Equalizer Over Different R-Modules

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July 2021

1 Introduction

1.1 Motivation

The motivation of this project is calculating the homology to assist the detection of areas that are uncovered by a set of a sensor networks [1]. Suppose we have a fence that surrounds multiple sensors inside; even though we can track sensors' locations within the fence, we have deficient information to know if sensors have full coverage. The locations of sensors are dependent upon the time $t \in \mathbb{R}$. To investigate the spatial structure of the sensor coverage, a set of sensors V_t can be connected by edges E_t to form a simplicial complex, where it forms a 1-dimensional region F_t that represents the sensor coverage.

1.2 The First Homology of a Directed Graph

Over time, suppose we have 1-dimensional regions $F_1, F_2, ..., F_i$, edges $e_1, e_2, ..., e_m$, and vertices $v_1, v_2, ...v_k$, that induce the following linear maps, where domains and codomains represent linear combinations over $\mathbb{R}$:

$$\partial_1 : \mathbb{R}[e_1, e_2, ..., e_m] \mapsto \mathbb{R}[v_1, v_2, ..., v_n]$$

$$\partial_2 : \mathbb{R}[F_1, F_2, ..., F_i] \mapsto \mathbb{R}[e_1, e_2, ..., e_j]$$

In a non-directed graph, where α_2 exsits, the first homology of the graph X can be written as

$$H_1(X; \mathbb{R}) := (\partial_1) / \operatorname{im}(\partial_2)$$

The above equation indicates that $im(\partial_2) \in (\partial_1)$. However, in the directed graph, where ∂_2 doesn't exist. The first homology of a directed graph can be written as

$$H_1(X; \mathbb{R}) := (\partial_1)$$

In our research, we are interested in finding the first homology of the directed graph X over $\mathbb{R}_+$, which is defined as follows.

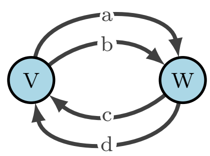
Definition 1. *Suppose $C_1 = \mathbb{R}_+[e_2, ..., e_m]$ and $C_0 = \mathbb{R}_+[v_1, v_2, ..., v_n]$, ∂^+ and ∂^- are two mappings that send C_1 to C_0. ∂^+ maps edges to vertices at their ending points and ∂^- maps vertices at edges' starting points.*

Definition 2. *Suppose* $E(\partial^+, \partial^-) = \{x \in C_1 \mid \partial^+(x) = \partial^-(x)\}$ *is an equalizer of two directed boundary maps* ∂^+ *and* ∂^-*. Notice that* $E(\partial^+, \partial^-) \subseteq \mathbb{R}_+[e_1, e_2, ..., e_m]$*. The first homology of X is defined as*

$$H_1(X; \mathbb{R}_+) := E(\partial^+, \partial^-)$$

Notice that if X is a directed graph when ∂^+ and ∂^- are well defined, we use $E(\partial^+, \partial^-) = \{x \in C_1 \mid \partial^+(x) = \partial^-(x)\}$ instead of $Ker(\partial^+, \partial^-) = \{x \in C_1 \mid (\partial^+ - \partial^-)x = 0\}$ to define $H_1(X; \mathbb{R}_+)$. Even though both of them appear to be identical, results from subtracting ∂^+ and ∂^- can contradict the criteria of *nonnegative linear combination* of edges. The proof can be constructed by giving a counterexample.

Proof. Suppose we have a directed graph X shown below.



Based on the directed graph X, $(\partial^+ - \partial^-)(a + 2c) = \partial^+(a) + \partial^+(2c) - \partial^-(a) - 2\partial^-(c) = w + 2v - v - 2w = v - w \notin C_0$ since C_0 requires coefficients to be positive real numbers. Therefore, $\partial^+ - \partial^-$ is not a map between C_1 and C_0 over $\mathbb{R}_+$. $\square$

2 Equalizer Between Module Over $\mathbb{R}_+$

In order to calculate the first homology of a directed graph X, it is essential to find the equalizer.

Definition 3 (Convex Cone)**.** *A set* C *is a convex cone if for any* $x_1, x_2 \in C$ *and* $\theta_1, \theta_2 \geqslant 0$*, we have* $\theta_1 x_1 + \theta_2 x_2 \in C$*.*

The *conic hull* of the set C is a collection of all *nonnegative linear combination* of $x_1, ..x_k \in C$. i.e. $\{\theta_1 x_1 + ... + \theta_k x_k \mid x_i \in C, \theta_i \geqslant 0, i = 1, ..., k\}$, which is also a convex cone [2]. We observe that both C_0 and C_1 are well-defined convex cones because they represent two *conic hulls* generated by vertices and edges. Hence, when $R = \mathbb{R}_+$, we can generalize our problem from finding $E(\partial^+, \partial^-)$ to any equalizer of two linear maps over a pair of convex cones.

Theorem 2.1. *Suppose* C *and* D *are convex cones with* $\hat{\alpha}, \hat{\beta} : C \mapsto D$*. Let* α *and* β *be their unique extensions that map between two vectors spaces* V *and* W*. i.e.* $\alpha, \beta : V \mapsto W$*. Then* $E(\hat{\alpha}, \hat{\beta}) = Ker(\alpha, \beta) \cap C$*.*

Proof. Notice that V and W are generated by basis of C and D over $\mathbb{R}$ instead of $\mathbb{R}_+$. We can illustrate their relationships using the following graph:

$$\begin{tikzcd}
V \arrow[r, shift left, "\alpha"] \arrow[r, shift right, "\beta"'] \arrow[d, "\subseteq"'] & W \arrow[d, "\subseteq"] \\
C \arrow[r, shift left, "\hat{\alpha}"] \arrow[r, shift right, "\hat{\beta}"'] & D
\end{tikzcd}$$

Since $C \subseteq V$, $E(\alpha, \beta) = \{x \in C \mid \hat{\alpha}(x) = \hat{\beta}(x)\} = \{x \in V \mid \alpha(x) = \beta(x)\} \cap C = Ker(\alpha - \beta) \cap C$ $\square$

Here, we already knew that finding an equalizer can be transformed into a problem that finds the intersection. To get an optimal result, our goal is to find the minimum presentation set for the equalizer. Since $E(\alpha, \beta) \subseteq C$, E can be treated as a polyhedron generated by a set of vectors over $\mathbb{R}_+$. A polyhedron can be represented by the sum of vertices, rays, and lines with Minkowski sum:

$$P = conv\{v_1, \dots, v_k\} + \sum_{i=1}^{m} \mathbb{R}_+ r_i + \sum_{j=1}^{n} \mathbb{R} l_j$$

In Minkowski sum, $conv\{v_1, \dots, v_k\}$ is a convex hull of vertices. Rays r_i finitely generate the polyhedron over $\mathbb{R}_+$, and lines l_i are unoriented over $\mathbb{R}$.

$Ker(\alpha - \beta)$ can be calculated by finding the null space of M_α and M_β. After finding such null space, we name its basis as $bKer$. In addition, we can write convex cone $C = \{\theta_1 x_1 + ... + \theta_k x_k \mid x_i \in C, \theta_i \geqslant 0, i = 1, ..., k\}$ as a finitely generated set by coefficients θ_i only. This collection of θ_i can be represented by standard basis vectors with *k-dimension*. Let *bCone* be such convex cone that has a form of $\{(1, 0, ..., 0), (0, 1, ..., 0)...(0, ..., 0, 1)\}$. With Minkowski sum, we can make $bKer$ as a polyhedron generated by unoriented lines l_j over $\mathbb{R}$ and *bCone* as a polyhedron generated by rays r_i over $\mathbb{R}_+$. Then, we would be able to find a generating set of the equalizer. Below is a code snippet that creates polyhedra and computes the equalizer in SAGE:

$$\begin{gathered} polyC = Polyhedron(rays = bCone, base\_ring = RDF) \\ polyK = Polyhedron(lines = bKer, base\_ring = RDF) \\ E = polyC.intersection(polyK) \end{gathered}$$

Fortunately, under the help of the polyhedron library in SAGE, we are able to get its dimensions and extreme rays directly.

3 Equalizer Between Module Over $\mathbb{N}$

We want to expand our problem to find an equalizer over different R-modules. Previously, we have visited methods to compute equalizer when $R = \mathbb{R}_+$ under the assistance of vector spaces over a field. Since $\mathbb{N}$ is a semiring, the module over $\mathbb{N}$ is a commutative monoid M. To mimic the construction of equalizer over convex cones, our goal is to embed the commutative monoid to its group completion. A commutative monoid has M its group completion $G(M) = M \times M/\sim = \{m - n \mid m, n \in M\}$ if and only if M is cancellative i.e. for all $a, b, c \in M$, $a + c = b + c$ implies $a = b$ [3][4].

Theorem 3.1. *Suppose M and N are cancellative commutative monoid and $G(M)$ and $G(N)$ are their corresponding group completions. Let group homomorphisms $f', g' : G(M) \to G(N)$ be unique extensions of $f, g : M \to N$. $E(f', g') = M \cap Ker(f - g)$*

Proof. First, we need to show that f' and g' are valid unique extensions of f and g. Notice that $G(N)$ is an abelian group under addition. Let $h_1, h_2 : M \to G(N)$, with $h1 = \eta_N \circ f$ and $h2 = \eta_N \circ g$. η_M and η_N are injections on $G(M)$. Based on the universal property of grothendieck group, there exists two unique homomorphisms $\tilde{h}_1$ and $\tilde{h}_2$ such that $h_1 = \tilde{h}_1 \circ \eta_M$ and $h_2 = \tilde{h}_2 \circ \eta_M$. Therefore, $\tilde{h}_1$ and $\tilde{h}_2$ are unique extensions of f, g, namely f', g'. We use the following commutative diagram to show above relationships.

$$
\begin{tikzcd}
M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] \arrow[d, "\eta_M"'] & N \arrow[d, "\eta_N"] \\
G(M) \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] & G(N)
\end{tikzcd}
$$

By definition, $E(f,g) = \{x \in M \mid f(x) = g(x)\} = \{x \in M \mid f'(x) = g'(x)\}$ since M is a submonoid of $G(M)$. This also equals to $\{x \in G(M) \mid f'(x) = g'(x)\} \cap M = \{x \in G(M) \mid (f' - g')(x) = e_M\} \cap M = Ker(f' - g') \cap M$. $\square$

Remark. *For any two commutative monoids M and N with homology f and g, $E = \{x \in M \mid f(x) = g(x)\}$ is also commutative monoid and e_M is its identity.*

For a more general case, when commutative monoids M and N are not necessarily cancellative, we can construct a basic framework that computes the equalizer using the following diagram:

$$
\begin{tikzcd}
\tilde{E} \arrow[r] \arrow[d] & \mathbb{N} \oplus ... \oplus \mathbb{N} \arrow[r, shift left, "f'"] \arrow[r, shift right, "g'"'] \arrow[d, two heads, "\eta_M"'] & \mathbb{N} \oplus ... \oplus \mathbb{N} \arrow[d, two heads, "\eta_N"] \\
E \arrow[r] & M \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & N
\end{tikzcd}
$$

In this diagram, two finitely generated commutative monoids M and N can be written as monoid presentations $< M \mid R_M >$ and $< N \mid R_N >$. There exists a unique surjective homomorphism that maps monoid generators $\mathbb{N} \oplus ... \oplus \mathbb{N}$ to M. Similarly, N also has a set of generators that can be obtained by a surjection. Here, we have several remaining questions that need further investigations. Can we lift our methodology from finding E to $\tilde{E}$ i.e.$E = \tilde{E}/\sim$? If we could formally define η_M, then E could be found by taking the surjection of $\tilde{E}$. Can we find the minimized generating set of E using integer programming, comparably to what we did in finding extreme rays in convex cones? This may be approachable since boundaries of optimization can be defined by equalizer's condition and all the generators are required to be positive.

4 Discussions

We have constructed both theoretical and computational methods to find an equalizer between convex cones. Our results from SAGE implementation gives a list of extreme rays that generates the equalizer. This result has been incorporated into a function that returns the first homology of a directed graph.

Previous research has shown that there are algorithms involving linear programming to find all extreme rays of convex cones with some complementarity conditions [5]. With this idea in hand, we are interested in seeing how integer programming can come into play to find the minimum presentation of a commutative monoid, an equalizer over $\mathbb{N}$.

So far, there are some difficulties to implement group completion of commutative monoid using an existing library, especially when it is defined under monoid presentations. The GAP(Groups, Algorithms, Programming) platform has the object to declare a presentation of the commutative monoid. Still, it is challenging to construct a Grothendieck group by accessing and altering instance variables within the monoid object.

Our research proved that an equalizer between two commutative monoids can be obtained from the kernel of two homomorphisms, when two monoids are cancellative. However, it remains unclear if there is a more general solution, regardless of the cancellation property. Future research can focus on questions we proposed above: can we find $\tilde{E}$ so that $E = \tilde{E}/\sim$?

Furthermore, we can expand our research to the category theory to investigate the limit that generalizes kernels and equalizers.

References

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