$$\mathcal{T}_i = \left\{ x_{T_i} \in \mathcal{X} \;\middle|\; \begin{array}{c} \exists \mathbf{u}_{T_i:T_\Phi\text{-}1} \in \mathcal{U}^{T_\Phi\text{-}T_i}, \forall \mathbf{w}_{T_i:T_\Phi\text{-}1} \in \mathcal{W}^{T_\Phi\text{-}T_i}, \\ \text{s.t. } \xi_f(x_{T_i}, \mathbf{u}_{T_i:T_\Phi\text{-}1}, \mathbf{w}_{T_i:T_\Phi\text{-}1}) \models \bigwedge_{i<j\le N} \Phi_j \end{array} \right\}.$$

$$\hat{\mathcal{T}}_i = \left\{ x_{T_i} \in \mathcal{X} \;\middle|\; \begin{array}{c} \exists u_{T_i} \forall w_{T_i} \in \mathcal{W} \; \exists u_{T_i+1} \forall w_{T_i+1} \in \bigoplus_{i=0}^{1} L^i \mathcal{W} \\ \cdots \exists u_{T_\Phi\text{-}1} \forall w_{T_\Phi\text{-}1} \in \bigoplus_{i=0}^{T_\Phi - T_i - 1} L^i \mathcal{W} \\ \text{s.t. } \xi_f(x_{T_i}, \mathbf{u}_{T_i:T_\Phi\text{-}1}, \mathbf{w}_{T_i:T_\Phi\text{-}1}) \models \bigwedge_{i<j\le N} \Phi_j \end{array} \right\}.$$

Proposition 1 *For each terminal set $\mathcal{T}_i$, we have $\hat{\mathcal{T}}_i \subseteq \mathcal{T}_i$.*

Proof: We will prove the above proposition by proving that if $x_{T_i} \in \hat{\mathcal{T}}_i$, then $x_{T_i} \in \mathcal{T}_i$.

Step 1:
Given that $x_{T_i} \in \hat{\mathcal{T}}_i$, assume that the existing control inputs are

$$\begin{aligned} \hat{u}_{T_i} &= h_0(x_{T_i}), \\ \hat{u}_{T_i+1} &= h_1(x_{T_i}, w_{T_i}), \\ &\cdots, \\ \hat{u}_{T_\Phi-1} &= h_{T_\Phi-T_i-1}(x_{T_i}, w_{T_i}, \cdots, w_{T_\Phi-2}), \end{aligned} \tag{1}$$

where $\{h_0, h_1, \cdots, h_{T_\Phi-1}\}$ is a kind of control strategy and $w_{T_i+p} \in \bigoplus_{i=0}^{p} L^i \mathcal{W}, p = 0, \cdots, T_\Phi - T_i - 2$. Recall that in $\hat{\mathcal{T}}_i$ case, the input $\hat{u}_{T_i+p}$ is decided after observing previous information which is captured by the elements in function h_p. There could be more than one strategy $\{h_0, h_1, \cdots, h_{T_\Phi-1}\}$, leading to more than one control input sequence $\mathbf{u}_{T_i:T_\Phi-1}$, and we denote by $\mathcal{H}$ the set of all the possible control strategy sequence for state x_{T_i}.

Step 2:
In set $\mathcal{H}$, there definitely exists at least one control strategy $\{h_0^*, h_1^*, \cdots, h_{T_\Phi-1}^*\}$ in which the specific strategy of every step will attenuate all the previous disturbance, i.e., for all $p \in \{1, \cdots, T_\Phi - T_i - 2\}$ and for all $w_{T_i+j} \in \bigoplus_{i=0}^{j} L^i \mathcal{W}, j = 0, \cdots, p-1$ we have

$$f(x_{T_i+p}, h_p^*(x_{T_i}, \underbrace{\mathbf{0}^n, \cdots, \mathbf{0}^n}_{p}) \in f(x_{T_i+p}, h_p^*(x_{T_i}, w_{T_i}, \cdots, w_{T_i+p-1})) \oplus \bigoplus_{i=1}^{p} L^i \mathcal{W}, \tag{2}$$

where x_{T_i+p} is a real state with disturbance at instant $T_i + p$ and $\bigoplus_{i=1}^{p} L^i \mathcal{W}$ is all the possible influences to x_{T_i+p+1} caused by disturbances of first p steps. The reason is as follows. (Of course, the attenuating thing is also what we want function h to do.) Suppose that there exists a control strategy $\{h_0', h_1', \cdots, h_{T_\Phi-1}'\} \in \mathcal{H}$ and in this strategy some functions $h_q', q \in Q \subseteq \{1, \cdots, T_\Phi - T_i - 2\}$ does not attenuate previous disturbance. It implies that under these function $h_q', q \in Q$, there exists a disturbance sequence $\{w_{T_i}', \cdots, w_{T_i+q-1}'\}$ such that

$$f(x_{T_i+q}, h_q'(x_{T_i}, \underbrace{\mathbf{0}^n, \cdots, \mathbf{0}^n}_{q}) \notin f(x_{T_i+q}, h_q'(x_{T_i}, w_{T_i}', \cdots, w_{T_i+q-1}')) \oplus \bigoplus_{i=1}^{q} L^i \mathcal{W}.$$

Under this condition, there definitely exists such a control strategy $\{h_0^*, h_1^*, \cdots, h_{T_\Phi-1}^*\}$ which will attenuate all the previous disturbances, where

$$h_j^* = \begin{cases} h_j'(x_{T_i}, w_{T_i}, \cdots, w_{T_i+j-1}) & \text{if } j \notin Q \\ h_j'(x_{T_i}, \underbrace{\mathbf{0}^n, \cdots, \mathbf{0}^n}_{j}). & \text{if } j \in Q \end{cases}$$

Step 3:
For such a state x_{T_i}, we claim that $x_{T_i} \in \mathcal{T}_i$ since there exists at least a control strategy $\{h_0^*, h_1^*, \cdots, h_{T_\Phi - 1}^*\}$ such that for all disturbance sequences the solution is right, i.e., the control inputs are

$$\begin{aligned} u_{T_i} &= h_0^*(x_{T_i}), \\ u_{T_i+1} &= h_1^*(x_{T_i}, \mathbf{0}^n), \\ &\cdots, \\ u_{T_\Phi - 1} &= h_{T_\Phi - T_i - 1}^*(x_{T_i}, \mathbf{0}^n, \cdots, \mathbf{0}^n). \end{aligned} \tag{3}$$

The proof is as follows. We first clarify two different control modes corresponding to $\mathcal{T}_i$ and $\hat{\mathcal{T}}_i$ respectively as follows:

- fixed control sequence $u_{T_i}, u_{T_i+1}, \cdots, u_{T_\Phi - 1}$ as (3) with random disturbance sequence $\forall \mathbf{w}_{T_i:T_\Phi\text{-}1} \in \mathcal{W}^{T_\Phi\text{-}T_i}$ and
- $\hat{u}_{T_i}, \forall w_{T_i} \in \mathcal{W}, \hat{u}_{T_i+1}, \forall w_{T_i+1} \in \bigoplus_{i=0}^{1} L^i \mathcal{W} \ \cdots \ \hat{u}_{T_\Phi\text{-}1}, \forall w_{T_\Phi\text{-}1} \in \bigoplus_{i=0}^{T_\Phi - T_i - 1} L^i \mathcal{W}$ where $\hat{u}_{T_i}, \hat{u}_{T_i+1}, \cdots, \hat{u}_{T_\Phi - 1}$ is defined in (1).

We introduce two symbols $\{x_{T_i+p+1}\}_p$ and $\{\hat{x}_{T_i+p+1}\}_p$ which represent the reachable sets starting from the same state x_{T_i+p} by applying above two strategy respectively with consideration of disturbance at instant $T_i + p$, i.e.,

$$\begin{cases} \{x_{T_i+p+1}\}_p = f(x_{T_i+p}, h_p(x_{T_i}, \underbrace{\mathbf{0}^n, \cdots, \mathbf{0}^n}_{p})) \oplus \mathcal{W} \\ \{\hat{x}_{T_i+p+1}\}_p = f(x_{T_i+p}, h_p(x_{T_i}, w_{T_i}, \cdots, w_{T_i+p})) \oplus L^p \mathcal{W} \oplus \cdots \oplus \mathcal{W}. \end{cases} \tag{4}$$

According to Equation (2), we have $\{x_{T_i+p+1}\}_p \subseteq \{\hat{x}_{T_i+p+1}\}_p$ obviously, which implies that with consideration of disturbance, starting from the same state, the reachable set after applying the former control way will be contained in that after applying the latter. It is easy to induce that the set of all the possible state sequence starting from x_{T_i} under the former control mode, denoted by $\{\mathbf{x}_{T_i:T_\Phi}\}$, is contained in that under the latter, denoted by $\{\hat{\mathbf{x}}_{T_i:T_\Phi}\}$, i.e., $\{\mathbf{x}_{T_i:T_\Phi}\} \subseteq \{\hat{\mathbf{x}}_{T_i:T_\Phi}\}$.

According to the assumption $x_{T_i} \in \hat{\mathcal{T}}_i$, we have $\forall \hat{\mathbf{x}}_{T_i:T_\Phi} \in \{\hat{\mathbf{x}}_{T_i:T_\Phi}\}$, $\hat{\mathbf{x}}_{T_i:T_\Phi} \models \bigwedge_{i<j\leq N} \Phi_j$. Then we can obtain $\forall \mathbf{x}_{T_i:T_\Phi} \in \{\mathbf{x}_{T_i:T_\Phi}\}, \mathbf{x}_{T_i:T_\Phi} \models \bigwedge_{i<j\leq N} \Phi_j$. Therefore, $x_{T_i} \in \mathcal{T}_i$ holds, i.e., the proposition is proved.