^{(i)} du) - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)} - \nu_u^{(i)})^2 du}\end{aligned}$$

since $(\sigma_u^{(i)} - \nu_u^{(i)})(\sigma_u^{(i)} + \nu_u^{(i)}) - (\sigma_u^{(i)} - \nu_u^{(i)})^2 = 2\nu_u^{(i)}(\sigma_u^{(i)} - \nu_u^{(i)})$.

By d-dim Girsanov's Theorem, the change of measure from \$ to €is

$$\begin{aligned} d\hat{B}_t^{(i)} &= d\tilde{B}_t^{(i)} - \nu_t^{(i)} dt \\ \frac{S_t}{N_t} &= \frac{S_0}{N_0} e^{\sum_{i=1}^d \int_0^t (\sigma_u^{(i)} - \nu_u^{(i)}) d\hat{B}_u^{(i)} - \frac{1}{2}\int_0^t \sum_{i=1}^d (\sigma_u^{(i)} - \nu_u^{(i)})^2 du} \\ d\frac{S_t}{N_t} &= \frac{S_t}{N_t} \sum_{i=1}^d (\sigma_t^{(i)} - \nu_t^{(i)}) d\hat{B}_t^{(i)} \end{aligned}$$

Example 10.3.1 Suppose

- $N_0 = 1$ in the bank account at time 0, and
- N_t in the bank account at time t.

Then we have $N_t = e^{\int_0^t R_u du} = \frac{1}{D_t} \Rightarrow D_t N_t = 1$.

Using bank account as numéraire, we have that $\frac{S_t}{N_t} = D_t S_t$ is a martingale under risk-neutral measure.

Remark 10.3.2 Quotient of martingale is a martingale with respect to the measure you change to using the denominator and is normalized to start at 1 as the Radon-Nikodym derivative.

In our case, $\{D_t S_t\}_{t \geq 0}$ and $\{D_t N_t\}_{t \geq 0}$ are martingales under risk-neutral measure.

Remark 10.3.3 Volatility subtracts component-by-component for each $\tilde{B}_t^{(i)}$.

In our case, rrom $\tilde{\mathbb{P}}$ to $\hat{\mathbb{P}}$, we have

$$\begin{aligned} d\hat{B}_t^{(i)} &= d\tilde{B}_t^{(i)} - \nu_t^{(i)} dt \\ \hat{\mathbb{E}}[X] &= \tilde{\mathbb{E}}\left[\frac{D_T N_T}{N_0} X\right] \quad \text{focusing on } [0, T] \\ d\frac{S_t}{N_t} &= \frac{S_t}{N_t} \sum_{i=1}^d (\sigma_t^{(i)} - \nu_t^{(i)}) d\hat{B}_t^{(i)} \end{aligned}$$

10.4 Foreign Exchange Rate

Define variables:

- S_t: Stock Price in USD\$.
- X_t: Price of €1 in USD\$.
- R_t: Interest rate of the US market.
- $R_t^{€}$: Interest rate of the Euro market.

€1 in Euro bank $\longrightarrow$ \$$\frac{X_t}{D_t^{€}}$ asset in US market, where €$\frac{1}{D_t^{€}} =$ €$e^{\int_0^t R_u^{€}} du$.

By definition of risk-neutral measure, in the US market, the discounted asset price process (using $D_t = e^{-\int_0^t R_u du}$) is a martingale, that is, $\{\frac{D_t}{D_t^{€}} X_t\}_{0\le t\le T}$ is a martingale.

Suppose we have

$$\begin{cases} dX_t = \gamma_t X_t dt + X_t \sigma_t^X \left(\rho_t d\tilde{B}_t^{(1)} + \sqrt{1-\rho_t^2} d\tilde{B}_t^{(2)}\right) \\ dS_t = R_t S_t dt + S_t \sigma_t^S d\tilde{B}_t^{(1)} \end{cases} \tag{$*$}$$

1. Focusing on X_t, what is γ_t?

   Since $\{\frac{D_t}{D_t^{€}} X_t\}_{0\le t\le T}$ is a martingale, $d\frac{D_t}{D_t^{€}} X_t$ will only contain $d\tilde{B}_t^{(1)}$ and $d\tilde{B}_t^{(2)}$. We have

$$\begin{aligned} \frac{D_t}{D_t^{€}} X_t &= X_t e^{\int_0^t (R_u^{€} - R_u) du} \\ d\frac{D_t}{D_t^{€}} X_t &= e^{\int_0^t (R_u^{€} - R_u) du} dX_t + X_t e^{\int_0^t (R_u^{€} - R_u) du} (R_t^{€} - R_t) dt + 0 \\ &= X_t e^{\int_0^t (R_u^{€} - R_u) du} \left[\left(\gamma_t + R_t^{€} - R_t\right) dt + \sigma_t^X \left(\rho_t d\tilde{B}_t^{(1)} + \sqrt{1-\rho_t^2} d\tilde{B}_t^{(2)}\right)\right] \end{aligned}$$

   Setting the drift term to zero, we get $\gamma_t = R_t - R_t^{€}$.

2. Change to € risk-neutral measure, that is, $\frac{S_t}{X_t}$.

   Consider martingales $D_t S_t$ and $\frac{D_t}{D_t^{€}} X_t$ under \$ risk-neutral measure, taking quotient, we have that $\frac{D_t S_t}{\frac{D_t}{D_t^{€}} X_t} = \frac{D_t^{€} S_t}{X_t}$ is a martingale under € risk-neutral measure. Then we define

$$\begin{aligned} & d\hat{B}_t^{(1)} = d\tilde{B}_t^{(1)} - \rho_t \sigma_t^X dt, \quad d\hat{B}_t^{(2)} = d\tilde{B}_t^{(2)} - \sqrt{1-\rho_t^2}\sigma_t^X dt \\ & d\left(\frac{S_t}{X_t}\right) = \frac{S_t}{X_t}\left(\frac{dS_t}{S_t} - \frac{dX_t}{X_t} + \frac{d[X,X]_t}{X_t^2} - \frac{d[S,X]_t}{S_t X_t}\right) \quad \text{by Itô's Quotient Rule} \\ & = \frac{S_t}{X_t}\left[R_t dt + \sigma_t^S d\tilde{B}_t^{(1)} - \gamma_t dt - \sigma_t^X\left(\rho_t d\tilde{B}_t^{(1)} + \sqrt{1-\rho_t^2} d\tilde{B}_t^{(2)}\right) + \frac{X_t^2 (\sigma_t^X)^2 dt}{X_t^2} - \frac{S_t X_t \sigma_t^S \sigma_t^X \rho_t dt}{S_t X_t}\right] \\ & = \frac{S_t}{X_t}\left[R_t^{€} dt + \left(\sigma_t^S - \sigma_t^X \rho_t\right) d\tilde{B}_t^{(1)} - \sigma_t^X \sqrt{1-\rho_t^2} d\tilde{B}_t^{(2)} + \left(\sigma_t^X\right)^2 dt - \sigma_t^S \sigma_t^X \rho_t dt\right] \\ & = \frac{S_t}{X_t}\left[R_t^{€} dt + \left(\sigma_t^S - \sigma_t^X \rho_t\right)\left(d\hat{B}_t^{(1)} + \rho_t \sigma_t^X dt\right) - \sigma_t^X \sqrt{1-\rho_t^2}\left(d\hat{B}_t^{(2)} + \sqrt{1-\rho_t^2}\sigma_t^X dt\right) + \left(\sigma_t^X\right)^2 dt - \sigma_t^S \sigma_t^X \rho_t dt\right] \\ & = \frac{S_t}{X_t}\left[R_t^{€} + \sigma_t^S \sigma_t^X \rho_t - \left(\sigma_t^X\right)^2 \rho_t^2 - \left(\sigma_t^X\right)^2 \left(1-\rho_t^2\right) + \left(\sigma_t^X\right)^2 - \sigma_t^S \sigma_t^X \rho_t\right] dt \\ & \quad + \frac{S_t}{X_t}\left[\left(\sigma_t^S - \sigma_t^X \rho_t\right) d\hat{B}_t^{(1)} - \sigma_t^X \sqrt{1-\rho_t^2} d\hat{B}_t^{(2)}\right] \\ & = R_t^{€} \frac{S_t}{X_t} dt + \frac{S_t}{X_t}\left[\left(\sigma_t^S - \sigma_t^X \rho_t\right) d\hat{B}_t^{(1)} - \sigma_t^X \sqrt{1-\rho_t^2} d\hat{B}_t^{(2)}\right]. \end{aligned}$$

3. $f(x) = \frac{1}{x}$, $f'(x) = -\frac{1}{x^2}$, $f''(x) = \frac{2}{x^3}$. Under $\hat{\mathbb{P}}^{€}$-measure, we have

$$\begin{aligned} d\left(\frac{1}{X_t}\right) &= -\frac{1}{X_t^2} dX_t + \frac{1}{2}\frac{2}{X_t^3}\left(dX_t\right)^2 \\ &= -\frac{1}{X_t^2}\left(\gamma_t X_t dt + X_t \sigma_t^X\left(\rho_t d\tilde{B}_t^{(1)} + \sqrt{1-\rho_t^2} d\tilde{B}_t^{(2)}\right)\right) + \frac{1}{X_t^3} X_t^2 \left(\sigma_t^X\right)^2 dt \\ &= \frac{1}{X_t}\left(R_t^{€} - R_t\right) dt + \frac{\sigma_t^X}{X_t}\left(\rho_t\left(d\hat{B}_t^{(1)} + \rho_t \sigma_t^X dt\right) + \sqrt{1-\rho_t^2}\left(d\hat{B}_t^{(2)} + \sqrt{1-\rho_t^2}\sigma_t^X dt\right) + \left(\sigma_t^X\right)^2 dt\right) \\ &= \frac{1}{X_t}\left(R_t^{€} - R_t\right) dt + \frac{\sigma_t^X}{X_t}\left(\rho_t d\hat{B}_t^{(1)} + \sqrt{1-\rho_t^2} d\hat{B}_t^{(2)}\right) \end{aligned}$$

Note that under $\tilde{\mathbb{P}}^{\$}$-measure, we have $dX_t = X_t(R_t - R_t^{€})dt + X_t\sigma_t^X \left(\rho_t d\tilde{B}_t^{(1)} + \sqrt{1-\rho_t^2}d\tilde{B}_t^{(2)}\right)$ from $(*)$.

4. Useful table:

Currency	**US Bank**	**Stock**	**Euro Bank**
\$ risk-neutral measure, $\tilde{B}_t$ Martingale, D_t	Time 0: \$1 Time t: $\frac{1}{D_t}$ $\{\$1\}_{0\le t\le T}$ is a martingale	Time 0: S_0 Time t: S_t $\{D_t S_t\}_{0\le t\le T}$ is a martingale	Time 0: X_0 Time t: $\frac{X_t}{D_t^{€}}$ $\{\frac{D_t}{D_t^{€}} X_t\}_{0\le t\le T}$ is a martingale
€ risk-neutral measure, $\hat{B}_t$ Martingale, $D_t^{€}$	Time 0: $\frac{1}{X_0}$ Time t: $\frac{1}{X_t D_t}$ $\{\frac{D_t^{€}}{D_t}\frac{1}{X_t}\}_{0\le t\le T}$ is a martingale	Time 0: $\frac{S_0}{X_0}$ Time t: $\frac{S_t}{X_t}$ $\{\frac{D_t^{€} S_t}{X_t}\}_{0\le t\le T}$ is a martingale	Time 0: €1 Time t: $\frac{€1}{D_t^{€}}$ $\{€1\}_{0\le t\le T}$ is a martingale

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