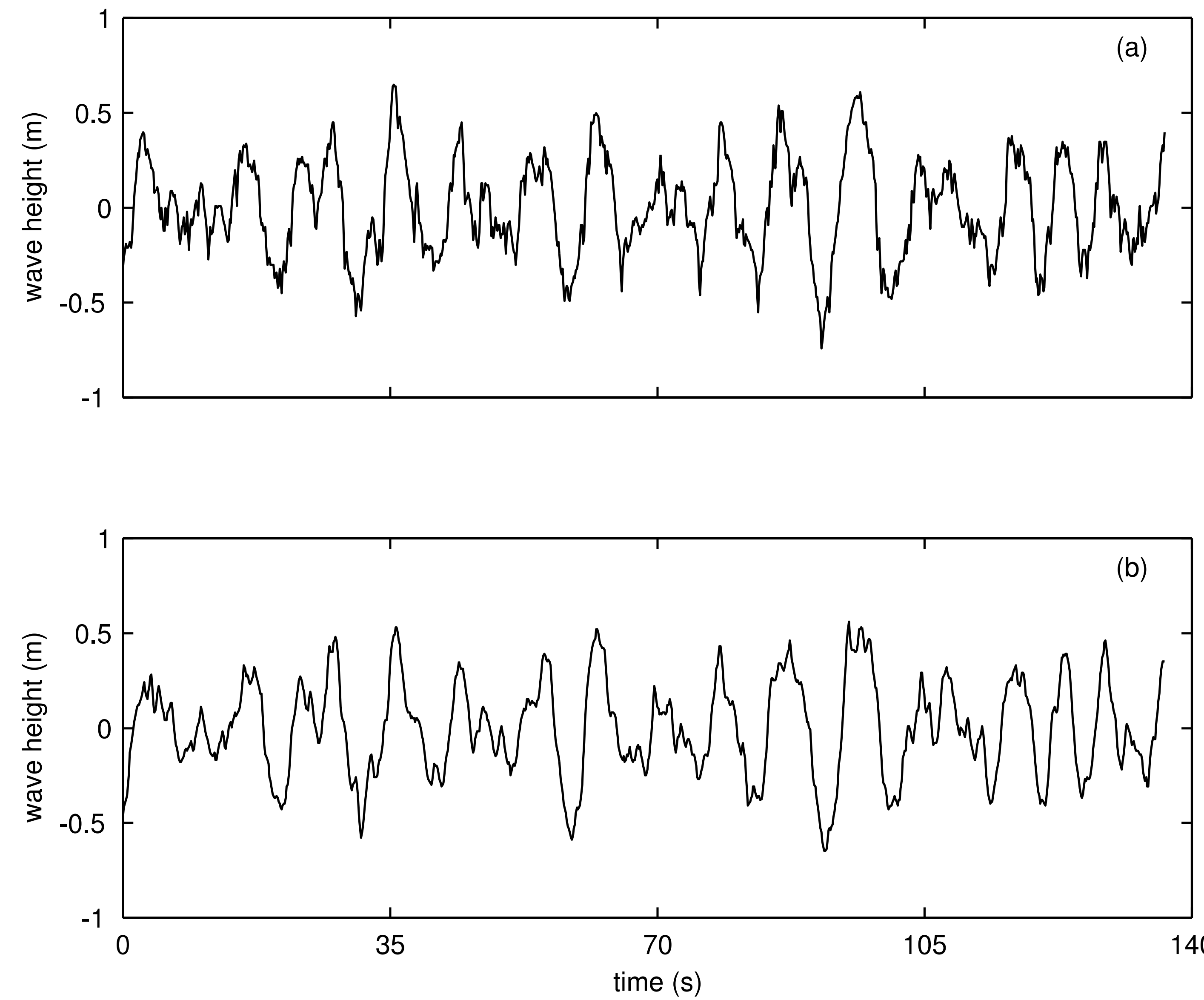
Bivariate Time Series



Simultaneous measurements of ocean waves versus time by two instruments of quite different design, (a) an infrared wave gauge, and (b) a wire wave gauge. There are $N = 1024$ data values in each series and the sample interval is $\delta_t = 4/30$s. (These series were derived from data supplied courtesy of Andy Jessup, Applied Physics Lab, University of Washington.

Bivariate Time Series

Definition: The two real-valued discrete time stochastic processes $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are said to be <u>jointly stationary stochastic processes</u> if $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are each, separately, second-order stationary processes, and $\text{cov}\{X_{1,t}, X_{2,t+\tau}\}$ is a function of τ only.

Then $\{X_{1,t}; X_{2,t}\}$ forms a <u>stationary bivariate process</u>.

Cross-covariance

The acvs are

$$
\begin{aligned}
s_{X_1,\tau} &= \mathrm{E}\{[X_{1,t} - \mu_{X_1}][X_{1,t+\tau} - \mu_{X_1}]\} \\
s_{X_2,\tau} &= \mathrm{E}\{[X_{2,t} - \mu_{X_2}][X_{2,t+\tau} - \mu_{X_2}]\}
\end{aligned}
$$

so that,

$$
\begin{aligned}
s_{X_1,0} &= \mathrm{var}\{X_{1,t}\} = \sigma^2_{X_1} \\
s_{X_2,0} &= \mathrm{var}\{X_{2,t}\} = \sigma^2_{X_2}.
\end{aligned}
$$

Cross-covariance

The cross-covariance sequence (ccvs) is given by

$$\begin{aligned} s_{X_1X_2,\tau} &= \mathrm{cov}\{X_{1,t}, X_{2,t+\tau}\} \\ &= \mathrm{E}\{[X_{1,t} - \mu_{X_1}][X_{2,t+\tau} - \mu_{X_2}]\}. \end{aligned}$$

The cross-correlation sequence (ccs) is

$$\rho_{X_1X_2,\tau} = \frac{s_{X_1X_2,\tau}}{\sqrt{s_{X_1,0}s_{X_2,0}}} = \frac{s_{X_1X_2,\tau}}{\sigma_{X_1}\sigma_{X_2}}.$$

Note that,

$$\begin{aligned} s_{X_2 X_1,\tau} &= \mathrm{cov}\{X_{2,t}, X_{1,t+\tau}\} \\ &= \mathrm{E}\{[X_{2,t} - \mu_{X_2}][X_{1,t+\tau} - \mu_{X_1}]\}. \end{aligned}$$

Hence,

$$\begin{aligned} s_{X_1 X_2,\tau} &= s_{X_2 X_1,-\tau} \quad \text{but} \\ s_{X_1 X_2,\tau} &\neq s_{X_1 X_2,-\tau} \quad \text{(unlike acvs)} \end{aligned}$$

The ccvs is generally quite asymmetric.

Estimation

Given

$$X_{1,1}, X_{1,2}, \ldots, X_{1,N}$$

$$X_{2,1}, X_{2,2}, \ldots, X_{2,N}$$

a natural estimator for the ccvs is

$$\hat{s}_{X_1 X_2,\tau} = \begin{cases} \frac{1}{N}\sum_{t=1}^{N-\tau}(X_{1,t} - \bar{X}_1)(X_{2,t+\tau} - \bar{X}_2) & \tau = 0, 1, 2, \ldots, N-1 \\ \frac{1}{N}\sum_{t=1-\tau}^{N}(X_{1,t} - \bar{X}_1)(X_{2,t+\tau} - \bar{X}_2) & \tau = -1, -2, \ldots, -(N-1), \end{cases}$$

so that the estimated ccs is

$$\hat{\rho}_{X_1 X_2,\tau} = \frac{\hat{s}_{X_1 X_2,\tau}}{\hat{\sigma}_{X_1}\hat{\sigma}_{X_2}}.$$

Linear filtering with noise

$$X_{2,t} = \sum_{u=-k}^{k} g_u X_{1,t-u} + \eta_t$$

where $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are zero mean stationary processes, $\{\eta_t\}$ is a zero mean (possible coloured) noise with variance σ_η^2, uncorrelated with $\{X_{1,t}\}$.
Then,

$$\begin{aligned}
s_{X_1X_2,\tau} &= \mathrm{cov}\{X_{1,t}, X_{2,t+\tau}\} \\
&= \mathrm{E}\{X_{1,t}X_{2,t+\tau}\} \\
&= \mathrm{E}\left\{X_{1,t}\left[\sum_{u=-k}^{k} g_u X_{1,t+\tau-u} + \eta_{t+\tau}\right]\right\} \\
&= \sum_{u=-k}^{k} g_u \mathrm{E}\{X_{1,t}, X_{1,t+\tau-u}\} \\
&= \sum_{u=-k}^{k} g_u s_{X_1,\tau-u}.
\end{aligned}$$

Since,

$$\begin{aligned}
\sigma_{X_2}^2 &= \mathrm{var}\{X_{2,t}\} = \mathrm{E}\{X_{2,t}^2\} \\
&= \mathrm{E}\left\{\left(\sum_{u=-k}^{k} g_u X_{1,t-u} + \eta_t\right)^2\right\} \\
&= \mathrm{E}\left\{\left(\sum_{u=-k}^{k} g_u X_{1,t-u}\right)^2\right\} + \mathrm{E}\{\eta_t^2\} \\
&= \mathrm{E}\left\{\sum_{u=-k}^{k} g_u X_{1,t-u} \sum_{v=-k}^{k} g_v X_{1,t-v}\right\} + \sigma_\eta^2 \\
&= \sum_{u=-k}^{k}\sum_{v=-k}^{k} g_u g_v \mathrm{E}\{X_{1,t-u}X_{1,t-v}\} + \sigma_\eta^2 \\
&= \sum_{u=-k}^{k}\sum_{v=-k}^{k} g_u g_v s_{X_1,u-v} + \sigma_\eta^2
\end{aligned}$$

the ccs is

$$\rho_{X_1X_2,\tau} = \frac{\sum_{u=-k}^{k} g_u s_{X_1,\tau-u}}{\sigma_{X_1}\sqrt{\sum_{u=-k}^{k}\sum_{v=-k}^{k} g_u g_v s_{X_1,u-v} + \sigma_\eta^2}}.$$

Cross-Spectra

Consider frequency domain characterization of the real-valued bivariate process $\{X_{1,t}; X_{2,t}\}$. Assume that $\{X_{1,t}\}$ and $\{X_{2,t}\}$ are both zero mean processes with spectral density functions

$$S_{X_j}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_j,\tau} e^{-i2\pi f\tau}; \quad |f| \leq 1/2, \;\; j = 1, 2.$$

Then the cross spectra are

$$S_{X_j X_k}(f) = \sum_{\tau=-\infty}^{\infty} s_{X_j X_k,\tau} e^{-i2\pi f\tau}; \quad |f| \leq 1/2, \;\; j \neq k = 1, 2,$$

assuming the ccvs is square summable.

Cross-Spectra

Note that for real processes $S^*_{X_j X_k}(f) = S_{X_j X_k}(-f)$.
Inverse Fourier transformation gives

$$s_{X_j X_k,\tau} = \int_{-1/2}^{1/2} S_{X_j X_k}(f) e^{i2\pi f\tau}\, df.$$

Now write

$$X_{j,t} = \int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ_{X_j}(f); \quad X_{k,t} = \int_{-1/2}^{1/2} e^{i2\pi f' t}\, dZ_{X_k}(f'),$$

so that,

$$\begin{aligned} s_{X_j X_k,\tau} &= \mathrm{cov}\{X_{j,t}, X_{k,t+\tau}\} \\ &= \mathrm{E}\{X_{j,t} X_{k,t+\tau}\} \\ &= \mathrm{E}\{X^*_{j,t} X_{k,t+\tau}\} \\ &= \mathrm{E}\left\{\int_{-1/2}^{1/2}\int_{-1/2}^{1/2} e^{-i2\pi ft} e^{i2\pi f'(t+\tau)} \mathrm{E}\{dZ^*_{X_j}(f) dZ_{X_k}(f')\}\right\}. \end{aligned}$$

Cross-specta

But this must be a function of τ only, so that $\mathrm{E}\{dZ^*_{X_j}(f)dZ_{X_k}(f')\} = 0$ for $f \neq f'$, i.e., $dZ^*_{X_j}$ and dZ_{X_k} are cross-orthogonal as well as individually orthogonal. Hence,

$$\begin{aligned} s_{X_j X_k,\tau} &= \int_{-1/2}^{1/2} e^{i2\pi f\tau}\mathrm{E}\{dZ^*_{X_j}(f)dZ_{X_k}(f)\} \\ \Rightarrow S_{X_j X_k}(f)\,df &= \mathrm{E}\{dZ^*_{X_j}(f)dZ_{X_k}(f)\} \\ \Rightarrow S^*_{X_k X_j}(f) &= S_{X_j X_k}(f). \end{aligned}$$

Spectral matrix

The complete spectral properties are given by the spectral matrix

$$S(f) = \begin{pmatrix} S_{X_1}(f) & S_{X_1X_2}(f) \\ S_{X_2X_1}(f) & S_{X_2}(f) \end{pmatrix}.$$

Since $S_{X_jX_k}(f)$ is a complex quantity we can write it as

$$S_{X_jX_k}(f) = |S_{X_jX_k}(f)|e^{i\theta_{X_jX_k}(f)},$$

where $|S_{X_jX_k}(f)|$ is the cross-amplitude spectrum
$\theta_{X_jX_k}(f)$ is the phase spectrum.

$\theta_{X_jX_k}(f)$ is defined only up to an integer multiple of 2π (since $e^{i2\pi} = e^{i4\pi} = \ldots = 1$).

Coherence

The quantity

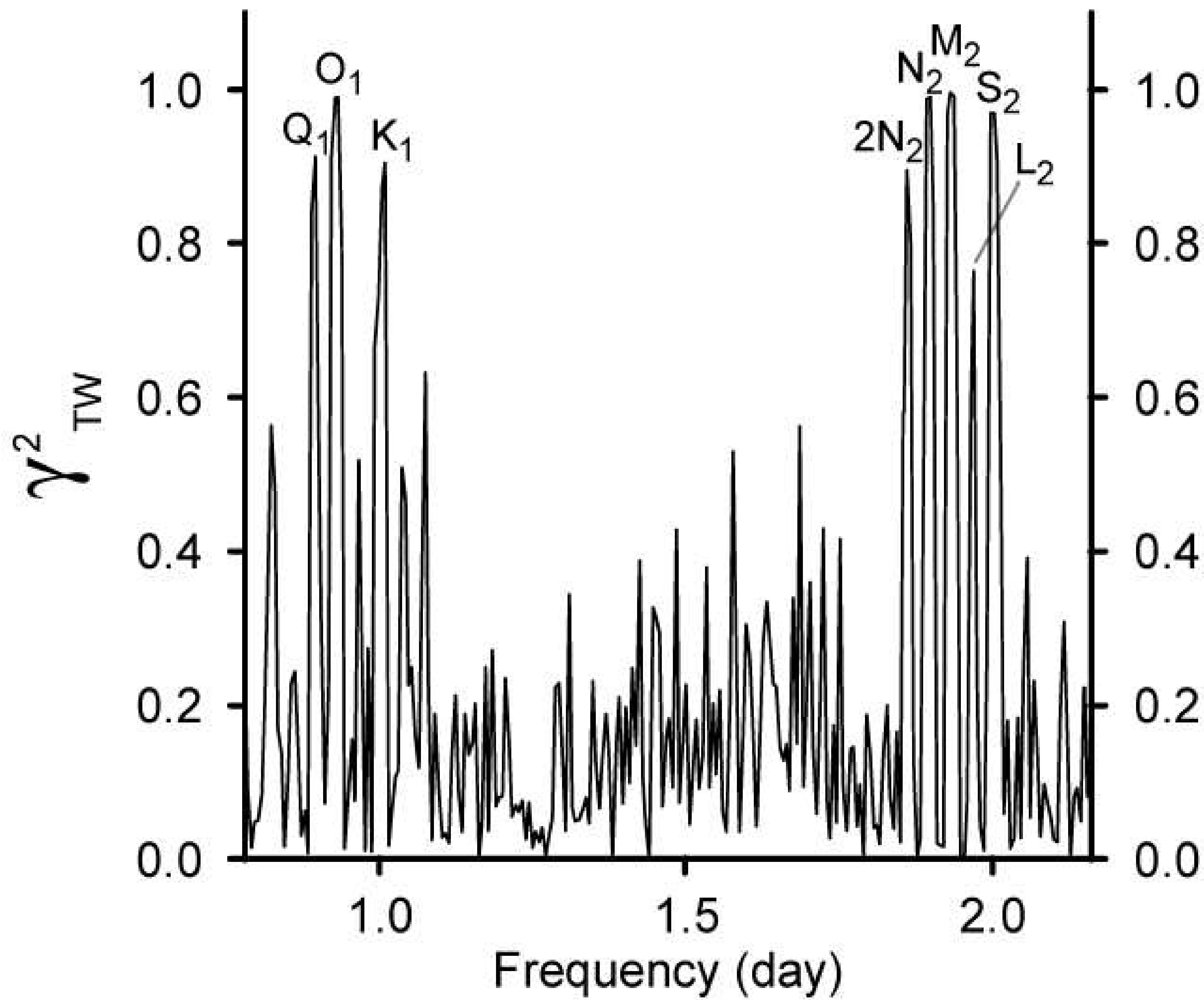
$$\gamma^2_{X_j X_k}(f) = \frac{|S_{X_j X_k}(f)|^2}{S_{X_j}(f) S_{X_k}(f)},$$

is called the magnitude squared coherence at f. It is a real valued coefficient such that

$$0 \leq \gamma^2_{X_j X_k}(f) \leq 1.$$

It measures the linear correlation between the components of $\{X_{j,t}\}$ and $\{X_{k,t}\}$ <u>at frequency f</u>.

Ocean Levels and Ground Well Levels

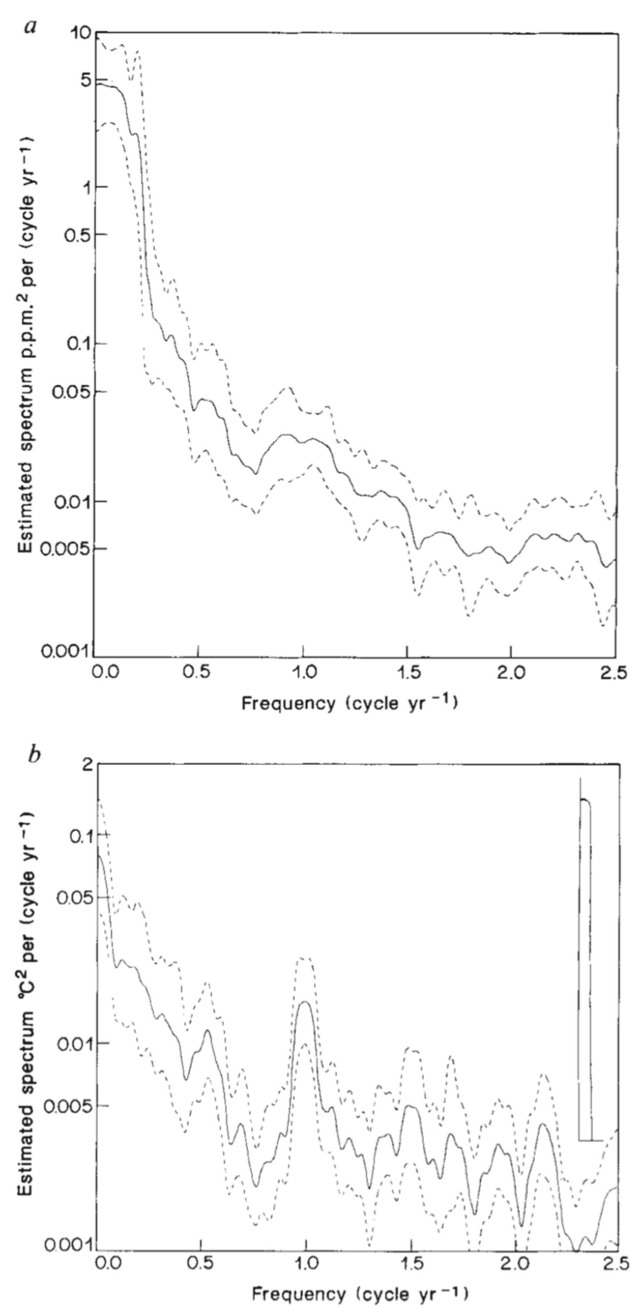


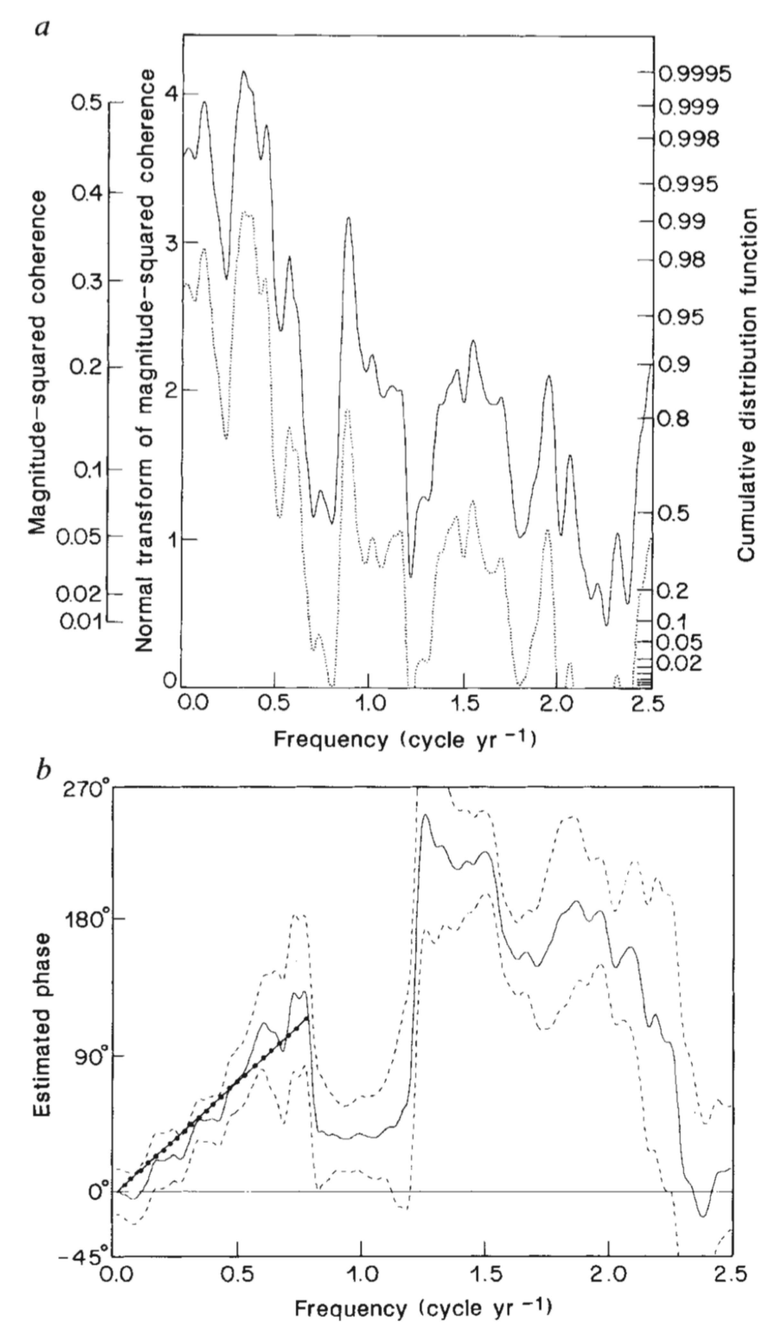
CO2 and Global Temp

Coherence established between atmospheric carbon dioxide and global temperature

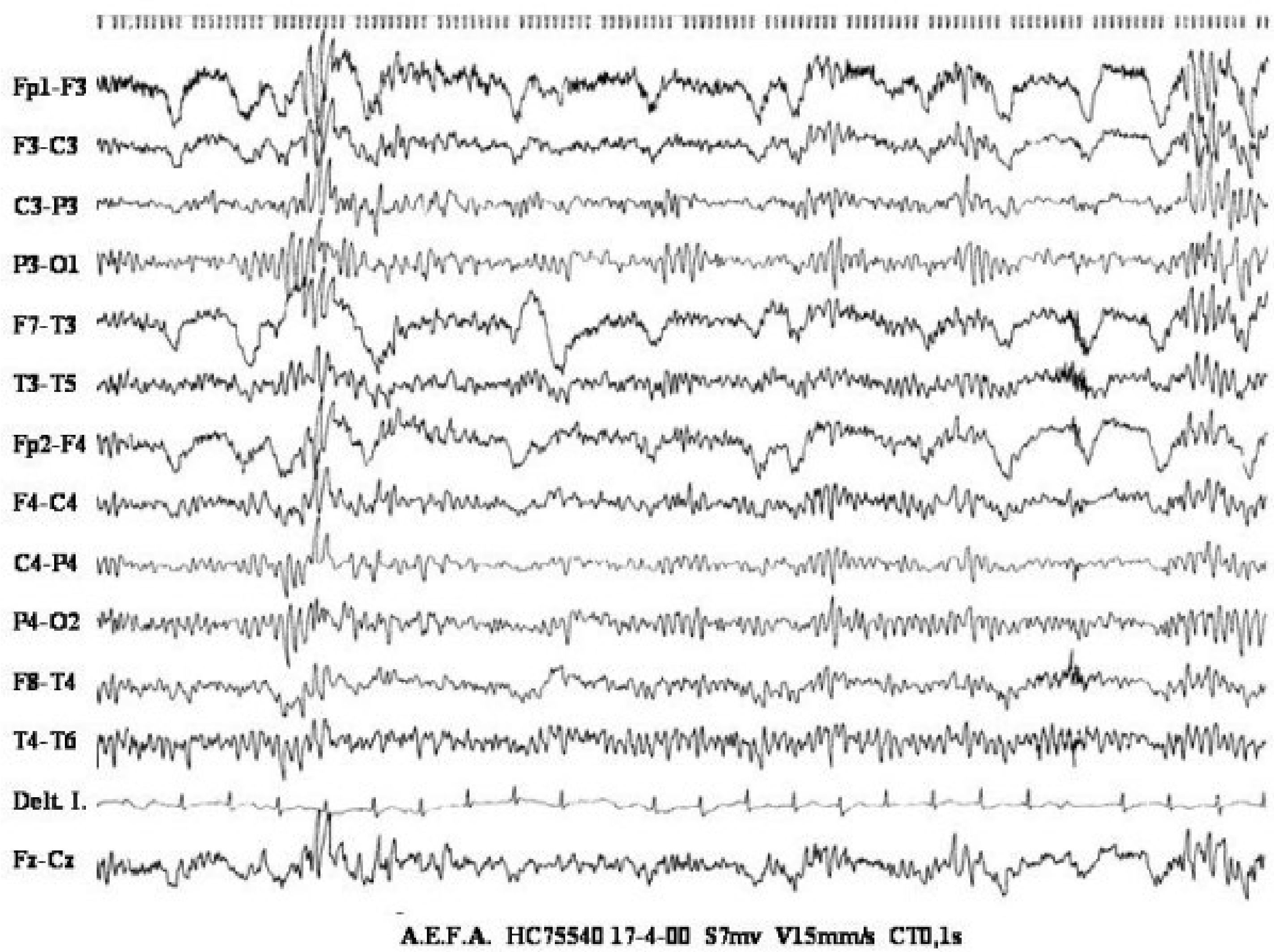
Cynthia Kuo, Craig Lindberg & David J. Thomson

Mathematical Sciences Research Center, AT&T Bell Labs, Murray Hill, New Jersey 07974, USA

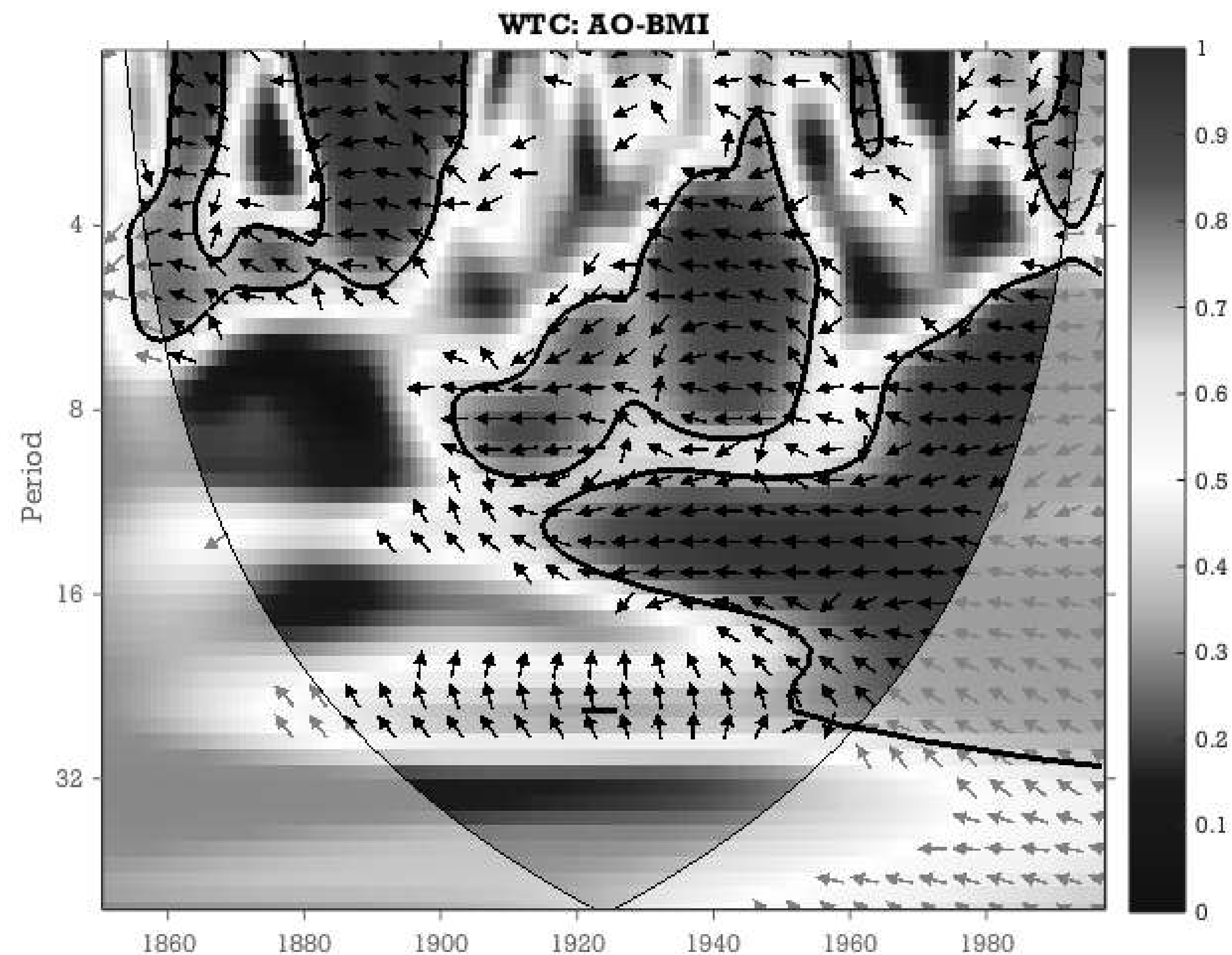




Multivariate time series - ECG

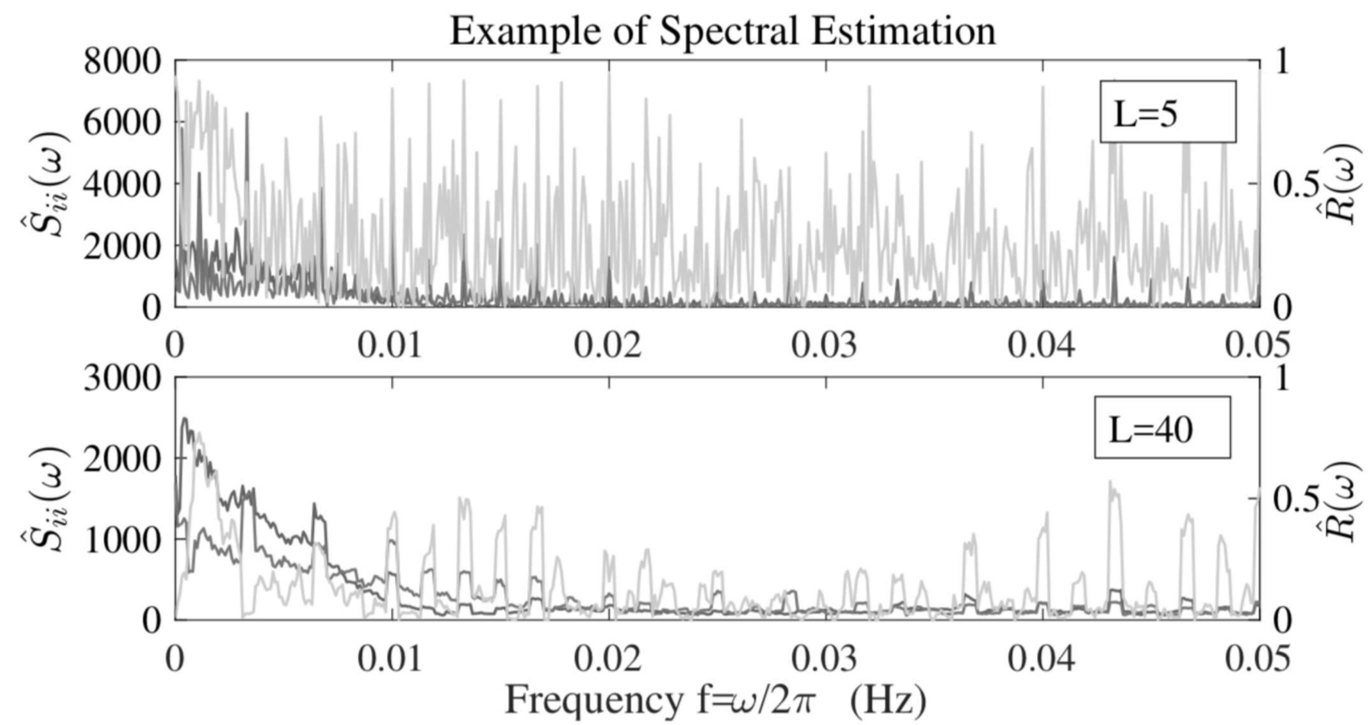


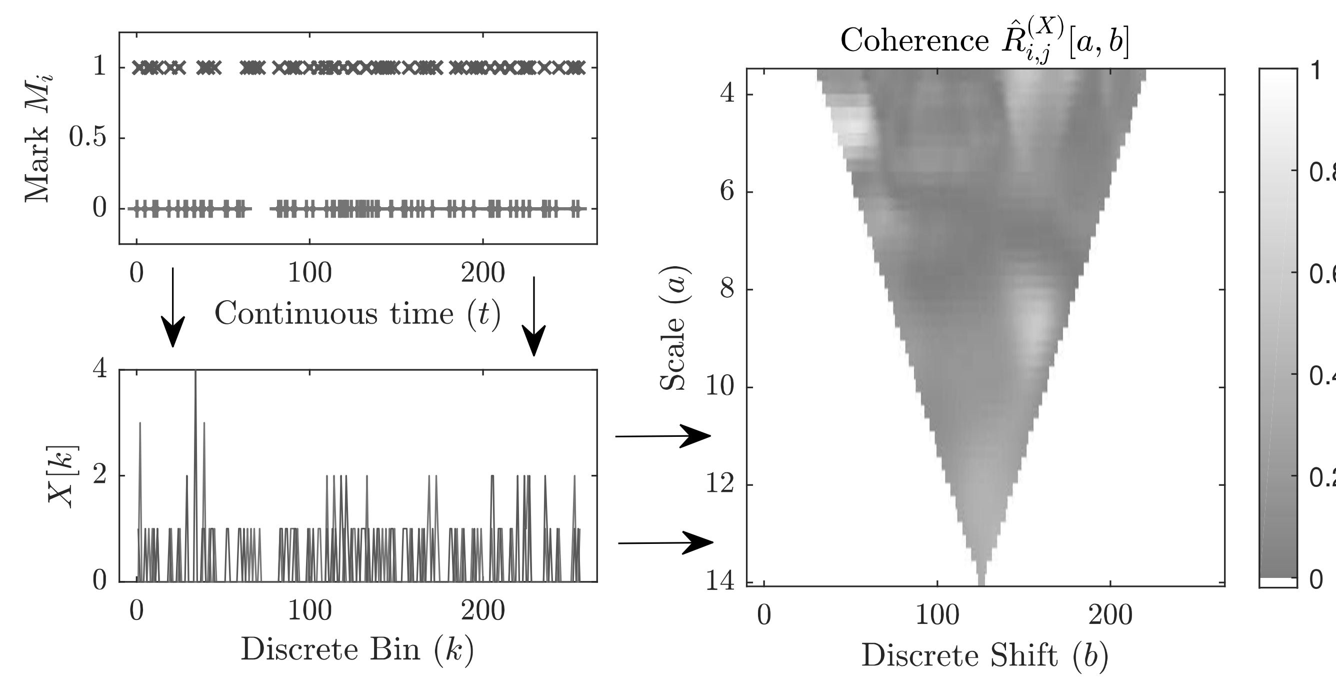
Non-stationary processes. Wavelet Coherence



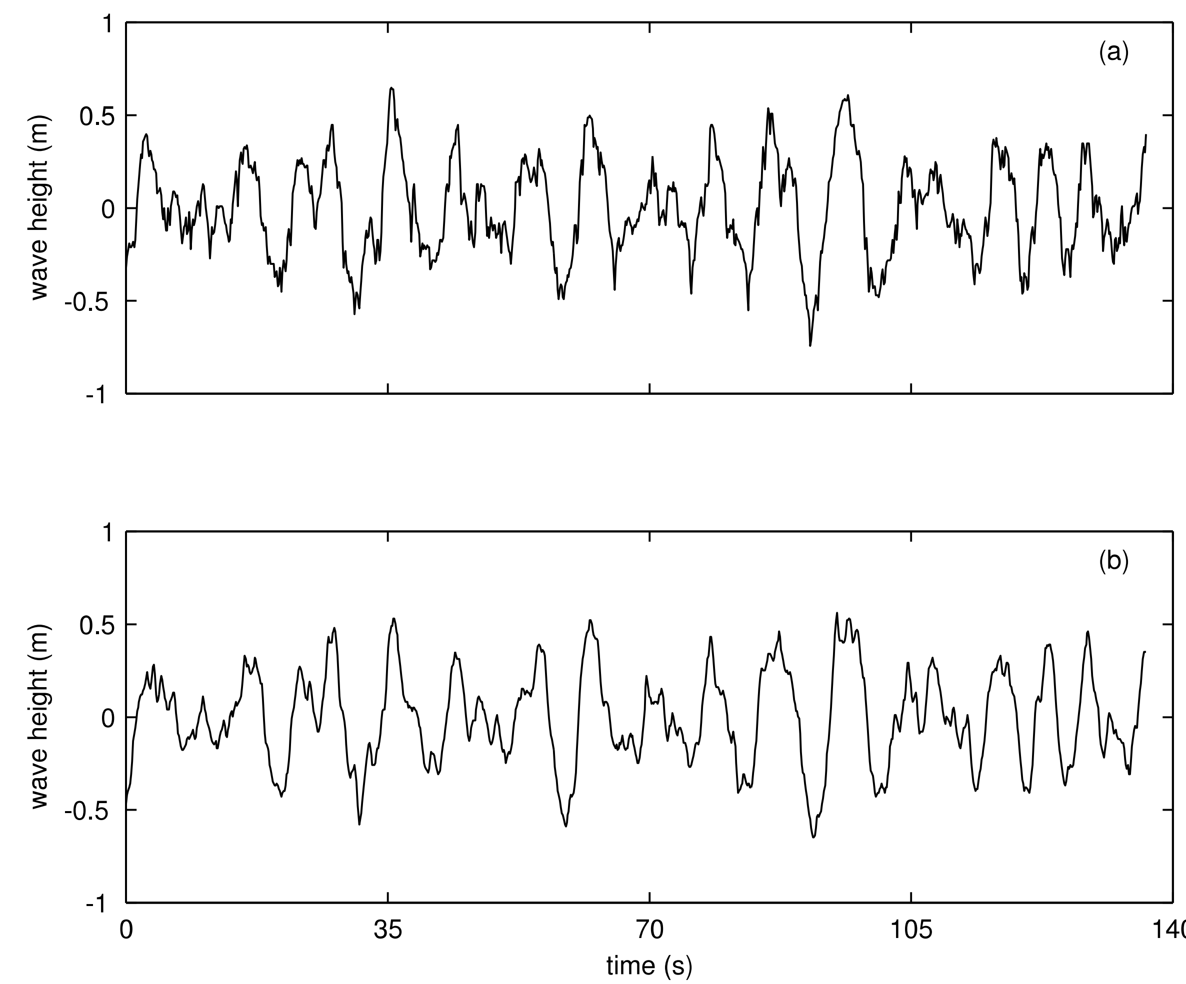
Arctic Oscillation index and the Baltic maximum sea ice extent record

Coherence and wavelet coherence for point processes

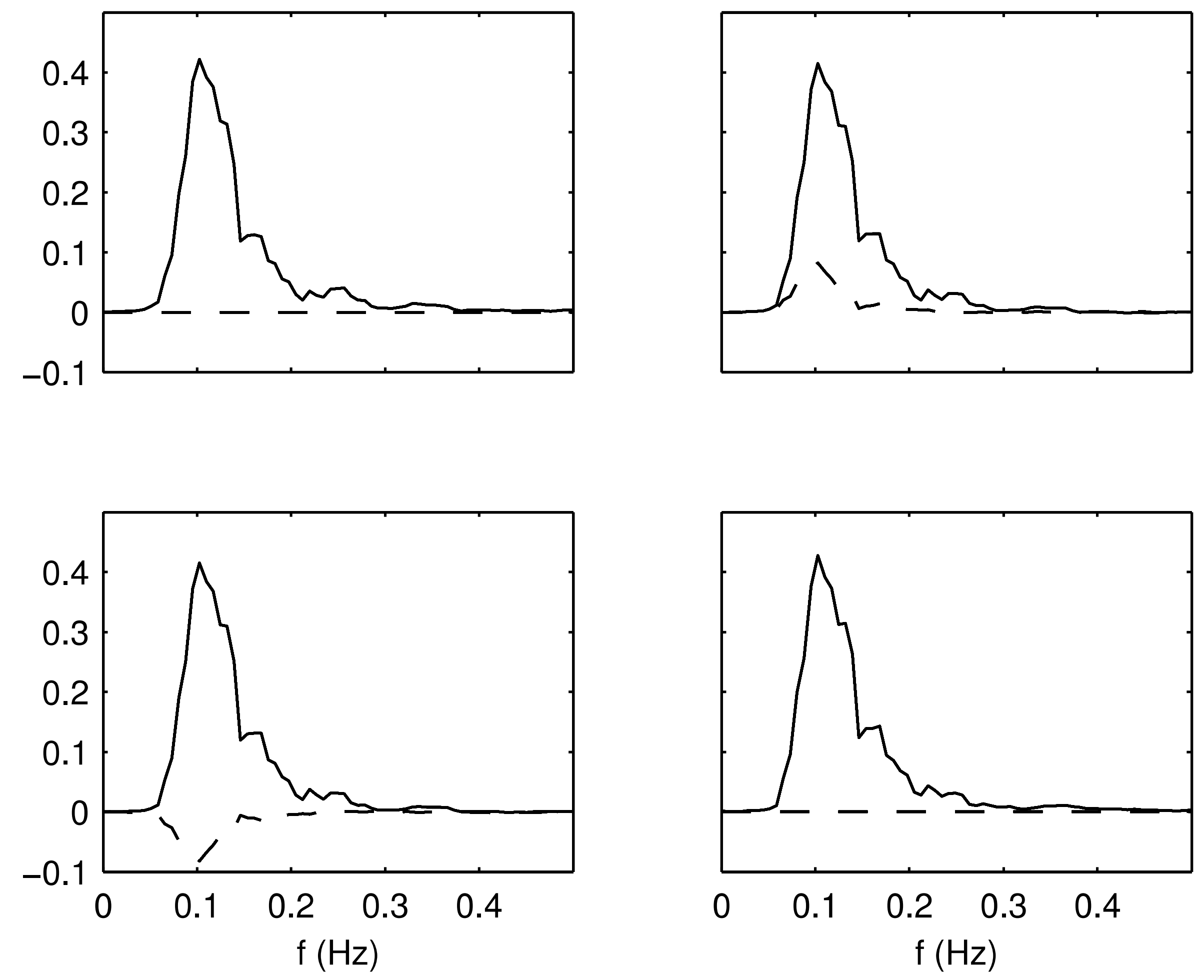




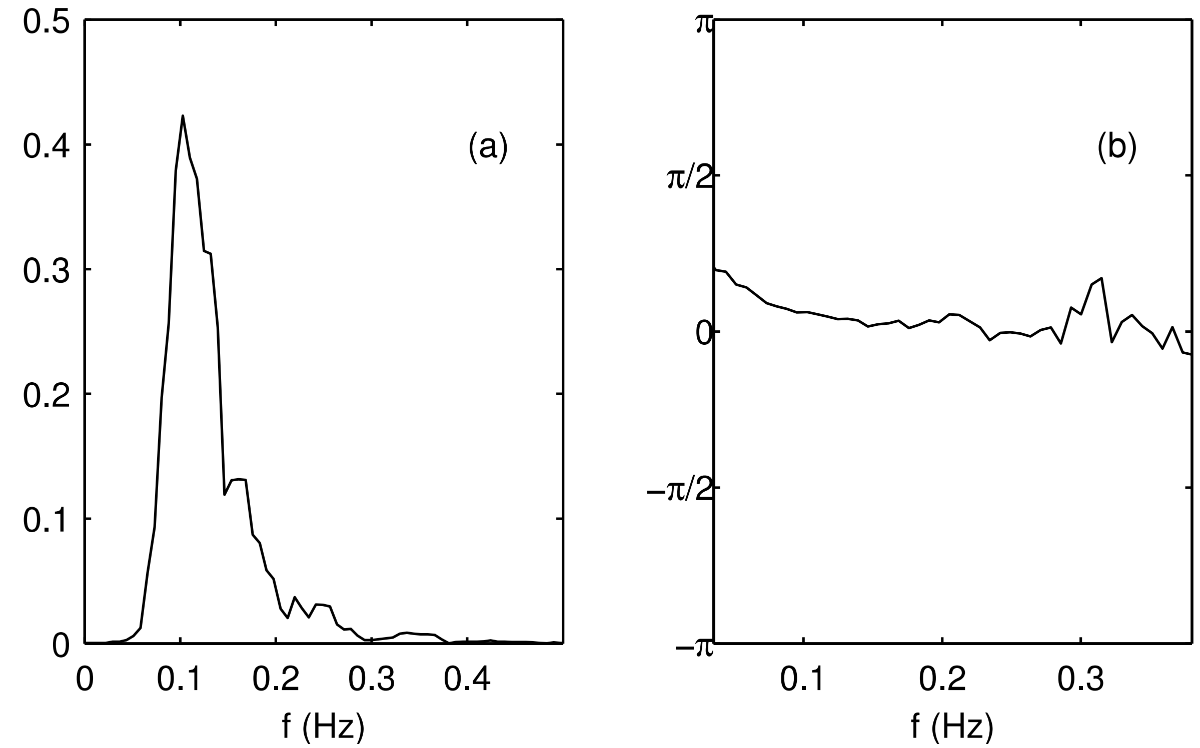
Example:



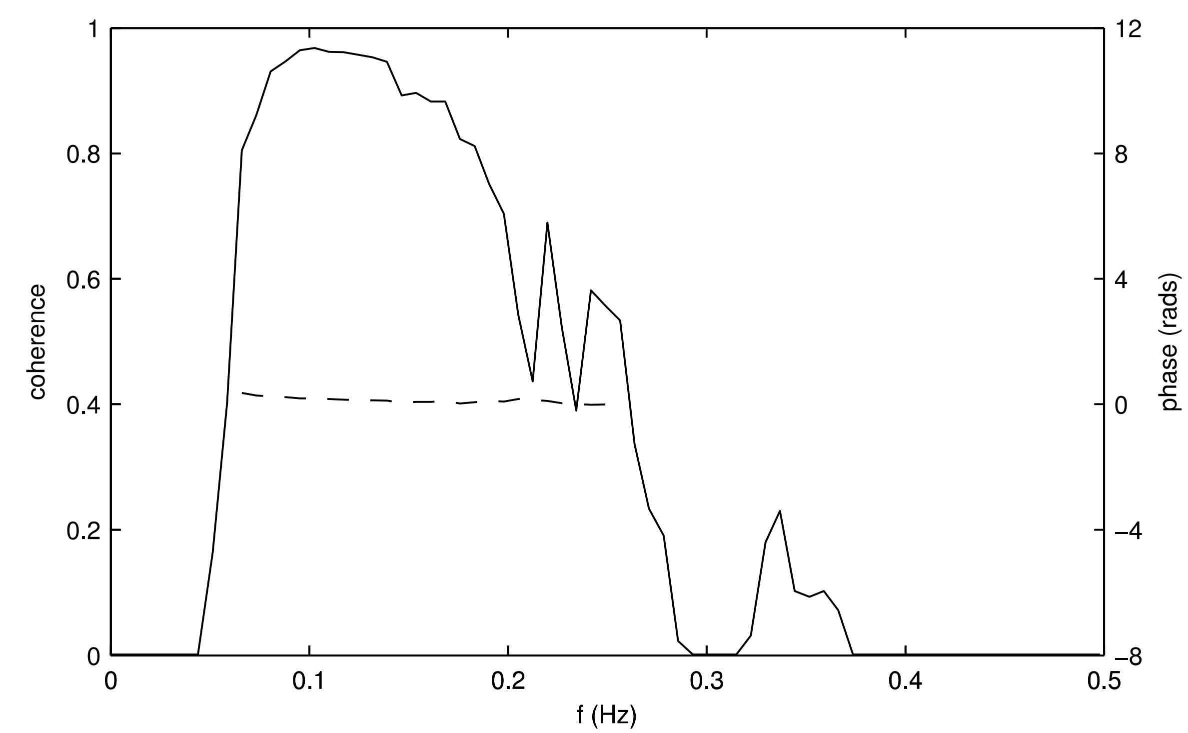
Simultaneous measurements of ocean waves versus time by two instruments of quite different design, (a) an infrared wave gauge, and (b) a wire wave gauge. There are $N = 1024$ data values in each series and the sample interval is $\delta_t = 4/30$s. (These series were derived from data supplied courtesy of Andy Jessup, Applied Physics Lab, University of Washington.)



The estimated spectral matrix, $\hat{\mathbf{S}}(\cdot)$, for the two ocean wave time series. The real and imaginary parts of $\hat{S}_{X_l X_m}(\cdot)$, $l, m = 1, 2$, are shown by solid and dashed lines, respectively.



(a) Estimated cross-amplitude spectrum $|\hat{S}_{X_1X_2}(\cdot)|$, in the interval $[0, 0.5]$ Hz and (b) estimated phase spectrum $\hat{\theta}_{X_1X_2}(\cdot)$, in the interval $[0.035, 0.38]$ Hz, for the two ocean wave time series .



Coherence estimate for the two ocean wave time series. Also shown (dashed line) is the estimated phase $\hat{\theta}_{X_1 X_2}(f)$ over frequencies for which the estimated ordinary coherence exceeds 0.5. The coherence between the datasets is highest around 0.1-0.2Hz, so this is the frequency range where the instruments behave most similarly (since they are measuring the same waves).

Linear filtering with noise

The model is

$$X_{2,t} = \sum_{u=-k}^{k} g_u X_{1,t-u} + \eta_t.$$

Then

$$\begin{aligned}
S_{X_1X_2}(f) &= \sum_{\tau=-\infty}^{\infty} s_{X_1X_2,\tau} e^{-i2\pi f\tau} \\
&= \sum_{u=-k}^{k} g_u \sum_{\tau=-\infty}^{\infty} s_{X_1,\tau-u} e^{-i2\pi f\tau} \\
&= \sum_{u=-k}^{k} g_u e^{-i2\pi fu} \sum_{\tau=-\infty}^{\infty} s_{X_1,\tau-u} e^{-i2\pi f(\tau-u)} \\
&= G(f) S_{X_1}(f).
\end{aligned}$$

We can write the model as:

$$\int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ_{X_2}(f) = \sum_{u=-k}^{k} g_u \int_{-1/2}^{1/2} e^{i2\pi f(t-u)}\, dZ_{X_1}(f) + \int_{-1/2}^{1/2} e^{i2\pi ft}\, dZ_\eta(f).$$

Hence,

$$dZ_{X_2}(f) = \sum_{u=-k}^{k} g_u e^{-i2\pi fu} dZ_{X_1}(f) + dZ_\eta(f).$$

Thus,

$$\mathrm{E}\{|dZ_{X_2}(f)|^2\} = \sum_{u=-k}^{k} g_u e^{-i2\pi fu} \sum_{v=-k}^{k} g_v e^{i2\pi fv} \mathrm{E}\{|dZ_{X_1}(f)|^2\} + \mathrm{E}\{|dZ_\eta(f)|^2\}$$

since cross-products have expectation zero.

Hence,

$$S_{X_2}(f) = |G(f)|^2 S_{X_1}(f) + S_\eta(f).$$

Then,

$$\begin{aligned} \gamma^2_{X_1X_2}(f) &= \frac{|G(f)|^2 S^2_{X_1}(f)}{S_{X_1}(f)[|G(f)|^2 S_{X_1}(f) + S_\eta(f)]} \\ &= \left[1 + \frac{S_\eta(f)}{|G(f)|^2 S_{X_1}(f)}\right]^{-1}. \end{aligned}$$

Now,

$$\begin{aligned} S_\eta(f) &= S_{X_2}(f) - |G(f)|^2 S_{X_1}(f) \\ &= S_{X_2}(f)\left[1 - \frac{|G(f)|^2}{S_{X_2}(f)} S_{X_1}(f)\right]. \end{aligned}$$

But,

$$\gamma^2_{X_1X_2}(f) = \frac{|G(f)|^2 S^2_{X_1}(f)}{S_{X_1}(f)S_{X_2}(f)} = \frac{|G(f)|^2 S_{X_1}(f)}{S_{X_2}(f)},$$

so,

$$S_\eta(f) = S_{X_2}(f)[1 - \gamma^2_{X_1X_2}(f)]$$

“noise” “total times unexplained proportion”

Bivariate autoregressive processes

A bivariate model arises as an extension to the univariate AR(p) process. Let

$$\boldsymbol{X}_t = \begin{pmatrix} X_{1,t} \\ X_{2,t} \end{pmatrix} \quad \text{and} \quad \boldsymbol{\epsilon}_t = \begin{pmatrix} \epsilon_{1,t} \\ \epsilon_{2,t} \end{pmatrix}.$$

The VAR(p) model can be expressed as

$$\begin{aligned} \boldsymbol{X}_t &= \phi_{1,p}\boldsymbol{X}_{t-1} + \ldots + \phi_{p,p}\boldsymbol{X}_{t-p} + \boldsymbol{\epsilon}_t, \\ \Phi(B)\boldsymbol{X}_t &= \boldsymbol{\epsilon}_t \end{aligned}$$

where,

$$\Phi(B) = \mathrm{I} - \phi_{1,p}B - \phi_{2,p}B^2 - \ldots - \phi_{p,p}B^p,$$

where I is the (2×2) identity matrix, and now $\{\phi_{i,p}\}$ are (2×2) matrices of parameters.

$\boldsymbol{\epsilon}_t$ is a bivariate white noise process, such that

$$\mathrm{E}\{\boldsymbol{\epsilon}_t\} = 0$$

and

$$\mathrm{E}\{\boldsymbol{\epsilon}_s \boldsymbol{\epsilon}_t^T\} = \begin{cases} \Sigma, & t = s \\ 0 & \text{otherwise} \end{cases}$$

and Σ is a (2×2) covariance matrix. Thus the elements of $\boldsymbol{\epsilon}_t$ may be correlated.

FACT:

The condition for stationarity is that the roots of the *determinantal polynomial*, $|\Phi(z)|$, lie outside the unit circle.