PLEASE HAND IN

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UNIVERSITY OF TORONTO
Faculty of Arts and Science

DECEMBER 2016 EXAMINATIONS

CSC 165 H1F
Instructor(s): Fairgrieve

Duration—3 hours

No Aids Allowed

You must earn at least 35% on this final examination in order to pass the course.

Student Number: |_|_|_|_|_|_|_|_|_|_|

Last (Family) Name(s): ______________________

First (Given) Name(s): ______________________

UTORid: ______________________

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*Do **not** turn this page until you have received the signal to start.*
In the meantime, please read the instructions below.

MARKING GUIDE

# 1:	______/10
# 2:	______/10
# 3:	______/12
# 4:	______/ 8
# 5:	______/ 8
# 6:	______/ 6
# 7:	______/ 8
# 8:	______/ 9
# 9:	______/10
# 10:	______/ 9
TOTAL:	______/90

This Final Examination paper consists of 10 questions on 20 pages (including this one), printed on both sides of the paper. When you receive the signal to start, please make sure that your copy of the paper is complete and fill in the identification section above.

Answer each question directly on the exam paper, in the space provided, and use the blank pages for rough work. If you need more space for one of your solutions, use a blank page and indicate clearly the part of your work that should be marked.

In your answers, you may use without proof any result covered in lectures, tutorials, homework, tests, or the textbook, as long as you give a clear statement of the result(s) you are using. You must justify all other facts required for your solutions.

Write up your solutions carefully! In particular, use notation and terminology correctly and explain what you are trying to do—part marks will be given for showing that you know the general structure of an answer, even if your solution is incomplete.

Definitions and formulas are on the second last page.

Good Luck!

Question 1. [10 MARKS]

Part (a) [2 MARKS]

Consider the statement: $\forall\, x \in \mathbb{N}\ \exists\, y \in \mathbb{N}\ (2x - y = 0)$

Is the statement True or False? Justify your conclusion.

Part (b) [2 MARKS]

Consider the statement: $\exists\, y \in \mathbb{N}\ \forall\, x \in \mathbb{N}\ (2x - y = 0)$

Is the statement True or False? Justify your conclusion.

Part (c) [2 MARKS]

Consider the statement: $\forall\, x \in \mathbb{N}\ \exists\, y \in \mathbb{N}\ (x - 2y = 0)$

Is the statement True or False? Justify your conclusion.

Part (d) [4 MARKS]

Find a formula involving only the propositional variables p and q and the connectives $\neg$ and $\Rightarrow$ that is equivalent to $p \wedge q$.

Question 2. [10 MARKS]

Part (a) [5 MARKS]

Prove that for all real numbers x and y there is a real number z such that $x + z = y - z$.

Proof.

$\square$

Part (b) [5 MARKS]

Prove that if $A \subseteq B$ and $A \nsubseteq C$ then $B \nsubseteq C$.

Proof.

$\square$

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each such answer with the appropriate question and part number.

Question 3. [12 MARKS]

The *harmonic numbers* are the numbers H_n for $n \geqslant 1$ defined by the formula

$$H_n = \sum_{i=1}^{n} \frac{1}{i}.$$

Part (a) [7 MARKS]

Prove that for all natural numbers n and m, if $n \geqslant m$ then $H_n - H_m \geqslant \dfrac{n-m}{n}$.

HINT: Let m be an arbitrary natural number and then prove that the rest of the statement is true for arbitrary n.

Proof.

$\square$

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each such answer with the appropriate question and part number.

Part (b) [5 MARKS]

Prove that for all natural numbers n, $H_{2^n} \geqslant 1 + \dfrac{n}{2}$.

HINT: You may assume that the statement in Part (a) is True even if you were not able to prove it.

Proof.

$\square$

Question 4. [8 MARKS]

A **bipartite graph** is a graph $G = (V, E)$ whose vertex set can be partitioned into two distinct nonempty subsets V_1 and V_2 such that the vertices in V_1 may have an edge with vertices in V_2 but no vertices in V_1 have an edge with vertices in V_1 and no vertices in V_2 have an edge with vertices in V_2. A **complete bipartite graph** is a **bipartite graph** that contains every edge that is allowed.

Part (a) [1 MARK]

For the case of $|V| = 4$, draw a sketch of a **complete** bipartite graph that has $|E| = 3$ and a sketch of another **complete** bipartite graph that has $|E| = 4$.

Part (b) [7 MARKS]

Prove that if $G = (V, E)$ is an **arbitrary** bipartite graph, then $|E| \leqslant \dfrac{|V|^2}{4}$.

Proof.

$\square$

Question 5. [8 MARKS]

Let $\mathcal{F}$ represent the set of all functions from $\mathbb{N}$ to $\mathbb{R}^{\geqslant 0}$. For example, the function $p(n)$ defined by $p(n) = n + 1$ is an element of $\mathcal{F}$. But the function $t(n)$ defined by $t(n) = \cos(n\pi)$ is **not** an element of $\mathcal{F}$, since $\cos(n\pi) < 0$ for odd n.

For each part of this question, put an "X" in the box next to **each** response that can be used to complete the given statement and make it True.

Part (a) [2 MARKS]

$\forall f, g \in \mathcal{F}\ f \in \Theta(g) \Rightarrow \dots$

- ☐ $\dots g \in \Omega(f)$
- ☐ $\dots g \notin \Theta(f)$
- ☐ $\dots (f + g) \in \Omega(g)$
- ☐ $\dots g \notin \mathcal{O}(f)$

Part (b) [2 MARKS]

$\exists f, g \in \mathcal{F}\ f \notin \mathcal{O}(g) \wedge \dots$

- ☐ $\dots g \notin \mathcal{O}(f)$
- ☐ $\dots f \in \Omega(g)$
- ☐ $\dots g \in \mathcal{O}(f)$
- ☐ $\dots g \in \Theta(f)$

Part (c) [2 MARKS]

$148n^2 + 165 \in \dots$

- ☐ $\dots \mathcal{O}(n^2)$
- ☐ $\dots \Theta(n^2)$
- ☐ $\dots \Omega(n)$
- ☐ $\dots \mathcal{O}(n^{148})$

Part (d) [2 MARKS]

$\dfrac{2n \log_{10} n}{3n + 1} \in \dots$

- ☐ $\dots \mathcal{O}(n)$
- ☐ $\dots \Theta(\log_2 n)$
- ☐ $\dots \mathcal{O}(\log_{16} n)$
- ☐ $\dots \Omega(\sqrt{n})$

Question 6. [6 MARKS]

Let $f : \mathbb{N} \rightarrow \mathbb{R}^{\geqslant 0}$ be the function defined by $f(n) = \left\lfloor \frac{n}{2} \right\rfloor$. Prove that $f(n) \in \Theta(n)$.

Proof.

$\square$

Question 7. [8 MARKS]

Let $f_1, f_2, h_1, h_2 : \mathbb{N} \to \mathbb{R}^{\geqslant 0}$.

Prove that if $f_1 \in \Theta(h_1)$, $f_2 \in \Theta(h_2)$, and $h_1 \in \mathcal{O}(h_2)$, then $(f_1 + f_2) \in \Theta(h_2)$.

Proof.

$\square$

Question 8. [9 MARKS]

Give a tight bound on the running time of each of the following algorithms. The algorithms are expressed in python functions. Write your final answer in the box below each algorithm. In the space below each algorithm, show your "discussion" work to justify each answer.

Part (a) [2 MARKS]

```
def f_1(n):
    """ (int) -> int
    """

    value = 1                    # line 1.
    for i in range(n):           # line 2.
        value = value * (i+1)    # line 3.
    return value                 # line 4.
```

$\Theta(\qquad)$

Part (b) [2 MARKS]

```
def f_2(A):
    """ (list of list of numbers) -> NoneType
    Precondition: len(A) == len(A[i]), for all i in range(len(A))
    """

    for i in range(len(A)):                              # line 1.
        for j in range(max(i-1,0),min(len(A),i+2)):      # line 2.
            if i == j:                                   # line 3.
                value = 2                                # line 4.
            else:                                        # line 5.
                value = -1                               # line 6.
            A[i][j] = value                              # line 7.
```

$\Theta(\qquad)$

Part (c) [2 MARKS]

```
def f_3(lst):
    """ (list of numbers) -> int
    Precondition: lst is a list that contains n > 0 numbers.
    """

    value = 0                         # line 1.
    index = 1                         # line 2.
    while index < len(lst):           # line 3.
        value = lst[index] - value    # line 4.
        index = index * 3             # line 5.
    return value                      # line 6.
```

$\Theta(\qquad\qquad)$

Part (d) [3 MARKS]

```
def f_4(lst):
    """ (list of numbers) -> int
    Precondition: lst is a list that contains n > 0 numbers.
    """

    value = 0                          # line 1.
    step = 1                           # line 2.
    index = 0                          # line 3.
    while index < len(lst):            # line 4.
        value = lst[index] - value     # line 5.
        index = index + step           # line 6.
        step = step + 1                # line 7.
    return value                       # line 8.
```

$\Theta(\qquad\qquad)$

Question 9. [10 MARKS]

Consider the algorithm mystery given in the following python function:

```
def mystery(A):
    """ (list of int) -> int

    Return a mystery function of the list A.

    >>> mystery([1, 2, 3])
    5
    >>> mystery([1])
    0
    """

    total = 0                               # line 1.
    for i in range(len(A)):                 # line 2.
        if A[i] % 2 == 0:                   # line 3.
            for j in range(i,len(A)):       # line 4.
                total = total + A[j]        # line 5.
    return total                            # line 6.
```

Let $Times_{\texttt{mystery},n} = \{$running time of executing `mystery(x)` $\mid$ input `x` has size $n\}$.

The worst-case running time, $WC_{\texttt{mystery}}(n)$, is the maximum value in $Times_{\texttt{mystery},n}$. And the best-case running time, $BC_{\texttt{mystery}}(n)$, is the minimum value in $Times_{\texttt{mystery},n}$.

Part (a) [2 MARKS]

Give an expression for $\Theta(WC_{\texttt{mystery}}(n))$ and state briefly the reasoning used to arrive at your answer.

Part (b) [2 MARKS]

Give an expression for $\Theta(BC_{\texttt{mystery}}(n))$ and state briefly the reasoning used to arrive at your answer.

Part (c) [1 MARK]

Write a description of a set of inputs to `mystery`, one for each input size, whose running time is exactly $WC_{\texttt{mystery}}(n)$. State briefly the reasoning used to arrive at your answer.

Part (d) [1 MARK]

Write a description of a set of inputs to `mystery`, one for each input size, whose running time is exactly $BC_{\texttt{mystery}}(n)$. State briefly the reasoning used to arrive at your answer.

Part (e) [4 MARKS]

Write a description of a set of inputs to `mystery`, one for each input size, whose running time is $\Theta(WC_{\texttt{mystery}}(n))$ but is also less that $WC_{\texttt{mystery}}(n)$. (You are being asked to describe a worst-case input family.) State briefly the reasoning used to arrive at your answer.

Question 10. [9 MARKS]

Consider the algorithm has_even given in the following python function:

```
def has_even(numbers):
    """ (list of int) -> bool

    Return True if and only if numbers contains an even number.

    >>> numbers = [ 1024 ]
    >>> has_even(numbers)
    True
    >>> numbers = [ 1, 3, 5, 7, 9 ]
    >>> has_even(numbers)
    False
    """

    for number in numbers:        # line 1.
        if number % 2 == 0:       # line 2.
            return True           # line 3.
    return False                  # line 4.
```

Part (a) [2 MARKS]

Define $S_n = \{\text{input lists numbers to has\_even} \mid \text{each element of numbers is an integer from } \{1, 2, \ldots, n\}\}$. Note that an integer is allowed to be repeated in an element of S_n. For example, $S_2 = \{[1,1], [1,2], [2,1], [2,2]\}$.

Consider the set of allowable inputs $\mathcal{I}_n = S_n$. Give an expression for $|\mathcal{I}_n|$. State briefly the reasoning used to arrive at your answer.

Part (b) [7 MARKS]

Calculate an *exact* expression for $Avg_{\texttt{has\_even}}(n)$ for the set of allowable inputs $\mathcal{I}_n = S_n$, where S_n is as defined in Part (a). Show your work.

Use the space on this "blank" page for scratch work, or for any answer that did not fit elsewhere.
Clearly label each such answer with the appropriate question and part number.

Standard Definitions

1. $A \subseteq B \iff \forall$ elements x $x \in A \Rightarrow x \in B$.

2. $A \not\subseteq B \iff \exists$ an element x $x \in A \wedge x \notin B$.

3. Let $\mathbb{N}$ represent the set of natural numbers. We have $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
   Let $\mathbb{Z}$ represent the set of integers. We have $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$.
   Let $\mathbb{R}$ represent the set of real numbers.
   Let $\mathbb{R}^{\geqslant 0}$ represent the set of non-negative real numbers.
   Let $\mathbb{R}^+$ represent the set of positive real numbers.

4. Let $\mathcal{F}$ represent the set of functions having domain $\mathbb{N}$ and range $\mathbb{R}^{\geqslant 0}$.
   For all functions $f, g \in \mathcal{F}$, $f \in \mathcal{O}(g)$ means $\exists\, c, n_0 \in \mathbb{R}^+\ \forall\, n \in \mathbb{N}\ n \geqslant n_0 \Rightarrow (f(n) \leqslant cg(n))$.
   For all functions $f, g \in \mathcal{F}$, $f \in \Omega(g)$ means $\exists\, c, n_0 \in \mathbb{R}^+\ \forall\, n \in \mathbb{N}\ n \geqslant n_0 \Rightarrow (cg(n) \leqslant f(n))$.
   For all functions $f, g \in \mathcal{F}$, $f \in \Theta(g)$ means
   $\exists\, c_0, c_1, n_0 \in \mathbb{R}^+\ \forall\, n \in \mathbb{N}\ n \geqslant n_0 \Rightarrow (c_0 g(n) \leqslant f(n) \leqslant c_1 g(n))$.

5. For all functions $f, g \in \mathcal{F}\ \forall\, n \in \mathbb{N}\ (f+g)(n) = f(n) + g(n)$.

Helpful Formulas

1. $p \Rightarrow q \iff \neg p \vee q$ $\qquad$ $\neg(p \wedge q) \iff \neg p \vee \neg q$
   $\neg(p \vee q) \iff \neg p \wedge \neg q$ $\qquad$ $\neg(p \Rightarrow q) \iff p \wedge \neg q$

2.
$$\begin{aligned} \lfloor x \rfloor &= \text{the largest integer less than or equal to } x \\ \lceil x \rceil &= \text{the smallest integer greater than or equal to } x \end{aligned}$$
$$y = \lfloor x \rfloor \text{ means that } y \in \mathbb{Z} \wedge y \leq x \wedge (\forall z \in \mathbb{Z}\ z \leq x \Rightarrow z \leq y)$$
$$y = \lceil x \rceil \text{ means that } y \in \mathbb{Z} \wedge y \geq x \wedge (\forall z \in \mathbb{Z}\ z \geq x \Rightarrow z \geq y)$$

3. $\forall\, m, n \in \mathbb{Z}\ (m \mid n \Leftrightarrow \exists\, k \in \mathbb{Z}\ n = km)$

4. $\forall\, G = (V, E)\ 0 \leqslant |E| \leqslant \dfrac{|V|(|V|-1)}{2}$ $\qquad$ $\forall\, G = (V, E)$ G is connected $\Rightarrow |E| \geqslant |V| - 1$
   $\forall\, G = (V, E)\ |E| > \dfrac{(|V|-1)(|V|-2)}{2} \Rightarrow$ G is connected

5. $\displaystyle\sum_{i=1}^{n} i = \frac{n(n+1)}{2}$ $\qquad$ $\displaystyle\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ $\qquad$ $\displaystyle\sum_{i=1}^{n} i^3 = \left(\frac{n(n+1)}{2}\right)^2$

6. $\displaystyle\sum_{i=0}^{n} \frac{1}{2^i} = \frac{2^{n+1}-1}{2^n}$ $\qquad$ $\displaystyle\sum_{i=1}^{n} \frac{i}{2^i} = 2 - \frac{1}{2^{n-1}} - \frac{n}{2^n}$

7. python range function

```
range([start], stop, [step]) -> list-like-object of int
Return the integers starting with start and ending with stop - 1 with step
specifying the amount to increment (or decrement).
If start is not specified, the list starts at 0.  If step is not specified,
the values are incremented by 1.
```

Please write nothing on this page.

Total Marks = 90

END