MMF1928 PRICING THEORY PROJECT 2 REPORT

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Abstract. This report explores delta-gamma hedging strategies within the Black-Scholes framework. Using a portfolio consisting of an at-the-money call option, the underlying asset, and a hedging option, we analyze the effectiveness of delta-gamma hedging in mitigating risk. Transaction costs are incorporated into the model to account for real-world trading frictions. The study employs a Monte Carlo simulation approach with 5,000 paths to evaluate profit and loss distributions under delta and delta-gamma hedging strategies, considering varying levels of the drift parameter μ. Additionally, we compare hedging performance for in-the-money and out-of-the-money asset paths, ensuring consistency across simulations. Finally, we examine the impact of misestimating real-world volatility on hedging outcomes, exploring scenarios where the actual volatility deviates from the assumed value. The results provide insights into the robustness and efficiency of delta-gamma hedging, particularly under varying market conditions and model assumptions.

1. Introduction

In this project, we will investigate delta-gamma hedging within the Black-Scholes model. Here is a link for reference: Delta-Gamma Hedging on Investopedia.

Assume that an asset price process $\{S_t\}_{t\geq 0}$ follows the Black-Scholes model. The asset's current price is \$10. We have sold 10,000 units of an at-the-money $\frac{1}{4}$ year (i.e., 63 days) call on this asset, and we wish to hedge it. Call this option g.

We may trade in an at-the-money call with a maturity of 0.3 year (call this option h), the stock, and the bank account. We will take transaction costs into account by assuming we are charged \$0.005 per share on equity transactions and \$0.005 per option on option transactions. We may only trade integer value of stocks and options.

The remaining model parameters are

$$\mu = 10\%, \quad \sigma = 25\%, \quad r = 5\%$$

and we hedge daily.

2. Methodology

2.1. **Delta Neutrality.** First, we do delta hedging with respect to selling the at-the-money 0.25 year call option g. Since the option is at-the-money, then $K_g = S_0 = \$10$. We have the following components:

(1) The SDE for the asset price process $\{S_t\}_{t\geq 0}$ following the Black-Scholes model is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t,$$

where W_t is a standard Brownian Motion under the real-world probability measure $\mathbb{P}$.

(2) The SDE for the bank account process $\{B_t\}_{t\geq 0}$ is given by

$$dB_t = rB_t dt \Longleftrightarrow B_t = B_0 e^{rt}, \quad B_0 = 1 \Longleftrightarrow B_{t+1} = B_t e^{r\Delta_t}$$

(3) The SDE for the option process $\{g_t\}_{t\geq 0}$ on the underlying asset S_t following the Black-Scholes model is given by

$$dg_t = \mu_t^g g_t dt + \sigma_t^g g_t dW_t = rg_t dt + \sigma g_t d\tilde{W}_t,$$

where $\tilde{W}_t$ is a standard Brownian Motion under the risk-neutral measure $\tilde{\mathbb{P}}$.

Now let us define our portfolio. Assume we hold α units in S, β units in B, and -10,000 units in g since we have sold 10,000 units of the at-the-money 0.25 year call option on the underlying asset at time $t = 0$. Let $\theta_t = (\alpha_t, \beta_t, -10,000)$. Then our portfolio is

$$V_t^\theta = \alpha_t S_t + \beta_t B_t - 10,000 g_t$$

We aim to construct a "delta-neutral" portfolio to hedge the risk of shorting the option by taking a position in the underlying asset and bank account. The idea is to approximate the option price at any time by a linear function using Taylor's series:

$$g(t, S_t + \Delta S_t) \approx g(t, S_t) + \frac{\partial g(t, S_t)}{\partial S_t}\Delta S_t$$

Delta of the option g is given by

$$\Delta^g = \frac{\partial g(t, S_t)}{\partial S_t} = g_s(t, s)|_{s=S_t}$$

Delta of the portfolio is

$$\Delta^V = \alpha_t \Delta^S + \beta_t \Delta^B - 10,000\Delta^g,$$

where $\Delta^S = \frac{\partial S_t}{\partial S_t} = 1$ and $\Delta^B = 0$, since the bank account is not affected by the underlying asset.

Setting the portfolio delta to zero, we have

$$\Delta^V = \alpha_t \cdot 1 + \beta_t \cdot 0 - 10,000\Delta^g = 0 \Rightarrow \alpha_t^* = 10,000\Delta^g,$$

where $\Delta = N(d_+)$, $d_+(t, S_t) = \frac{\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, and $N(\cdot)$ is the cumulative distribution function of standard normal distribution, since the underlying asset price follows the Black-Scholes model.

2.2. **Delta Hedging.** With knowing that

$$\alpha_t^* = 10,000\Delta^g$$

the *delta hedging* evolution of the portfolio is as follows:

At $t = t_0$:

We sell 10,000 units of option g_{t_0} and receive $10,000 g_{t_0}$.

We buy α_{t_0} units of stock S_{t_0} and pay $\alpha_{t_0} S_{t_0}$.

Subtracting the initial transaction cost, the initial bank account position is

$$B_{t_0} = 10,000 g_{t_0} - \alpha_{t_0} S_{t_0} - \alpha_{t_0}\$0.005$$

At $t = t_1$:

The rebalanced bank account positions is

$$(10,000g_{t_0} - \alpha_{t_0}S_{t_0} - \alpha_{t_0}\$0.005)e^{r(t_1-t_0)} - (\alpha_{t_1} - \alpha_{t_0})S_{t_1} - |\alpha_{t_1} - \alpha_{t_0}|\$0.005$$

At $t = t_n$ (Terminal Value):

The value of holding in the stock is $\alpha_{t_{n-1}}S_{t_n}$.

We owe the option payoff $10,000g_{t_n}$.

The bank account position is $B_{t_n} = B_{t_{n-1}}e^{r(t_n-t_{n-1})}$.

Subtracting the final transaction cost, the final portfolio value (profit and loss) is

$$\begin{aligned} &B_{t_n} + \alpha_{t_{n-1}}S_{t_n} - 10,000g_{t_n} - \alpha_{t_{n-1}}\$0.005 \\ &= B_{t_{n-1}}e^{r(t_n-t_{n-1})} + \alpha_{t_{n-1}}S_{t_n} - 10,000g_{t_n} - \alpha_{t_{n-1}}\$0.005 \end{aligned}$$

2.3. **Delta and Gamma Neutrality.** Then, we do delta-gamma hedging by adding the consideration on purchasing the at-the-money 0.3 year call option h, the stock, and the bank account.

The second-order Taylor series is

$$g(t, S_t + \Delta S_t) \approx g(t, S_t) + \frac{\partial g(t, S_t)}{\partial S_t}\Delta S_t + \frac{1}{2}\frac{\partial^2 g(t, S_t)}{\partial S_t^2}(\Delta S_t)^2$$

Then the gamma of an option g is given by

$$\Gamma^g = \frac{\partial^2 g(t, S_t)}{\partial S_t^2} = g_{SS}(t, S_t)$$

It is the second-order partial derivative of option price with respect to the underlying asset, which is equivalent to

$$\Gamma^g = \frac{\partial(\partial g(t, S_t))}{\partial S_t^2} = \frac{\partial \Delta^g}{\partial S_t}$$

We aim to construct a "delta-gamma neutral" portfolio to hedge against second-order changes in the option price. Our original portfolio and the gamma of the original portfolio are

$$\begin{aligned} V_t^\theta &= \alpha_t S_t + \beta_t B_t - 10,000g_t \\ \Gamma^V &= \alpha_t \Gamma^S + \beta_t \Gamma^B - 10,000\Gamma^g \\ &= \alpha_t \cdot 0 + \beta_t \cdot 0 - 10,000\Gamma^g \\ &= -10,000\Gamma^g \end{aligned}$$

because $\Gamma^S = \frac{\partial^2 S_t}{\partial S_t^2} = 0$ and $\Gamma^B = 0$.

Since $\Gamma^V = -10,000\Gamma^g$, we do not have gamma-neutrality with only S, B, and g in the portfolio. Hence, we introduce a second option $h(t, S_t)$ to the portfolio. Now our portfolio holds $\theta_t = (\alpha_t, \beta_t, -10,000, \eta_t)$ in (S_t, B_t, g_t, h_t). The current portfolio becomes

$$\begin{aligned} V_t^\theta &= \alpha_t S_t + \beta_t B_t - 10,000g_t + \eta_t h_t \\ \Delta^V &= \alpha_t \cdot 1 + \beta_t \cdot 0 - 10,000\Delta^g + \eta_t \Delta^h \end{aligned}$$

To arrive at delta-neutral, setting the portfolio delta equal to zero, we have

$$\Delta^V = \alpha_t \cdot 1 + \beta_t \cdot 0 - 10,000\Delta^g + \eta_t \Delta^h = 0 \Rightarrow \alpha_t + \eta_t \Delta^h = 10,000\Delta^g \quad (*)$$

We get another equation from gamma-neutrality, which is

$$\begin{aligned}\Gamma^V &= \alpha_t \Gamma^S + \beta_t \Gamma^B - 10,000\Gamma^g + \eta_t \Gamma^h \\ &= \alpha_t \cdot 0 + \beta_t \cdot 0 - 10,000\Gamma^g + \eta_t \Gamma^h\end{aligned}$$

To arrive at gamma-neutral, setting the portfolio gamma equal to zero, we have

$$\begin{aligned}\Gamma^V = \alpha_t \cdot 0 + \beta_t \cdot 0 - 10,000\Gamma^g + \eta_t \Gamma^h = 0 &\Rightarrow \eta_t \Gamma^h = 10,000\Gamma^g \\ &\Rightarrow \eta_t^* = \frac{10,000\Gamma^g}{\Gamma^h}\end{aligned}$$

Substituting into $(*)$, we get

$$\alpha_t^* = 10,000\Delta^g - \frac{10,000\Gamma^g}{\Gamma^h}\Delta^h,$$

where $\Delta = N(d_+)$ and $\Gamma(t,x) = \frac{1}{x\sigma\sqrt{T-t}} N(d_+)$, $d_+(t, S_t) = \frac{\ln\frac{S_t}{K} + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}$, and $N(\cdot)$ is the cumulative distribution function of the standard normal distribution, since the underlying asset price follows the Black-Scholes model.

2.4. **Delta-Gamma Hedging.** With knowing that

$$\eta_t^* = \frac{10,000\Gamma^g}{\Gamma^h},$$

and

$$\alpha_t^* = 10,000\Delta^g - \frac{10,000\Gamma^g}{\Gamma^h}\Delta^h$$

the *delta-gamma hedging* evolution of the portfolio is as follows:

At $t = t_0$:

We sell 10,000 units of option g_{t_0} and receive $10,000g_{t_0}$.

We buy α_{t_0} units of stock S_{t_0} and pay $\alpha_{t_0}S_{t_0}$.

We buy η_{t_0} units of option h_{t_0} and pay $\eta_{t_0}h_{t_9}$.

Subtracting the initial transaction cost, the initial bank account position is

$$B_{t_0} = 10,000g_{t_0} - \alpha_{t_0}S_{t_0} - \eta_{t_0}h_{t_0} - (\alpha_{t_0} + \eta_{t_0})\$0.005$$

At $t = t_1$:

The rebalanced bank account positions is

$$\begin{aligned}&(10,000g_{t_0} - \alpha_{t_0}S_{t_0} - \eta_{t_0}h_{t_0} - (\alpha_{t_0} + \eta_{t_0})\$0.005)e^{r(t_1-t_0)} \\ &- (\alpha_{t_1} - \alpha_{t_0})S_{t_1} - (\eta_{t_1} - \eta_{t_0})h_{t_1} \\ &- (|\alpha_{t_1} - \alpha_{t_0}| + |\eta_{t_1} - \eta_{t_0}|)\,\$0.005\end{aligned}$$

At $t = t_n$ (Terminal Value):

The value of holding in the stock S is $\alpha_{t_{n-1}} S_{t_n}$.

We owe the option g payoff $10,000 g_{t_n}$.

The value of holding in the option h is $\eta_{t_{n-1}} h_{t_n}$

The bank account position is $B_{t_n} = B_{t_{n-1}} e^{r(t_n - t_{n-1})}$.

Subtracting the final transaction cost, the final portfolio value (profit and loss) is

$$\begin{aligned} &B_{t_n} + \alpha_{t_{n-1}} S_{t_n} - 10,000 g_{t_n} + \eta_{t_{n-1}} h_{t_n} - (\alpha_{t_{n-1}} + \eta_{t_{n-1}})\$0.005 \\ &= B_{t_{n-1}} e^{r(t_n - t_{n-1})} + \alpha_{t_{n-1}} S_{t_n} - 10,000 g_{t_n} + \eta_{t_{n-1}} h_{t_n} - (\alpha_{t_{n-1}} + \eta_{t_{n-1}})\$0.005 \end{aligned}$$

3. Results

3.1. **Question 1.** Compare the profit and loss distributions for both delta hedging and delta-gamma hedging, using 5,000 simulated paths. How do they vary as μ varies?

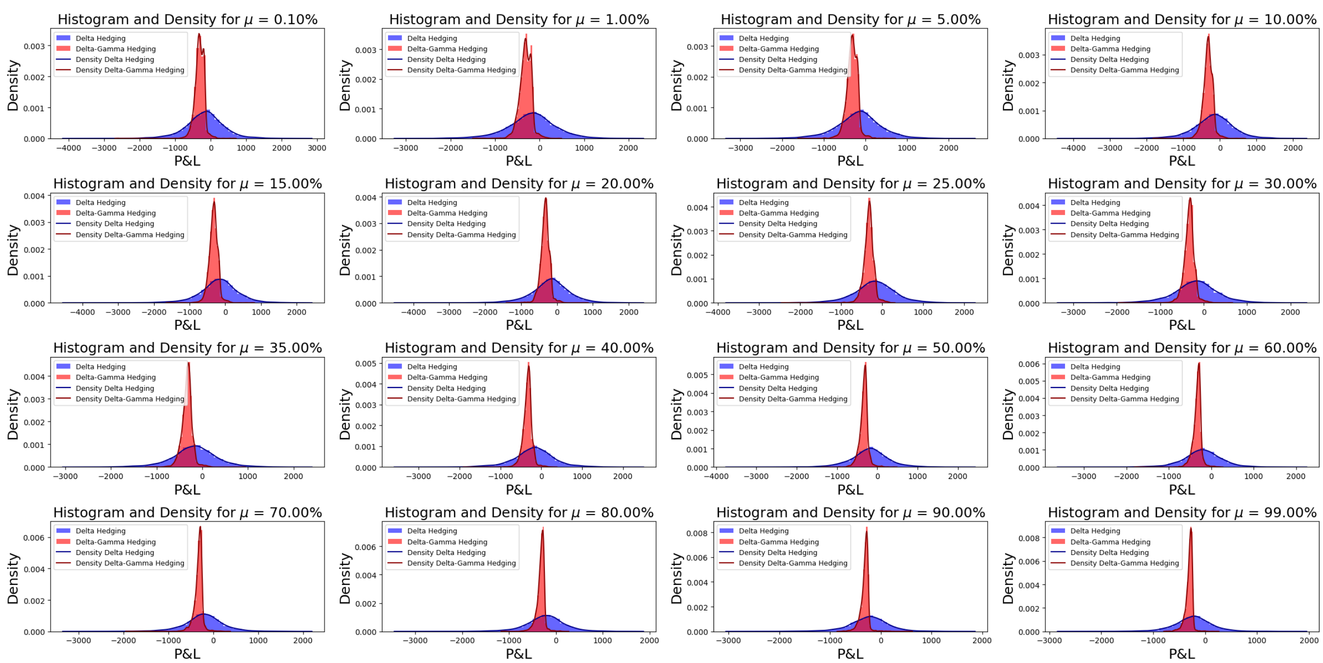

Figure 1. Histogram and Density Plots of Delta Hedging vs. Delta-Gamma Hedging Profit and Loss Distributions as μ varies for 0.001, 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99

From the histogram and density subplots across various drift rates (μ) in Figure 1, we compare the profit and loss distributions for delta and delta-gamma hedging. The profit and loss distributions for delta hedging exhibit wider tails, suggesting higher exposure to extreme losses, while the profit and loss distributions for delta-gamma hedging exhibit narrower tails, suggesting lower exposure to extreme losses and demonstrating a consistent ability to limit downside risk for different μ values. Overall, delta-gamma hedging consistently produces more concentrated distributions with reduced variability compared to delta hedging.

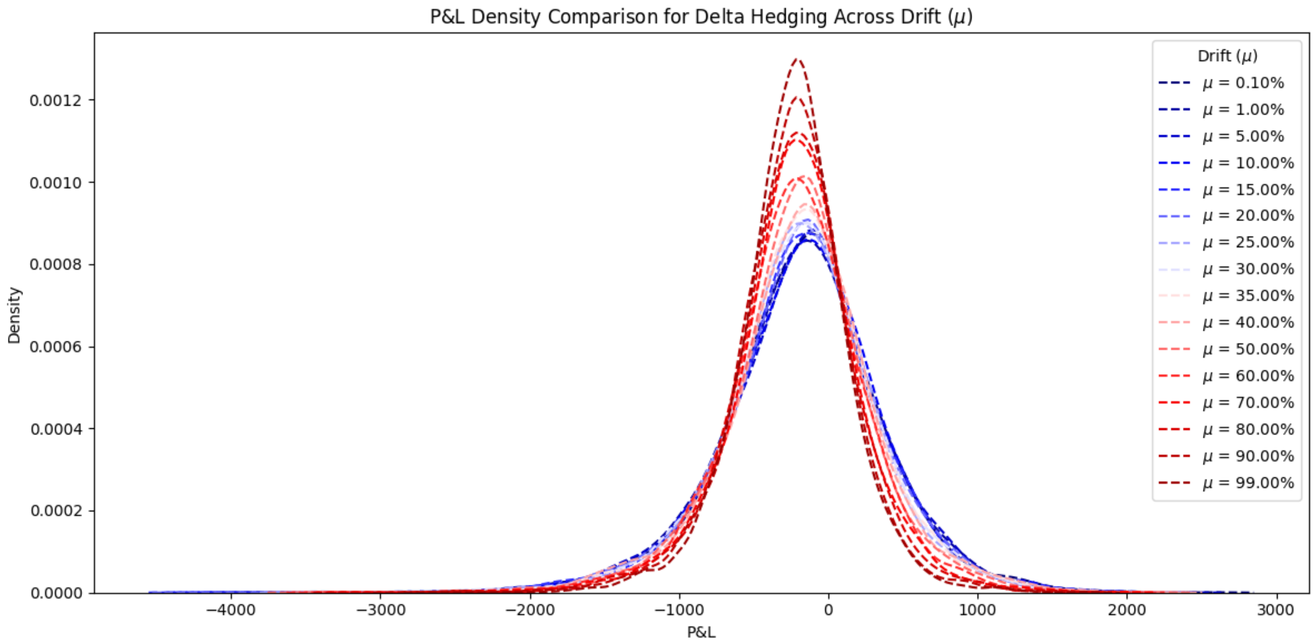

FIGURE 2. Density Plot of Delta Hedging Profit and Loss Distributions as μ varies for 0.001, 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99

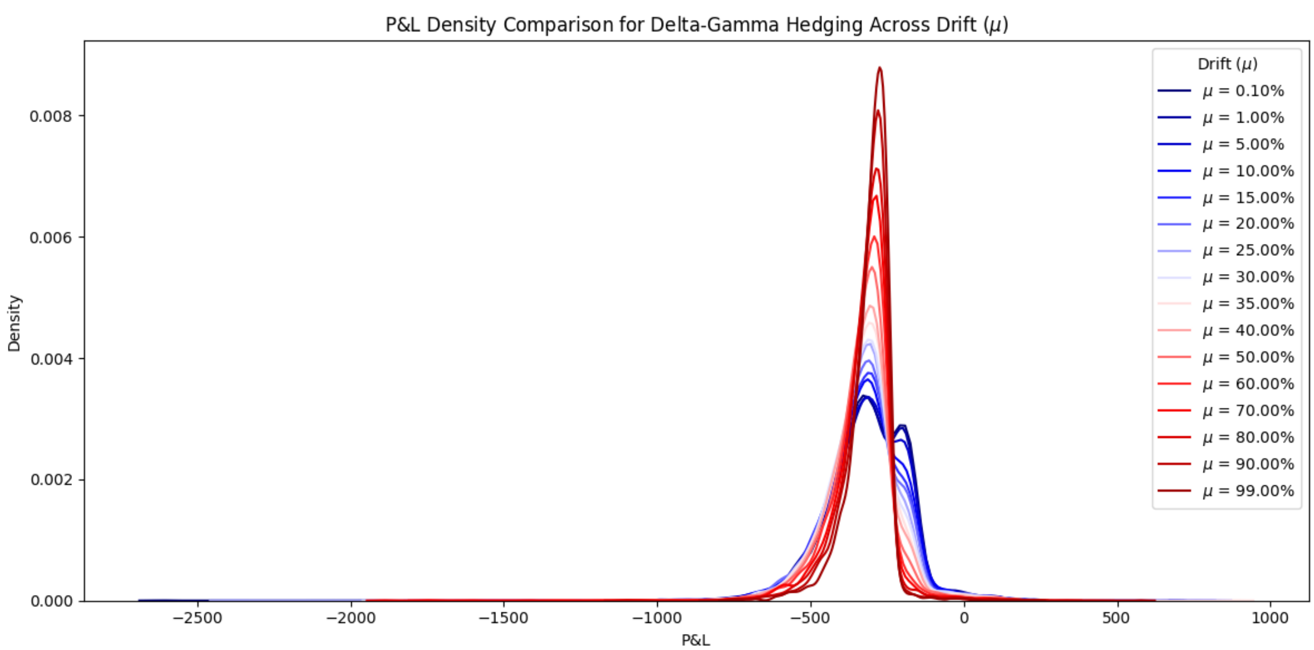

FIGURE 3. Density Plot of Delta-Gamma Hedging Profit and Loss Distributions as μ varies for 0.001, 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99

In Figure 2, we plot the density of profit and loss distributions using the delta hedging method across different μ values.

As μ increases, the peak of the profit and loss distributions becomes higher and the tails of the distributions move inward. This indicates that a larger proportion of outcome concentrate near the mean of the profit and loss distributions, reflecting reduced dispersion around the central tendency.

As μ increases, the mean of the profit and loss distributions shifts leftward, showing that the delta hedging strategy yields increasingly negative results in high-drift scenarios.

Combining these two observations that as μ increases the distribution becomes more concentrated around a mean that shifts leftward, we notice that while the variability in outcomes decreases, the strategy consistently underperforms, with the likelihood of more negative profit and loss outcomes increasing. The narrowing tails and higher peak highlight reduced risk of extreme outcomes, but the leftward shift in the mean emphasizes the strategy's inability to adapt effectively to higher drift environments, leading to systematically poorer results. This suggests that while the strategy becomes more predictable in high-drift scenarios, it does so at the expense of profitability.

In Figure 3, we plot the density of profit and loss distributions using the delta-gamma hedging method across different μ values.

As μ increases, the peak of the profit and loss distributions becomes higher and the tails becomes narrower, which is similar to what we observe in delta hedging. This indicates a growing concentration of outcomes around the mean of the distribution with less dispersion and a reduced likelihood of extreme profit and loss.

As μ increases, the mean of the profit and loss distributions exhibit a slight rightward shift, which is in contrast to delta hedging, showing that the delta-gamma hedging strategy yields decreasingly negative results with higher drifts.

These two effects together imply that the delta-gamma hedging strategy demonstrates increased predictability, reduced risk, and improved performance as μ increases, making delta-gamma hedging a superior strategy compared to delta hedging, especially in environments with higher drift.

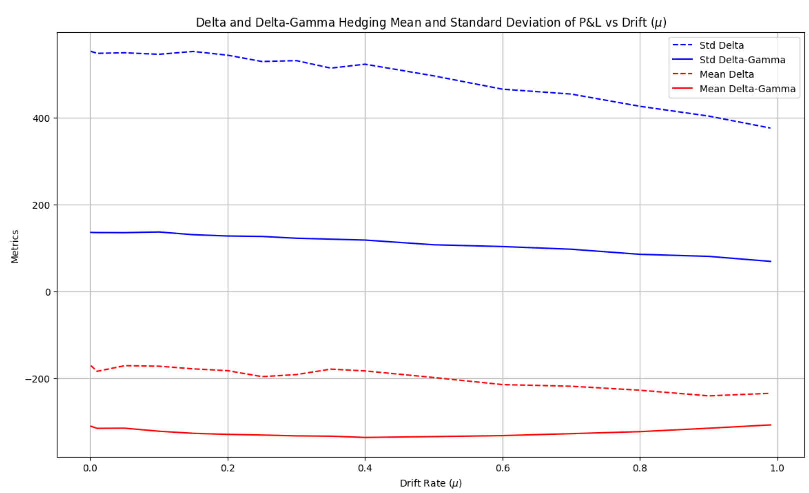

FIGURE 4. Mean and Standard Deviation for Delta and Delta-Gamma Hedging vs. Drift

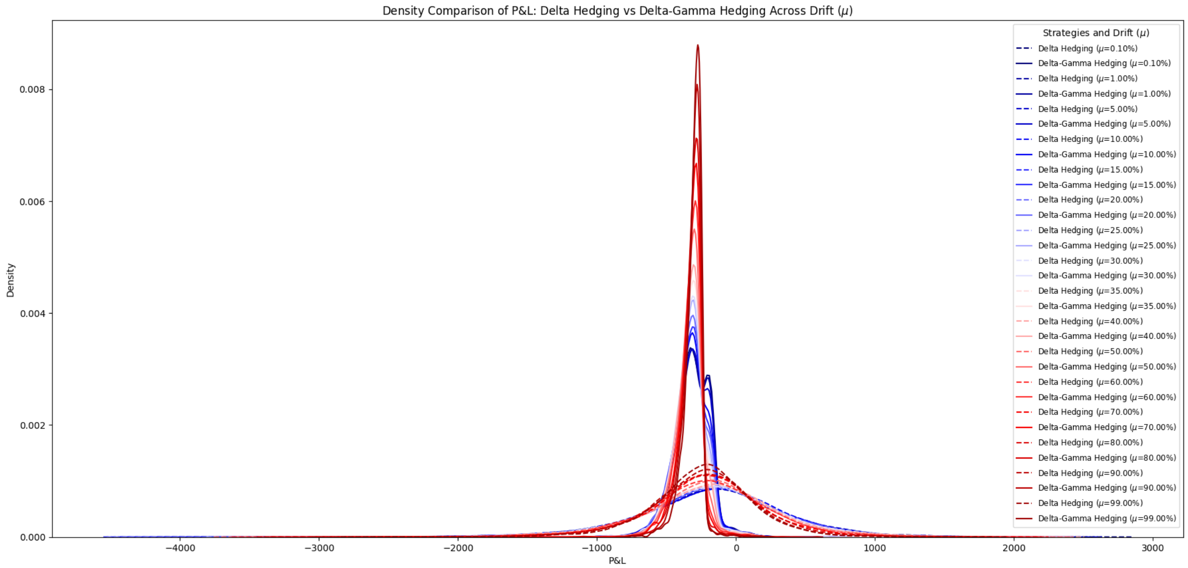

FIGURE 5. Density Plot of Delta Hedging vs. Delta-Gamma Hedging Profit and Loss Distributions as μ varies for 0.001, 0.01, 0.05, 0.1, 0.15, 0.2, 0.25, 0.3, 0.35, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 0.99

In Figure 5, we plot the density of profit and loss distributions using both the delta hedging method and the delta-gamma hedging method together across different μ values.

As μ increases, for both delta and delta-gamma hedging, the peaks of profit and loss distributions become sharper and taller, indicating a higher concentration of outcomes near the mean.

As μ increases, for both delta and delta-gamma hedging, the tails of profit and loss distributions become narrower, reflecting reduced likelihoods of extreme profits or losses.

As μ increases, for delta hedging (dashed lines), the means shifts leftward, signifying increasingly negative profit and loss outcomes in higher drift scenarios; for delta-gamma hedging (solid lines), the mean shifts slightly rightward, indicating improved profit and loss outcomes in higher drift scenarios. This can also be evidenced in Figure 4, where mean line of delta hedging keeps decreasing while mean line of delta-gamma hedging has a slight increase trend.

In terms of the overall shape, the profit and loss distributions for delta-gamma hedging exhibit less variability and are more concentrated around the mean compared to delta hedging at all μ values. This is evident from the narrower spread of the delta-gamma hedging distributions and from Figure 4, where the standard deviation line of delta hedging is above of delta-gamma hedging.

However, the observation that the mean profit and loss of delta-gamma hedging is consistently lower than that of delta hedging seems counterintuitive, given delta-gamma hedging's improved performance and superior stability. There are several reasons accounting for the lower mean of delta-gamma hedging, including dual-layer transaction costs and non-linearity. The lower mean profit and loss of delta-gamma hedging reflects the trade-off between stability and profitability. Delta-gamma hedging sacrifices some average returns to achieve significantly reduced variability and risk, making it a more robust strategy for risk-averse investors. Delta hedging, in contrast, allows for higher average returns but with greater risk and variability.

To conclude, delta-gamma hedging consistently outperforms delta hedging by reduced magnitude of negative outcomes in environments with higher drifts and also provides less variability and more stable profit and loss distributions compared to delta hedging. Therefore, delta-gamma hedging is a more robust strategy than delta hedging.

3.2. **Question 2.** Plot the position we hold in the asset and the hedging option (when delta-gamma hedging) for two sample paths: 1) one that ends in-the-money, and 2) one that ends out-of-the-money. Set the random number seed so that the asset sample paths when delta and delta-gamma hedging are the same so that we can compare them.

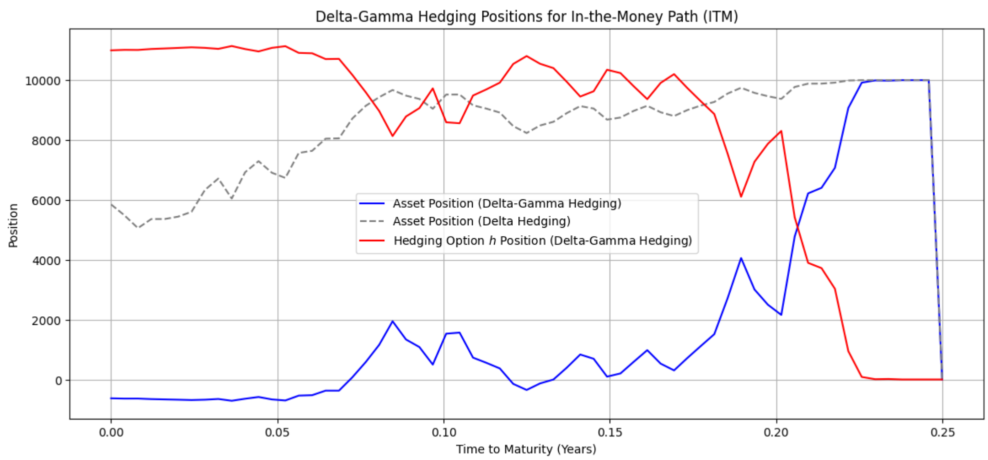

FIGURE 6. Delta-Gamma Hedging Positions for In-the-Money Path

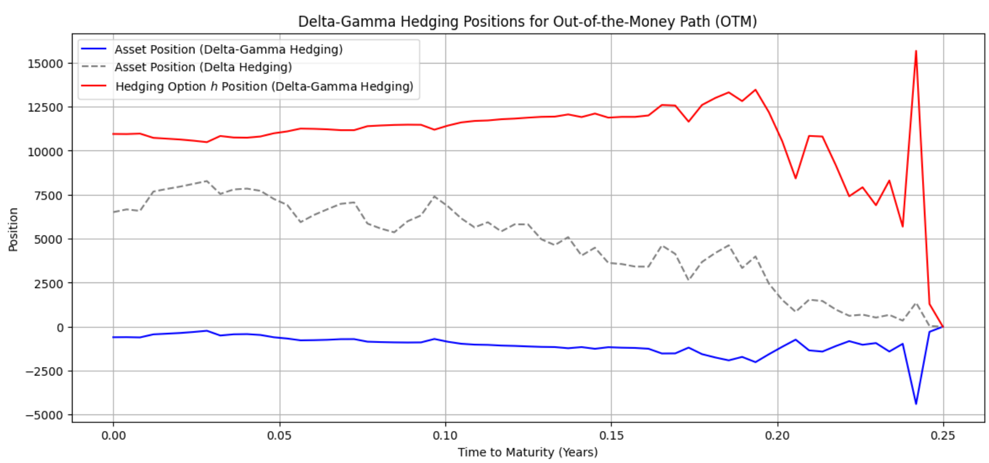

FIGURE 7. Delta-Gamma Hedging Positions for Out-of-the-Money Path

In Figure 6, we plot the position we hold in the asset position for delta hedging (gray dashed line) and the position we hold in the asset (blue line) and the hedging option h (red line) for delta-gamma hedging along a sample path that ends in-the-money (ITM), where the asset price at maturity exceeds the strike price ($S_T > K$). In Figure 7, we plot along a sample path that ends out-of-the-money (OTM), where the asset price at maturity is less than or equal to the strike price ($S_T \leq K$). The random seed for sample paths is set as 0 to ensure reproducibility.

Compare Delta and Delta-Gamma Hedging for Both ITM and OTM Paths

Since delta hedging relies solely on the underlying asset to neutralize first-order risk (delta risk) and does not involve additional instruments to manage higher-order risks (gamma risk), this strategy results in a relatively *stable* asset position with fewer adjustments. Also, as it fully accounts for the changing delta of the option g, the asset positions under delta hedging are consistently *higher* than those in delta-gamma hedging. However, this simplicity limits its effectiveness in scenarios with high gamma sensitivity or significant market volatility.

Delta-gamma hedging involves managing both delta and gamma risks by dynamically adjusting the position in the underlying asset and introducing positions in the hedging option h. The asset position line *fluctuates* more significantly compared to delta hedging, reflecting the need for continuous adjustments to maintain delta neutrality while offsetting the impact of gamma risk. Then, the hedging option h position shows a high initial value, gradually decreasing as the option approaches maturity. This reflects the declining gamma sensitivity as the option nears expiration, reducing the need for gamma hedging. Moreover, the number of asset positions in delta-gamma hedging is typically *smaller* compared to delta hedging because part of the gamma exposure is managed through the hedging option h. The complementary role of the hedging option reduces the reliance on the underlying asset for risk management. The asset position and the hedging option h positions work together to maintain delta and gamma neutrality. The combined strategy adapts dynamically to changing market conditions, particularly as the time to maturity shortens.

Differences in Delta Hedging Between ITM and OTM Paths

For ITM paths, delta increases gradually from approximately 0.5 to 1 and the number of asset positions for delta hedging increases gradually from approximately 5,000 to 10,000, as the option becomes deeper in-the-money closer to maturity. This results in higher asset positions as maturity approaches.

For OTM paths, delta decreases from approximately 0.5 to 0 and the number of asset positions for delta hedging decreases gradually from approximately 5,000 to 0, as the option becomes less likely to expire in-the-money. This results in a gradual reduction in asset positions over time.

Differences in Delta Hedging Between ITM and OTM Paths

For ITM paths, delta-gamma hedging exhibits fluctuations in both the asset and hedging option positions before nearing expiration. As gamma sensitivity decreases over time, the hedging option positions gradually declines from initially above 10,000 to 0 at expiration. Simultaneously, the asset position increases from initially below 0 to 10,000 to fully account for the delta as expiration approaches. Near expiration, the positions stabilize, with the asset positions converging to 10,000 and the hedging option positions converging to 0.

For OTM paths, delta-gamma hedging exhibits fluctuations in both positions before expiration, where the behavior differs due to the lower gamma sensitivity of OTM options. The asset position remains below and near to 0 throughout most of the option's life, reflecting the low delta of OTM options. However, there is a sudden decrease in the asset position near expiration as the option's value collapses to 0. The hedging option positions remain above and near to 10,000 for most of the option's life but also experiences large fluctuations near expiration, reflecting the volatility in gamma sensitivity as the option approaches maturity.

Conclusion

Both delta hedging and delta-gamma hedging exhibit distinct behaviors for ITM and OTM paths due to differences in delta and gamma sensitivity. Delta-gamma hedging provides more precise risk management by incorporating the hedging option h, allowing for better handling of gamma sensitivity. However, this comes at the cost of increased complexity and frequent adjustments, particularly for ITM paths where gamma sensitivity is higher. Delta hedging, while simpler, is more stable but less effective in managing higher-order risks, particularly for ITM options with significant gamma sensitivity. OTM paths are generally more stable for both strategies due to the lower sensitivity of OTM options, but delta-gamma hedging still provides improved precision compared to delta hedging.

3.3. **Question 3.** Suppose that the real-world $\mathbb{P}$ volatility is $\sigma \in \{20\%, 22\%, \ldots, 30\%\}$, but we still sold the option using $\sigma = 25\%$, and hedge assuming that volatility is 25%. We compare again the delta and delta-gamma hedging cases.

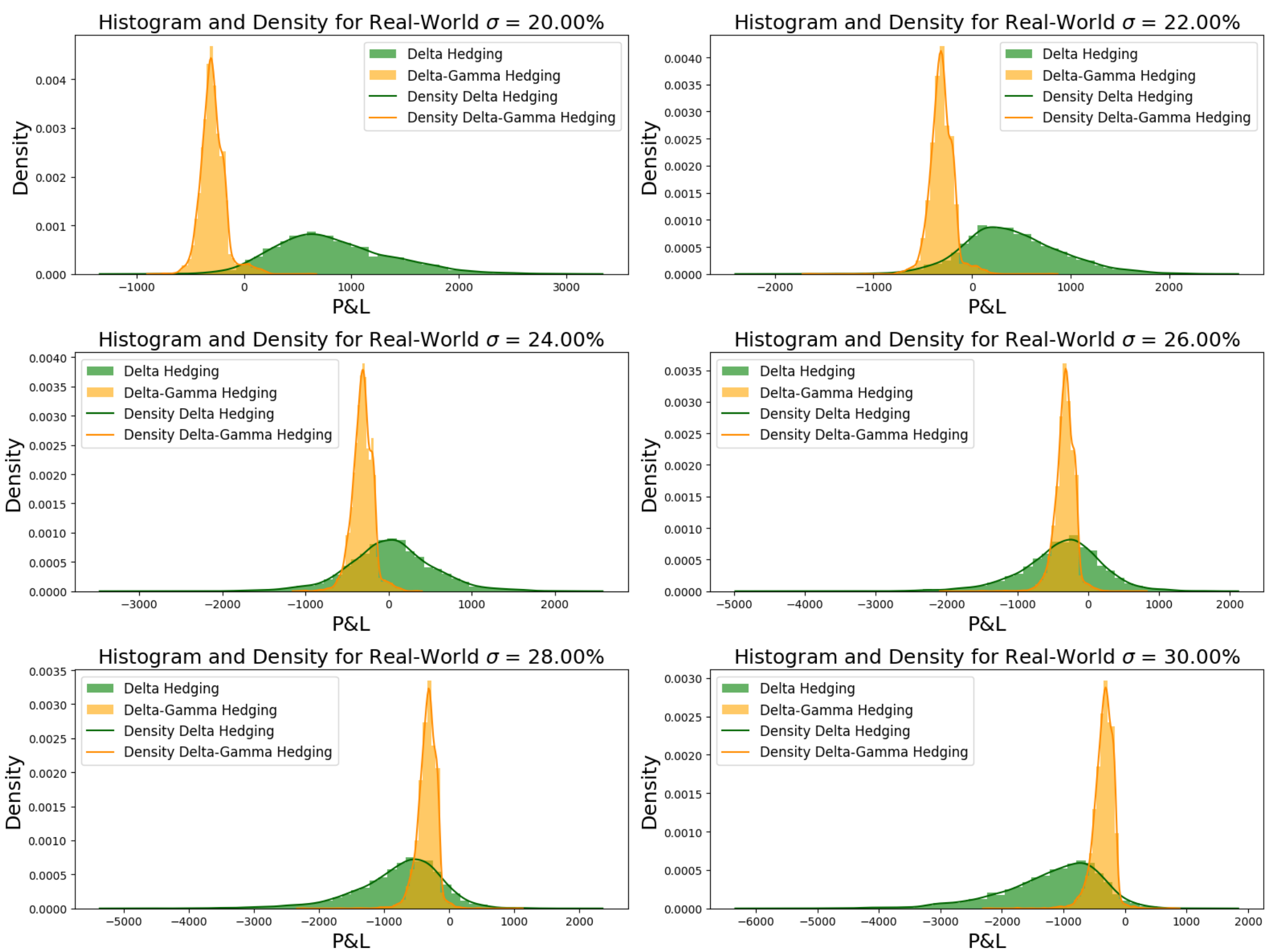

Figure 8. Histogram and Density Plots of Delta Hedging vs. Delta-Gamma Hedging Profit and Loss Distributions as σ varies for 0.2, 0.22, 0.24, 0.26, 0.28, 0.3

In Figure 8, we present the histogram and density plots of delta and delta-gamma hedging profit and loss distributions. As discussed in Question 1, the delta-gamma hedging profit and loss distributions remain more concentrated compared to the delta hedging strategy. As real-world volatility increases, both delta and delta-gamma hedging profit and loss distributions transition from exhibiting a long right tail to a long left tail. This implies that with higher real-world volatility, both strategies are more likely to experience extreme losses rather than profits. Additionally, while the mean value changes in the delta hedging profit and loss distributions are not easily observed in the histograms, the relative mean value of the delta-gamma hedging profit and loss distributions with respect to the center of the delta hedging profit and loss distributions clearly shifts from left to right as real-world volatility increases. This highlights the improved performance of delta-gamma hedging in a volatility-increasing environment.

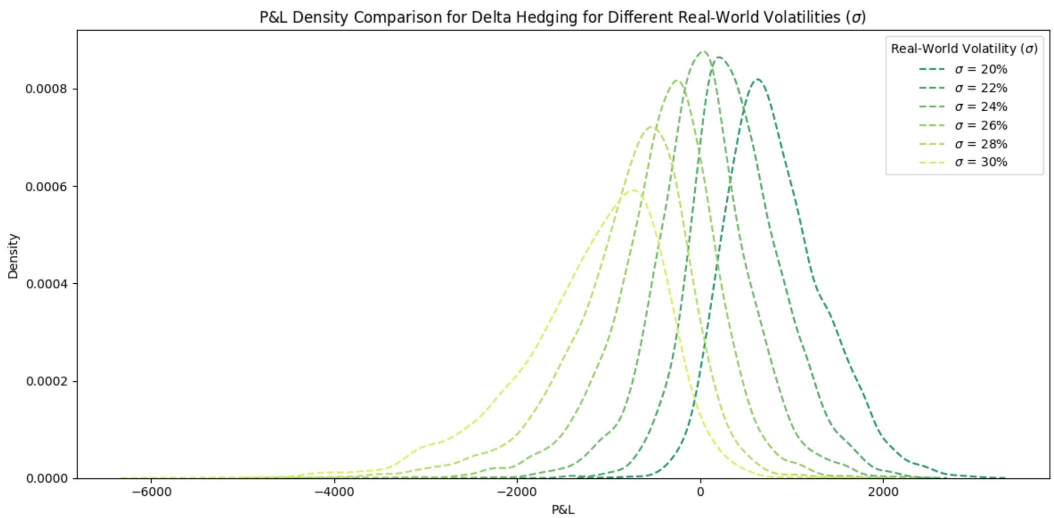

FIGURE 9. Density Plot of Delta Hedging Profit and Loss Distributions as σ varies for 0.2, 0.22, 0.24, 0.26, 0.28, 0.3

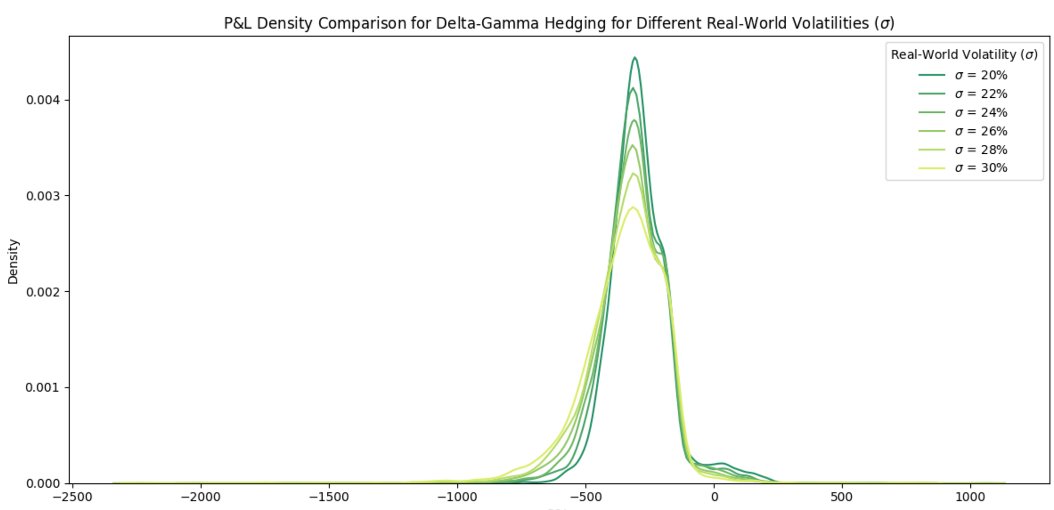

FIGURE 10. Density Plot of Delta-Gamma Hedging Profit and Loss Distributions as σ varies for 0.2, 0.22, 0.24, 0.26, 0.28, 0.3

In Figure 9, we plot the density of profit and loss distributions using the delta hedging method for different real-world volatilities.

In this figure, we can clearly observe that the profit and loss distributions shift leftward as real-world volatility increases, indicating increasing losses when the real-world volatility deviates from the assumed volatility ($\sigma = 25\%$). This reflects the poor performance of delta hedging under model misspecification since delta hedging often assumes constant volatility as specified in the Black-Scholes model.

The density spreads widen significantly as the real-world σ increases, highlighting the rising standard deviation of delta hedging outcomes under higher real-world volatility.

In Figure 10, we plot the density of profit and loss distributions using the delta-gamma hedging method for different real-world volatilities.

In this figure, we can also observe that the profit and loss distributions shift leftward as real-world volatility increases, but the shift is less pronounced than the delta hedging method, showing the stability of delta-gamma hedging when the real-world volatility deviates from the assumed volatility. This suggests that delta-gamma hedging is less sensitive to model misspecification regarding volatility.

Delta-gamma hedging produces more peaked density curves compared to delta hedging, demonstrating its effectiveness in limiting the dispersion of profit and loss outcomes. The density spreads also widen as the real-world σ increases, but the degree of widening tails is less than delta hedging.

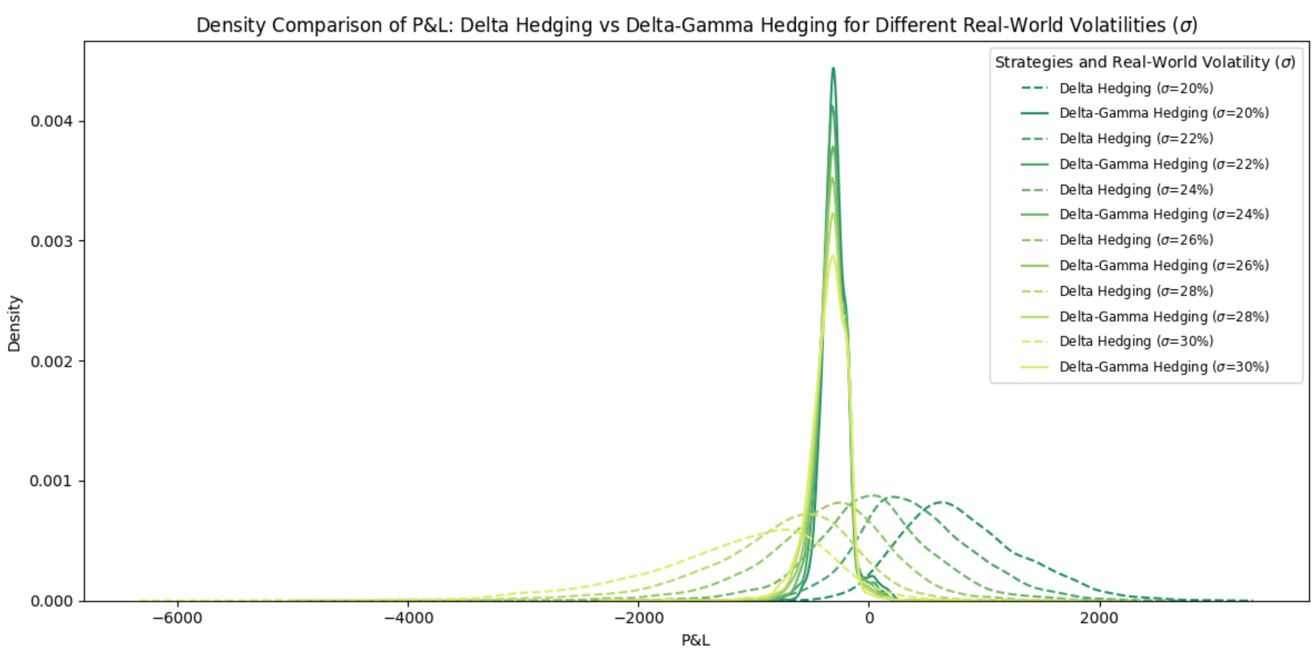

FIGURE 11. Density Plot of Delta Hedging vs. Delta-Gamma Hedging Profit and Loss Distributions as σ varies for 0.2, 0.22, 0.24, 0.26, 0.28, 0.3

In Figure 11, we plot the density of profit and loss distributions using the delta hedging method and the delta-gamma hedging method together for different real-world volatilities for the convenience of comparison.

Delta-gamma hedging consistently outperforms delta hedging in terms of profit and loss stability across all values of real-world volatility. The delta-gamma density peaks remain significantly sharper than those of delta hedging.

As real-world volatility deviates further from the assumed $\sigma = 25\%$, the divergence between delta and delta-gamma hedging outcomes becomes more pronounced. Delta hedging displays greater susceptibility to extreme losses compared to delta-gamma hedging.

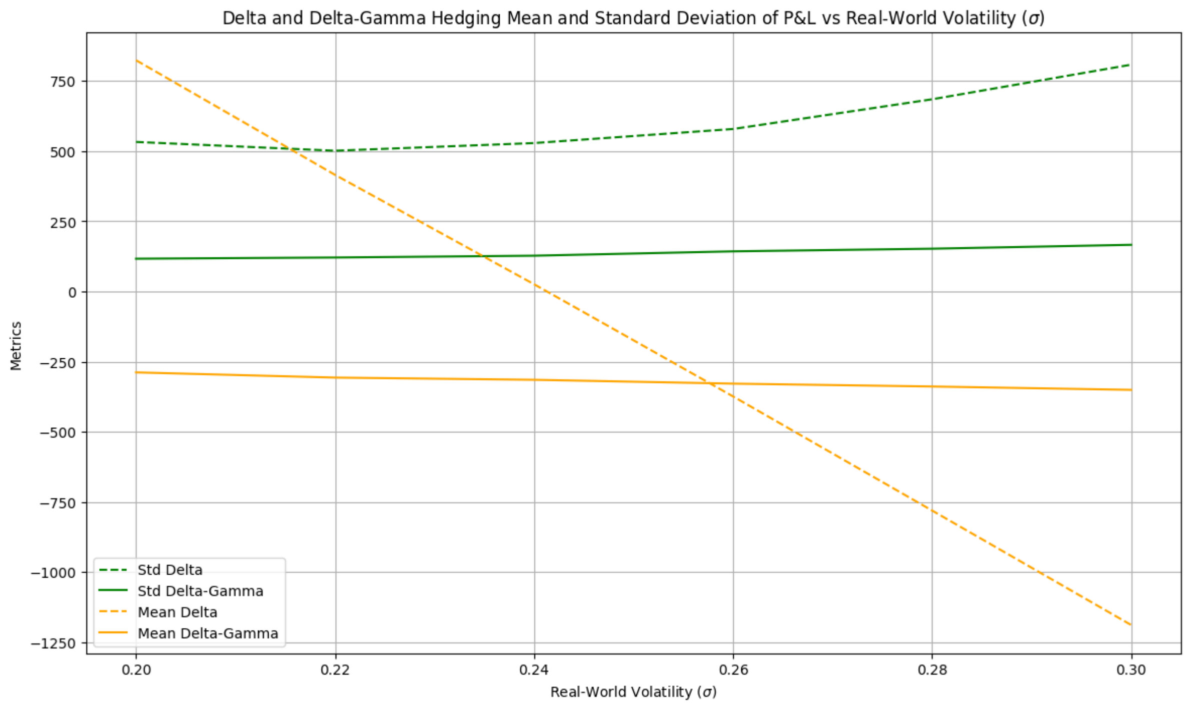

FIGURE 12. Mean and Standard Deviation for Delta and Delta-Gamma Hedging vs. Real-World Sigma

In Figure 12, we also plot the mean and standard deviation statistics metrics for both delta hedging and delta-gamma hedging methods.

For delta hedging, the mean profit and loss decreases explicitly as real-world volatility increases, reflecting systematic underperformance when volatility assumptions are inaccurate. Simultaneously, the standard deviation of profit and loss has a clear rising trend, indicating heightened risk.

For delta-gamma hedging, the mean profit and loss remains relatively stable across varying real-world volatilities, showcasing its robustness against volatility misspecification. The standard deviation increases slightly but at a much slower rate compared to delta hedging.

The standard deviation of delta hedging surpasses that of delta-gamma hedging at all levels of real-world volatility, confirming the latter's superior risk management capabilities.

Delta hedging is highly sensitive to model misspecification in volatility, with increasing losses and risk as real-world volatility deviates from assumptions. This strategy struggles to adapt to higher-order risks and significant market volatility changes.

Delta-gamma hedging demonstrates remarkable resilience to changes in real-world volatility, with consistently narrower distributions and more stable mean outcomes. This highlights the effectiveness of managing both delta and gamma risks for mitigating the impact of market conditions.

Overall, delta-gamma hedging provides better profit and loss stability and risk control than delta hedging, especially under scenarios of elevated volatility or model misspecification. However, this strategy requires more sophisticated instruments and frequent rebalancing to maintain neutrality.

4. Conclusions

This study explored delta and delta-gamma hedging strategies within the Black-Scholes framework, focusing on managing risk for an at-the-money call option portfolio. Through simulations of profit and loss distributions across various drift rates (μ), real-world volatilities (σ), and sample paths (ITM and OTM), we provided a comparative analysis of the two hedging strategies.

The results demonstrated that delta-gamma hedging consistently outperforms delta hedging in terms of stability and risk management. Delta-gamma hedging effectively reduces profit and loss variability by incorporating second-order risk (gamma sensitivity) into the hedging strategy. This results in narrower profit and loss distributions, lower standard deviation, and improved robustness under varying market conditions and model misspecification. However, this strategy introduces additional complexity, higher transaction costs, and frequent rebalancing requirements, especially in ITM scenarios where gamma sensitivity is higher.

On the other hand, delta hedging relies solely on the underlying asset to neutralize first-order risk (delta risk), leading to a simpler and more stable strategy. However, its inability to account for gamma sensitivity results in wider profit and loss distributions and higher susceptibility to extreme losses, particularly under significant market volatility or model misspecification.

In environments where the real-world volatility deviates from the assumed value, delta hedging exhibited greater sensitivity to model misspecification, with profit and loss distributions shifting significantly toward losses as volatility increases. Delta-gamma hedging, in contrast, maintained more consistent performance, showcasing its ability to adapt to such scenarios.

In conclusion, while delta-gamma hedging provides superior precision and risk control, delta hedging remains a viable alternative for lower-complexity scenarios or for investors willing to tolerate higher risk for simplicity. The choice between the two strategies ultimately depends on the investor's risk tolerance, market expectations, and capacity for managing hedging complexities.

Contribution

Xinyi Shen and Zhuolin Zhou drafted, discussed, and completed this project together.

Xinyi Shen and Zhuolin Zhou both contributed to Python programming for this project.

Xinyi Shen mainly contributed to transforming the delta and delta-gamma hedging logic to methodology and formatting the Python code outputs to observations in the report.

Zhuolin Zhou mainly contributed to proofreading, correcting errors, and improving the methodology and result writing in the report.

Signatures

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Xinyi Shen (Team Member 1)	Zhuolin Zhou (Team Member 2)