CSE253 Homework 1

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Abstract

The abstract paragraph should be indented 1/2 inch (3 picas) on both left and right-hand margins. Use 10 point type, with a vertical spacing of 11 points. The word **Abstract** must be centered, bold, and in point size 12. Two line spaces precede the abstract. The abstract must be limited to one paragraph.

1 Problems from Bishop

1.1

$$\begin{aligned} \prod_{i=1}^{d}\int_{-\infty}^{\infty} e^{-x_i^2}dx_i &= S_d\int_0^{\infty} e^{-r^2}r^{d-1}dr \\ S_d &= \frac{\prod_{i=1}^{d}\int_{-\infty}^{\infty} e^{-x_i^2}dx_i}{\int_0^{\infty} e^{-r^2}r^{d-1}dr} \end{aligned} \tag{1}$$

According to (1.41), we have

$$\int_{-\infty}^{\infty} exp\{-\frac{2}{\lambda}x^2\}dx = (\frac{2\pi}{\lambda})^{1/2} \tag{2}$$

In (2), we set $\lambda = 2$, we can get

$$\begin{aligned} \int_{-\infty}^{\infty} e^{-x^2}dx &= \pi^{1/2} \\ \prod_{i=1}^{d}\int_{-\infty}^{\infty} e^{-x_i^2}dx_i &= \pi^{d/2} \end{aligned} \tag{3}$$

According to (1.44), we have

$$\Gamma(x) \equiv \int_0^{\infty} u^{x-1}e^{-u}du \tag{4}$$

In (4), we set $u = r^2$, we can get

$$\Gamma(x) \equiv \int_0^{\infty} r^{2(x-1)}e^{-r^2}dr^2 \equiv 2\int_0^{\infty} r^{2x-1}e^{-r^2}dr \tag{5}$$

In (5), set $d = 2x$

$$\Gamma(\frac{d}{2}) \equiv 2\int_0^{\infty} r^{d-1}e^{-r^2}dr \tag{6}$$

Put (3) and (6) into (1),we can get

$$\begin{aligned} S_d &= \frac{\prod_{i=1}^{d}\int_{-\infty}^{\infty} e^{-x_i^2}dx_i}{\int_0^{\infty} e^{-r^2}r^{d-1}dr} \\ &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \end{aligned} \tag{7}$$

When $d = 2$,

$$\begin{aligned} S_2 &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \\ &= \frac{2\pi}{\Gamma(1)} \\ &= 2\pi \end{aligned} \tag{8}$$

The area of circle is πr^2

When $d = 3$,

$$\begin{aligned} S_3 &= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \\ &= \frac{2\pi^{3/2}}{\Gamma(\frac{3}{2})} \\ &= 4\pi \end{aligned} \tag{9}$$

The area of sphere is $\frac{4}{3}\pi r^3$

1.2

For a hypersphere with radius a and dimension d, the surface $S = S_d a^{d-1}$. Assuming a small cube in unit sphere with length of side L. Then the area is L^{d-1}. If the radius is a, then the length of new cube is aL, the new corresponding area is aL^{d-1} .

So

$$\begin{aligned} V_d &= \int_0^a S_d r^{d-1} dr \\ &= \frac{S_d}{d} \int_0^a r^d dr \\ &= \frac{s_d a^d}{d} \end{aligned} \tag{10}$$

$$\begin{aligned} \frac{volume\ of\ Sphere}{volume\ of\ Cube} &= \frac{2\pi^{\frac{d}{2}} a^d}{\Gamma(d/2) d (2a)^d} \\ &= \frac{\pi^{\frac{d}{2}}}{d2^{d-1}\Gamma(d/2)} \end{aligned} \tag{11}$$

When $d \to \infty$

$$\begin{aligned} \frac{volume\ of\ Sphere}{volume\ of\ Cube} &= \frac{\pi^{\frac{d}{2}}}{d2^{d-1}\Gamma(d/2)} \\ &= \frac{\pi e}{d/2-1}^{d/2-1/2} \frac{e^{-1/2}}{d2^{d-1/2}} (substitute\ \Gamma(d/2)) \\ &= 0 \cdot 0 (d \to \infty) \\ &= 0 \end{aligned} \tag{12}$$

Next, calculating ratio of the distance from the centre of the hypercube to one of the corners, divided by the perpendicular distance to ' one of the edges,

$$\begin{aligned} \frac{Dis(corner)}{Dis(edge)} &= \frac{\sqrt{\sum_{i=1}^{d} a^2}}{a} \\ &= \sqrt{d} \end{aligned} \tag{13}$$

It is straight-forward to see when $d \to \infty$, this value goes to ∞

1.3

$$\begin{aligned} f &= 1 - \frac{V_{a-\epsilon}}{V_a} \\ &= 1 - \frac{(a-\epsilon)^d}{a^d} \\ &= 1 - (1 - \frac{\epsilon}{a})^d \end{aligned} \tag{14}$$

When $d \to \infty$

$$\begin{aligned}\lim_{d\to\infty} f &= \lim_{d\to\infty} 1-(1-\frac{\epsilon}{a})^d \\ &= 1-0 \\ &= 1\end{aligned} \tag{15}$$

When $\epsilon/a - 0.01$,

$f_{d=2} = 1 - 0.99^2 = 0.0199$
$f_{d=10} = 1 - 0.99^10 = 0.0956$
$f_{d=1000} = 1 - 0.99^1000 = 0.99996$

When lies inside the radius a/2,

$f_{d=2} = 0.5^2 = 0.25$
$f_{d=10} = 1 - 0.5^10 = 0.095618$
$f_{d=1000} = 1 - 0.5^1000 \approx 0.00000$

1.4

In Problem 1.2, we have show that for a hypersphere with radius a and dimension d, the surface $S = S_d a^{d-1}$.

So,

$$\begin{aligned}\int_{shell} p(x)dx &= p(r)S_d r^{d-1}\epsilon \\ &= \rho(r)\epsilon\end{aligned} \tag{16}$$

Next, we calculate the single maximum point,

$$\begin{aligned}\rho(r) &= \frac{S_d r^{d-1}}{(2\pi\sigma^2)^{1/2}} exp(-\frac{r^2}{2\sigma^2}) \\ &\propto r^{d-1} exp(-\frac{r^2}{2\sigma^2}) = f(r)\end{aligned} \tag{17}$$

We want to find the value of r to make f(r) reach maximum value, so

$$\begin{aligned}\frac{df(r)}{dr} &= [(d-1)r^{d-2} - r^{d-1}\frac{r}{\sigma^2}]exp(-\frac{r^2}{2\sigma^2}) = 0 \\ \Rightarrow \hat{r} &= \sigma\sqrt{d-1} \\ \Rightarrow \hat{r} &\approx \sigma\sqrt{d}(d \gg 1)\end{aligned} \tag{18}$$

From equation (17), we have

$$\begin{aligned}\rho(r) &\propto r^{d-1} exp(-\frac{r^2}{2\sigma^2}) \\ &= exp((d-1)\ln r - \frac{r^2}{2\sigma^2})\end{aligned} \tag{19}$$

So,

$$\begin{aligned}\frac{\rho(\hat{r}+\epsilon)}{\rho(\hat{r})} &= exp^{(d-1)\ln(1+\frac{\epsilon}{\hat{r}}) - \frac{2\epsilon\hat{r}+\epsilon^2}{2\sigma^2}} \\ &= exp^{(d-1)(\frac{\epsilon}{\hat{r}} - \frac{\epsilon^2}{2\hat{r}^2}) - \frac{2\epsilon\hat{r}+\epsilon^2}{2\sigma^2}} \\ &= exp^{-\frac{\epsilon^2}{\sigma^2}}(substitute\ (d-1)\ with(\frac{\hat{r}}{\sigma})^2)\end{aligned} \tag{20}$$

2 Perceptron

2.1

$$\begin{aligned} y(x) &= 0 \\ \Rightarrow \quad \omega^T x - \theta &= 0 \\ \Rightarrow \quad \omega_1 x_1 + \omega_2 x_2 - \theta &= 0 \end{aligned} \tag{21}$$

The decision boundary line is $\omega^T x = \theta$

$$\begin{aligned} l &= \frac{ax_0+by_0+c}{\sqrt{a^2+b^2}} \\ &= \frac{c}{\sqrt{a^2+b^2}} \\ &= \frac{-\theta}{\|\omega\|} \\ &= \frac{-\omega_0}{\|\omega\|} \end{aligned} \tag{22}$$

2.2

(a)
Learning rule: If output is 1 and should be 0, then lower weights to active inputs and raise the threshold; If output is 0 and should be 1, then raise weights to active inputs and lower the threshold.

(b)

x_1	x_2	Output	Teacher	w_1	w_2	Threshold (θ)
1	1	1	0	-1	-1	1
0	1	0	1	-1	0	0
1	0	0	1	0	0	-1
1	1	1	0	-1	-1	0
0	1	0	1	-1	0	-1
1	1	1	0	-2	-1	0
1	0	0	1	-1	-1	-1

$w_1 = -1, w_2 = -1, \theta = -1$

(c)
It is not unique.

For example, $w_1 = -2, w_2 = -1, \theta = -2$

2.3

(i.)

The implementation is in the appendix.

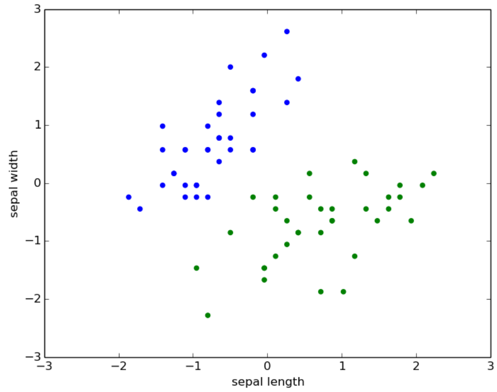
The advantage of the z score transformation is that it takes into account both the mean value and the variability in a set of raw data.

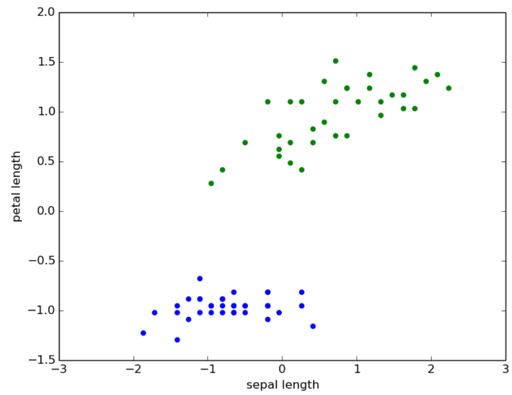
(ii.)

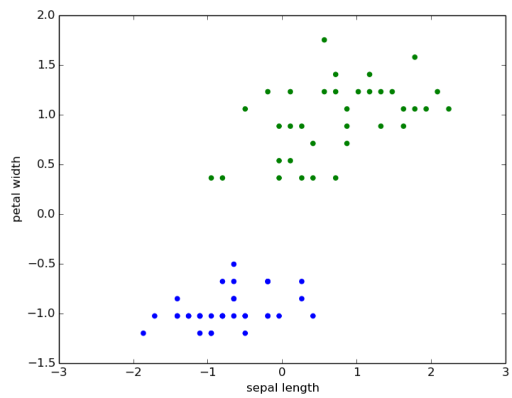
Blue colour stand for setosa, red colour stands for versicolor.

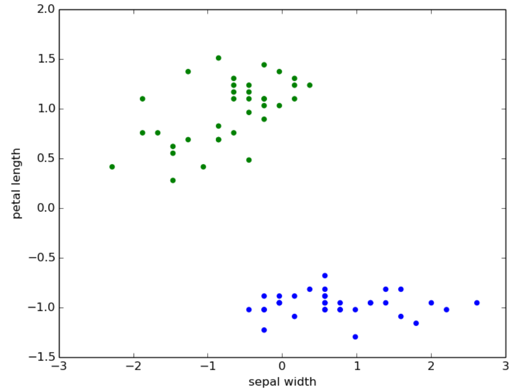
Classes linearly separable in each of the feature spaces. Because it is easy to find a boundary line in each figure below.

(iii.)
My source code is in the appendix.

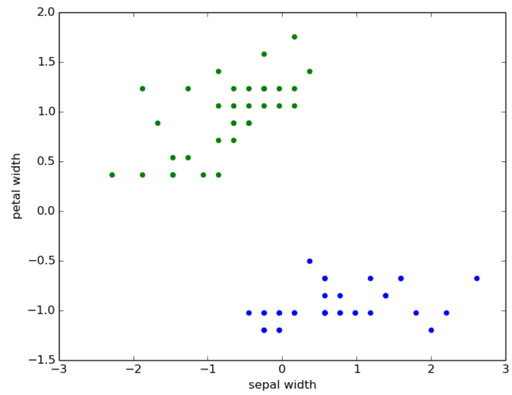

sepal width
sepal length


petal length
sepal length

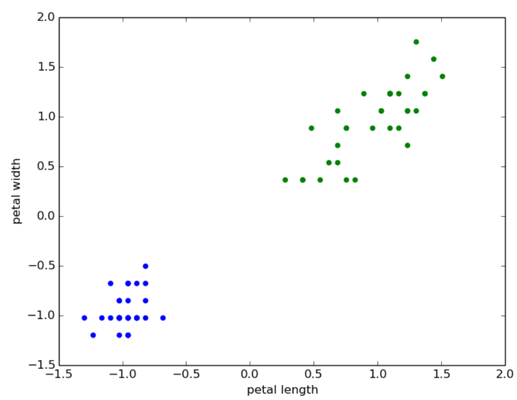

petal width
sepal length



petal length
sepal width



petal width
sepal width



petal width
petal length

(iv. v.)

learning_rate	error rate
.25	0.0%
.5	0.0%
1.0	0.0%
2.0	0.0%

For this test set, our model and prediction is very prefect. I run many times and each iteration always choose a random point and try different learning rate, all get 0.0% error rate. So I think the data must be linearly separable. That is to say, the test data is very good.

3 Logistic and Softmax Regression

3.1

$$\begin{aligned} -\frac{\partial E^n(\omega)}{\partial \omega_j} &= \frac{t^n}{g_w(x^n)}\frac{\partial g_w(x^n)}{\partial \omega_j} - \frac{(1-t^n)}{1-g_w(x^n)}\frac{\partial g_w(x^n)}{\partial \omega_j} \\ &= t^n g_w(-x^n)x_j^n - (1-t^n)g_w(x^n)x_j^n \\ &= x_j(-t^n(1-y^n) + (1-t^n)y^n) \\ &= (t^n - y^n)x_j \end{aligned} \tag{23}$$

3.2

First, $\ln(\frac{a}{b}) = \ln a - \ln b \Rightarrow \ln y_k^n = a_k^n - \ln \sum_{k'} exp(a_{k'}^n)$

Then,

$$\begin{aligned} -\frac{\partial E^n(\omega)}{\partial \omega_{jk}} &= \sum_{m=1}^{c} t_c^n \frac{\partial \ln y_m^n}{\partial \omega_{jk}} \\ &= \sum_{m=1}^{c} t_c^n \frac{\partial a_m^n - \ln \sum_{k'} exp(a_{k'}^n)}{\partial \omega_{jk}} \\ &= t_k^n(x_j^n - \frac{exp(a_k^n)x_j^n}{\sum_{k'} exp(a_{k'}^n)}) + \sum_{m \not\equiv k} t_m^n(-\frac{exp(a_k^n)x_j^n}{\sum_{k'} exp(a_{k'}^n)}) \\ &= t_k^n x_j^n - \frac{exp(a_k^n)x_j^n}{\sum_{k'} exp(a_{k'}^n)} \sum_{m=1}^{c} t_m^n \\ &= t_k^n x_j^n - \frac{exp(a_k^n)x_j^n}{\sum_{k'} exp(a_{k'}^n)} \\ &= (t_k^n - y_k^n)x_j^n \end{aligned} \tag{24}$$

3.3

The implementation is in appendix reference http://g.sweyla.com/blog/2012/mnist-numpy/.

3.4

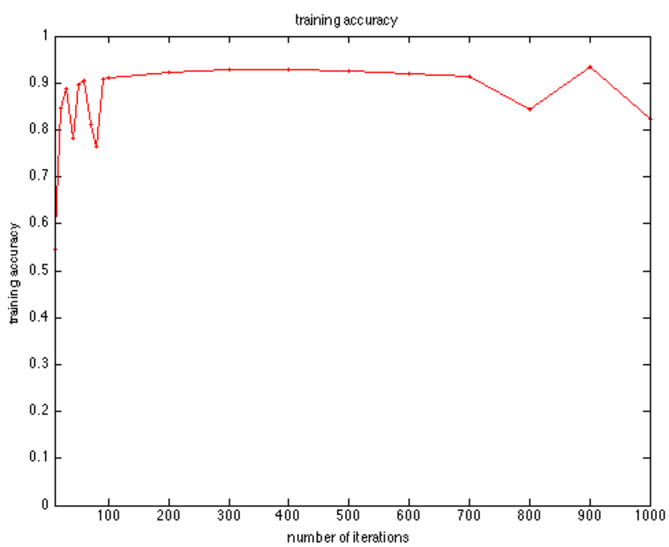
(a)

Number	Accuracy
0	98.3%
1	98.95%
2	97.3%
3	97.05%
4	97.4%
5	92.8%
6	97.8%
7	96.4%
8	91.9%
9	94.2%

(b)
The overall accuracy is 82.25 %

4 Softmax Regression via Gradient Descent

(a)



(b)

The test accuracy on the test set is : 87.9%

(c)
The test accuracy is higher than the one-vs-all logistic regression approach.

5 Appendix

5.1 Perceptron Source Code

```
import numpy
import random
import matplotlib.pyplot as plt

train_data = []
```

```python
label = ["sepal␣length", "sepal␣width", "petal␣length", "petal␣width"]
means = []
stds = []

class trainModel:

    def __init__ (self, learning_rate, dimension):
        self.learning_rate = learning_rate
        self.dimension = dimension
        self.threshold = 0
        self.weights = numpy.array([0 for i in range(dimension)])

    def train(self, train_data):
        self.ERROR = 0
        Itertimes = 0
        while Itertimes < 1000: #and not self.check(train_data):
            row = train_data[random.randint(0, len(train_data)-1)]
            teacher = row[self.dimension]
            x = row[:self.dimension]
            output = self.predict(x)
            self.weights = self.weights + self.learning_rate * (teacher - output) * x
            self.threshold = self.threshold - self.learning_rate * (teacher - output)
            Itertimes = Itertimes + 1

    def predict(self, x):
        v = sum(x[:self.dimension] * self.weights)

        if v >= self.threshold:
            return 1
        else:
            return 0

    def check(self, data):
        err = 0
        for row in data:
            v = self.predict(row)
            if v != row[self.dimension]:
                err = err + 1
        if err == self.ERROR and err < 3:
            return True
        else:
            self.ERROR = err
            return False

def read(filepath):
    file = open(filepath,"r")
    data = []
    while 1:
        line = file.readline()
        if not line:
            break
        line = line.strip('\n')
        row = line.split(',')
        for i in xrange(4):
                row[i] = float(row[i])
        row[4] = 1 if row[4] == "Iris-setosa" else 0
        data.append(row)
    return data
```

```
def zscore(data):
    mydata = numpy.vstack(data)
    for i in xrange(4):
        mean = numpy.mean(mydata.T[i])
        std = numpy.std(mydata.T[i])
        means.append(mean)
        stds.append(std)
        for j in xrange(len(mydata.T[i])):
            mydata[j][i] = (mydata[j][i] - mean)/std
    return mydata

def plot_data():
    setosa_data = []
    versicolor_data = []
    for ele in train_data:
        if ele[4] == 1:
            setosa_data.append(ele)
        else:
            versicolor_data.append(ele)
    setosa_data = numpy.vstack(setosa_data)
    versicolor_data = numpy.vstack(versicolor_data)

    # plot my data
    for i in xrange(3):
        for j in xrange(3 - i):
            plt.scatter(setosa_data.T[i], setosa_data.T[i+j+1], color = 'blue')
            plt.scatter(versicolor_data.T[i], versicolor_data.T[i+j+1], color = 'gree
            plt.xlabel(label[i])
            plt.ylabel(label[i+j+1])
            plt.show()

def Runtest(learning_rate):
    print "Running␣as␣Learning␣Rate␣:␣", learning_rate
    model = trainModel(learning_rate, 4)
    model.train(train_data)
    data = read("iris_test.data")
    mydata = numpy.vstack(data)
    for i in xrange(4):
        for j in xrange(len(mydata.T[i])):
            mydata[j][i] = (mydata[j][i] - means[i])/stds[i]

    miss = 0
    for ele in mydata:
        label = model.predict(ele)
        if label != ele[-1]:
            miss = miss+1

    print "ERROR␣rate␣:␣", miss * 100.0 / len(mydata), "%"


train_data = read("iris_train.data")
train_data = zscore(train_data)

Runtest(2)
Runtest(1)
Runtest(.5)
Runtest(.25)
```

5.2 Logistic and Softmax Regression Source Code

```
import os, struct
import numpy as np
from array import array as pyarray
from numpy import append, array, int8, uint8, zeros


def load_mnist(dataset="training", num = 20000, digits=np.arange(10), path="."):
    """
    Loads MNIST files into 3D numpy arrays

    Adapted from: http://abel.ee.ucla.edu/cvxopt/_downloads/mnist.py
    """

    if dataset == "training":
        fname_img = os.path.join(path, 'train-images.idx3-ubyte')
        fname_lbl = os.path.join(path, 'train-labels.idx1-ubyte')
    elif dataset == "testing":
        fname_img = os.path.join(path, 't10k-images.idx3-ubyte')
        fname_lbl = os.path.join(path, 't10k-labels.idx1-ubyte')
    else:
        raise ValueError("dataset must be 'testing' or 'training'")

    flbl = open(fname_lbl, 'rb')
    magic_nr, size = struct.unpack(">II", flbl.read(8))
    lbl = pyarray("b", flbl.read())
    flbl.close()

    fimg = open(fname_img, 'rb')
    magic_nr, size, rows, cols = struct.unpack(">IIII", fimg.read(16))
    img = pyarray("B", fimg.read())
    fimg.close()

    ind = [ k for k in range(size) if lbl[k] in digits ]
    N = num

    images = zeros((N, rows*cols+1), dtype=uint8)
    labels = zeros(N, dtype=int8)
    for i in range(num):
        feature  = img[ ind[i]*rows*cols : (ind[i]+1)*rows*cols ]
        feature.insert(0, 1)
        images[i] = array(feature)
        labels[i] = lbl[ind[i]]

    return images, labels


class Logistic:

    def __init__ (self, step, numlabel, dimension):
        self.step = step
        self.numlabel = numlabel
        self.weights = np.zeros(dimension)

    def sigmoid(self, data):
        return 1.0 / (1 + np.exp(-1 * (self.weights.dot(data))))
```

```
    def train(self, traindata, label):
        Itertime = 0
        teacher = np.zeros(len(label))
        for i in range(len(label)):
            if label[i] == self.numlabel:
                teacher[i] = 1

        while Itertime < 2000:
            output = 1.0 / (1 + np.exp(-1.0 * (self.weights.dot(traindata.T) ) ) )
            self.weights = self.weights + self.step * ( (teacher - output).dot(traind
            Itertime = Itertime + 1

    def predict(self, x):
        return self.sigmoid(x)


class softmax:

    def __init__(self, step, dimension, nclass, data, labels):
        self.step = step
        self.dimension = dimension
        self.nclass = nclass
        self.data = data
        self.labels = labels

        self.weights = np.zeros((nclass, dimension))
        self.teacher = np.zeros((nclass, len(labels)))
        for i in range(len(labels)):
            self.teacher[labels[i]][i] = 1


    def train(self, Round):
        Itertime = 0
        while Itertime < Round:
            print Itertime
            output = self.predict(self.data)
            self.weights = self.weights + self.step * (self.teacher - output).dot(se
            Itertime = Itertime + 1

    def predict(self, data):
        numerator = np.exp(self.weights.dot(data.T))
        denominator = np.sum(numerator, axis = 0)
        return numerator/denominator


def Lrun1():
    for i in range(10):
        model = Logistic(10e-8, i, 785)
        print "Classify number",i
        model.train(traindata, trainlabels)
        N = 0
        for index in range(len(testdata)):
            v = model.predict(testdata[index])
            if v >= 0.5 and testlabels[index] == i:
                N += 1
            if v < 0.5 and testlabels[index] != i:
                N += 1
        print "Classify number ",i," accuracy is : " , (N/2000.0)*100, "%"
```

```
def Lrun2():
    models = []
    for i in range(10):
        model = Logistic(10e-9, i, 785)
        model.train(traindata, trainlabels)
        print "trainging", i, "finished"
        models.append(model)
        N = 0

    for index in range(len(testdata)):
        data = testdata[index]
        m = -1.0
        ind = -1
        for j in range(10):
            v = models[j].predict(data)
            if v > m:
                m = v
                ind = j
        if testlabels[index] == ind:
            N = N + 1
    print "Overall Accuracy is : ", (N/2000.0)*100, "%"

def Srun1():
    for i in range(2,11):
        model = softmax(10e-9, 785, 10, traindata, trainlabels)
        model.train(100*i)
        output = model.predict(traindata)
        output = np.argmax(output, axis = 0)
        N = 0
        for j in range(len(trainlabels)):
            if output[j] == trainlabels[j]:
                N = N + 1
        print "Iteration : ", i*100, " Accuracy is : ", (N/20000.0)*100, "%"

def Srun2():

    model = softmax(10e-9, 785, 10, traindata, trainlabels)
    model.train(500)
    output = model.predict(testdata)
    output = np.argmax(output, axis = 0)
    N = 0
    for j in range(len(testlabels)):
        if output[j] == testlabels[j]:
            N = N + 1
    print " Accuracy is : ", (N*1.0/len(testlabels))*100, "%"




# read data
traindata, trainlabels = load_mnist('training', 20000)
testdata, testlabels = load_mnist('testing', 2000)

# run logistic
# Lrun1()
# Lrun2()

# run softmax
# Srun1()
```

```
Srun2()
```