CSC165H1: Problem Set 4

Due December 6, 2017 before 10pm

1.(a)

WTS: $\forall\, G = (V, E) \in \mathcal{G}, 2 \mid |\{v \mid 2 \mid (d(v) + 1)\ \wedge\ v \in V\}|$

Proof:

I will prove this by contradiction.

Assume that this statement is false that is:

Assume that $\exists G = (V, E) \in \mathcal{G},\ 2 \mid (|\{v \mid 2 \mid (d(v) + 1)\ \wedge\ v \in V\}| + 1)$, Let G be that graph

- First, I want to prove that $\sum_{v \in V} d(v) = 2|E|$:
  According to the definition of edge: each pair (v_1, v_2), consists of two distinct vertices is called an edge of the graph. Hence each edge is connected with 2 vertices, and according to the definition of (degree of v, d(v)).$\mathrm{d(v)}: |\{(\mathrm{v},\mathrm{u}) \mid (\mathrm{u},\mathrm{v}) \in \mathrm{E}\}|$, for $\mathrm{v} \in \mathrm{V}, \mathrm{G} = (\mathrm{V}, \mathrm{E})$. It is the magnitude of edges stemming from the vertex. Hence we can conlude that $\sum_{v \in V} d(v) = 2|E|$.
- Second, I want to prove that the sum of odd numbers of odd number is odd. And the sum of even numbers is always even, and the sum of an odd number and a even number is odd:
  1. the sum of even numbers is always even:
     let $a_1, a_2, \cdots a_p$, be a set of even number. So $2|a_1\ \wedge\ 2|a_2 \wedge \ldots \wedge 2|a_p$ According to the definition 2.1, they can be written as $2b_1, 2b_2, \ldots 2b_p$ with $b_1, b_2, \cdots, b_p$ are integers. Hence $a_1 + a_2 + \cdots + a_p = 2(b_1 + b_2 + \cdots + b_p)$. Since $2|2(b_1 + b_2 + \cdots + b_p)$, the sum of $a_1, a_2, \cdots a_p$ is a even number. Hence the sum of even numbers is always even
  2. the sum of an odd number and a even number is odd:
     let a1 be an odd number,2|(a1-1) ,hence it can be written as 2b1+1 with b1 be an integer.
     Let a2 be a even number,2|a2, hence it can be written as 2b2 with b2 be an integer. Hence a1+a2=2b1+1+2b2=2(b1+b2)+1. Since 2|2(b1+b2)+1+1, the sum of a1 and a2 is an odd number. Hence the sum of an odd number and a even number is odd
  3. the sum of two odd numbers is even:
     let a1, a2 be two odd numbers. So 2|(a1-1) and 2|(a2-1) , hence they can be written as 2b1+1,2b2+1 with b1,b2 are integers.
     Hence a1+a2 =2(b1+b2+1). Since 2|2(b1+b2+1), the sum of a1,a2 is a even number. Hence the sum of two odd numbers is even
  4. the sum of odd numbers of odd number is odd:
     let a1,a2,⋯ap, be a set of odd number. We can write their sum as (a1+a2)+(a3+a4)+⋯+(ap-2+ap-1)+ap
     since the sum of two odd numbers is even, the sum of a1 and a2, the sum of a3+a4⋯ are all even number. So (a1+a2)+(a3+a4)+⋯+(ap-2+ap-1) is a even number. Since the sum of an odd number and a even number is odd.
     Hence(a1+a2)+(a3+a4)+⋯+(ap-2+ap-1)+ap is an odd number.
     Hence the sum of odd numbers of odd number is odd

- Let $V_1 = \{v \mid 2 \mid (d(v) + 1)\ \wedge\ v \in V\}$, and $V_2 = \{v \mid 2 \mid d(v)\ \wedge\ v \in V\}$
  And according to our assumption: $2 \mid (|V_1| + 1)$

Hence $\sum_{v\in V} d(v)$=$\sum_{v\in V_1} d(v) + \sum_{v\in V_2} d(v)$ **is an odd number**, (since the sum of odd numbers of odd number is odd. And the sum of even numbers is always even, and the sum of an odd number and a even number is odd. We have proved it in the above)

- Since we have proved that $\sum_{v\in V} d(v) = 2|E|$ and since $|E| \in \mathbb{N}$, Thus $\sum_{v\in V} d(v)$ must **be a even number**

So then we have deduced both $\sum_{v\in V} d(v)$ **is an odd number** and $\sum_{v\in V} d(v)$ must **be a even number** which is our contradiction.

Hence we have proved that for any graph G = (V, E) the number of vertices with odd degree is even. ■

(b)

$\{(v_1, v_3), (v_2, v_3), (v_2, v_4), (v_{3,} v_4)\}$

Proof:

According to the definition of (degree of v, d(v)).$\mathrm{d(v)}: |\{(\mathrm{v}, \mathrm{u})|(\mathrm{u}, \mathrm{v}) \in \mathrm{E}\}|, \text{for } \mathrm{v} \in \mathrm{V}, \mathrm{G} = (\mathrm{V}, \mathrm{E})$, we can find $0 \le \mathrm{d}(v_4) \le 3$ (to be adjacent with all of v_1, v_2, v_3), and I will prove that $\mathrm{d}(v_4)$ can only be 2 by 4 different cases :

Case1. $\mathrm{d}(v_4) \neq 3$,

Since we have proven in a that $\sum_{v\in V} d(v) = 2|E|$ and since $|\mathrm{E}| \in \mathbb{N}$, $\sum_{v\in V} d(v)$ must **be a even number**

If $\mathrm{d}(v_4) = 3$, then $\sum_{v\in V} d(v) = \mathrm{d}(v_1) + \mathrm{d}(v_2) + \mathrm{d}(v_3) + \mathrm{d}(v_4) = 1 + 2 + 3 + 3 = 9$, which is an odd.

Hence $\mathrm{d}(v_4) \neq 3$

Case2. $\mathrm{d}(v_4) \neq 1$,

Since we have proven in a that $\sum_{v\in V} d(v) = 2|E|$ and since $|\mathrm{E}| \in \mathbb{N}$, $\sum_{v\in V} d(v)$ must **be a even number**

If $\mathrm{d}(v_4) = 1$, then $\sum_{v\in V} d(v) = \mathrm{d}(v_1) + \mathrm{d}(v_2) + \mathrm{d}(v_3) + \mathrm{d}(v_4) = 1 + 2 + 3 + 1 = 7$, which is an odd.

Hence $\mathrm{d}(v_4) \neq 1$

Case3. $\mathrm{d}(v_4) \neq 0$

Since $\mathrm{d}(v_3) = 3$, according to the definition of the degree of a vertex, and since there is only 4 vertices v_1, v_2, v_3, v_4, so v_3 must be adjacent with every vertices that is not v_3 itself. Hence v_4 must be adjacent with at least 1 vertice. Thus $\mathrm{d}(v_4) > 0$.

Hence $\mathrm{d}(v_4) \neq 0$

Case4. $\mathrm{d}(v_4) = 2$

We can construct a graph matches with these 4 requires ($\mathrm{d}(v_1) = 1\,, \mathrm{d}(v_2) = 2, \mathrm{d}(v_3) = 3, \mathrm{d}(v_4) = 2$)

The graph is described as follows:

v_1 is only adjacent with v_3,

v_2 is only adjacent with v_3 *and* v_4,

v_3 is adjacent with v_1 *and* v_2 *and* v_4,

v_4 is only adjacent with v_3 *and* v_2

Hence $\mathrm{d}(v_1) = 1\,, \mathrm{d}(v_2) = 2, \mathrm{d}(v_3) = 3, \mathrm{d}(v_4) = 2$ and we match the 4 requires.

Hence $\mathrm{d}(v_4)$ can be 2 .

In all $\mathbf{d}(v_4)$ can only be 2

Now we will prove that the graph(with its edge be $\{(v_1, v_3), (v_2, v_3), (v_2, v_4), (v_{3,}v_4)\}$) we described above is the only one possible:
Since we have proved that $\mathrm{d}(v_4)$ can only be 2 as above. We now have the degree of the 4 vertices that is $\mathrm{d}(v_1) = 1, \mathrm{d}(v_2) = 2, \mathrm{d}(v_3) = 3, \mathrm{d}(v_4) = 2$.

- Since $\mathrm{d}(v_3) = 3$ according to the definition of the degree of a vertex, and since there is only 4 vertices v_1, v_2, v_3, v_4, so v_3 must be adjacent with every vertices that is not v_3 itself.
- Since $\mathrm{d}(v_1) = 1$ hence v_1 can only be adjacent with only one vertices from v_2, v_3, v_4, and we have proved that v_3 must be adjacent with every vertices that is not v_3 itself above. Hence v_1 can only be adjacent with v_3
- For v_2 *and* v_4, since $\mathrm{d}(v_2) = 2, \mathrm{d}(v_4) = 2$, they can only be adjacent with 2 vertices from v_1, v_3, v_4 *and* v_2, v_3, v_1 respectively. And since v_1 can only be adjacent with v_3, v_2 *and* v_4 can only be adjacent with 2 vertices from v_3, v_4 *and* v_2, v_3, hence v_2 is only adjacent with v_3 *and* v_4 and v_4 is only adjacent with v_3 *and* v_2

And we get the only possible graph (with its edge be $\{(v_1, v_3), (v_2, v_3), (v_2, v_4), (v_{3,}v_4)\}$) from the description above.
Hence we have proved that the graph (with its edge be $\{(v_1, v_3), (v_2, v_3), (v_2, v_4), (v_{3,}v_4)\}$) is the only one possible■

(c)
WTS: $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G},\ (\exists\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) = \mathrm{n}) \Longrightarrow |\mathrm{V}| \geq \mathrm{n} + 1$
Proof:
I will prove this by contrapositive that is $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G}, |\mathrm{V}| < \mathrm{n} + 1 \Longrightarrow \forall\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq \mathrm{n}$
Let $\mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G}$, assume that $|\mathrm{V}| < \mathrm{n} + 1$, we want to prove that $\forall\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq \mathrm{n}$
Since $|\mathrm{V}| < \mathrm{n} + 1$, we can assume that $|\mathrm{V}| \leq \mathrm{n}$
And according to the definition of (degree of v, d(v)).$\mathrm{d}(\mathrm{v}): |\{(\mathrm{v}, \mathrm{u}) | (\mathrm{u}, \mathrm{v}) \in \mathrm{E}\}|, \text{for } \mathrm{v} \in \mathrm{V}, \mathrm{G} = (\mathrm{V}, \mathrm{E})$ and the definition of edge (a set E of pairs of objects ,where each pair (v_1, v_2) consists of two distinct vertices is called an edge of the graph).
We can find that $\forall\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) < \mathrm{n}$, hence $\forall\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq \mathrm{n}$
Hence $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G}, |\mathrm{V}| < \mathrm{n} + 1 \Longrightarrow \forall\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq \mathrm{n}$
Thus we have proven that $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G},\ (\exists\, \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) = \mathrm{n}) \Longrightarrow |\mathrm{V}| \geq \mathrm{n} + 1$■

(d)
WTS: $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G}, (\forall \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) = 2) \Longrightarrow \mathrm{G\ has\ a\ cycle}$
Proof:
I will prove this by contrapositive, that is $\forall\, \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G}, \mathbf{G\ does\ not\ have\ a\ cycle} \Longrightarrow (\exists \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq 2)$
$\mathrm{Let} \mathrm{G} = (\mathrm{V}, \mathrm{E}) \in \mathcal{G},$ assume that G does not have a cycle. We want to show that $\exists \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq 2$

- For |V|=0, so G does not have a circle (since a circle must have at least 3 vertices) and the whole statement $\exists \mathrm{v} \in \mathrm{V}, \mathrm{d}(\mathrm{v}) \neq 2$ is wrong. Hence, the statement is true for |V|=0.

- For |V|=1, so G does not have a circle (since a circle must have at least 3 vertices) and the only degree exists is equal to zero. Hence, the statement is true for |V|=1.
- For |V|=2, so G does not have a circle (since a circle must have at least 3 vertices) and $\forall v \in V, d(v) \leq 1$ Hence, the statement is true for |V|=2.
- For |V|>2:
  Let u be an arbitrary vertex in V. Let v be a vertex in G that is at the maximum possible distance from u, i.e., the path between v and u has maximum possible length (compared to paths between u and any other vertex). We will prove that v has exactly one neighbor.
  Let P be the shortest path between v between u. We know that v has at least one neighbor: the vertex immediately before it on P. v cannot be adjacent to any other vertex on P, as otherwise G would have a cycle (and we have assumed before that G does not have a circle). Also, v cannot be adjacent to any other vertex w not on P, as otherwise we could extend P to include w, and this would create a longer path.
  And so v has exactly one neighbor (the one on P immediately before v). Thus d(v)=1.
  And we have proved that $\exists v \in V, d(v) \neq 2$.

Hence we have proved that $\forall\ G = (V, E) \in \mathcal{G}, (\forall v \in V, d(v) = 2) \Longrightarrow \text{G has a cycle}$■

(e)

WTS: $\forall\ G = (V, E) \in \mathcal{G}, (\forall v \in V, d(v) \geq |V| - 3 \land |V| > 4) \Longrightarrow \text{G is connected}$

Proof:

I will prove this by contradiction.

Assume that this statement is false that is:

$\exists\ G = (V, E) \in \mathcal{G}, (\forall v \in V, d(v) \geq |V| - 3 \land |V| > 4) \land \text{G is not connected}$, let G be that graph

Since G is not connected, there must exist a pair of vertices $u, v \in V$, that are not connected. Let u, v be that value.

Since u, v are not connected, there is no path from u to v.

And according to the definition of path:

$\text{Let } G = (V, E) \text{ and let } u, v \in V, \text{a path between u and v is a sequence of distinct vertices } v_0, v_1, v_2, \dots, v_k \in V \textit{ which satisfy the following properties}:$

$1. v_0 = u, v_k = v$

$2. \textit{Each consecutive pair of vertices are adjacent.}$

Hence there must not have a vertice which is adjacent to u and adjacent to v.

Therefore, the neighbours of u and v must be different.

And u, v are not adjacent to each other.

Let A be the set of all neighbours of u and let B be the set of all neighbours of v

Hence $A \cap B = \emptyset$ and since $A \in V \text{ and } B \in V \text{ and } u \notin A \text{ and } u \notin B \text{ and } v \notin A \text{ and } v \notin B \text{ and } u \in V \text{ and } v \in V$ we can conclude that $|A| + |B| + 2 \leq |V|$

We also know from the definition of (degree of v, d(v)).$d(v){:}\ |\{(v, u) | (u, v) \in E\}|, \text{for } v \in V, G = (V, E)$ and the definition of edge (a set E of pairs of objects ,where each pair (v_1, v_2) consists of two distinct vertices is called an edge of the graph)., that |A|=d(u) and |B|=d(v)

We know from the assumption that $\forall v \in V, d(v) \geq |V| - 3$ and $u, v \in V$

Hence $d(u) \geq |V| - 3$ and $d(v) \geq |V| - 3$, that is $|A| \geq |V| - 3$ and $|B| \geq |V| - 3$
Hence $|A| + |B| + 2 \geq 2|V| - 4$
Since we have assumed that $|V| > 4$ in the assumption, we can conclude that
$|A| + |B| + 2 \geq 2|V| - 4 > |V|$
So then we have deduced both $|A| + |B| + 2 \leq |V|$ and $|A| + |B| + 2 > |V|$, which is our contradiction.
Hence, we have proved that $\forall\, G = (V, E) \in \mathcal{G}, (\forall v \in V, d(v) \geq |V| - 3 \wedge |V| > 4) \Longrightarrow$
$G \text{ is connected} \blacksquare$

(f)
From (e), we know that $\forall v \in V, d(v) \geq |V| - 3$, assume n=|V|
We also proved before that $\sum_{v \in V} d(v) = 2|E|$

Hence we can conclude that $|E| = \frac{1}{2}\sum_{v \in V} d(v) \geq \frac{|V| \times (|V|-3)}{2} = \frac{n(n-3)}{2} = \frac{n^2-3n}{2}$ and G in this situation with the situation in (e) is connected.
We also know from the textbook that $\text{Let } n \in \mathbb{Z}^+. \text{For all graphs } G = (V.E), \text{if } |V| = n \text{ and } |E| \geq \frac{(n-1)(n-2)}{2} + 1, \text{then G is connected.}$

Since $\frac{(n-1)(n-2)}{2} + 1 = \frac{n^2-3n+2}{2} + 1 = \frac{n^2-3n}{2} + 2$

Since $\frac{n^2-3n}{2} + 2$ is not equal to $\frac{n^2-3n}{2}$, we can find that the statement will slightly change after specifying the minimum degree.

And we will prove roughly that when $|E| = \frac{n^2-3n}{2}$ and $|E| = \frac{n^2-3n}{2} + 1$ in the condition (e), the graph is still connected.

$|E| = \frac{n^2-3n}{2} + 1$: in the class the counter example we chose for example 6.6 is Let G = (V ,E) be the graph defined as follows first $V = \{v_1, v_2, \dots, v_n\}$ second $E = \{(v_i, v_j) | i, j \in \{1, \dots, n\text{-}1\} \text{ and } i < j\}$.That is E consists of all edges between the first n-1 vertices, and has no edges connected to v_n.
And we can find this is the only counter-example for example 6.6.
But in (e), we have stated that $\forall v \in V, d(v) \geq |V| - 3 \wedge |V| > 4$ hence $\forall v \in V, d(v) \geq 1$
So there will not be any point that stays alone without any neighbors. Hence, there will be no such counter-example. Hence, G is connected when $|E| = \frac{n^2-3n}{2} + 1$ in the situation of (e)

$|E| = \frac{n^2-3n}{2}$: thinking like what we do in example 6.6. Let G = (V ,E) be the graph defined as follows first $V = \{v_1, v_2, \dots, v_n\}$ second $E = \{(v_i, v_j) | i, j \in \{1, \dots, n\text{-}2\} \text{ and } i < j\}$.That is E consists of all edges between the first n-2 vertices, and has no edges connected to v_n and $v_{n\text{-}1}$. The |E| in this graph is $\frac{(n-2)(n-3)}{2}$, and we know that $\frac{n^2-3n}{2} - \frac{(n-2)(n-3)}{2} = n\text{-}3 > 1$ since $|V| > 4$, hence one of v_n and $v_{n\text{-}1}$ must be connected with graph G. Hence the worst case for $|E| = \frac{\mathrm{n}^2\text{-}3\mathrm{n}}{2}$ is actually E consists of all edges (except 1)between the first n-1 vertices, and has no edges connected to $\mathrm{v_n}$, which is quite similar as the worst graph in $|E| = \frac{\mathrm{n}^2\text{-}3\mathrm{n}}{2} + 1$.

But in (e), we have stated that $\forall \mathrm{v} \in \mathrm{V}, \mathrm{d(v)} \geq |\mathrm{V}| - 3 \wedge |\mathrm{V}| > 4$ hence $\forall \mathrm{v} \in \mathrm{V}, \mathrm{d(v)} \geq 1$
So there will not be any point that stays alone without any neighbors. Hence, there will be no such counter-example. Hence, G is connected when $|E| = \frac{\mathrm{n}^2\text{-}3\mathrm{n}}{2}$ in the situation of (e)

2(a)

Proof:

WTS: $\forall \mathrm{n} \in \mathbb{N}^+, there\ is\ a\ unique\ binary\ representation\ of\ n\ in\ the\ following\ form: n = \sum_{i=0}^{p} b_i\, 2^i, where$ p is the smallest integer such that $2^{p+1} > n$, p is non-negative, and $b_0, b_1, \ldots, b_p \in \{0,1\}$

I will first prove that $\forall \mathrm{n} \in \mathbb{N}^+, there\ is\ a\ unique\ binary\ representation\ of\ n\ in\ the\ following\ form: n = \sum_{i=0}^{p} b_i\, 2^i$

Then prove that p is the smallest integer such that $2^{p+1} > n$

First prove $\forall \mathbf{n} \in \mathbb{N}^+, \boldsymbol{there\ is\ a\ unique\ binary\ representation\ of\ n\ in\ the\ following\ form: n = \sum_{i=0}^{p} b_i\, 2^i}$:

Rather than proving the statement as written, we will prove an equivalent statement that is more amenable to using our technique of induction.

$$\forall \mathrm{m} \in \mathbb{N}^+, (\forall \mathrm{n} \in \mathbb{N}^+, n \leq m \Longrightarrow (\exists!\, p \in \mathbb{N}, \exists!\, b_0, b_1, \ldots, b_{p\text{-}1} \in \{0,1\}, b_p = 1 \wedge n = \sum_{i=0}^{p} b_i\, 2^i))$$

We define the predicate P(m) to be the part after the $\forall \mathrm{m} \in \mathbb{N}$, which can be translated as "every natural number less than m has and only has a binary representation." We'll prove by induction on m that $\forall \mathrm{m} \in \mathbb{N}^+, \mathrm{P(m)}$.

Base case (m=1): let $\mathrm{n} \in \mathbb{N}^+$ and assume that $n \leq m = 1$, hence n=1.

We want to show that $\exists!\, p \in \mathbb{N}, \exists!\, b_0, b_1, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i\, 2^i$ with $b_p = 1$

First.

let p=0 and let $b_0 = 1$, (they meet the require that $p \in \mathbb{N}\ and\ b_0, b_1, \ldots, b_p \in \{0,1\}$)

$\sum_{i=0}^{p} b_i\, 2^i = b_0 2^0 = 1 = n$, hence $\exists p \in \mathbb{N}, \exists b_0, b_1, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i\, 2^i$

Second.

for p=0 there is only one possible value for b_0 which is 1 since if $b_0 = 0, \sum_{i=0}^{p} b_i\, 2^i = 0$, that is not equal to n=1.

Can p be other value other than 0 ? let p be another value such as a other than 0. Since $p \in \mathbb{N}$, a>0, hence $2^p = 2^a \geq 2^1 = 2$, and since there are no leading zeros in binary representation of n, $b_p = b_a = 1$. Hence $\sum_{i=0}^{p} b_i\, 2^i \geq b_p 2^p \geq 2 > 1 = n$. Hence p can not be any other value other than 0

Hence $\exists! p \in \mathbb{N}, \exists! b_0, b_1, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i 2^i$

Hence p(1) is true.

Inductive step.

Let $\mathrm{m} \in \mathbb{N}^+$, and assume that P(m) is true, that is $\forall \mathrm{n} \in \mathbb{N}^+, n \leq m \Longrightarrow (\exists! p \in \mathbb{N}, \exists! b_0, b_1, \ldots, b_p \in \{0,1\}, b_p = 1 \wedge n = \sum_{i=0}^{p} b_i 2^i)$, We want to prove that P(m+ 1) is true, that is $\forall \mathrm{n} \in \mathbb{N}^+, n \leq m+1 \Longrightarrow (\exists! p \in \mathbb{N}, \exists! b_0, b_1, \ldots, b_p \in \{0,1\}, b_p = 1 \wedge\ n = \sum_{i=0}^{p} b_i 2^i)$

Let $\mathrm{n} \in \mathbb{N}^+$, and assume that $n \leq m+1$, we want to show that $\exists! p \in \mathbb{N}, \exists! b_0, b_1, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i 2^i$ with $b_p = 1$

- For $n \leq m$, then by the induction hypothesis n has and only has a binary representation. So we'll further assume that n = m+ 1 for the rest of this proof.
- For n=m+1. We'll divide up the rest of the proof into two cases, depending on whether n is either even or odd.
  - Case 1: assume n is even, i.e., there exists $\mathrm{k} \in \mathbb{N}$, such that n = 2k.

    First, we prove the existence.

    By one of our earlier properties of divisibility, we know that since k | n, k < n. Therefore by the induction hypothesis there exists $p \in \mathbb{N}\ and\ b_0, b_1, \ldots, b_p \in \{0,1\}$ such that $\mathrm{k} = \sum_{i=0}^{p} b_i 2^i$. Then $\mathrm{n} = 2\sum_{i=0}^{p} b_i 2^i = \sum_{i=0}^{p} b_i 2^{i+1}$

    Let p'=p+1, and let $b'_0 = 0$, and for all $\mathrm{i} \in \{1,2,\ldots,\mathrm{p}+1\}$, let $b'_i = b_{i-1}$. Then n=$\sum_{i=0}^{p'} b'_i 2^i$, since $b_p = 1 \text{ so } b'_p = 1$

    Now we gonna to prove the uniqueness of the $binary\ representation$:

    We will prove this by contradiction, assume that there are least 2 different ways for the $binary\ representation$ of n.

    Hence we can represent n as $\sum_{i=0}^{p} b_i 2^i$ and $\sum_{i=0}^{q} a_i 2^i$ with $p \in \mathbb{N}\ and\ b_0, b_1, \ldots, b_p \in \{0,1\}$ and $q \in \mathbb{N}\ and\ a_0, a_1, \ldots, a_p \in \{0,1\}$ and they are different in some elements,

    Since n is even, $b_0 = a_0 = 0$, hence we can write the n as $\sum_{i=1}^{p} b_i 2^i$ and $\sum_{i=1}^{q} a_i 2^i$,

    Therefore $k = \frac{1}{2}\sum_{i=1}^{p} b_i 2^i = \frac{1}{2}\sum_{i=1}^{q} a_i 2^i$

    Let p'=p-1, , and for all $\mathrm{i} \in \{0,1,2,\ldots,\mathrm{p}-1\}$, let $b'_i = b_{i+1}$. Then

    $$\frac{1}{2}\sum_{i=1}^{p} b_i 2^i = \sum_{i=0}^{p'} b'_i 2^i$$

    Let q'=q-1, , and for all $\mathrm{i} \in \{0,1,2,\ldots,\mathrm{q}-1\}$, let $a'_i = a_{i+1}$. Then

    $$\frac{1}{2}\sum_{i=1}^{q} a_i 2^i = \sum_{i=0}^{q'} a'_i 2^i$$

    Since $\sum_{i=0}^{p} b_i 2^i$ and $\sum_{i=0}^{q} a_i 2^i$ are two different ways for the $binary\ representation$ of n.

    $\sum_{i=0}^{p'} b'_i 2^i\ and \sum_{i=0}^{q'} a'_i 2^i$ therefore must also be two different ways for the $binary\ representation$ of k (since $b_0 = a_0 = 0$, we do not need to consider their difference into the whole representation and since since $b_p = 1 \text{ so } b'_p = 1, \text{since } b_{\mathrm{q}} = 1\ so\ b'_{\mathrm{q}} = 1$)

Since n=m+1 with $\mathrm{m} \in \mathbb{N}^{+}$ and $\mathrm{n} \in \mathbb{N}^{+}$

Hence k=$\frac{1}{2}n$=$\frac{1}{2}(m+1)>0$

Since $\mathrm{m} \in \mathbb{N}^{+}$ hence $\mathrm{m} \geq 1 \Longrightarrow 2\mathrm{m} \geq \mathrm{m}+1 \Longrightarrow \mathrm{m} \geq \frac{1}{2}(m+1)$=k

Thus we can find that $0<\mathrm{k} \leq \mathrm{m}$ and $\mathrm{k} \in \mathbb{N}$

Therefore, according to the induction hypothesis we can conclude that k only have one way for the $binary\ representation$

So then we have deduced both k only have one way for the $binary\ representation$ and k must also be two different way for the $binary\ representation$ which is our contradiction.

Hence we have proved the uniqness.

Therefore we have proved that $\exists!\, p \in \mathbb{N}, \exists!\, b_0, b_1, \ldots, b_p \in \{0,1\}, n=\sum_{i=0}^{p} b_i\, 2^i$ with $b_p=1$

- ✧ Case 2: assume n is odd, i.e., there exists $\mathrm{k} \in \mathbb{N}$, such that n = 2k+1.

First, we prove the existence.

Similar to the previous case, by the induction hypothesis, there exists $p \in \mathbb{N}\ and\ b_0, b_1, \ldots, b_p \in \{0,1\}$ such that $\mathrm{k}=\sum_{i=0}^{p} b_i\, 2^i$. Then $\mathrm{n}=2\sum_{i=0}^{p} b_i\, 2^i+1=\sum_{i=0}^{p} b_i\, 2^{i+1}+1$

Let p'=p+1, and let $b'_0=1$, and for all $\mathrm{i} \in \{1,2,\ldots,\mathrm{p}+1\}$, let $b'_i=b_{i-1}$. Then n=$\sum_{i=0}^{p'} b'_i\, 2^i$ since $b_p=1$ so $b'_p=1$

Now we gonna to prove the uniqueness of the $binary\ representation$:

We will prove this by contradiction, assume that there are least 2 different ways for the $binary\ representation$ of n.

Hence we can represent n as $\sum_{i=0}^{p} b_i\, 2^i$ and $\sum_{i=0}^{q} a_i\, 2^i$ with $p \in \mathbb{N}\ and\ b_0, b_1, \ldots, b_p \in \{0,1\}$ and $q \in \mathbb{N}\ and\ a_0, a_1, \ldots, a_p \in \{0,1\}$ and they are different in some elements,

Since n is odd, $b_0=a_0=1$, hence we can write the n-1 as $\sum_{i=1}^{p} b_i\, 2^i$ and $\sum_{i=1}^{q} a_i\, 2^i$,

Therefore $k=\frac{1}{2}\sum_{i=1}^{p} b_i\, 2^i=\frac{1}{2}\sum_{i=1}^{q} a_i\, 2^i$

Let p'=p-1, , and for all $\mathrm{i} \in \{0,1,2,\ldots,\mathrm{p}-1\}$, let $b'_i=b_{i+1}$. Then

$$\frac{1}{2}\sum_{i=1}^{p} b_i\, 2^i=\sum_{i=0}^{p\prime} b'_i\, 2^i$$

Let q'=q-1, , and for all $\mathrm{i} \in \{0,1,2,\ldots,\mathrm{q}-1\}$, let $a'_i=a_{i+1}$. Then

$$\frac{1}{2}\sum_{i=1}^{q} a_i\, 2^i=\sum_{i=0}^{q\prime} a'_i\, 2^i$$

Since $\sum_{i=0}^{p} b_i\, 2^i$ and $\sum_{i=0}^{q} a_i\, 2^i$ are two different ways for the $binary\ representation$ of n.

$\sum_{i=0}^{p\prime} b'_i\, 2^i\ and \sum_{i=0}^{q\prime} a'_i\, 2^i$ therefore must also be two different ways for the $binary\ representation$ of k (since $b_0=a_0=1$, we do not need to

consider their difference into the whole representation and since $b_p = 1$ so $b'_p = 1$ and since $b_q = 1$ so $b'_q = 1$)
Since n=m+1 with $m \in \mathbb{N}^+$ and $n \in \mathbb{N}^+$

Hence k=$\frac{1}{2}(n-1)$=$\frac{1}{2}m > 0$

Since $m \in \mathbb{N}^+$ hence $m \geq 1 \Longrightarrow 2m \geq m+1 \Longrightarrow m \geq \frac{1}{2}(m+1) > \frac{1}{2}m$=k

Thus we can find that $0 < k \leq m$ and $k \in \mathbb{N}$
Therefore, according to the induction hypothesis we can conclude that k only have one way for the *binary representation*
So then we have deduced both k only have one way for the *binary representation* and k must also be two different way for the *binary representation* which is our contradiction.
Hence we have proved the uniqness.

Therefore we have proved that $\exists! p \in \mathbb{N}, \exists! b_0, b_1, \ldots, b_p \in \{0,1\}, n = \sum_{i=0}^{p} b_i 2^i$ with $b_p = 1$

Hence we have proved that $\forall n \in \mathbb{N}^+$, *there is a unique binary representation of n in the following form*: $n = \sum_{i=0}^{p} b_i 2^i$

Second, we will prove that p is the smallest integer such that $2^{p+1} > n$:
Let a be the smallest integer such that $2^{a+1} > n$, if p>a, then $2^{p+1} \geq 2 \times 2^{a+1}$, then $2^p \geq 2^{a+1} > n$.(since 2>0)
since there are no leading zeros in binary representation of n, $b_p = 1$.
Hence $\sum_{i=0}^{p} b_i 2^i \geq b_p 2^p > n$. Hence $p \leq a$.
Since a is the smallest integer such that $2^{a+1} > n$, hence $2^{a-1+1} \leq n$, that is $2^a \leq n$
If p<a then $\sum_{i=0}^{p} b_i 2^i < 2^0 + 2^1 + \ldots + 2^p < 2^p + 2^p = 2 \times 2^p \leq 2^a \leq n$ (we will prove that $2^0 + 2^1 + \ldots + 2^{p-1} < 2^p$ $(p > 0)$ *at below, if* $p = 0$ *it is certainly true*)
Hence $\sum_{i=0}^{p} b_i 2^i < n$. Hence p must not smaller than a .
Therefore p can only be equal to a which is the smallest integer such that $2^{p+1} > n$

(Proof for $2^0 + 2^1 + \ldots + 2^{p-1} < 2^p$ $(p > 0)$:
We will prove that $\forall n \in \mathbb{N}^+, 2^0 + 2^1 + \ldots + 2^{n-1} < 2^n$
We will prove this by induction on n
Base case： n=1, then$2^0 + 2^1 + \ldots + 2^{n-1} = 1$ and $2^n = 2$, since 2>1, the statement is true for n=1
Induction step: let $n \in \mathbb{N}^+$assume $2^0 + 2^1 + \ldots + 2^{n-1} < 2^n$, we want to show that $2^0 + 2^1 + \ldots + 2^n < 2^{n+1}$
Since $2^{n+1} = 2^n + 2^n > 2^n + 2^0 + 2^1 + \ldots + 2^{n-1}$ by induction hypothesis
Hence $2^0 + 2^1 + \ldots + 2^n < 2^{n+1}$
Therefore we have shown that $2^0 + 2^1 + \ldots + 2^{p-1} < 2^p$ $(p > 0)$)

In all we have proven that $\forall n \in \mathbb{N}^+$, *there is a unique binary representation of n in the following form*: $n = \sum_{i=0}^{p} b_i 2^i$, *where* p is the smallest integer such that $2^{p+1} > n$, p is non-negative, and

$b_0, b_1, \ldots, b_p \in \{0,1\}$

$\blacksquare$

(b)

Proof:

Since we have proven that $\forall \mathrm{n} \in \mathbb{N}^+, there\ is\ a\ unique\ binary\ representation\ of\ n\ in\ the\ following\ form: n = \sum_{i=0}^{p} b_i 2^i, where$ p is the smallest integer such that $2^{p+1} > n$, p is non-negative, and $b_0, b_1, \ldots, b_p \in \{0,1\}$

We only need to prove that $there\ is\ a\ unique\ binary\ representation\ for\ 0$

First , since 0=0, and 0 is a $binary\ representation$, so there exist a $binary\ representation\ for\ 0$

Second , can there be any other $binary\ representation\ for\ 0$?

We first assume yes, that there is aother $binary\ representation\ for\ 0$, let this representation be a. since there must be a 1 in a (otherwise it will be 0 since the bits are either 1 or 0)

Hence a=$\mathrm{x} \times 2^x + y \times 2^y + \ldots z \times 2^z > 0 \times 2^x + 0 \times 2^y + \ldots 0 \times 2^z = 0$, hence a>0. Therefore our first assumption is false.

Therefore, $there\ is\ a\ unique\ binary\ representation\ for\ 0$

Hence we have proved that there is a unique binary representation of every natural number $\blacksquare$

Explaination about why it wasn't just possible to make the domain of the previous proof “every number $\mathrm{n} \in \mathbb{N}$":

If the domain of the previous proof “every number $n \in \mathbb{N}$", then $there\ is\ a\ unique\ binary\ representation\ of\ 0\ in\ the\ following\ form: n = \sum_{i=0}^{p} b_i 2^i, where$ p is the smallest integer such that $2^{p+1} > 0$, p is non-negative, and $b_0, b_1, \ldots, b_p \in \{0,1\}$

Since we know that the smallest integer such that $2^{p+1} > 0$, with p is non-negative is p=0.

And we also know that there are no leading zeros in binary representation of n, so $b_0 = 1$

Hence $\sum_{i=0}^{p} b_i 2^i = 2^0 = 1 \neq 0$. Thus the whole statement is not true for n=0

Therefore, it wasn't possible to make the domain of the previous proof “every number $n \in \mathbb{N}$"

3(a)

Proof:

We first want to calculate an exact expression for

$$Avg_{search}(n) = \frac{1}{|I'_n|} \sum_{(lst,x) \in I'_n} running\ time\ of\ search(lst, x)$$

Note that $|I'_n| = 10 \times 10^n$, since we have 10 choices of x, and we select number from 1 to 10 to form the list.

the running time of search (lst, x) is the number of loop iterations performed, and this is exactly equal to the position that x appears in lst plus 1

Using this allows us to obtain a final expression for $Avg_{search}(n)$

$$Avg_{search}(n) = \frac{1}{|I'_n|} \sum_{(lst,x)\in I'_n} running\ time\ of\ search(lst, x)$$

$$= \frac{10\times[\sum_{i=0}^{n-1} 9^i\times 10^{n-i-1}\times(i+1)+9^n\times(n+1)]}{10\times 10^n}$$

$$= \frac{\sum_{i=0}^{n-1} 9^i\times 10^{n-i-1}\times(i+1)+9^n\times(n+1)}{10^n}$$

$$= \frac{\frac{1}{9}\times 10^n\times\sum_{i=1}^{n}(\frac{9}{10})^i\times i+9^n\times(n+1)}{10^n}$$

$$=\frac{1}{9}\times\sum_{i=1}^{n}(\frac{9}{10})^i\times i+(\frac{9}{10})^n\times(n+1)$$

$$=\frac{1}{9}\times(\frac{n\times(\frac{9}{10})^n}{-\frac{1}{10}}+\frac{\frac{9}{10}-(\frac{9}{10})^{n+1}}{(-\frac{1}{10})^2})+(\frac{9}{10})^n\times(n+1)$$ (by the formula provided)

$$= -\mathrm{n}\times\left(\frac{9}{10}\right)^{n-1}+10+(n-9)\left(\frac{9}{10}\right)^n$$

$$= 10-\left(\frac{9}{10}\right)^n(9+\frac{1}{9}n)$$

We gonna prove that $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\in\Theta(1)$

First, prove that $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\in O(1)$, that is $\exists \mathrm{c}, n_0\in\mathbb{R}^+, \forall n\in\mathbb{N}, n\geq n_0 \Longrightarrow 10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\leq c$:

Let c=10, $n_0=1$, let $n\in\mathbb{N},\ and\ assume\ that\ n\geq n_0$

Hence $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\leq 10=c$ since $\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)>0, with\ n>0$

Therefore, we have proved that $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\in O(1)$

Second, prove that $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\in\Omega(1)$, that is $\exists \mathrm{c}, n_0\in\mathbb{R}^+, \forall n\in\mathbb{N}, n\geq n_0 \Longrightarrow$ $10-\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\geq c$:

Let c=1, $n_0=100$, let $n\in\mathbb{N},\ and\ assume\ that\ n\geq n_0$

From calculation we get to know that $\left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)$=$0.0005341792442948161 < 1, when\ n=100$

$$we\ can\ also\ written\ \left(\frac{9}{10}\right)^n\left(9+\frac{1}{9}n\right)\ as\ \frac{\left(9+\frac{1}{9}n\right)}{\left(\frac{10}{9}\right)^n}\ for\ easy\ understanding$$

Since We also know that the derivative of $\left(\frac{10}{9}\right)^n=\left(\frac{10}{9}\right)^n\times ln(\frac{10}{9})>\left(\frac{10}{9}\right)^n\times ln(\frac{10}{9})>\left(\frac{10}{9}\right)^{100}\times ln(\frac{10}{9})\approx 3966.6773744073>\frac{1}{9}$, so $\left(\frac{10}{9}\right)^n$ grows much faster than $\left(9+\frac{1}{9}n\right)$

when $n \geq n_0 = 100$

Hence $\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right)$ (when $n \geq n_0 = 100$) will be smaller than $\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right)$=0.0005341792442948161 $< 1, when\, n = 100$

Hence $\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) < 1$, when $n \geq n_0$

Thus -$\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) > -1$

So $10-\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) > 9 > 1$, when $n \geq n_0$

Hence we have prove that $10-\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) \in \Omega(1)$

Since $10-\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) \in \Omega(1)\ and\ 10-\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) \in O(1)$, so $10-\left(\frac{9}{10}\right)^n \left(9+\frac{1}{9}n\right) \in \Theta(1)$. Hence the average-case running time of Search on this set of inputs is $\Theta(1)$■

(b)
We want to calculate an exact expression for

$$Avg_{search}(n) = \frac{1}{|I'_n|}\sum_{(lst,x)\in I'_n} running\ time\ of\ search(lst,x)$$

Note that $|I'_n| = 500 \times 500^n$, since we have 500 choices of x, and we select number from 1 to 500 to form the list.
the running time of search (lst, x) is the number of loop iterations performed, and this is exactly equal to the position that x appears in lst plus 1
Using this allows us to obtain a final expression for $Avg_{search}(n)$

$$Avg_{search}(n) = \frac{1}{|I'_n|}\sum_{(lst,x)\in I'_n} running\ time\ of\ search(lst,x)$$

$$= \frac{500\times[\sum_{i=0}^{n-1} 499^i\times 500^{n-i-1}\times(i+1)+499^n\times(n+1)]}{500\times 500^n}$$

$$= \frac{500\times[\sum_{i=0}^{n-1} 499^i\times 500^{n-i-1}\times(i+1)+499^n\times(n+1)]}{500^{n+1}}$$

$$= \frac{1}{499}\times\left(\frac{n\times(\frac{499}{500})^n}{-\frac{1}{500}} + \frac{\frac{499}{500}-(\frac{499}{500})^{n+1}}{(-\frac{1}{500})^2}\right) + \left(\frac{499}{500}\right)^n \times (n+1) \text{ (by the formula provided)}$$

$$= 500-\left(\frac{499}{500}\right)^n\left(499+\frac{1}{499}n\right)$$

And this is the answer.

(c)
$\mathrm{WC_{counter}(n)} \in \Theta(\mathrm{n})$

Proof:

In order to prove $\mathrm{WC}_{\mathrm{counter}}(\mathrm{n}) \in \Theta(\mathrm{n})$, we will first prove $\mathrm{WC}_{\mathrm{counter}}(\mathrm{n}) \in \mathrm{O}(\mathrm{n})$, then prove $\mathrm{WC}_{\mathrm{counter}}(\mathrm{n}) \in \Omega(\mathrm{n})$

(since in the question we assumed that n>0, hence there will cerntainly have iterations, and we won't consider the situation that n=0 and the worst running time of it be zero here.)

First, prove that $\mathrm{WC}_{\mathrm{counter}}(\mathrm{n}) \in \mathrm{O}(\mathrm{n})$:

Since n=s+u, and the function will be changed with the value of s and u ,we can divide the proof into two cases:

- u=0 and s=n:

  let's record the value of s and u for every 8 iterations:

  $$\mathrm{s} = \text{n-k} \longrightarrow \text{n-k} \to \text{n-k} \to \text{n-k} \to \text{n-k} \to \text{n-k} \to \text{n-k} \to \text{n-k} \to \text{n-k-1}$$
  $$\mathrm{u} = 0 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 6$$

  hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

  Then for any natural number k, after 7k iterations s will be n-k, and u will be 0.

  We also know from the question that the loop terminates when $\mathrm{s} + \mathrm{u} \leq 0$, and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

  So the loop terminates when $\mathrm{s} + \mathrm{u} = \text{n-k} + 0 \leq 0$, i.e.$\mathrm{k} \geq \mathrm{n}$.

  So then the loop will run for at most 7n iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n steps.

- $0 < \mathrm{u} < 7, \mathrm{s} = \text{n-u}$:

  let's record the value of s and u for the first u iterations:

  $$u = u \to u - 1 \to u - 2 \to \cdots \to 0$$
  $$\mathrm{s} = \text{n-u} \longrightarrow \text{n-u} \to \ldots \to \text{n-u}$$

  let's record the value of s and u for the next every 8 iterations:

  $$\mathrm{s} = \text{n-u-k} \longrightarrow \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k} \to \text{n-u-k-1}$$
  $$\mathrm{u} = 0 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 6$$

  hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

  Then for any natural number k, after 7k iterations s will be n-u-k, and u will be 0.

  We also know from the question that the loop terminates when $\mathrm{s} + \mathrm{u} \leq 0$, and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

  So the loop terminates when $\mathrm{s} + \mathrm{u} = \text{n-u-k} + 0 \leq 0$, i.e.$\mathrm{k} \geq \text{n-u}$.

  So then the loop will run for at most 7(n-u)+u iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n-6u steps.

since $\forall \mathrm{n} \in \mathbb{N}$, 7n-6u<7n (since u>0)

hence the upper bound of the worst-case running time of this function is 7n

since from the properties of Big-oh, Omega and theta, the Thereom 5.6 states that

for all $f: \mathbb{N} \to \mathbb{R}^{\geq 0}$ and all $a \in \mathbb{R}^{+}, a \times f \in \Theta(f)$, hence the upper bound of the worst-case running time of this function $\in \Theta(n)$.
Therefore, $WC_{counter}(n) \in O(n)$

Second, prove that $WC_{counter}(n) \in \Omega(n)$
We will prove a matching lower bound on the worst-case running time of this function, by finding an input family whose asymptotic runtime matches the bound we found in the previous part.
The input family we found for every natural number n is s=n and u=0:
let's record the value of s and u for every 8 iterations:

$$s = n\text{-}k \longrightarrow n\text{-}k \to n\text{-}k \to n\text{-}k \to n\text{-}k \to n\text{-}k \to n\text{-}k \to n\text{-}k \to n\text{-}k\text{-}1$$
$$u = 0 \to 6 \to 5 \to 4 \to 3 \to 2 \to 1 \to 0 \to 6$$

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.
Then for any natural number k, after 7k iterations s will be n-k, and u will be 0.
We also know from the question that the loop terminates when $s + u \leq 0$, and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)
So the loop terminates when $s + u = n\text{-}k + 0 \leq 0$, i.e.$k \geq n$.
So then the loop will run for 7n iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n steps.
since from the properties of Big-oh, Omega and theta, the Thereom 5.6 states that for all $f: \mathbb{N} \to \mathbb{R}^{\geq 0}$ and all $a \in \mathbb{R}^{+}, a \times f \in \Theta(f)$, hence the running time of this input family is $\Theta(n)$
Therefore the worst-case running time of this function $\in \Omega(n)$

In all we have proved that $WC_{counter}(n) \in O(n)$ and $WC_{counter}(n) \in \Omega(n)$
Hence $WC_{counter}(n) \in \Theta(n)$ ■

(d)
$BC_{counter}(n) \in \Theta(n)$ ($BC_{counter}(n)$ means the best-case run-time of counter with different input n)
Proof:
In order to prove $BC_{counter}(n) \in \Theta(n)$, we will first prove $BC_{counter}(n) \in \Omega(n)$, then prove $BC_{counter}(n) \in O(n)$
(since in the question we assumed that n>0, hence there will cerntainly have iterations, and we won't consider the situation that n=0 and the best running time of it be zero here.)
First, prove that $BC_{counter}(n) \in \Omega(n)$:
We can find from the question that after each iteration, either s will be decreased by 1 or u will be decreased by 1. Since we ignored other conditions (such that u will turned to 6 in the next iteration when it is 0 and s is positive meanwhile, since these will actually increasing th

number of iterations), so after k iteration $s + u \geq n\text{-}k$
We also know from the question that the loop terminates when $s + u \leq 0$, since $s + u \geq n\text{-}k$, so when k=n, $s + u \geq 0$, hence the loop must run for at least n iterations.
Since $n \in \Theta(n)$, the lower bound of the best-case running time of this function $\in \Theta(n)$ hence $BC_{counter}(n) \in \Omega(n)$.
Second, prove that $BC_{counter}(n) \in O(n)$
We will prove a matching upper bound on the best-case running time of this function, by finding an input family whose asymptotic runtime matches the bound we found in the previous part.
The input family we found for every natural number n is s=n-1 and u=1 (since we assumed that n>0 in the problem, hence this input family is suitable for all n):
let's record the value of s and u for the first iteration:

$$u = 1 \rightarrow 0$$

$$s = n\text{-}1 \longrightarrow n\text{-}1$$

let's record the value of s and u for the next every 8 iterations:

$$s = n\text{-}6\text{-}k \longrightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k \rightarrow n\text{-}6\text{-}k\text{-}1$$

$$u = 0 \rightarrow 6 \rightarrow 5 \rightarrow 4 \rightarrow 3 \rightarrow 2 \rightarrow 1 \rightarrow 0 \rightarrow 6$$

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.
Then for any natural number k, after 7k iterations s will be n-1-k, and u will be 0.
We also know from the question that the loop terminates when $s + u \leq 0$, and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)
So the loop terminates when $s + u = n\text{-}1\text{-}k + 0 \leq 0$, i.e.$k \geq n\text{-}1$.
So then the loop will run for at most 7(n-1)+1 iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n-6 steps.
We will prove that 7n-6$\in \Theta(n)$, that is $\exists a, b, n' \in \mathbb{R}^+, \forall n \in \mathbb{N}, n \geq n' \Longrightarrow a \times n \leq 7n\text{-}6 \leq b \times n$
Let a=6, b=8, n'=100, so $a, b, n' \in \mathbb{R}^+$
Let $n \in \mathbb{N}$, and assume that $n \geq n' = 100$
Since n≥100
So $7n\text{-}6 = 6n + n\text{-}6 \geq 6n + 100\text{-}6 > 6n$ hence $a \times n \leq 7n\text{-}6$
$8n > 8n\text{-}6 > 7n\text{-}6$, hence $7n\text{-}6 \leq b \times n$
Hence we have proved that 7n-6$\in \Theta(n)$
hence the running time of this input family is $\Theta(n)$
Therefore the best-case running time of this function $\in O(n)$
In all we have proved that $BC_{counter}(n) \in \Omega(n)$ and $BC_{counter}(n) \in O(n)$
Hence $BC_{counter}(n) \in \Theta(n)$■

(e)
$Avg_{counter}(n) \in \Theta(n)$

Explanation:

From the definition of average run-time, we got to know that

$$Avg_{\text{counter}}(n) = \frac{\sum_{t \in T_{f,n}} t}{|I_{f,n}|}$$

Where $I_{f,n} = \{i | i \text{ is an input to f } \wedge |i| = n\}$

And $T_{f,n} = \{t | \exists x \in I_{f,n}, t = RT_f(x)\}$

Hence $\frac{BC_{\text{counter}}(n) \times |I_{f,n}|}{|I_{f,n}|} \leq \frac{\sum_{t \in T_{f,n}} t}{|I_{f,n}|} \leq \frac{WC_{\text{counter}}(n) \times |I_{f,n}|}{|I_{f,n}|}$

Therefore $BC_{\text{counter}}(n) \leq Avg_{\text{counter}}(n) \leq WC_{\text{counter}}(n)$ so $Avg_{\text{counter}}(n) \in \Omega(BC_{\text{counter}}(n))$ and $Avg_{\text{counter}}(n) \in O(WC_{\text{counter}}(n))$ (since the statement before is always true no matter what n is, so $Avg_{\text{counter}}(n)$ is absolutely dominated by $WC_{\text{counter}}(n)$ and $BC_{\text{counter}}(n)$ is absolutely dominated by $Avg_{\text{counter}}(n)$ so they meet the requirement for big-oh and omega, for omega we use the properties of Big-oh, Omega, Theta theorem 5.3, for all f,g: $\mathbb{N} \to \mathbb{R}^{\geq 0}, g \in O(f)$ if and only if $f \in \Omega(g)$)

We have also proved before that $WC_{\text{counter}}(n) \in \Theta(n)$ and $BC_{\text{counter}}(n) \in \Theta(n)$

Hence $WC_{\text{counter}}(n) \in O(n)$ and $BC_{\text{counter}}(n) \in \Omega(n)$

Therefore we can write $BC_{\text{counter}}(n) \leq Avg_{\text{counter}}(n) \leq WC_{\text{counter}}(n)$ as $Avg_{\text{counter}}(n) \in \Omega(n)$ and $Avg_{\text{counter}}(n) \in O(n)$ (according to the properties of Big-oh, Omega, Theta theorem 5.4 that for all f,g,h: $\mathbb{N} \to \mathbb{R}^{\geq 0}$, if $f \in O(g)$ and $g \in O(h)$, then $f \in O(h)$. Moreover, the statement is still true if you replace Big-oh with Omega, , or if you replace Big-oh with theta.)

Since $Avg_{\text{counter}}(n) \in \Omega(n)$ and $Avg_{\text{counter}}(n) \in O(n)$, $Avg_{\text{counter}}(n) \in \Theta(n)$.