CSC236 Fall 2018

Assignment#1: induction

1.

(a) Yes.The proof is the following.

Proof:

Assume P(234), that is: every bipartite graph on 234 vertices has no more than $\frac{234^2}{4}$ edges.

I will use this to prove P(235), that is: every bipartite graph on 235 vertices has no more than $\frac{235^2}{4}$ edges.

Let G = (V, E) be a bipartite graph on 235 vertices

Let V_1, V_2 be such that $\mathrm{V} = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$

Thus $|V_1| + |V_2| = |V| = 235$

Assume $|V_1| \geq |V_2|$ (without loss of generality)

Then $|V_2| \leq \left\lfloor \frac{|V|}{2} \right\rfloor$

$\Rightarrow |V_2| \leq \left\lfloor \frac{235}{2} \right\rfloor$

$\Rightarrow |V_2| \leq 117$

Let k be an arbitrary vertex in V_1

Let x be the number of edges incident with k

Since every edge incident with x must have the other endpoint in V_2, and $|V_2| \leq 117$

Then $\mathrm{x} \leq 117$

Taking away k from G results in a new bipartite graph $G' = (V', E')$

Thus $|E'| = |\mathrm{E}| - \mathrm{x}$ (Since takes away k also takes away the x edges that are incident with k.)

And $|V'| = |V| - 1 = 235 - 1 = 234$, which means G' is a bipartite graph on 234 vertices

Also,

$$|E'| = |\mathrm{E}| - \mathrm{x}$$
$$\Rightarrow |\mathrm{E}| = x + |E'|$$
$$\Rightarrow |\mathrm{E}| \leq 117 + |E'| \quad \text{(since } \mathrm{x} \leq 117\text{)}$$
$$\Rightarrow |\mathrm{E}| \leq 117 + \frac{234^2}{4} \quad \text{(since P(234))}$$
$$\Rightarrow |\mathrm{E}| \leq \frac{234^2 + 468}{4}$$
$$\Rightarrow |\mathrm{E}| \leq \frac{235^2}{4} \quad \text{(since } 235^2 \geq 234^2 + 468\text{)}$$

■

(b)

No, I cannot directly use P(235) to prove P(236)
Explanation:

Assume P(235), that is: every bipartite graph on 235 vertices has no more than $\frac{235^2}{4}$ edges.

Since the number of edges must be an integer, we can say that: every bipartite graph on 235 vertices has no more than $\left\lfloor\frac{235^2}{4}\right\rfloor$ edges.

I will use this to prove P(236), that is: every bipartite graph on 236 vertices has no more than $\frac{236^2}{4}$ edges.

Let G = (V, E) be a bipartite graph on 236 vertices
Let V_1, V_2 be such that $\mathrm{V} = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$
Thus $|V_1| + |V_2| = |V| = 236$
Assume $|V_1| \geq |V_2|$ (without loss of generality)

Then $|V_2| \leq \left\lfloor\frac{|V|}{2}\right\rfloor$

$\Rightarrow |V_2| \leq \left\lfloor\frac{236}{2}\right\rfloor$

$\Rightarrow |V_2| \leq 118$

Let k be an arbitrary vertex in V_1
Let x be the number of edges incident with k
Since every edge incident with x must have the other endpoint in V_2, and $|V_2| \leq 118$
Then $\mathrm{x} \leq 118$
Taking away k from G results in a new bipartite graph $G' = (V', E')$
Thus $|E'| = |\mathrm{E}| - \mathrm{x}$ (Since takes away k also takes away the x edges that are incident with k.)

And $|V'| = |V| - 1 = 236 - 1 = 235$, which means G' is a bipartite graph on 235 vertices
Also,

$$|E'| = |\mathrm{E}| - \mathrm{x}$$
$$\Rightarrow |\mathrm{E}| = x + |E'|$$
$$\Rightarrow |\mathrm{E}| \leq 118 + |E'| \quad \text{(since } \mathrm{x} \leq 118\text{)}$$

By P(235) and the fact that the number of edges must be an integer, we know: every bipartite graph on 235 vertices has no more than $\left\lfloor\frac{235^2}{4}\right\rfloor$ edges.

Then:

$$|\mathrm{E}| \leq 118 + |E'|$$
$$\Rightarrow |\mathrm{E}| \leq 118 + \left\lfloor\frac{235^2}{4}\right\rfloor$$
$$\Rightarrow |\mathrm{E}| \leq 118 + 13806 \quad \text{(since } \frac{235^2}{4} = 13806.25\text{)}$$
$$\Rightarrow |\mathrm{E}| \leq 13924$$

$$\Rightarrow |E| \le \frac{236^2}{4} \quad \text{(since } \frac{236^2}{4} = 13924\text{)}$$

From the above proof we can say that P(236) holds, by assuming P(235) and using the fact that the number of edges must be an integer.

So, I'm actually using the assumption that: every bipartite graph on 235 vertices has no more than $\left\lfloor \frac{235^2}{4} \right\rfloor$ edges, not P(235). That is P(235) is not a necessary part of my proof.

Therefore, I cannot prove P(236) by directly using P(235).

■

(c)

Proof:

P(n): Every bipartite graph on n vertices has no more than $\frac{n^2}{4}$ edges.

Wants to show: $\forall \mathrm{n} \in \mathbb{N}, \mathrm{P(n)}$

Define H(n): Every bipartite graph on n vertices has no more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

Since $\left\lfloor \frac{n^2}{4} \right\rfloor \le \frac{n^2}{4}$

Then $\forall \mathrm{n} \in \mathbb{N}, \mathrm{H(n)} \Rightarrow \mathrm{P(n)}$

I will prove that $\forall \mathrm{n} \in \mathbb{N}, \mathrm{H(n)}$ by simple induction.

<u>Base case:</u>

Every bipartite graph on 0 vertices has 0 edges, which is no more than $\left\lfloor \frac{0^2}{4} \right\rfloor$ edges.

This verifies H(0).

<u>Inductive step:</u>

Let $\mathrm{n} \in \mathbb{N}$

Assume H(n), that is: every bipartite graph on n vertices has no more than $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges.

I will show that H(n+1) follows, that is: every bipartite graph on (n+1) vertices has no more than $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ edges.

Let G = (V, E) be a bipartite graph on (n+1) vertices.

Let V_1, V_2 be such that $\mathrm{V} = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$

Thus $|V_1| + |V_2| = |V| = n + 1$

Assume $|V_1| \ge |V_2|$ (without loss of generality)

Then $|V_2| \le \left\lfloor \frac{|V|}{2} \right\rfloor$

$$\Rightarrow |V_2| \le \left\lfloor \frac{n+1}{2} \right\rfloor$$

Let k be an arbitrary vertex in V_1

Let x be the number of edges incident with k

Since every edge incident with x must have the other endpoint in V_2, and $|V_2| \leq \left\lfloor\frac{n+1}{2}\right\rfloor$

Then $\mathrm{x} \leq \left\lfloor\frac{n+1}{2}\right\rfloor$

Taking away k from G results in a new bipartite graph $G' = (V', E')$

Thus $|E'| = |\mathrm{E}| - \mathrm{x}$ (Since takes away k also takes away the x edges that are incident with k.)

And $|V'| = |V| - 1 = n + 1 - 1 = n$, which means G' is a bipartite graph on n vertices

Also,

$$|E'| = |\mathrm{E}| - \mathrm{x}$$
$$\Rightarrow |\mathrm{E}| = x + |E'|$$

Case 1: (n+1) is odd

When (n+1) is odd, $\left\lfloor\frac{n+1}{2}\right\rfloor = \frac{n}{2}$

Thus,

$$\mathrm{x} \leq \left\lfloor\frac{n+1}{2}\right\rfloor$$
$$\Rightarrow \mathrm{x} \leq \frac{n}{2}$$

Then,

$$|\mathrm{E}| = x + |E'|$$
$$\Rightarrow |\mathrm{E}| \leq x + \left\lfloor\frac{n^2}{4}\right\rfloor \quad \text{(since H(n))}$$
$$\Rightarrow |\mathrm{E}| \leq \frac{n}{2} + \left\lfloor\frac{n^2}{4}\right\rfloor \quad \left(\text{since } \mathrm{x} \leq \frac{n}{2}\right)$$
$$\Rightarrow |\mathrm{E}| \leq \left\lfloor\frac{n}{2} + \frac{n^2}{4}\right\rfloor \quad \left(\text{since } \frac{n}{2} \text{ is an integer, } \left\lfloor\frac{n}{2} + \frac{n^2}{4}\right\rfloor = \frac{n}{2} + \left\lfloor\frac{n^2}{4}\right\rfloor\right)$$
$$\Rightarrow |\mathrm{E}| \leq \left\lfloor\frac{n^2 + 2n}{4}\right\rfloor$$
$$\Rightarrow |\mathrm{E}| \leq \left\lfloor\frac{n^2+2n+1}{4}\right\rfloor \quad \left(\text{since } n^2 + 2n + 1 > n^2 + 2n\right)$$
$$\Rightarrow |\mathrm{E}| \leq \left\lfloor\frac{(n+1)^2}{4}\right\rfloor$$

Case 2: (n+1) is even

When (n+1) is even, $\left\lfloor\frac{n+1}{2}\right\rfloor = \frac{n+1}{2}$

Thus,

$$\mathrm{x} \leq \left\lfloor\frac{n+1}{2}\right\rfloor$$
$$\Rightarrow \mathrm{x} \leq \frac{n+1}{2}$$

Also, when (n+1) is even, n is odd

Then, by H(n):

$$|E'| \le \left\lfloor \frac{n^2}{4} \right\rfloor$$

$$\Rightarrow |E'| \le \left\lfloor \frac{n^2 - 1 + 1}{4} \right\rfloor$$

$$\Rightarrow |E'| \le \left\lfloor \frac{(n-1)(n+1)}{4} + 0.25 \right\rfloor$$

Since n is odd, n+1 and n-1 are all even

Let $k_1, k_2 \in \mathbb{Z}$ such that $\mathrm{n} + 1 = 2k_1, n - 1 = 2k_2$

Then $(\mathrm{n}+1)(\mathrm{n}-1) = (2k_1)(2k_2) = 4k_1k_2$

Thus, $\frac{(n-1)(n+1)}{4} = \frac{4k_1k_2}{4} = k_1k_2$

Since $k_1, k_2 \in \mathbb{Z}$, then $k_1k_2 \in \mathbb{Z}$

Thus, $\left\lfloor \frac{(n-1)(n+1)}{4} + 0.25 \right\rfloor = \lfloor k_1k_2 + 0.25 \rfloor = k_1k_2 = \lfloor k_1k_2 \rfloor = \left\lfloor \frac{(n-1)(n+1)}{4} \right\rfloor = \left\lfloor \frac{n^2-1}{4} \right\rfloor$

Then,

$$|E'| \le \left\lfloor \frac{(n-1)(n+1)}{4} + 0.25 \right\rfloor$$

$$\Rightarrow |E'| \le \left\lfloor \frac{n^2 - 1}{4} \right\rfloor$$

Then,

$$|\mathrm{E}| = x + |E'|$$

$$\Rightarrow |\mathrm{E}| \le x + \left\lfloor \frac{n^2 - 1}{4} \right\rfloor$$

$$\Rightarrow |\mathrm{E}| \le \frac{n+1}{2} + \left\lfloor \frac{n^2-1}{4} \right\rfloor \quad \text{(since } \mathrm{x} \le \frac{n+1}{2}\text{)}$$

$$\Rightarrow |\mathrm{E}| \le \left\lfloor \frac{n+1}{2} + \frac{n^2-1}{4} \right\rfloor \quad \text{(since } \frac{n+1}{2} \text{ is an integer, } \left\lfloor \frac{n+1}{2} + \frac{n^2-1}{4} \right\rfloor = \frac{n+1}{2} + \left\lfloor \frac{n^2-1}{4} \right\rfloor\text{)}$$

$$\Rightarrow |\mathrm{E}| \le \left\lfloor \frac{n^2 + 2n + 1}{4} \right\rfloor$$

$$\Rightarrow |\mathrm{E}| \le \left\lfloor \frac{(n+1)^2}{4} \right\rfloor$$

So, I've proven H(n+1)

In all, I've proven that $\forall \mathrm{n} \in \mathbb{N}, \mathrm{H}(\mathrm{n})$,

that is, $\forall \mathrm{n} \in \mathbb{N}, \mathrm{P}(\mathrm{n})$ (since $\forall \mathrm{n} \in \mathbb{N}, \mathrm{H}(\mathrm{n}) \Rightarrow \mathrm{P}(\mathrm{n})$)

■

2.(a)Yes, I can prove P(29) by assuming P(3), the proof is the following:

Proof:

- Assume P(3) that is f(3) is a multiple of 4, so $\exists k \in \mathbb{Z}, f(3) = 4k, let\ k\ be\ that\ value.$
- I am going to use this to prove P(29) that is f(29) is a multiple of 4. So we are going to show that $\exists k_1 \in \mathbb{Z}, \mathrm{f}(29) = 4k_1$.
- Let $k_1 = k(4k+1)$. Since 29>0 by the question we know that:

f(29)=$[\mathrm{f}(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor)$

=$[\mathrm{f}(3)]^2 + f(3)$

=$\mathrm{f}(3)[\mathrm{f}(3) + 1]$

=$4\mathrm{k} \times (4\mathrm{k} + 1)$ # since $\mathrm{f}(3) = 4\mathrm{k}$

=$4 \times \mathrm{k}(4\mathrm{k} + 1)$

=$4k_1$ # since $k_1 = k(4k+1)$

So f(29) is a multiple of 4. So we can prove P(29) by assuming P(3).■

(b)No, I cannot prove P(29) directly by P(4), the proof is the following:

Proof:

- Assume P(4) that is f(4) is a multiple of 4.

Since 3>0 , so according to the question f(3)=$[\mathrm{f}(\lfloor \log_3 3 \rfloor)]^2 + f(\lfloor \log_3 3 \rfloor)$

=$[\mathrm{f}(1)]^2 + f(1)$

Since 4>0, so according to the question f(4)=$[\mathrm{f}(\lfloor \log_3 4 \rfloor)]^2 + f(\lfloor \log_3 4 \rfloor)$

=$[\mathrm{f}(1)]^2 + f(1)$

Hence we can conclude that f(3)=f(4) (since $[\mathrm{f}(1)]^2 + f(1)$=$[\mathrm{f}(1)]^2 + f(1)$)

We also know from assumption that f(4) is a multiple of 4, so f(3) is also a multiple of 4. (This is actually P(3)), So $\exists k \in \mathbb{Z}, f(3) =$

$4k, let\ k\ be\ that\ value.$

- I am going to use this to prove P(29) that is f(29) is a multiple of 4. So we are going to show that $\exists k_1 \in \mathbb{Z}, \mathrm{f}(29) = 4k_1$.
- Let $k_1 = k(4k+1)$. Since 29>0 by the question we know that:

f(29)=$[\mathrm{f}(\lfloor \log_3 29 \rfloor)]^2 + f(\lfloor \log_3 29 \rfloor)$

=$[\mathrm{f}(3)]^2 + f(3)$

=$\mathrm{f}(3)[\mathrm{f}(3)+1]$

=$4\mathrm{k} \times (4\mathrm{k} + 1)$ # since $\mathrm{f}(3) = 4\mathrm{k}$

=$4 \times \mathrm{k}(4\mathrm{k}+1)$

=$4k_1$ # since $k_1 = k(4k+1)$

So f(29) is a multiple of 4. So P(29) is proved to be true.

From the above proof we can conclude that the value of f(29) actually depends on the value of f(3) rather than f(4). That is P(29) actually depends on P(3) rather than P(4). So although we can indirectly prove P(29) by assuming P(4), P(4) is not a necessary part of our proof for P(29). Hence I cannot prove P(29) directly by P(4). ■

(c)WTS:$\forall \mathrm{n} \in \mathbb{N}, \mathrm{n} > 0 \Longrightarrow \mathrm{P}(\mathrm{n})$

- I am going to prove this by complete induction.
- Inductive step: Let n be a natural number greater than 0. Assume P(1) and…P(n-1). I will show that P(n) follows, that is f(n) is a multiple of 4.

  There are two cases to consider: n<3 and $\mathrm{n} \geq 3$.

  1. Base case: $\mathrm{n} \in \{1,2\}$: So $\mathrm{f}(\mathrm{n}) = f(0)^2 + f(0) = 3^2 + 3 = 12 = 4 \times 3$. So

P(n) follows in this case.

2. Case $n \geq 3$:

- Since $n \geq 3$, so $\log_3 n \geq 1$, hence $\lfloor \log_3 n \rfloor \geq 1$.
- Since it is clear in calculus that $\log_3 n < n$. Since $\lfloor \log_3 n \rfloor \leq \log_3 n$, we know that $\lfloor \log_3 n \rfloor < n$.

In all, $1 \leq \lfloor \log_3 n \rfloor < n$. Also we know by definition that $\lfloor \log_3 n \rfloor$ is an integer. So we may use the assumption P($\lfloor \log_3 n \rfloor$) （since we assumed P(1) and…P(n-1) before）, in other words, $f(\lfloor \log_3 n \rfloor)$ is a multiple of 4. Let k be an integer such that $f(\lfloor \log_3 n \rfloor) = 4k$.

So $f(n) = [f(\lfloor \log_3 n \rfloor)]^2 + f(\lfloor \log_3 n \rfloor)$ #since n>0

$= (4k)^2 + 4k$ #since $f(\lfloor \log_3 n \rfloor) = 4k$

$= 4[4(k^2) + k]$

So P(n) follows in this case（since with k be an integer, $4(k^2) + k$ will also be an integer, so we can conclude that f(n) is a multiple of 4）

So P(n) follows in both possible cases.

In all we've proven that $\forall n \in \mathbb{N}, n > 0 \Longrightarrow P(n)$. ■

3.

Proof by contradiction:

Assume, for the sake of contradiction, the negation of what we are proving, that is:

$$\exists x, y, z \in \mathbb{N}^+, 5x^3 + 50y^3 = 3z^3$$

Define $Z = \{z \in \mathbb{N}^+ \mid \exists x, y \in \mathbb{N}^+, 5x^3 + 50y^3 = 3z^3\}$

By our assumption Z is non-empty, so by the Principle of Well-Ordering it has a smallest element.

Let $z_0 \in Z$ be the smallest element of Z

Let $x_0, y_0 \in \mathbb{N}^+$ such that $5x_0^3 + 50y_0^3 = 3z_0^3$

Then:

$$5x_0^3 + 50y_0^3 = 3z_0^3$$
$$\Rightarrow 3z_0^3 = 5(x_0^3 + 10y_0^3)$$
$$\Rightarrow 5 \mid 3z_0^3$$
$$\Rightarrow 5 \mid z_0^3 \quad \text{(since } 5 \nmid 3 \text{ and 5 is a prime number)}$$
$$\Rightarrow 5 \mid z_0 \quad \text{(by clue for A1 Q3)}$$

Let $z_1 \in \mathbb{N}^+, 5z_1 = z_0$

Then:

$$5x_0^3 + 50y_0^3 = 3(5^3)(z_1^3)$$
$$\Rightarrow x_0^3 + 10y_0^3 = 3(5^2)(z_1^3) \quad \text{(divide through by 5)}$$
$$\Rightarrow x_0^3 = 3(5^2)(z_1^3) - 2(5y_0^3)$$
$$\Rightarrow 5 \mid x_0^3$$
$$\Rightarrow 5 \mid x_0 \quad \text{(by clue for A1 Q3)}$$

Let $x_1 \in \mathbb{N}^+, 5x_1 = x_0$

Then:

$$5^3x_1^3 + 10y_0^3 = 3(5^2)(z_1^3)$$
$$\Rightarrow 5^2x_1^3 + 2y_0^3 = 3(5)(z_1^3) \quad \text{(divide through by 5)}$$
$$\Rightarrow 2y_0^3 = 3(5)(z_1^3) - 5^2x_1^3$$
$$\Rightarrow 5 \mid 2y_0^3$$
$$\Rightarrow 5 \mid y_0^3 \quad \text{(since } 5 \nmid 2 \text{ and 5 is a prime number)}$$
$$\Rightarrow 5 \mid y_0 \quad \text{(by clue for A1 Q3)}$$

Let $y_1 \in \mathbb{N}^+, 5y_1 = y_0$

Then:

$$5^2x_1^3 + 2(5^3)y_1^3 = 3(5)(z_1^3)$$
$$\Rightarrow 5x_1^3 + 2(5^2)y_1^3 = 3z_1^3 \quad \text{(divide through by 5)}$$
$$\Rightarrow 5x_1^3 + 50y_1^3 = 3z_1^3$$
$$\Rightarrow z_1 \in Z$$

----><---- contradiction!

$z_1 < z_0$, but z_0 is the smallest element of Z

Since assuming that $\exists x, y, z \in \mathbb{N}^+, 5x^3 + 50y^3 = 3z^3$ leads to a contradiction, the assumption is false.

■

4.

(a) WTS: $\forall t \in T, \text{left\_count}(t) \leq 2^{max\_left\_surplus(t)} - 1$

Proof: Define P(t): $\text{left\_count}(t) \leq 2^{max\_left\_surplus(t)} - 1$. So we will prove that $\forall t \in T, P(t)$. I will prove this by structural induction.

- Basis: Let t be "⋆", since there is no "(" in "⋆", hence left_count(t) = 0.

  Since there is no "(" and no ")" in "⋆", so the left surplus for all prefixes in "⋆"

is 0 (since 0-0=0), hence max_left_surplus(t) = 0

Since $0 \leq 2^0 - 1$, hence $\text{left\_count(t)} \leq 2^{max\_left\_surplus(t)} - 1$. Hence P("*") holds.

- Inductive step: Let $t_1, t_2 \in T$. Assume P(t_1) and P(t_2). We will show that P(($t_1 t_2$)), that is $\text{left\_count}((t_1 t_2)) \leq 2^{max\_left\_surplus((t_1 t_2))} - 1$.

$$\textbf{left\_count}((\boldsymbol{t_1 t_2})) = \textbf{left\_count}(\boldsymbol{t_1}) + \textbf{left\_count}(\boldsymbol{t_2}) + \mathbf{1}$$

(since according to the structure of $(t_1 t_2)$ and left_count() function return the number of "(")

$$\leq \mathbf{2}^{\boldsymbol{max\_left\_surplus(t_1)}} - \mathbf{1} + \mathbf{2}^{\boldsymbol{max\_left\_surplus(t_2)}} - \mathbf{1} + \mathbf{1}$$

(by P(t_1) and P(t_2))

$$\leq \mathbf{2}^{\boldsymbol{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))}} + \mathbf{2}^{\boldsymbol{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))}} - \mathbf{1}$$

(since $max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2)) > max\_left\_surplus(t_1) and$

$max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2)) > max\_left\_surplus(t_2)$

hence $2^{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))} > 2^{max\_left\_surplus(t_1)}$

and $2^{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))} > 2^{max\_left\_surplus(t_2)}$*)*

$$= \mathbf{2}^{\boldsymbol{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))}} \times \mathbf{2} - \mathbf{1}$$

$$= \mathbf{2}^{\boldsymbol{max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2))+1}} - \mathbf{1}$$

$$= \mathbf{2}^{\boldsymbol{max\_left\_surplus((t_1 t_2))}} - \mathbf{1}.$$

(since $max\_left\_surplus((t_1 t_2)) = max(max\_left\_surplus(t_1),\ max\_left\_surplus(t_2)) + 1$*)*

So P(($t_1 t_2$)) follows.

In all we've proven that $\forall \text{t} \in \text{T}, \text{left\_count(t)} \leq 2^{max\_left\_surplus(t)} - 1$ ■

(b)WTS:$\forall \text{t} \in \text{T}, \text{double\_count(t)} = \begin{cases} 0 & if\ t = "*" \\ left\_count(t) - 1 & otherwise \end{cases}$

Proof: Define P(t): $\text{double\_count(t)} = \begin{cases} 0 & if\ t = "*" \\ left\_count(t) - 1 & otherwise \end{cases}$. So we will prove that $\forall \text{t} \in \text{T}, \text{P(t)}.$ I will prove this by structural induction.

- Basis: Let t be "*", since there is no "(" and no ")" in "*", hence there is no "((" and no "))" in "*". Hence according to the definition of double_count function $\textbf{double\_count(t)} = \mathbf{0}.$ Hence P("*") holds.

- Inductive step: Let $t_1, t_2 \in T$. Assume P(t_1) and P(t_2). We will show that

  P(($t_1 t_2$)), that is $\text{double\_count}((t_1 t_2)) =$
  $\begin{cases} 0 & if\ t = \text{"}*\text{"} \\ left\_count((t_1 t_2)) - 1 & otherwise \end{cases}$.

  I will show that in 4 cases.

  1. Case 1, $t_1 = \text{"}*\text{"}\ and\ t_2 = \text{"}*\text{"}$.

     - Then $(t_1 t_2) = \text{"}(**)\text{"}\ by\ definition.$ So $(t_1 t_2) \neq \text{"}*\text{"}, hence\ we\ want\ to\ show\ that$ $\text{double\_count}((t_1 t_2)) = \text{left\_count}((t_1 t_2)) - 1.$
     - Since $\text{double\_count}((**)) = 0$ and $\text{left\_count}((**)) = 1$,hence

       $\text{double\_count}((**)) = \text{left\_count}((**)) - 1$. That is

       $\text{double\_count}((t_1 t_2)) = \text{left\_count}((t_1 t_2)) - 1$

     So P(($t_1 t_2$)) follows in this case.

  2. Case 2, $t_1 = \text{"}*\text{"}\ and\ t_2 \neq \text{"}*\text{"}$.

     - Then $(t_1 t_2) = \text{"}(*\, t_2)\text{"}\ by\ definition.$ So $(t_1 t_2) \neq \text{"}*\text{"}, hence\ we\ want\ to\ show\ that$ $\text{double\_count}((t_1 t_2)) = \text{left\_count}((t_1 t_2)) - 1.$
     - $\text{double\_count}(\ (t_1 t_2)) = \text{double\_count}(\ t_2) + 1$. Since there is

       another ")" in the right hand side of t_2 and we know that since

       $t_2 \neq \text{"}*\text{"}$, there must be a ")" in the rightmost t_2. Hence there is an

       additional "))" in $\text{"}(*\, t_2)\text{"}$ rather than t_2.

     - $\text{left\_count}((t_1 t_2))$= $\text{left\_count}(t_2) + 1$, since we add a bracket

       around t_1 and t_2, and $\text{left\_count}(t_1) = \text{left\_count}(\text{"}*\text{"}) = 0.$

     Hence $\text{double\_count}(\ (t_1 t_2)) = \text{double\_count}(\ t_2) + 1$

     = $\text{left\_count}(t_2)$ – 1 +1 (#by P(t_2) and $t_2 \neq \text{"}*\text{"}$)

     = $\text{left\_count}(t_2)$

$=$ left_count$((t_1 t_2)) - 1$

(#since left_count$((t_1 t_2))=$ left_count$(t_2) + 1$)

So P$((t_1 t_2))$ follows in this case.

3. Case 3, $t_1 \neq " * "$ and $t_2 = " * "$.

- Then $(t_1 t_2) = "(t_1 *)"$ by $definition.$ So $(t_1 t_2) \neq " * "$, $hence$ we $want$ to $show$ $that$ double_count$((t_1 t_2)) =$ left_count$((t_1 t_2)) - 1$.
- double_count$((t_1 t_2)) =$ double_count$(t_1) + 1$. Since there is another "(" in the left hand side of t_1 and we know that since $t_1 \neq " * "$, there must be a "(" in the leftmost t_1. Hence there is an additional "((" in $"(t_1 *)"$ rather than t_1.
- left_count$((t_1 t_2))=$ left_count$(t_1) + 1$, since we add a bracket around t_1 and t_2, and left_count$(t_2) =$ left_count$(" * ") = 0$.

Hence double_count$((t_1 t_2)) =$ double_count$(t_1) + 1$

$=$ left_count$(t_1) - 1 + 1$ (#by P(t_1) and $t_1 \neq " * "$)

$=$ left_count(t_1)

$=$ left_count$((t_1 t_2)) - 1$

(#since left_count$((t_1 t_2))=$ left_count$(t_1) + 1$)

So P$((t_1 t_2))$ follows in this case.

4. Case 4, $t_1 \neq " * "$ and $t_2 \neq " * "$.

- Then $(t_1 t_2) = "(t_1 t_2)"$ by $definition.$ So $(t_1 t_2) \neq " * "$, $hence$ we $want$ to $show$ $that$ double_count$((t_1 t_2)) =$ left_count$((t_1 t_2)) - 1$.
- double_count$\big((t_1 t_2)\big) =$ double_count$(t_1) +$ double_count$(t_2) + 2$.

  Since there is another "(" in the left hand side of t_1 and we know that since $t_1 \neq$ " $*$ ", there must be a "(" in the leftmost t_1. Hence there is an additional "((" in $(t_1 t_2)$. Also, since there is another ")" in the right hand side of t_2 and we know that since $t_2 \neq$ " $*$ ", there must be a ")" in the rightmost t_2. Hence there is an additional "))" in $(t_1 t_2)$.

- $\text{left\_count}((t_1 t_2)) = \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1$, since we add a bracket around t_1 and t_2, there is an additional "(" in $(t_1 t_2)$.

Hence $\text{double\_count}((t_1 t_2)) = \text{double\_count}(t_1) + \text{double\_count}(t_2) + 2$

$= \text{left\_count}(t_1) - 1 + \text{left\_count}(t_2) - 1 + 2$ (#by P(t_1) and $t_1 \neq$ " $*$ " and P(t_2) and $t_2 \neq$ " $*$ ")

$= \text{left\_count}(t_1) + \text{left\_count}(t_2)$

$= \text{left\_count}((t_1 t_2)) - 1$

(#since $\text{left\_count}((t_1 t_2)) = \text{left\_count}(t_1) + \text{left\_count}(t_2) + 1$)

So P($(t_1 t_2)$) follows in this case.

In all, P($(t_1 t_2)$) follows.

In all we've proven that $\forall t \in T, \text{double\_count}(t) = \begin{cases} 0 & if\ t = \text{" * "} \\ left\_count(t) - 1 & otherwise \end{cases}$ ■