CSC236 A3

1.

WTS： term(x) terminates.

Proof:

Let $x \in \mathbb{N}$.

Let x_i be x after the ith iteration and y_i be y after the ith iteration.

Define p(i): after the ith iteration of the loop (if it occurs), x_i *is an integer and* $x_{i+1} = x_i - 1$ and $y_i = (x_i)^3$ and $y_i \geq 0$. I will prove that $\forall i \in \mathbb{N}, p(i)$ using simple induction on i.

Base case: $x_0 = x, y_0 = x^3 = (x_0)^3$ (by initialization). Since $x \in \mathbb{N}$, $x \geq 0$, hence $y_0 = x^3 \geq 0$. By code (line4) we know that if there is a iteration then $x_1 = x_0 - 1$. Since $x \in \mathbb{N}$, x is an integer, hence $x_0 = x$ is also an integer. So p(0) follows.

Inductive step:

Let $i \in \mathbb{N}$ and assume p(i), that is x_i *is an integer and* $x_{i+1} = x_i - 1$ and $y_i = (x_i)^3$ and $y_i \geq 0$. Show that p(i+1) follows. If there is an (i+1)th loop iteration. Then by code, $x_{i+1} = x_i - 1$,

$$y_{i+1} = y_i - 3 \times x_{i+1} \times x_{i+1} - 3 \times x_{i+1} - 1$$
$$= (x_i)^3 - 3 \times x_{i+1} \times x_{i+1} - 3 \times x_{i+1} - 1$$
$$= (x_{i+1} + 1)^3 - 3 \times x_{i+1} \times x_{i+1} - 3 \times x_{i+1} - 1$$
$$= (x_{i+1})^3 + 3(x_{i+1})^2 + 3(x_{i+1}) + 1 - 3 \times x_{i+1} \times x_{i+1} - 3 \times x_{i+1} - 1$$

$= (x_{i+1})^3$ (# by induction hypothesis we know that $y_i = (x_i)^3$). If there is a next iteration, then by code (line4), we know that $x_{i+2} = x_{i+1} - 1$. Since there is an (i+1)th loop iteration, by the loop condition we know that $y_i \neq 0$, and by induction hypothesis we know that $y_i \geq 0$ hence $y_i > 0$, and by induction hypothesis we know that $y_i = (x_i)^3$, hence $x_i > 0$, since $x_{i+1} = x_i - 1$, hence $x_{i+1} \geq 0$, hence $y_{i+1} = (x_{i+1})^3 \geq 0$. Since by induction hypothesis, x_i *is an integer*, $x_{i+1} = x_i - 1$ is also an integer. So p(i+1) follows.

Hence we've proved that $\forall i \in \mathbb{N}$, after the ith iteration of the loop (if it occurs), x_i *is an integer and* $x_{i+1} = x_i - 1$ and $y_i = (x_i)^3$ and $y_i \geq 0$. Then we will prove termination by this loop iteration.

Try the sequence $\{y_i\}$, since by loop iteration we know that

$x_i\ is\ an\ integer\ and\ y_i = (x_i)^3\ and\ y_i \geq 0$. Hence $y_i\ is\ an\ integer\ and\ y_i \geq 0.$

Hence $y_i \in \mathbb{N}$. Hence each element of the sequence is a natural number. It remains to show that the sequence is strictly decreasing. Suppose that there is an (i+1)th iteration of the loop, then by loop iteration we know that $y_i = (x_i)^3\ and\ x_{i+1} = x_i - 1$, hence $y_{i+1} = (x_{i+1})^3 = (x_i - 1)^3 < (x_i)^3 = y_i$ since $(x)^3$ is monotonic increasing. So, the sequence is strictly decreasing.

Since a strictly decreasing sequence in $\mathbb{N}$ is finite, and hence has a last (smallest) element. Thus, the loop terminates.

Hence, we've proved that term(x) terminates.

■

2.

(a)

Here is my specification for $M_a = \{Q, \Sigma = \{\mathrm{a}, \mathrm{b}\}, \delta, Q_0, F\}$ that accepts L_a:

$$\{\mathrm{Q} = \{\mathrm{A}, \mathrm{A}_D\},$$

$$\Sigma = \{\mathrm{a}, \mathrm{b}\},$$

$$\delta =$$

δ	a	b
A	A	A_D
A_D	A_D	A_D

$$Q_0 = \mathrm{A},$$

$$\mathrm{F} = \{\mathrm{A}\}\}$$

Prove that M_a accepts L_a:

Define Σ^* as the smallest set such that:

(a) $\varepsilon \in \Sigma^*$

(b) $s \in \Sigma^* \Rightarrow sa \in \Sigma^* \wedge sb \in \Sigma^*$

Define $P(s)$ as:

$$P(s):\ \delta^*(A,s) = \begin{cases} A & if\ s\ contains\ only\ as \\ A_D & if\ s\ contains\ at\ least\ one\ b \end{cases}$$

I will prove that $\forall s \in \Sigma^*, P(s)$ by structural induction.

<u>Base case:</u>

ε contains only as and $\delta^*(A,\varepsilon) = A$ so the implication in the first line of the invariant is true in this case. Also, since ε does not contain any bs, the implication in the second line of the invariant is vacuously true. So $P(\varepsilon)$ holds.

<u>Inductive step:</u>

Let $s \in \Sigma^*$, assume $P(s)$. I will show that $P(sa)$ and $P(sb)$ follow. There are two cases to consider:

<u>Case sa:</u> Then

$$\begin{aligned} \delta^*(A,sa) &= \delta(\delta^*(A,s),a) \\ &= \begin{cases} \delta(A,a) & if\ s\ contains\ only\ as \\ \delta(A_D,a) & if\ s\ contains\ at\ least\ one\ b \end{cases} && \text{\# by } P(s) \\ &= \begin{cases} A & if\ sa\ contains\ only\ as \\ A_D & if\ sa\ contains\ at\ least\ one\ b \end{cases} && \text{\# one more } a \end{aligned}$$

So $P(sa)$ follows.

<u>Case sb:</u> Then

$$\begin{aligned} \delta^*(A,sb) &= \delta(\delta^*(A,s),b) \\ &= \begin{cases} \delta(A,b) & if\ s\ contains\ only\ as \\ \delta(A_D,b) & if\ s\ contains\ at\ least\ one\ b \end{cases} && \text{\# by } P(s) \\ &= A_D && \text{\# add one } b \text{ cause the set contain at least one } b \end{aligned}$$

sb contains at least one b so the implication in the second line of the invariant is true in this case. Also, since s does not contain only as, the implication in the first line of the invariant is vacuously true. So $P(sb)$ follows.

The first line of the invariant ensures that all strings contain only as are accepted.

The contrapositive of the second line of the invariant ensures that any string that does not drive the machine to state A_D does not contain any bs, in other words all strings that drive the machine to state A contain only as.

Notice that a string contains only as is the same to say that this string $= a^k$, where $k \in \mathbb{N}$.

So M_a accepts L_a.

■

(b)

Here is my specification for $M_b = \{Q, \Sigma = \{\mathrm{a}, \mathrm{b}\}, \delta, Q_0, F\}$ that accepts L_b:

$$\{\mathrm{Q} = \{\mathrm{B}, \mathrm{B}_D\},$$

$$\Sigma = \{\mathrm{a}, \mathrm{b}\},$$

$$\delta =$$

δ	a	b
B	B_D	B
B_D	B_D	B_D

$$Q_0 = \mathrm{B},$$

$$\mathrm{F} = \{\mathrm{B}\}\}$$

Prove that M_b accepts L_b:

Define Σ^* as the smallest set such that:

(a) $\varepsilon \in \Sigma^*$

(b) $s \in \Sigma^* \Rightarrow sa \in \Sigma^* \wedge sb \in \Sigma^*$

Define $P(s)$ as:

$$P(s):\ \delta^*(B, s) = \begin{cases} B & \text{if } s \text{ contains only } bs \\ B_D & \text{if } s \text{ contains at least one } a \end{cases}$$

I will prove that $\forall s \in \Sigma^*, P(s)$ by structural induction.

<u>Base case:</u>

ε contains only bs and $\delta^*(B, \varepsilon) = B$ so the implication in the first line of the invariant is true in this case. Also, since ε does not contain any as, the implication in the second line of the invariant is vacuously true. So $P(\varepsilon)$ holds.

<u>Inductive step:</u>

Let $s \in \Sigma^*$, assume $P(s)$. I will show that $P(sa)$ and $P(sb)$ follow. There are two cases to consider:

Case sb: Then

$$\delta^*(B, sb) = \delta(\delta^*(B, s), b)$$

$$= \begin{cases} \delta(B, b) & if\ s\ contains\ only\ bs \\ \delta(B_D, b) & if\ s\ contains\ at\ least\ one\ a \end{cases} \quad \text{\# by } P(s)$$

$$= \begin{cases} B & if\ sb\ contains\ only\ bs \\ B_D & if\ sb\ contains\ at\ least\ one\ a \end{cases} \quad \text{\# one more b}$$

So $P(sb)$ follows.

Case sa: Then

$$\delta^*(B, sa) = \delta(\delta^*(B, s), a)$$

$$= \begin{cases} \delta(B, a) & if\ s\ contains\ only\ bs \\ \delta(B_D, a) & if\ s\ contains\ at\ least\ one\ a \end{cases} \quad \text{\# by } P(s)$$

$$= B_D \quad \text{\# add one a cause the set contain at least one a}$$

sa contains at least one a so the implication in the second line of the invariant is true in this case. Also, since s does not contain only bs, the implication in the first line of the invariant is vacuously true. So $P(sa)$ follows.

The first line of the invariant ensures that all strings contain only bs are accepted.

The contrapositive of the second line of the invariant ensures that any string that does not drive the machine to state B_D does not contain any as, in other words all strings that drive the machine to state B contain only bs.

Notice that a string contains only bs is the same to say that this string $= b^j$, where $j \in \mathbb{N}$.

So M_b accepts L_b.

■

(c)

Here is my specification for $M_2 = \{Q, \Sigma = \{\mathrm{a}, \mathrm{b}\}, \delta, Q_0, F\}$ that accepts L_2:

$$\{\mathrm{Q} = \{\mathrm{E}, \mathrm{O}\},$$

$$\Sigma = \{\mathrm{a}, \mathrm{b}\},$$

$$\delta =$$

δ	a	b
E	O	O
O	E	E

$$Q_0 = \text{E},$$

$$\text{F} = \{\text{E}\}\}$$

Prove that M_2 accepts L_2:

Define Σ^* as the smallest set such that:

(a) $\varepsilon \in \Sigma^*$

(b) $s \in \Sigma^* \Rightarrow sa \in \Sigma^* \wedge sb \in \Sigma^*$

Define $P(s)$ as:

$$P(s):\ \delta^*(E, s) = \begin{cases} E & if\ |s|\ is\ even \\ O & if\ |s|\ is\ odd \end{cases}$$

I will prove that $\forall s \in \Sigma^*, P(s)$ by structural induction.

<u>Base case:</u>

$|\varepsilon| = 0$, an even number, and $\delta^*(E, \varepsilon) = E$ so the implication in the first line of the invariant is true in this case. Also, since $|\varepsilon|$ is not odd, the implication in the second line of the invariant is vacuously true. So $P(\varepsilon)$ holds.

<u>Inductive step:</u>

Let $s \in \Sigma^*$, assume $P(s)$. Let $c \in \{\text{a}, \text{b}\}$. I will show that $P(sc)$ follows.

$$\delta^*(E, sc) = \delta(\delta^*(E, s), c)$$

$$= \begin{cases} \delta(E, c) & if\ |s|\ is\ even \\ \delta(O, c) & if\ |s|\ is\ odd \end{cases} \quad \text{\# by } P(s)$$

$$= \begin{cases} O & if\ |sc|\ is\ odd \\ E & if\ |sc|\ is\ even \end{cases} \quad \text{\# one more element}$$

So $P(sc)$ follows.

The first line of the invariant ensures that all strings with an even number of elements are accepted.

The contrapositive of the second line of the invariant ensures that any string that does not drive the machine to state O does not have an odd number of elements, in other words all strings that drive the machine to state E have an even number of elements.

Notice that a string x with an even number of elements is the same to say that $|x|$ is even.

So M_2 accepts L_2.

■

(d)

Here is my specification for $M_{a|b} = \{Q, \Sigma = \{\mathrm{a}, \mathrm{b}\}, \delta, Q_0, F\}$ that accepts $L_a \cup L_b$:

$$\{\mathrm{Q} = \{(\mathrm{A}, \mathrm{B}), (\mathrm{A}, \mathrm{B}_D), (\mathrm{A}_D, B), (\mathrm{A}_D, \mathrm{B}_D)\},$$

$$\Sigma = \{\mathrm{a}, \mathrm{b}\},$$

$$\delta =$$

δ	a	b
(A, B)	$(\mathrm{A}, \mathrm{B}_D)$	(A_D, B)
$(\mathrm{A}, \mathrm{B}_D)$	$(\mathrm{A}, \mathrm{B}_D)$	$(\mathrm{A}_D, \mathrm{B}_D)$
(A_D, B)	$(\mathrm{A}_D, \mathrm{B}_D)$	(A_D, B)
$(\mathrm{A}_D, \mathrm{B}_D)$	$(\mathrm{A}_D, \mathrm{B}_D)$	$(\mathrm{A}_D, \mathrm{B}_D)$

$$Q_0 = (\mathrm{A}, \mathrm{B}),$$

$$\mathrm{F} = \{(\mathrm{A}, \mathrm{B}), (\mathrm{A}, \mathrm{B}_D), (\mathrm{A}_D, B)\}\}$$

Prove that $M_{a|b}$ accepts $L_a \cup L_b$:

Denote the states for M_a as Q_a, the states for M_b as Q_b, their respective transition functions as δ_a and δ_b, and the transition function for $M_{a|b}$ as $\delta_{a|b}$. Inspection of $\delta_{a|b}$ shows that if $(q_a, q_b, c) \in Q_a \times Q_b \times \Sigma$, then $\delta_{a|b}\big((q_a, q_b), c\big) = \big(\delta_a(q_a, c), \delta_b(q_b, c)\big)$. Thus, the following invariant follows by simply taking conjunctions of the invariants of the component machines, for any $s \in \Sigma^*$:

$$P(s):\ \delta^*\big((A, B), s\big)$$

$$= \begin{cases} (\mathrm{A}, \mathrm{B}) & \mathit{if\ s\ contains\ only\ as} \wedge \mathit{s\ contains\ only\ bs} \\ (\mathrm{A}, \mathrm{B}_D) & \mathit{if\ s\ contains\ only\ as} \wedge \mathit{s\ contains\ at\ least\ one\ a} \\ (\mathrm{A}_D, B) & \mathit{if\ s\ contains\ at\ least\ one\ b} \wedge \mathit{s\ contains\ only\ bs} \\ (\mathrm{A}_D, \mathrm{B}_D) & \mathit{if\ s\ contains\ at\ least\ one\ b} \wedge \mathit{s\ contains\ at\ least\ one\ a} \end{cases}$$

The implication on the first line ensures that all strings contain only as and only bs end up in state (A, B).

The implication on the second line ensures that all strings contain only as and at least one a end up in state $(\mathrm{A}, \mathrm{B}_D)$.

The implication on the third line ensures that all strings contain at least one b and only bs end up in state $(\mathrm{A}_D, \mathrm{B})$.

The contrapositive of the implication on the fourth line ensure that any string that does not drive the machine to state $(\mathrm{A}_D, \mathrm{B}_D)$ must contain only as, or only bs, or only as and only bs.

Hence $M_{a|b}$ accepts $L_a \cup L_b$.

■

(e)

Here is my specification for $M_{a|b\ even} = \{Q, \Sigma = \{\mathrm{a}, \mathrm{b}\}, \delta, Q_0, F\}$ that accepts $(L_a \cup L_b) \cap L_2$:

$$\{\mathrm{Q} = \left\{\begin{matrix} \big((\mathrm{A}, \mathrm{B}), E\big), \big((\mathrm{A}, \mathrm{B}), O\big), \big((\mathrm{A}, \mathrm{B}_D), E\big), \big((\mathrm{A}, \mathrm{B}_D), O\big), \\ \big((\mathrm{A}_D, B), E\big), \big((\mathrm{A}_D, B), O\big), \big((\mathrm{A}_D, \mathrm{B}_D), E\big), \big((\mathrm{A}_D, \mathrm{B}_D), O\big) \end{matrix}\right\},$$

$$\Sigma = \{\mathrm{a}, \mathrm{b}\},$$

$$\delta =$$

δ	a	b
$\big((\mathrm{A}, \mathrm{B}), E\big)$	$\big((\mathrm{A}, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}_D, B), O\big)$
$\big((\mathrm{A}, \mathrm{B}), O\big)$	$\big((\mathrm{A}, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}_D, B), E\big)$
$\big((\mathrm{A}, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), O\big)$
$\big((\mathrm{A}, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), E\big)$
$\big((\mathrm{A}_D, B), E\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}_D, B), O\big)$
$\big((\mathrm{A}_D, B), O\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}_D, B), E\big)$
$\big((\mathrm{A}_D, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), O\big)$
$\big((\mathrm{A}_D, \mathrm{B}_D), O\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), E\big)$	$\big((\mathrm{A}_D, \mathrm{B}_D), E\big)$

$$Q_0 = \big((\mathrm{A}, \mathrm{B}), E\big),$$

$$\mathrm{F} = \{\big((\mathrm{A}, \mathrm{B}), E\big), \big((\mathrm{A}, \mathrm{B}_D), E\big), \big((\mathrm{A}_D, B), E\big)\}\}$$

(Notice that state $\big((\mathrm{A}, \mathrm{B}), O\big)$ is unreachable, just keep it for the sake of proof).

Prove that $M_{a|b\ even}$ accepts $(L_a \cup L_b) \cap L_2$:

Denote the states for $M_{a|b}$ as $Q_{a|b}$, the states for M_2 as Q_2, their respective transition functions as $\delta_{a|b}$ and δ_2, and the transition function for $M_{a|b\ even}$ as $\delta_{a|b\ even}$. Inspection of $\delta_{a|b\ even}$ shows that if $(q_{a|b}, q_2, c) \in Q_{a|b} \times Q_2 \times \Sigma$, then $\delta_{a|b\ even}\left((q_{a|b}, q_2), c\right) = \left(\delta_{a|b}(\ q_{a|b}, c), \delta_2(\ q_2, c)\right)$. Thus, the following invariant follows by simply taking conjunctions of the invariants of the component machines, for any $s \in \Sigma^*$:

$$P(s):\ \delta^*\left(\left((A,B),E\right),s\right)$$

$$= \begin{cases} \left((\mathrm{A},\mathrm{B}),E\right) & if\ s\ contains\ only\ as \wedge s\ contains\ only\ bs \wedge |s|\ is\ even \\ \left((\mathrm{A},\mathrm{B}),O\right) & if\ s\ contains\ only\ as \wedge s\ contains\ only\ bs \wedge |s|\ is\ odd \\ \left((\mathrm{A},\mathrm{B}_D),E\right) & if\ s\ contains\ only\ as \wedge s\ contains\ at\ least\ one\ a \wedge |s|\ is\ even \\ \left((\mathrm{A},\mathrm{B}_D),O\right) & if\ s\ contains\ only\ as \wedge s\ contains\ at\ least\ one\ a \wedge |s|\ is\ odd \\ \left((\mathrm{A}_D,B),E\right) & if\ s\ contains\ at\ least\ one\ b \wedge s\ contains\ only\ bs \wedge |s|\ is\ even \\ \left((\mathrm{A}_D,B),O\right) & if\ s\ contains\ at\ least\ one\ b \wedge s\ contains\ only\ bs \wedge |s|\ is\ odd \\ \left((\mathrm{A}_D,\mathrm{B}_D),E\right) & if\ s\ contains\ at\ least\ one\ b \wedge s\ contains\ at\ least\ one\ a \wedge |s|\ is\ even \\ \left((\mathrm{A}_D,\mathrm{B}_D),O\right) & if\ s\ contains\ at\ least\ one\ b \wedge s\ contains\ at\ least\ one\ a \wedge |s|\ is\ odd \end{cases}$$

The implication on the first line ensures that all strings contain only as and only bs, and have an even number of elements end up in state $\left((\mathrm{A},\mathrm{B}),E\right)$.

The implication on the third line ensures that all strings contain only as and at least one a, and have an even number of elements end up in state $\left((\mathrm{A},\mathrm{B}_D),E\right)$.

The implication on the fifth line ensures that all strings contain at least one b and only bs, and have an even number of elements end up in state $\left((\mathrm{A}_D,B),E\right)$.

The contrapositive of the implications on the other lines ensure that any string that does not drive the machines to one of those 5 states must (contain only as and have an even number of elements), or (contain only bs and have an even number of elements), or (contain only as and only bs, and have an even number of elements).

Hence $M_{a|b\ even}$ accepts $(L_a \cup L_b) \cap L_2$.

■

3.

Explanation:

Show that $\mathbf{L_0} = \boldsymbol{Rev}(\mathbf{L_0})$:

Here is my specification for $M_0 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts $L_0 \cup \{\varepsilon\}$:

$\{\mathrm{Q} = \{\mathrm{q0}_0, \mathrm{q0}_1, \mathrm{q0}_2\},$

$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$

$\delta =$

δ	0	1	2	3	4	5	6	7	8	9
$\mathrm{q0}_0$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$
$\mathrm{q0}_1$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$
$\mathrm{q0}_2$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$	$\mathrm{q0}_0$	$\mathrm{q0}_1$	$\mathrm{q0}_2$

$Q_0 = \mathrm{q0}_0,$

$\mathrm{F} = \{\mathrm{q0}_0\}\}$

By swapping starting and accepting states and reversing all transitions of M_0, here is my specification for $M^R{}_0 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts $Rev(L_0) \cup \{\varepsilon\}$:

$\{\mathrm{Q} = \{\mathrm{Rq0}_0, \mathrm{Rq0}_1, \mathrm{Rq0}_2\},$

$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$

$\delta =$

δ	0	1	2	3	4	5	6	7	8	9
$\mathrm{Rq0}_0$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$
$\mathrm{Rq0}_1$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$
$\mathrm{Rq0}_2$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$	$\mathrm{Rq0}_1$	$\mathrm{Rq0}_0$	$\mathrm{Rq0}_2$

$Q_0 = \mathrm{Rq0}_0,$

$\mathrm{F} = \{\mathrm{Rq0}_0\}\}$

From the above we can see:

1. Both M_0, $M^R{}_0$ have three states; M_0, $M^R{}_0$ have the same Σ;

2. The accepting state and starting state of M_0 are the same state;

3. The accepting state and starting state of $M^R{}_0$ are the same state;

4. In any of the three states of M_0 or $M^R{}_0$, (0, 3, 6, 9) keeps the machine in the same state, (1, 4, 7) drives the machine to one of the remaining two states, and (2, 5, 8) drives the machine to the other one of the remaining two states.

5. 1*4*7* makes M_0 and $M^R{}_0$ go in a cycle of three different states.

6. 2*5*8* makes M_0 and ${M^R}_0$ go in a cycle of three different states. This cycle has an opposite direction compare to the cycle made by 1*4*7*.

Thus, M_0 and ${M^R}_0$ are actually the same machine.

Since M_0 accepts $L_0 \cup \{\varepsilon\}$ and ${M^R}_0$ accepts $Rev(L_0) \cup \{\varepsilon\}$, we can say that M_0 accepts $Rev(L_0) \cup \{\varepsilon\}$ and ${M^R}_0$ accepts $L_0 \cup \{\varepsilon\}$.

Therefore, $\mathrm{L}_0 = Rev(\mathrm{L}_0)$.

Show that $\mathbf{L_1} = \boldsymbol{Rev}(\mathbf{L_1})$:

Here is my specification for $M_1 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts L_1:

$$\{\mathrm{Q} = \{\mathrm{q1_0}, \mathrm{q1_1}, \mathrm{q1_2}\},$$

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$\delta =$$

δ	0	1	2	3	4	5	6	7	8	9
$\mathrm{q1_0}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$
$\mathrm{q1_1}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$
$\mathrm{q1_2}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$	$\mathrm{q1_0}$	$\mathrm{q1_1}$	$\mathrm{q1_2}$

$$Q_0 = \mathrm{q1_0},$$

$$\mathrm{F} = \{\mathrm{q1_1}\}\}$$

By swapping starting and accepting states and reversing all transitions of M_1, here is my specification for ${M^R}_1 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts $Rev(L_1)$:

$$\{\mathrm{Q} = \{\mathrm{Rq1_0}, \mathrm{Rq1_1}, \mathrm{Rq1_2}\},$$

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$\delta =$$

δ	0	1	2	3	4	5	6	7	8	9
$\mathrm{Rq1_0}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$
$\mathrm{Rq1_1}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$
$\mathrm{Rq1_2}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$	$\mathrm{Rq1_1}$	$\mathrm{Rq1_0}$	$\mathrm{Rq1_2}$

$$Q_0 = \mathrm{Rq1_1},$$

$$\mathrm{F} = \{\mathrm{Rq1_0}\}\}$$

From the above we can see:

1. Both M_1, ${M^R}_1$ have three states; M_1, ${M^R}_1$ have the same Σ;

2. The accepting state and starting state of M_1 are different states, and they are adjacent;

3. The accepting state and starting state of ${M^R}_1$ are different states, and they are adjacent;

4. In any of the three states of M_1 or ${M^R}_1$, (0, 3, 6, 9) keeps the machine in the same state, (1, 4, 7) drives the machine to one of the remaining two states, and (2, 5, 8) drives the machine to the other one of the remaining two states.

5. 1*4*7* makes M_1 and ${M^R}_1$ go in a cycle of three different states.

6. 2*5*8* makes M_1 and ${M^R}_1$ go in a cycle of three different states. This cycle has an opposite direction compare to the cycle made by 1*4*7*.

Thus, M_1 and ${M^R}_1$ are actually the same machine.

Since M_1 accepts L_1 and ${M^R}_1$ accepts $Rev(L_1)$, we can say that M_1 accepts $Rev(L_1)$ and ${M^R}_1$ accepts L_1.

Therefore, $\mathrm{L}_1 = Rev(\mathrm{L}_1)$.

Show that $\mathbf{L}_2 = \boldsymbol{Rev}(\mathbf{L}_2)$:

Here is my specification for $M_2 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts L_2:

$$\{\mathrm{Q} = \{\mathrm{q2}_0, \mathrm{q2}_1, \mathrm{q2}_2\},$$

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$\delta =$$

δ	0	1	2	3	4	5	6	7	8	9
$\mathrm{q2}_0$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$
$\mathrm{q2}_1$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$
$\mathrm{q2}_2$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$	$\mathrm{q2}_0$	$\mathrm{q2}_1$	$\mathrm{q2}_2$

$$Q_0 = \mathrm{q2}_0,$$

$$\mathrm{F} = \{\mathrm{q2}_2\}\}$$

By swapping starting and accepting states and reversing all transitions of M_2, here is my specification for ${M^R}_2 = \{Q, \Sigma, \delta, Q_0, F\}$ that accepts $Rev(L_2)$:

$$\{Q = \{Rq2_0, Rq2_1, Rq2_2\},$$

$$\Sigma = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\},$$

$$\delta =$$

δ	0	1	2	3	4	5	6	7	8	9
$Rq2_0$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$
$Rq2_1$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$
$Rq2_2$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$	$Rq2_1$	$Rq2_0$	$Rq2_2$

$$Q_0 = Rq2_2,$$

$$F = \{Rq2_0\}\}$$

From the above we can see:

1. Both M_2, $M^R{}_2$ have three states; M_2, $M^R{}_2$ have the same Σ;

2. The accepting state and starting state of M_2 are different states, and they are adjacent;

3. The accepting state and starting state of $M^R{}_2$ are different states, and they are adjacent;

4. In any of the three states of M_2 or $M^R{}_2$, (0, 3, 6, 9) keeps the machine in the same state, (1, 4, 7) drives the machine to one of the remaining two states, and (2, 5, 8) drives the machine to the other one of the remaining two states.

5. 1*4*7* makes M_2 and $M^R{}_2$ go in a cycle of three different states.

6. 2*5*8* makes M_2 and $M^R{}_2$ go in a cycle of three different states. This cycle has an opposite direction compare to the cycle made by 1*4*7*.

Thus, M_2 and $M^R{}_2$ are actually the same machine.

Since M_2 accepts L_2 and $M^R{}_2$ accepts $Rev(L_2)$, we can say that M_2 accepts $Rev(L_2)$ and $M^R{}_2$ accepts L_2.

Therefore, $L_2 = Rev(L_2)$.

■

4.

(a)

WTS: $\forall r \in RE, \exists r' \in RE, Rev(L(r)) = L(r')$

Let RE be the set of regular expressions over the alphabet ∑={0,1}, Define p(r): $\exists r' \in RE, Rev(L(r)) = L(r')$, I will show that $\forall \mathrm{r} \in \mathrm{RE}, \mathrm{p(r)}$ by structural induction on r. Let $\mathrm{r} \in \mathrm{RE}$

Basis: Let r∈ $\{\emptyset, \varepsilon, 0{,}1\}$,

For r= $\emptyset$, then $r' = \emptyset$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{\}$, hence $Rev(L(r)) = \{\}$. And $L(r') = \{\}$, hence $Rev(L(r)) = L(r')$.

For r= ε, then $r' = \varepsilon$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{\varepsilon\}$, hence $Rev(L(r)) = \{\varepsilon\}$ (since $\varepsilon^R = \varepsilon$). And $L(r') = \{\varepsilon\}$, hence $Rev(L(r)) = L(r')$.

For r= 0, then $r' = 0$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{0\}$, hence $Rev(L(r)) = \{0\}$ (since $0^R = 0$). And $L(r') = \{0\}$, hence $Rev(L(r)) = L(r')$.

For r= 1, then $r' = 1$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{1\}$, hence $Rev(L(r)) = \{1\}$ (since $1^R = 1$). And $L(r') = \{1\}$, hence $Rev(L(r)) = L(r')$.

So p(r) holds.

Inductive step:

Let t, s∈ RE, assume p(t), p(s), that is $\exists t' \in RE, Rev(L(t)) = L(t')$, $\exists s' \in RE, Rev(L(s)) = L(s')$. Let t', s' be those regular expression. I will show that (t+s), (ts), (t^*) follows. And I will prove this in 3 cases.

1. To show that (t+s) follows.

   Take $r' = t' + s'$, since $t' \in RE$ and $s' \in RE$, hence by the definition of RE, we know that $t' + s' \in RE$. And we will show that $\forall \mathrm{x} \in Rev(L(\mathrm{t+s})), x \in L(t' + s')$ and $\forall \mathrm{y} \in L(t' + s'), y \in Rev(L(\mathrm{t+s}))$.

   First, show that $\forall \mathrm{x} \in Rev(L(\mathrm{t+s})), x \in L(t' + s')$, let x $\in Rev(L(\mathrm{t+s}))$, then $\exists\ \mathrm{xx} \in L(\mathrm{t+s}), x = xx^R$, let xx be that regular expression, since xx∈ $L(\mathrm{t+s})$, xx must be one of the element of $L(\mathrm{t})\ or\ L(\mathrm{s})$, hence xx^Rmust be one of the element of $L(t')\ or\ L(s')$ (since by inductive hypothesis $Rev(L(t)) = L(t')$

and $Rev(L(s)) = L(s')$), hence $xx^R \in L(t') \cup L(s') = L(t' + s')$, hence $x \in L(t' + s')$.

Second, show that $\forall y \in L(t' + s'), y \in Rev(L(\mathrm{t} + \mathrm{s}))$, let $y \in L(t' + s') = L(t') \cup L(s')$, hence y must be one of the element of $L(t')$ *or* $L(s')$. Then $\exists$ yy $\in L(\mathrm{t})$ *or* $L(\mathrm{s})$, y=yy^R, let yy be that regular expression (since by inductive hypothesis $Rev(L(t)) = L(t')$ and $Rev(L(s)) = L(s')$), hence yy $\in L(\mathrm{t}) \cup L(\mathrm{s}) = L(t + s)$, hence $yy^R \in Rev(L(\mathrm{t} + \mathrm{s}))$, that is $y \in Rev(L(\mathrm{t} + \mathrm{s}))$.

Hence $\forall x \in Rev(L(\mathrm{t} + \mathrm{s})), x \in L(t' + s')$ and $\forall y \in L(t' + s'), y \in Rev(L(\mathrm{t} + \mathrm{s}))$. That is $Rev(L(s + t)) = L(t' + s')$.

So (t+s) follows.

2. To show that (ts) follows.

Take $r' = s't'$, since $t' \in RE$ and $s' \in RE$, hence by the definition of RE, we know that $s't' \in RE$. And we will show that $\forall x \in Rev(L(\mathrm{ts})), x \in L(s't')$ and $\forall y \in L(s't'), y \in Rev(L(\mathrm{ts}))$.

First, show that $\forall x \in Rev(L(\mathrm{ts})), x \in L(s't')$, let $x \in Rev\big(L(\mathrm{ts})\big)$, then $\exists$xx $\in L(\mathrm{ts}), x = xx^R$, let xx be that regular expression, since xx$\in L(\mathrm{ts})$, xx must be a concatenation of one of the element of $L(\mathrm{t})$ *and* one of the element of $L(\mathrm{s})$, Let's show it as

$t_1 s_1$ (*with* t_1 *be* one of the element of $L(\mathrm{t})$, s_1 *be* one of the element of $L(\mathrm{s})$. *let them be their value*)

Hence $xx^R = {s_1}^R {t_1}^R$ *by the definition of reverse of string. We also know that* ${s_1}^R$ must be one of the element of $L(s')$, ${t_1}^R$ must be one of the element of $L(t')$ (since by inductive hypothesis $Rev(L(t)) = L(t')$ and $Rev(L(s)) = L(s')$), hence ${s_1}^R {t_1}^R \in L(s')L(t') = L(s't')$, hence $x = xx^R = {s_1}^R {t_1}^R \in L(s't')$.

Second, show that $\forall y \in L(s't'), y \in Rev(L(\mathrm{ts}))$, let $y \in L(s't') = L(s')L(t')$, hence y must be a concatenation of one of the element of $L(t')$ *and* one of the element of $L(s')$. Let show it as ${s_2}^R {t_2}^R$, let them be their value. Then $\exists\ s_2 \in\ L(\mathrm{s})\ and\ t_2 \in\ L(\mathrm{t})$, s_2=${s_2}^R, t_2 = {t_2}^R\ and\ (t_2 s_2)^R = {s_2}^R {t_2}^R$, let s_2, t_2 be that regular expression (since by inductive hypothesis $Rev(L(t)) = L(t')$ and $Rev(L(s)) = L(s')$), hence $t_2 s_2 \in\ L(\mathrm{t})L(\mathrm{s}) = L(ts)$, hence $(t_2 s_2)^R \in Rev(L(\mathrm{ts}))$, that is $y = {s_2}^R {t_2}^R \in Rev(L(\mathrm{t}+\mathrm{s}))$.

Hence $\forall \mathrm{x} \in Rev(L(\mathrm{ts})), x \in L(\ s't')$ and $\forall \mathrm{y} \in L(\ s't'), y \in Rev(L(\mathrm{ts}))$.

So (ts) follows.

3. To show that (t^*) follows.

Take $r' = t'^*$, since $t' \in RE$, hence by the definition of RE, we know that $t'^* \in RE$. And we will show that $\forall \mathrm{x} \in Rev(L(t^*)), x \in L(t'^*)$ and $\forall \mathrm{y} \in L(t'^*), y \in Rev(L(t^*))$.

First, show that $\forall \mathrm{x} \in Rev(L(t^*)), x \in L(t'^*)$, let $\mathrm{x} \in Rev(L(t^*))$, then $\exists\ \mathrm{xx} \in L(t^*), x = \mathrm{xx}^R$, let xx be that regular expression, since $\mathrm{xx} \in L(t^*)$, xx must be a star of some of the elements of $L(\mathrm{t})$,let those elements be t_1, t_2, t_3(since the string are always finite, we only suppose 3 to be the number of the element, and this will cover all the cases) and $\mathrm{xx} = t_1 t_2 t_3$. And by the definition of the reverse of the string, we know that $\mathrm{xx}^R = {t_3}^R {t_2}^R {t_1}^R$, and ${t_3}^R, {t_2}^R,\ {t_1}^R$ must be one of the element of $L(t')$ (since by inductive hypothesis $Rev(L(t)) = L(t')$), hence ${t_3}^R {t_2}^R {t_1}^R \in L(t')L(t')L(t') =\ L(t'^*)$, hence $x = \mathrm{xx}^R = {t_3}^R {t_2}^R {t_1}^R \in L(t'^*)$.

Second, show that $\forall \mathrm{y} \in L(t'^*), y \in Rev(L(t^*))$, let $\mathrm{y} \in L(t'^*)$, hence y must be one of the element of $L(t'^*)$, hence y must be a concatenation of some elements of $L(t')$, let them be ${t_6}^R, {t_5}^R,\ {t_4}^R$(since the string are always finite, we only suppose 3 to be the number of the element, and this will cover all the cases).Then y= ${t_6}^R\ {t_5}^R\ {t_4}^R$. And $\exists\ t_4, t_5, t_6 \in\ L(\mathrm{t})$, t_4=${t_4}^R$, t_5=${t_5}^R$, t_6=${t_6}^R$, let t_4, t_5, t_6 be that regular expression (since by inductive hypothesis $Rev(L(t)) = L(t')$), and by the definition of the reverse of the string ${t_6}^R\ {t_5}^R\ {t_4}^R = (t_4\ t_5 t_6)^R$, and since $t_4, t_5, t_6 \in\ L(\mathrm{t}), t_4\ t_5 t_6 \in\ L(t^*)$, hence $(t_4\ t_5 t_6)^R \in Rev(\ L(t^*))$. That is y= ${t_6}^R\ {t_5}^R\ {t_4}^R = (t_4\ t_5 t_6)^R \in Rev(\ L(t^*))$.

Hence $\forall \mathrm{x} \in Rev(L(t^*)), x \in L(t'^*)$ and $\forall \mathrm{y} \in L(t'^*), y \in Rev(L(t^*))$. That is

$Rev(L(t^*)) = L(t'^*)$.

So (t^*) follows.

Hence we've proved that $\forall \mathrm{r} \in \mathrm{RE}, \exists r' \in RE, Rev(L(r)) = L(r')$■

(b) WTS: $\forall \mathrm{r} \in \mathrm{RE}, \exists r' \in RE, Prefix(L(r)) = L(r')$

Let RE be the set of regular expressions over the alphabet ∑={0,1}, Define

p(r): $\exists r' \in RE, Prefix(L(r)) = L(r')$, I will show that $\forall \mathrm{r} \in \mathrm{RE}, \mathrm{p(r)}$ by structural induction on r. Let $\mathrm{r} \in \mathrm{RE}$

Basis: Let r$\in \{\emptyset, \varepsilon, 0,1\}$,

For r= $\emptyset$, then $r' = \emptyset$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{\}$, hence $Prefix(L(r)) = \{\}$ $according\ to\ the\ definition\ of\ Prefix(L)$. And $L(r') = \{\}$, hence $Prefix(L(r)) = L(r')$.

For r= ε, then $r' = \varepsilon$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{\varepsilon\}$, hence $Prefix(L(r)) = \{\varepsilon\}$

($according\ to\ the\ definition\ of\ Prefix(L), \varepsilon\varepsilon = \varepsilon$). And $L(r') = \{\varepsilon\}$, hence $Prefix(L(r)) = L(r')$.

For r= 0, then $r' = 0 + \varepsilon$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{0\}$, hence $Prefix(L(r)) = \{0, \varepsilon\}$

($according\ to\ the\ definition\ of\ Prefix(L), 0\varepsilon = 0, \varepsilon 0 = 0$). And $L(r') = \{0, \varepsilon\}$, hence $Prefix(L(r)) = L(r')$.

For r= 1, then $r' = 1 + \varepsilon$, hence by the definition of RE, we know that $r' \in RE$, and $L(r) = \{1\}$, hence $Prefix(L(r)) = \{1, \varepsilon\}$

($according\ to\ the\ definition\ of\ Prefix(L), 1\varepsilon = 1, \varepsilon 1 = 1$). And $L(r') = \{1, \varepsilon\}$, hence $Prefix(L(r)) = L(r')$.

So p(r) holds.

Inductive step:

Let t, s$\in \mathrm{RE}$, assume p(t), p(s), that is $\exists t' \in RE, Prefix(L(t)) = L(t')$, $\exists s' \in RE, Prefix(L(s)) = L(s')$. Let t', s' be those regular expression. Hence we can write the above as $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')\ and\ \{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s')$. I will show that (t+s), (ts), (t^*) follows. And I will prove this in 3 cases.

1. To show that (t+s) follows.

   Take $r' = t' + s'$, since $t' \in RE$ and $s' \in RE$, hence by the definition of RE, we know that $t' + s' \in RE$. And we will show that $\forall \mathrm{x} \in Prefix(L(\mathrm{t}+\mathrm{s})), x \in L(t' + s')$ and $\forall \mathrm{x} \in L(t' + s'), x \in Prefix(L(\mathrm{t}+\mathrm{s}))$.

   First, show that $\forall \mathrm{x} \in Prefix(L(\mathrm{t}+\mathrm{s})), x \in L(t' + s')$, let $\mathrm{x} \in Prefix(L(\mathrm{t}+\mathrm{s})) = \{\mathrm{x} \in \sum^*: xy \in L(t+s), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(t) \cup L(s), for\ some\ y \in \sum^*\}$, then $\exists\ y \in \sum^*, xy \in L(t) \cup L(s)$, let y be that value, since $xy \in L(t) \cup L(s)$, xy must be one of the element of $L(\mathrm{t})\ or\ L(\mathrm{s})$, if xy is a element of L(t), then $xy \in L(t)$ then $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(t) \cup L(s), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t') \in L(t') \cup L(s') = L(t' + s')$ (by inductive hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$) ,if xy is a element of L(s), then $xy \in L(s)$ then $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(t) \cup L(s), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s') \in L(t') \cup L(s') = L(t' + s')$ (by inductive hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s')$)

   Hence $x \in L(t' + s')$.

   Second, show that $\forall \mathrm{x} \in L(t' + s'), x \in Prefix(L(\mathrm{t}+\mathrm{s}))$, let $\mathrm{x} \in L(t' + s') = L(t') \cup L(s') = \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} \cup \{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\}$ (by inductive hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$, $\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s')$), hence x must be one of the element of $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}\ or\ \{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\}$. If x is an element of $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}\ then\ x \in \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(t) \cup L(s), for\ some\ y \in \sum^*\}$, that is $x \in Prefix(L(\mathrm{t}+\mathrm{s}))$. If x is an element of $\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\}\ then\ x \in \{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(t) \cup$

$L(s), for\ some\ y \in \sum^*\}$, that is $x \in Prefix(L(\mathrm{t}+\mathrm{s}))$. Hence $x \in Prefix(L(\mathrm{t}+\mathrm{s}))$.

Hence $\forall \mathrm{x} \in Prefix(L(\mathrm{t}+\mathrm{s})), x \in L(t'+s')$ and $\forall \mathrm{x} \in L(t'+s'), x \in Prefix(L(\mathrm{t}+\mathrm{s}))$. That is $Prefix(L(s+t)) = L(t'+s')$.

So (t+s) follows.

2. To show that (ts) follows.

Take $r' = t' + ts'$, since $\mathrm{t} \in RE$ and $t' \in RE$ and $s' \in RE$, hence by the definition of RE, we know that $t' + ts' \in RE$. And we will show that $\forall \mathrm{x} \in Prefix(L(\mathrm{ts})), x \in L(t'+ts')$ and $\forall \mathrm{x} \in L(t'+ts'), x \in Prefix(L(\mathrm{ts}))$.

First, show that $\forall \mathrm{x} \in Prefix(L(\mathrm{ts})), x \in L(t'+ts')$, let $\mathrm{x} \in Prefix\big(L(\mathrm{ts})\big) = \{\mathrm{x} \in \sum^*: xy \in L(ts), for\ some\ y \in \sum^*\} = \{\mathrm{x} \in \sum^*: xy \in L(t)L(s), for\ some\ y \in \sum^*\}$, then $\exists y \in \sum^*, xy \in L(t)L(s)$, let y be that value, since $xy \in L(t)L(s)$, xy must be a concatenation of one of the element of $L(\mathrm{t})$ and **one of the element of** $L(\mathrm{s})$,

Let's show it as

$t_1 s_1$ ($with\ t_1\ be$ **one of the element of** $L(\mathrm{t}), s_1\ be$ **one of the element of** $L(\mathrm{s})$**.**
$let\ them\ be\ their\ value$)

Hence $xy = t_1 s_1$, if $|x|<|t_1|$, then x is some first part of t_1 then $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$. (by induction hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$)

If $|\mathrm{x}| \geq |t_1|$, then x is the concatenation of t_1 with some first part of s_1, then $\mathrm{x} \in t_1\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} \in L(\mathrm{t})\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(\mathrm{t})L(s') = L(ts')$ (by induction hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s')$)

Hence $\mathrm{x} \in L(t') \cup L(ts') = L(t'+ts')$.

Second, show that $\forall \mathrm{x} \in L(t'+ts'), x \in Prefix(L(\mathrm{ts}))$, let $\mathrm{x} \in L(t'+ts') = L(t') \cup L(\mathrm{t})L(s') = \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} \cup L(t)\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\}$ by inductive hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$, $\{\mathrm{x} \in \sum^*: xy \in L(s), for\ some\ y \in \sum^*\} = L(s')))$,

If $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}$, then $\exists\ y \in \sum^*, xy \in L(t)$, let y be that value, let $s' \in L(s)$, then xys $\in L(t)L(s)$, since RE is over $\sum, s \in RE, hence\ s' \in L(s)\ is\ over \sum, hence\ s' \in \sum^*$. hence $ys' \in \sum^*$, hence x $\in$ {x $\in \sum^*: xy \in L(t)L(s), for\ some\ y \in \sum^*$}={x $\in \sum^*: xy \in L(ts), for\ some\ y \in \sum^*$}, hence $x \in Prefix(L(\mathrm{ts}))$.

If x $\in L(t)$\{x $\in \sum^*: xy \in L(s), for\ some\ y \in \sum^*$\}, then $\exists\ y \in \sum^*, xy \in L(t)$\{x $\in \sum^*: xy \in L(s), for\ some\ y \in \sum^*$\}$y \in L(t)L(s) = L(ts)$, hence x $\in$ \{x $\in \sum^*: xy \in L(ts), for\ some\ y \in \sum^*$\}, hence $x \in Prefix(L(\mathrm{ts}))$.

So (ts) follows.

3. To show that (t^*) follows.

Take $r' = \mathrm{t}^*t'$, since t, $t' \in RE$, hence by the definition of RE, we know that $\mathrm{t}^*t' \in RE$. And we will show that $\forall$x $\in Prefix(L(t^*)), x \in L(\mathrm{t}^*t')$ and $\forall$x $\in L(\mathrm{t}^*t'), x \in Prefix(L(t^*))$.

First, show that $\forall$x $\in Prefix(L(t^*)), x \in L(\mathrm{t}^*t')$, let x $\in Prefix(L(t^*)) =$ \{x $\in \sum^*: xy \in L(t^*), for\ some\ y \in \sum^*$\}, , hence $\exists y \in \sum^*, xy \in L(t^*), let\ y\ be\ that\ value\ and\ suppose\ xy = t_1t_2t_3$(since the string are always finite, we only suppose 3 to be the number of the element, and it will cover all the cases).

if |x|<|t_1|, then x is some first part of t_1 then x $\in$ \{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(t')$. (by induction hypothesis \{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(t')$)

If |t_2|+|t_1|>|x| $\geq |t_1|$, then x is the concatenation of t_1 with some first part of t_2, then x $\in t_1$\{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $\in L(\mathrm{t})$\{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(\mathrm{t})L(t') = L(tt')$ (by induction hypothesis \{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(t')$)

If |x| $\geq |t_2| + |t_1|$, then x is the concatenation of t_1, t_2 with some first part of t_3, then x $\in t_1t_2$\{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $\in L(\mathrm{t})L(\mathrm{t})$\{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(\mathrm{t})L(\mathrm{t})L(t') = L(t^2t')$ (by induction hypothesis \{x $\in \sum^*: xy \in L(t), for\ some\ y \in \sum^*$\} $= L(t')$)

Hence x $\in L(t') \cup L(tt') \cup L(t^2t') = (\varepsilon \cup L(t) \cup L(t^2))L(t') \in L(\mathrm{t}^*)L(t') =$

$L(\mathrm{t}^*t')$

Second, we will show that $\forall \mathrm{x} \in L(\mathrm{t}^*t'), x \in Prefix(L(t^*))$.Let $\mathrm{x} \in L(\mathrm{t}^*t') =$ $L(\mathrm{t}^*)L(t')$= $(\varepsilon \cup L(t) \cup L(t^2)\ldots)L(t')$= $L(t') \cup L(tt') \cup L(t^2t')$ .... Since the string are always finite, we only suppose the first 3 sets to be the constraint of x, and it will cover all the cases, that is $\mathrm{x} \in L(t') \cup L(tt') \cup L(t^2t')$=$L(t') \cup L(t)L(t') \cup L(\mathrm{t})\mathrm{L}(\mathrm{t})\mathrm{L}(t') = \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} \cup L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} \cup L(t)L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}$ (by inductive hypothesis $\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\} = L(t')$) ,

If $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}$, then $\exists\ y \in \sum^*, xy \in L(t)$, let y be that value, then xy $\in L(t) \in (\mathrm{L}(\mathrm{t}))^* = L(\mathrm{t}^*)$, hence $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(\mathrm{t}^*), for\ some\ y \in \sum^*\}$, hence $x \in Prefix(L(\mathrm{t}^*))$.

If $\mathrm{x} \in L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}$, then $\exists\ y \in \sum^*, xy \in L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}y \in L(t)L(t) = (\mathrm{L}(\mathrm{t}))^2 \in (\mathrm{L}(\mathrm{t}))^* = L(\mathrm{t}^*)$, hence $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(\mathrm{t}^*), for\ some\ y \in \sum^*\}$, hence $x \in Prefix(L(\mathrm{t}^*))$.

If $\mathrm{x} \in L(t)L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}$, then $\exists\ y \in \sum^*, xy \in L(t)L(t)\{\mathrm{x} \in \sum^*: xy \in L(t), for\ some\ y \in \sum^*\}y \in L(t)L(t)L(t) = (\mathrm{L}(\mathrm{t}))^3 \in (\mathrm{L}(\mathrm{t}))^* = L(\mathrm{t}^*)$, hence $\mathrm{x} \in \{\mathrm{x} \in \sum^*: xy \in L(\mathrm{t}^*), for\ some\ y \in \sum^*\}$, hence $x \in Prefix(L(\mathrm{t}^*))$.

So (t^*) follows.

Hence we've proved that $\forall \mathrm{r} \in \mathrm{RE}, \exists r' \in RE, Prefix(L(r)) = L(r')$ ■

(c)

WTS: If $\mathrm{r} \in \mathrm{RE}$ does not contain the Kleene star, then |L(r)| is finite.

Proof:

Let RE be the set of regular expressions over the alphabet $\sum$={0,1}, Define p(r): **If r does not contain the Kleene star, then |L(r)| is finite.**, I will show that $\forall \mathrm{r} \in \mathrm{RE}, \mathrm{p}(\mathrm{r})$ by structural induction on r. Let $\mathrm{r} \in \mathrm{RE}$

Basis: Let r$\in \{\emptyset, \varepsilon, 0, 1\}$, For r= $\emptyset$, $\emptyset$ obviously does not contain the Kleene star. And L(r)={}, hence |L(r)|=0, hence $|\mathrm{L(r)}|$ is finite.

For r= ε, ε obviously does not contain the Kleene star. And L(ε)={ ε}, hence |L(r)|=1, hence $|\mathrm{L(r)}|$ is finite.

For r= 0, 0 obviously does not contain the Kleene star. And L(0)={ 0}, hence |L(r)|=1, hence $|\mathrm{L(r)}|$ is finite.

For r= 1, 1 obviously does not contain the Kleene star. And L(1)={ 1}, hence |L(r)|=1, hence $|\mathrm{L(r)}|$ is finite.

So p(r) holds.

Inductive step:

Let t, s$\in$ RE, assume p(t), p(s), that is

$\text{If t does not contain the Kleene star, then } |\mathrm{L(t)}| \text{ is finite}$,

$\text{If s does not contain the Kleene star, then } |\mathrm{L(s)}| \text{ is finite}$. I will show that (t+s), (ts), (t^*) follows. And I will prove this in 3 cases.

1. To show that (t+s) follows.

   I will show that (t+s) follows in 4 cases.

   - $\text{t does not contain the Kleene star and s does not contain the Kleene star}$

     Hence by the induction hypothesis, we know that $|\mathrm{L(t)}|$ is finite and $|\mathrm{L(s)}|$ is finite. ($\text{If t does not contain the Kleene star, then } |\mathrm{L(t)}| \text{ is finite}$, $\text{If s does not contain the Kleene star, then } |\mathrm{L(s)}| \text{ is finite}$.)

     Since

     $\text{t does not contain the Kleene star and s does not contain the Kleene star}$,

     t+s must also does not contain the Kleene star.

     And $|\mathrm{L}(t+s)| = |\mathrm{L}(t) \cup \mathrm{L}(s)| \leq |\mathrm{L}(t)| + |\mathrm{L}(s)|$ must be a finite number since $|\mathrm{L(t)}|$ is finite and $|\mathrm{L(s)}|$ is finite.

     Hence p(r) holds in this case.

   - $\text{t contains the Kleene star and s does not contain the Kleene star}$

     Then t+s must also contain the Kleene star.

     Since the assumption is false, p(r) is vacuously true in this case.

- t does not contain the Kleene star and s contains the Kleene star

  Then t+s must also contain the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

- t contains the Kleene star and s contains the Kleene star

  Then t+s must also contain the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

Hence (t+s) follows.

2. To show that (ts) follows.

   I will show that (ts) follows in 4 cases.

- t does not contain the Kleene star and s does not contain the Kleene star

  Hence by the induction hypothesis, we know that $|\mathrm{L(t)}|$ is finite and $|\mathrm{L(s)}|$ is finite. (If t does not contain the Kleene star, then $|\mathrm{L(t)}|$ is finite, If s does not contain the Kleene star, then $|\mathrm{L(s)}|$ is finite.)

  Since

  t does not contain the Kleene star and s does not contain the Kleene star,

  ts must also does not contain the Kleene star.

  And $|\mathrm{L(ts)}| = |\mathrm{L(t)L(s)}| = |\{\mathrm{xy}|\mathrm{x} \in \mathrm{L(t)}, \mathrm{y} \in \mathrm{L(s)}\}| = |\mathrm{L(t)}| \times |\mathrm{L(s)}|$ must be a finite number since $|\mathrm{L(t)}|$ is finite and $|\mathrm{L(s)}|$ is finite.

  Hence p(r) holds in this case.

- t contains the Kleene star and s does not contain the Kleene star

  Then ts must also contain the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

- t does not contain the Kleene star and s contains the Kleene star

  Then ts must also contain the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

- t contains the Kleene star and s contains the Kleene star

  Then ts must also contain the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

Hence (ts) follows.

3. To show that (t^*) follows.

   Since (t^*) itself contains the Kleene star.

  Since the assumption is false, p(r) is vacuously true in this case.

Hence (t^*) follows.

Hence we've proved that If $\mathrm{r} \in \mathrm{RE}$ does not contain the Kleene star, then |L(r)| is finite.■

5.

(a)

WTS: any DFA that accepts L_{R4} has at least nine states, not including dead states.

Proof:

Let $\sum$={a,b,c} and $L_{R4} = \{\mathrm{x} \in \sum^* | \ |x| \ = \ 4 \ \wedge \ x = x^R\}$, I will prove that any DFA that accepts L_{R4} has at least nine states, not including dead states by contradiction. The proof is the following.

Assume, for the sake of contradiction, the negation of what we are proving, that is there is a DFA that accepts L_{R4} has less than 9 states not including dead states. That it be that value.

Then for this DFA, first ignoring the dead state, we know that if we choose 9 strings over $\sum$, then there must be at least two strings that will end with the same states since we've assumed that this DFA has less than 9 states.

Let $x_0 = \mathcal{E}$, $x_1 = a$, $x_2 = aa$, $x_3 = aaa$, $x_4 = aaaa$, $x_5 = b$, $x_6 = ccc$, $x_7 = bbb$ and $x_8 = c$, be these 9 strings. Since each of them can be transferred into a string of L_{R4} by concatenating some string after them, hence by the definition of DFA, none of them are in the dead state. Then at least two strings of them will end with the same states as proved above.

For the strings end with the same states, let s be a string over $\sum$, then the concatenation of these strings with s must also end in the same state, since s=s and these strings end with the same states.

Hence these concatenations must be both accepted or both rejected.

Since at least two strings of $x_0, x_1, x_2, x_3,\ x_4, x_5, x_6, x_7, x_8$ will end with the same states, that means there are two strings from $x_0, x_1, x_2, x_3,\ x_4, x_5, x_6, x_7, x_8$, for any string s over $\sum$, the concatenations of these two with s must always be both accepted or both rejected. (1)

We also know that:

<u>Pair 1 x_0 and x_1:</u>

Choose s = aaa.

Then x_0s = aaa, rejected; x_1s = aaaa, accepted.

<u>Pair 2 x_0 and x_2:</u>

Choose s = aa.

Then x_0s = aa, rejected; x_2s = aaaa, accepted.

<u>Pair 3x_0 and x_3:</u>

Choose s = a.

Then x_0s = a, rejected; x_3s = aaaa, accepted.

<u>Pair 4 x_0 and x_4:</u>

Choose s = ε.

$x_0 s$ = ε is rejected; $x_4 s$ = aaaa is accepted.

<u>Pair 5 x_0 and x_5:</u>

Choose s = bbb.

Then x_0s = bbb, rejected; x_5s = bbbb, accepted.

<u>Pair 6 x_0 and x_6:</u>

Choose s = c.

Then x_0s = c, rejected; x_6s = cccc, accepted.

<u>Pair 7x_0 and x_7:</u>

Choose s = b.

Then x_0s = b, rejected; x_7s = bbbb, accepted.

Pair 8 x_0 and x_8:

Choose s = ccc.

Then x_0s = ccc, rejected; x_8s = cccc, accepted.

Pair 9 x_1 and x_2:

Choose s = aa.

Then x_1s = aaa, rejected; x_2s = aaaa, accepted.

Pair 10 x_1 and x_3:

Choose s = a.

Then x_1s = a, rejected; x_3s = aaaa, accepted.

Pair 11 x_1 and x_4:

Choose s = ε.

$x_1 s$ = a is rejected; $x_4 s$ = aaaa is accepted.

Pair 12 x_1 and x_5:

Choose s = bbb.

Then x_1s = abbb, rejected; x_5s = bbbb accepted.

Pair 13 x_1 and x_6:

Choose s = c.

Then x_1s = ac, rejected; x_6s = cccc accepted.

Pair 14 x_1 and x_7:

Choose s = b.

Then x_1s = ab, rejected; x_7s = bbbb accepted.

Pair 15 x_1 and x_8:

Choose s = ccc.

Then x_1s = accc, rejected; x_8s = cccc accepted.

Pair 16 x_2 and x_3:

Choose s = a.

Then x_2s = aaa, rejected; x_3s = aaaa, accepted.

Pair 17 x_2 and x_4:

Choose s = ε.

$x_2 s$ = aa is rejected; $x_4 s$ = aaaa is accepted.

Pair 18 x_2 and x_5:

Choose s = bbb.

Then x_2s = aabbb, rejected; x_5s = bbbb accepted.

Pair 19 x_2 and x_6:

Choose s = c.

Then x_2s = aac, rejected; x_6s = cccc accepted.

Pair 20 x_2 and x_7:

Choose s = b.

Thenx_2s = aab, rejected; x_7s = bbbb accepted.

Pair 21 x_2 and x_8:

Choose s = ccc.

Then x_2s = aaccc, rejected; x_8s = cccc accepted.

Pair 22 x_3 and x_4:

Choose s = ε.

$x_3 s$ = aaa is rejected; $x_4 s$ = aaaa is accepted.

Pair 23 x_3 and x_5:

Choose s = bbb.

Then x_3s = aaabbb, rejected; x_5s = bbbb accepted.

Pair 24 x_3 and x_6:

Choose s = c.

Then x_3s = aaac, rejected; x_6s = cccc accepted.

Pair 25 x_3 and x_7:

Choose s = b.

Then x_3s = aaab, rejected; x_7s = bbbb accepted.

Pair 26 x_3 and x_8:

Choose s = ccc.

Then x_3s = aaaccc, rejected; x_8s = cccc accepted.

Pair 27 x_4 and x_5:

Choose s = ε.

x_4s = aaaa is accepted; x_5s = b is rejected.

Pair 28 x_4 and x_6:

Choose s = ε.

x_4s = aaaa is accepted; x_6s = ccc is rejected.

Pair 29 x_4 andx_7:

Choose s = ε.

x_4s = aaaa is accepted; x_7s = bbb is rejected.

Pair 30 x_4 and x_8:

Choose s = ε.

x_4s = aaaa is accepted; x_8s = c is rejected.

Pair 31 x_5 andx_6:

Choose s = c.

Then x_5s = bc, rejected; x_6s = cccc accepted.

Pair 32 x_5 and x_7:

Choose s = b.

Then x_5s = bb, rejected; x_7s = bbbb accepted.

Pair 33 x_5 and x_8:

Choose s = ccc.

Then x_5s = bccc, rejected; x_8s = cccc accepted.

Pair 34 x_6 and x_7:

Choose s = b.

Then x_6s = cccb, rejected; x_7s = bbbb accepted.

Pair 35 x_6 and x_8:

Choose s = ccc.

Then x_6s = cccccc, rejected; x_8s = cccc accepted.

Pair 36 x_7 and x_8:

Choose s = ccc.

Then x_7s = bbbccc, rejected; x_8s = cccc accepted.

Hence for any two strings of $x_0, x_1, x_2, x_3,\ x_4, x_5, x_6, x_7, x_8$, there is a string s over $\sum$, make the concatenations of these two with s be one accepted and one rejected.

---><--- contradiction! The conclusion above is exactly the negation of (1) which we've assumed before. Since assuming that there is a DFA that accepts L_{R4} has less than 9 states leads to a contradiction, the assumption is false. Hence we've proved that any DFA that accepts L_{R4} has at least nine states, not including dead states.

■

(b)

There does not exist a DFA that accepts $L_R = \{x \in \Sigma^* |\ x = x^R\}$, the proof is the following:

Let $\sum = \{\mathrm{a, b, c}\}$, let $\mathrm{n} \in \mathbb{N}$

I will first prove that for $L_{Ra} = \{x \in \Sigma^* |\, |x| = n\ \land\ x = x^R\}$, any DFA that accepts L_{Ra} has at least $3^{\left\lfloor \frac{n}{2} \right\rfloor}$ states, not including the dead states. The proof is the following.

I will prove that for any 2 different prefixes of length $\left\lfloor \frac{n}{2} \right\rfloor$, there is a string s over $\sum$, the concatenation of these two prefixes with s will one be rejected one be accepted. We will prove this in 2 cases.

1.n is an even number. Then $\left\lfloor \frac{n}{2} \right\rfloor = \frac{n}{2}$. Let x, y be any 2 different prefixes of length $\frac{n}{2}$, let s=x^R, then xs=xx^R and |xs|=n, which is obviously accepted, while ys=yx^R is rejected since $\mathrm{y} \neq \mathrm{x}, y^R \neq x^R$, hence ys is not reversible. Hence the concatenation of these two prefixes with s are one rejected one accepted.

2. n is an odd number. Then $\left\lfloor\frac{n}{2}\right\rfloor = \frac{n-1}{2}$. Let x, y be any 2 different prefixes of length $\frac{n-1}{2}$, let s=$\mathrm{a}(x^R)$, then xs = x$\mathrm{a}(x^R)$ and |xs| = | x$\mathrm{a}(x^R)| = 1 + \frac{n-1}{2} + \frac{n-1}{2} = n$, which is obviously accepted, while ys = y $\mathrm{a}(x^R)$ is rejected since $\mathrm{y} \neq \mathrm{x}, y^R \neq x^R$, hence ys is not reversible. Hence the concatenation of these two prefixes with s are one rejected one accepted.

Hence we've proved that for any 2 different prefixes of length $\left\lfloor\frac{n}{2}\right\rfloor$, there is a string s over $\sum$, the concatenation of these two prefixes with s will one be rejected one be accepted. Hence the strings of these $3^{\left\lfloor\frac{n}{2}\right\rfloor}$ prefixes (since $\sum = \{\mathrm{a}, \mathrm{b}, \mathrm{c}\}$, hence the number of prefixes of length $\left\lfloor\frac{n}{2}\right\rfloor$ is $3^{\left\lfloor\frac{n}{2}\right\rfloor}$) are all end in different states as proved in (a), otherwise two of them end in the same states, for any string s over $\sum$, the concatenation of those 2 with s will end in the same states which is a contradiction of what we proved before that any 2 different prefixes of length $\left\lfloor\frac{n}{2}\right\rfloor$, there is a string s over $\sum$, the concatenation of these two prefixes with s will one be rejected one be accepted.

Since the strings of these $3^{\left\lfloor\frac{n}{2}\right\rfloor}$ prefixes are all end in different states, and none of them are dead state since all of these prefixes can be leaded into a string of L_{Ra} by concatenate with a string s over $\sum$ as showed in the prove above, hence there must be at least $3^{\left\lfloor\frac{n}{2}\right\rfloor}$ states, not including the dead states.

Hence, we've proved that for $L_{Ra} = \{x \in \Sigma^* \mid |x| = n \ \wedge \ x = x^R\}$, any DFA that accepts L_{Ra} has at least $3^{\left\lfloor\frac{n}{2}\right\rfloor}$ states, not including the dead states.

<u>Generalize:</u>

Let $L_R = \{x \in \Sigma^* | x = x^R\}$.

For the sake of contradiction, assume that there exist a DFA that accepts L_R.

Then from what we've proven above, this DFA has at least $3^{\left\lfloor\frac{n}{2}\right\rfloor}$ states, not including the dead states.

I will show that this assumption for the general case is false.

We define L_R by removing $|x| = n$ from L_{Rn}. From the above proof, we didn't use the length of x to prove whether or not a state is accepted or rejected. Thus, removing $|x| = n$ does not affect the least number of states the DFA has. So, for L_R, $|x|$ does not have an upper bound, which means n does not have an upper bound.

As $n \to \infty^+, 3^{\left\lfloor \frac{n}{2} \right\rfloor} \to \infty^+$. ----><---- contradiction, since a DFA should have a finite number of states (by the definition of DFA).

Therefore, I've shown that there does not exist a DFA that accepts $L_R = \{x \in \Sigma^* | x = x^R\}$.

■