Stationarity

The class of all stochastic processes is too large to work with in practice. We consider only the subclass of stationary processes.

COMPLETE/STRONG/STRICT stationarity

$\{X_t\}$ is said to be completely stationary if, for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n \in T$, and for any τ such that $t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau \in T$ are also contained in the index set, the joint cdf of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ is the same as that of $\{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}$ i.e.,

$$\mathrm{F}_{t_1,t_2,\ldots,t_n}(a_1, a_2, \ldots, a_n) = \mathrm{F}_{t_1+\tau,t_2+\tau,\ldots,t_n+\tau}(a_1, a_2, \ldots, a_n),$$

so that the probabilistic structure of a completely stationary process is invariant under a shift in time.

SECOND-ORDER/WEAK/COVARIANCE stationarity

$\{X_t\}$ is said to be second-order stationary if, for all $n \geq 1$, for any $t_1, t_2, \ldots, t_n \in T$, and for any τ such that $t_1 + \tau, t_2 + \tau, \ldots, t_n + \tau \in T$ are also contained in the index set, all the joint moments of orders 1 and 2 of $\{X_{t_1}, X_{t_2}, \ldots, X_{t_n}\}$ exist, are finite, and equal to the corresponding joint moments of $\{X_{t_1+\tau}, X_{t_2+\tau}, \ldots, X_{t_n+\tau}\}$.

SECOND-ORDER/WEAK/COVARIANCE stationarity

Hence,

$$\mathrm{E}\{X_t\} \equiv \mu \quad ; \quad \mathrm{var}\{X_t\} \equiv \sigma^2 \ \ (= \mathrm{E}\{X_t^2\} - \mu^2),$$

are constants independent of t.
If we let $\tau = -t_1$,

$$\begin{aligned} \mathrm{E}\{X_{t_1} X_{t_2}\} &= \mathrm{E}\{X_{t_1+\tau} X_{t_2+\tau}\} \\ &= \mathrm{E}\{X_0 X_{t_2-t_1}\}, \end{aligned}$$

and with $\tau = -t_2$,

$$\begin{aligned} \mathrm{E}\{X_{t_1} X_{t_2}\} &= \mathrm{E}\{X_{t_1+\tau} X_{t_2+\tau}\} \\ &= \mathrm{E}\{X_{t_1-t_2} X_0\}. \end{aligned}$$

SECOND-ORDER/WEAK/COVARIANCE stationarity

Hence, $\mathsf{E}\{X_{t_1}X_{t_2}\}$ is a function of the absolute difference $|t_2 - t_1|$ only, similarly, for the covariance between X_{t_1} & X_{t_2}:

$$\mathrm{cov}\{X_{t_1}, X_{t_2}\} = \mathsf{E}\{(X_{t_1} - \mu)(X_{t_2} - \mu)\} = \mathsf{E}\{X_{t_1}X_{t_2}\} - \mu^2.$$

For a discrete time second-order stationary process $\{X_t\}$ we define the autocovariance sequence (acvs) by

$$s_\tau \equiv \mathrm{cov}\{X_t, X_{t+\tau}\} = \mathrm{cov}\{X_0, X_\tau\}.$$

Properties and Notation

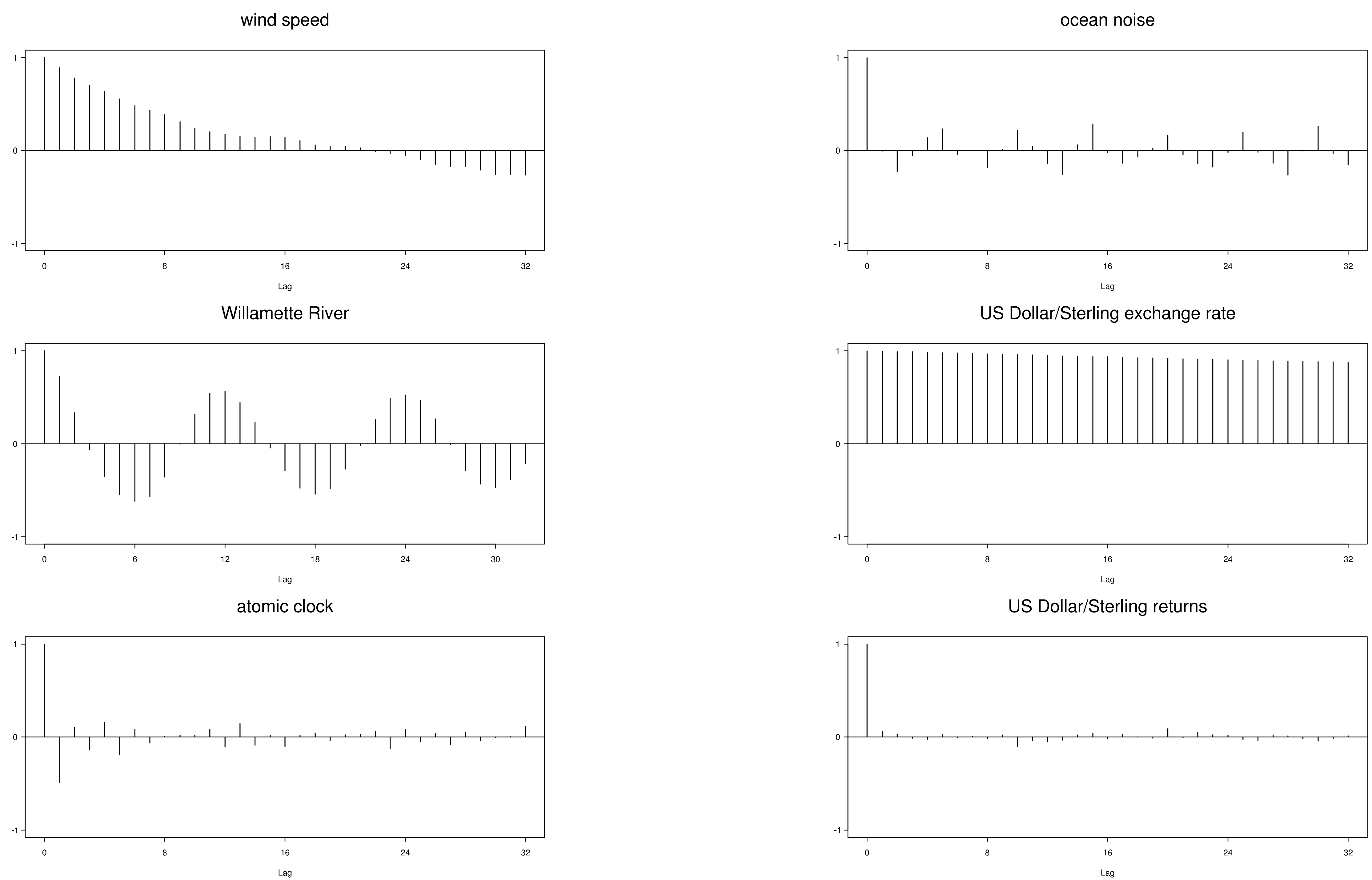
1. τ is called the lag.
2. $s_0 = \sigma^2$ and $s_{-\tau} = s_\tau$.

3. The autocorrelation sequence (acs) is given by

$$\rho_\tau = \frac{s_\tau}{s_0} = \frac{\mathrm{cov}\{X_t, X_{t+\tau}\}}{\sigma^2}.$$

The sample or estimated autocorrelation sequence (acs), $\{\hat{\rho}_\tau\}$, for each of our time series are given in Figs. 6 and 7.

Example autocorrelation sequences



Worked example - variogram

For the stationary process $\{X_t\}$ with mean μ, acvs $\{s_\tau\}$ and variance $s_0 = \sigma^2$, show that the variogram, defined as

$$v_\tau := \mathrm{E}\{(X_{t+\tau} - X_t)^2\}/2$$

has an upper bound of $2\sigma^2$.

[We shall see how to compute these in Chapter 4.] Note e.g., that for the Willamette river data X_t and X_{t+6} seem to be negatively correlated, while X_t and X_{t+12} seem positively correlated (consistent with the river flow varying with a period of roughly 12 months).

4. Since ρ_τ is a correlation coefficient, $|s_\tau| \le s_0$.

5. The sequence $\{s_\tau\}$ is positive semidefinite, i.e., for all $n \ge 1$, for any $t_1, t_2, \ldots, t_n$ contained in the index set, and for any set of nonzero real numbers $a_1, a_2, \ldots, a_n$

$$\sum_{j=1}^{n} \sum_{k=1}^{n} s_{t_j - t_k} a_j a_k \ge 0.$$

Proof

Let

$$\boldsymbol{a} = (a_1, a_2, \ldots, a_n)^{\mathsf{T}}, \quad \boldsymbol{V} = (X_{t_1}, X_{t_2}, \ldots, X_{t_n})^{\mathsf{T}}$$

and let Σ be the variance-covariance matrix of $\boldsymbol{V}$. Its j, k-th element is given by $s_{t_j - t_k} = \mathrm{E}\{(X_{t_j} - \mu)(X_{t_k} - \mu)\}$. Define the r.v.

$$w = \sum_{j=1}^{n} a_j X_{t_j} = \boldsymbol{a}^{\mathsf{T}} \boldsymbol{V}.$$

Then

$$0 \leq \mathrm{var}\{w\} = \mathrm{var}\{\boldsymbol{a}^{\mathsf{T}} \boldsymbol{V}\} = \boldsymbol{a}^{\mathsf{T}} \mathrm{var}\{\boldsymbol{V}\} \boldsymbol{a} = \boldsymbol{a}^{\mathsf{T}} \Sigma \boldsymbol{a} = \sum_{j=1}^{n} \sum_{k=1}^{n} s_{t_j - t_k} a_j a_k.$$

6. The variance-covariance matrix of equispaced X's, $(X_1, X_2, \ldots, X_N)^\mathsf{T}$ has the form

$$\begin{bmatrix} s_0 & s_1 & \cdots & s_{N-2} & s_{N-1} \\ s_1 & s_0 & \cdots & s_{N-3} & s_{N-2} \\ \vdots & & \ddots & & \\ s_{N-2} & s_{N-3} & \cdots & s_0 & s_1 \\ s_{N-1} & s_{N-2} & \cdots & s_1 & s_0 \end{bmatrix}$$

which is known as a symmetric Toeplitz matrix – all elements on a diagonal are the same. Note the matrix has only N unique elements, $s_0, s_1, \ldots, s_{N-1}$.

7. A stochastic process $\{X_t\}$ is called Gaussian if, for all $n \geq 1$ and for any $t_1, t_2, \ldots, t_n$ contained in the index set, the joint cdf of $X_{t_1}, X_{t_2}, \ldots, X_{t_n}$ is multivariate Gaussian.
   - 2nd-order stationary <u>Gaussian</u> $\Rightarrow$ complete stationarity (since MVN completely characterized by 1st and 2nd moments). It is not true in general that 2nd-order stationary $\Rightarrow$ complete stationarity.
   - Complete stationarity $\Rightarrow$ 2nd-order stationary in general.

8. The simple term "stationary" will be taken to mean second-order stationary unless stated otherwise.

Examples of discrete stationary processes

[1] **White noise process**

Also known as a purely random process. Let $\{X_t\}$ be a sequence of uncorrelated r.v.s such that

$$\mathrm{E}\{X_t\} = \mu, \qquad \mathrm{var}\{X_t\} = \sigma^2 \qquad \forall t$$

and

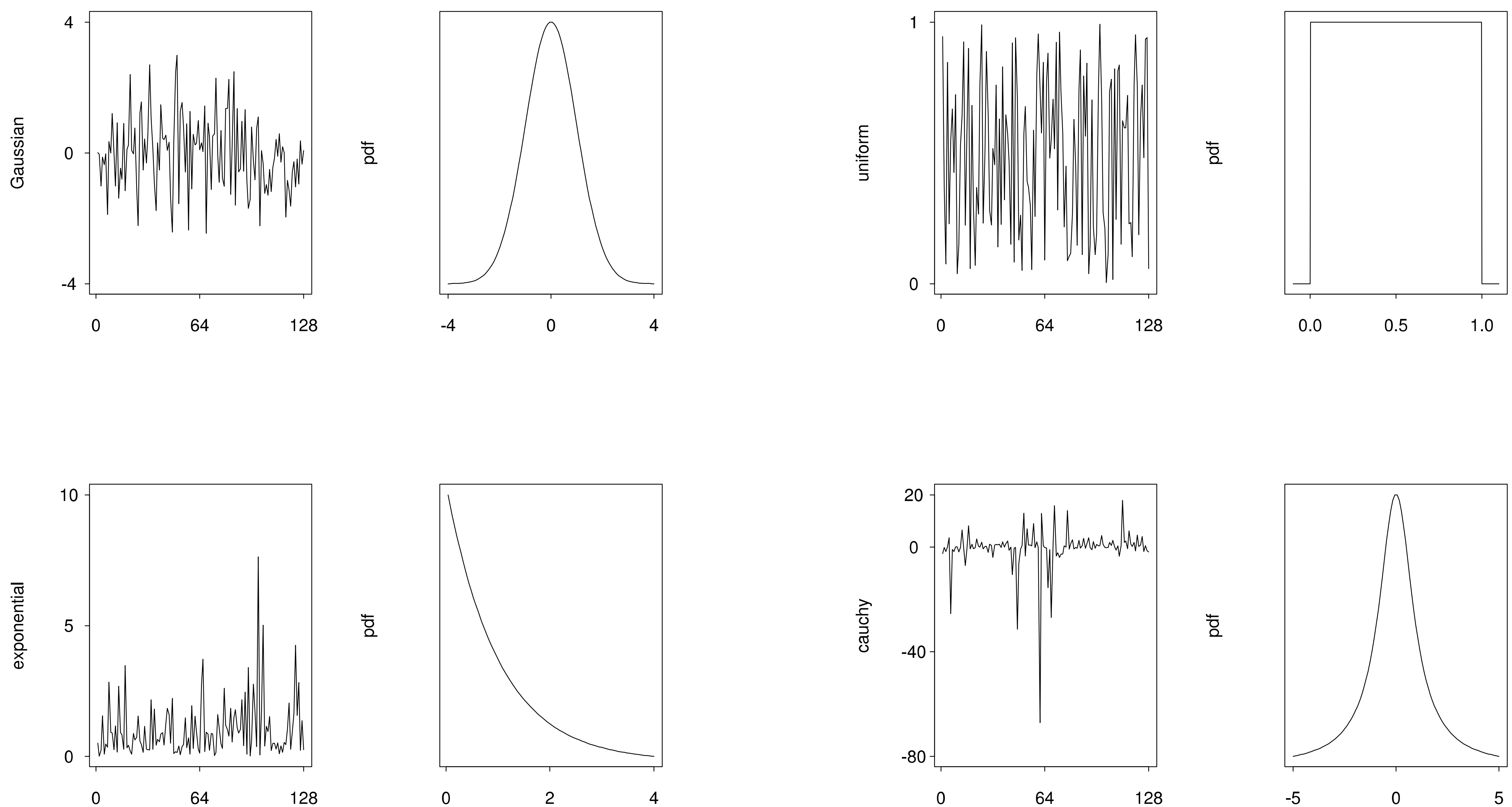
$$s_\tau = \begin{cases} \sigma^2 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases} \qquad \text{or} \qquad \rho_\tau = \begin{cases} 1 & \tau = 0 \\ 0 & \tau \neq 0 \end{cases}$$

forms a basic building block in time series analysis.

Often special notation is used $X_t \equiv \epsilon_t$ for white noise, i.e. $\{\epsilon_t\}$ is a white noise process.

White noise examples

Very different realizations of white noise can be obtained for different distributions of $\{X_t\}$. Examples are given in Figures 8 and 9 for processes with (a) Gaussian, (b) exponential, (c) uniform and (d) truncated Cauchy distributions.



Examples of discrete stationary processes

[2]<u>***q*-th order moving average process**</u> MA(q)

X_t can be expressed in the form

$$\begin{aligned} X_t &= \mu - \theta_{0,q}\epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q} \\ &= \mu - \sum_{j=0}^{q} \theta_{j,q}\epsilon_{t-j}, \end{aligned}$$

where μ and $\theta_{j,q}$'s are constants ($\theta_{0,q} \equiv -1, \theta_{q,q} \neq 0$), and $\{\epsilon_t\}$ is a zero-mean white noise process with variance σ_ϵ^2.

W.l.o.g. assume $\mathrm{E}\{X_t\} = \mu = 0$.
Then $\mathrm{cov}\{X_t, X_{t+\tau}\} = \mathrm{E}\{X_t X_{t+\tau}\}$.
Recall: $\mathrm{cov}(X, Y) = \mathrm{E}\{(X - \mathrm{E}\{X\})(Y - \mathrm{E}\{Y\})\}$.

MA(q)

Since $\mathrm{E}\{\epsilon_t \epsilon_{t+\tau}\} = 0 \;\; \forall\, \tau \neq 0$ we have for $\tau \geq 0$.

$$\begin{aligned}
\mathrm{cov}\{X_t, X_{t+\tau}\} &= \sum_{j=0}^{q} \sum_{k=0}^{q} \theta_{j,q} \theta_{k,q} \mathrm{E}\{\epsilon_{t-j} \epsilon_{t+\tau-k}\} \\
&= \sigma_\epsilon^2 \sum_{j=0}^{q-\tau} \theta_{j,q} \theta_{j+\tau,q} \qquad (k = j + \tau) \\
&\equiv s_\tau ,
\end{aligned}$$

which does not depend on t. Since $s_\tau = s_{-\tau}$, $\{X_t\}$ is a stationary process with acvs given by

$$s_\tau = \begin{cases} \sigma_\epsilon^2 \sum_{j=0}^{q-|\tau|} \theta_{j,q} \theta_{j+|\tau|,q} & |\tau| \leq q \\ 0 & |\tau| > q \end{cases}$$

N.B. No restrictions were placed on the $\theta_{j,q}$'s to ensure stationarity, though obviously, $|\theta_{j,q}| < \infty, \; j = 1, \ldots, q$.

Examples (see Figures 10 and 11)

$$X_t = \epsilon_t - \theta_{1,1}\epsilon_{t-1} \qquad \text{MA(1)}$$

$\boxed{\text{acvs:}}$

$$s_\tau = \sigma_\epsilon^2 \sum_{j=0}^{1-|\tau|} \theta_{j,1}\theta_{j+|\tau|,1} \quad |\tau| \le 1,$$

so,

$$\begin{aligned} s_0 &= \sigma_\epsilon^2(\theta_{0,1}\theta_{0,1} + \theta_{1,1}\theta_{1,1}) \\ &= \sigma_\epsilon^2(1 + \theta_{1,1}^2); \end{aligned}$$

and,

$$\begin{aligned} s_1 &= \sigma_\epsilon^2\theta_{0,1}\theta_{1,1} \\ &= -\sigma_\epsilon^2\theta_{1,1}. \end{aligned}$$

Examples

$\boxed{\text{acs:}}$

$$\rho_\tau = \frac{s_\tau}{s_0}.$$

$$\rho_0 = 1.0;\ \ \rho_1 = \frac{-\theta_{1,1}}{1+\theta_{1,1}^2};\ \ \rho_2 = \rho_3 = \cdots = 0.$$

Specific Examples

(a) $\theta_{1,1} = 1.0, \sigma_\epsilon^2 = 1.0$,
we have,

$$s_0 = 2.0;\ s_1 = -1.0;\ s_2 = s_3 = \cdots = 0.0,$$

giving,

$$\rho_0 = 1.0;\ \rho_1 = -0.5;\ \rho_2 = \rho_3 = \cdots = 0.0.$$

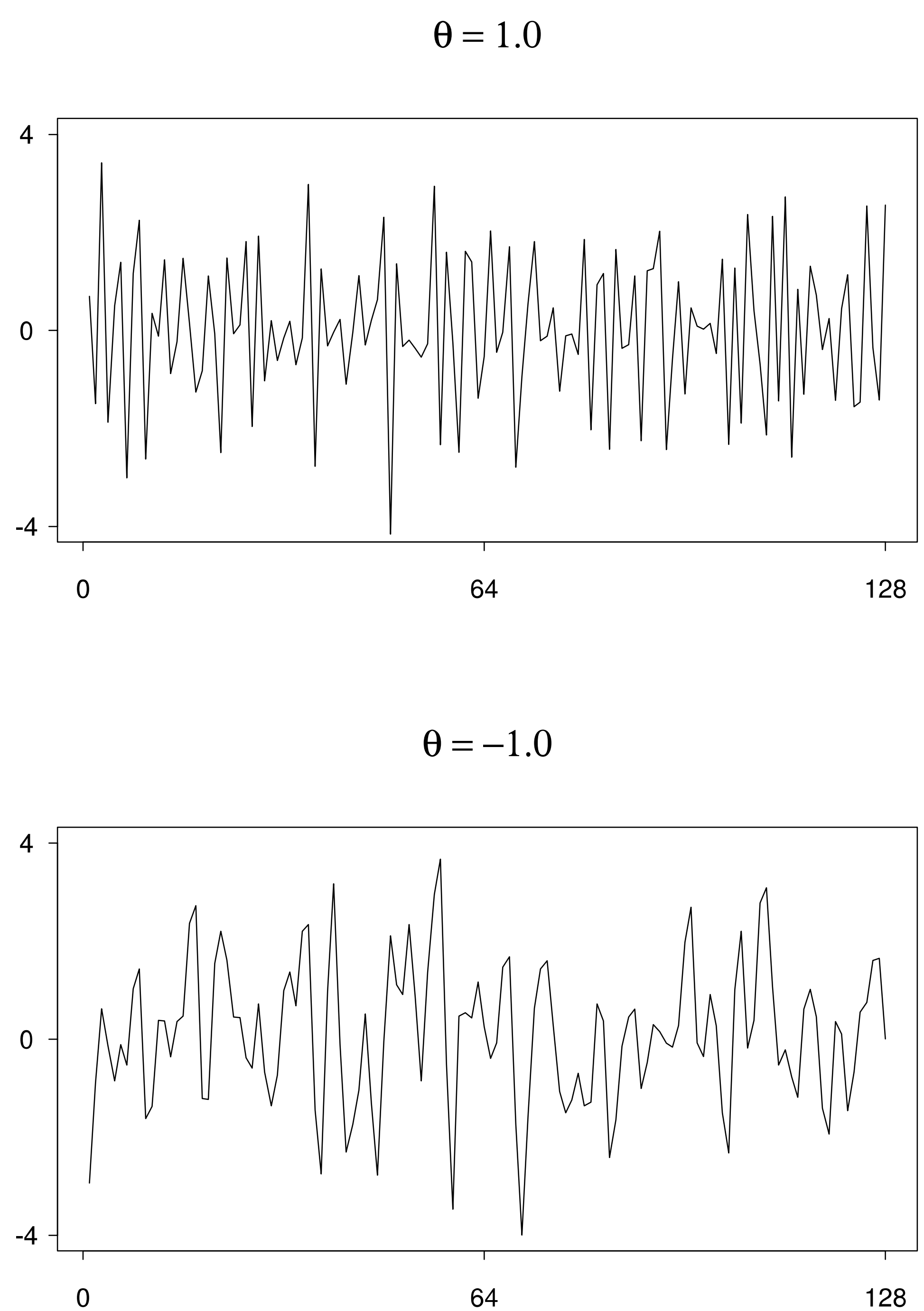
(b) $\theta_{1,1} = -1.0, \sigma_\epsilon^2 = 1.0$,
we have,

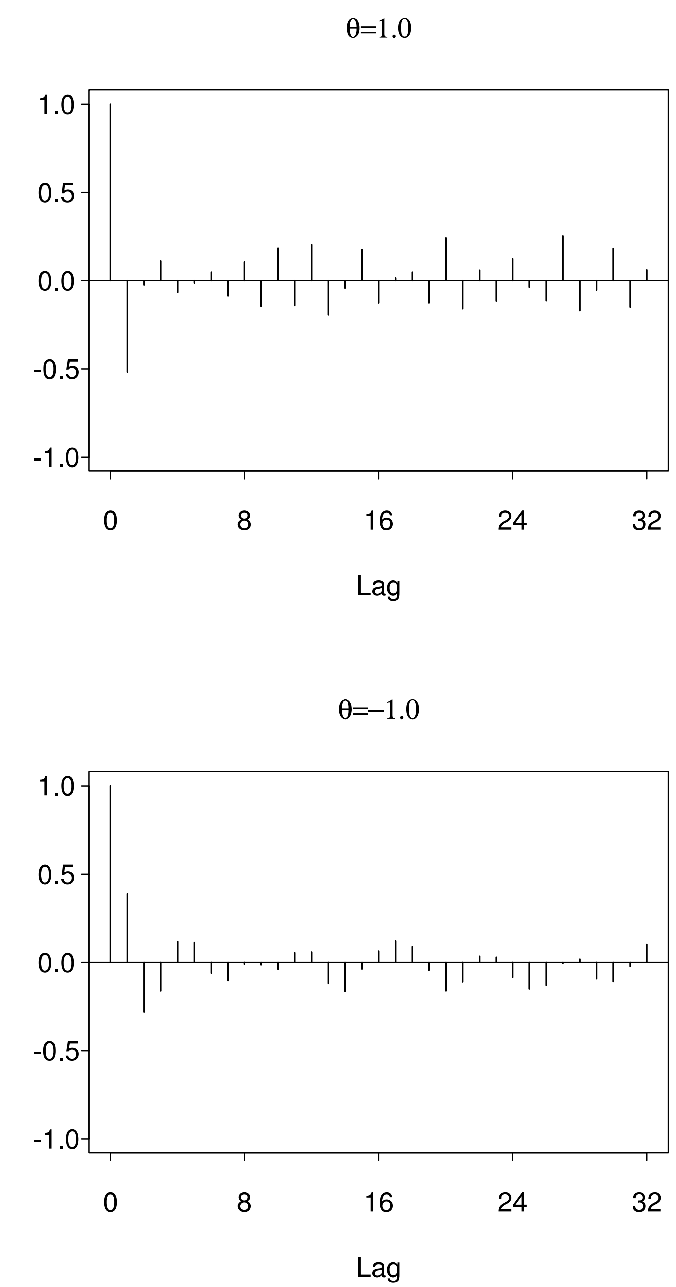
$$s_0 = 2.0;\ s_1 = 1.0;\ s_2 = s_3 = \cdots = 0.0,$$

giving,

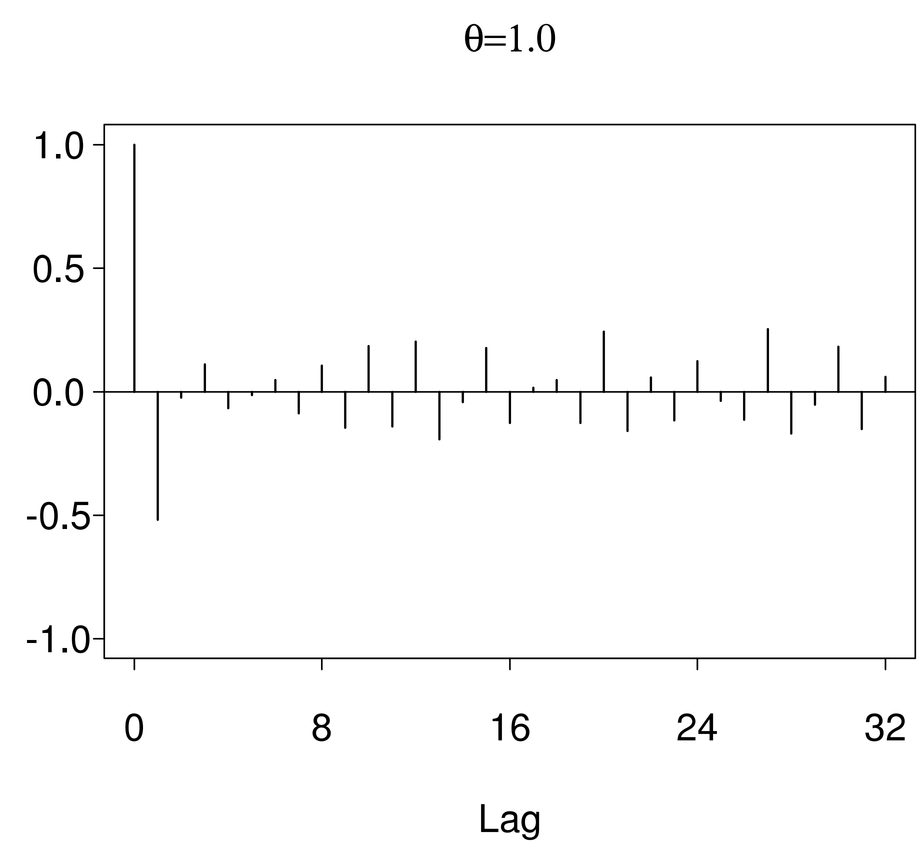
$$\rho_0 = 1.0;\ \rho_1 = 0.5;\ \rho_2 = \rho_3 = \cdots = 0.0.$$

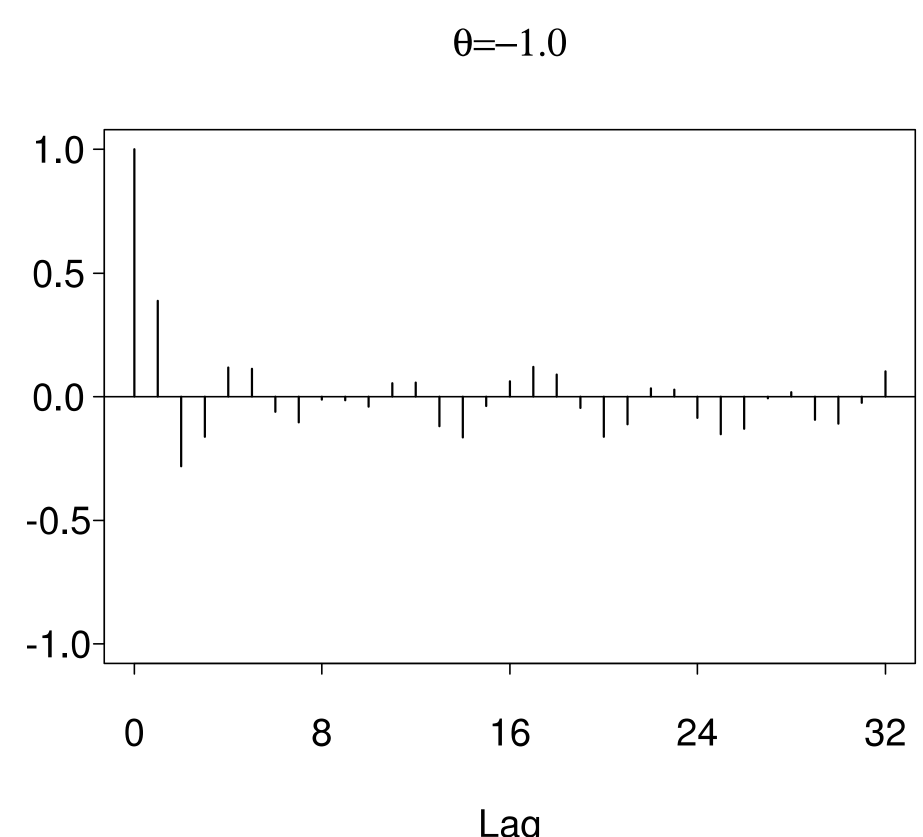
Realisations and SAMPLE acs





SAMPLE autocorrelation sequences





Note: if we replace $\theta_{1,1}$ by $\theta_{1,1}^{-1}$ the model becomes

$$X_t = \epsilon_t - \frac{1}{\theta_{1,1}}\epsilon_{t-1}$$

and the autocorrelation becomes

$$\rho_1 = \frac{-\frac{1}{\theta_{1,1}}}{1 + \left(\frac{1}{\theta_{1,1}}\right)^2} = \frac{-\theta_{1,1}}{\theta_{1,1}^2 + 1},$$

i.e., is <u>unchanged!!!</u>

We cannot identify the MA(1) process uniquely from its autocorrelation!

Examples of discrete stationary processes

[3] ***p*-th order autoregressive process** AR(p)

$\{X_t\}$ is expressed in the form

$$X_t = \phi_{1,p} X_{t-1} + \phi_{2,p} X_{t-2} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t,$$

where $\phi_{1,p}, \phi_{2,p}, \ldots, \phi_{p,p}$ are constants ($\phi_{p,p} \neq 0$) and $\{\epsilon_t\}$ is a zero mean white noise process with variance σ_ϵ^2.

In contrast to the parameters of an MA(q) process, the $\{\phi_{k,p}\}$ must satisfy certain conditions for $\{X_t\}$ to be a stationary process – i.e., not all AR(p) processes are stationary (more later).

Examples (Figures 12 and 13)

$$\begin{aligned} X_t &= \phi_{1,1}X_{t-1} + \epsilon_t \qquad \text{AR(1) – Markov process} \qquad (1) \\ &= \phi_{1,1}\{\phi_{1,1}X_{t-2} + \epsilon_{t-1}\} + \epsilon_t \\ &= \phi_{1,1}^2 X_{t-2} + \phi_{1,1}\epsilon_{t-1} + \epsilon_t \\ &= \phi_{1,1}^3 X_{t-3} + \phi_{1,1}^2\epsilon_{t-2} + \phi_{1,1}\epsilon_{t-1} + \epsilon_t \\ &\vdots \\ &= \sum_{k=0}^{\infty} \phi_{1,1}^k \epsilon_{t-k}. \end{aligned}$$

Here we take the initial condition $X_{-N} = 0$ and let $N \to \infty$.

AR(1)

Note $\mathrm{E}\{X_t\} = 0$.

$$\mathrm{var}\{X_t\} = \mathrm{var}\left\{\sum_{k=0}^{\infty} \phi_{1,1}^{k} \epsilon_{t-k}\right\} = \sum_{k=0}^{\infty} \mathrm{var}\{\phi_{1,1}^{k} \epsilon_{t-k}\} = \sigma_\epsilon^2 \sum_{k=0}^{\infty} \phi_{1,1}^{2k}.$$

For $\mathrm{var}\{X_t\} < \infty$ we must have $|\phi_{1,1}| < 1$, in which case

$$\mathrm{var}\{X_t\} = \frac{\sigma_\epsilon^2}{1 - \phi_{1,1}^2}.$$

AR(1)

To find the form of the acvs, we notice that for $\tau > 0$, $X_{t-\tau}$ is a linear function of $\epsilon_{t-\tau}, \epsilon_{t-\tau-1}, \ldots$ and is therefore uncorrelated with ϵ_t. Hence

$$\mathrm{E}\{\epsilon_t X_{t-\tau}\} = 0,$$

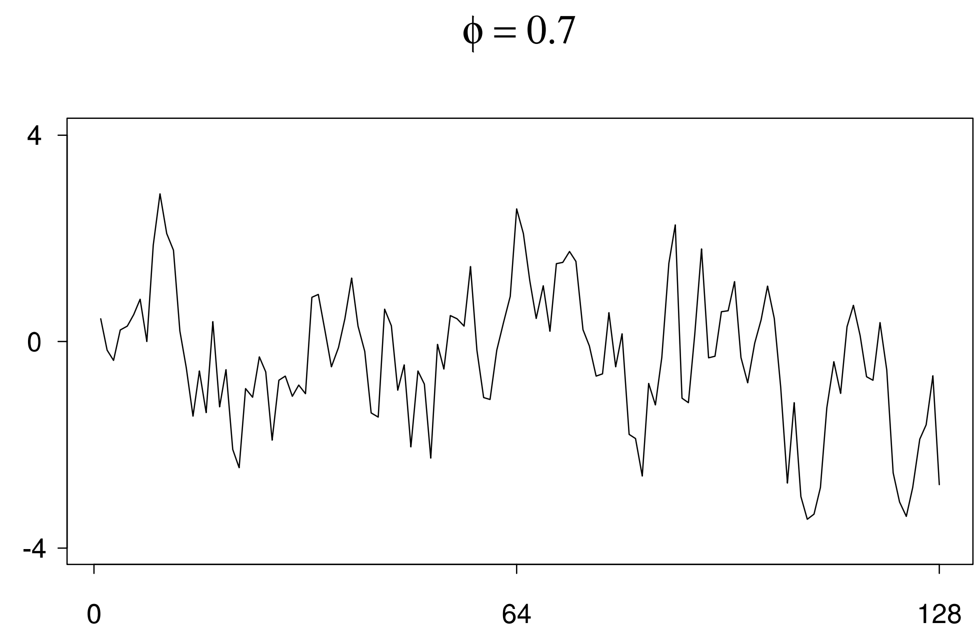
so, assuming stationarity and multiplying the defining equation (1) by $X_{t-\tau}$:

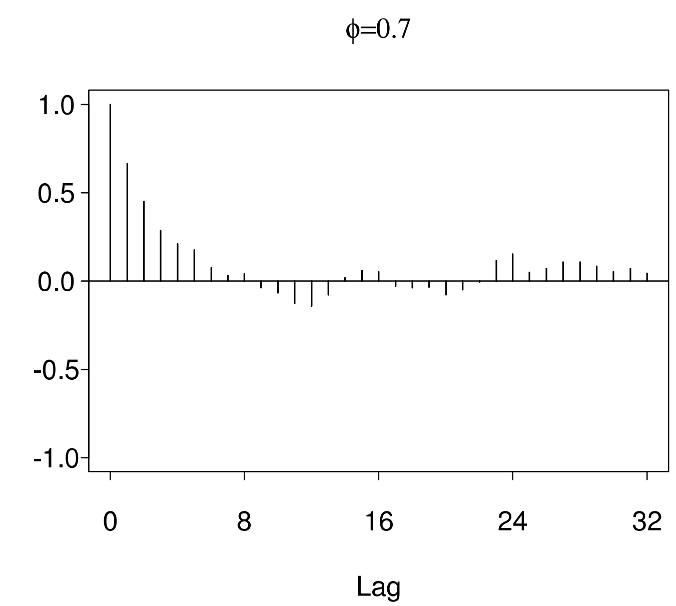
$$\begin{aligned}
X_t X_{t-\tau} &= \phi_{1,1} X_t X_{t-\tau} + \epsilon_t X_{t-\tau} \\
\Rightarrow \mathrm{E}\{X_t X_{t-\tau}\} &= \phi_{1,1} \mathrm{E}\{X_{t-1} X_{t-\tau}\} \\
\text{i.e., } s_\tau &= \phi_{1,1} s_{\tau-1} = \phi_{1,1}^2 s_{\tau-2} = \ldots = \phi_{1,1}^\tau s_0 \\
\Rightarrow \rho_\tau &= \frac{s_\tau}{s_0} = \phi_{1,1}^\tau .
\end{aligned}$$

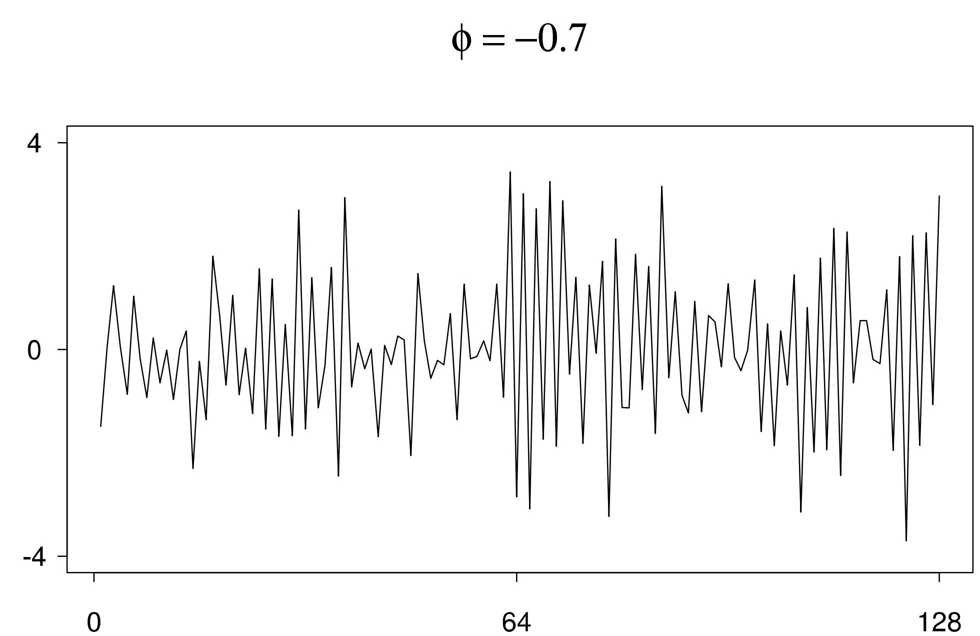
But ρ_τ is an even function of τ, so we obtain an exponentially decaying sequence

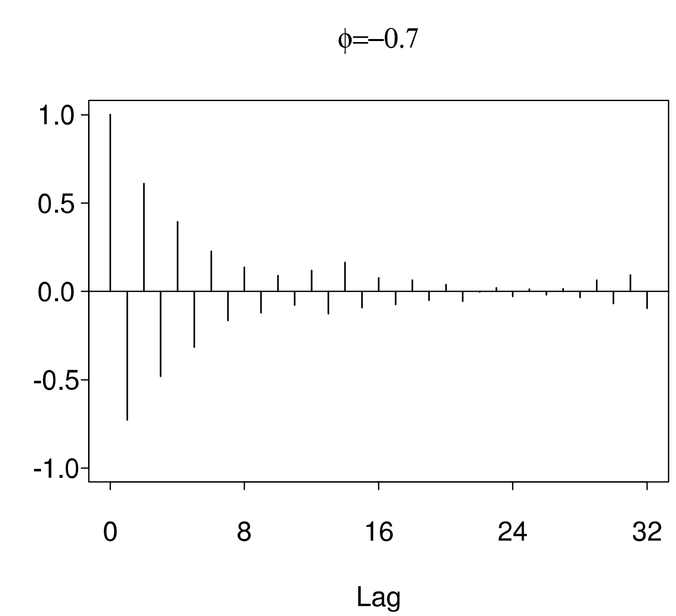
$$\rho_\tau = \phi_{1,1}^{|\tau|} \qquad \tau = 0, \pm 1, \pm 2, \ldots .$$

Realisations and SAMPLE acs









Examples of discrete stationary processes

[4]**(p, q)'th order autoregressive-moving average process**
ARMA(p, q)

Here $\{X_t\}$ is expressed as

$$X_t = \phi_{1,p} X_{t-1} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

where the $\phi_{j,p}$'s and the $\theta_{j,q}$'s are all constants ($\phi_{p,p} \neq 0; \theta_{q,q} \neq 0$) and again $\{\epsilon_t\}$ is a zero mean white noise process with variance σ_ϵ^2.

The ARMA class is important as many data sets may be approximated in a more parsimonious way (meaning fewer parameters are needed) by a mixed ARMA model than by a pure AR or MA process.

Worked Example

Examples of discrete stationary processes

[5]***p*'th order autoregressive conditionally heteroscedastic model** ARCH(p)

Standard linear models were found to be inappropriate for describing the dependence of financial log-return series of stock indices, share prices, exchange rates etc. New multiplicative noise models were developed. One such is the ARCH(p) model.

ARCH(p)

Assume we have a time series $\{X_t\}$ that has a variance (volatility) that changes through time,

$$X_t = \sigma_t \varepsilon_t \tag{2}$$

where here $\{\varepsilon_t\}$ is a sequence of independent and identically distributed (iid) r.v.s with zero mean and unit variance. (This is stronger than simply uncorrelated). Here, σ_t represents the local conditional standard deviation of the process.

NOTE: σ_t is not observable.

ARCH(p)

$\{X_t\}$ is ARCH(p) if it satisfies equation (2) and

$$\sigma_t^2 = \alpha + \beta_{1,p} X_{t-1}^2 + \ldots + \beta_{p,p} X_{t-p}^2, \tag{3}$$

where $\alpha > 0$ and $\beta_{j,p} \geq 0, j = 1, \ldots, p$ (to ensure σ_t^2 is positive).

EXAMPLE: ARCH(1)

$$\sigma_t^2 = \alpha + \beta_{1,1} X_{t-1}^2$$

Define,

$$v_t = X_t^2 - \sigma_t^2, \quad \Rightarrow \quad \sigma_t^2 = X_t^2 - v_t.$$

So $X_t^2 = \sigma_t^2 + v_t$ and the model can be written as

$$X_t^2 = \alpha + \beta_{1,1} X_{t-1}^2 + v_t,$$

i.e., as an AR(1) model for $\{X_t^2\}$. The errors, $\{v_t\}$, have zero mean, but as $v_t = \sigma_t^2(\epsilon_t^2 - 1)$ the errors are heteroscedastic.

Examples of discrete time stationary processes

[6]**Harmonic with random amplitude** (see Figures 14 and 14a)

Here $\{X_t\}$ is expressed as

$$X_t = \epsilon_t \cos(2\pi f_0 t + \phi)$$

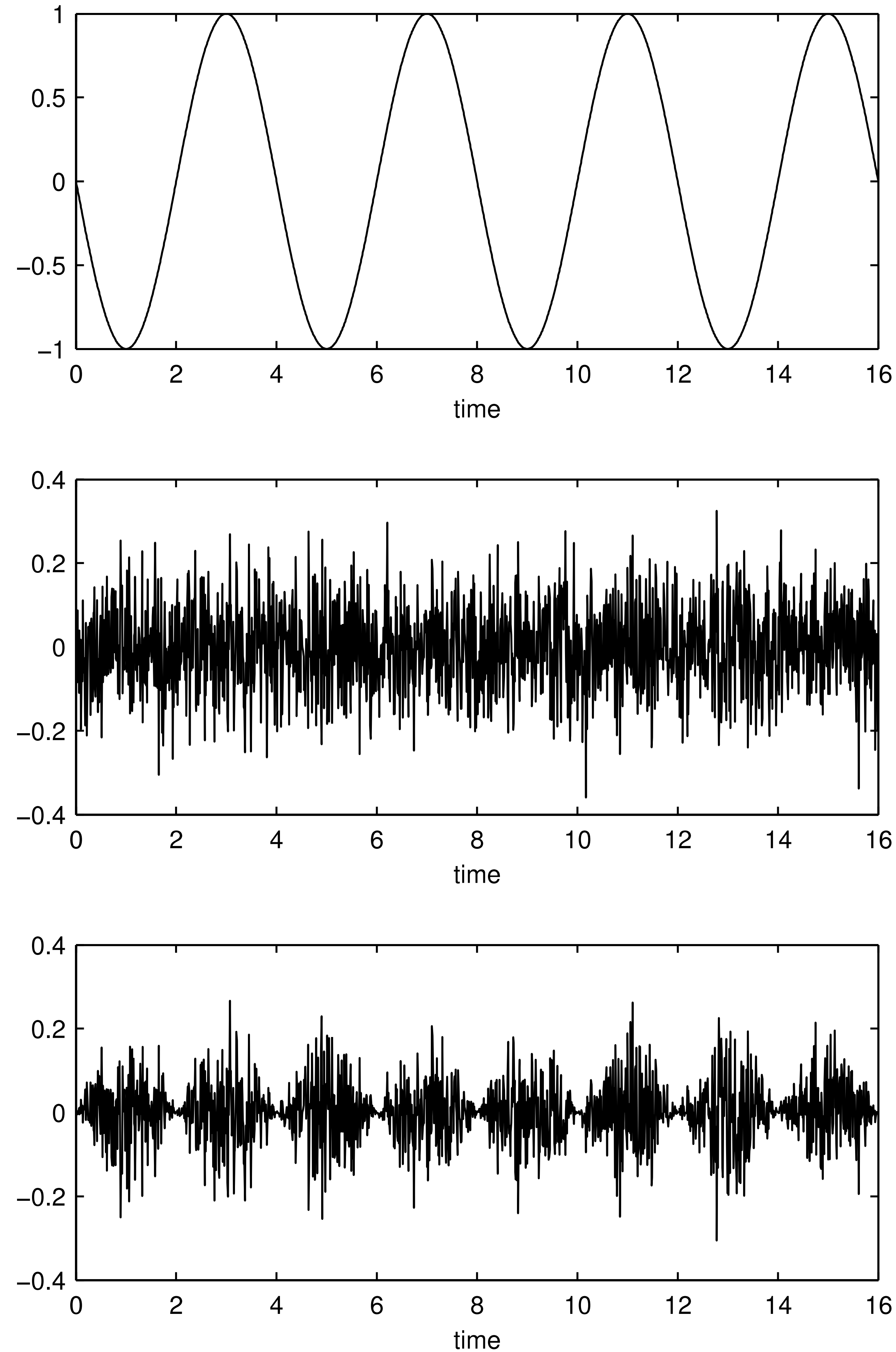
f_0 is a fixed frequency and $\{\epsilon_t\}$ is zero mean white noise with variance σ_ϵ^2.

Harmonic with random amplitude: case (a)

ϕ is constant.

$$\begin{aligned}
\mathrm{E}\{X_t\} &= \mathrm{E}\{\epsilon_t \cos(2\pi f_0 t + \phi)\} \\
&= \mathrm{E}\{\epsilon_t\} \cos(2\pi f_0 t + \phi) = 0. \\
\mathrm{var}\{X_t\} &= \mathrm{E}\{X_t^2\} \\
&= \mathrm{E}\{\epsilon_t^2\} \cos^2(2\pi f_0 t + \phi) \\
&= \sigma_\epsilon^2 \cos^2(2\pi f_0 t + \phi).
\end{aligned}$$

So the variance depends on t and the process is nonstationary.



time
time
time

Harmonic with random amplitude: case (b)

$\phi \sim U[-\pi, \pi]$ and indep. of $\{\epsilon_t\}$.

$$\mathrm{E}\{X_t\} = \mathrm{E}\{\epsilon_t \cos(2\pi f_0 t + \phi)\} = \mathrm{E}\{\epsilon_t\}\mathrm{E}\{\cos(2\pi f_0 t + \phi)\} = 0.$$

$$\begin{aligned} \mathrm{cov}\{X_t, X_{t+\tau}\} &= \mathrm{E}\{X_t X_{t+\tau}\} \\ &= \mathrm{E}\{\epsilon_t \epsilon_{t+\tau}\}\mathrm{E}\left\{\cos(2\pi f_0 t + \phi)\cos(2\pi f_0 (t+\tau) + \phi)\right\} \end{aligned}$$

Harmonic with random amplitude: case (b)

Since $\{\epsilon_t\}$ is white noise we have,

$$E\{\epsilon_t \epsilon_{t+\tau}\} = \begin{cases} \sigma_\epsilon^2 & \text{if } \tau = 0, \\ 0 & \text{if } \tau \neq 0, \end{cases}$$

So, for $\tau \neq 0$, $\text{cov}\{X_t, X_{t+\tau}\} = 0$, while for $\tau = 0$,

$$\text{cov}\{X_t, X_t\} = s_0 = \sigma_\epsilon^2 E\{\cos^2(2\pi f_0 t + \phi)\}.$$

Harmonic with random amplitude: case (b)

Now,

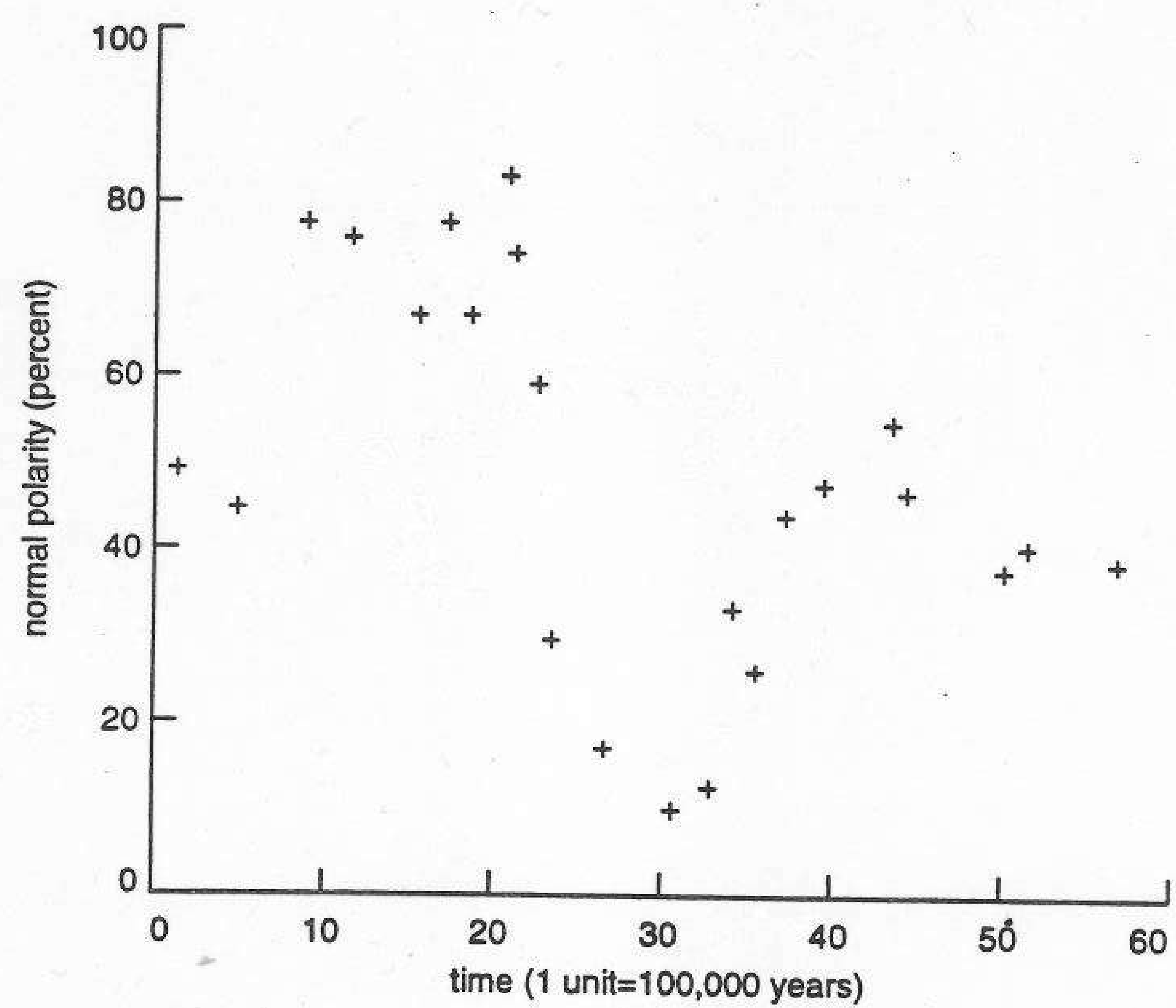
$$\begin{aligned} \mathrm{E}\{\cos^2(2\pi f_0 t + \phi)\} &= \int_{-\pi}^{\pi} \cos^2(2\pi f_0 t + \phi) \frac{1}{2\pi}\, \mathrm{d}\phi \\ &= \frac{1}{2} \int_{-\pi}^{\pi} [1 + \cos(4\pi f_0 t + 2\phi)]\, \frac{1}{2\pi}\, \mathrm{d}\phi \\ &= \frac{1}{2}. \end{aligned}$$

So,

$$s_0 = \sigma_\epsilon^2 / 2,$$

and the process is stationary.

The random phase idea is illustrated in Figure 14a: the random point at which data collection started corresponds to breaking-in to the ‘sinusoidal-like’ behaviour at a random point, which equates to a random phase.



normal polarity (percent)
100
80
60
40
20
0
0
10
20
30
40
50
60
time (1 unit=100,000 years)

Trend removal and seasonal adjustment

There are certain, quite common, situations where the observations exhibit a <u>trend</u> – a tendency to increase or decrease slowly steadily over time – or may fluctuate in a <u>periodic/seasonal</u> manner. The model is modified to

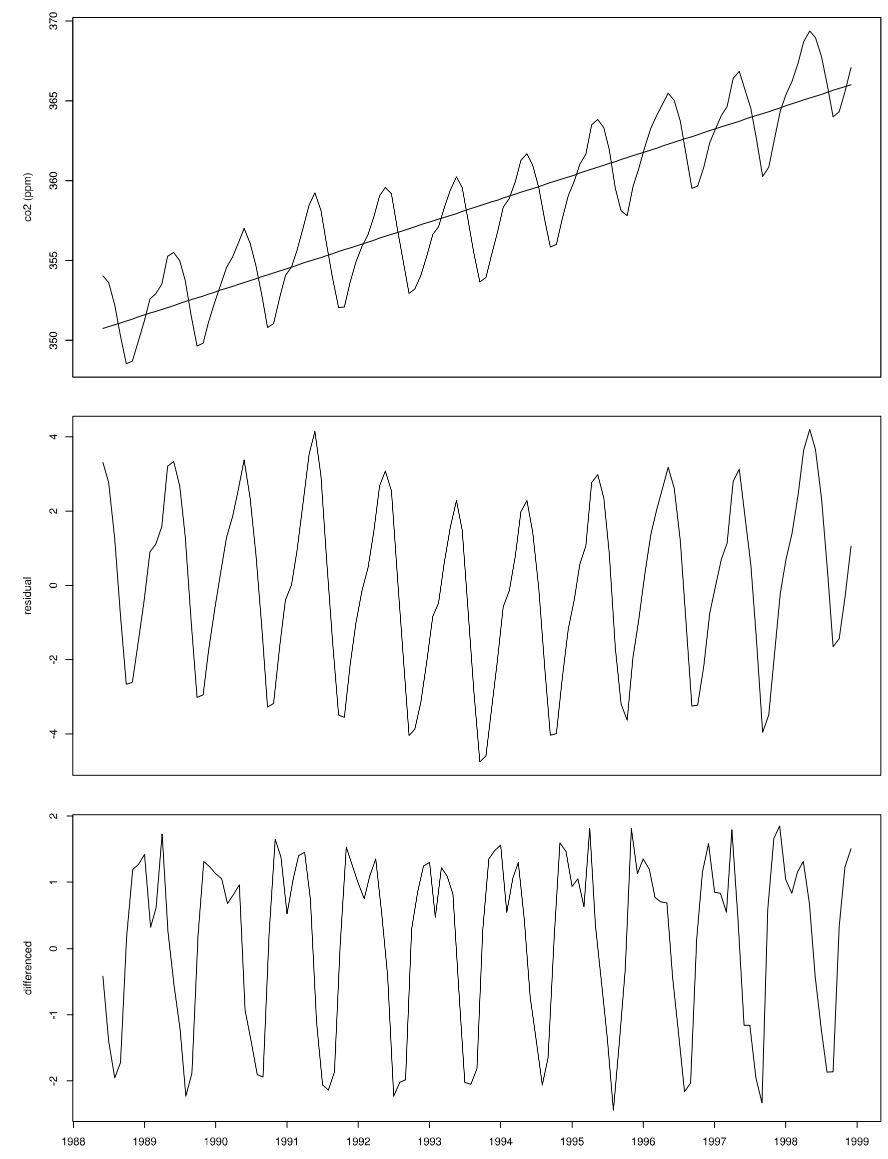
$$X_t = \mu_t + Y_t$$

μ_t = time dependent mean.
Y_t = zero mean stationary process.

Example: CO_2 data

X_t = monthly atmospheric CO_2 concentrations expressed in parts per million (ppm) derived from in situ air samples collected at Mauna Loa observatory, Hawaii. Monthly data from May 1988 – December 1998, giving $N = 128$.

The data is plotted in Figure 15. We can see both a trend and periodic/seasonal effects.



Trend Adjustment

Represent a simple linear trend by $\alpha + \beta t$. So take $X_t = \alpha + \beta t + Y_t$. At least two possible approaches:

(a) Estimate α and β by least squares, and work with the residuals

$$\hat{Y}_t = X_t - \hat{\alpha} - \hat{\beta} t.$$

For the CO_2 data these are shown in the middle plot of figure 15.

Trend Adjustment

(b) Take first differences:

$$\begin{aligned} X_t^{(1)} = X_t - X_{t-1} &= \alpha + \beta t + Y_t - (\alpha + \beta(t-1) + Y_{t-1}) \\ &= \beta + Y_t - Y_{t-1}. \end{aligned}$$

For the CO_2 data these are shown in the bottom plot of figure 15.

Note

if $\{Y_t\}$ is stationary so is $\{Y_t^{(1)}\}$

In the case of linear trend, if we difference again:

$$\begin{aligned} X_t^{(2)} &= X_t^{(1)} - X_{t-1}^{(1)} = (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\ &= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\ &= Y_t - 2Y_{t-1} + Y_{t-2}, \quad (\equiv Y_t^{(1)} - Y_{t-1}^{(1)} = Y_t^{(2)}), \end{aligned}$$

so that the effect of $\mu_t (= \alpha + \beta t)$ has been completely removed.

If μ_t is a polynomial of degree $(d-1)$ in t, then dth differences of μ_t will be zero ($d = 2$ for linear trend). Further,

$$\begin{aligned} X_t^{(d)} &= \sum_{k=0}^{d} \binom{d}{k} (-1)^k X_{t-k} \\ &= \sum_{k=0}^{d} \binom{d}{k} (-1)^k Y_{t-k}. \end{aligned}$$

Difference and Backward Shift Operators

There are other ways of writing this. Define the difference operator

$$\Delta = (1 - B)$$

where $BX_t = X_{t-1}$ is the *backward shift operator* (sometimes known as the *lag operator* L – especially in econometrics). Then,

$$X_t^{(d)} = \Delta^d X_t = \Delta^d Y_t.$$

For example, for $d = 2$:

$$\begin{aligned}
X_t^{(2)} &= (1 - B)^2 X_t = (1 - B)(X_t - X_{t-1}) \\
&= (X_t - X_{t-1}) - (X_{t-1} - X_{t-2}) \\
&= (\beta + Y_t - Y_{t-1}) - (\beta + Y_{t-1} - Y_{t-2}) \\
&= (Y_t - Y_{t-1}) - (Y_{t-1} - Y_{t-2}) \\
&= (1 - B)^2 Y_t = \Delta^2 Y_t.
\end{aligned}$$

This notation can be incorporated into the ARMA set up. Recall if $\{X_t\}$ is ARMA(p, q),

$$X_t = \phi_{1,p} X_{t-1} + \ldots + \phi_{p,p} X_{t-p} + \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q},$$

$$\begin{aligned} X_t - \phi_{1,p} X_{t-1} - \ldots - \phi_{p,p} X_{t-p} &= \epsilon_t - \theta_{1,q}\epsilon_{t-1} - \ldots - \theta_{q,q}\epsilon_{t-q} \\ (1 - \phi_{1,p} B - \phi_{2,p} B^2 - \ldots - \phi_{p,p} B^p) X_t &= (1 - \theta_{1,q} B - \theta_{2,q} B^2 - \ldots - \theta_{q,q} B^q)\epsilon_t \\ \Phi(B) X_t &= \Theta(B)\epsilon_t. \end{aligned}$$

Here

$$\begin{aligned} \Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - \ldots - \phi_{p,p}B^p \\ \text{and} \quad \Theta(B) &= 1 - \theta_{1,q}B - \theta_{2,q}B^2 - \ldots - \theta_{q,q}B^q \end{aligned}$$

are known as the *associated* or *characteristic polynomials*. Further, we can generalize the class of ARMA models to include differencing to account for certain types of non-stationarity, namely, X_t is called ARIMA(p, d, q) if

$$\begin{aligned} \Phi(B)(1 - B)^d X_t &= \Theta(B)\epsilon_t, \\ \text{or} \quad \Phi(B)\Delta^d X_t &= \Theta(B)\epsilon_t. \end{aligned}$$

Seasonal adjustment

The model is

$$X_t = \nu_t + Y_t$$

where

$$\begin{aligned} \nu_t &= \text{ seasonal component,} \\ Y_t &= \text{ zero mean stationary process.} \end{aligned}$$

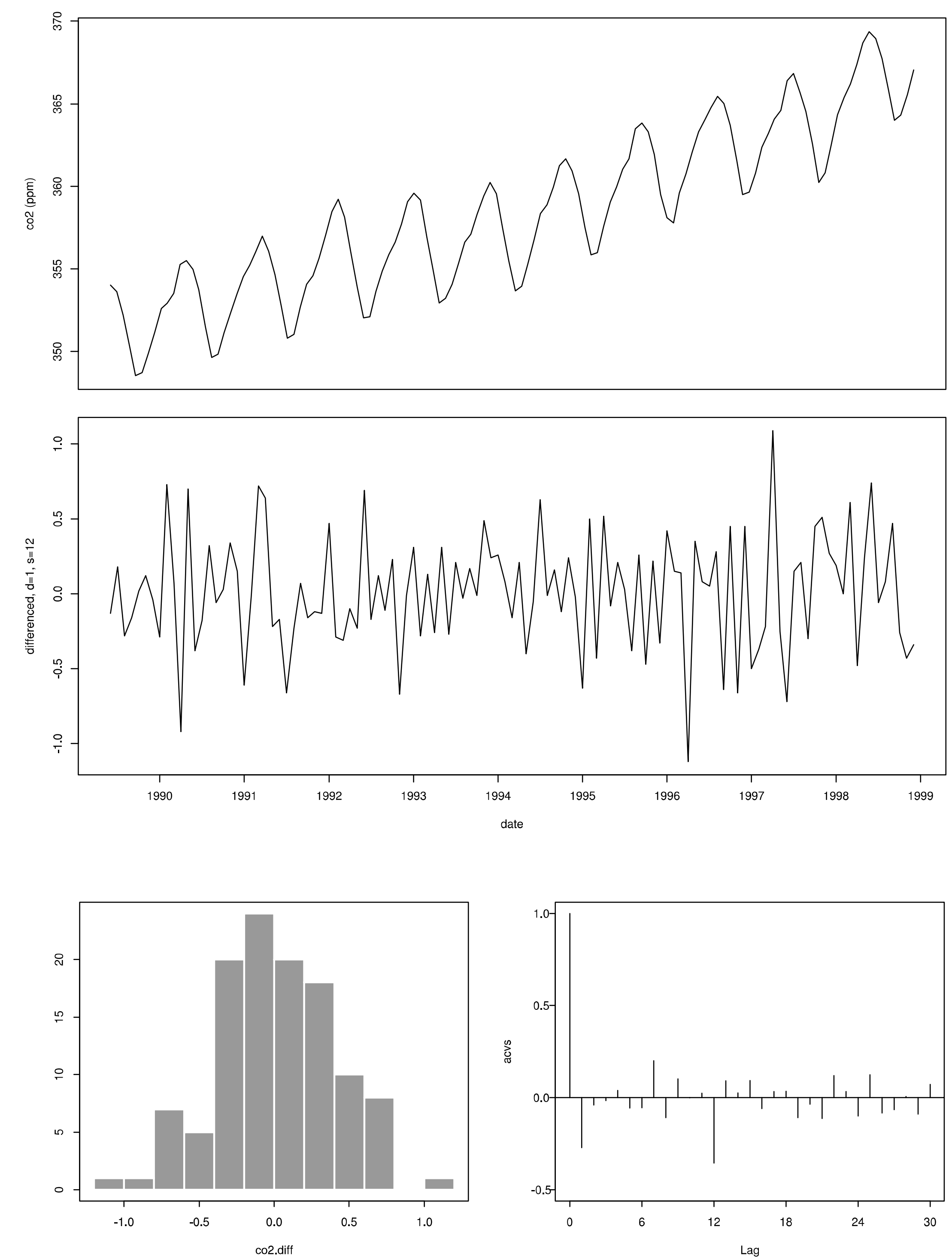
Presuming that the seasonal component maintains a constant pattern over time with period s, there are again several approaches to removing ν_t. A popular approach used by Box & Jenkins is to use the operator $(1 - B^s)$:

$$\begin{aligned} X_t^{(s)} &= (1 - B^s)X_t = X_t - X_{t-s} \\ &= (\nu_t + Y_t) - (\nu_{t-s} + Y_{t-s}) \\ &= Y_t - Y_{t-s} \end{aligned}$$

since ν_t has period s (and so $\nu_{t-s} = \nu_t$).

Figure 16 shows this technique applied to the CO_2 data – most of the seasonal structure and trend has been removed by applying the seasonal operator after the differencing operator:

$$(1 - B^{12})(1 - B)X_t.$$



General Linear Process

Consider a process of the form

$$X_t = \sum_{k=-\infty}^{\infty} g_k \epsilon_{t-k},$$

where $\{\epsilon_t\}$ is a purely random process, and $\{g_k\}$ is a given sequence of real-valued constants satisfying $\sum_{k=\infty}^{-\infty} g_k^2 < \infty$, which ensures that $\{X_t\}$ has finite variance.
Now we know $|\rho_\tau| \leq 1$, so

$$|s_\tau| = |\text{cov}\{X_t, X_{t+\tau}\}| \leq \sigma_X^2 = \sigma_\epsilon^2 \sum_k g_k^2 < \infty,$$

so the covariance is bounded also.

General Linear Process

If

$$g_{-1}, g_{-2}, \ldots = 0,$$

then we obtain what is called the General Linear Process

$$X_t = \sum_{k=0}^{\infty} g_k \epsilon_{t-k},$$

where X_t depends only on present and past values $\epsilon_t, \epsilon_{t-2}, \epsilon_{t-2}, \ldots$ of the purely random process.

z-polynomial

Introduce the "z-polynomial"

$$G(z) = \sum_{k=0}^{\infty} g_k z^k,$$

where $z \in \mathbb{C}$. Note $X_t = G(B)\epsilon_t$.

We will be dealing with z-polynomials of the form

$$G(z) = \frac{G_1(z)}{G_2(z)}, \quad \text{say.}$$

z-polynomial

Call the roots of $G_2(z)$ (the "poles" of $G(z)$) in the complex plane $z_1, z_2, \ldots, z_p$, where the zeros are ordered so that $z_1, \ldots, z_k$ are inside and $z_{k+1}, \ldots, z_p$ are outside the unit circle $|z| = 1$. Then,

$$
\begin{aligned}
\frac{1}{G_2(z)} = \sum_{j=1}^{p} \frac{A_j}{z - z_j} &= \sum_{j=1}^{k} \frac{A_j}{z} \times \frac{1}{\left(1 - \frac{z_j}{z}\right)} + \sum_{j=k+1}^{p} \frac{A_j}{z_j} \times \frac{-1}{\left(1 - \frac{z}{z_j}\right)} \\
&= \sum_{j=1}^{k} \frac{A_j}{z} \sum_{l=0}^{\infty} \left(\frac{z_j}{z}\right)^l - \sum_{j=k+1}^{p} \frac{A_j}{z_j} \sum_{l=0}^{\infty} \left(\frac{z}{z_j}\right)^l
\end{aligned}
$$

z-polynomial

Replace z by the backshift operator B and apply to $\{\epsilon_t\}$:

$$\begin{aligned}\left\{\frac{1}{G_2(B)}\right\}\epsilon_t &= \left\{\sum_{j=1}^{k} A_j B^{-1}\sum_{l=0}^{\infty} z_j^l B^{-l} - \sum_{j=k+1}^{p} A_j z_j^{-1}\sum_{l=0}^{\infty} z_j^{-l} B^l\right\}\epsilon_t \\ &= \sum_{j=1}^{k} A_j \sum_{l=0}^{\infty} z_j^l \epsilon_{t+l+1} - \sum_{j=k+1}^{p} A_j \sum_{l=0}^{\infty} \underbrace{z_j^{-l-1}}_{\text{outside}} \epsilon_{t-l}.\end{aligned}$$

Hence, if all the roots of $G_2(z)$ are <u>outside</u> the unit circle (i.e. all the poles of $G(z)$ are <u>outside</u> the unit circle) only past and present values of $\{\epsilon_t\}$ are involved and the General Linear Process exists.

z-polynomial

Another way of stating this is that

$$G(z) < \infty \qquad |z| \leq 1$$

i.e., $G(z)$ is analytic inside and on the unit circle.
So, all the

$$\begin{cases} \text{poles of } G(z) \text{ lie outside the unit circle} \\ \text{roots (zeros) of } G^{-1}(z) \text{ lie outside the unit circle} \end{cases}$$

Consider the MA(q) model

$$X_t = \Theta(B)\epsilon_t,$$

then,

$$\Theta^{-1}(B)X_t = \epsilon_t$$

and in general, the expansion of $\Theta^{-1}(B)$ is a polynomial of infinite order. Similarly, consider the AR(p) model

$$\Phi(B)X_t = \epsilon_t,$$

then,

$$X_t = \Phi^{-1}(B)\epsilon_t$$

Hence,

$$\begin{aligned} \text{MA (finite order)} &\equiv \text{AR (infinite order)} \\ \text{AR (finite order)} &\equiv \text{MA (infinite order)} \end{aligned}$$

provided the infinite order expansions exist!

Invertibility

Consider inverting the general linear process into autoregressive form

$$
\begin{aligned}
X_t &= \sum_{k=0}^{\infty} g_k \epsilon_{t-k} \\
&= \sum_{k=0}^{\infty} g_k B^k \epsilon_t \\
X_t &= G(B)\epsilon_t \\
\Rightarrow \quad G^{-1}(B) X_t &= \epsilon_t
\end{aligned}
$$

Invertibility

The expansion of $G^{-1}(B)$ in powers of B gives the required autoregressive form. provided $G^{-1}(B)$ admits a power series expansion

$$G^{-1}(z) = \sum_{k=0}^{\infty} h_k z^k$$

i.e. if $G^{-1}(z)$ is analytic, $|z| \leq 1$. Thus the model is invertible if

$$G^{-1}(z) < \infty, \quad |z| \leq 1.$$

$\Rightarrow$ All the poles of $G^{-1}(z)$ are outside the unit circle.

For the MA(q) process, $G(z) = \Theta(z)$, and so the invertibility condition is that $\Theta(z)$ has no <u>roots</u> inside or on the unit circle; i.e. all the <u>roots</u> of $\Theta(z)$ lie outside the unit circle.

Example

Consider the following process

$$X_t = \epsilon_t - 1.3\epsilon_{t-1} + 0.4\epsilon_{t-2}$$

Writing this in B notation:

$$\begin{aligned} X_t &= (1 - 1.3B + 0.4B^2)\epsilon_t \\ &= \Theta(B)\epsilon_t \end{aligned}$$

to check if invertible, find roots of $\Theta(z) = 1 - 1.3z + 0.4z^2$,

$$\begin{aligned} 1 - 1.3z + 0.4z^2 &= 0 \\ 4z^2 - 13z + 10 &= 0 \\ (4z-5)(z-2) &= 0 \end{aligned}$$

roots of $\Theta(z)$ are $z = 2$ and $z = 5/4$, which are both <u>outside</u> the unit circle $\Rightarrow$ invertible.

Stationarity

For the AR(p) process

$$\begin{aligned}\Phi(B)X_t &= \epsilon_t \\ \Rightarrow X_t &= \Phi^{-1}(B)\epsilon_t = G(B)\epsilon_t,\end{aligned}$$

so that $G(z) = \Phi^{-1}(z)$. Thus the model is stationary if

$$G(z) < \infty, \quad |z| \leq 1.$$

$\Rightarrow$ All the poles of $G(z)$ are outside the unit circle.

Hence the requirement for stationarity is that all the <u>roots</u> of $G^{-1}(z) = \Phi(z)$ must lie outside the unit circle.

For the MA(q) process

$$X_t = \Theta(B)\epsilon_t = G(B)\epsilon_t$$

and since $G(B) = \Theta(B)$ is a polynomial of finite order $G(z) < \infty$, $|z| \leq 1$, automatically.

SUMMARY

	AR(p)	MA(q)	ARMA(p, q)
Stationarity	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$	Always stationary	Roots of $\Phi(z)$ outside $\lvert z\rvert \leq 1$
Invertibility	Always invertible	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$	Roots of $\Theta(z)$ outside $\lvert z\rvert \leq 1$

Example

Determine whether the following model is stationary and/or invertible,

$$X_t = 1.3X_{t-1} - 0.4X_{t-2} + \epsilon_t - 1.5\epsilon_{t-1}.$$

Writing in B notation:

$$(1 - 1.3B + 0.4B^2)X_t = (1 - 1.5B)\epsilon_t$$

we have

$$\Phi(z) = 1 - 1.3z + 0.4z^2$$

with roots $z = 2$ and $5/4$ (from previous example), so the roots of $\Phi(z) = 0$ both lie outside the unit circle, therefore model is stationary, and

$$\Theta(z) = 1 - 1.5z,$$

so the root of $\Theta(z) = 0$ is given by $z = 2/3$ which lies inside the unit circle and the model is not invertible.

Directionality and Reversibility

Consider again the general linear model

$$
\begin{aligned}
X_t &= \sum_{k=0}^{\infty} g_k \epsilon_{t-k} \\
&= \sum_{k=0}^{\infty} g_k B^k \epsilon_t \\
&= G(B)\epsilon_t
\end{aligned}
$$

The reversed form is clearly,

$$
\begin{aligned}
X_t &= \sum_{k=0}^{\infty} g_k \epsilon_{t+k} \\
&= \sum_{k=0}^{\infty} g_k B^{-k} \epsilon_t \\
&= G\left(\frac{1}{B}\right) \epsilon_t,
\end{aligned}
$$

with some stationarity condition.

Now consider the ARMA(p, q) model given by

$$\Phi(B)X_t = \Theta(B)\epsilon_t,$$

where,

$$\begin{aligned}\Phi(B) &= 1 - \phi_{1,p}B - \phi_{2,p}B^2 - \ldots - \phi_{p,p}B^p \\ \Theta(B) &= 1 - \theta_{1,q}B - \theta_{2,q}B^2 - \ldots - \theta_{q,q}B^q.\end{aligned}$$

The reversed form of the ARMA(p, q) model is,

$$\begin{aligned}
\Phi\left(\frac{1}{B}\right) X_t &= \Theta\left(\frac{1}{B}\right) \epsilon_t, \\
\left(1 - \phi_{1,p}\frac{1}{B} - \phi_{2,p}\frac{1}{B^2} - \ldots - \phi_{p,p}\frac{1}{B^p}\right) X_t &= \left(1 - \theta_{1,q}\frac{1}{B} - \theta_{2,q}\frac{1}{B^2} - \ldots - \theta_{q,q}\frac{1}{B^q}\right) \epsilon_t \\
\frac{1}{B^p}(B^p - \phi_{1,p}B^{p-1} - \ldots \phi_{p,p})X_t &= \frac{1}{B^q}(B^q - \theta_{1,q}B^{q-1} - \ldots - \theta_{q,q})\epsilon_t \\
\Phi^R(B)X_t &= B^{p-q}\Theta^R(B)\epsilon_t
\end{aligned}$$

where,

$$\begin{aligned}
\Phi^R(B) &= B^p - \phi_{1,p}B^{p-1} - \phi_{2,p}B^{p-2} - \ldots - \phi_{p,p} \\
\Theta^R(B) &= B^q - \theta_{1,q}B^{q-1} - \theta_{2,q}B^{q-2} - \ldots - \theta_{q,q}.
\end{aligned}$$

For example, for the ARMA(1,1) model,

$$(1 - \phi_{1,1} B) X_t = (1 - \theta_{1,1} B) \epsilon_t,$$

reversed form is

$$(B - \phi_{1,1}) X_t = (B - \theta_{1,1}) \epsilon_t.$$

Now $\Phi(z) = 1 - \phi_{1,1} z$, and a root is the solution of $1 - \phi_{1,1} z = 0$, i.e.,

$$|z| = \left| \frac{1}{\phi_{1,1}} \right| > 1 \Rightarrow |\phi_{1,1}| < 1.$$

But, $\Phi^R(z) = z - \phi_{1,1}$, and so a root is the solution of $z - \phi_{1,1} = 0$, i.e., $z = \phi_{1,1}$. But, since for stationarity $|\phi_{1,1}| < 1$ we have

$$|z| = |\phi_{1,1}| < 1,$$

so the root of $\Phi^R(z)$ is <u>inside</u> the unit circle.

Hence the standard assumption for stationarity (roots outside the unit circle) has within it an assumption of <u>directionality</u>. [N.B. only if the roots of $\Phi(z)$ are on the unit circle is model ALWAYS non-stationary].

Figure 17 shows two time series which have different characteristics when time reversed.