CSC165H1: Problem Set 4

Due December 6, 2017 before 10pm

1.(a)

WTS:

Proof:

I will prove this by contradiction.

Assume that this statement is false that is:

Assume that

* First, I want to prove that :

According to the definition of edge: each pair , consists of two distinct vertices is called an edge of the graph. Hence each edge is connected with 2 vertices, and according to the definition of (degree of v, d(v)).. It is the magnitude of edges stemming from the vertex. Hence we can conlude that

* Second, I want to prove that the sum of odd numbers of odd number is odd. And the sum of even numbers is always even, and the sum of an odd number and a even number is odd:

1. the sum of even numbers is always even:

let ,,…, be a set of even number. So 2|According to the definition 2.1, they can be written as 2,2,..2 with ,,…, are integers.

Hence ++…+=2(++…+). Since 2|2(++…+), the sum of ,,… is a even number. Hence the sum of even numbers is always even

1. the sum of an odd number and a even number is odd:

let a1 be an odd number,2|(a1-1) ,hence it can be written as 2b1+1 with b1 be an integer.

Let a2 be a even number,2|a2, hence it can be written as 2b2 with b2 be an integer.

Hence a1+a2=2b1+1+2b2=2(b1+b2)+1. Since 2|2(b1+b2)+1+1, the sum of a1 and a2 is an odd number. Hence the sum of an odd number and a even number is odd

1. the sum of two odd numbers is even:

let a1, a2 be two odd numbers. So 2|(a1-1) and 2|(a2-1) , hence they can be written as 2b1+1,2b2+1 with b1,b2 are integers.

Hence a1+a2 =2(b1+b2+1). Since 2|2(b1+b2+1), the sum of a1,a2 is a even number. Hence the sum of two odd numbers is even

1. the sum of odd numbers of odd number is odd:

let a1,a2,…ap, be a set of odd number. We can write their sum as (a1+a2)+(a3+a4)+…+(ap-2+ap-1)+ap

since the sum of two odd numbers is even, the sum of a1 and a2, the sum of a3+a4… are all even number. So (a1+a2)+(a3+a4)+…+(ap-2+ap-1) is a even number. Since the sum of an odd number and a even number is odd. Hence(a1+a2)+(a3+a4)+…+(ap-2+ap-1)+ap is an odd number.

Hence the sum of odd numbers of odd number is odd

* Let , and

And according to our assumption:

Hence = **is an odd number**, (since the sum of odd numbers of odd number is odd. And the sum of even numbers is always even, and the sum of an odd number and a even number is odd. We have proved it in the above)

* Since we have proved that and since , Thus must **be a even number**

So then we have deduced both  **is an odd number** and must **be a even number** which is our contradiction.

Hence we have proved that for any graph G = (V, E) the number of vertices with odd degree is even.

(b)

{(}

Proof:

According to the definition of (degree of v, d(v))., we can find (to be adjacent with all of ), and I will prove that can only be 2 by 4 different cases :

**Case1**.,

Since we have proven in a that and since , must **be a even number**

If , then , which is an odd.

Hence

**Case2**.,

Since we have proven in a that and since , must **be a even number**

If , then , which is an odd.

Hence

**Case3**.

Since , according to the definition of the degree of a vertex, and since there is only 4 vertices , so must be adjacent with every vertices that is not itself. Hence must be adjacent with at least 1 vertice. Thus .

Hence

**Case4**.

We can construct a graph matches with these 4 requires ()

The graph is described as follows:

is only adjacent with ,

is only adjacent with ,

is adjacent with ,

is only adjacent with

Hence and we match the 4 requires.

Hence can be 2 .

**In all can only be 2**

Now we will prove that the graph(with its edge be {(}) we described above is the only one possible:

Since we have proved that can only be 2 as above. We now have the degree of the 4 vertices that is .

* Since according to the definition of the degree of a vertex, and since there is only 4 vertices , so must be adjacent with every vertices that is not itself.
* Since hence can only be adjacent with only one vertices from , and we have proved that must be adjacent with every vertices that is not itself above. Hence can only be adjacent with
* For , since , they can only be adjacent with 2 vertices from respectively. And since can only be adjacent with, can only be adjacent with 2 vertices from , hence is only adjacent with and is only adjacent with

And we get the only possible graph (with its edge be {(}) from the description above.

Hence we have proved that the graph (with its edge be {(}) is the only one possible

(c)

WTS:

Proof:

I will prove this by contrapositive that is

Let , assume that , we want to prove that

Since , we can assume that

And according to the definition of (degree of v, d(v)). and the definition of edge (a set E of pairs of objects ,where each pair () consists of two distinct vertices is called an edge of the graph).

We can find that , hence

Hence

Thus we have proven that

(d)

WTS:

Proof:

I will prove this by contrapositive, that is

assume that G does not have a cycle. We want to show that

* For |V|=0, so G does not have a circle (since a circle must have at least 3 vertices) and the whole statement is wrong. Hence, the statement is true for |V|=0.
* For |V|=1, so G does not have a circle (since a circle must have at least 3 vertices) and the only degree exists is equal to zero. Hence, the statement is true for |V|=1.
* For |V|=2, so G does not have a circle (since a circle must have at least 3 vertices) and Hence, the statement is true for |V|=2.
* For |V|>2:

Let u be an arbitrary vertex in V. Let v be a vertex in G that is at the maximum possible distance from u, i.e., the path between v and u has maximum possible length (compared to paths between u and any other vertex). We will prove that v has exactly one neighbor.

Let P be the shortest path between v between u. We know that v has at least one neighbor: the vertex immediately before it on P. v cannot be adjacent to any other vertex on P, as otherwise G would have a cycle (and we have assumed before that G does not have a circle). Also, v cannot be adjacent to any other vertex w not on P, as otherwise we could extend P to include w, and this would create a longer path.

And so v has exactly one neighbor (the one on P immediately before v). Thus d(v)=1.

And we have proved that .

Hence we have proved that

(e)

WTS:

Proof:

I will prove this by contradiction.

Assume that this statement is false that is:

, let G be that graph

Since G is not connected, there must exist a pair of vertices , that are not connected. Let u, v be that value.

Since u, v are not connected, there is no path from u to v.

And according to the definition of path:

Hence there must not have a vertice which is adjacent to u and adjacent to v.

Therefore, the neighbours of u and v must be different.

And u, v are not adjacent to each other.

Let A be the set of all neighbours of u and let B be the set of all neighbours of v

Hence and since we can conclude that

We also know from the definition of (degree of v, d(v)). and the definition of edge (a set E of pairs of objects ,where each pair () consists of two distinct vertices is called an edge of the graph)., that |A|=d(u) and |B|=d(v)

We know from the assumption that and

Hence , that is

Hence

Since we have assumed that in the assumption, we can conclude that

So then we have deduced both and , which is our contradiction.

Hence, we have proved that

(f)

From (e), we know that , assume n=|V|

We also proved before that

Hence we can conclude that and G in this situation with the situation in (e) is connected.

We also know from the textbook that

Since

**Since is not equal to , we can find that the statement will slightly change after specifying the minimum degree.**

And we will prove roughly that when and in the condition (e), the graph is still connected.

in the class the counter example we chose for example 6.6 is Let G = (V ,E) be the graph defined as follows first second .That is E consists of all edges between the first n-1 vertices, and has no edges connected to.

And we can find this is the only counter-example for example 6.6.

But in (e), we have stated that hence

So there will not be any point that stays alone without any neighbors. Hence, there will be no such counter-example. Hence, G is connected when in the situation of (e)

thinking like what we do in example 6.6. Let G = (V ,E) be the graph defined as follows first second .That is E consists of all edges between the first n-2 vertices, and has no edges connected to. The |E| in this graph is , and we know that , hence one of must be connected with graph G. Hence the worst case for is actually E consists of all edges (except 1)between the first n-1 vertices, and has no edges connected to, which is quite similar as the worst graph in .

But in (e), we have stated that hence

So there will not be any point that stays alone without any neighbors. Hence, there will be no such counter-example. Hence, G is connected when in the situation of (e)

2(a)

Proof:

WTS:p is the smallest integer such that , p is non-negative, and

I will first prove that

Then prove that p is the smallest integer such that

**First prove:**

Rather than proving the statement as written, we will prove an equivalent statement that is more amenable to using our technique of induction.

We define the predicate P(m) to be the part after the , which

can be translated as “every natural number less than m has and only has a binary

representation.” We’ll prove by induction on m that .

**Base case (m=1)**: let and assume that , hence n=1.

We want to show that with

First.

let p=0 and let , (they meet the require that )

, hence

Second.

for p=0 there is only one possible value for which is 1 since if , , that is not equal to n=1.

Can p be other value other than 0 ? let p be another value such as a other than 0. Since , a>0, hence , and since there are no leading zeros in binary representation of n, . Hence . Hence p can not be any other value other than 0

Hence

Hence p(1) is true.

**Inductive step.**

Let , and assume that P(m) is true, that is , We want to prove that P(m+ 1) is true, that is

Let , and assume that , we want to show that with

* For , then by the induction hypothesis n has and only has a binary representation. So we’ll further assume that n = m+ 1 for the rest of this proof.
* For n=m+1. We’ll divide up the rest of the proof into two cases, depending on whether

n is either even or odd.

* Case 1: assume n is even, i.e., there exists , such that n = 2k.

First, we prove the existence.

By one of our earlier properties of divisibility, we know that since k | n,

k < n. Therefore by the induction hypothesis there exists such that . Then

Let p’=p+1, and let , and for all , let . Then n=, since

Now we gonna to prove the uniqueness of the :

We will prove this by contradiction, assume that there are least 2 different ways for the of n.

Hence we can represent n as and with and and they are different in some elements,

Since n is even, , hence we can write the n as

and ,

Therefore

Let p’=p-1, , and for all , let . Then

Let q’=q-1, , and for all , let . Then

Since and are two different ways for the of n.

therefore must also be two different ways for the of (since , we do not need to consider their difference into the whole representation and since since )

Since n=m+1 with and

Hence k==

Since hence =k

Thus we can find that and

Therefore, according to the induction hypothesis we can conclude that k only have one way for the

So then we have deduced both k only have one way for the and k must also be two different way for the which is our contradiction.

Hence we have proved the uniqness.

Therefore we have proved that with

* Case 2: assume n is odd, i.e., there exists , such that n = 2k+1.

First, we prove the existence.

Similar to the previous case, by the induction hypothesis, there exists such that . Then

Let p’=p+1, and let , and for all , let . Then n= since

Now we gonna to prove the uniqueness of the :

We will prove this by contradiction, assume that there are least 2 different ways for the of n.

Hence we can represent n as and with and and they are different in some elements,

Since n is odd, , hence we can write the n-1 as

and ,

Therefore

Let p’=p-1, , and for all , let . Then

Let q’=q-1, , and for all , let . Then

Since and are two different ways for the of n.

therefore must also be two different ways for the of (since , we do not need to consider their difference into the whole representation and since )

Since n=m+1 with and

Hence k==

Since hence =k

Thus we can find that and

Therefore, according to the induction hypothesis we can conclude that k only have one way for the

So then we have deduced both k only have one way for the and k must also be two different way for the which is our contradiction.

Hence we have proved the uniqness.

Therefore we have proved that with

Hence we have proved that

**Second, we will prove that p is the smallest integer such that :**

Let a be the smallest integer such that , if p>a, then , then.(since 2>0)

since there are no leading zeros in binary representation of n, .

Hence . Hence .

Since a is the smallest integer such that , hence , that is

If p<a then (we will prove that

Hence . Hence p must not smaller than a .

Therefore p can only be equal to a which is the smallest integer such that

(Proof for :

We will prove that

We will prove this by induction on n

Base case： n=1, then and , since 2>1, the statement is true for n=1

Induction step: let assume , we want to show that

Since by induction hypothesis

Hence

Therefore we have shown that )

In all we have proven that p is the smallest integer such that , p is non-negative, and

(b)

Proof:

Since we have proven that p is the smallest integer such that , p is non-negative, and

We only need to prove that

First , since 0=0, and 0 is a , so there exist a

Second , can there be any other ?

We first assume yes, that there is aother , let this representation be a. since there must be a 1 in a (otherwise it will be 0 since the bits are either 1 or 0)

Hence a=, hence a>0. Therefore our first assumption is false.

Therefore,

Explaination about why it wasn't just possible to make the domain of the previous proof “every number :

If the domain of the previous proof “every number , then p is the smallest integer such that , p is non-negative, and

Since we know that the smallest integer such that , with p is non-negative is p=0.

And we also know that there are no leading zeros in binary representation of n, so

Hence . Thus the whole statement is not true for n=0

Therefore, it wasn't possible to make the domain of the previous proof “every number

3(a)

Proof:

We first want to calculate an exact expression for

Note that , since we have 10 choices of x, and we select number from 1 to 10 to form the list.

the running time of search (lst, x) is the number of loop iterations performed, and this is exactly equal to the position that x appears in lst plus 1

Using this allows us to obtain a final expression for

=

(by the formula provided)

We gonna prove that

First, prove that , that is :

Let c=10, , let

Hence since

Therefore, we have proved that

Second, prove that , that is :

Let c=1, , let

From calculation we get to know that =

Since We also know that the derivative of , so grows much faster than when

Hence (when will be smaller than =

Hence , when

Thus -

So , when

Hence we have prove that

Since , so . Hence the average-case running time of Search on this set of

inputs is

(b)

We want to calculate an exact expression for

Note that , since we have 500 choices of x, and we select number from 1 to 500 to form the list.

the running time of search (lst, x) is the number of loop iterations performed, and this is exactly equal to the position that x appears in lst plus 1

Using this allows us to obtain a final expression for

(by the formula provided)

And this is the answer.

(c)

Proof:

In order to prove , we will first prove , then prove

(since in the question we assumed that n>0, hence there will cerntainly have iterations, and we won’t consider the situation that n=0 and the worst running time of it be zero here.)

First, prove that :

Since n=s+u, and the function will be changed with the value of s and u ,we can divide the proof into two cases:

* u=0 and s=n:

let’s record the value of s and u for every 8 iterations:

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

Then for any natural number k, after 7k iterations s will be n-k, and u will be 0.

We also know from the question that the loop terminates when , and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

So the loop terminates when , i.e..

So then the loop will run for at most 7n iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n steps.

let’s record the value of s and u for the first u iterations:

let’s record the value of s and u for the next every 8 iterations:

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

Then for any natural number k, after 7k iterations s will be n-u-k, and u will be 0.

We also know from the question that the loop terminates when , and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

So the loop terminates when , i.e..

So then the loop will run for at most 7(n-u)+u iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n-6u steps.

since , 7n-6u<7n (since u>0)

hence the upper bound of the worst-case running time of this function is 7n

since from the properties of Big-oh, Omega and theta, the Thereom 5.6 states that , hence the upper bound of the worst-case running time of this function .

Therefore,

Second, prove that

We will prove a matching lower bound on the worst-case running time of this function, by finding an input family whose asymptotic runtime matches the bound we found in the previous part.

The input family we found for every natural number n is s=n and u=0:

let’s record the value of s and u for every 8 iterations:

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

Then for any natural number k, after 7k iterations s will be n-k, and u will be 0.

We also know from the question that the loop terminates when , and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

So the loop terminates when , i.e..

So then the loop will run for 7n iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n steps.

since from the properties of Big-oh, Omega and theta, the Thereom 5.6 states that , hence the running time of this input family is

Therefore the worst-case running time of this function

In all we have proved that and

Hence

(d)

( means the best-case run-time of counter with different input n)

Proof:

In order to prove , we will first prove , then prove

(since in the question we assumed that n>0, hence there will cerntainly have iterations, and we won’t consider the situation that n=0 and the best running time of it be zero here.)

First, prove that :

We can find from the question that after each iteration, either s will be decreased by 1 or u will be decreased by 1. Since we ignored other conditions (such that u will turned to 6 in the next iteration when it is 0 and s is positive meanwhile, since these will actually increasing th number of iterations), so after k iteration

We also know from the question that the loop terminates when , since , so when k=n, , hence the loop must run for at least n iterations.

Since , the lower bound of the best-case running time of this function

hence

Second, prove that

We will prove a matching upper bound on the best-case running time of this function, by finding an input family whose asymptotic runtime matches the bound we found in the previous part.

The input family we found for every natural number n is s=n-1 and u=1 (since we assumed that n>0 in the problem, hence this input family is suitable for all n):

let’s record the value of s and u for the first iteration:

let’s record the value of s and u for the next every 8 iterations:

hence we can find that s stay the same in the first 7 iterations and minus one in the eighth iteration. While u change from 6 to 0 in the first 7 iterations and change back to 6 in the eighth iteration.

Then for any natural number k, after 7k iterations s will be n-1-k, and u will be 0.

We also know from the question that the loop terminates when , and it is only when u=0 (after 7k iterations) that s+u can be smaller than zero (since u will be 6 or 5,4,3,2,1 after other iterations not 7k and the sum of s and u will always be larger than their sum in the end of 7(k-1) iterations)

So the loop terminates when , i.e..

So then the loop will run for at most 7(n-1)+1 iterations, since each iterations take 1 step (we take line 6,7,8,9,10,11 as one step), the total runtime is 7n-6 steps.

We will prove that 7n-6, that is

Let a=6, b=8, n’=100, so

Let

Since n100

So hence

, hence

Hence we have proved that 7n-6

hence the running time of this input family is

Therefore the best-case running time of this function

In all we have proved that and

Hence

(e)

Explanation:

From the definition of average run-time, we got to know that

Where

And

Hence

Therefore so and ( since the statement before is always true no matter what n is, so and so they meet the requirement for big-oh and omega, for omega we use the properties of Big-oh, Omega, Theta theorem 5.3, for all f,g: )

We have also proved before that and

Hence and

Therefore we can write as

and (according to the properties of Big-oh, Omega, Theta theorem 5.4 that for all f,g,h:, if

Since and , .